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Homological Algebra of Gorenstein Rings

Author
Henningsen, Harrison

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HOMOLOGICAL ALGEBRA OF GORENSTEIN RINGS

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Harrison Hunter Henningsen

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The Thesis of Harrison Hunter
Henningsen is approved:

Johan Steen, Chair

Beren Sanders

Junecue Suh

Quentin Williams
Acting Vice Provost and Dean of Graduate Studies

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Abstract

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Harrison Hunter Henningsen

We study three triangulated categories associated to a Gorenstein ring, that is, a right- and left-noetherian ring with finite right and left injective dimension. After a survey of exact categories and cotorsion theory, we discuss the homological algebra of the category of finitely generated modules over a Gorenstein ring, concluding that the subcategory of maximal Cohen-Macaulay (MCM) modules is both a Frobenius category and a cotorsion class. Immediate corollaries include a triangulated structure on the projective stabilization of the subcategory of MCM modules and structure theorems for finitely generated modules. Then we describe two triangulated categories equivalent to the stable category of MCM modules. The homotopy category of acyclic complexes of projective modules is the first, making precise a connection observed between MCM modules and existence of projective co-resolutions. The second, the singularity category, is a Verdier quotient of the bounded derived category that allows us to study modules up to MCM approximation.

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Introduction

In the beginning, the tentative program of study was commutative algebra. I had little experience in the field, but I found it interesting: a set of tools designed largely in service of algebraic geometry which, in the last half century, has become a subject unto itself. I proposed a research project with faint direction to Johan, the topic not more specific than just commutative algebra, and to my absolute delight he agreed to advise the project. He cautioned at our first meeting that the project would likely lie in the intersection of commutative and homological algebra, a compromise I made readily. However, there is a gravity inherent to an advisor's research domain, and I felt the pull as the project drifted quickly in the direction of homological algebra, an adjustment I soon welcomed.

My perspective shifted with the focus of the project to a more unfamiliar style of argument: that of category theory. Homomorphisms of individual objects made way for more macroscopic tools like functors, and soon the language of category theory became the primary dialect. We noticed that several propositions can be rephrased in terms of cotorsion theory—diversion or new direction? My fluency in working on a categorical level improved as we explored the problem more, and it was not long before the "commutative" assumption was dropped in favor of increased generality. What results is an investigation of the homological algebra of Gorenstein rings inspired by a program to reinterpret and modernize Buchweitz's outstanding unpublished treatise [3].

Outline

We construct three triangulated categories associated to a Gorenstein ring, that is, a noetherian ring with finite injective dimension, all of decidedly different flavors. Breaking from tradition, most of our effort is directed toward the category of modules over a Gorenstein ring S, where we unravel several resulting rich structures. Particular focus is placed on the subcategory of maximal Cohen-Macaulay modules, the full subcategory consisting of objects for which $\mathrm{Hom}_S(-, S)$ is an exact duality. We examine maximal Cohen-Macaulay modules from several perspectives to highlight a tapestry of remarkable properties. In this way and despite their top-line billing, Gorenstein rings serve more as a technical assumption to facilitate the study of maximal Cohen-Macaulay modules.

Chapter 1 surveys two specialized topics we utilize throughout. First we detail exact categories, which can be viewed as a natural generalization of abelian categories. The usual short exact sequences are replaced with a carefully axiomatized structure designed to mimic the familiar properties enjoyed by an abelian category, and we may accomplish much the same homological algebra with decidedly weaker structure. We discuss the stabilization of an additive category, and specialize to the case of Frobenius categories to draw conclusions about triangulated categories. Then we present cotorsion pairs, a formalism we use to unify many structural results. Such a pair is comprised of two Ext-orthogonal subcategories, which we equip with more structure as we prepend adjectives. Cotorsion pairs serve as a natural context for discussing approximation, and upon stabilizing, the approximations can be made functorial. We propose a torsion theory for additive categories suitable for the stabilized categories in consideration and quickly derive results concerning approximating subcategories that arise from cotorsion pairs.

In Chapter 2, we begin our main line of inquiry by studying modules. Our first application of cotorsion pairs concerns the subcategory of finitely generated projective modules, offering a concrete example of cotorsion theory in a familiar context. After a review of some homological algebra, we introduce the subcategories of maximal Cohen-Macaulay modules and modules of finite projective dimension, partners in a more intricate cotorsion pair. We introduce lemmas detailing connections between the pair argued in a more classical homological-algebraic fashion, slowly building toward more macroscopic structural results. In the remaining two sections, we appeal to the theory established in Chapter 1 to prove the projective stabilization of the subcategory of maximal Cohen-Macaulay modules is triangulated and derive structure theorems for any finitely generated module over a Gorenstein ring. We conclude the chapter with a discussion of additional consequences, including functorial approximation.

Chapter 3 adds variety by introducing other categories associated to the same Gorenstein ring. We examine the category of chain complexes through the new lens of exact structure, deriving the homotopy category of complexes as the natural triangulated stabilization. The triangulated subcategory of acyclic complexes of projective modules is introduced to formalize a connection between maximal Cohen-Macaulay modules and projective co-resolutions noted in the previous chapter, made precise now by an equivalence of categories. Then we treat Verdier localization, a technique for formally inverting a class of morphisms in a category, at which point we can introduce the derived category with its natural triangulated structure. The third category of interest is the singularity category, a Verdier quotient of (a triangulated subcategory of) the derived category, which carries its own inherited triangulation. By studying the singularity category, we gain a better understanding of MCM approximation. We lastly show the singularity category is equivalent to the other two, furnishing three perspectives on the homological algebra of Gorenstein rings.

Notation, conventions, and assumptions

Throughout R will denote a unital ring, and we reserve the symbol S for a Gorenstein ring. We denote by $\mathsf{Mod}\, R$ the category of right R-modules and by $\mathsf{mod}\, R$ the full additive subcategory of finitely generated right R-modules. We write $\mathbf{Z}$ for the integers, that is, the initial object in the category of unital rings, and according let $\mathsf{Ab} := \mathsf{Mod}\, \mathbf{Z}$ be the category of abelian groups. For a category $\mathcal{X}$, write $\mathsf{Proj}(\mathcal{X})$ and $\mathsf{Inj}(\mathcal{X})$ for the full subcategories of projective objects and injective objects respectively; as shorthand, we will write $\mathsf{proj}(R)$ for the full subcategory of projective modules in $\mathsf{mod}\, R$. Lastly, we regard biproducts as column vectors.

We expect the reader to be familiar with graduate level algebra, e.g., rings and modules, and a fluency with homological algebra would be welcome, but the necessary concepts for the latter are briefly recalled throughout. Many of our results are formulated in the language of category theory, so we assume the reader has at least some familiarity with the basic concepts of categories, functors, equivalences, and adjoints; see [13] for an excellent account. Most of the categorical language is supported by more classically algebraic proofs, so the text should be fairly accessible to an algebraically-inclined graduate student.

CHAPTER 1

Preliminaries

We begin with a survey of preliminary notions, namely exact categories and cotorsion theory. Building much of the machinery at the outset leaves later arguments streamlined. For example, Theorem 1.1.19, the headlining result of the chapter, says that every stabilized Frobenius category is triangulated, a technical and lengthy verification. We use this result to show directly that the stable category of maximal Cohen-Macaulay modules is triangulated, a fact that historically has been verified indirectly, e.g., in [3].

1.1. Exact categories

We treat exact categories, with the goal of proving stabilized Frobenius categories are triangulated. This will reduce the hassle, or at least bulk, of proving a category is triangulated, shifting the task to proving each category of interest satisfies the hypotheses laid out in this section. It is our hope that each of the resulting proofs (that each category under scrutiny is Frobenius) feels less like a routine verification of axioms and more like an exploration of appreciably different exact structures. The reference of choice for exact categories is [4].

1.1.1. Definitions. The function of an exact structure on an additive category is to recover the usual homological algebra of short exact sequences in the absence of guaranteed kernels/cokernels. By prescribing a class of well-behaved sequences, we can do much of the work usually accomplished with short exact sequences. Exact structures furthermore highlight what conditions we *need* for many standard lemmas of homological algebra—i.e., the short five lemma, snake lemma, horseshoe lemma, existence of projective resolution and uniqueness up to homotopy equivalence, etc.—showing that hypotheses we often impose, hypotheses predicated on working in an abelian category, are needlessly restrictive.

Definition 1.1.1 ([4, Definition 2.1]). Let $\mathcal{X}$ be an additive category. A pair of composable morphisms (i, p) in $\mathcal{X}$ is called a **kernel-cokernel pair** if i is a kernel of p and p is a cokernel of i.

Fix a class $\mathcal{E}$ of kernel-cokernel pairs on $\mathcal{X}$. We call a morphism i an **inflation** if there exists a morphism p such that $(i, p) \in \mathcal{E}$. Symmetrically, a morphism p for which there exists an i such that $(i, p) \in \mathcal{E}$ is a **deflation**. Together, a kernel-cokernel pair in $\mathcal{E}$ is called a **conflation**. We write a conflation as $A \hookrightarrow B \twoheadrightarrow C$, that is, the arrow $\hookrightarrow$ is an inflation and $\twoheadrightarrow$ is a deflation.

An **exact structure** on $\mathcal{X}$ is an isomorphism closed class $\mathcal{E}$ of kernel-cokernel pairs subject to the following axioms.

E0 For all objects A in $\mathcal{X}$, the identity on A is an inflation.

E1 The class of inflations is closed under composition.

E2 The pushout of an inflation along an arbitrary morphism exists and is itself an inflation.

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \overset{\exists}{\hookrightarrow} & D \end{array}$$

E0$^{\text{op}}$ For all objects A in $\mathcal{X}$, the identity on A is a deflation.

E1$^{\text{op}}$ The class of deflations is closed under composition.

E2$^{\text{op}}$ The pullback of a deflation along an arbitrary morphism exists and is itself a deflation.

$$\begin{array}{ccc} D & \overset{\exists}{\dashrightarrow\!\!\!\!\rightarrow} & C \\ \downarrow & \lrcorner & \downarrow \\ B & \twoheadrightarrow & A \end{array}$$

We call the pair $(\mathcal{X}, \mathcal{E})$, or sometimes just $\mathcal{X}$ when $\mathcal{E}$ is understood, an **exact category**. Quillen is often credited with formalizing the notion of exact categories, whence the more attributive terminology **Quillen exact category** derives.

REMARK 1.1.2. Given a kernel-cokernel pair (i, p), we have that i is a monomorphism and p is an epimorphism. This yields the alternate nomenclature **admissible monomorphism** and **admissible epimorphism** in place of inflation and deflation. In the literature, one may also encounter **short exact sequence** as a substitute for conflation. Looking to avoid overloading terminology, we reserve the term short exact sequence for the familiar notion in an abelian category.

Immediately we might ask the question: If the exact structure $\mathcal{E}$ is externally prescribed, what parameters do the axioms ensure? For example, a more tractable question: Is there a class of conflations belonging to every exact structure? The smallest exact structure, in the sense that every other exact structure contains it, is the class of split exact sequences $A \hookrightarrow A \oplus B \twoheadrightarrow B$, the maps being canonical inclusion into the first component and canonical projection from the second respectively.

Let $\mathcal{E}$ be an exact structure on an additive category $\mathcal{X}$. To prove $\mathcal{E}$ contains the split exact sequences, first note that E0 (resp. E0$^{\text{op}}$) implies $A \to 0$ (resp. $0 \to A$) is a deflation (resp. an inflation) for any A in $\mathcal{X}$. Taking the pushout of $0 \hookrightarrow B$ along $0 \hookrightarrow A$ shows that $A \hookrightarrow A \oplus B$ is an inflation (E2). Essentially dually, the pullback of $A \twoheadrightarrow 0$ along $B \twoheadrightarrow 0$ shows $A \oplus B \twoheadrightarrow B$ is a deflation (E2$^{\text{op}}$). Gluing the squares together,

$$\begin{array}{ccccc} 0 & \hookrightarrow & B & & \\ \downarrow & \ulcorner & \downarrow & & \\ A & \hookrightarrow & A \oplus B & \twoheadrightarrow & B \\ & & \downarrow & \lrcorner & \downarrow \\ & & A & \twoheadrightarrow & 0 \end{array}$$

and upon recognizing that the canonical inclusion and projection form a kernel-cokernel pair, we conclude that the middle row, the desired sequence, belongs to $\mathcal{E}$. Notice that the symmetry of the axioms allowed us to freely use duality arguments. Formally, $(\mathcal{X}, \mathcal{E})$ is an exact category if and only if $(\mathcal{X}^{\text{op}}, \mathcal{E}^{\text{op}})$ is an exact category.

Any additive category is an exact category with the exact structure given by the split exact sequences. This trivial structure is usually not particularly insightful, though we will require it in Chapter 3. On the other end of the spectrum are abelian categories, with the class of all short exact sequences a priori an exact structure. Here we have an interesting exact structure at the expense of generality, that is, we gave up on the category being only additive. What will prove most fruitful is something in the middle.

Our scope is not so broad however. We restrict our attention to a class of exact categories that possess both enough projectives and injectives, and that furthermore cannot tell the difference between the two. To unpack, define an **exact functor** $F\colon (\mathcal{X}, \mathcal{E}) \to (\mathcal{X}', \mathcal{E}')$ between exact categories as an additive functor such that $F(\mathcal{E}) \subset \mathcal{E}'$. An object P in an exact category is **projective** if $\operatorname{Hom}_{\mathcal{X}}(P, -)$ is an exact functor from $\mathcal{X}$ to the category of abelian groups. As the category of abelian groups is an abelian category, we impose on it the naturally inherited exact structure consisting of all short exact sequences. We say an exact category $(\mathcal{X}, \mathcal{E})$ has **enough projectives** if, for each object A in $\mathcal{X}$, there exists a projective P and a deflation $P \twoheadrightarrow A$. Dually, an object I is called **injective** if $\mathrm{Hom}_{\mathcal{X}}(-, I)$ is an exact functor from $\mathcal{X}$ to the category of abelian groups, and $(\mathcal{X}, \mathcal{E})$ is said to have **enough injectives** if, for each object A in $\mathcal{X}$, there exists an injective I and an inflation $A \hookrightarrow I$. Note that projectives in an exact category possess many familiar properties, with dual statements about injectives being immediate.

LEMMA 1.1.3 ([4, Proposition 11.3]). *For an object P in an exact category $\mathcal{X}$, the following are equivalent.*

- $\mathrm{Hom}_{\mathcal{X}}(P, -)$ *is an exact functor from $\mathcal{X}$ to the category of abelian groups, i.e., P is projective.*
- *for all deflations $A \twoheadrightarrow B$ and every map $P \to B$, there exists a lift $P \to A$ (making the usual triangle commute).*
- $\mathrm{Hom}_{\mathcal{X}}(P, -)$ *sends deflations to surjections.*
- *every deflation $A \twoheadrightarrow P$ has a right inverse.*

DEFINITION 1.1.4. An exact category is called **Frobenius** if it has enough projectives and injectives, and if the classes of projective objects and injective objects coincide.

EXAMPLE 1.1.5. As mentioned before, any additive category is an exact category with respect to the split exact structure. Here every object is both projective and injective, so trivially the category is Frobenius, but this is not insightful in practice.

For any right- and left-noetherian ring R, the category $\mathsf{mod}\, R$ is abelian, thus exact—the exact structure is simply all short exact sequences—so $\mathsf{mod}\, R$ is Frobenius if and only if projective R-modules and injective R-modules coincide. By [15, Theorem 4.2.4], the latter is equivalent to R being injective as a right and left module over itself, i.e., that R is (right and left) self-injective. Such a ring is called quasi-Frobenius.

Historically, the definition of quasi-Frobenius rings was not motivated by the observation above, but rather as a generalization of Frobenius algebras. A finite dimensional algebra Λ over a field k is called a Frobenius algebra if it is isomorphic to its k-linear dual as Λ-modules:

$$\Lambda \cong \operatorname{Hom}_k(\Lambda, k).$$

Naturally, any Frobenius k-algebra Λ is quasi-Frobenius, since we have natural isomorphisms of functors

$$\operatorname{Hom}_\Lambda(-, \Lambda) \cong \operatorname{Hom}_\Lambda(-, \operatorname{Hom}_k(\Lambda, k)) \cong \operatorname{Hom}_k(- \otimes_\Lambda \Lambda, k)$$

by the Yoneda Lemma and tensor-hom adjunction, and the rightmost functor is exact since k is self-injective. Heller was the first to systematically study Frobenius categories in [7, Section 3] (according to [3, Section 8.1]), with modules over a Frobenius algebra being the motivating example, whence the name.

An important example of Frobenius algebras is group algebras: For any field k and any finite group G, the group algebra kG is Frobenius ([15, Proposition 4.2.6]), from which it follows that the category $\mathsf{mod}\, kG$ is Frobenius. As we will see in the next chapter, the Frobenius structure on $\mathsf{mod}\, kG$ facilitates the study of representation theory.

Example 1.1.6. Another nice example that more readily permits computation is the category of finitely generated modules over the truncated polynomial ring $\Lambda := k[x]/(x^n)$ for some $n \geq 1$. In this case, the only nontrivial indecomposable finitely generated Λ-modules are those of the form $k[x]/(x^i)$ for $1 \leq i \leq n$; this follows from the structure theorem for finitely generated modules over a principal ideal domain, in this case $k[x]$, and accounting for the vanishing of x^n. Already we know that Λ is a projective module, and as Λ is indecomposable, we conclude that finite rank free Λ-modules are the only finitely generated projective modules.

We show Λ is self-injective by way of Baer's Criterion: For any ring R, a right R-module Q is injective if and only if for every right ideal I of R, any R-module homomorphism $I \to Q$ can be lifted to an R-module homomorphism $R \to Q$. Note that Λ is a local ring with maximal ideal $(x)/(x^n)$. Any ideal of Λ is of the form $(x^i)/(x^n)$, so a Λ-module homomorphism

$$(x^i)/(x^n) \to \Lambda$$

is determined by the image of a generator $x^i \mapsto f(x)$, which we may furthermore assume is given by $x^i \mapsto x^j$: Writing $f(x) = x^j \cdot g(x)$ with $x \nmid g(x)$, we have that $g(x) \notin (x)/(x^n)$, and thus is a unit, say with inverse $h(x)$, so $x^i \cdot h(x)$ also generates $(x^i)/(x^n)$ and maps to x^j. Additionally, $i \leq j$, since otherwise $0 = x^{n-i} \cdot x^i \mapsto x^{n-i} \cdot x^j \neq 0$ by Λ-linearity. Then $1 \mapsto x^j - i$ is a lift; in the sequel, we write $\cdot x^\ell$ for $1 \mapsto x^\ell$.

$$\begin{array}{ccc} (x^i)/(x^n) & \xrightarrow{x^i \mapsto x^j} & \Lambda \\ \big\downarrow & \nearrow_{\cdot x^{j-i}} & \\ \Lambda & & \end{array}$$

Therefore Λ is self-injective. Clearly Λ is noetherian (it's artinian!), so Λ is quasi-Frobenius, hence $\mathsf{mod}\,\Lambda$ is Frobenius.

As a practical matter, we need the notion of an admissible morphism to work in exact categories as we do in abelian categories.

Definition 1.1.7. A morphism $f\colon A \to B$ is **admissible** if it factors as a deflation followed by an inflation.

Admissible morphisms require that image and coimage are isomorphic, as in abelian categories, and encode the familiar factorization over the image.

Furthermore, the factorization of an admissible morphism is unique up to unique isomorphism ([4, Lemma 8.4]). We caution that the composition of admissible morphisms is not admissible in general. To learn more about an admissible morphism, we study its analysis. The **analysis** of an admissible morphism is a diagram

$$\begin{array}{ccccccccc} & & A & & \xrightarrow{\quad f \quad} & & B & & \\ & {\scriptstyle k}\nearrow & & \searrow{\scriptstyle p} & & \nearrow{\scriptstyle i} & & \searrow{\scriptstyle c} & \\ K & & & & I & & & & C \end{array}$$

such that k is a kernel and c is a cokernel. A sequence of composable admissible morphisms

$$\begin{array}{ccccccccc} A' & & \xrightarrow{\quad f \quad} & & A & & \xrightarrow{\quad f' \quad} & & A'' \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & I & & & & I' & & \end{array}$$

is called **acyclic** or **exact** if $I \hookrightarrow A \twoheadrightarrow I'$ is a conflation. The definition can be extended to longer sequences by applying it to every successive morphism pair in the sequence. Note that we can discuss complexes in any additive category—a complex is just a $\mathbf{Z}$-indexed sequence $\cdots \to X^{i-1} \xrightarrow{d_{i-1}} X^i \xrightarrow{d_i} X^{i+1} \to \cdots$ such that $d_i d_{i-1} = 0$ for all i—so it is the *acyclicity* that requires an exact structure. Now that we have detailed the machinery required to work with acyclic complexes like projective resolutions, we will make mention of it rarely, operating as one generally would in abelian categories.

1.1.2. Stabilization. The results of this section, and many subsequent headlining results, concern stabilized categories, which allow us to study objects and morphisms up to some subcategory. Examples include projective stabilization, which is used in the representation theory of finite groups, and homotopy stabilization, which makes the category of chain complexes more flexible. We start with a brief treatment of stabilization.

Definition 1.1.8. Let $\mathcal{X}$ be an additive category, ω an isomorphism closed full subcategory of $\mathcal{X}$, and A, B, objects in $\mathcal{X}$. Write $\omega(A, B)$ for the set of morphisms from A to B that factor through an object in ω. Note that $\omega(A, B)$ is a subgroup of $\mathrm{Hom}_{\mathcal{X}}(A, B)$, and that the collection of all $\omega(A, B)$ forms an ideal, call it $\mathcal{I}_\omega$, of $\mathcal{X}$. The **ω-stabilization** of $\mathcal{X}$ is the quotient $\mathcal{X}/\mathcal{I}_\omega$, or $\mathcal{X}/\omega$ for short. In $\mathcal{X}/\omega$, objects are the same as those in $\mathcal{X}$, and morphisms from A to B are given by the image of morphisms in the quotient $\mathrm{Hom}_{\mathcal{X}}(A, B)/\omega(A, B)$. There is an additive functor $q\colon \mathcal{X} \to \mathcal{X}/\mathcal{I}_\omega$ that is identity on objects and sends morphisms to their equivalence class that is universal in the following sense: If $F\colon \mathcal{X} \to \mathcal{D}$ is any additive functor to any additive category $\mathcal{D}$ such that $F(f) \cong 0$ for all $f \in \mathcal{I}_\omega$, then there exists a unique $\widetilde{F}\colon \mathcal{X}/\mathcal{I}_\omega \to \mathcal{D}$ such that $\widetilde{F}q = F$.

