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Sharp Inequalities for Weyl Operators and Heisenberg Groups

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Introduction

We study inequalities in harmonic analysis in the context of non-commutative non-compact locally compact groups. Our main result is the determination of the best constant in the Hausdorff-Young inequality for Heisenberg groups. We also obtain the somewhat surprising fact that the resulting sharp inequality does not admit any extremal functions. These results are obtained after a detailed study of the operators which occur in the Fourier decomposition of the regular representation of the Heisenberg groups. These are called Weyl operators and are of independent interest. We also obtain bounds for the best constants in the Hausdorff-Young inequality and in Young's inequality on semi-direct product groups, including non-unimodular groups. In particular, for real nilpotent groups of dimension n those best constants are shown to be dominated by the corresponding best constants for $\mathbb{R}^n$. Although some of our preliminary lemmas are valid for all values of $p \in (1, 2)$ the methods we use for our main results require that p belong to the sequence $4/3, 6/5, 8/7, \ldots$, i.e. that p', the conjugate index, be an even integer.

The contents of this paper are as follows. In Section 1 we discuss Weyl operators and determine, for p' even, the best constant in a Hausdorff-Young type inequality (Theorem 1). We also show the non-existence of extremal functions for this inequality. In Section 2 we prove some general results for locally compact groups which includes a form of Young's inequality for convolution appropriate for non-unimodular groups. This is applied to arbitrary semi-direct products. Then using a duality argument which relates the inequalities of Young and of Hausdorff-Young we obtain bounds for the Hausdorff-Young inequality (Theorem 2) on unimodular semi-direct product groups (for p' even). An interesting consequence of these results is that for a connected simply connected real nilpotent Lie group of dimension n, the best constants in the inequalities of Young and Hausdorff-Young are dominated by the corresponding best constants for $\mathbb{R}^n$. In Section 3 we show (Theorem 3), using the theory of Weyl operators developed

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in Section 1, that for the Heisenberg groups, the best constant for the Hausdorff-Young inequality is the same as the corresponding one for $\mathbb{R}^n$ (p' even) but that there are no extremal functions. This can be contrasted with the classical case $\mathbb{R}^n$ where Gaussian functions are extremal functions. In Section 4 we use the methods of the present paper to improve on a previous estimate for the "$ax+b$" group (Theorem 4).

We now set down some of the notation which will be used throughout. If $\mathscr{H}$ is a complex Hilbert space, $\mathscr{B}(\mathscr{H})$ will denote the Banach space of bounded linear operators on $\mathscr{H}$, with the operator norm. Our inequalities will be stated in terms of the Banach spaces $C_r(\mathscr{H})$, $1 \leqq r < \infty$, consisting of elements T of $\mathscr{B}(\mathscr{H})$ for which

$$\|T\|_r = \|T\|_{C_r(\mathscr{H})} = [\operatorname{trace}(T^*T)^{r/2}]^{1/r} < \infty . \tag{0.1}$$

We let $\mathscr{F}$ be the Fourier transform on $\mathbb{R}^n$ defined by

$$\mathscr{F}f(y) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot y} dx, \quad \text{for} \quad f \in L^1(\mathbb{R}^n) \tag{0.2}$$

where dx denotes n-dimensional Lebesgue measure and $x \cdot y$ is the Euclidean inner product. We also denote by $\mathscr{F}$ the extension of the Fourier transform to the Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 \leqq p \leqq 2$, given by the Plancherel theorem and the Hausdorff-Young theorem. Thus by results of Babenko (for p' an even integer) [1] and Beckner (in general) [2],

$$\|\mathscr{F}f\|_{L^{p'}(\mathbb{R}^n)} \leqq A_p^n \|f\|_{L^p(\mathbb{R}^n)}, \quad \frac{1}{p} + \frac{1}{p'} = 1, 1 \leqq p \leqq 2 . \tag{0.3}$$

Also, by results of Beckner [2] and Brascamp and Lieb [3],

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leqq (A_{r'} A_p A_q)^n \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} , \tag{0.4}$$

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \ r, p, q \in (1, \infty) .$$

In (0.3) and (0.4) and throughout this paper

$$A_m = \left(\frac{m^{1/m}}{m'^{1/m'}}\right)^{\frac{1}{2}}, \quad \frac{1}{m} + \frac{1}{m'} = 1 . \tag{0.5}$$

As we are concerned with the best possible constants in our inequalities, our methods, like those used in the proofs of (0.3) and (0.4) do not involve the Riesz Convexity theorem.

Finally, if X is a locally compact topological space, $\mathscr{K}(X)$ denotes the continuous complex valued functions on X with compact support.

1. Weyl Operators

For $x \in \mathbb{R}^n$, consider the unitary operators $U(x)$ and $V(x)$ on $L^2(\mathbb{R}^n)$ defined by

$$(U(x)f)(z) = f(x+z), \quad f \in L^2(\mathbb{R}^n), \quad z \in \mathbb{R}^n, \tag{1.1}$$

$$(V(x)f)(z) = e^{2\pi i x z} f(z), \quad f \in L^2(\mathbb{R}^n), \quad z \in \mathbb{R}^n . \tag{1.2}$$

It is well known and easy to verify that U and V are each n-parameter unitary groups which are unitarily equivalent (as groups) to each other via the Fourier Plancherel transform, i.e.

$$\mathscr{J}V(x)\mathscr{J}^{-1}=U(x), \quad x\in\mathbb{R}^n, \tag{1.3}$$

and

$$\mathscr{J}U(x)\mathscr{J}^{-1}=V(-x), \quad x\in\mathbb{R}^n \tag{1.4}$$

and which satisfy the following commutation relation:

$$U(x)V(y)=e^{2\pi ixy}V(y)U(x), \quad x,y\in\mathbb{R}^n. \tag{1.5}$$

For a measurable function F on $\mathbb{R}^{2n}$, the Weyl operator corresponding to F is the operator on $L^2(\mathbb{R}^n)$

$$K_F=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(x,y)U(x)V(y)dxdy. \tag{1.6}$$

The operator K_F certainly exists as an element of $\mathscr{B}(L^2(\mathbb{R}^n))$ if $F\in L^1(\mathbb{R}^{2n})$ and in fact

$$\|K_F\|_{\mathscr{B}(L^2(\mathbb{R}^n))}\leqq\|F\|_{L^1(\mathbb{R}^{2n})}. \tag{1.7}$$

A routine calculation shows that K_F is an integral operator

$$(K_Ff)(z)=\int_{\mathbb{R}^n}k_F(z,x)f(x)dx, \quad f\in L^2(\mathbb{R}^n), \quad z\in\mathbb{R}^n, \tag{1.8}$$

with kernel k_F given by

$$k_F(z,x)=F(x-z,\hat{x})=\int_{\mathbb{R}^n}F(x-z,y)e^{2\pi ixy}dy, \quad z,x\in\mathbb{R}^n. \tag{1.9}$$

Therefore

$$\|K_F\|_{C_2(L^2(\mathbb{R}^n))}=\|k_F\|_{L^2(\mathbb{R}^{2n})}=\|F\|_{L^2(\mathbb{R}^{2n})}. \tag{1.10}$$

If we apply standard interpolation theory (e.g. Reed and Simon [11, p. 44]) to (1.7) and (1.10) we obtain

$$\|K_F\|_{C_{p'}(L^2(\mathbb{R}^n))}\leqq\|F\|_{L^p(\mathbb{R}^{2n})}, \quad F\in L^p(\mathbb{R}^{2n}), \tag{1.11}$$

where $p\in(1,2)$ and $\frac{1}{p}+\frac{1}{p'}=1$.

Thus for each $p\in(1,2)$ and $n=1,2,\ldots$, if we let

$$w_{p,n}=\sup\{\|K_F\|_{C_{p'}(L^2(\mathbb{R}^n))}:\|F\|_{L^p(\mathbb{R}^{2n})}=1\},$$

(1.11) states that $w_{p,n}\leqq 1$. It has been pointed out (Russo [14]) that $w_{p,n}<1$ for all p and n. The following theorem gives a complete analysis of the inequality

$$\|K_F\|_{p'}\leqq w_{p,n}\|F\|_p, \tag{1.12}$$

for certain values of p.

Theorem 1. *Let $p\in(1,2)$ be of the form $p=2k/(2k-1)$ for some integer $k\geqq 2$. Then*

(i) $\|K_F\|_{C_{p'}(L^2(\mathbb{R}^n))}\leqq A_p^{2n}\|F\|_{L^p(\mathbb{R}^{2n})}$, $F\in L^p(\mathbb{R}^{2n})$;

(ii) A_p^{2n} *is the smallest constant in* (i), *i.e.* $w_{p,n}=A_p^{2n}$;

(iii) $A_p^{2n}=\sup_{a,b>0}\{\|K_{F_{a,b}}\|_{p'}/\|F_{a,b}\|_p\}$ *where* $F_{a,b}(x,y)=\exp\{-a\|x\|^2-b\|y\|^2\}$ *for* x, $y\in\mathbb{R}^n$;

(iv) *There are no extremal functions for* (i), *i.e. if equality holds in* (i) *for some* $F\in L^p(\mathbb{R}^{2n})$, *then* $F=0$ *a.e.*

Remark 1.1. If we think of the map $F\to K_F$ as an operator valued analog of the Fourier transform on $\mathbb{R}^{2n}$, then, in view of (0.3), (i) and (ii) are not surprising. However in (0.3) equality holds for any Gaussian function so that extremal functions exist for (0.3).