Remark 1.1.9. So as not to abandon vocabulary unaccompanied by definition, we mention ideals. A left/right/two sided ideal in a locally small pre-additive category $\mathcal{Y}$ is a collection $\mathcal{I}$ of morphisms closed under left/right/two sided composition; all ideals considered herein are two sided. The collection $\mathcal{I}$ is subject to the constraint that $\mathcal{I} \cap \mathrm{Hom}_{\mathcal{Y}}(A, B)$ is a subgroup of $\mathrm{Hom}_{\mathcal{Y}}(A, B)$ for any two objects A, B in $\mathcal{Y}$. We write $\mathcal{I}(A, B)$ for the subgroup $\mathcal{I} \cap \mathrm{Hom}_{\mathcal{Y}}(A, B)$. For example, suppose $\mathcal{I}$ is a right ideal of $\mathcal{Y}$ and A, B, C objects in $\mathcal{Y}$. Then for any $f \in \mathrm{Hom}_{\mathcal{Y}}(A, B)$ and any $g \in \mathcal{I}(B, C)$, we have $gf \in \mathcal{I}(A, C) \subset \mathcal{I}$. Quotients by an ideal behave precisely as described in the last definition. Note that a locally small pre-additive category with one object can be viewed as a ring, namely the endomorphism ring of that object, and in this case the ideals of the pre-additive category are the ideals of the ring. Consequently, one can think of a locally small pre-additive category as a ring with many objects.

Suppose $\mathcal{X}$ is an exact category with enough projectives. For each object A in $\mathcal{X}$, we have a conflation $K \hookrightarrow P \twoheadrightarrow A$ with P projective. Set $\Omega A := K$ and write $\mathsf{Proj}(\mathcal{X})$ for the full subcategory of projective objects in $\mathcal{X}$. Then the assignment Ω defines an additive endofunctor, called the **syzygy functor**, on the projective stabilization $\underline{\mathcal{X}} := \mathcal{X}/\mathsf{Proj}(\mathcal{X})$. Syzygies on a projectively stabilized category were studied by Heller in the case of abelian categories (see [7]), from which our more general treatment derives.

PROPOSITION 1.1.10. *The assignment* $A \mapsto \Omega A$ *is functorial in* $\underline{\mathcal{X}}$.

PROOF. The plan of attack is to prove a general fact about lifting of the zero map and then apply this to show the assignment of objects and morphisms is well-defined. Suppose we have two conflations $K \hookrightarrow P \twoheadrightarrow A$ and $L \hookrightarrow Q \twoheadrightarrow B$ in $\mathcal{X}$ with P, Q projective. Then the zero map $A \to B$ lifts to a map $\varphi \colon P \to Q$, making the right square commute.

$$\begin{array}{ccccc} K & \xhookrightarrow{i} & P & \twoheadrightarrow & A \\ \downarrow{\scriptstyle \exists!\psi} & & \downarrow{\scriptstyle \exists\varphi} & & \downarrow{\scriptstyle 0} \\ L & \xhookrightarrow[j]{} & Q & \xrightarrow[\pi]{} & B \end{array}$$

In addition, as $\pi\varphi i = 0$, there exists a map $\psi \colon K \to L$ making the left square commute. Note that ψ is unique with respect to the choice of φ, but φ need not be unique. Then commutativity of the right square implies that φ factors uniquely through j, i.e., there exists $\lambda \colon P \to L$ such that $\varphi = j\lambda$.

$$\begin{array}{ccccc} K & \xhookrightarrow{i} & P & \twoheadrightarrow & A \\ \downarrow{\scriptstyle \psi} & {\scriptstyle \exists!\lambda} & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 0} \\ L & \xhookrightarrow[j]{} & Q & \xrightarrow[\pi]{} & B \end{array}$$

Take note that only one of these triangles commutes a priori, that is, there is no guarantee that $\psi = \lambda i$. In this case however, the other triangle does indeed commute: $j\psi = \varphi i = j\lambda i$, and j is a monomorphism, so $\psi = \lambda i$. Therefore ψ factors over a projective, so the image of ψ in $\underline{\mathcal{X}}$ is 0.

Now for an object A in $\mathcal{X}$, pick two conflations $K \hookrightarrow P \twoheadrightarrow A$ and $L \hookrightarrow Q \twoheadrightarrow A$ for A with P, Q projective. Consider the following diagram.

$$\begin{array}{ccccc} K & \hookrightarrow & P & \twoheadrightarrow & A \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \| \\ L & \hookrightarrow & Q & \twoheadrightarrow & A \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} & & \| \\ K & \hookrightarrow & P & \twoheadrightarrow & A \end{array}$$

We want to show the composition $g_0 f_0$ factors over a projective, so look at the difference between it and the identity map.

$$\begin{array}{ccccc} K & \hookrightarrow & P & \twoheadrightarrow & A \\ \downarrow{\scriptstyle 1-g_0 f_0} & & \downarrow{\scriptstyle 1-gf} & & \downarrow{\scriptstyle 0} \\ K & \hookrightarrow & P & \twoheadrightarrow & A \end{array}$$

The diagram above is the situation from the last paragraph, so $g_0 f_0$ equals the identity on K in $\underline{\mathcal{X}}$. Similarly for L, and we find $K \cong L$ in $\underline{\mathcal{X}}$, hence the assignment on objects is well-defined.

For morphisms, we appeal again to our work extending the zero morphism. Suppose we are given two conflations $K \hookrightarrow P \twoheadrightarrow A$ and $L \hookrightarrow Q \twoheadrightarrow B$ in $\mathcal{X}$ with P, Q projective and a map $h \colon A \to B$. Assume furthermore that we have two lifts $f, g \colon P \to Q$ that suit the diagram, and $f_0, g_0 \colon K \to L$ the induced maps. Take the difference to get a diagram extending the zero map,

$$\begin{array}{ccccc} K & \hookrightarrow & P & \twoheadrightarrow & A \\ \downarrow{\scriptstyle f_0-g_0} & & \downarrow{\scriptstyle f-g} & & \downarrow{\scriptstyle 0} \\ L & \hookrightarrow & Q & \twoheadrightarrow & B \end{array}$$

and we conclude $f_0 = g_0$ in $\underline{\mathcal{X}}$. The rest, that Ω preserves identities, compositions, etc., follows from the same tactic as above. $\square$

EXAMPLE 1.1.11. Let R be a right-noetherian ring, so $\mathsf{mod}\, R$ is an abelian (and thus exact) category with enough projectives. Given a right R-module M and two short exact sequences $N \hookrightarrow P \twoheadrightarrow M$ and $N' \hookrightarrow P' \twoheadrightarrow M$ with P, P' projective modules, Schanuel's Lemma says $N \oplus P \cong N' \oplus P'$. Notice that N and N' are both syzygies of M, isomorphic up to a projective module. Projectively stabilizing $\mathsf{mod}\, R$ removes projective summands, so the lemma amounts to saying that taking syzygies is indeed unique in $\mathsf{mod}\, R/\,\mathsf{proj}(R)$, as should be expected from the last proposition. More generally, Schanuel's Lemma holds for any exact category with enough projectives, so we can be assured (or perhaps impressed!) that the result depends only on properties of exact categories, rather than the specific characteristics of modules over a noetherian ring.

Additionally, returning to Example 1.1.6, consider $\Lambda = k[x]/(x^n)$. Computing syzygies of the nontrivial indecomposable Λ-modules $k[x]/(x^i)$, $1 \leq i \leq n$, is immediate upon considering the short exact sequence

$$0 \to k[x]/(x^{n-i}) \to \Lambda \to k[x]/(x^i) \to 0$$

with the injection and surjection given by $\cdot x^i$ and reduction modulo x^i respectively. Furthermore, symmetry in i and $n-i$ implies that higher syzygies, that is, syzygies of syzygies, can be computed with the same short exact sequence, and accordingly higher syzygies of $k[x]/(x^i)$ are 2-periodic:

$$\underbrace{\Omega \cdots \Omega}_{j \text{ times}} k[x]/(x^i) \cong \begin{cases} k[x]/(x^{n-i}) & j \text{ odd}, \\ k[x]/(x^i) & j \text{ even}. \end{cases}$$

This isomorphism, of course, is in $\mathsf{mod}\, \Lambda/\,\mathsf{proj}(\Lambda)$.

Dual to syzygies, if $\mathcal{X}$ is an exact category with enough injectives, $\mathsf{Inj}(\mathcal{X})$ the full subcategory of injective objects in $\mathcal{X}$, then we can form conflations

$A \hookrightarrow I \twoheadrightarrow C$ out of A with I injective and set $\Sigma A := C$. The assignment Σ defines an additive endofunctor on the injective stabilization $\overline{\mathcal{X}} := \mathcal{X}/\mathsf{Inj}(\mathcal{X})$ of $\mathcal{X}$, called the **cosyzygy functor**. As is ever the case with formal duality, we have a dual result to the last, the verification for which we omit.

PROPOSITION 1.1.12. *The assignment $A \mapsto \Sigma A$ is functorial in $\overline{\mathcal{X}}$.*

Our notation hints at the upcoming investigation of triangulated categories, and moreover has historical grounds: In topology, given a space X, one may consider both the suspension ΣX and loop space ΩX. The interaction of Σ and Ω motivated the development of much of the theory we utilize herein. It is known, for example, that the suspension (cosyzygy) and loop space (syzygy) functors are adjoint, e.g., Eckmann–Hilton duality in the homotopy category of topological spaces, but in our investigation we ask for more. We want to study when Σ and Ω define inverse auto-equivalences, and the natural setting to do so is Frobenius categories. If $\mathcal{F}$ is a Frobenius category, then the projective stabilization $\underline{\mathcal{F}}$ and the injective stabilization $\overline{\mathcal{F}}$ coincide—or at least there is an *isomorphism* of categories identifying the two—so we get two endofunctors Ω and Σ acting on the stabilization. For consistency, we fix the notation $\underline{\mathcal{F}}$ for the projective/injective stabilization of $\mathcal{F}$.

PROPOSITION 1.1.13. *The functors Ω and Σ define inverse auto-equivalences on the projective/injective stabilization of any Frobenius category.*

PROOF. Let $\mathcal{F}$ be a Frobenius category, $\underline{\mathcal{F}}$ the projective/injective stabilization of $\mathcal{F}$, Ω the syzygy and Σ the cosyzygy functors on $\underline{\mathcal{F}}$. The claim boils down to the functoriality of Ω and Σ. Note that we write, for example, ΩA for an arbitrary fixed representative in $\mathcal{F}$ of the unique object in $\underline{\mathcal{F}}$. Consider $f\colon A \to B$ in $\mathcal{F}$. Using the conflations $\Omega A \hookrightarrow P \twoheadrightarrow A$

and $\Omega B \hookrightarrow Q \twoheadrightarrow B$, with P, Q projective/injective, the map f induces $\Omega f\colon \Omega A \to \Omega B$ in $\mathcal{F}$. We can compare $\Omega A \hookrightarrow I \twoheadrightarrow \Sigma\Omega A$ to $\Omega A \hookrightarrow P \twoheadrightarrow A$, where I is projective/injective, and the identity map on ΩA induces a map $\varepsilon_A\colon \Sigma\Omega A \to A$, giving the following commutative diagram in $\mathcal{F}$.

$$\begin{array}{ccccc}
\Omega A & \hookrightarrow & I & \twoheadrightarrow & \Sigma\Omega A \\
\| & & \downarrow{\scriptstyle \exists} & & \downarrow{\scriptstyle \exists! \varepsilon_A} \\
\Omega A & \hookrightarrow & P & \twoheadrightarrow & A \\
\downarrow{\scriptstyle \Omega f} & & \downarrow & & \downarrow{\scriptstyle f} \\
\Omega B & \hookrightarrow & Q & \twoheadrightarrow & B
\end{array}$$

Functoriality of Σ implies the residue of ε_A in $\underline{\mathcal{F}}$ is an isomorphism. On the other hand, the conflation $\Omega B \hookrightarrow J \twoheadrightarrow \Sigma\Omega B$ gives the next commutative diagram in $\mathcal{F}$ (again, J is projective/injective).

$$\begin{array}{ccccc}
\Omega A & \hookrightarrow & I & \twoheadrightarrow & \Sigma\Omega A \\
\downarrow{\scriptstyle \Omega f} & & \downarrow{\scriptstyle \exists} & & \downarrow{\scriptstyle \exists! \Sigma\Omega f} \\
\Omega B & \hookrightarrow & J & \twoheadrightarrow & \Sigma\Omega B \\
\| & & \downarrow{\scriptstyle \exists} & & \downarrow{\scriptstyle \exists! \varepsilon_B} \\
\Omega B & \hookrightarrow & Q & \twoheadrightarrow & B
\end{array}$$

Again, the residue of ε_B in $\underline{\mathcal{F}}$ is an isomorphism. Combining the last two diagrams, we get the following situation.

$$\begin{array}{ccccc}
\Omega A & \hookrightarrow & I & \twoheadrightarrow & \Sigma\Omega A \\
\downarrow{\scriptstyle \Omega f} & & \downarrow\downarrow & & {\scriptstyle f\varepsilon_A}\downarrow\downarrow{\scriptstyle \varepsilon_B(\Sigma\Omega f)} \\
\Omega B & \hookrightarrow & Q & \twoheadrightarrow & B
\end{array}$$

The rightmost pair of vertical maps both arise from Ωf, depending on different choices of the middle vertical map. As we know, the residue of the right vertical map depends only on the left vertical map, not the middle vertical map, so $f\varepsilon_A = \varepsilon_B(\Sigma\Omega f)$ in $\underline{\mathcal{F}}$. Therefore there exists a natural isomorphism $\varepsilon\colon \Sigma\Omega \to \mathbf{1}_{\underline{\mathcal{F}}}$. Similarly, one can find a natural isomorphism $\mathbf{1}_{\underline{\mathcal{F}}} \to \Omega\Sigma$, proving Ω and Σ define inverse auto-equivalences on $\underline{\mathcal{F}}$. $\square$

EXAMPLE 1.1.14. Here is an intuitive description of the last proposition, at least on the level of objects. Recall Schanuel's Lemma (Example 1.1.11), and consider the dual statement: Given an exact category $\mathcal{X}$ with enough injectives and two conflations $A \hookrightarrow I \twoheadrightarrow C$ and $A \hookrightarrow I' \twoheadrightarrow C'$ with I, I' injective, then $I \oplus C \cong I' \oplus C'$ in $\mathcal{X}$. Suppose $\mathcal{X}$ is Frobenius and $A \in \mathcal{X}$. Schanuel's Lemma implies that any fixed lift of $\Omega\Sigma A$ can be made isomorphic (in $\mathcal{X}$) to A upon adding projective and injective summands to both; the dual of Schanuel's Lemma implies the same about $\Sigma\Omega A$ and A. It follows that Ω and Σ are inverses on objects in the projective/injective stabilization of $\mathcal{X}$.

Building on our recurring example, consider once more $\Lambda = k[x]/(x^n)$. Recall (Example 1.1.6) that Λ is self-injective, so the argument from Example 1.1.11 yields the cosyzygies of an indecomposable Λ-module:

$$\underbrace{\Sigma \cdots \Sigma}_{j \text{ times}} k[x]/(x^i) \cong \begin{cases} k[x]/(x^{n-i}) & j \text{ odd}, \\ k[x]/(x^i) & j \text{ even}. \end{cases}$$

Now we can concretely verify $\Sigma\Omega(k[x]/(x^i)) \cong k[x]/(x^i)$ in $\mathsf{mod}\,\Lambda/\,\mathsf{proj}(\Lambda)$, either by the argument from the last paragraph or using the formulas for syzygies and cosyzygies of $k[x]/(x^i)$.

1.1.3. Triangulation of stabilized Frobenius categories. We now turn our attention to triangulated categories, the goal being to show the stabilization of a Frobenius category with respect to its class of projective objects—or, equivalently, injective objects—is a triangulated category (called an **algebraic triangulated category**). Recall the definition, as appears in [12, Chapter 1] or [15, Definition 10.2.1]:

Definition 1.1.15. Let $\mathcal{T}$ be an additive category equipped with an additive auto-equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$. Generally, Σ is called the **suspension** or **shift** functor. A **triangle** in $\mathcal{T}$ is a sequence of objects and maps (X, Y, Z, f, g, h) in $\mathcal{T}$ that takes the following shape:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Often, as is shorthand in the literature, we may replace ΣX by $X[1]$, and more generally $\Sigma^i X$ by $X[i]$ for all $i \in \mathbf{Z}$. A morphism between triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') is a triple (u, v, w) of morphisms in $\mathcal{T}$ that make the following diagram commute.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

We call the category $\mathcal{T}$ a **triangulated category** if it has a class Δ of so-called **distinguished** triangles and the following four axioms hold.

TR1 Δ is isomorphism closed. Every triangle of the form

$$X \xrightarrow{id} X \to 0 \to \Sigma X$$

belongs to Δ. Any morphism $f\colon X \to Y$ can be embedded into a distinguished triangle, i.e., for any such f there exists maps g, h and an object Z so that $(X, Y, Z, f, g, h) \in \Delta$.

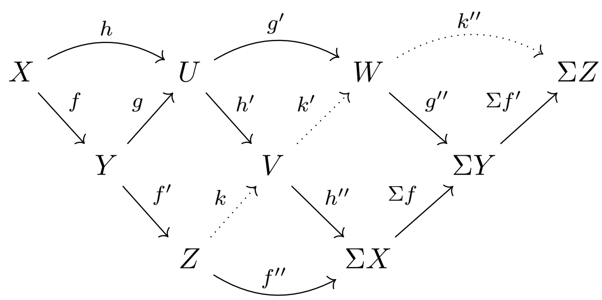
TR2 If $(X, Y, Z, f, g, h) \in \Delta$, then $(Y, Z, \Sigma X, g, h, -\Sigma f) \in \Delta$.

TR3 Given a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

of distinguished triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') with maps $u\colon X \to X'$ and $v\colon Y \to Y'$ so that $f'u = vf$, there exists a map $w\colon Z \to Z'$ making (u, v, w) a morphism of triangles.

TR4 Suppose we are given dishinguished triangles (X, Y, Z, f, f', f''), (Y, U, W, g, g', g''), and (X, U, V, h, h', h'') with the stipulation $gf = h$. Then there exist maps $k\colon Z \to V$, $k'\colon V \to W$, and $k''\colon W \to \Sigma Z$ so that (Z, V, W, k, k', k'') is a distinguished triangle and the following diagram commutes.



Remark 1.1.16. A category that satisfies TR1, TR2, and TR3 is called **pre-triangulated**. The remaining axiom, TR4, is called the **octahedral axiom**, or sometimes **Verdier's axiom**, after Jean-Louis Verdier. While similar axioms were circulating around two years before Verdier's treatment, it was he who devised TR4, which allowed a thorough examination of the derived category in his Ph.D. thesis (published posthumously in [14]). The braid diagram for Verdier's axiom is due to [11] (according to [9, Section 2.3]), however initially, the diagram was non-planar, envisioned by Verdier as an octahedron; see [15, Definition 10.2.1] for an illustration. The axioms have been scrutinized thoroughly in the last half century, and there is still some debate over whether TR4 follows from the other three. Some treatments state TR2 as a biconditional, but [11] showed that one direction of TR2, along with TR3, follow from the rest. While this streamlines matters,

we forgo the shortcut in the upcoming proof, as we will use facts about pre-triangulated categories in the verification of TR4.

Definition 1.1.17 ([12, Definition 1.1.7]). Given a triangulated category $\mathcal{T}$ and an abelian category $\mathcal{A}$, a functor $H\colon \mathcal{T} \to \mathcal{A}$ is **homological** if every distinguished triangle

$$X \to Y \to Z \to \Sigma X$$

is sent to an exact sequence

$$HX \to HY \to HZ.$$

Moreover, axiom TR2 implies that this exact sequence can be extended to a long exact sequence

$$\cdots \to H(\Sigma^{-1}Z) \to HX \to HY \to HZ \to H(\Sigma X) \to \cdots$$

in $\mathcal{A}$. If H is instead contravariant, we call it **cohomological**.

Lemma 1.1.18 (Triangulated 5 Lemma). *Suppose $\mathcal{T}$ is a pre-triangulated category and we have the following morphism of triangles.*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

If two of u, v, w are isomorphisms, then so is the third.

Proof. In light of TR2 (and since Σ is an auto-equivalence), we assume u and v are isomorphisms, and we accept one well known fact without proof: For any object $W \in \mathcal{T}$, the covariant functor $H_W(-) := \mathrm{Hom}_{\mathcal{T}}(W, -)$

is homological. Applying $H_W(-)$ to the situation gives the following commutative diagram of abelian groups.

$$\begin{array}{ccccccccc} H_W(X) & \longrightarrow & H_W(Y) & \longrightarrow & H_W(Z) & \longrightarrow & H_W(\Sigma X) & \longrightarrow & H_W(\Sigma Y) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle H_W(w)} & & \downarrow & & \downarrow \\ H_W(X') & \longrightarrow & H_W(Y') & \longrightarrow & H_W(Z') & \longrightarrow & H_W(\Sigma X') & \longrightarrow & H_W(\Sigma Y') \end{array}$$

Every vertical morphism except $H_W(w)$ is necessarily an isomorphism, so by the 5 Lemma (for abelian groups), we conclude that $H_W(w)$ is an isomorphism. But this holds for all objects W, so the Yoneda Lemma implies w is an isomorphism. □

Pre-triangulated categories allow us to study homology/cohomology in categories that fail to be abelian by replacing short exact sequences with triangles. One of the properties maintained in this substitution is vanishing composition, which is to say, given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

the compositions gf, hg, and $(-\Sigma f)h$ are all 0. For $gf = 0$, compare the above triangle to the distinguished (by TR1) triangle $X = X \to 0 \to \Sigma X$ and apply TR3; the other two follow similarly by TR2. In a pre-triangulated category, any representable covariant (resp. contravariant) functor is homological (resp. cohomological), a fact we utilized in the proof of Lemma 1.1.18 above. Searching for examples of triangulated categories, we arrive at the next result classifying a broad class of triangulated categories. The theorem will furthermore be central to arguments in the following chapters.

THEOREM 1.1.19. *The projective/injective stabilization of a Frobenius category is triangulated.*

We will use the several diagram lemmas in the proof of the theorem. The verification of both amounts to the usual game of cat and mouse, so we omit proofs and refer the intent reader to [4].

LEMMA 1.1.20. *In a commutative square*

$$\begin{array}{ccc} A & \hookrightarrow & C \\ \downarrow & & \downarrow \\ B & \hookrightarrow & D \end{array}$$

with inflations for horizontal arrows, the following are equivalent:

(1) *The square is a pushout.*

(2) *The square fits into a commutative diagram*

$$\begin{array}{ccccc} A & \hookrightarrow & C & \twoheadrightarrow & E \\ \downarrow & & \downarrow & & \| \\ B & \hookrightarrow & D & \twoheadrightarrow & E \end{array}$$

where the rows are conflations.

LEMMA 1.1.21 (3×3 Lemma). *Consider a commutative diagram*

$$\begin{array}{ccccc} A' & \hookrightarrow & B' & \twoheadrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \hookrightarrow & B'' & \twoheadrightarrow & C'' \end{array}$$

with columns and outer rows conflations. If $gf = 0$, then the middle row is a conflation.

PROOF OF THEOREM 1.1.19. Let $\mathcal{F}$ be a Frobenius category, $\underline{\mathcal{F}}$ the projective/injective stabilization, and Σ the cosyzygy functor on $\underline{\mathcal{F}}$. By Proposition 1.1.13, Σ is an auto-equivalence. We begin by exhibiting standard distinguished triangles.