Before going into the proof of Theorem 1 we give some elementary properties of Weyl operators. For $z=(x,y)\in\mathbb{R}^{2n}$ let $W(z)=U(x)V(y)$. Then (1.5) implies that, with $z_i=(x_i,y_i)$, $x_i,y_i\in\mathbb{R}^n$, $i=1,2$

$$W(z_1+z_2)=e^{2\pi i x_2\cdot y_1}W(z_1)W(z_2) \tag{1.13}$$

and therefore W is a projective representation of the Abelian group $\mathbb{R}^{2n}$ corresponding to the 2-cocycle $\omega:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{T}$ defined by

$$\begin{aligned}&\omega(z_1,z_2)=\exp\{2\pi i x_2\cdot y_1\}\,,\\ &z_i=(x_i,y_i)\in\mathbb{R}^{2n},\quad i=1,2\,.\end{aligned} \tag{1.14}$$

We can turn $L^1(\mathbb{R}^{2n})$ into an involutive non-commutative Banach algebra by introducing a product $F\underset{\omega}{*}G$ and involution $F^{\#}$ determined by the rules

$$K_FK_G=K_{F\underset{\omega}{*}G},\quad (K_F)^*=K_{F^{\#}}. \tag{1.15}$$

From (1.6) and (1.15) we obtain explicit formulae for these operations, namely

$$F^{\#}(x,y)=\bar{F}(-x,-y)e^{-2\pi i x\cdot y}, \tag{1.16}$$

$$F\underset{\omega}{*}G(x,y)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(x-z,y-w)G(z,w)e^{-2\pi i z\cdot(y-w)}dz\,dw. \tag{1.17}$$

In (1.17) we note that $e^{-2\pi i z\cdot(y-w)}=\omega(z_1,z_2)$ where $z_1=(z-x,w-y)$ and $z_2=(z,w)$ and this explains our notation $F\underset{\omega}{*}G$ since we could rewrite (1.17) as

$$F\underset{\omega}{*}G(\alpha)=\int_{\mathbb{R}^{2n}}F(\alpha-\beta)G(\beta)\omega(\beta-\alpha,\beta)d\beta\,. \tag{1.18}$$

From (1.17) we get

$$|F\underset{\omega}{*}G|\leqq|F|*|G| \tag{1.19}$$

where the $*$ on the right side of (1.19) denotes the usual convolution for functions on $\mathbb{R}^{2n}$.

We proceed now to the proof of Theorem 1. Fix an integer $k\geqq 2$ and let $p=\dfrac{2k}{(2k-1)}$ and let $F\in L^1(\mathbb{R}^{2n})\cap L^2(\mathbb{R}^{2n})$. For $1\leqq j\leqq k$, let $F_j=F$ if j is odd and let

$F_j = F^{\#}$ if j is even. Then let $G^{(k)} = F_k \underset{\omega}{*} F_{k-1} \underset{\omega}{*} \cdots \underset{\omega}{*} F_1$. We have $p' = 2k$ and thus

$$\begin{aligned}\|K_F\|_{p'}^{p'} &= \|K_F\|_{2k}^{2k} = \operatorname{tr}[((K_F)^* K_F)^k] = \operatorname{tr}[(K_{G^{(k)}})^* K_{G^{(k)}}]\\ &= \|K_{G^{(k)}}\|_{C_2(L^2(\mathbb{R}^n))}^2 = \|G^{(k)}\|_{L^2(\mathbb{R}^{2n})}^2 = \|F_k \underset{\omega}{*} \cdots \underset{\omega}{*} F_1\|_2^2\\ &\leqq \| |F_k| * \cdots * |F_1| \|_2^2 = \| |F_k|^\wedge |F_{k-1}|^\wedge \cdots |F_1|^\wedge \|_2^2\end{aligned}$$

$$\leqq (\| |F_k|^\wedge \|_{p'} \cdots \| |F_1|^\wedge \|_{p'})^2 \left(\text{since } k \cdot \frac{1}{p'} = \frac{1}{2}\right) \leqq \left(\prod_{j=1}^{k} A_p^{2n} \| |F_j| \|_p\right)^2$$

[by (0.3)] $= A_p^{4kn} \|F\|_p^{2k}$. Therefore $\|K_F\|_{p'} \leqq A_p^{2n} \|F\|_p$ and (i) is proved. To prove (iii) we begin by observing that by repeated use of (1.17) we can write

$$G^{(k)}(z_k, w_k) = \int_{\mathbb{R}^{n(2k-2)}} \left\{\prod_{j=1}^{k} F_j(z_j - z_{j-1}, w_j - w_{j-1})\right\} \cdot \exp\left\{-2\pi i \sum_{j=1}^{k-1} z_j(w_{j+1} - w_j)\right\} dz dw \tag{1.20}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_{k-1}) \in \mathbb{R}^{n(k-1)}$ and $\mathbf{w} = (w_1, \ldots, w_{k-1}) \in \mathbb{R}^{n(k-1)}$ and we have set $z_0 = w_0 = 0$. Then using (1.20) we have

$$\|G^{(k)}\|_2^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G^{(k)}(z_k, w_k) \bar{G}^{(k)}(z_k, w_k) dz_k dw_k \tag{1.21}$$

$$\begin{aligned}&= \int_{\mathbb{R}^{n(4k-2)}} \left\{\prod_{j=1}^{k} F_j(z_j - z_{j-1}, w_j - w_{j-1})\right\} \exp\left\{-2\pi i \sum_{j-1}^{k-1} z_j(w_{j+1} - w_j)\right\}\\ &\quad \cdot \left\{\prod_{j=1}^{k} \bar{F}_j(z'_j - z'_{j-1}, w'_j - w'_{j-1})\right\}\\ &\quad \cdot \exp\left\{2\pi i \sum_{j=1}^{k-1} z'_j(w'_{j+1} - w'_j)\right\} dz dw dz' dw' dz_k dw_k\end{aligned}$$

where

$$z_k = z'_k,\ w_k = w'_k,\ \mathbf{z} = (z_1, \ldots, z_{k-1}),\ \mathbf{w} = (w_1, \ldots, w_{k-1}),\ \mathbf{z}' = (z'_1, \ldots, z'_{k-1})$$

and $\mathbf{w}' = (w'_1, \ldots, w'_{k-1})$. We now consider the Gaussian function $F_{a,b,c}$ defined by

$$F_{a,b,c}(x, y) = \exp\{-a\|x\|^2 - b\|y\|^2 + icx \cdot y\}, \qquad x, y \in \mathbb{R}^n, \tag{1.22}$$

$a > 0$, $b > 0$, $c \in \mathbb{R}$. We note that $F^{\#}_{a,b,c} = F_{a,b,-(c+2\pi)}$. We then use $F_{a,b,c}$ for our F in (1.21), make the change of variables $z_j \to \frac{z_j}{\sqrt{a}}$, $w_j \to \frac{w_j}{\sqrt{b}}$ for $1 \leqq j \leqq k$ and $z'_j \to \frac{z'_j}{\sqrt{a}}$, $w'_j \to \frac{w'_j}{\sqrt{b}}$ for $1 \leqq j \leqq k$. Then $\|G^{(k)}\|_2^2 = (ab)^{\frac{(1-2k)}{2}\frac{n}{2}} I(a, b, c)$ where $I(a, b, c)$ is an integral which by dominated convergence will approach $I(1, 1, 0) = \|F_{1,1,0} * \cdots * F_{1,1,0}\|_2^2$ (k fold convolution) as $ab \to \infty$ with c fixed. As pointed out by Beckner [2] and Brascamp and Lieb [3] or as can be seen by an explicit calculation

$$\|F_{1,1,0} * \cdots * F_{1,1,0}\|_2^2 = (A_p^{2kn} \|F_{1,1,0}\|_p^k)^2.$$

Now

$$\|F_{a,b,c}\|_p = \left(\frac{\pi}{p}\right)^{n/2p} \Big/ (ab)^{n/4p} \tag{1.23}$$

so in particular $\|F_{1,1,0} * \ldots * F_{1,1,0}\|_2^2 = A_p^{4kn}\left(\frac{\pi}{p}\right)^{kn/p}$. Thus

$$\begin{aligned}
\|K_{F_{a,b,c}}\|_{p'}^{p'} / \|F_{a,b,c}\|_p^{p'} &= \|G^{(k)}\|_2^2 \left(\frac{p}{\pi}\right)^{p'n/2p} (ab)^{p'n/4p} \\
&= (ab)^{\left(\frac{1-2k}{2}\right)\frac{n}{2}} I(a,b,c)\left(\frac{p}{\pi}\right)^{p'n/2p} (ab)^{p'n/4p} \\
&= I(a,b,c)\left(\frac{p}{\pi}\right)^{p'n/2p} \\
&\cdot \left(\text{since}\, \frac{p'n}{4p} + \frac{(1-2k)}{2}\cdot\frac{n}{2} = 0\right) \\
&\to A_p^{4kn}\left(\frac{\pi}{p}\right)^{kn/p}\left(\frac{p}{\pi}\right)^{p'n/2p} = A_p^{4kn} = A_p^{2np'} .
\end{aligned}$$

This completes the proof of (iii) and hence of (ii).

Remark 1.2. For later use let us note that we have proved the following lemma:

Lemma 1.3. *Let $F_{a,b,c}(x,y) = \exp\{-a\|x\|^2 - b\|y\|^2 + icxy\}$, and let $p = 2k/(2k-1)$. Then for any $\epsilon > 0$, there exists a positive number M such that*

$$\|K_{F_{a,b,c}}\|_{p'} \geqq (A_p^{2n} - \epsilon)\|F_{a,b,c}\|_p$$

whenever $ab \geqq M$, and c is fixed.

Remark 1.4. To give an example of what the integral (1.21) looks like an explicit calculation shows that

$$\frac{\|K_{F_{a,b,0}}\|_4}{\|F_{a,b,0}\|_{4/3}} = \left[A_{4/3}^2\left(\frac{ab}{\pi^2+ab}\right)^{\frac{1}{8}}\right]^n \to A_{4/3}^{2n} \quad \text{as} \quad ab \to \infty .$$

We now complete the proof of Theorem 1 by proving (iv). For convenience we use the notation $Y_{p,q}^{(r)}$ for the best constant in Young's inequality for $\mathbb{R}^{2n}$. Thus, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and according to (0.4)

$$Y_{p,q}^{(r)} = (A_{r'}A_pA_q)^{2n} . \tag{1.24}$$

Suppose now that $F \in L^p(\mathbb{R}^{2n})$, with $p = \frac{2k}{(2k-1)}$ and that equality holds in (i). We shall call F an extremal function and show that $F = 0$ a.e. The formula $\|K_F\|_{p'}^{p'} = \|G^{(k)}\|_2^2$, used previously for $F \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})$, is valid for $F \in L^p(\mathbb{R}^{2n})$ because the integral defining $G^{(k)}$ is dominated by $|F_k| * \ldots * |F_1|$ which is in $L^2(\mathbb{R}^{2n})$ by

Young's inequality since $F_j \in L^p(\mathbb{R}^{2n})$, $1 \leqq j \leqq k$. We now apply (1.19) and (0.4) to obtain

$$\|G^{(k)}\|_2 \leqq \| \, |F_k| * |G^{(k-1)}| \, \|_2 \leqq Y^{(2)}_{p,q_1} \|F_k\|_p \|G^{(k-1)}\|_{q_1} \tag{1.25}$$

where $\frac{1}{p} + \frac{1}{q_1} = 1 + \frac{1}{2}$ so that $q_1 = \frac{2k}{k+1}$.

Similarly

$$\|G^{(k-1)}\|_{q_1} \leqq \| \, |F_{k-1}| * |G^{(k-2)}| \, \|_{q_1} \leqq Y^{(q_1)}_{p,q_2} \|F_{k-1}\|_p \|G^{(k-2)}\|_{q_2} \tag{1.26}$$

where $\frac{1}{p} + \frac{1}{q_2} = 1 + \frac{1}{q_1}$ so that $q_2 = \frac{2k}{k+2}$.

Continuing in this way we are led to

$$\|G^{(k)}\|_2 \leqq Y^{(q_0)}_{p,q_1} Y^{(q_1)}_{p,q_2} \dots Y^{(q_{k-3})}_{p,q_{k-2}} \|F\|_p^{k-2} \|F^\# \underset{\omega}{*} F\|_{q_{k-2}} \tag{1.27}$$

where $q_0 = 2$, $q_j = \frac{2k}{k+j}$ for $1 \leqq j \leqq k-1$. In particular $q_{k-1} = p$ and

$$\|F^\# \underset{\omega}{*} F\|_{q_{k-2}} \leqq \| \, |F^\#| * |F| \, \|_{q_{k-2}} \leqq Y^{(q_{k-2})}_{p,p} \|F\|_p^2 \, . \tag{1.28}$$

Combining (1.27) and (1.28) we have

$$\|G^{(k)}\|_2 \leqq \left(\prod_{j=1}^{k-1} Y^{(q_{j-1})}_{p,q_j} \right) \|F\|_p^k \, . \tag{1.29}$$

But $\prod_{j=1}^{k-1} Y^{(q_{j-1})}_{p,q_j} = A_p^{2kn}$ [by applying (1.24) the product telescopes].

Since F is extremal we have equality in (1.25)–(1.29). In particular $\|F^\# \underset{\omega}{*} F\|_{q_{k-2}} = \| \, |F^\#| * |F| \, \|_{q_{k-2}}$ and by (1.19) we have

$$|F^\# \underset{\omega}{*} F| = |F^\#| * |F| \text{ a.e.} \tag{1.30}$$

If we write out the integrals in (1.30) we see that we can apply Hewitt and Stromberg [6, (12.29)] to obtain a null set $N \subset \mathbb{R}^{2n}$ and for each $(x, y) \in \mathbb{R}^{2n} - N$ a null set $N_{(x,y)}$ and a real number $\theta(x, y)$ such that

$$\begin{aligned} &F^\#(x-z, y-w) F(z, w) e^{-2\pi i z(y-w)} \\ &= e^{i\theta(x,y)} |F^\#(x-z, y-w) F(z, w)| \\ &\quad \text{for all} \quad (z, w) \in \mathbb{R}^{2n} - N_{(x,y)} \, . \end{aligned} \tag{1.31}$$

Our aim is to show that $F = 0$ a.e. For this purpose let S_F be the set where F is not zero. We shall show that S_F is a null set. We can write $F = |F| U$ where $|U| = 1$ and then cancel all non-zero terms in (1.31). The result is

$$\bar{U}(z-x, w-y) U(z, w) = e^{-2\pi i x(w-y) + i\theta(x,y)} \tag{1.32}$$

for $(x, y) \in \mathbb{R}^{2n} - N$, and $(z, w) \in (\mathbb{R}^{2n} - N_{(x,y)}) \cap S_F \cap ((x, y) + S_F)$.

Consider now the function $\tilde{F}(x, y) = \bar{F}(y, -x)$, $x, y \in \mathbb{R}^n$. Using the properties of Weyl operators it is easily seen that $(\mathcal{J} K_F \mathcal{J}^{-1})^* = K_{\tilde{F}}$. This shows that if F is an extremal function, then so is $\tilde{F}$. We note that $\tau(S_{\tilde{F}}) = S_F$ where $\tau(x, y) = (y, -x)$ and that we can write $\tilde{F} = |\tilde{F}| V$ where $|V| = 1$ and V can be chosen such that

$$V(x, y) = \bar{U}(y, -x) \quad \text{for all} \quad (x, y) \in \mathbb{R}^{2n} \, . \tag{1.33}$$

Since (1.32) is valid for any extremal function we apply it to our $\tilde{F}$ to get null sets N' and $N'_{(x,y)}$ such that

$$\bar{V}(z-x, w-y)V(z,w) = e^{-2\pi i x(w-y)+i\psi(x,y)} \tag{1.34}$$

for $(x,y)\in\mathbb{R}^{2n}-N'$ and $(z,w)\in(\mathbb{R}^{2n}-N'_{(x,y)})\cap S_{\tilde{F}}\cap((x,y)+S_{\tilde{F}})$.

We now combine (1.32), (1.33) and (1.34). The result is

$$e^{-2\pi i x(w-y)+i\theta(x,y)} = e^{-2\pi i y(z-x)-i\psi(-y,x)} \tag{1.35}$$

for $(x,y)\in(\mathbb{R}^{2n}-N)\cap\tau(\mathbb{R}^{2n}-N')$ and

$$(z,w)\in(\mathbb{R}^{2n}-N_{(x,y)})\cap\tau(\mathbb{R}^{2n}-N'_{(-y,x)})\cap S_F\cap((x,y)+S_F)\equiv M_{(x,y)}\,.$$

We claim that $M_{x,y}$ is a null set for a.e. (x,y). To see this fix $(x,y)\in(\mathbb{R}^{2n}-N)\cap\tau(\mathbb{R}^{2n}-N')\cap(\mathbb{R}^{2n}-\{(0,0)\})$. Then (1.35) implies that

$$M_{(x,y)}\subset\bigcup_{k=-\infty}^{\infty}\{(z,w): x\cdot w+y\cdot z=k+C_{x,y}\}$$

with $C_{x,y}\in\mathbb{R}$. Since (x,y) is a non-zero vector in $\mathbb{R}^{2n}$, each set in the union has $2n$-dimensional measure zero and the same holds for $M_{(x,y)}$. The fact that S_F is a null set is now a simple consequence of the following lemma.