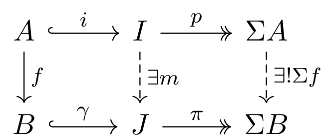
Given a morphism $f\colon A \to B$ and conflation $A \hookrightarrow I \twoheadrightarrow \Sigma A$ with I injective, take the pushout of $i\colon A \hookrightarrow I$ along f, and extend to the following commutative diagram by Lemma 1.1.20.

$$\begin{array}{ccccc} A & \overset{i}{\hookrightarrow} & I & \overset{p}{\twoheadrightarrow} & \Sigma A \\ \downarrow{\scriptstyle f} & \ulcorner & \downarrow{\scriptstyle g} & & \Vert \\ B & \overset{j}{\hookrightarrow} & \operatorname{cone}(f) & \overset{q}{\twoheadrightarrow} & \Sigma A \end{array}$$

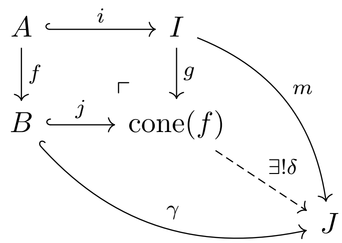
Note that, while the cosyzygy object ΣA is unique up to isomorphism in $\underline{\mathcal{F}}$ (by Proposition 1.1.12), it is not well-defined in $\mathcal{F}$. By ΣA we mean any fixed object of $\mathcal{F}$ that satisfies the diagram and whose image in $\underline{\mathcal{F}}$ is isomorphic to ΣA. Wherever the specific choice does not matter, we retain the abuse of notation. We designate diagrams of the form $A \xrightarrow{f} B \to \operatorname{cone}(f) \to \Sigma A$ arising from the construction above as prototypical distinguished triangles, and we stipulate that any diagram isomorphic to a prototypical distinguished triangle is itself distinguished—an isomorphism of distinguished triangles is a morphism (u, v, w) of distinguished triangles, as in Definition 1.1.15, where u, v, w are all isomorphisms. Now we verify the axioms using diagram arguments in $\mathcal{F}$.

Most of TR1 is free by construction. All we must show is that $A = A \to 0 \to \Sigma A$ is distinguished. Working in $\mathcal{F}$, embed A into an injective I and take the pushout along $A = A$; the pushout is necessarily I. Complete the diagram to include cokernels and excise the triangle. Since $I \cong 0$ in $\underline{\mathcal{F}}$, we are done. Therefore it suffices to check the remaining axioms only on prototypical distinguished triangles.

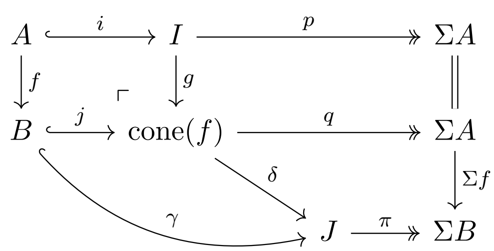
For TR2, consider the distinguished triangle $A \xrightarrow{f} B \to \operatorname{cone}(f) \to \Sigma A$. As with $A \hookrightarrow I \twoheadrightarrow \Sigma A$, embed B into an injective J via γ and exhibit the cokernel $\pi\colon J \twoheadrightarrow \Sigma B$. Then since $i : A \hookrightarrow I$ is an inflation and J is injective, there exists a map $m\colon I \to J$ such that $\gamma f = mi$, which induces a unique map Σf on the cokernels so that $(\Sigma f)p = \pi m$.



Looking at the pushout of i along f, we induce a unique map $\delta\colon \operatorname{cone}(f) \to J$ such that $m = \delta g$ and $\gamma = \delta j$ by the universal property of the pushout.



Extend the diagram to include ΣA and ΣB.



Then $\pi\delta g = \pi m$ by the pushout property, $\pi m = (\Sigma f)p$ by an earlier commutative diagram, and $(\Sigma f)p = (\Sigma f)qg$, as was shown at the start of the proof, thus $\pi\delta g = (\Sigma f)qg$. Additionally, $\pi\delta j = \pi\gamma = 0$, so the pushout property implies there exists a unique map $\operatorname{cone}(f) \to \Sigma B$ that respects all the equalities just found. But there are two such maps, namely $\pi\delta$ and $(\Sigma f)q$, so they must be equal. Now consider the following commutative diagram.

$$\begin{tikzcd} B \arrow[r, hook, "\gamma"] \arrow[d, hook, "j"] & J \arrow[r, two heads, "\pi"] \arrow[d, hook, "\binom{1}{0}"] & \Sigma B \arrow[d, equal] \\ \operatorname{cone}(f) \arrow[r, "\binom{\delta}{q}"] \arrow[d, two heads, "q"] & J \oplus \Sigma A \arrow[r, "(\pi\ -\Sigma f)"] \arrow[d, two heads, "(0\ 1)"] & \Sigma B \\ \Sigma A \arrow[r, equal] & \Sigma A & \end{tikzcd}$$

Here the columns and outer rows are conflations, and the composition in the middle row is 0, so the middle row is a conflation by Lemma 1.1.21. Moreover, Lemma 1.1.20 implies that the upper left square is a pushout. Therefore

$$B \xrightarrow{\quad j \quad} \operatorname{cone}(f) \xrightarrow{\binom{\delta}{q}} J \oplus \Sigma A \xrightarrow{(\pi\ -\Sigma f)} \Sigma B$$

is a (prototypical) distinguished triangle by construction, and we can compare it to the rotation of the triangle with which we started. Canonical inclusion into and projection out of the second component $\Sigma A \to J \oplus \Sigma A$ and $J \oplus \Sigma A \to \Sigma A$ yield the following diagram.

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, equal] & \operatorname{cone}(f) \arrow[r, "\binom{\delta}{q}"] \arrow[d, equal] & J \oplus \Sigma A \arrow[r, "(\pi\ -\Sigma f)"] \arrow[d, bend left, "(0\ 1)"] & \Sigma B \arrow[d, equal] \\ B \arrow[r, "j"] & \operatorname{cone}(f) \arrow[r, "q"] & \Sigma A \arrow[r, "-\Sigma f"] \arrow[u, bend left, "\binom{0}{1}"] & \Sigma B \end{tikzcd}$$

While the diagram fails to commute in $\mathcal{F}$, upon projecting down to $\underline{\mathcal{F}}$, we get an isomorphism of triangles. Therefore the rotated triangle

$$B \xrightarrow{j} \operatorname{cone}(f) \xrightarrow{q} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$$

is distinguished.

For TR3, we are given two distinguished triangles arising the following diagrams.

$$\begin{array}{ccccc} A & \overset{i}{\hookrightarrow} & I & \overset{p}{\twoheadrightarrow} & \Sigma A \\ \downarrow f & \ulcorner & \downarrow \hat{f} & & \| \\ B & \overset{j}{\hookrightarrow} & \operatorname{cone}(f) & \overset{q}{\twoheadrightarrow} & \Sigma A \end{array} \qquad \begin{array}{ccccc} C & \overset{i'}{\hookrightarrow} & J & \overset{p'}{\twoheadrightarrow} & \Sigma C \\ \downarrow g & \ulcorner & \downarrow \hat{g} & & \| \\ D & \overset{j'}{\hookrightarrow} & \operatorname{cone}(g) & \overset{q'}{\twoheadrightarrow} & \Sigma C \end{array}$$

Suppose there exists maps $u\colon A \to C$ and $v\colon B \to D$ such that $vf = gu$ in $\underline{\mathcal{F}}$. We want a map $w\colon \operatorname{cone}(f) \to \operatorname{cone}(g)$ such that (u, v, w) is a morphism of triangles. As $vf - gu = 0$ in $\underline{\mathcal{F}}$, there exists an injective I_0 that factors $vf - gu$ in $\mathcal{F}$, i.e., we have maps $\varepsilon\colon A \to I_0$ and $\beta\colon I_0 \to D$ with $\beta\varepsilon = vf - gu$. Note that ε necessarily factors through I since $i\colon A \hookrightarrow I$ is an inflation, giving the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{vf-gu} & D \\ \downarrow i & \searrow \varepsilon & \uparrow \beta \\ I & \overset{\eta}{\dashrightarrow} & I_0 \end{array}$$

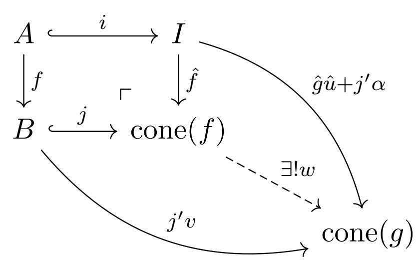
Thus, instead of factoring $vf - gu$ over I_0, we can factor over I, and letting $\alpha := \beta\eta$, we get $vf - gu = \alpha i$. Additionally, as I in injective, we can induce maps $\hat{u}\colon I \to J$ and $\Sigma u\colon \Sigma A \to \Sigma C$ making the following diagram commute.

$$\begin{array}{ccccc} A & \overset{i}{\hookrightarrow} & I & \overset{p}{\twoheadrightarrow} & \Sigma A \\ \downarrow u & & \downarrow \exists \hat{u} & & \downarrow \exists! \Sigma u \\ C & \overset{i'}{\hookrightarrow} & J & \overset{p'}{\twoheadrightarrow} & \Sigma C \end{array}$$

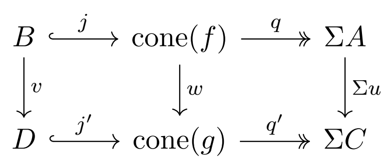
Now we can induce the required map w by studying the pushout of i along f. We have maps $j'v\colon B \to \operatorname{cone}(g)$ and $(\hat{g}\hat{u} + j'\alpha)\colon I \to \operatorname{cone}(g)$, and

$$\begin{aligned} (\hat{g}\hat{u} + j'\alpha)i &= \hat{g}\hat{u}i + j'\alpha i \\ &= \hat{g}i'u + j'(vf - gu) \\ &= j'gu + j'vf - j'gu = j'vf, \end{aligned}$$

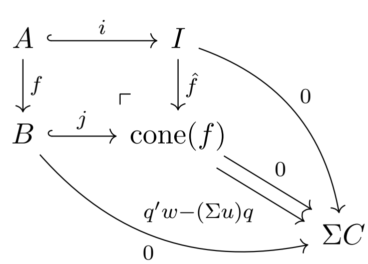
so by the pushout property there exists a unique map $w\colon \operatorname{cone}(f) \to \operatorname{cone}(g)$ such that $wj = j'v$ and $w\hat{f} = \hat{g}\hat{u} + j'\alpha$, in other words, making the following diagram commute.



Now that we have a candidate map w, we must show that (u, v, w) is a morphism of triangles. This is surprisingly straightforward, as we can actually show the following diagram commutes in $\mathcal{F}$.



From the construction of w we get that the left square commutes. To show the right square commutes, we appeal to the pushout of i along f once again. We want that $q'w - (\Sigma u)q = 0$, so if we can show that $(q'w - (\Sigma u)q)j = 0 = (q'w - (\Sigma u)q)\hat{f}$, then there are two maps, namely 0 and $q'w - (\Sigma u)q$, from $\operatorname{cone}(f)$ to ΣC satisfying the following diagram, so uniqueness (from the pushout property) implies they are equal.



Two computations complete the proof of TR3. Since both (j, q) and (j', q') are kernel-cokernel pairs and by construction of w,

$$\begin{aligned}(q'w - (\Sigma u)q)j &= q'wj - (\Sigma u)qj \\ &= q'j'v - 0 = 0,\end{aligned}$$

and

$$\begin{aligned}(q'w - (\Sigma u)q)\hat{f} &= q'w\hat{f} - (\Sigma u)q\hat{f} \\ &= q'(\hat{g}\hat{u} + j'\alpha) - (\Sigma u)p \\ &= q'\hat{g}\hat{u} + q'j'\alpha - (\Sigma u)p \\ &= p'\hat{u} - (\Sigma u)p \\ &= (\Sigma u)p - (\Sigma u)p = 0.\end{aligned}$$

Therefore $w\colon \operatorname{cone}(f) \to \operatorname{cone}(g)$ is a map in $\mathcal{F}$ such that (u, v, w) is a morphism of triangles in $\underline{\mathcal{F}}$.

We have shown that $\underline{\mathcal{F}}$ is pre-triangulated. For the octahedral axiom, suppose we are given three (prototypical, by TR1) distinguished triangles

$$\begin{aligned}&A \xrightarrow{f} B \xrightarrow{f'} \operatorname{cone}(f) \xrightarrow{f''} \Sigma A, \\ &A \xrightarrow{h} C \xrightarrow{h'} \operatorname{cone}(h) \xrightarrow{h''} \Sigma A, \\ &B \xrightarrow{g} C \xrightarrow{g'} \operatorname{cone}(g) \xrightarrow{g''} \Sigma B,\end{aligned}$$

such that $gf = h$. For the first two triangles, we may assume they arise from the following commutative diagrams.

$$\begin{array}{ccccc} A & \overset{i}{\hookrightarrow} & I & \overset{p}{\twoheadrightarrow} & \Sigma A \\ \downarrow{\scriptstyle f} & \ulcorner & \downarrow{\scriptstyle \hat{f}} & & \Vert \\ B & \overset{f'}{\hookrightarrow} & \operatorname{cone}(f) & \overset{f''}{\twoheadrightarrow} & \Sigma A \end{array} \qquad \begin{array}{ccccc} A & \overset{i}{\hookrightarrow} & I & \overset{p}{\twoheadrightarrow} & \Sigma A \\ \downarrow{\scriptstyle h=gf} & \ulcorner & \downarrow{\scriptstyle \hat{h}} & & \Vert \\ D & \overset{h'}{\hookrightarrow} & \operatorname{cone}(h) & \overset{h''}{\twoheadrightarrow} & \Sigma C \end{array}$$

For the third triangle, which comes from a diagram looking like the following, we need to make a replacement.

$$\begin{array}{ccccc} B & \hookrightarrow & J & \twoheadrightarrow & \Sigma B \\ \downarrow{\scriptstyle g} & \ulcorner & \downarrow & & \| \\ C & \hookrightarrow & \operatorname{cone}(g) & \twoheadrightarrow & \Sigma B \end{array}$$

Using the embedding (inflation) $\ell\colon \operatorname{cone}(f) \hookrightarrow K$, where K is injective, we get a map $\ell f'\colon B \hookrightarrow K$, which, as the composition of inflations, is itself an inflation (axiom E1). Accordingly, there must be a corresponding deflation $K \twoheadrightarrow B'$. We may now assume we have the following commutative diagram.

$$\begin{array}{ccccc} B & \xhookrightarrow{\ell f'} & K & \xrightarrow{p'} & B' \\ \downarrow{\scriptstyle g} & \ulcorner & \downarrow{\scriptstyle \hat{g}} & & \| \\ C & \xhookrightarrow{g'} & E & \xrightarrow{g''} & B' \end{array}$$

By Proposition 1.1.12 and Lemma 1.1.18, the triangles $B \to C \to \operatorname{cone}(g) \to \Sigma B$ and $B \to C \to E \to B'$ are isomorphic in $\underline{\mathcal{F}}$, so we assume from now on that E and B' are replaced by $\operatorname{cone}(g)$ and ΣB respectively in the diagram above. After all, the symbols $\operatorname{cone}(g)$ and ΣB represent one arbitrary lift of unique (up to isomorphism) objects in $\underline{\mathcal{F}}$, so we may as well choose the most advantageous lift. Consider the resulting diagram.

$$\begin{array}{ccccc} A & \xhookrightarrow{i} & I & \xrightarrow{p} & \Sigma A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \ell\hat{f}} & & \downarrow{\scriptstyle \exists!\Sigma f} \\ B & \xhookrightarrow{\ell f'} & K & \xrightarrow{p'} & \Sigma B \end{array}$$

The left square commutes automatically, as it is the result of applying ℓ to the pushout of i along f. Factoring through the cokernel yields a unique map Σf making the right square commute, so we have $(\Sigma f)p = p'\ell\hat{f}$.

We want maps $k\colon \operatorname{cone}(f) \to \operatorname{cone}(h)$ and $k'\colon \operatorname{cone}(h) \to \operatorname{cone}(g)$ so that

$$\operatorname{cone}(f) \xrightarrow{k} \operatorname{cone}(h) \xrightarrow{k'} \operatorname{cone}(g) \xrightarrow{(\Sigma f')g''} \Sigma \operatorname{cone}(f)$$

is a distinguished triangle compatible with the diagram given in Definition 1.1.15. As before, we use the property of the pushout to induce the required maps. First, we have maps $\hat{h}\colon I \to \operatorname{cone}(h)$ and $h'g\colon B \to \operatorname{cone}(h)$ such that $\hat{h}i = h'h = h'gf$. Now upon considering the pushout of i along f, we find there exists a unique map $k\colon \operatorname{cone}(f) \to \operatorname{cone}(h)$ such that $k\hat{f} = \hat{h}$ and $kf' = h'g$. Next we have maps $\hat{g}\ell\hat{f}\colon I \to \operatorname{cone}(g)$ and $g'\colon C \to \operatorname{cone}(g)$ such that

$$\hat{g}\ell\hat{f}i = \hat{g}\ell f'f = g'gf = g'h,$$

so looking at the pushout of i along h gives a unique map $k'\colon \operatorname{cone}(h) \to \operatorname{cone}(g)$ such that $k'\hat{h} = \hat{g}\ell\hat{f}$ and $k'h' = g'$.

Let $k'' := (\Sigma f')g''$. For clarity, we record our progress using the following diagram of the octahedral axiom.

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{f'} & \operatorname{cone}(f) & \xrightarrow{f''} & \Sigma A \\
\| & & \downarrow{\scriptstyle g} & (1) & \downarrow{\scriptstyle k} & (2) & \| \\
A & \xrightarrow{h} & C & \xrightarrow{h'} & \operatorname{cone}(h) & \xrightarrow{h''} & \Sigma A \\
\downarrow{\scriptstyle f} & & \| & (3) & \downarrow{\scriptstyle k'} & (4) & \downarrow{\scriptstyle \Sigma f} \\
B & \xrightarrow{g} & C & \xrightarrow{g'} & \operatorname{cone}(g) & \xrightarrow{g''} & \Sigma B \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle h'} & & \| & & \downarrow{\scriptstyle \Sigma f'} \\
\operatorname{cone}(f) & \xrightarrow{k} & \operatorname{cone}(h) & \xrightarrow{k'} & \operatorname{cone}(g) & \xrightarrow{k''} & \Sigma \operatorname{cone}(f)
\end{array}$$

In the last paragraph, we showed squares (1) and (3) commute, so it remains to show that the other two commute and that the bottom row is a distinguished triangle. For square (2), consider the map $(f'' - h''k)\colon \operatorname{cone}(f) \to \Sigma A$ and argue by the pushout of i along f. Since

$$(f'' - h''k)\hat{f} = f''\hat{f} - h''k\hat{f} = p - h''\hat{h} = p - p = 0$$

and

$$(f'' - h''k)f' = f''f' - h''kf' = 0 - h''kf' = h''h'g = 0,$$

the pushout property implies $f'' - h''k = 0$, so square (2) commutes. Then square (4) again boils down to using the pushout diagram of i along h. As

$$((\Sigma f)h'' - g''k')\hat{h} = (\Sigma f)h''\hat{h} - g''k'\hat{h} = (\Sigma f)p - g''\hat{g}\ell\hat{f} = p'\ell\hat{f} - p'\ell\hat{f} = 0$$

and

$$((\Sigma f)h'' - g''k')h' = (\Sigma f)h''h' - g''k'h' = 0 - g''g' = 0,$$

the pushout property gives $(\Sigma f)h'' - g''k' = 0$, that is, square (4) commutes.

To show the bottom row of the last diagram is a distinguished triangle, we start by showing $k'k = \hat{g}\ell$. Note that

$$(k'k - \hat{g}\ell)\hat{f} = k'k\hat{f} - \hat{g}\ell\hat{f} = k'\hat{h} - k'\hat{h} = 0$$

and

$$(k'k - \hat{g}\ell)f' = k'kf' - \hat{g}\ell f' = k'h'g - g'g = g'g - g'g = 0,$$

so as $\operatorname{cone}(f)$ is the pushout of i along f, we find $k'k = \hat{g}\ell$. Now consider the following commutative diagram.

$$\begin{array}{ccccc}
A & \xrightarrow{i} & I & & \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \hat{f}} & & \\
B & \xrightarrow{f'} & \operatorname{cone}(f) & \xrightarrow{\ell} & K \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \hat{g}} \\
C & \xrightarrow{h'} & \operatorname{cone}(h) & \xrightarrow{k'} & \operatorname{cone}(g)
\end{array}$$

The upper left square is a pushout, and since $h = gf$ and $\hat{h} = k\hat{f}$, the left rectangle is a pushout. By the pasting lemma for pushouts, the lower left square is a pushout. Then the bottom rectangle is a pushout because $g' = k'h'$, so the pasting lemma implies that the lower right square is a pushout. Therefore, as it arises from a pushout diagram, $\operatorname{cone}(f) \to \operatorname{cone}(h) \to \operatorname{cone}(g) \to \Sigma\operatorname{cone}(f)$ is a distinguished triangle. This completes the proof. □

1.2. Cotorsion theory

In Chapter 2, we take the opportunity to discuss some structural aspects of certain module categories, with particular focus placed on cotorsion pairs and approximation. We outline the general theory of cotorsion pairs in this section, which allow us to decompose an abelian category into "orthogonal" subcategories, in a certain sense. We also treat approximation theory, which generalizes notions like projective covers and injective hulls, and we propose a weakened torsion theory suitable for any additive category.

1.2.1. Cotorsion pairs. Let $\mathcal{A}$ be an abelian category. We open with some definitions.

DEFINITION 1.2.1. A pair $(\mathcal{C}, \mathcal{D})$ of classes of objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{A}$ is called a **cotorsion pair** if $\mathcal{C}$ and $\mathcal{D}$ are mutually Ext^1-orthogonal, that is

(1) $C \in \mathcal{C}$ if and only if $\mathrm{Ext}^1(C, D) = 0$ for all $D \in \mathcal{D}$, and
(2) $D \in \mathcal{D}$ if and only if $\mathrm{Ext}^1(C, D) = 0$ for all $C \in \mathcal{C}$.

In a cotorsion pair $(\mathcal{C}, \mathcal{D})$, $\mathcal{C}$ is called a **cotorsion class** and $\mathcal{D}$ is called a **cotorsion-free class**. Without any issue, we may write $\mathcal{C}$ and $\mathcal{D}$ for the isomorphism closed full subcategories of $\mathcal{A}$ containing the object classes of the same name.

Note that the two conditions defining cotorsion pairs is stronger than requiring $\mathrm{Ext}^1(\mathcal{C}, \mathcal{D}) = 0$. In this case, we would have

$$\mathcal{D} \subseteq \mathcal{C}^{\perp_1} := \{A \in \mathcal{A} \colon \mathrm{Ext}^1(\mathcal{C}, A) = 0\} \text{ and}$$
$$\mathcal{C} \subseteq {}^{\perp_1}\mathcal{D} := \{A \in \mathcal{A} \colon \mathrm{Ext}^1(A, \mathcal{D}) = 0\}.$$

Instead, we are requiring that the cotorsion class and cotorsion-free class mutually determine the other, that is, $\mathcal{C}^{\perp_1} = \mathcal{D}$ and $\mathcal{C} = {}^{\perp_1}\mathcal{D}$.