Lemma 1.5. *Let B be a Lebesgue measurable set in $\mathbb{R}^{\ell}$ such that $B\cap(v+B)$ is a null set for a.e. $v\in\mathbb{R}^{\ell}$. Then B is a null set.*

Proof. We can reduce to the case that B is bounded. Then $\chi_B\in L^1(\mathbb{R}^{\ell})$ and the assumption is equivalent to $\chi_{-B}*\chi_B=0$ a.e. Since $(\chi_{-B})^\wedge=\overline{(\chi_B)^\wedge}$ we have $|(\chi_B)^\wedge|^2=0$ and therefore $\chi_B=0$, so B is a null set.

This completes the proof of Theorem 1.

2. Inequalities on Locally Compact Groups

Let G be a locally compact group and denote a right Haar measure on G by dx or d_Rx. Convolution and norms are defined with respect to d_Rx as follows:

$$f*g(x)=\int_G f(xy^{-1})g(y)d_Ry\,, \tag{2.1}$$

$$\|f\|_p=\left\{\int_G |f(x)|^p d_Rx\right\}^{\frac{1}{p}},\quad 1\leqq p<\infty\,. \tag{2.2}$$

The symbol Δ, or Δ_G when necessary, denotes the modular function of G. It follows from a careful application of Hölder's inequality that

$$\|f\Delta^{-1/q'}*g\|_q\leqq\|f\|_1\|g\|_q,\quad 1\leqq q\leqq\infty\,, \tag{2.3}$$

$$\|f\Delta^{-1/q'}*g\|_\infty\leqq\|f\|_{q'}\|g\|_q,\quad 1\leqq q\leqq\infty \tag{2.4}$$

where as usual $\frac{1}{q}+\frac{1}{q'}=1$. Thus for fixed $g\in L^q(G)$, the map $f\to f\Delta^{-1/q'}*g$ on simple functions is of type $(1,q)$ with norm $\leqq\|g\|_q$ and of type (q',∞) of norm $\leqq\|g\|_q$. By the Riesz-Thorin Theorem this map is of type (p_t,q_t) where $0\leqq t\leqq 1$,

$\frac{1}{p_t}=\frac{1-t}{1}+\frac{t}{q'}, \frac{1}{q_t}=\frac{1-t}{q}+\frac{t}{\infty}$. If we set $r=q_t$ and $p=p_t$ we obtain a generalization of Young's inequality which we state as a lemma.

Lemma 2.1. *Let G be a locally compact group with modular function Δ. If $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$ and $p,q,r\in[1,\infty]$ and if norms and convolution are defined relative to a right Haar measure, then*

$$\left\|f\Delta^{-\frac{1}{q'}}*g\right\|_r \leqq \|f\|_p \|g\|_q . \tag{2.5}$$

Remark 2.2. There is a corresponding inequality which uses left Haar measure and which takes the form $\|f*g\Delta^{1/p'}\|_r \leqq \|f\|_p\|g\|_q, \frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. This can be proved in the same way and in this case the convolution is given by $f*g(x) = \int_G f(y)g(y^{-1}x)d_Ly$ where d_Lx is a left Haar measure. In this paper we shall always use right Haar measures.

For any locally compact group G and $p,q,r\in[1,\infty]$ we define (using right Haar measure) when $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$,

$$Y_{p,q}^{(r)}(G)=\sup_{\substack{f\neq 0\\ g\neq 0}} \frac{\|f\Delta^{-1/q'}*g\|_r}{\|f\|_p\|g\|_q} \tag{2.6}$$

Thus $Y_{p,q}^{(r)}(G)$, which we shall sometimes write as $Y_{p,q}$ is the best constant in (2.5) and is $\leqq 1$. By repeated application of (2.5) we can obtain

Corollary 2.3. *Let G be a locally compact group. If $(k-1)+\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_k}$ for some $k\geqq 2$ and $r,p_1,\cdot,p_k\in[1,\infty]$ then*

$$\left\|\mathop{*}_{i=1}^{k} f_i\Delta^{-\sum_{j=i+1}^{k}\frac{1}{p'_j}}\right\|_r \leqq \prod_{i=1}^{k}\left\|f_i\right\|_{p_i} \tag{2.7}$$

We then define $Y^{(r)}_{p_1,p_2,\cdot,p_k}(G)$ to be the best constant in (2.7). The special case of (2.7) in which $p_1=p_2=\ldots=p_k=p=\frac{2k}{2k-1}$, and thus $p'=2k$, will be used in Section 4. It states that

$$\left\|f_1\Delta^{-\frac{(k-1)}{2k}}*f_2\Delta^{-\frac{(k-2)}{2k}}*\ldots*f_{k-1}\Delta^{-\frac{1}{2k}}*f_k\right\|_2 \leqq Y^{(2)}_{p,\ldots,p}(G)\prod_{i=1}^{k}\|f_i\|_p \tag{2.8}$$

where $p=\frac{2k}{(2k-1)}$.

We shall now consider Young's inequality (Lemma 2.1) for semi-direct products. For direct products, the inequality we obtain in Lemma 2.4 in easily seen to be an equality. We conjecture this to be also the case for semi-direct products.

For certain indices and certain groups this is a consequence of Lemma 2.6 and Theorem 3.

Consider now a pair X, A of locally compact groups together with a homomorphism $a \to \tau_a$ of A into the group of automorphisms of X. We suppose that $(x,a) \to \tau_a(x)$ is continuous from $X \times A$ to X so that the semi-direct product group $G = X \circledS A$ becomes a locally compact topological group with the product topology and multiplication $(x,a)(y,b) = (x\tau_a(y), ab)$. We shall write $a(x)$ instead of $\tau_a(x)$. We recall (Hewitt and Ross [5, (15.29)]) that a right Haar measure on G is the product of the right Haar measures $d_R x$ on X and $d_R a$ on A and that the modular functions of the groups X, A, and $G = X \circledS A$ are related by

$$\Delta_G(x,a) = \delta(a)\Delta_A(a)\Delta_X(x), \quad (x,a) \in G, \tag{2.9}$$

where δ is (a homomorphism of A) defined by

$$\int_X f(a(x)) d_R x = \delta(a) \int_X f(x) d_R x, \quad f \in \mathcal{K}(X). \tag{2.10}$$

In particular, $\Delta_X(a(x)) = \Delta_X(x)$ and

$$\| f(a(\cdot)) \|_p = \delta(a)^{1/p} \| f \|_p, \quad 1 \leqq p < \infty. \tag{2.11}$$

Lemma 2.4. *If $G = X \circledS A$ is a semi-direct product of locally compact groups X, A, then*

$$Y_{p,q}^{(r)}(G) \leqq Y_{p,q}^{(r)}(X) Y_{p,q}^{(r)}(A) \tag{2.12}$$

whenever $r, p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

Proof.

$$\begin{aligned} (f\Delta_G^{-1/q'} * g)(y,b) &= \int_X \int_A f\Delta_G^{-1/q'}((y,b)(x,a)^{-1}) g(x,a) dx da \\ &= \int\int f\Delta_G^{-1/q'}(yba^{-1}(x^{-1}), ba^{-1}) g(x,a) dx da \\ &= \int\int f\Delta_G^{-1/q'}(yb(x^{-1}), ba^{-1}) g(a(x), a) dx \delta(a)^{-1} da. \end{aligned}$$

Therefore

$$\begin{aligned} &\| f\Delta_G^{-1/q'} * g \|_r^r \\ &= \int_B \int_Y \left| \int_A \int_X f\Delta_G^{-1/q'}(b(b^{-1}(y)x^{-1}), ba^{-1}) g(a(x), a) dx \delta(a)^{-1} da \right|^r dy db \\ &= \int_B \int_Y \left| \int_A \int_X f\Delta_G^{-1/q'}(b(yx^{-1}), ba^{-1}) g(a(x), a) dx \delta(a)^{-1} da \right|^r dy \delta(b)^{-1} db \\ &\leqq \int_B \int_Y \left\{ \int_A \left| \int_X f\Delta_G^{-1/q'}(b(yx^{-1}), ba^{-1}) g(a(x), a) dx \right| \delta(a)^{-1} da \right\}^r dy \delta(b)^{-1} db \\ &\leqq \int_B \left[\int_A \left\{ \int_Y \left| \int_X f\Delta_G^{-1/q'}(b(yx^{-1}), ba^{-1}) g(a(x), a) dx \right|^r dy \right\}^{\frac{1}{r}} \delta(a)^{-1} da \right]^r \delta(b)^{-1} db \\ &= \int_B \left[\int_A \left\{ \int_Y \left| \int_X f(b(yx^{-1}), ba^{-1}) \Delta_X(b(yx^{-1}))^{-1/q'} g(a(x), a) dx \right|^r dy \right\}^{\frac{1}{r}} \right. \\ &\quad \left. \cdot \Delta_A(ba^{-1})^{-1/q'} \delta(ba^{-1})^{-1/q'} \delta(a)^{-1} da \right]^r \delta(b)^{-1} db. \end{aligned}$$