The easiest example, available in every abelian category, is when $\mathcal{D} = \mathcal{A}$. Requiring $C \in \mathcal{C}$ such that $\mathrm{Ext}^1(C, D) = 0$ for all $D \in \mathcal{A}$ amounts to saying that C is projective, hence $(\mathsf{Proj}(\mathcal{A}), \mathcal{A})$ is a cotorsion pair in $\mathcal{A}$. Dually, with $\mathcal{C} = \mathcal{A}$, we find that $(\mathcal{A}, \mathsf{Inj}(\mathcal{A}))$ is a cotorsion pair.

Cotorsion pairs are a tool to decompose a category into orthogonal parts. We can immediately demonstrate several properties of cotorsion and cotorsion-free classes in $\mathcal{A}$.

Lemma 1.2.2. *Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair in $\mathcal{A}$. Then $\mathcal{C}$ and $\mathcal{D}$ are closed under extensions and direct summands. Also, $\mathcal{C}$ contains all the projective objects in $\mathcal{A}$ and $\mathcal{D}$ contains all the injective objects in $\mathcal{A}$.*

Proof. Take a short exact sequence $C \hookrightarrow A \twoheadrightarrow C'$ with $C, C' \in \mathcal{C}$ and apply $\mathrm{Ext}^1(-, D)$ with $D \in \mathcal{D}$, giving an exact sequence

$$\mathrm{Ext}^1(C', D) \longrightarrow \mathrm{Ext}^1(A, D) \longrightarrow \mathrm{Ext}^1(C, D)$$

with the outer terms vanishing, hence $A \in \mathcal{C}$, showing $\mathcal{C}$ is closed under extensions. Let C' be a summand of $C \in \mathcal{C}$. Since Ext commutes with direct sums, $\mathrm{Ext}^1(C, D) = 0$ implies $\mathrm{Ext}^1(C', D) = 0$ for all $D \in \mathcal{D}$, so $C' \in \mathcal{C}$. Therefore $\mathcal{C}$ is closed under retracts. Lastly, for any projective P, $\mathrm{Ext}^1(P, -)$ vanishes on $\mathcal{A}$, so in particular on $\mathcal{D}$, hence $P \in {}^{\perp_1}\mathcal{D} = \mathcal{C}$. The rest follows similarly for $\mathcal{D}$. □

What we encounter in later inquiry will not be pedestrian cotorsion pairs. A more restrictive species of cotorsion pair, which we now christen, appears in Chapter 2.

Definition 1.2.3. A cotorsion pair $(\mathcal{C}, \mathcal{D})$ in $\mathcal{A}$ is called **hereditary** if $\mathrm{Ext}^i(\mathcal{C}, \mathcal{D}) = 0$ for all $i > 0$.

LEMMA 1.2.4. *If* $(\mathcal{C}, \mathcal{D})$ *is a hereditary cotorsion pair in* $\mathcal{A}$*, then* $\mathcal{C}$ *is closed under kernels of epimorphisms and* $\mathcal{D}$ *is closed under cokernels of monomorphisms.*

PROOF. Let $p\colon C \twoheadrightarrow C'$ be an epimorphism of objects in $\mathcal{C}$. For any $D \in \mathcal{D}$, the short exact sequence $\ker p \hookrightarrow C \twoheadrightarrow C'$ induces the exact sequence

$$\operatorname{Ext}^i(C, D) \longrightarrow \operatorname{Ext}^i(\ker p, D) \longrightarrow \operatorname{Ext}^{i+1}(C', D).$$

The outer terms vanish for all $i > 0$, so $\operatorname{Ext}^i(\ker p, D) = 0$ for all $D \in \mathcal{D}$, thus $\ker p \in \mathcal{C}$. Similarly for $\mathcal{D}$. □

The converse is harder, but requiring that $\mathcal{A}$ has enough projectives or injectives does the trick. The conditions are not coupled however, and it is quite reasonable to have a hereditary cotorsion pair $(\mathcal{C}, \mathcal{D})$ with, for example, $\mathcal{C}$ closed under kernels of epimorphisms and $\mathcal{D}$ not closed under cokernels of monomorphisms.

LEMMA 1.2.5. *Let* $(\mathcal{C}, \mathcal{D})$ *be a cotorsion pair in* $\mathcal{A}$*. If* $\mathcal{C}$ *is closed under kernels of epimorphisms and* $\mathcal{A}$ *has enough projectives, then* $(\mathcal{C}, \mathcal{D})$ *is hereditary. If* $\mathcal{D}$ *is closed under cokernels of monomorphisms and* $\mathcal{A}$ *has enough injectives, then* $(\mathcal{C}, \mathcal{D})$ *is hereditary.*

PROOF. Suppose $\mathcal{C}$ is closed under kernels of epimorphisms and $\mathcal{A}$ has enough projectives. For $C \in \mathcal{C}$, we always have an epimorphism $p\colon P \twoheadrightarrow C$ from some projective P. With $D \in \mathcal{D}$, we get an exact sequence

$$\operatorname{Ext}^i(\ker p, D) \longrightarrow \operatorname{Ext}^{i+1}(C, D) \longrightarrow \operatorname{Ext}^{i+1}(P, D)$$

for all $i > 0$. Then $\operatorname{Ext}^1(\ker p, D) = 0$ because $\mathcal{C}$ is closed under kernels of epimorphisms and $\operatorname{Ext}^2(P, D) = 0$ because P is projective, so $\operatorname{Ext}^2(C, D) =$

0 by exactness at $i = 1$. Induction implies $(\mathcal{C}, \mathcal{D})$ is hereditary. The situation with $\mathcal{D}$ is formally dual. □

DEFINITION 1.2.6. A cotorsion pair $(\mathcal{C}, \mathcal{D})$ in $\mathcal{A}$ is called **complete** if, for any object $A \in \mathcal{A}$, there exist short exact sequences

$$A \hookrightarrow \mathsf{d}A \twoheadrightarrow \mathsf{c}'A \quad \text{and} \quad \mathsf{d}'A \hookrightarrow \mathsf{c}A \twoheadrightarrow A$$

with $\mathsf{c}'A, \mathsf{c}A \in \mathcal{C}$ and $\mathsf{d}A, \mathsf{d}'A \in \mathcal{D}$.

REMARK 1.2.7. Warning: the assignments $\mathsf{c}', \mathsf{c} \colon \mathcal{A} \to \mathcal{C}$ and $\mathsf{d}, \mathsf{d}' \colon \mathcal{A} \to \mathcal{D}$ are not functorial in general. We will show what alterations are necessary to make these choices uniquely in Section 1.2.2 (Proposition 1.2.15), however these sequences are still plenty useful without functoriality.

LEMMA 1.2.8. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in $\mathcal{A}$, $C \in \mathcal{C}$, and $D \in \mathcal{D}$. Then $f \colon C \to D$ factors over an object in $\mathcal{C} \cap \mathcal{D}$.*

PROOF. As the cotorsion pair is complete, we can write a short exact sequence $\mathsf{d}'D \hookrightarrow \mathsf{c}D \twoheadrightarrow D$ for D. Now apply $\operatorname{Ext}^i(C, -)$ with $i = 0, 1$, giving the exact sequence

$$\operatorname{Hom}(C, \mathsf{c}D) \to \operatorname{Hom}(C, D) \to \operatorname{Ext}^1(C, \mathsf{d}D) \to \operatorname{Ext}^1(C, \mathsf{c}D) \to \operatorname{Ext}^1(C, D).$$

First, $\operatorname{Ext}^1(C, \mathsf{d}D)$ vanishes, so $\operatorname{Hom}(C, \mathsf{c}D) \to \operatorname{Hom}(C, D)$ is epi, that is, f factors through $\mathsf{c}D \in \mathcal{C}$. But $\operatorname{Ext}^1(C, D)$ vanishes too (or equivalently, $\mathcal{D}$ is extension closed), so $\operatorname{Ext}^1(C, \mathsf{c}D) = 0$, i.e., $\mathsf{c}D \in \mathcal{C} \cap \mathcal{D}$. □

With complete cotorsion pairs, we get additional data about how Ext-orthogonal classes decompose our category. For example, if $(\mathcal{C}, \mathcal{D})$ is a complete cotorsion pair in $\mathcal{A}$, then it follows from the definition that every object in $\mathcal{A}$ is isomorphic to the quotient of an object in $\mathcal{C}$ by an object in $\mathcal{D}$. Additionally, the existence of such short exact sequences provides room to weaken earlier hypotheses.

LEMMA 1.2.9 ([2, Remark V.3.2]). *Suppose $\mathcal{C}$ and $\mathcal{D}$ are two isomorphism closed full subcategories in an abelian category $\mathcal{A}$ such that*

(1) $\mathrm{Ext}^1(\mathcal{C}, \mathcal{D}) = 0$,
(2) *$\mathcal{C}$ and $\mathcal{D}$ are closed under direct summands, and*
(3) *for any object $A \in \mathcal{A}$, there exist short exact sequences*

$$A \hookrightarrow \mathsf{d}A \twoheadrightarrow \mathsf{c}'A \quad \textit{and} \quad \mathsf{d}'A \hookrightarrow \mathsf{c}A \twoheadrightarrow A$$

with $\mathsf{c}'A, \mathsf{c}A \in \mathcal{C}$ and $\mathsf{d}A, \mathsf{d}'A \in \mathcal{D}$.

Then $(\mathcal{C}, \mathcal{D})$ is a complete cotorsion pair in $\mathcal{A}$.

PROOF. As was discussed earlier, $\mathrm{Ext}^1(\mathcal{C}, \mathcal{D}) = 0$ implies $\mathcal{D} \subseteq \mathcal{C}^{\perp_1}$. Consider a short exact sequence $A \hookrightarrow \mathsf{d}A \twoheadrightarrow \mathsf{c}'A$ with $A \in \mathcal{C}^{\perp_1}$, so the sequence splits. Then A is a summand of $\mathsf{d}A$, and since $\mathcal{D}$ is closed under summands, $A \in \mathcal{D}$. Therefore $\mathcal{C}^{\perp_1} = \mathcal{D}$. Similarly, using the other short exact sequence with $A \in {}^{\perp_1}\mathcal{D}$, we conclude $\mathcal{C} = {}^{\perp_1}\mathcal{D}$. □

1.2.2. Approximation theory. We are often interested in approximating objects in an additive category $\mathcal{X}$ by a subclass of objects with nice properties. The analogy is projective covers and injective hulls of modules, which supply approximations of arbitrary modules by modules of a particular breed.

DEFINITION 1.2.10. Let $\mathcal{C}$ be an isomorphism closed full subcategory of an additive category $\mathcal{X}$.

(1) A **right $\mathcal{C}$-approximation** of an object $A \in \mathcal{X}$ is a morphism $f\colon C \to A$ with the property that any other map $g\colon C' \to A$ factors via $h\colon C' \to C$, i.e., $g = fh$.

(2) If every object $A \in \mathcal{X}$ has a right $\mathcal{C}$-approximation, then $\mathcal{C}$ is called a **contravariantly finite** subcategory of $\mathcal{X}$.

(3) Dually, a **left $\mathcal{C}$-approximation** of an object $A \in \mathcal{X}$ is a morphism $f\colon A \to C$ with the property that any other map $g\colon A \to C'$ factors via $h\colon C \to C'$, i.e., $g = hf$.

(4) If every object $A \in \mathcal{X}$ has a left $\mathcal{C}$-approximation, then $\mathcal{C}$ is called a **covariantly finite** subcategory of $\mathcal{X}$.

(5) We say $\mathcal{C}$ is a **functorially finite** subcategory of $\mathcal{X}$ if it is both contravariantly finite and covariantly finite.

REMARK 1.2.11. Notice that we do not require approximations to be unique! Many authors get around this by fixing assignments for each object in $\mathcal{X}$, but making such choices functorial would certainly be more robust. We will show that appropriately stabilizing $\mathcal{X}$ makes the selection functorial when the approximations come from complete cotorsion pairs.

Recall from Definition 1.2.6 that a complete cotorsion pair $(\mathcal{C}, \mathcal{D})$ in an abelian category $\mathcal{A}$ comes equipped with assignments of objects in $\mathcal{C}$ and $\mathcal{D}$ for each object in $\mathcal{A}$. Approximations naturally arise from complete cotorsion pairs.

PROPOSITION 1.2.12. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in $\mathcal{A}$. Then $\mathcal{C}$ is a contravariantly finite subcategory of $\mathcal{A}$ and $\mathcal{D}$ is a covariantly finite subcategory of $\mathcal{A}$.*

PROOF. Fix an object $A \in \mathcal{A}$ and short exact sequences $\mathsf{d}'A \hookrightarrow \mathsf{c}A \twoheadrightarrow A$ and $A \hookrightarrow \mathsf{d}A \twoheadrightarrow \mathsf{c}'A$. We have to show if $C \in \mathcal{C}$ is any object with a morphism $f\colon C \to A$, then f factors over $\mathsf{c}A$. The short exact sequence $\mathsf{d}'A \hookrightarrow \mathsf{c}A \twoheadrightarrow A$ yields an exact sequence

$$\operatorname{Hom}(C, \mathsf{c}A) \to \operatorname{Hom}(C, A) \to \operatorname{Ext}^1(C, \mathsf{d}'A)$$

where the last term vanishes. Thus $\operatorname{Hom}(C, \mathsf{c}A) \to \operatorname{Hom}(C, A)$ is epi, so $f\colon C \to A$ factors through $\mathsf{c}A$.

Dually, if $D \in \mathcal{D}$ is any object with $g\colon A \to D$, we need that g factors over $\mathsf{d}A$. Using $A \hookrightarrow \mathsf{d}A \twoheadrightarrow \mathsf{c}'A$, we get an exact sequence

$$\operatorname{Hom}(\mathsf{d}A, D) \to \operatorname{Hom}(A, D) \to \operatorname{Ext}^1(\mathsf{c}'A, D).$$

Once more the last term vanishes, so $\operatorname{Hom}(\mathsf{d}A, D) \to \operatorname{Hom}(A, D)$ is epi, completing the proof. □

Let $f\colon A' \to A$ and suppose we are given right $\mathcal{C}$-approximations $C' \to A'$ and $C \to A$. We want to narrow down what it would take to make the induced map $C' \to C$ unique. Say we have two maps g and h, for which the following diagram commutes.

$$\begin{array}{ccc} C' & \longrightarrow & A' \\ {\scriptstyle g}\downarrow\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ C & \longrightarrow & A \end{array}$$

Then seeking such restrictions, and in light of the fact that $\mathcal{A}$ is abelian, it suffices to study $g - h$. In the case approximations come from complete cotorsion pairs, we get the following result.

LEMMA 1.2.13. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in $\mathcal{A}$ and $f\colon A' \to A$ a morphism in $\mathcal{A}$. Suppose we fix short exact sequences $\mathsf{d}'A' \hookrightarrow \mathsf{c}A' \twoheadrightarrow A'$ and $\mathsf{d}'A \hookrightarrow \mathsf{c}A \twoheadrightarrow A$, i.e., right $\mathcal{C}$-approximations with kernels in $\mathcal{D}$. If $g, h\colon \mathsf{c}A' \to \mathsf{c}A$ are two maps induced by f, then $g - h$ factors through $\mathcal{C} \cap \mathcal{D}$.*

PROOF. We have the commutative diagram

$$\begin{array}{ccccc} & & \mathsf{c}A' & \twoheadrightarrow & A' \\ & \swarrow{\scriptstyle j} & \downarrow{\scriptstyle g-h} & & \downarrow{\scriptstyle 0} \\ \mathsf{d}'A & \hookrightarrow & \mathsf{c}A & \twoheadrightarrow & A \end{array}$$

with $g - h$ in the kernel of $\mathsf{c}A \twoheadrightarrow A$, inducing the unique dashed diagonal map $j\colon \mathsf{c}A' \to \mathsf{d}'A$. By Lemma 1.2.8, j factors over an object in $\mathcal{C} \cap \mathcal{D}$, so $g - h$ does as well. $\square$

We get a dual statement about left $\mathcal{D}$-approximations with cokernels in $\mathcal{C}$, which follow immediately from the last lemma by formal duality.

LEMMA 1.2.14. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in $\mathcal{A}$ and $f\colon A' \to A$ a morphism in $\mathcal{A}$. Suppose we fix short exact sequences $A' \hookrightarrow \mathsf{d}A' \twoheadrightarrow \mathsf{c}'A'$ and $A \hookrightarrow \mathsf{d}A \twoheadrightarrow \mathsf{c}'A$, i.e., left $\mathcal{D}$-approximations with cokernels in $\mathcal{C}$. If $g, h\colon \mathsf{d}A' \to \mathsf{d}A$ are two maps induced by f, then $g - h$ factors through $\mathcal{C} \cap \mathcal{D}$.*

If we can remove maps that factor through $\mathcal{C} \cap \mathcal{D}$, the assignments given in Definition 1.2.6 would become functorial. This is accomplished by stabilizing $\mathcal{A}$ (Definition 1.1.8) with respect to $\mathcal{C} \cap \mathcal{D}$.

PROPOSITION 1.2.15. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in $\mathcal{A}$ and $\omega := \mathcal{C} \cap \mathcal{D}$. The assignments of short exact sequences from Definition 1.2.6 define functors $\mathsf{c}\colon \mathcal{A} \to \mathcal{C}/\omega$ and $\mathsf{d}\colon \mathcal{A} \to \mathcal{D}/\omega$. These functors descend to functors on $\mathcal{A}/\omega$, which we also call c and d.*

PROOF. Lemmas 1.2.13 and 1.2.14 quickly dispose of the first claim. It is worth remarking that the choice of short exact sequence does not matter, for if we are given two sequences $\tilde{\mathsf{d}'}A \hookrightarrow \tilde{\mathsf{c}}A \twoheadrightarrow A$ and $\mathsf{d}'A \hookrightarrow \mathsf{c}A \twoheadrightarrow A$, then the identity map on A induces maps $\alpha\colon \tilde{\mathsf{c}}A \to \mathsf{c}A$ and $\beta\colon \mathsf{c}A \to \tilde{\mathsf{c}}A$. Lemma 1.2.13 implies that both $id_{\tilde{\mathsf{c}}A} - \beta\alpha$ and $id_{\mathsf{c}A} - \alpha\beta$ factor through ω, that is,

$\mathsf{c}A \cong \tilde{\mathsf{c}}A$ in $\mathcal{C}/\omega$. Similarly, Lemma 1.2.14 ensures that the assignment of an object in $\mathcal{D}/\omega$ does not depend on the choice of sequence.

For the second claim, take $A \in \mathcal{C} \cap \mathcal{D}$ with short exact sequences $A \hookrightarrow \mathsf{d}A \twoheadrightarrow \mathsf{c}'A$ and $\mathsf{d}'A \hookrightarrow \mathsf{c}A \twoheadrightarrow A$. Since $\mathcal{C}$ and $\mathcal{D}$ are extension closed, we find $\mathsf{d}A \in \mathcal{C}$ and $\mathsf{c}A \in \mathcal{D}$, hence both are in $\mathcal{C} \cap \mathcal{D}$. Thus $\mathsf{c}A$ and $\mathsf{d}A$ are 0 in their respective quotients, proving we get functors on $\mathcal{A}/\omega$. $\square$

PROPOSITION 1.2.16. *Let* $(\mathcal{C}, \mathcal{D})$ *be a complete cotorsion pair in* $\mathcal{A}$ *and* $\omega := \mathcal{C} \cap \mathcal{D}$. *Then* $\mathcal{C}/\omega$ *is a contravariantly finite and* $\mathcal{D}/\omega$ *a covariantly finite subcategory of* $\mathcal{A}/\omega$.

PROOF. This follows from Propositions 1.2.12 and 1.2.15. $\square$

In summary, if $(\mathcal{C}, \mathcal{D})$ is a complete cotorsion pair in an abelian category $\mathcal{A}$, then $\mathcal{C}$ is a contravariantly finite and $\mathcal{D}$ a covariantly finite subcategory of $\mathcal{A}$ (Proposition 1.2.12). If we stabilize $\mathcal{A}$ with respect to $\omega := \mathcal{C} \cap \mathcal{D}$, then we get functors $\mathsf{c}\colon \mathcal{A}/\omega \to \mathcal{C}/\omega$ and $\mathsf{d}\colon \mathcal{A}/\omega \to \mathcal{D}/\omega$ (Proposition 1.2.15). Moreover, c and d respectively define right $\mathcal{C}/\omega$-approximations and left $\mathcal{D}/\omega$-approximations of objects in $\mathcal{A}/\omega$ (Proposition 1.2.16), and the assignment of an approximation is now functorial.

To finish, we generalize the syzygy and cosyzygy functors from Section 1.1.2. Let $\mathcal{C}$ be a contravariantly finite subcategory of an abelian category $\mathcal{A}$, and form the stable category $\mathcal{A}/\mathcal{C}$ as in Definition 1.1.8. For each object $A \in \mathcal{A}$, we have a right $\mathcal{C}$-approximation $f\colon C \to A$, and we set $\Omega A := \ker f$. The assignment Ω descends to an endofunctor on $\mathcal{A}/\mathcal{C}$, which we call a syzygy functor. Dually, for a covariantly finite subcategory $\mathcal{D}$ of $\mathcal{A}$, define a cosyzygy functor—either denoted by $\mho$ (traditional) or Σ (to align with Section 1.1.2)—on $\mathcal{A}/\mathcal{D}$ by taking cokernels of left $\mathcal{D}$-approximations.

1.2.3. Torsion pairs. In this section we propose a torsion theory suitable for additive categories. Our approach distills the classical approach down to the existence of certain adjoint functors, eliminating the need for the full structure of an abelian category. What's more, when specializing back to abelian categories, our theory recovers the expected data, though it fails to specialize to a similar notion for triangulated categories. We remark on the latter at the end of the section and propose an alteration that suitably generalizes both the abelian and triangulated cases.

Our motivation for greater generality is a connection to cotorsion pairs. We will find that cotorsion pairs immediately give rise to torsion-pair-like structures, but in a category that is only additive in general. Extending the notion of torsion pairs to additive categories allows us to draw conclusions about adjoint functors from cotorsion structure, furnishing much richer structure with little extra work. We begin with a review of torsion pairs for abelian categories.

Definition 1.2.17 ([2, Definition I.1.1]). Let $\mathcal{A}$ be an abelian category. A **torsion pair** in $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of isomorphism closed full subcategories of $\mathcal{A}$ with the following properties.

(1) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$.

(2) For any $Z \in \mathcal{A}$, there exists a short exact sequence

$$0 \to \mathsf{t}Z \to Z \to \mathsf{f}Z \to 0$$

in $\mathcal{A}$ with $\mathsf{t}Z \in \mathcal{T}$ and $\mathsf{f}Z \in \mathcal{F}$.