Now

$$\left\{\int_Y \left|\int_X f(b(yx^{-1}), ba^{-1}) \Delta_X(b(yx^{-1}))^{-1/q'} g(a(x), a) dx\right|^r dy\right\}^{\frac{1}{r}}$$
$$= \|(f(b(\cdot), ba^{-1})\Delta_X(\cdot)^{-1/q'}) * g(a(\cdot), a)\|_r \quad [\text{since } \Delta_X(a(x)) \equiv \Delta_X(x)]$$
$$\leqq Y_{p,q}^{(r)}(X) \|f(b(\cdot), ba^{-1})\|_p \|g(a(\cdot), a)\|_q$$
$$= Y_{p,q}^{(r)}(X) \delta(b)^{1/p} \|f(\cdot, ba^{-1})\|_p \delta(a)^{1/q} \|g(\cdot, a)\|_q .$$

Thus

$$\|f\Delta^{-1/q'} * g\|_r^r$$
$$\leqq \int_B \left[\int_A Y_{p,q}^{(r)}(X) \|f(\cdot, ba^{-1})\|_p \|g(\cdot, a)\|_q \Delta_A(ba^{-1})^{-1/q'}\right.$$
$$\left.\cdot \delta(ba^{-1})^{-1/q'} \delta(a)^{-1}\right]^r \delta(b)^{-1} db$$
$$= [Y_{p,q}^{(r)}(X)]^r \int_B \left[\int_A \|f(\cdot, ba^{-1})\|_p \Delta_A(ba^{-1})^{-1/q'} \|g(\cdot, a)\|_q da\right]^r db$$
$$\left(\text{since } \frac{1}{q} + \frac{1}{q'} - 1 = 0 = -1 - \frac{r}{q'} + \frac{r}{p}\right)$$
$$= [Y_{p,q}^{(r)}(X)]^r \|\varphi \Delta_A^{-1/q'} * \psi\|_r^r$$
$$[\text{where } \varphi(a) = \|f(\cdot, a)\|_p \text{ and } \psi(a) = \|g(\cdot, a)\|_q]$$
$$\leqq [Y_{p,q}^{(r)}(X) Y_{p,q}^{(r)}(A) \|\varphi\|_p \|\psi\|_q]^r$$
$$= [Y_{p,q}^{(r)}(X) Y_{p,q}^{(r)}(A) \|f\|_p \|g\|_q]^r, \quad \text{as required.}$$

Corollary 2.5. *If $G = X \circledS A$ is a semi-direct product and*

$$(k-1) + \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k}$$

for some $k \geqq 2$ and $r, p_1, p_2, \ldots, p_k \in [1, \infty]$, then

$$Y_{p_1, p_2, \ldots, p_k}^{(r)}(G) \leqq Y_{p_1, \ldots, p_k}^{(r)}(X) Y_{p_1, \ldots, p_k}^{(r)}(A) .$$

Corollary 2.5′. *Let Γ be a connected simply connected real nilpotent Lie group of dimension n. Then*

$$Y_{p_1, \ldots, p_k}^{(r)}(\Gamma) \leqq Y_{p_1, \ldots, p_k}^{(r)}(\mathbb{R}^n) .$$

It will result from Theorem 3 (in Section 3) that equality holds in Corollary 2.5′ if Γ is a Heisenberg group and if $p_1 = p_2 = \ldots = p_k = 2k/(2k-1)$ for some $k \geqq 2$.

We turn now to a discussion of the Hausdorff Young theorem for a locally compact group. The generalization to all locally compact Abelian groups of the inequality (0.3) with constant 1 is well known [5, (31.20)]. So is the corresponding inequality for compact non-Abelian groups [5, (31.22)]. In 1958 Kunze [9] extended the Riesz-Thorin Theorem to the setting of operator algebras and thereby was able to prove a Hausdorff-Young theorem for any locally compact unimodular group G. This theorem took the form

$$\|L_f\|_{p'} \leqq \|f\|_p, \quad 1 \leqq p \leqq 2, \quad \frac{1}{p}+\frac{1}{p'}=1, \quad f \in L^p(G), \tag{2.13}$$

where L_f is the operator of convolution by f on the left in the Hilbert space $L^2(G)$ and the norm $\|L_f\|_{p'}$ is defined by using a generalized trace cononically constructed from the group. This result was new even for compact groups and subsumed the usual Hausdorff-Young theorem in case G was Abelian. A concrete realization of (2.13) can be given if the unimodular group G is separable and of Type I (Lipsman [10, Theorem 2.2]). In this case letting $\hat{G}$ denote the space of unitary equivalence classes of continuous irreducible unitary representations of G, there is a measure μ_G on $\hat{G}$ such that

$$\int_G |f(x)|^2 dx = \int_{\hat{G}} \|U_\lambda(f)\|_2^2 d\mu_G(\lambda), \quad f \in L^1(G) \cap L^2(G), \tag{2.14}$$

where U_λ is an element of $\lambda \in \hat{G}$,

$$U_\lambda(f) = \int_G f(x) U_\lambda(x) dx, \quad \text{for} \quad f \in L^1(G),$$

and $\|U_\lambda(f)\|_2^2 = \operatorname{tr}[U_\lambda(f)^* U_\lambda(f)]$.

The Hausdorff-Young inequality (2.13) now asserts

$$\left\{\int_{\hat{G}} \|U_\lambda(f)\|_{p'}^{p'} d\mu_G(\lambda)\right\}^{1/p'} \leqq \left\{\int_G |f(x)|^p dx\right\}^{1/p}. \tag{2.15}$$

We shall identify L_f with the family $\{U_\lambda(f)\}_{\lambda \in \hat{G}}$ and consider it to be the Fourier transform of f. We define $A_p(G)$, for $1 \leqq p \leqq 2$ to be the best constant in inequality (2.13) or in case G is separable and of Type I in inequality (2.15).Thus $A_p(G) = \sup\limits_{f \neq 0} \dfrac{\|L_f\|_{p'}}{\|f\|_p}$ and (2.13) implies $A_p(G) \leqq 1$ always.

Lemma 2.6. *Let G be a locally compact unimodular group and let $p = 2k/(2k-1)$ for some integer $k \geqq 2$. Let $Y_p(G) = Y^{(2)}_{p_1, p_2, \ldots, p_k}(G)$ if $p_1 = p_2 = \ldots = p_k = p$. Then*

$$A_p(G) = Y_p(G)^{1/k}.$$

Proof. Since $p = 2k/(2k-1)$ we have $p' = 2k$ and thus if

$$\begin{aligned} f \in \mathscr{K}(G), \ \|L_f\|_{p'}^{p'} &= \|L_f\|_{2k}^{2k} = \|L_{f_k * f_{k-1} * \ldots * f_1}\|_2^2 \\ &= \|f_k * \ldots * f_1\|_2^2 \leqq (Y_p(G)\|f\|_p^k)^2 \end{aligned}$$

where $f = f_1 = f_3 = \ldots$ and $f^* = f_2 = f_4 = \ldots$.

Thus $\|L_f\|_{p'} \leqq Y_p(G)^{2/p'}\|f\|_p = Y_p(G)^{1/k}\|f\|_p$ and this proves $A_p(G) \leqq Y_p(G)^{1/k}$. On the other hand if $f_1, f_2, \ldots, f_k \in \mathscr{K}(G)$, then

$$\begin{aligned} \|f_1 * \ldots * f_k\|_2 &= \|L_{f_1 * \ldots * f_k}\|_2 = \|L_{f_1} L_{f_2} \ldots L_{f_1}\|_2 \\ &\leqq \|L_{f_1}\|_{p'} \|L_{f_2}\|_{p'} \ldots \|L_{f_k}\|_{p'} \quad \left(\text{since } \frac{k}{p'} = \frac{1}{2}\right) \\ &\leqq \prod_{j=1}^k A_p(G)\|f_j\|_p = (A_p(G))^k \prod_{j=1}^k \|f_j\|_p \end{aligned}$$

and this shows that $Y_p(G) \leqq A_p(G)^k$.

Remark 2.7. In the proof of Lemma 2.6 we have used the following facts from Kunze [9]:

(i) equality holds in (2.13) for $p=2$.

(ii) $\|L_f\|_r = m_G((L_f^* L_f)^{r/2})$, $1 \leqq r < \infty$, where m_G is the generalized trace referred to above.

(iii) Hölder's inequality for operators.