With the assignment of a short exact sequence we immediately get a functor $\mathsf{t}\colon \mathcal{A} \to \mathcal{T}$ that is right adjoint of the inclusion $\mathsf{i}\colon \mathcal{T} \to \mathcal{A}$. Symmetrically we get a functor $\mathsf{f}\colon \mathcal{A} \to \mathcal{F}$ that is left adjoint of the inclusion $\mathsf{j}\colon \mathcal{F} \to \mathcal{A}$. Indeed, if $0 \to \mathsf{t}Z \to Z \to \mathsf{f}Z \to 0$ and $0 \to \mathsf{t}'Z \to Z \to \mathsf{f}'Z \to 0$ are two short exact sequences for an object Z, then since $\mathrm{Hom}_{\mathcal{A}}(\mathsf{t}Z, \mathsf{f}'Z) = 0$, there exists a unique $\varphi\colon \mathsf{t}Z \to \mathsf{t}'Z$ making the left square below commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{t}Z & \longrightarrow & Z & \longrightarrow & \mathsf{f}Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \exists!\varphi} & & \| & & \downarrow{\scriptstyle \exists!\psi} & & \\ 0 & \longrightarrow & \mathsf{t}'Z & \longrightarrow & Z & \longrightarrow & \mathsf{f}'Z & \longrightarrow & 0 \end{array}$$

Similarly, there exists a unique $\psi\colon \mathsf{f}Z \to \mathsf{f}'Z$ making the right square above commute. In both cases, it is quick to observe that φ and ψ are isomorphisms (induce maps in the other direction), so specifying a short exact sequence determines, for each object Z, objects in $\mathcal{T}$ and $\mathcal{F}$ that are unique up to unique isomorphism. Now suppose we have a map $g\colon Z \to Z'$. As $\mathrm{Hom}_{\mathcal{A}}(\mathsf{t}Z, \mathsf{f}Z') = 0$, there exists unique maps $\mathsf{t}g\colon \mathsf{t}Z \to \mathsf{t}Z'$ and $\mathsf{f}g\colon \mathsf{f}Z \to \mathsf{f}Z'$ making the left and right squares below respectively commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{t}Z & \longrightarrow & Z & \longrightarrow & \mathsf{f}Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \exists!\mathsf{t}g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \exists!\mathsf{f}g} & & \\ 0 & \longrightarrow & \mathsf{t}Z' & \longrightarrow & Z' & \longrightarrow & \mathsf{f}Z' & \longrightarrow & 0 \end{array}$$

Therefore the assignments $\mathsf{t}\colon \mathcal{A} \to \mathcal{T}$ and $\mathsf{f}\colon \mathcal{A} \to \mathcal{F}$ define functors; that these functors are adjoints to inclusion is a quick verification. Moreover, $\mathsf{t}Z \to Z$ is the component at Z of the counit of the adjunction $\mathsf{i} \dashv \mathsf{t}$ and $Z \to \mathsf{f}Z$ is the component at Z of the unit of the adjunction $\mathsf{f} \dashv \mathsf{j}$. Inspired by these adjoints, we extend the notion of torsion pairs to any additive category.

Definition 1.2.18. A **torsion pair** in an additive category $\mathcal{X}$ is a pair $(\mathcal{T}, \mathcal{F})$ of isomorphism closed full subcategories $\mathcal{T}$ and $\mathcal{F}$, the **torsion class** and **torsion-free class** respectively, such that

- T1 $\mathrm{Hom}_{\mathcal{X}}(\mathcal{T}, \mathcal{F}) = 0$, and
- T2 the canonical inclusions $\mathcal{T} \hookrightarrow \mathcal{X}$ and $\mathcal{F} \hookrightarrow \mathcal{X}$ admit right and left adjoints respectively.

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an additive category $\mathcal{X}$. To see the generalized notion specializes back to the classical, simply notice that when $\mathcal{X}$ is abelian, T1 implies $\mathcal{T}$ is closed under extensions of right exact sequences: If $X \to Z \to X' \to 0$ is exact with $X, X' \in \mathcal{T}$, then

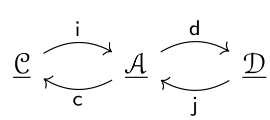
$$0 \to \operatorname{Hom} \mathcal{X}(X', Y) \to \operatorname{Hom}_{\mathcal{X}}(Z, Y) \to \operatorname{Hom}_{\mathcal{X}}(X, Y)$$

is exact for every $Y \in \mathcal{F}$. As the outer terms vanish, $\operatorname{Hom}_{\mathcal{X}}(Z, Y) = 0$ for all Y, thus $Z \in \mathcal{T}$. Similarly, T1 implies $\mathcal{F}$ is closed under extensions of left exact sequences. Then T2 and [2, Proposition I.1.2] imply that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in the sense of Definition 1.2.17.

The main purpose of extending the notion of torsion pairs to additive categories is to serve abelian categories stabilized by the intersection of complete cotorsion pairs. Given a pair of isomorphism closed full subcategories $(\mathcal{C}, \mathcal{D})$ in an abelian category $\mathcal{A}$, we can stabilize $\mathcal{A}$ (in the sense of Definition 1.1.8) with respect to $\mathcal{C} \cap \mathcal{D}$. The resulting stabilization is not abelian in general, but when $(\mathcal{C}, \mathcal{D})$ is a complete cotorsion pair, we find that the stabilization is endowed with a torsion pair.

Proposition 1.2.19. *Let $(\mathcal{C}, \mathcal{D})$ be a complete cotorsion pair in an abelian category $\mathcal{A}$ and $\omega := \mathcal{C} \cap \mathcal{D}$. Then $(\mathcal{C}/\omega, \mathcal{D}/\omega)$ is a torsion pair in $\mathcal{A}/\omega$.*

Proof. For brevity, write $\underline{\mathcal{A}}$, $\underline{\mathcal{C}}$, and $\underline{\mathcal{D}}$ for the ω-stabilization of $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$ respectively. First, $\operatorname{Hom}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) = 0$ by Lemma 1.2.8. For the adjoints, the setup is as follows.



We have inclusions i, j, i.e., full embeddings, and the approximation functors c, d from Proposition 1.2.15. It is immediate from Lemma 1.2.8 that $\mathrm{Hom}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) = 0$, so it remains to show (i, c) and (d, j) are adjoint pairs; we will demonstrate only the former, as the latter is dual. Since $\underline{\mathcal{C}}$ is a full subcategory of $\underline{\mathcal{A}}$ and i is a full embedding, it suffices to demonstrate

$$\mathrm{Hom}_{\underline{\mathcal{A}}}(C, \mathsf{c}A) \to \mathrm{Hom}_{\underline{\mathcal{A}}}(C, A)$$

is an isomorphism for $C \in \underline{\mathcal{C}}$ and $A \in \underline{\mathcal{A}}$. Fix a short exact sequence

$$0 \to \mathsf{d}'A \to \mathsf{c}A \xrightarrow{p} A \to 0$$

in $\mathcal{A}$ as in Definition 1.2.6. Applying $\mathrm{Hom}_{\mathcal{A}}(C, -)$, we get an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(C, \mathsf{d}'A) \to \mathrm{Hom}_{\mathcal{A}}(C, \mathsf{c}A) \xrightarrow{p_*} \mathrm{Hom}_{\mathcal{A}}(C, A) \to \mathrm{Ext}^1_{\mathcal{A}}(C, \mathsf{d}'A)$$

where the last term vanishes, hence p_* is an epimorphism. We should indeed expect this, since $\mathsf{c}A$ is a right $\mathcal{C}$-approximation of A, so any map $C \to A$ will factor through $\mathsf{c}A$. It remains to show every $f \in \ker p_*$ vanishes in $\underline{\mathcal{A}}$. By the above exact sequence, $\ker p_* \cong \mathrm{Hom}_{\mathcal{A}}(C, \mathsf{d}'A)$, so Lemma 1.2.8 implies the residue of f in $\underline{\mathcal{A}}$ is 0. Hence $p*$ descends to an isomorphism

$$\mathrm{Hom}_{\underline{\mathcal{A}}}(C, \mathsf{c}A) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathcal{A}}}(C, A),$$

with naturality in both components immediate from construction. Therefore (i, c) is an adjoint pair. □

Remark 1.2.20. Similar to Definition 1.2.17, there is a notion of torsion pairs for triangulated categories: A **torsion pair** in a triangulated category $\mathcal{U}$ with suspension Σ is a pair $(\mathcal{T}, \mathcal{F})$ of isomorphism closed full subcategories with the following properties.

(1) $\mathrm{Hom}_{\mathcal{U}}(\mathcal{T}, \mathcal{F}) = 0$.

(2) $\Sigma(\mathcal{T}) \subset \mathcal{T}$ and $\Sigma^{-1}(\mathcal{F}) \subset \mathcal{F}$.

(3) For any $Z \in \mathcal{U}$, there exists a distinguished triangle

$$\mathsf{t}Z \to Z \to \mathsf{f}Z \to \Sigma(\mathsf{t}Z)$$

in $\mathcal{U}$ with $\mathsf{t}Z \in \mathcal{T}$ and $\mathsf{f}Z \in \mathcal{F}$.

See [2, Section I.2] for details.

The issue with Definition 1.2.18 is that it yields no control over Σ. We could adapt our definition to suit *suspended* additive categories, that is, additive categories endowed with an additive auto-equivalence Σ, subject to mild assumptions. If $\mathcal{X}$ is one such category and $(\mathcal{T}, \mathcal{F})$ is a proposed torsion pair, we could require—in addition to the stipulations of Definition 1.2.18—that $\Sigma(\mathcal{T}) \subset \mathcal{T}$, $\Sigma^{-1}(\mathcal{F}) \subset \mathcal{F}$, and that the adjoints commute with Σ. Taking Σ to be the identity on $\mathcal{X}$, this version can be applied to any additive category, but now we have the flexibility to choose a nontrivial suspension, as in the case of triangulated categories.

CHAPTER 2

Modules over a Gorenstein ring

Recall that, for any ring R, the category of right R-modules $\mathsf{Mod}\, R$ is abelian and has enough injectives and projectives. In general, the full subcategory of finitely generated right R-modules $\mathsf{mod}\, R$ is only additive, but when R is noetherian, $\mathsf{mod}\, R$ is abelian. By **noetherian** ring we mean a ring that is right- and left-noetherian. In this chapter, we assume modules are finitely generated; we entreat the reader to do as much if the language, against our better efforts, somewhere lacks. We furthermore assume that all rings have unity.

2.1. Projectively stable modules

Let R be a noetherian ring, and let $\mathsf{proj}(R)$ be the full subcategory of $\mathsf{mod}\, R$ comprised of (finitely generated) projective right R-modules.

Definition 2.1.1. The **projectively stabilized category** of finitely generated right R-modules is the stabilization $\mathsf{mod}\, R/ \mathsf{proj}(R)$, which we will abbreviate by $\underline{\mathsf{mod}}\, R$. For subcategories of $\mathsf{mod}\, R$ containing $\mathsf{proj}(R)$, we denote the stable subcategory in the same way. If M and N are isomorphic objects in $\underline{\mathsf{mod}}\, R$, we say they are **projectively stably equivalent** modules.

A classical result of Auslander and Bridger, [1, Proposition 1.44], is that M and N are projectively stably equivalent if and only if there exists projectives P and Q such that $M \oplus P \cong N \oplus Q$ in $\mathsf{mod}\, R$. Compare this result to Schanuel's Lemma (Example 1.1.11) concerning syzygies. Indeed, by Proposition 1.1.10, there exists an additive endofunctor Ω on $\underline{\mathsf{mod}}\, R$ given by taking syzygies, and Schanuel's Lemma details how to connect isomorphism classes of syzygies in $\underline{\mathsf{mod}}\, R$ to isomorphism classes of lifts in $\mathsf{mod}\, R$.

PROPOSITION 2.1.2. *For a noetherian ring R, the pair $(\mathsf{proj}(R), \mathsf{mod}\, R)$ is a complete hereditary cotorsion pair in $\mathsf{mod}\, R$. In addition, the subcategory $\mathsf{proj}(R)$ is a contravariantly finite subcategory of $\mathsf{mod}\, R$.*

PROOF. Immediately we have that $(\mathsf{proj}(R), \mathsf{mod}\, R)$ is a hereditary cotorsion pair, for $\mathrm{Hom}(P, -)$ is exact if and only if P is projective. To show the pair is complete, consider any right R-module M. Take a projective cover of M to get one of the short exact sequences: $\ker(P \twoheadrightarrow M) \hookrightarrow P \twoheadrightarrow M$. For the other, just map M identically onto itself and then the trivial module: $M \hookrightarrow M \twoheadrightarrow 0$. The last claim follows from Proposition 1.2.12 and what we have just shown. □

2.2. Some homological algebra

2.2.1. Dimension theory. Let R be a noetherian ring. One homological invariant we may assign to any R-module, including R itself, is its injective dimension. For a right R-module M, the **injective dimension** of M is the minimum length of resolutions of M by injective right R-modules. If M has no injective resolutions of finite length, we say the injective dimension of M is infinite. To emphasize that we are working with right modules, we may say *right* injective dimension, and write $\mathrm{idim}(M_R)$ for clarity. When there is no concern for confusion, we may just write $\mathrm{idim}(M)$. Predictably, there is an analogous definition for left modules, and we write $\mathrm{idim}({}_R M)$ for the left injective dimension of a left R-module M.

By looking at projective resolutions of a right R-module M, we can assign to M a **projective dimension**. We define the projective dimension of M to be the minimum length of a resolution of M by projective right R-modules, and if no finite resolution exists we say the projective dimension is infinite. Like before, we may emphasize *right* projective dimension, written $\mathrm{pdim}(M_R)$, with $\mathrm{pdim}(M)$ as shorthand. Furthermore, $\mathrm{pdim}(M_R) \leq d$ if and only if $\mathrm{Ext}^i_R(M, N) = 0$ for all right R-modules N; see [10, Appendix B] for details.

Finally, there is weak dimension: Introduced by Cartan and Eilenberg as an exercise, [5, Exercise VI.5.3], the **weak dimension**, sometimes **flat dimension**, of a right R-module M is the greatest integer n such that $\mathrm{Tor}^R_n(M, N) \neq 0$ for some left R-module N. If no such maximum integer exists, the weak dimension of M is infinite. The terminology flat dimension comes from flat resolutions. It turns out the weak dimension is the minimum length of a resolution

$$\cdots \to F_1 \to F_0 \to M \to 0$$

of M by flat right R-modules. We write $\mathrm{wdim}(M_R)$, or simply $\mathrm{wdim}(M)$, for the weak dimension. Notice that, since projectives are flat, $\mathrm{wdim}(M_R) \leq \mathrm{pdim}(M_R)$. In addition and by definition, M has weak dimension at most n if and only if $\mathrm{Tor}^R_{n+1}(M, N) = 0$ for all left R-modules N. Symmetric definitions, notations, and criterions exist for left modules.

To better understand right- or left-noetherian rings, we give several results relating the three homological invariants. These results will be used later and further illustrate the utility of the noetherian hypothesis. In both proofs, we utilize the following construction from Cartan and Eilenberg.

LEMMA 2.2.1 ([5, Proposition VI.5.3]). *Let R be a left-noetherian ring. There exists an isomorphism*

$$\operatorname{Tor}_i^R(\operatorname{Hom}_S(B,I),A) \xrightarrow{\sim} \operatorname{Hom}_S(\operatorname{Ext}_R^i(A,B),I),$$

for all $i \geq 0$, where A is a finitely generated left R-module, I is an injective right S-module, and B is any (R,S)-bimodule.

LEMMA 2.2.2. *For a left-noetherian ring R and any finitely generated left R-module A,* $\operatorname{wdim}({}_RA) = \operatorname{pdim}({}_RA)$. *Symmetrically, for a right-noetherian ring R and any finitely generated right R-module A,* $\operatorname{wdim}(A_R) = \operatorname{pdim}(A_R)$.

PROOF. Suppose $\operatorname{wdim}({}_RA) = d$. It is enough to show $\operatorname{pdim}({}_RA) \leq d$, which amounts to showing $\operatorname{Ext}_R^{d+1}(A,B) = 0$ for all left R-modules B, equivalently, for all $(R,\mathbf{Z})$-bimodules B. Note that $\operatorname{Ext}_R^{d+1}(A,B)$ is a right $\mathbf{Z}$-module and $\mathbf{Q}/\mathbf{Z}$ is an injective cogenerator of $\mathsf{Mod}\,\mathbf{Z}$, so

$$\operatorname{Hom}_{\mathbf{Z}}(\operatorname{Ext}_R^{d+1}(A,B),\mathbf{Q}/\mathbf{Z}) = 0 \iff \operatorname{Ext}_R^{d+1}(A,B) = 0.$$

From Lemma 2.2.1 we get

$$\operatorname{Tor}_{d+1}^R(\operatorname{Hom}_{\mathbf{Z}}(B,\mathbf{Q}/\mathbf{Z}),A) \cong \operatorname{Hom}_{\mathbf{Z}}(\operatorname{Ext}_R^{d+1}(A,B),\mathbf{Q}/\mathbf{Z}),$$

with the lefthand side vanishing because $\operatorname{wdim}({}_RA) = d$. It follows that $\operatorname{Ext}_R^{d+1}(A,B) = 0$, hence $\operatorname{wdim}({}_RA) = \operatorname{pdim}({}_RA)$.

Suppose now R is right-noetherian and A is any finitely generated right R-module. Recall that, for any ring R, a right R-module is a left R^{op}-module. With this in mind, R^{op} is left-noetherian and A is a finitely generated left R^{op}-module. The argument from the preceding paragraph shows $\operatorname{wdim}({}_{R^{\mathrm{op}}}A) = \operatorname{pdim}({}_{R^{\mathrm{op}}}A)$, and recognizing $\operatorname{wdim}({}_{R^{\mathrm{op}}}A) = \operatorname{wdim}(A_R)$ and $\operatorname{pdim}({}_{R^{\mathrm{op}}}A) = \operatorname{pdim}(A_R)$ completes the proof. $\square$

Lemma 2.2.3. *For a noetherian ring R,*

$$\operatorname{idim}({}_RR) = \sup\{\operatorname{wdim}(I_R) : I_R \textit{ is an injective right } R\textit{-module}\}.$$

Symmetrically,

$$\operatorname{idim}(R_R) = \sup\{\operatorname{wdim}({}_RI) : {}_RI \textit{is an injective left } R\textit{-module}\}.$$

Proof. We follow [8, Proposition 1]. Throughout the proof, A always denotes a finitely generated left R-module and I is always an injective right R-module. Note that, from Lemma 2.2.1, we have an isomorphism

$$\operatorname{Tor}_i^R(I, A) \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Ext}_R^i(A, R), I)$$

for all $i > 0$, using the fact that R is an (R, R)-bimodule.

First we will bound the weak dimension of every injective right R-module by the left injective dimension of R. Since Tor commutes with direct limits in the right component, it suffices to show, for each I, that $\operatorname{Tor}_{d+1}^R(I, A) = 0$ for all finitely generated A. Assume $\operatorname{idim}({}_RR) = d$ and A is arbitrary. Then $\operatorname{Ext}_R^{d+1}(A, R) = 0$, so for any I, $\operatorname{Tor}_{d+1}^R(I, A) = 0$ too. This shows $\operatorname{idim}({}_RR) \geq \operatorname{wdim}(I_R)$ for all I, thus

$$\operatorname{idim}({}_RR) \geq \sup\{\operatorname{wdim}(I_R) \colon {}_RI \text{ is an injective right } R\text{-module}\}.$$

Now suppose that $\operatorname{wdim}(I_R) \leq d$ for any I. We find

$$0 = \operatorname{Tor}_{d+1}^R(I, A) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^{d+1}(A, R), I)$$

for all A, so taking I to be an injective cogenerator shows $\operatorname{Ext}_R^{d+1}(A, R) = 0$. Hence $\operatorname{idim}({}_RR) \leq d$, demonstrating the other required inequality. For right injective dimension of R and left weak dimension of injective R-modules, working over R^{op} gives the symmetric result. □

For an (R, R)-bimodule M, we may discuss both the right and left injective dimension of M, and they need not coincide in general. However, we have a result due to Zaks that simplifies matters for R, viewed as a bimodule over itself.

Proposition 2.2.4 ([16, Lemma A]). *For any noetherian ring R, if the right and left injective dimensions are finite, then they are equal.*

2.2.2. Gorenstein rings and duality. In light of the last proposition, we may refer unambiguously to *the* injective dimension of R—no mention of right/left—so we make the following definition.

Definition 2.2.5. A (not necessarily commutative) noetherian ring R is called **Gorenstein** if the injective dimension of R is finite.

Remark 2.2.6. Please note the symmetry in the definition: R is Gorenstein if and only if R^{op} is Gorenstein. As an aside, this terminology differs from [3], which favors *strongly* Gorenstein. The reasoning follows from analogy to the commutative case: In commutative algebra, a commutative noetherian ring R is called Gorenstein if the localization $R_{\mathfrak{p}}$ is a Gorenstein local ring—[10, Definition following Theorem 18.1]—for every prime $\mathfrak{p}$ of R. The issue is that, for the preceding condition to imply R has finite injective dimension, we require that R have finite Krull dimension, so indeed requiring R has finite injective dimension at the outset is more restrictive. For the sanity of both the author and reader, as we encounter no confounding examples throughout, we maintain simpler nomenclature.

Example 2.2.7. We should start with some noncommutative and not-necessarily-commutative examples. Clearly every quasi-Frobenius ring (Example 1.1.5) is Gorenstein. In particular, group algebras over a field are

Gorenstein. On the other hand, the right global dimension of a ring R is defined to be

$$\sup\{\mathrm{pdim}(M) : M \in \mathsf{Mod}\, R\} = \sup\{\mathrm{idim}(M) : M \in \mathsf{Mod}\, R\};$$

these are equal by [15, Theorem 4.1.2], where it is also remarked that right global dimension equals its left analogue when R is right- and left-noetherian. From the righthand side, we see that any right- and left-noetherian ring with finite global dimension must be Gorenstein.

Example 2.2.8. Turning to commutative algebra, if R is a commutative Gorenstein ring, then the polynomial ring $R[x]$ is Gorenstein ([10, Exercise 18.3]). If R is a commutative local Gorenstein ring, then the completion $\widehat{R}$ is Gorenstein—this is in fact a biconditional; see [10, Theorem 18.3]. One class of commutative rings that are all Gorenstein is regular local rings. Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$. We say R is a regular local ring if the Krull dimension of R equals the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over the residue field $R/\mathfrak{m}$. By [10, Theorem 21.1.iii] and the next example, every regular local ring is Gorenstein, though the resulting theory is not that interesting: A theorem of Serre says a commutative noetherian local ring is regular if and only if it has finite global dimension (see [10, Theorem 19.2]), and the case of finite global dimension does not furnish an insightful theory.

Example 2.2.9. Having defined regular local rings, we give a more substantial example. Let R be a commutative noetherian ring and M an R-module. Following [10, Section 16], an element $x \in R$ is M-regular if $xm \neq 0$ for all nonzero $m \in M$. A sequence $(x_1, \ldots, x_n)$ of elements of R is called an M-sequence if it satisfies two conditions: First, x_1 is M-regular, x_2 is

(M/x_1M)-regular, and in general x_{i+1} is $(M/(x_1, \ldots, x_i)M)$-regular; second, we require $(x_1, \ldots, x_n)M \neq M$. A ring S is a complete intersection if there exists a regular local ring R and an R-sequence $(x_1, \ldots, x_n)$ such that $S \cong R/(x_1, \ldots, x_n)$; when $n = 1$, S is called a hypersurface. As the name suggests, complete intersection rings were initially studied in the context of geometry, though of more immediate interest to us is the fact that every complete intersection ring is Gorenstein ([10, Theorem 21.3]).

Let k be a field, and recall the truncated polynomial ring $\Lambda = k[x]/(x^n)$ from Example 1.1.6. Note that $k[x]$ is a regular local ring, that x^n is not a zero divisor of $k[x]$, and that $(x^n) \neq k[x]$. By definition, Λ is a complete intersection ring, in particular a hypersurface. The ring

$$k[[x, y, z]]/(x^2 - y^2, y^2 - z^2, xy, yz, zx),$$

a quotient of the formal power series ring over k in three indeterminants, is a Gorenstein ring that is not a complete intersection ring ([10, Exercise 21.3]).

Example 2.2.10. Buchweitz gives a recipe for Gorenstein rings of arbitrary injective dimension in [3, Proposition 8.3.1]. Let R be a regular local ring with maximal ideal $\mathfrak{m}$. Suppose S is a finite and flat R-algebra via some morphism $R \to S$ such that $S \otimes_R R/\mathfrak{m}$ is quasi-Frobenius. Then S is Gorenstein and $\mathrm{idim}(S)$ equals the Krull dimension of R.

We review some duality theory.

Definition 2.2.11. For a right R-module M, the **dual module** M^* is the left R-module, equivalently right R^{op}-module, $\mathrm{Hom}_R(M, R)$. For the sake of visibility and clarity, temporarily make the convention that $\odot$ is R-action and $\odot^{\mathrm{op}}$ is R^{op}-action. Then for $m \in M$, $f \in M^*$, and $r \in R$, the action is given by

$$r \odot f(m) = f \odot^{\mathrm{op}} r(m) := f(r \odot^{\mathrm{op}} m) = f(m \odot r).$$

A module M is called **reflexive** if $M \cong M^{**}$.

LEMMA 2.2.12. *If P is a finitely generated projective right R-module, then the dual module P^* is a projective left R-module. Furthermore, P is reflexive.*

PROOF. If P is projective, it is a direct summand of a free module, i.e., there exists another module Q such that $P \oplus Q \cong R^n$. Then

$$\begin{aligned} P^* \oplus Q^* &= \mathrm{Hom}_R(P, R) \oplus \mathrm{Hom}_R(Q, R) \\ &\cong \mathrm{Hom}_R(P \oplus Q, R) \\ &\cong \mathrm{Hom}_R(R^n, R) \cong R^n \end{aligned}$$

so P^* is a direct summand of a free module.

Maintaining $P \oplus Q \cong R^n$, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker f & \longrightarrow & \ker g & \longrightarrow & \ker h & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P & \longrightarrow & R^n & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & P^{**} & \longrightarrow & (R^n)^{**} & \longrightarrow & Q^{**} & \longrightarrow & 0 \end{array}$$

where the rows are split short exact sequences, and f, g, and h are the natural evaluation maps. As g is an isomorphism, the first row implies f and h are injective. It follows from the snake lemma that $\ker h \cong \operatorname{coker} f$, so f is an isomorphism. □

Following [6], we define the Auslander-Bridger transpose for a module over a noetherian ring, a useful tool when studying duality.

DEFINITION 2.2.13. Let R be a noetherian ring and M a finitely generated right R-module. If

$$\cdots \to P_1 \xrightarrow{f} P_0 \to M \to 0$$

is a resolution of M by finitely generated projective modules, then

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{coker} f^* \to 0$$

is an exact sequence. Call $\operatorname{coker} f^*$ the **Auslander-Bridger transpose** of M, denoted $\operatorname{Tr} M$.

REMARK 2.2.14. While $\operatorname{Tr} M$ certainly depends on the projective resolution chosen, it is unique up to projectively stable equivalence, which is good enough for computing Ext. To demonstrate the latter, if $\cdots \to P_1 \xrightarrow{f} P_0 \to M \to 0$ and $\cdots \to Q_1 \xrightarrow{g} Q_0 \to M \to 0$ are two projective resolutions of the right R-module M, then there exists projective left R-modules P and Q such that $\operatorname{coker} f^* \oplus P \cong \operatorname{coker} g^* \oplus Q$, hence

$$\begin{aligned}\operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{coker} f^*, R) &\cong \operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{coker} f^* \oplus P, R)\\ &\cong \operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{coker} g^* \oplus Q, R) \cong \operatorname{Ext}^i_{R^{\mathrm{op}}}(\operatorname{coker} g^*, R)\end{aligned}$$

for all $i > 0$.

We use the following later.

LEMMA 2.2.15 ([6, Lemma 2.5]). *Let M be a right R-module. We have an exact sequence*

$$0 \to \operatorname{Ext}^1_{R^{\mathrm{op}}}(\operatorname{Tr} M, R) \to M \to M^{**} \to \operatorname{Ext}^2_{R^{\mathrm{op}}}(\operatorname{Tr} M, R) \to 0.$$

PROOF. A priori we have the natural evaluation map $\tau\colon M \to M^{**}$. Take a resolution of M by finitely generated projective right R-modules

$$\begin{array}{ccccccccc} \cdots \longrightarrow P_1 & & \xrightarrow{\quad f \quad} & & P_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \\ & {\scriptstyle \pi_1}\searrow & & \nearrow{\scriptstyle i_1} & \\ & & \Omega M & & \end{array}$$

and dualize to get an exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow M^* \xrightarrow{\varepsilon^*} P_0^* & & \xrightarrow{\qquad} & & P_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0 \\ & {\scriptstyle \pi_2}\searrow & & \nearrow & \\ & & C & & \end{array}$$

where $C := \operatorname{coker}(M^* \to P_0^*)$. From the short exact sequence

$$0 \to C \to P_1^* \to \operatorname{Tr} M \to 0$$

we get the exact sequence $0 \to (\operatorname{Tr} M)^* \to P_1^{**} \to C^* \to \operatorname{Ext}^1(\operatorname{Tr} M, R) \to 0 \to \operatorname{Ext}^1(C, R) \to \operatorname{Ext}^2(\operatorname{Tr} M, R) \to 0$, from which it follows $\operatorname{Ext}^1(C, R) \cong \operatorname{Ext}^2(\operatorname{Tr} M, R)$. Dualizing the short exact sequence

$$0 \to M^* \xrightarrow{\varepsilon^*} P_0^* \to C \to 0$$

gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Omega M & \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \xrightarrow{\qquad} & 0 \\ & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle \sigma_0} & & \Big\downarrow{\scriptstyle \tau} & & \\ 0 \longrightarrow & C^* & \longrightarrow & P_0^{**} & \xrightarrow{\varepsilon^{**}} & M^{**} & \longrightarrow \operatorname{Ext}^1(C, R) \longrightarrow & 0 \end{array}$$

with exact rows. Here σ_0 is an isomorphism (Lemma 2.2.12) and g is induced by the diagram (i.e., factoring through $\ker \varepsilon^{**}$). As the first row is exact and σ_0 is an isomorphism (so in particular injective), it follows that g is injective. The snake lemma implies $\ker \tau \cong \operatorname{coker} g$. Moreover, $\operatorname{coker} \varepsilon^{**} \cong \operatorname{coker} \tau\varepsilon\sigma_0^{-1}$ by commutativity, and since both σ_0^{-1} and ε are surjections, $\operatorname{coker} \tau\varepsilon\sigma_0^{-1} \cong$

$\operatorname{coker}\tau$. But $\operatorname{coker}\varepsilon^{**} \cong \operatorname{Ext}^1(C, R)$, so we have shown

$$\operatorname{coker}\tau \cong \operatorname{Ext}^1(C, R) \cong \operatorname{Ext}^2(\operatorname{Tr} M, R).$$

To show $\ker\tau \cong \operatorname{Ext}^1(\operatorname{Tr} M, R)$, consider the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & P_1 & \xrightarrow{g\pi_1} & C^* & \longrightarrow & \operatorname{Ext}^1(\operatorname{Tr} M, R) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \|{\scriptstyle id} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & \Omega M & \xrightarrow{g} & C^* & \longrightarrow & \operatorname{coker} g & \longrightarrow & 0
\end{array}$$

Note the sequence $C^* \to \operatorname{Ext}^1(\operatorname{Tr} M, R) \to 0$, lifted from an earlier exact sequence, and remark that $\operatorname{coker}\pi_1 = \operatorname{coker} id = \ker id = 0$. The snake lemma implies h is an isomorphism, so as $\ker\tau \cong \operatorname{coker} g$, we get the desired exact sequence. □

DEFINITION 2.2.16. For a noetherian ring R, we say a finitely generated right R-module M is **Gorenstein dimension zero**, sometimes **Gorenstein projective**, if M is reflexive and

$$\operatorname{Ext}^i_R(M, R) = 0 \quad \text{and} \quad \operatorname{Ext}^i_{R^{\mathrm{op}}}(M^*, R) = 0$$

for all $i > 0$. For many purposes, modules of Gorenstein dimension zero behave like projectives.

2.3. Two subcategories

From now on, let S be a Gorenstein ring. Unless otherwise stated, we make the convention of working with right modules, so by the symbols idim, pdim, and wdim we mean right injective dimension, right projective dimension, and right weak dimension respectively.

Many of the arguments in this section and the next use a technique called dimension shifting. We introduce it here as a lemma.

LEMMA 2.3.1. *If*

$$0 \to N \to P_{j-1} \to P_{j-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

is an exact sequence in $\mathsf{mod}\, S$ *with* P_k *projective,* $0 \le k \le j$*, then*

$$\operatorname{Ext}^i_S(N, S) \cong \operatorname{Ext}^{i+j}_S(M, S).$$

PROOF. Consider the short exact sequence $0 \to N \to P \to M \to 0$ with P a projective module. Applying Ext yields a long exact sequence

$$\cdots \to \operatorname{Ext}^i_S(P, S) \to \operatorname{Ext}^i_S(N, S) \to \operatorname{Ext}^{i+1}(M, S) \to \operatorname{Ext}^{i+1}_S(P, S) \to \cdots$$

with the outer terms vanishing. Therefore $\operatorname{Ext}^i_S(N, S) \cong \operatorname{Ext}^{i+1}_S(M, S)$, and the general result follows immediately. □

2.3.1. Maximal Cohen-Macaulay modules.

DEFINITION 2.3.2. A finitely generated right S-module M is **maximal Cohen-Macaulay**, abbreviated **MCM**, if $\operatorname{Ext}^i_S(M, S) = 0$ for all $i > 0$. We write $\mathsf{MCM}(S)$ for the full subcategory of maximal Cohen-Macaulay modules in $\mathsf{mod}\, S$.

EXAMPLE 2.3.3. Let S be a quasi-Frobenius ring (Example 1.1.5). Then $\operatorname{Ext}^i(M, S) = 0$ for all finitely generated S-modules M and all $i > 0$, so every S-module is MCM. In some respects, this may look like a trivial case, but as we shall see for a general Gorenstein ring S, the subcategory $\mathsf{MCM}(S)$ has structure that $\mathsf{mod}\, S$ lacks. Therefore when S is quasi-Frobenius, we can leverage the additional structure on $\mathsf{mod}\, S$ to learn more about *all* finitely generated S-modules. This is particularly important in the case of group algebras for a finite group over a field; see Example 2.5.3.

Here are some basic facts about $\mathsf{MCM}(S)$.

LEMMA 2.3.4. $\mathsf{MCM}(S)$ *is closed under extensions and kernels of epimorphisms. Additionally,* $\mathsf{proj}(S)$ *is a full subcategory of* $\mathsf{MCM}(S)$ *and* S *is an injective object in* $\mathsf{MCM}(S)$.

PROOF. Let $L \hookrightarrow M \twoheadrightarrow N$ be a short exact sequence in $\mathsf{mod}\, S$. Apply $\mathrm{Ext}^*(-, S)$ to get an exact sequence

$$\mathrm{Ext}^1(N, S) \to \mathrm{Ext}^1(M, S) \to \mathrm{Ext}^1(L, S) \to \mathrm{Ext}^2(N, S) \to \cdots$$

If L and N are MCM, then $\mathrm{Ext}^i(L, S)$ and $\mathrm{Ext}^i(N, S)$ vanish for all $i > 0$, so $\mathrm{Ext}^i(M, S)$ must vanish too, as is evident in the case $i = 1$ above, thus $\mathsf{MCM}(S)$ is extension closed. If M and N are MCM, then since $\mathrm{Ext}^i(L, S)$ is sandwiched between $\mathrm{Ext}^i(M, S)$ and $\mathrm{Ext}^{i+1}(N, S)$, we conclude the former must vanish when the latter two do (again see $i = 1$ above), hence $\mathsf{MCM}(S)$ is closed under kernels of epimorphisms.

That $\mathsf{proj}(S)$ is a full subcategory of $\mathsf{MCM}(S)$ is obvious, for if P is projective, then $\mathrm{Hom}(P, -)$ is exact, so $\mathrm{Ext}^i(P, S) = 0$ for all $i > 0$. It follows that S is an object in $\mathsf{MCM}(S)$ (since S is projective), but perhaps more interesting is that S is an injective object of $\mathsf{MCM}(S)$. Surely, as the functor $\mathrm{Hom}(-, S)$ is exact on $\mathsf{MCM}(S)$, we conclude S is an injective object in $\mathsf{MCM}(S)$. □

LEMMA 2.3.5 ([3, Lemma 4.2.2.iv]). *Any finitely generated right* S*-module admits a finite resolution by MCM modules with length at most* $\mathrm{idim}(S) + 1$, *and all but the last module in such a resolution can be chosen to be projective.*

PROOF. Let $d := \mathrm{idim}(S)$. Most of this boils down to the fact that, for finitely generated modules over a noetherian ring, we can choose all modules in a projective resolution to be finitely generated. Let N be any finitely generated right S-module and

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & & P_2 & & \longrightarrow & & P_1 & & \longrightarrow & & P_0 & & \\ & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ & N_3 & & & & N_2 & & & & N_1 & & & & N \end{array}$$

a resolution of N by finitely generated projective modules. Here N_{j+1} is the kernel of the next epimorphism $P_j \twoheadrightarrow N_j$. Truncate the resolution at $d-1$ to get a finite resolution

$$0 \to \ker\delta \to P_{d-1} \xrightarrow{\delta} P_{d-2} \to \cdots \to P_0 \to N \to 0$$

of length $d+1$. A priori the P_j are projective ($j = 0, \ldots, d-1$), hence MCM, so all we need is that $\ker\delta$ is MCM too. By dimension shifting (Lemma 2.3.1),

$$\operatorname{Ext}^i(\ker\delta, S) \cong \operatorname{Ext}^{i+d}(N, S) = 0$$

for all $i > 0$ because $\operatorname{idim}(S) = d$. Therefore $\ker\delta$ is MCM. □

Duality theory is particularly nice over Gorenstein rings.

Lemma 2.3.6. *The functors*

$$\operatorname{Hom}_S(-, S)\colon\ \mathsf{MCM}(S) \to \mathsf{MCM}(S^{\mathrm{op}})$$

and

$$\operatorname{Hom}_{S^{\mathrm{op}}}(-, S^{\mathrm{op}})\colon\ \mathsf{MCM}(S^{\mathrm{op}}) \to \mathsf{MCM}(S)$$

are exact. Moreover, a right S-module M is MCM if and only if the dual module M^* is a MCM right S^{op}-module, and MCMs are reflexive.

Proof. By Lemma 2.3.4, S is an injective object in $\mathsf{MCM}(S)$, so the first functor $\operatorname{Hom}_S(-, S)$ is exact; dually, we find $\operatorname{Hom}_{S^{\mathrm{op}}}(-, S^{\mathrm{op}})$ is exact. Fix a resolution by finitely generated projective right S-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

for M and dualize, giving a co-resolution

$$0 \to M^* \to P_0^* \to P_1^* \to P_2^* \to \cdots$$

of M^* by projective left S-modules (Lemma 2.2.12). We thus realize M^* as a syzygy module of arbitrarily high order, so in particular

$$\operatorname{Ext}^i(M^*, S) \cong \operatorname{Ext}^{i+j}(\operatorname{coker}(P_{j-2}^* \to P_{j-1}^*), S)$$

for all $i > 0$ by dimension shifting (Lemma 2.3.1). However, the left injective dimension of S is finite, so taking $j > \operatorname{idim}(S)$ shows that M^* is MCM. Dualizing again, we find M^{**} is MCM.

If we can show $M \cong M^{**}$, then we are done. But notice that, by the same argument as in the last paragraph, the Auslander-Bridger transpose $\operatorname{Tr} M$ is a syzygy module of arbitrarily high order, hence a MCM right S^{op}-module. By Lemma 2.2.15, we are done. □

Corollary 2.3.7. *An S-module is MCM if and only if it admits a projective co-resolution.*

Proof. If an S-module M admits a projective co-resolution, then M is a syzygy of arbitrarily high order, as was shown in the last proof, so M must be MCM. Conversely, if M is MCM, then M is reflexive. Take a projective resolution of the dual module M^* (in $\mathsf{mod}\, S^{\mathrm{op}}$) and dualize to produce a projective co-resolution of $M \cong M^{**}$ containing only finitely generated projective right S-modules by Lemma 2.2.12. □

Remark 2.3.8. The last two results yield additional characterizations of MCM modules. Lemma 2.3.6 implies that the subcategory $\mathsf{MCM}(S)$ is precisely the collection of Gorenstein dimension zero objects in $\mathsf{mod}\, S$; this is the characterization of $\mathsf{MCM}(S)$ often found in the literature. Alternatively, we can join a co-resolution of a MCM module M to a projective resolution of M through the connecting map $P_0 \twoheadrightarrow M \cong M^{**} \hookrightarrow P^*_{-1}$, giving what is called a **complete resolution** of M.

$$\underbrace{\cdots \to P_2 \to P_1 \to P_0}_{\text{resolution of } M} \to \underbrace{P^*_{-1} \to P^*_{-2} \to P^*_{-3} \to \cdots}_{\text{co-resolution of } M \cong M^{**}}$$

Corollary 2.3.7 says that we can characterize MCM modules as those S-modules that admit a complete resolution. We study complete resolutions more in Chapter 3 (for example, see Definition 3.1.20).

2.3.2. Modules of finite projective dimension.

DEFINITION 2.3.9. For a Gorenstein ring S, we write $\mathsf{fpd}(S)$ for the full subcategory of $\mathsf{mod}\, S$ comprised of modules of finite projective dimension. In the interest of brevity, we may refer to a module of finite projective dimension as a **fpd** module.

PROPOSITION 2.3.10. *Suppose S is a Gorenstein ring and U a finitely generated S-module. Then* $\operatorname{pdim}(U) < \infty$ *if and only if* $\operatorname{idim}(U) < \infty$.

PROOF. Let U be a finitely generated S-module. Note that as S has finite (right) injective dimension, all finite rank free modules must too. Recall (e.g., [10, Appendix B]) that U has finite injective dimension, say $\operatorname{idim}(U) \leq d$, if and only if $\operatorname{Ext}^{d+1}(N, U) = 0$ for all S-modules N. If $\operatorname{pdim}(U) = 0$, i.e., if U is projective, then there exists Q such that $U \oplus Q \cong S^n$ for some integer n. We know there exists $d \in \mathbf{Z}$ such that $\operatorname{Ext}^{d+1}(N, S^n) = 0$ for all N, so

$$0 = \operatorname{Ext}^{d+1}(N, S^n) \cong \operatorname{Ext}^{d+1}(N, U \oplus Q) \cong \operatorname{Ext}^{d+1}(N, U) \oplus \operatorname{Ext}^{d+1}(N, Q).$$

Thus $\operatorname{Ext}^{d+1}(N, U) = 0$ for all N, equivalently, $\operatorname{idim}(U)$ is finite. Proceed by induction on $\operatorname{pdim}(U)$. Assuming the claim holds for modules of projective dimension at most $d-1$, suppose $\operatorname{pdim}(U) = d$, and write a projective resolution for U:

$$0 \to P_d \to \cdots \to P_1 \to P_0 \xrightarrow{\pi} U \to 0.$$

Then $\operatorname{pdim}(\ker \pi) \leq d-1$, so $\operatorname{idim}(\ker \pi) < \infty$. Apply Ext to the short exact sequence

$$0 \to \ker \pi \to P_0 \to U \to 0$$

to get a long exact sequence

$$\cdots \to \operatorname{Ext}^i(N, P_0) \to \operatorname{Ext}^i(N, U) \to \operatorname{Ext}^{i+1}(N, \ker \pi) \to \cdots$$

for $i > 0$ and all N. Since P_0 and $\ker \pi$ have finite injective dimension, the outer terms vanish for $i \gg 0$, therefore U has finite injective dimension.

To show the converse, first notice that injectives have finite projective dimension. Indeed, Lemma 2.2.3 and the fact that S has finite left injective dimension imply injectives have finite weak dimension. Lemma 2.2.2 assures that the weak dimension and projective dimension coincide, so injectives have finite projective dimension. The rest follows by induction on $\operatorname{idim}(U)$. Assuming the hypothesis holds for modules with injective dimension at most $d-1$, suppose $\operatorname{idim}(U) = d$. Take an injective resolution

$$0 \to U \xrightarrow{\iota} I^0 \to I^1 \to \cdots \to I^d \to 0$$

of U. Then $\operatorname{idim}(\operatorname{coker} \iota) \leq d-1$, and the short exact sequence

$$0 \to U \to I^0 \to \operatorname{coker} \iota \to 0,$$

yields the long exact sequence

$$\cdots \to \operatorname{Ext}^i(I^0, N) \to \operatorname{Ext}^i(U, N) \to \operatorname{Ext}^{i+1}(\operatorname{coker} \iota, N) \to \cdots$$

for $i > 0$ and all N. As both I^0 and $\operatorname{coker}\iota$ have finite projective dimension, we conclude—analogously to the previous paragraph—U must have finite projective dimension as well. □

2.4. Connections

To further understand the makeup of $\mathsf{mod}\, S$, we give several results that characterize the relationship between $\mathsf{MCM}(S)$ and $\mathsf{fpd}(S)$. The first concerns the projective dimension of MCMs, and the other two present criteria relating MCM modules and modules of finite projective dimension.