Theorem 2. *If $G = X ⓢ A$ is a semi-direct product of locally compact unimodular groups X and A and if G is unimodular, then for $p = \frac{2k}{(2k-1)}$, k an integer $\geqq 2$, we have*

$$A_p(G) \leqq A_p(X) A_p(A) .$$

Proof. By Lemma 2.6 and Corollary 2.5

$$A_p(G)^k = Y_p(G) \leqq Y_p(X) Y_p(A) = A_p(X)^k A_p(A)^k .$$

We now give some examples to which Theorem 2 applies. First let $G = \mathbb{R} \times H$ be a direct product where H is an arbitrary unimodular group. By Theorem 2 we have $A_p(\mathbb{R} \times H) \leqq A_p(\mathbb{R}) A_p(H)$ for p' an even integer. As noted previously, since this is a direct product we have equality. On the other hand it is proved in Russo [12, Theorem 2] that equality holds here for *all* $p \in (1,2)$. Next consider a semi-direct product $\mathbb{R}^n ⓢ K$ where K is compact. By Theorem 2, for p' even $A_p(\mathbb{R}^n ⓢ K) \leqq A_p(\mathbb{R}^n)$ since $A_p(K) = 1$. For this example the proof of [12, Theorem 4] shows that equality holds for p' an even integer. Therefore Theorem 2 gives no new information for these examples. Consider next a connected simply connected real nilpotent Lie group Γ. A consequence of [14, Proposition 12] is that for all $p \in (1,2)$, $A_p(\Gamma) \leqq A_p^{\ell}$ where ℓ is the dimension of the center of Γ. This can be improved using Theorem 2.

Corollary 2.8. *Let Γ be a connected simply connected real nilpotent Lie group of dimension n. Then if $p = 2k/(2k-1)$ for some integer $k \geqq 2$, $A_p(\Gamma) \leqq A_p^n$.*

Proof. If $n=1$ (or 2) Γ is the Abelian group $\mathbb{R}$ (or $\mathbb{R}^2$) and $A_p(\mathbb{R}) = A_p$ (and $A_p(\mathbb{R}^2) = A_p^2$). If $n > 2$, write $\Gamma = \Gamma' ⓢ \Delta$ where Γ' has dimension $n-1$ and $\Delta \simeq \mathbb{R}$. Then Theorem 2 and the induction hypothesis gives the corollary.

We shall show in the next section that if for example, $n=3$, then equality holds in Corollary 2.8.

Two other interesting groups for which Theorem 2 gives a specific bound less than 1 are the inhomogeneous Lorentz groups and the Oscillator group (see Kleppner and Lipsman [8]). Other examples of important semi-direct products can be found in Wolf [15]. We note that, according to Fournier [4], $A_p(G) < 1$ if G has no compact open subgroups.

3. Heisenberg Groups

In this section Γ will denote the Heisenberg group of dimension $2n+1$, $n \geqq 1$. Thus the points of Γ are triples $\gamma = (x, y, t)$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$ and the group multiplication is

$$(x,y,t)(x',y',t')=(x+x',y+y',t+t'-xy'). \tag{3.1}$$

It is known that for each $\lambda \neq 0$ in $\mathbb{R}$ there is an irreducible unitary representation U_λ of Γ on $L^2(\mathbb{R}^n)$ given by

$$(U_\lambda(x,y,t)f)(\theta)=\exp[i\lambda(t-y\cdot\theta)]f(\theta+x) \tag{3.2}$$

for $(x,y,t)\in\Gamma$, $f\in L^2(\mathbb{R}^n)$, and $\theta\in\mathbb{R}^n$; and that

$$\int_\Gamma |\varphi(\gamma)|^2 d\gamma=(2\pi)^{-n-1}\int_{\lambda\neq 0}\|U_\lambda(\varphi)\|_2^2|\lambda|^n d\lambda \tag{3.3}$$

for $\varphi\in L^1(\Gamma)\cap L^2(\Gamma)$. Here $d\gamma$ is Lebesgue measure on $\mathbb{R}^{2n+1}$,

$$U_\lambda(\varphi)=\int_\Gamma \varphi(\gamma)U_\lambda(\gamma)d\gamma, \quad \text{and} \quad \|U_\lambda(\varphi)\|_2^2=\operatorname{tr}(U_\lambda(\varphi)^*U_\lambda(\varphi)).$$

The Hausdorff-Young theorem for Γ, which can be obtained from (3.3) by Riesz convexity, is the statement, for $1\leqq p\leqq 2$ and $\frac{1}{p}+\frac{1}{p'}=1$, that

$$\left((2\pi)^{-n-1}\int_{\lambda\neq 0}\|U_\lambda(\varphi)\|_{p'}^{p'}|\lambda|^n d\lambda\right)^{1/p'}\leqq\left(\int_\Gamma|\varphi(\gamma)|^p d\gamma\right)^{1/p} \tag{3.4}$$

for $\varphi\in L^1(\Gamma)\cap L^2(\Gamma)$ [cf. (2.15)]. The inequality (3.4) then extends to all $\varphi\in L^p$ and we shall write it as

$$\|L_\varphi\|_{p'}\leqq\|\varphi\|_p, \quad \varphi\in L^p(\Gamma), \quad 1\leqq p\leqq 2. \tag{3.5}$$

By using (3.2) to determine the kernel of the integral operator $U_\lambda(\varphi)$ one can verify (3.3) directly. Also by using this kernel it was shown in [13] that

$$\|L_\varphi\|_{p'}\leqq A_p^{n+1}\|\varphi\|_p, \quad \varphi\in L^p(\Gamma), \quad 1<p<2, \tag{3.6}$$

and in particular that $A_p(\Gamma)<1$.

Using the theory of Weyl operators developed in Section 1 we will prove, for p' an even integer, the best possible inequality of the type (3.5).

Theorem 3. *Let $p\in(1,2)$ be of the form $p=2k/(2k-1)$ for some integer $k\geqq 2$ and let Γ be the Heisenberg group of dimension $2n+1$. Then*

(i) $\|L_\varphi\|_{p'}\leqq A_p^{2n+1}\|\varphi\|_p$, *for* $\varphi\in L^p(\Gamma)$,

(ii) A_p^{2n+1} *is the smallest constant in* (i), *or in the notation of Section* 2, $A_p(\Gamma)=A_p^{2n+1}$.

(iii) $A_p^{2n+1}=\sup_{a,b>0}\{\|L\varphi_{a,b}\|_{p'}/\|\varphi_{a,b}\|_p\}$, *where*

$$\varphi_{a,b}(x,y,t)=\exp\{-a\|x\|^2-b\|y\|^2-\pi t^2\},$$

(iv) *there are no extremal functions in* (i), *i.e. if equality holds in* (i) *for some* $\varphi\in L^p(\Gamma)$ *then* $\varphi=0$ *a.e.*

Proof. A comparison of (3.2) with (1.1) and (1.2) shows that

$$U_\lambda(x,y,t)=e^{i\lambda(t+x\cdot y)}U(x)V\left(-\frac{\lambda y}{2\pi}\right), \quad \text{for} \quad (x,y,t)\in\Gamma \tag{3.7}$$

and therefore that

$$U_\lambda(\varphi)=\left(\frac{2\pi}{|\lambda|}\right)^n \iint e^{-2\pi i x\cdot y}\left[\int \varphi\left(x, -\frac{2\pi y}{\lambda}, t\right)e^{i\lambda t}dt\right] \cdot U(x)V(y)dxdy$$
$$=\left(\frac{2\pi}{|\lambda|}\right)^n K_{G_\lambda}, \tag{3.8}$$

where G_λ is defined by

$$G_\lambda(x,y)=e^{-2\pi i x\cdot y}\varphi\left(x, -\frac{2\pi}{\lambda}y, \frac{\hat{\lambda}}{2\pi}\right), \tag{3.9}$$

for $(x,y)\in\mathbb{R}^{2n}$.

Therefore, for every $p\in(1,2)$,

$$\|L_\varphi\|_{p'}^{p'}=(2\pi)^{-n-1}\int_{\lambda\neq 0}\|U_\lambda(\varphi)\|_{p'}^{p'}|\lambda|^n d\lambda$$
$$=(2\pi)^{-n-1+p'n}\int_{\lambda\neq 0}\|K_{G_\lambda}\|_{p'}^{p'}|\lambda|^{n(1-p')}d\lambda. \tag{3.10}$$

Now assume that p' is an even integer. Then by Theorem 1, $\|K_{G_\lambda}\|_{p'}\leqq A_p^{2n}\|G_\lambda\|_p$ and thus by (3.10)

$$\|L_\varphi\|_{p'}^{p'}\leqq(2\pi)^{-n-1+p'n}A_p^{2np'}\int_{\lambda\neq 0}\|G_\lambda\|_p^{p'}|\lambda|^{n(1-p')}d\lambda. \tag{3.11}$$

By (3.9)

$$\|G_\lambda\|_p^{p'}=\left(\iint|G_\lambda(x,y)|^p dxdy\right)^{p'/p}$$
$$=\left(\iint\left|\varphi\left(x,-\frac{2\pi}{\lambda}y,\frac{\hat{\lambda}}{2\pi}\right)\right|^p dxdy\right)^{p'/p}$$
$$=\left(\iint\left|\varphi\left(x,y,\frac{\hat{\lambda}}{2\pi}\right)\right|^p\left(\frac{|\lambda|}{2\pi}\right)^n dxdy\right)^{p'/p}. \tag{3.12}$$

Thus by (3.11)

$$\|L_\varphi\|_{p'}^{p'}\leqq(2\pi)^{-n-1+np'}A_p^{2np'} \cdot\int\left[\iint\left|\varphi\left(x,y,\frac{\hat{\lambda}}{2\pi}\right)\right|^p\left(\frac{|\lambda|}{2\pi}\right)^n dxdy\right]^{p'/p}|\lambda|^{n(1-p')}d\lambda$$
$$=A_p^{2np'}\int\left[\iint|\varphi(x,y,\hat{\lambda})|^p dxdy\right]^{p'/p}d\lambda$$
$$\leqq A_p^{2np'}\left(\iint\left(\int_{\lambda\neq 0}|\varphi(x,y,\hat{\lambda})|^{p'}d\lambda\right)^{p/p'}dxdy\right)^{p'/p}$$
$$\leqq A_p^{2np'}\left(\iint\left(A_p^p\int|\varphi(x,y,t)|^p dt\right)dxdy\right)^{p'/p}$$
$$=A_p^{(2n+1)p'}\|\varphi\|_p^{p'}, \tag{3.13}$$

and (i) is proved.