LEMMA 2.4.1 ([3, Lemma 5.1.1.iv]). *If* $\operatorname{pdim}(M) < \infty$ *for some MCM module* M*, then* M *is projective.*

PROOF. Let M be a MCM module. Clearly if $\operatorname{pdim}(M) = 0$ then M is projective. Assume the claim holds for MCM modules of projective dimension at most $d-1$, and let M be MCM with $\operatorname{pdim}(M) = d$. Consider a projective resolution of M:

$$0 \to P_d \to \cdots \to P_1 \to P_0 \xrightarrow{\pi} M \to 0.$$

First, we argue $\ker \pi$ is MCM. Starting with the short exact sequence

$$0 \to \ker \pi \to P_0 \to M \to 0,$$

we get a long exact sequence

$$\cdots \to \operatorname{Ext}^i(P_0, S) \to \operatorname{Ext}^i(\ker \pi, S) \to \operatorname{Ext}^{i+1}(M, S) \to \cdots$$

for $i > 0$. The outer terms vanish, so $\ker \pi$ is MCM. By the hypothesis, we conclude $\ker \pi$ is projective, since $\operatorname{pdim}(\ker \pi) \leq d - 1$. Then dualize the short exact sequence to get

$$0 \to M^* \to P_0^* \to \ker \pi^* \to 0,$$

a short exact sequence of MCM right S^{op}-modules. For any right S^{op}-module N, we have the induced exact sequence

$$\mathrm{Ext}^1_{S^{\mathrm{op}}}(P_0^*, N) \to \mathrm{Ext}^1_{S^{\mathrm{op}}}(M^*, N) \to \mathrm{Ext}^2_{S^{\mathrm{op}}}(\ker \pi^*, N)$$

with outer terms vanishing. Therefore $\mathrm{Ext}^1_{S^{\mathrm{op}}}(M^*, N) = 0$ for all N, so M^* is a finitely generated projective right S^{op}-module. Lemma 2.2.12 implies M^{**} is a projective left S^{op}-module, hence a projective right S-module, and the fact that MCMs are reflexive (Lemma 2.3.6) finishes the job. □

Example 2.4.2. We take a moment to discuss an implication of the last lemma in two exceptional cases. In general, we have

$$\mathsf{proj}(S) \subset \mathsf{MCM}(S) \subset \mathsf{mod}\, S,$$

but what happens when one of these inclusions is actually equality?

Let S be a right- and left-noetherian ring of finite global dimension (Example 2.2.7). The last lemma implies that every MCM module in $\mathsf{mod}\, S$ is projective, i.e., that $\mathsf{proj}(S) = \mathsf{MCM}(S)$. This leads to a rather trivial theory. On the other hand, if S is quasi-Frobenius (Example 1.1.5), then we have seen in Example 2.3.3 that $\mathsf{MCM}(S) = \mathsf{mod}\, S$, and the last lemma says that the only S-modules of finite projective dimension are projective modules. As we detail the properties of the category $\mathsf{MCM}(S)$, we will see that this yields a rich theory.

Suppose now that S is a quasi-Frobenius ring of finite global dimension. The conclusions above yield $\mathsf{proj}(S) = \mathsf{MCM}(S) = \mathsf{mod}\, S$, that is, every S-module is projective and injective. We call such a ring semisimple, though these will not significantly contribute to our investigation—their triviality makes them wholly unenlightening in our context.

LEMMA 2.4.3 ([3, Lemma 5.1.1.i]). *Let M be a right S-module. Then M is MCM if and only if* $\operatorname{Ext}^i(M, U) = 0$ *for all* $i > 0$ *and all finitely generated right S-modules U of finite projective dimension.*

PROOF. One direction is by definition. For the other, we use induction on the projective dimension of U. If U is projective, then there exists Q such that $U \oplus Q \cong S^n$, and

$$0 = \operatorname{Ext}^i(M, S^n) \cong \operatorname{Ext}^i(M, U) \oplus \operatorname{Ext}^i(M, Q)$$

for all $i > 0$, hence $\operatorname{Ext}^i(M, U) = 0$. Now assuming this holds for modules of projective dimension at most $d - 1$, we suppose $\operatorname{pdim}(U) = d$. Take a projective resolution of U

$$0 \to P_d \to \cdots \to P_1 \to P_0 \xrightarrow{\pi} U \to 0$$

and note that $\operatorname{pdim}(\ker \pi) \leq d - 1$. From the short exact sequence

$$0 \to \ker \pi \to P_0 \to U \to 0,$$

we get a long exact sequence

$$\cdots \to \operatorname{Ext}^i(M, P_0) \to \operatorname{Ext}^i(M, U) \to \operatorname{Ext}^{i+1}(M, \ker \pi) \to \cdots$$

for $i > 0$, and as the outer terms vanish, we conclude $\operatorname{Ext}^i(M, U) = 0$ for all $i > 0$. □

LEMMA 2.4.4 ([3, Lemma 5.1.1.ii]). *Let U be a right S-module. Then U has finite projective dimension if and only if* $\operatorname{Ext}^i(M, U) = 0$ *for all* $i > 0$ *and all MCM modules M.*

PROOF. Suppose first that $\operatorname{Ext}^i(M, U) = 0$ for all $i > 0$ and all MCM modules M. By Proposition 2.3.10, it suffices to show U has finite injective dimension. Let V be an arbitrary S-module, $d := \operatorname{idim}(S)$, and

$$0 \to M \to P_{d-1} \to \cdots \to P_1 \to P_0 \to V \to 0$$

a finite resolution of V by MCM modules, per Lemma 2.3.5. Dimension shifting (Lemma 2.3.1) and our hypothesis together show

$$\operatorname{Ext}^{i+d}(V, U) \cong \operatorname{Ext}^i(M, U) = 0$$

for all $i > 0$. In particular, there exists a positive integer j such that $\operatorname{Ext}^{j+1}(V, U) = 0$ for every V, so U has finite injective dimension (by the criterion introduced in the beginning of the proof of Proposition 2.3.10).

Conversely, if U is projective, then there exists Q such that $U \oplus Q \cong S^n$, and for all MCM modules M,

$$0 = \operatorname{Ext}^i(M, S^n) \cong \operatorname{Ext}^i(M, U) \oplus \operatorname{Ext}^i(M, Q)$$

for all $i > 0$, thus $\operatorname{Ext}^i(M, U) = 0$. The rest follows by induction on projective dimension of U, as has become routine for claims of this flavor. □

From the proof of Lemma 2.4.4, we see that every projective is an injective object in $\mathsf{MCM}(S)$. That is, while an object of $\mathsf{proj}(S)$ may not, and likely is not, injective in $\mathsf{mod}\, S$, when we restrict our attention to the full subcategory of MCM modules, all projectives are in fact injective objects. We can rephrase the above as the containment $\mathsf{proj}(S) \subset \mathsf{Inj}(\mathsf{MCM}(S))$, and we can show even more.

LEMMA 2.4.5. *In* $\mathsf{MCM}(S)$*, the projective objects and injective objects coincide. Furthermore, projective objects in* $\mathsf{MCM}(S)$ *are precisely projective modules in* $\mathsf{mod}\, S$*.*

PROOF. We have shown (proof of Lemma 2.4.4) that projective S-modules are injective in $\mathsf{MCM}(S)$, however, it is not yet clear that all projective objects in $\mathsf{MCM}(S)$ arise as projective S-modules. We claim if P is an object in $\mathsf{Proj}(\mathsf{MCM}(S))$, then P is in $\mathsf{proj}(S)$ too. To show $\mathrm{Ext}^1(P, N) = 0$ for all N in $\mathsf{mod}\, S$, it suffices to consider a short exact sequence

$$0 \to M_1 \to M_0 \to N \to 0$$

with N any finitely generated S-module and M_0, M_1 MCM. We have an exact sequence

$$\mathrm{Ext}^1(P, M_0) \to \mathrm{Ext}^1(P, N) \to \mathrm{Ext}^2(P, M_1)$$

where the outer terms vanish, so the middle one must too, hence P is in $\mathsf{proj}(S)$. In general, write a finite resolution

$$0 \to M_d \to M_{d-1} \to \cdots \to M_1 \to M_0 \to N \to 0$$

of N by MCM modules using Lemma 2.3.5, and apply the above result, starting with $0 \to M_d \to M_{d-1} \to \mathrm{coker}(M_d \to M_{d-1}) \to 0$ and continuing down the line, to conclude $\mathrm{Ext}^1(P, N) = 0$.

The current state of affairs is $\mathsf{proj}(S) = \mathsf{Proj}(\mathsf{MCM}(S)) \subset \mathsf{Inj}(\mathsf{MCM}(S))$. To demonstrate the other containment, suppose I is injective in $\mathsf{MCM}(S)$, so $\mathrm{Ext}^i(M, I) = 0$ for all $i > 0$ and all MCM modules M. By Lemma 2.4.4, I has finite projective dimension, so Lemma 2.4.1 implies I is projective in $\mathsf{mod}\, S$, i.e., belongs to $\mathsf{proj}(S)$. Therefore $\mathsf{proj}(S) = \mathsf{Proj}(\mathsf{MCM}(S)) = \mathsf{Inj}(\mathsf{MCM}(S))$. $\square$

2.5. Frobenius structure and triangulation

While the structural results of the last few sections are insightful, they do not directly shed light on the heart of our investigation. We now make headway on some headlining results by focusing on the category $\mathsf{MCM}(S)$.

The category of MCM S-modules inherits an exact structure from $\mathsf{mod}\, S$, obtained by designating all short exact sequences of MCM modules as conflations. If $M \twoheadrightarrow N$ is an epimorphism of MCM modules, then it is a deflation by Lemma 2.3.4. For inflations, consider a monomorphism $f\colon L \hookrightarrow M$ of MCM modules. In $\mathsf{mod}\, S$, we can extend to include the cokernel $L \hookrightarrow M \twoheadrightarrow C$, and C is MCM if and only if $f^*\colon \operatorname{Hom}(M,S) \to \operatorname{Hom}(L,S)$ is an epimorphism; this characterizes inflations.

Proposition 2.5.1. *The category* $\mathsf{MCM}(S)$ *is Frobenius.*

Proof. To show $\mathsf{MCM}(S)$ is an exact category, we only really need to show E1, E2, and E2$^{\mathrm{op}}$, as the others are immediate. If $f\colon L \hookrightarrow M$ and $g\colon M \hookrightarrow N$ are two inflations, then f^* and g^* are epimorphisms. It follows that $f^*g^* = (gf)^*$ is an epimorphism, proving E1.

Now let $i\colon L \hookrightarrow M$ be an inflation and $f\colon L \to N$ an arbitrary morphism. Take the pushout, say K, in $\mathsf{mod}\, S$.

$$\begin{array}{ccc} L & \xhookrightarrow{\ i\ } & M \\ \big\downarrow{\scriptstyle f} & \ulcorner & \big\downarrow{\scriptstyle \hat{f}} \\ N & \xrightarrow{\ j\ } & K \end{array}$$

We must show K is MCM and that $j\colon N \to K$ is an inflation. Note that we have a short exact sequence

$$0 \to L \xrightarrow{\left(\begin{smallmatrix} i \\ -f \end{smallmatrix}\right)} M \oplus N \to K \to 0,$$

so K is MCM if and only if

$$\begin{pmatrix} i^* & -f^* \end{pmatrix} \colon \operatorname{Hom}(M,S) \oplus \operatorname{Hom}(N,S) \to \operatorname{Hom}(L,S)$$

is an epimorphism. Since i is an inflation, i^* is an epimorphism, so $\begin{pmatrix} i^* & -f^* \end{pmatrix}$ is an epimorphism too. Therefore K is MCM. Now j is a priori a monomorphism (this is true in any abelian category), but we need to show it is a kernel of some deflation $q\colon K \twoheadrightarrow C$ in $\mathsf{MCM}(S)$. As i is an inflation, we can exhibit a cokernel $p\colon M \twoheadrightarrow C$ in $\mathsf{MCM}(S)$, and along with the zero map $N \to C$, we induce a unique map $q\colon K \to C$ such that $q\hat{f} = p$ and $qj = 0$.

$$\begin{array}{ccccc} L & \xhookrightarrow{\ i\ } & M & & \\ \Big\downarrow f & \ulcorner & \Big\downarrow \hat{f} & \searrow^{p} & \\ N & \xrightarrow{\ j\ } & K & \overset{\exists! q}{\dashrightarrow} & C \\ & & \xrightarrow{\quad 0 \quad} & & \end{array}$$

First, $q\hat{f} = p$ implies q is an epimorphism, hence a deflation, so it is enough to show q is a cokernel of j. Let $h\colon K \to X$ be any map such that $hj = 0$. To see that h factors uniquely through q, notice that, by commutativity, $h\hat{f}i = 0$, so there exists a unique map $r\colon C \to X$ with the property that $h\hat{f} = rp$.

$$\begin{array}{ccccc} L & \xhookrightarrow{\ i\ } & M & \xrightarrow{\ p\ } & C \\ \Big\downarrow f & \ulcorner & \Big\downarrow \hat{f} & \nearrow^{q} & \Big\downarrow \exists! r \\ N & \xrightarrow{\ j\ } & K & \xrightarrow{\ h\ } & X \end{array}$$

We must show $h = rq$. By construction of q and h, $h\hat{f} = rq\hat{f}$ and $hj = 0 = rqj$. Then there are two maps $h - rq$ and 0 from the pushout K to X compatible with the pushout diagram, but uniqueness of maps from the pushout implies $h - rq = 0$. Therefore q is a cokernel of j, or put another way, $0 \to N \to K \to C \to 0$ is a short exact sequence of MCM modules in $\mathsf{mod}\, S$. Hence j is an inflation, verifying E2.

For E2$^{\mathrm{op}}$, let $q\colon N \twoheadrightarrow K$ be a deflation and $g\colon M \to K$ any morphism. Take the pullback in $\mathsf{mod}\, S$.

$$\begin{array}{ccc} L & \xrightarrow{p} & M \\ \downarrow & \lrcorner & \downarrow g \\ N & \overset{q}{\twoheadrightarrow} & K \end{array}$$

As p is the pullback of an epimorphism, p is an epimorphism (as is the case in any abelian category). Then we get a short exact sequence

$$0 \to L \to M \oplus N \xrightarrow{(g\ -q)} K \to 0.$$

Both $M \oplus N$ and K are MCM modules, so as $\mathsf{MCM}(S)$ is closed under kernels of epimorphisms (Lemma 2.3.4), L is an MCM module. Thus $p\colon L \twoheadrightarrow M$ is a deflation, showing E2$^{\mathrm{op}}$. Therefore $\mathsf{MCM}(S)$ is an exact category.

To show $\mathsf{MCM}(S)$ is Frobenius, we note that $\mathsf{mod}\, S$ has enough projectives, so $\mathsf{MCM}(S)$ must have enough too. Then Corollary 2.3.7 and Lemma 2.4.5 imply that $\mathsf{MCM}(S)$ has enough injectives and that $\mathsf{Proj}(\mathsf{MCM}(S)) = \mathsf{Inj}(\mathsf{MCM}(S))$. Therefore $\mathsf{MCM}(S)$ is a Frobenius category. □

Corollary 2.5.2. *The category* $\underline{\mathsf{MCM}}(S)$ *is triangulated.*

Proof. By Theorem 1.1.19 and Proposition 2.5.1. □

Example 2.5.3. From the perspective of representation theory, the last corollary is quite good news, as the triangulated structure aids the computation of group cohomology. Indeed, if G is a finite group and k is a field, we know that the group algebra kG is self-injective, hence $\mathsf{MCM}(kG) = \mathsf{mod}\, kG$. Group cohomology of G with coefficients in a kG-module M is given by

$$H^n(G; M) \cong \mathrm{Ext}^n_{kG}(k, M)$$

for $n \geq 0$; see [15, Exercise 6.1.2]. The righthand side is computed in the stable category $\underline{\mathsf{mod}}\, kG$, that is, writing $\underline{\mathrm{Hom}}$ for the stable Hom,

$$\mathrm{Ext}^n_{kG}(k, M) \cong \underline{\mathrm{Hom}}_{kG}(\Omega^n k, M).$$

In this way, the natural context for studying group cohomology is the stable category $\underline{\mathsf{mod}}\, kG$.

The cohomology ring of G is the graded ring

$$H^\star(G) := \bigoplus_{n \geq 0} H^n(G; k) \cong \bigoplus_{n \geq 0} \mathrm{Ext}^n_{kG}(k, k).$$

While Ext is only defined in nonnegative degrees, we can use the triangulated structure on $\underline{\mathsf{mod}}\, kG$ to extend the definition. That is to say, taking syzygies is an auto-equivalence on $\underline{\mathsf{mod}}\, kG$, so negative syzygies are just positive cosyzygies, i.e., we have a natural isomorphism

$$\underline{\mathrm{Hom}}_{kG}(\Omega^n M, N) \cong \underline{\mathrm{Hom}}_{kG}(M, \Sigma^n N)$$

for all $n \in \mathbf{Z}$. For any integer n, we define the n^{th} Tate cohomology group of G to be

$$\widehat{H}^n(G) := \underline{\mathrm{Hom}}_{kG}(k, \Sigma^n k).$$

A seemingly different definition is given in [3, Definition 6.1.1], though it follows from our work in Section 3.2.2 that our definition is equivalent. The Tate cohomology ring is then the graded ring

$$\widehat{H}^\star(G) := \bigoplus_{n \in \mathbf{Z}} \underline{\mathrm{Hom}}_{kG}(k, \Sigma^n k).$$

See Remark 3.2.6 for another application of the triangulation of $\underline{\mathsf{mod}}\, kG$.

One note regarding the theory just described: Maschke's Theorem says that kG is semisimple when the characteristic of k does not divide the order of G. In this case every kG-module is projective (by definition; recall Example 2.4.2), from which it follows that the triangulated category $\underline{\mathsf{mod}}\,kG$ is trivial. Hence the theory is only really interesting when the characteristic of k divides the order of G.

2.6. Cotorsion structure

Let S be a Gorenstein ring. As is evident from Lemmas 2.4.3 and 2.4.4, the Ext-orthogonal complement to MCM modules are modules of finite projective dimension. Using the language of Section 1.2, we will unify the numerous preceding lemmas into statements about a cotorsion structure of the category $\mathsf{mod}\,S$.

PROPOSITION 2.6.1. *For a Gorenstein ring S, the pair* $(\mathsf{MCM}(S), \mathsf{fpd}(S))$ *is a complete hereditary cotorsion pair in* $\mathsf{mod}\,S$.

PROOF. That $(\mathsf{MCM}(S), \mathsf{fpd}(S))$ is a hereditary cotorsion pair in $\mathsf{mod}\,S$ follows from Lemmas 2.4.3 and 2.4.4. Now let N be an arbitrary finitely generated right S-module. Our approach assumes a bit more of the reader, so we encourage a review of chain complexes (Section 3.1.1) and quasi-isomorphisms (Remark 3.1.16) if necessary. We will construct both required short exact sequences at once, and we start by taking a projective resolution P of N.

$$\cdots \to P_2 \to P_1 \to P_0 \to N \to 0.$$

Dualizing, we get a complex (not acyclic!) $P^* := \mathrm{Hom}_S(P, S)$

$$\begin{array}{ccccccccc} & & P_0^* & \longrightarrow \cdots \longrightarrow & P_{m-1}^* & & \longrightarrow & & P_m^* \longrightarrow \cdots \\ & \nearrow & & & & \searrow & & \nearrow & \\ N^* & & & & & & K & & \end{array}$$

with bounded homology. Indeed,

$$H_n P^* = H_n \,\mathrm{Hom}_{S^{\mathrm{op}}}(P^*, S^{\mathrm{op}}) = \mathrm{Ext}^n_{S^{\mathrm{op}}}(N^*, S^{\mathrm{op}}) = 0$$

for $n > \mathrm{idim}(S^{\mathrm{op}})$. Suppose m is the cutoff for nonzero homology, that is, $H_m P^* \neq 0$ and $H_n P^* = 0$ for all $n > m$. Truncate P^* at degree m by replacing P^*_m with $K := \ker(P^*_m \to P^*_{m+1})$, giving a finite complex $\tau_{\geq m} P^*$ of projectives quasi-isomorphic to P^*:

$$0 \to P^*_0 \to P^*_1 \to \cdots \to P^*_{m-1} \to K \to 0.$$

The cost of truncating is that K may not be projective. To remedy this, take a resolution G of $\tau_{\geq m} P^*$ by projective right S^{op}-modules. This amounts to taking a projective resolution F of K, labelled

$$\cdots \to F_{m-2} \to F_{m-1} \to F_m \to K \to 0$$

for notational convenience, and setting G to be the complex

$$\cdots \to F_{-1} \to P^*_0 \oplus F_0 \to \cdots \to P^*_{m-1} \oplus F_{m-1} \to F_m \to 0.$$

Dualizing again, we get a bounded below complex G^* of projective right S-modules that is quasi-isomorphic to $P \cong P^{**}$. The picture is as follows.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & G^*_1 & \xrightarrow{d_1} & G^*_0 & \xrightarrow{d_0} & G^*_{-1} & \longrightarrow & \cdots \end{array}$$

Now G^* has homology only in degree 0, in particular, $H_0 G^* \cong N$, so we have exact sequences

$$0 \to \mathrm{im}(d_1) \to \ker(d_0) \to N \to 0$$

and

$$0 \to N \to \mathrm{coker}(d_1) \to \mathrm{im}(d_0) \to 0.$$

We claim $\ker(d_0)$ and $\mathrm{im}(d_0)$ are MCM, while $\mathrm{im}(d_1)$ and $\mathrm{coker}(d_1)$ are of finite projective dimension. The latter follows since G^* is bounded below, so

$$0 \to G_m^* \to \cdots \to G_1^* \to \mathrm{im}(d_1) \to 0$$

and

$$0 \to G_m^* \to \cdots \to G_0^* \to \mathrm{coker}(d_1) \to 0$$

are the required finite projective resolutions. To see $\ker(d_0)$ and $\mathrm{im}(d_0)$ are MCM, notice that the righthand side of G^* is a projective co-resolution of both. Dimension shifting (Lemma 2.3.1) gives

$$\mathrm{Ext}^i(\ker(d_0), S) \cong \mathrm{Ext}^{i+j}(\ker(d_{-j}), S)$$

and

$$\mathrm{Ext}^i(\mathrm{im}(d_0), S) \cong \mathrm{Ext}^{i+j}(\mathrm{im}(d_{-j}), S),$$

and taking $j > \mathrm{idim}(S)$ makes both 0 for all $i > 0$, hence $\ker(d_0)$ and $\mathrm{im}(d_0)$ are MCM. Therefore the cotorsion pair $(\mathsf{MCM}(S), \mathsf{fpd}(S))$ is complete. □

We may now apply everything we learned about complete hereditary cotorsion pairs in Section 1.2 to further describe modules over a Gorenstein ring. First, the preceding proposition implies that we can realize a finitely generated S-module as either the quotient of a MCM module by a fpd module or as the kernel of an epimorphism from a fpd module to a MCM module. Furthermore, it is immediate from Proposition 1.2.12 that $\mathsf{MCM}(S)$ is a contravariantly finite subcategory of $\mathsf{mod}\, S$ and $\mathsf{fpd}(S)$ is a covariantly finite subcategory of $\mathsf{mod}\, S$. This yields the notions of MCM approximation and fpd approximation: For any finitely generated S-module N, there exists a MCM module M that right approximates N as well as or better than any other MCM module and similarly an fpd module U with a left approximation of N; see Definition 1.2.10. Briefly, a right approximation of N by $\mathsf{MCM}(S)$ is a map $M \to N$ that factors any other map into N from a MCM module. In fact, we can refine this map to be a surjection by adding on a projective cover of N to M. Dually, we have a map from N into a module U of finite projective dimension that factors any other map from N into a fpd module, and by adding an injective envelope to U, we can make this map an injection. Note that these assignments of approximations are not functorial in general, but upon projectively stabilizing, we obtain functorial approximations.