Suppose now that $\|L_\varphi\|_{p'}=A_p^{2n+1}\|\varphi\|_p$ for some $\varphi\in L^p(\Gamma)$. By (3.11) and (3.13) we must have $\|K_{G_\lambda}\|_{p'}=A_p^{2n}\|G_\lambda\|_p$ for a.e. λ. By Theorem 1 $G_\lambda=0$ a.e., for a.e. λ so from (3.12)

$$\|\varphi\|_2^2=\iiint|\varphi(x,y,t)|^2dxdydt=\iiint|\varphi(x,y,\hat{\lambda})|^2dxdyd\lambda$$
$$=(2\pi)^{-1}\iiint\left|\varphi\left(x,y,\frac{\hat{\lambda}}{2\pi}\right)\right|^2dxdyd\lambda=0\,,\quad\text{so}\quad\varphi=0\quad\text{a.e.}$$

To prove (iii) and therefore (ii) let $\varepsilon>0$ and choose N so that

$$\int_{|\lambda|\leqq N}\exp\left[-p'\frac{\lambda^2}{4\pi}\right]d\lambda>\int_{-\infty}^{\infty}\exp\left[-\frac{p'\lambda^2}{4\pi}\right]d\lambda-\varepsilon=\frac{2\pi}{(p')^{\frac{1}{2}}}-\varepsilon\,, \tag{3.14}$$

and let M be given by Lemma 1.3 with $c=-2\pi$. Then fix a and b such that

$$ab>\frac{MN^2}{4\pi^2}\,. \tag{3.15}$$

Write φ for $\varphi_{a,b}$. By (3.9)

$$\begin{aligned}G_\lambda(x,y)&=e^{-2\pi ix\cdot y}\varphi\left(x,-\frac{2\pi}{\lambda}y,\frac{\hat{\lambda}}{2\pi}\right)\\&=e^{-2\pi ix\cdot y}e^{-a\|x\|^2-b\frac{4\pi^2}{\lambda^2}\|y\|^2}e^{-\frac{\lambda^2}{4\pi}}\\&=e^{-\lambda^2/4\pi}H_\lambda(x,y)\end{aligned} \tag{3.16}$$

where we have put

$$H_\lambda(x,y)=e^{-2\pi ix\cdot y}e^{-a\|x\|^2}e^{-4b\pi^2\|y\|^2/\lambda^2}\,. \tag{3.17}$$

By (3.16)

$$K_{G_\lambda}=e^{-\lambda^2/4\pi}K_{H_\lambda}$$

and thus

$$\|K_{G_\lambda}\|_{p'}^{p'}=e^{-p'\lambda^2/4\pi}\|K_{H_\lambda}\|_{p'}^{p'} \tag{3.18}$$

By computation from (3.17)

$$\|H_\lambda\|_p^{p'}=\left(\frac{|\lambda|}{p}\right)^{np'/p}(4ab)^{-\frac{np'}{2p}} \tag{3.19}$$

and by Lemma 1.3

$$\|K_{H_\lambda}\|_{p'}^{p'}\geqq(A_p^{2n}-\varepsilon)^{p'}\|H_\lambda\|_p^{p'}\quad\text{provided}\quad\frac{ab4\pi^2}{\lambda^2}\geqq M\,. \tag{3.20}$$

Thus

$$\begin{aligned}\|L_\varphi\|_{p'}^{p'}&=(2\pi)^{-n-1}\int_{\lambda\neq0}\|U_\lambda(\varphi)\|_{p'}^{p'}|\lambda|^n d\lambda\\&\geqq(2\pi)^{-n-1}\int_{0<|\lambda|\leqq N}\|U_\lambda(\varphi)\|_{p'}^{p'}|\lambda|^n d\lambda\\&=(2\pi)^{-n-1+np'}\int_{0<|\lambda|\leqq N}\|K_{G_\lambda}\|_{p'}^{p'}|\lambda|^{n(1-p')}d\lambda\end{aligned} \tag{3.21}$$

[by (3.8)].

By (3.18), (3.20), and (3.19) we have

$$\|K_{G_\lambda}\|_p^p = e^{-\lambda^2 p'/4\pi}\|K_{H_\lambda}\|_{p'}^{p'} \geqq e^{-\lambda^2 p'/4\pi}(A_p^{2n}-\varepsilon)^{p'}\|H_\lambda\|_p^{p'}$$

$$= e^{-\lambda^2 p'/4\pi}(A_p^{2n}-\varepsilon)^{p'}\left(\frac{|\lambda|}{p}\right)^{np'/p}(4ab)^{-np'/2p} \tag{3.22}$$

provided $\frac{4\pi^2 ab}{\lambda^2} \geqq M$. By (3.15) $4\pi^2 ab > MN^2$ so if $|\lambda| \geqq N$ we have $\frac{4\pi^2 ab}{\lambda^2} \geqq \frac{MN^2}{\lambda^2} \geqq M$. Thus (3.22) holds for any $0<|\lambda|\leqq N$ and using (3.22) in (3.21) yields

$$\|L_\varphi\|_{p'}^{p'} \geqq (2\pi)^{-n-1+np'}(A_p^{2n}-\varepsilon)^{p'} \cdot (4ab)^{-np'/2p} \int_{0<|\lambda|\leqq N} e^{-\frac{\lambda^2 p'}{4\pi}} d\lambda\, p^{-np'/p}$$

$$\geqq (2\pi)^{-n-1+np'}(A_p^{2n}-\varepsilon)^{p'}(4ab)^{-np'/2p}\left(\frac{2\pi}{(p')^{\frac{1}{2}}}-\varepsilon\right)p^{-np'/p} \tag{3.23}$$

[by (3.14)].

But

$$\|\varphi\|_p^{p'} = (\pi)^{np'/p}(ab)^{-\frac{np'}{2p}} p^{-(n+\frac{1}{2})p'/p}\,. \tag{3.24}$$

Putting (3.24) in (3.23) we get

$$\|L_\varphi\|_{p'/\|\varphi\|_p} \geqq (2\pi)^{-1/p'}\left(\frac{2\pi}{(p')^{\frac{1}{2}}}-\varepsilon\right)^{1/p'} p^{\frac{1}{2}p}(A_p^{2n}-\varepsilon)$$

and this completes the proof since $A_p = \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{\frac{1}{2}}$.

4. The a $x+b$ Group

Let G be the group of matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a>0$ and $b\in\mathbb{R}$, under matrix multiplication. Denote the group element $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ by (b,a). If we set $N=\{(b,1):b\in\mathbb{R}\}$ and $K=\{(0,a):a\in\mathbb{R}_+^*\}$ then $G=NK$ is a semi-direct product with N normal in G. The product in G is $(b,a)(b',a')=(b+ab',aa')$ and the Haar measures are $\frac{dadb}{a^2}$ (left) and $\frac{dadb}{a}$ (right). The modular function is $\Delta(b,a)=a^{-1}$.

For each $\lambda\in\mathbb{R}$ there is a one-dimensional unitary representation π_λ of G given by $\pi_\lambda(b,a)=a^{i\lambda}$. Two infinite dimensional irreducible continuous unitary representations of G are given by $\pi_\pm$ defined on $\mathscr{H}=L^2\left(\mathbb{R}_+^*,\frac{dt}{t}\right)$ by

$$(\pi_\pm(b,a)\xi)(t) = e^{\mp 2\pi i b t}\xi(at),\ \xi\in\mathscr{H},\ t\in\mathbb{R}_+^*\,. \tag{4.1}$$

Every continuous unitary irreducible representation of G is unitarily equivalent to one of $\pi_\pm$, π_λ. For questions of harmonic analysis on G only the two infinite dimensional representations $\pi_\pm$ enter.

Consider the unbounded densely defined operator δ in $\mathscr{H}$ given by $\mathscr{D}=\{\xi\in\mathscr{H}: t^{\frac{1}{2}}\xi(t)\in\mathscr{H}\}$ and $\delta\xi(t)=t^{\frac{1}{2}}\xi(t)$, for $\xi\in\mathscr{D}$.

For $f\in L^1(G)$ let $\mathscr{I}_\pm(f)=\pi_\pm(f)=\iint f(b,a)\pi_\pm(b,a)\frac{dbda}{a}$ (right Haar measure). Finally set $\mathscr{P}_\pm(f)=\delta\mathscr{I}_\pm(f)$ and $\mathscr{P}(f)=(\mathscr{P}_+(f),\mathscr{P}_-(f))$ for f locally integrable.

The Plancherel theorem for G is the assertion that $f\to\mathscr{P}(f)$, defined initially for locally integrable f, extends to an isometric mapping of $L^2(G)$ onto the space $L^2(\pm)$ of pairs of Hilbert-Schmidt operators on $\mathscr{H}$, i.e.

$$\|\mathscr{P}(f)\|_{L^2(\pm)}\equiv(\|\mathscr{P}_+(f)\|_2^2+\|\mathscr{P}_-(f)\|_2^2)^{\frac{1}{2}}=\|f\|_{L^2(G)}\,. \tag{4.2}$$

This is proved by Khalil [7] using left Haar measure.

Now let $r\in[2,\infty)$ and define, for locally integrable f,

$$\mathscr{I}_{r,\pm}(f)=\delta^{2/r}\pi_\pm(f)\,. \tag{4.3}$$

Note that formally $\mathscr{I}_{\infty,\pm}=\mathscr{I}_\pm$ and $\mathscr{I}_{2,\pm}=\mathscr{P}_\pm$. The Hausdorff-Young inequality for G is the assertion

$$\|\mathscr{I}_{p'}(f)\|_{L^{p'}(\pm)}\equiv(\|\mathscr{I}_{p',+}(f)\|_{p'}^{p'}+\|\mathscr{I}_{p',-}(f)\|_{p'}^{p'})^{\frac{1}{p'}}\leqq\|f\|_{L^p(G)}\,. \tag{4.4}$$

This can be proved by using the extension of the Riesz-Thorin interpolation theorem in which the linear operator varies analytically on a complex parameter. However, by using the Hausdorff-Young theorem for integral operators one of us has shown [14] that $\|\mathscr{I}_{p'}(f)\|_{L^{p'}(\pm)}\leqq A_p\|f\|_{L^p(G)}$ for $f\in L^p(G)$ where $A_p=[p^{1/p}/(p')^{1/p'}]^{\frac{1}{2}}$. By using our results on Young's inequality for non-unimodular groups we can obtain the following theorem:

Theorem 4. *Let $p=2k/(2k-1)$ for some $k=2,3,4,\ldots$ and let G be the $ax+b$ group. Then*

$$\|\mathscr{I}_{p'}(f)\|_{p'}\leqq A_p^2\|f\|_{L^p(G)} \quad \text{for} \quad f\in L^p(G)\,.$$

We first establish some properties of the transform $\mathscr{I}_{r,\pm}$. We note first that by (4.1)

$$\begin{aligned}(\pi_\pm(f)\xi)(t)&=\iint f(b,a)e^{\mp 2\pi ibt}\xi(at)dba^{-1}da\\&=\int(\int f(b,at^{-1})e^{\mp 2\pi ibt}db)\xi(a)a^{-1}da\end{aligned}$$

so that the kernel of $\pi_\pm(f)=\mathscr{I}_\pm(f)$ is

$$k_{\pm,f}(t,a)=f(\widehat{\mp t},at^{-1}) \quad \text{for} \quad t,a\in\mathbb{R}_+^*\,. \tag{4.5}$$

A similar calculation shows that the kernel of $\mathscr{I}_{r,\pm}(f)$ is

$$k_{r,\pm,f}(t,a)=t^{1/r}f(\widehat{\mp t},at^{-1}), \quad \text{for} \quad t,a\in\mathbb{R}_+^*\,. \tag{4.6}$$

Using (4.6) it is easy to check that

$$(\mathscr{I}_{r,\pm}(f))^*=\mathscr{I}_{r,\pm}(\tilde{f}\Delta^{1/r'}),\ \frac{1}{r}+\frac{1}{r'}=1 \tag{4.7}$$

[where generally $\tilde{f}(x)=\bar{f}(x^{-1})$], and that

$$\mathscr{I}_{r,\pm}\left(g\Delta^{-\frac{1}{r}}*f\right)=\pi_{\pm}(g)\mathscr{I}_{r,\pm}(f)\,. \tag{4.8}$$

Then using (4.8) and an induction argument one can show that for $k\geqq 2$

$$\mathscr{I}_{r,\pm}(f_1)\cdots\mathscr{I}_{r,\pm}(f_k)$$
$$=\mathscr{I}_{r/k,\pm}\left(f_1\Delta^{-\frac{(k-1)}{r}}*f_2\Delta^{-\frac{(k-2)}{r}}*\ldots*f_{k-1}\Delta^{-\frac{1}{r}}*f_k\right). \tag{4.9}$$

We omit the proofs of (4.7) and (4.8) except to remind the reader that the convolution is given by (2.1) and thus

$$f*g(b,a)=\int_{\mathbb{R}_+^*}\int_{\mathbb{R}} f(b-a\bar{\alpha}^1\beta, a\bar{\alpha}^1)g(\beta,\alpha)d\beta\bar{\alpha}^1 d\alpha\,. \tag{4.10}$$

The inductive step for (4.9) is the following:

$$\mathscr{I}_{r,\pm}(f_1)\cdots\mathscr{I}_{r,\pm}(f_{k+1})$$
$$=\mathscr{I}_{r,\pm}(f_1)[\mathscr{I}_{r,\pm}(f_2)\cdots\mathscr{I}_{r,\pm}(f_{k+1})]$$
$$=\mathscr{I}_{r,\pm}(f_1)\mathscr{I}_{\frac{r}{k},\pm}(g_k)\ \left(\text{where } g_k=f_2\Delta^{-\frac{(k-1)}{r}}*\ldots*f_k\Delta^{-1/r}*f_{k+1}\right)$$
$$=\delta^{2/r}\pi_{\pm}(f_1)\mathscr{I}_{\frac{r}{k},\pm}(g_k)=\delta^{2/r}\mathscr{I}_{\frac{r}{k},\pm}(f_1\Delta^{-k/r}*g_k)$$
$$=\delta^{2/r}\delta^{2k/r}\pi_{\pm}(f_1\Delta^{-k/r}*g_k)$$
$$=\mathscr{I}_{r/(k+1),\pm}\left(f_1\Delta^{-k/r}*f_2\Delta^{-\frac{(k-1)}{r}}*\ldots*f_k\Delta^{-1/r}*f_{k+1}\right)$$

as required. Finally we note that

$$\|\tilde{f}\Delta^{1/r}\|_r=\|f\|_r \quad \text{for} \quad 1\leqq r<\infty\,. \tag{4.11}$$

We can now prove Theorem 4. Fix k such that $p=2k/(2k-1)$. Then $p'=2k$ and

$$\|\mathscr{I}_{p'}(f)\|_{p'}^{p'}=\sum_{\pm}\|\mathscr{I}_{p',\pm}(f)\|_{p'}^{p'}=\sum_{\pm}\|\mathscr{I}_{2k,\pm}(f)\|_{2k}^{2k}\,. \tag{4.12}$$

For any operator T, $\|T\|_{2k}^{2k}=\|S_k\|_2^2$ where S_k is either $T^*T\ldots T^*T$ or $TT^*\ldots T^*T$ (k factors) according as k is even or odd. Applying this to $T=\mathscr{I}_{2k,\pm}(f)$ and using (4.7) and (4.9) we see that, setting

$$g_k=\left(f_1\Delta^{-\frac{(k-1)}{2k}}*f_2\Delta^{-\frac{(k-1}{2k}}*\ldots*f_{k-1}\Delta^{-1/2k}*f_k\right) \tag{4.13}$$

where $f_1=f_3\ldots=f_k=f$, $\quad f_2=f_4=\ldots=f_{k-1}=\tilde{f}\Delta^{1/p}$ (k odd), and $f_1=f_3=\ldots=f_{k-1}=\tilde{f}\Delta^{1/p}$, $f_2=f_4=\ldots=f_k=f$ (k even) we have $S_k=\mathscr{I}_{2,\pm}(g_k)$. Thus from (4.12)

$$\|\mathscr{I}_{p'}(f)\|_{p'}^{p'}=\sum_{\pm}\|S_k\|_2^2=\sum_{\pm}\|\mathscr{I}_{2,\pm}(g_k)\|_2^2=\|\mathscr{I}_2(g_k)\|_2^2=\|g_k\|_2^2$$
$$\leqq\left(Y_p(G)\prod_{j=1}^{k}\|f_j\|_p\right)^2\leqq(Y_p(N)Y_p(K)\|f\|_p^k)^2$$

by (2.8) and Corollary 2.5. Since $Y_p(N)=Y_p(K)=Y_p(\mathbb{R})=A_p^k$ and $p'=2k$ we have $\|\mathscr{I}_p(f)\|_{p'}\leqq A_p^2\|f\|_p$ and the proof is complete.

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