PROPOSITION 2.6.2. *Let* $\underline{\mathsf{mod}}\,S$*,* $\underline{\mathsf{fpd}}(S)$*, and* $\underline{\mathsf{MCM}}(S)$ *be the projective stabilizations of* $\mathsf{mod}\,S$*,* $\mathsf{fpd}(S)$*, and* $\mathsf{MCM}(S)$ *respectively. Then* $\underline{\mathsf{MCM}}(S)$ *is a contravariantly finite subcategory of* $\underline{\mathsf{mod}}\,S$*,* $\underline{\mathsf{fpd}}(S)$ *is a covariantly finite subcategory of* $\underline{\mathsf{mod}}\,S$*, and* $(\underline{\mathsf{MCM}}(S), \underline{\mathsf{fpd}}(S))$ *is a torsion pair in* $\underline{\mathsf{mod}}\,S$*.*

PROOF. By Lemma 2.4.1, $\mathsf{MCM}(S) \cap \mathsf{fpd}(S) = \mathsf{proj}(S)$, and the first two claims follow from Proposition 1.2.16. The last claim is immediate by Proposition 1.2.19. □

We can be more explicit about how the approximations M and U arise. Maintaining the notation from the proof of Proposition 2.6.1, let $M := \ker(d_0)$ and $U := \operatorname{coker}(d_1)$. We have a short exact sequence

$$0 \to M \to N \oplus G_0^* \to U \to 0.$$

Here G_0^* is doing double duty, ensuring that the first and second maps are respectively monic and epic. Then in $\underline{\mathsf{mod}}\,S$, G_0^* vanishes, and we get functorial right and left approximations of N by M and U.

The picture is as follows, with i, j the canonical inclusions and c, d the adjoint approximations.

$$\underline{\mathsf{MCM}}(S) \underset{\mathsf{c}}{\overset{\mathsf{i}}{\rightleftarrows}} \underline{\mathsf{mod}}\,S \underset{\mathsf{j}}{\overset{\mathsf{d}}{\rightleftarrows}} \underline{\mathsf{fpd}}(S)$$

Of particular interest is the functor c and the notion of functorial MCM approximation. The polarity in our discussion comes from Corollary 2.5.2, which says $\underline{\mathsf{MCM}}(S)$ is triangulated. In the next chapter we will relate $\underline{\mathsf{MCM}}(S)$ to other triangulated categories, and we will see that one of these is connected to $\underline{\mathsf{MCM}}(S)$ through MCM approximation.

CHAPTER 3

Categories equivalent to $\underline{\mathsf{MCM}}(S)$

We turn our attention to the construction of several triangulated categories equivalent to the projectively stable category of maximal Cohen-Macaulay modules. Throughout, constructions arising from chain complexes, such as the homotopy category of complexes and the derived category, guide the narrative. We declare once and for all that S is a Gorenstein ring.

3.1. The homotopy category of acyclic complexes of projectives

We begin with the homotopy category of acyclic complexes of finitely generated projective right S-modules. This category, written $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$, is not only equivalent to $\underline{\mathsf{MCM}}(S)$, but is intimately connected to one characterization of maximal Cohen-Macaulay modules. In examining the ties between $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ and $\underline{\mathsf{MCM}}(S)$, we will shine a light on how $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ shapes the discussion of MCM modules, illuminating earlier sections where $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ lingered implicit.

3.1.1. Exact structure on the category of chain complexes.

Definition 3.1.1. Let $\mathcal{X}$ be an additive category. A **chain complex** with values in $\mathcal{X}$ is a diagram of the form

$$\cdots \xrightarrow{d^X_{n+2}} X_{n+1} \xrightarrow{d^X_{n+1}} X_n \xrightarrow{d^X_n} X_{n-1} \xrightarrow{d^X_{n-1}} \cdots$$

where X_i are objects in $\mathcal{X}$ and $d^X_i d^X_{i+1} = 0$ for all $i \in \mathbf{Z}$. One writes $X_\bullet$, or even X, in place of the diagram above for brevity. The maps d^X_i

are called **differentials**, and the superscript recording where differential resides is often dropped. A morphism of chain complexes $f_\bullet \colon X_\bullet \to Y_\bullet$ is a set of morphisms $\{f_i \colon X_i \to Y_i\}_{i\in\mathbf{Z}}$ such that $d^Y_{i+1}f_{i+1} = f_i d^X_{i+1}$ for all $i \in \mathbf{Z}$; again, one may write $f \colon X \to Y$ as shorthand. The decoration may be exchanged, whence we would write

$$\cdots \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \cdots$$

instead—note the indices ascend from left to right now. Collecting chain complexes and their morphisms, we define the **category of chain complexes** with values in $\mathcal{X}$, denoted by $\mathsf{Ch}(\mathcal{X})$. This is an additive category.

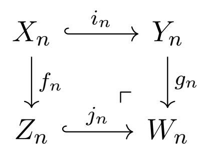
Note that any additive category has a minimal exact structure given by the split exact sequences. A conflation can be thought of as complex, and in this way, isomorphism classes of diagrams of the form

$$X \xhookrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} X \oplus Z \xrightarrow{(0\ 1)} Z$$

comprise the exact structure. Then endowing $\mathcal{X}$ with this minimal exact structure, we can impose on $\mathsf{Ch}(\mathcal{X})$ an exact structure given by degree-wise split exact sequences, that is $X_\bullet \to Y_\bullet \to Z_\bullet$ is a conflation in $\mathsf{Ch}(\mathcal{X})$ if and only if $X_i \to Y_i \to Z_i$ is a conflation in $\mathcal{X}$ for all $i \in \mathbf{Z}$. From now on, we suppress the dot subscript on complexes and their morphisms.

LEMMA 3.1.2. *Let $\mathcal{X}$ be an additive category equipped with the split exact structure. Then $\mathsf{Ch}(\mathcal{X})$ is an exact category with respect to the degree-wise split exact structure.*

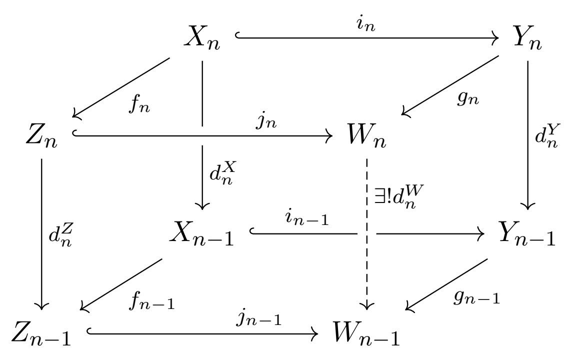
PROOF. All we really need to check is E2, and E2$^{\mathrm{op}}$ will follow similarly. Given an inflation $i \colon X \hookrightarrow Y$ and an arbitrary $f \colon X \to Z$, take pushouts degree-wise. At each level, we have an inflation $j_n \colon Z_n \hookrightarrow W_n$ and a morphism $g_n \colon Y_n \to W_n$ making the following (pushout square) commute.



All that we need are the differentials $d_n^W \colon W_n \to W_{n-1}$ compatible with g and j. Consider the maps $g_{n-1}d_n^Y \colon Y_n \to W_{n-1}$ and $j_{n-1}d_n^Z \colon Z_n \to W_{n-1}$. Then

$$g_{n-1}d_n^Y i_n = g_{n-1}i_{n-1}d_n^X = j_{n-1}f_{n-1}d_n^X = j_{n-1}d_n^Z f_n,$$

so there exists a unique $d_n^W \colon W_n \to W_{n-1}$ such that $d_n^W g_n = g_{n-1}d_n^Y$ and $d_n^W j_n = j_{n-1}d_n^Z$ by the universal property of the pushout.



So long as W is a complex, the induced map $j \colon Z \to W$ is an inflation, so we must show $d_{n-1}^W d_n^W = 0$. To see this, we can construct a map $W_n \to W_{n-2}$ via the universal property of the pushout. Taking

$$g_{n-2}d_{n-1}^Y d_n^Y \colon Y_n \to W_{n-2} \quad \text{and} \quad j_{n-2}d_{n-1}^Z d_n^Z \colon Z_n \to W_{n-2},$$

a chase similar to the one above verifies the required equality, and we induce a *unique* map $W_n \to W_{n-2}$. But $d_{n-1}^Y d_n^Y = 0 = d_{n-1}^Z d_n^Z$, so the induced map is the zero map too, and uniqueness implies $d_{n-1}^W d_n^W = 0$. Therefore the pushout of i along f exists and is an inflation. $\square$

Now we can examine projective and injective objects in $\mathsf{Ch}(\mathcal{X})$. Recall that a complex P is projective if, for all deflations $Y \twoheadrightarrow Z$, the induced map $\mathrm{Hom}_{\mathsf{Ch}(\mathcal{X})}(P, Y) \to \mathrm{Hom}_{\mathsf{Ch}(\mathcal{X})}(P, Z)$ is a surjection. On the other hand, a complex I is injective if, for all inflations $X \hookrightarrow Y$, the induced map $\mathrm{Hom}_{\mathsf{Ch}(\mathcal{X})}(Y, I) \to \mathrm{Hom}_{\mathsf{Ch}(\mathcal{X})}(X, I)$ is a surjection. To start, consider complexes of the form $0 \to A = A \to 0$, which we denote by P_A. These will serve as our prototype projective/injective objects, as is verified in the lemma.

Lemma 3.1.3. *The complex P_A is both projective and injective in* $\mathsf{Ch}(\mathcal{X})$.

Proof. We will prove that P_A is projective, and as the only fact we need about deflations is that they are degree-wise split, the proof readily carries over to show P_A is injective. Let $a\colon P_A \to Z$ be a morphism of complexes, and assume without loss of generality that we index as follows.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & = & A & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_1} & & \downarrow & & \\ \cdots & \longrightarrow & Z_3 & \longrightarrow & Z_2 & \xrightarrow{d_2^Z} & Z_1 & \longrightarrow & Z_0 & \longrightarrow & \cdots \end{array}$$

Remark that $a_1 = d_2^Z a_2$ by assumption. Now let $f\colon Y \twoheadrightarrow Z$ be a deflation, so f is a degree-wise split epimorphism, say with splitting $s_i\colon Z_i \to Y_i$ for each i, so $f_i g_i = id_{Z_i}$. Note that the splitting is only degree-wise and need not be a morphism of complexes. Define $b\colon P_A \to Y$ by $b_2 := g_2 a_2$ and $b_1 := d_2^Y g_2 a_2$ (and $b_i = 0$ everywhere else). We have the following situation.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & = & A & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle b_2} & & \downarrow{\scriptstyle b_1} & & \downarrow & & \\ \cdots & \longrightarrow & Y_3 & \longrightarrow & Y_2 & \xrightarrow{d_2^Y} & Y_1 & \longrightarrow & Y_0 & \longrightarrow & \cdots \\ & & {\scriptstyle g_3}\uparrow\,\downarrow\!\!\!\downarrow{\scriptstyle f_3} & & {\scriptstyle g_2}\uparrow\,\downarrow\!\!\!\downarrow{\scriptstyle f_2} & & {\scriptstyle g_1}\uparrow\,\downarrow\!\!\!\downarrow{\scriptstyle f_1} & & {\scriptstyle g_0}\uparrow\,\downarrow\!\!\!\downarrow{\scriptstyle f_0} & & \\ \cdots & \longrightarrow & Z_3 & \longrightarrow & Z_2 & \xrightarrow[d_2^Z]{} & Z_1 & \longrightarrow & Z_0 & \longrightarrow & \cdots \end{array}$$

By construction, b is a morphism of complexes: $b_1 = d_2^Y g_2 a_2 = d_2^Y b_2$. Then $f_2 b_2 = f_2 g_2 a_2 = a_2$ and

$$f_1 b_1 = f_1 d_2^Y g_2 a_2 = d_2^Z f_2 g_2 a_2 = d_2^Z a_2 = a_1,$$

so $fb = a$, showing $\mathrm{Hom}(P_A, Y) \to \mathrm{Hom}(P_A, Z)$ is surjective. □

One more operation we need is translation. The translation functor is an additive auto-equivalence $(-)[1] : \mathsf{Ch}(\mathcal{X}) \to \mathsf{Ch}(\mathcal{X})$ that shifts complexes against the differential, swaps the sign of the differential, and carries morphisms along with it. More generally we can speak of a $\mathbf{Z}$-indexed family of translation functors $(-)[i]$ given by $X[i]_n := X_{n-i}$ and $d_n^{X[i]} := (-1)^i d_{n-i}^X$ for a complex X. For a morphism $f\colon X \to Y$, the result is $f[i]_n := f_{n-i}$.

From the prototype projectives, we can construct more elaborate projectives, in particular projective covers. Given a complex X, we make the convention that P_{X_0} is concentrated in degrees 0 and -1 so we can get the alignment correct; see below for P_{X_0}, $P_{X_1}[1]$, and X (top to bottom).

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X_0 & \xrightarrow{1} & X_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cdots & \longrightarrow & 0 & \longrightarrow & X_1 & \xrightarrow{-1} & X_1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X_{-1} & \longrightarrow & X_{-2} & \longrightarrow & \cdots
\end{array}$$

The coproduct of projectives is still projective, so define the projective cover of X by

$$P_X := \coprod_{i \in \mathbf{Z}} P_{X_i}[i].$$

There should be no confusion of the notation as it will be clear when the subscript is an object in $\mathcal{X}$ or a complex. By construction, the differential on P_X is

$$d_i^{P_X} = \begin{pmatrix} 0 & (-1)^i \\ 0 & 0 \end{pmatrix} : X_{i+1} \oplus X_i \to X_i \oplus X_{i-1},$$

and we can define $b\colon P_X \to X$ by

$$b_i := \begin{pmatrix} d^X_{i+1} & (-1)^i \end{pmatrix} : X_{i+1} \oplus X_i \to X_i,$$

which is a morphism of complexes since $d^X_i b_i = b_{i-1} d^{P_X}_i$. We make the following claim.

PROPOSITION 3.1.4. *The map $b\colon P_X \to X$ is a deflation. Consequently, $\mathsf{Ch}(\mathcal{X})$ has enough projectives.*

PROOF. To justify the claim, we must show b fits into a degree-wise split exact sequence, so we will construct a sequence

$$\widetilde{X} \xrightarrow{\binom{1}{0}} \widetilde{P_X} \xrightarrow{(0\ 1)} X \tag{3.1.1}$$

that is surely degree-wise split exact and show it is isomorphic to a sequence containing b. We start by proposing a candidate sequence for b: Define the complex $\widetilde{X}$ by $\widetilde{X}_i := X_{i+1}$ and $d^{\widetilde{X}}_i := d^X_{i+1}$. Remark that $\widetilde{X}$ is just $X[-1]$ without the negative sign on the differentials. Furthermore, define $f\colon \widetilde{X} \to P_X$ by

$$f_i := \begin{pmatrix} (-1)^{i+1} \\ d^X_{i+1} \end{pmatrix} : X_{i+1} \to X_{i+1} \oplus X_i.$$

Then $d^{P_X}_i f_i = f_{i-1} d^{\widetilde{X}}_i$, so f is a morphism of complexes, and $bf = 0$ as one would hope. Having established our candidate

$$\widetilde{X} \xrightarrow{f} P_X \xrightarrow{b} X, \tag{3.1.2}$$

we must show it is in fact degree-wise split exact.

Let $\widetilde{P_X}$ be the complex with $(\widetilde{P_X})_i = (P_X)_i$ and differential

$$d_i^{\widetilde{P_X}} := \begin{pmatrix} d_{i+1}^X & (-1)^i \\ 0 & d_i^X \end{pmatrix} : X_{i+1} \oplus X_i \to X_i \oplus X_{i-1}.$$

Notice that $\widetilde{P_X}$ fits into the sequence (3.1.1), where both inclusion into the first component and projection from the second are morphisms of complexes. Define $\varphi\colon \widetilde{P_X} \to P_X$ by

$$\varphi_i := \begin{pmatrix} (-1)^{i+1} & 0 \\ d_{i+1}^X & (-1)^i \end{pmatrix} : X_{i+1} \oplus X_i \to X_{i+1} \oplus X_i,$$

a morphism of complexes since $d_i^{P_X} \varphi_i = \varphi_{i-1} d_i^{\widetilde{P_X}}$. Perhaps surprisingly, φ is an involution! Indeed, $(\varphi_i)^2$ is the identity on $X_{i+1} \oplus X_i$ for all i, so right and left multiplying appropriately with the last equation gives $\varphi_{i-1} d_i^{P_X} = d_i^{\widetilde{P_X}} \varphi_i$, i.e., that $\varphi\colon P_X \to \widetilde{P_X}$ is a morphism of complexes. It follows that φ is an isomorphism. We claim the candidate sequence (3.1.2) is isomorphic to (3.1.1), so evaluate the following diagram.

$$\begin{array}{ccccc} X_{i+1} & \xrightarrow{f_i} & X_{i+1} \oplus X_i & \xrightarrow{b_i} & X_i \\ \| & & \varphi_i \uparrow \; \downarrow \varphi_i & & \| \\ X_{i+1} & \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} & X_{i+1} \oplus X_i & \xrightarrow{(0\ 1)} & X_i \end{array}$$

Both squares commute for both directions of φ_i for all i, so (3.1.2) is a degree-wise split exact sequence. Therefore b is a deflation, and as we can form a projective cover P_X for each object X in $\mathsf{Ch}(\mathcal{X})$, we conclude $\mathsf{Ch}(\mathcal{X})$ has enough projectives. $\square$

On the other hand, the prototype complexes P_A are injective, so a product of these is still injective. Define the injective hull of a complex X to be

$$I_X := \prod_{i \in \mathbf{Z}} P_{X_i}[i+1],$$

which has differential

$$d_i^{I_X} = \begin{pmatrix} 0 & (-1)^i \\ 0 & 0 \end{pmatrix} : X_i \oplus X_{i-1} \to X_{i-1} \oplus X_{i-2}$$

by construction. Additionally, we have a morphism of complexes $f\colon X \to I_X$ given by

$$a_i := \begin{pmatrix} (-1)^{i+1} \\ d_i^X \end{pmatrix} : X_i \to X_i \oplus X_{i-1},$$

and we find the following.

PROPOSITION 3.1.5. *The map $a\colon X \to I_X$ is an inflation. Consequently,* $\mathsf{Ch}(\mathcal{X})$ *has enough injectives.*

The proof follows mutatis mutandis from the last, so is omitted. Note that these projective covers and injective hulls coincide—i.e., for each X, there exists Y such that $P_X \cong I_Y$—as we have an isomorphism

$$\coprod_{i\in\mathbf{Z}} P_{X_i}[i] \cong \prod_{i\in\mathbf{Z}} P_{X_i}[i]$$

for each X (there are only two summands in each degree). Having shown $\mathsf{Ch}(\mathcal{X})$ has enough projectives and enough injectives, one could hope that all projectives and injectives coincide, making $\mathsf{Ch}(\mathcal{X})$ Frobenius. In fact this is the case, and we connect projectives and injectives through contractible complexes.

DEFINITION 3.1.6. Let $f, g\colon X \to Y$ be two morphisms of complexes. A **(chain) homotopy** between f and g is a set of morphisms $\{h_i\colon X_i \to Y_{i+1}\}_{i\in\mathbf{Z}}$ subject to

$$f_i - g_i = h_{i-1}d_i^X + d_{i+1}^Y h_i$$

for all $i \in \mathbf{Z}$. We may write $f \sim g$ if they are homotopic. A chain map is called **null-homotopic** if it is homotopic to the zero map. We call $f\colon X \to$

Y is a **homotopy equivalence** if there exists $g\colon Y \to X$ with gf homotopic to the identity on X and fg homotopic to the identity on Y, and in this case, we say X and Y are **homotopy equivalent**. A complex is **contractible** if its identity map is null-homotopic, so contractible complexes are homotopy equivalent to the zero complex.

One obstacle we face in homological algebra is constructions that fail to be unique in some way. For example, we speak not of *the* projective resolution of an object, only *a* projective resolution, because there is no way to choose projective resolutions uniquely. Projective resolutions are complexes, and isomorphism of complexes is too strong a notion to capture the behavior here. Homotopy does a better job at expressing this similarity, as two projective resolutions of the same object are homotopy equivalent (see Lemma 3.1.22). Lastly, we detail a construction fundamental to the study of chain complexes.

DEFINITION 3.1.7. For a morphism of complexes $f\colon X \to Y$, define the **mapping cone** of f by $\operatorname{cone}(f) := Y \oplus X[1]$ with differential

$$d_i^f := \begin{pmatrix} d_i^Y & f_{i-1} \\ 0 & -d_{i-1}^X \end{pmatrix} : Y_i \oplus X_{i-1} \to Y_{i-1} \oplus X_{i-2}.$$

REMARK 3.1.8. There are differing conventions for the definition of mapping cone which the above definition generally betrays. Usually, the cone of a morphism $f\colon X \to Y$ is defined to be $X[1] \oplus Y$ with differential either

$$\begin{pmatrix} -d_{i-1}^X & 0 \\ f_{i-1} & d_i^Y \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -d_{i-1}^X & 0 \\ -f_{i-1} & d_i^Y \end{pmatrix}$$

depending on a sign convention. The terminology "mapping cone" comes from topology, where the suspension of a space X is glued to Y along f, perhaps motivating (by analogy) the traditional order of the summands.

We swap the order of the summands so that the inclusion into the cone and projection from the cone agree with our established shape of a degree-wise split exact sequence

$$Y_i \xrightarrow{\binom{1}{0}} Y_i \oplus X_{i-1} \xrightarrow{(0\ 1)} X_{i-1}.$$

We feel this ordering is more natural for our discussion.

We have actually seen a mapping cone before. In the proof of Proposition 3.1.4, $\widetilde{P_X}$ is the cone of the alternating sign map $\pm 1\colon X[-1] \to \widetilde{X}$ given by

$$X[-1]_i = X_{i+1} \xrightarrow{(-1)^{i+1}} X_{i+1} = \widetilde{X}_i.$$

We use cones again in the proof of the next result.

PROPOSITION 3.1.9. *A complex is projective if and only if it is contractible if and only if it is injective, so* $\mathsf{Ch}(\mathcal{X})$ *is Frobenius.*

PROOF. First note that $\mathsf{Ch}(\mathcal{X})$ has enough projectives and enough injectives by Propositions 3.1.4 and 3.1.5. For the first biconditional, suppose C is a contractible complex and $p\colon X \twoheadrightarrow C$ a deflation. Using the projective cover $b\colon P_C \twoheadrightarrow C$ defined by

$$b_i := \begin{pmatrix} d^C_{i+1} & (-1)^i \end{pmatrix} : C_{i+1} \oplus C_i \to C_i,$$

we induce a map $q\colon P_C \to X$ so that $pq = b$. Thus, in order to find a splitting of p, it suffices to find a splitting of b. Since C is contractible, there exists a homotopy $\{h_i\colon C_i \to C_{i+1}\}_{i\in\mathbf{Z}}$ such that $d^C_{i+1}h_i + h_{i-1}d^C_i = 1$ for all i. Using this homotopy, define a splitting map $s\colon C \to P_C$ by

$$s_i := \begin{pmatrix} h_i \\ (-1)^i h_{i-1} d^C_i \end{pmatrix} : C_i \to C_{i+1} \oplus C_i.$$

Then, as $d_i^{P_C} s_i = s_{i-1} d_i^C$, we see s is a morphism of complexes, and $bs = 1$:

$$b_i s_i = d_{i+1}^C h_i + h_{i-1} d_i^C = 1$$

for all i. Lastly, qs is a splitting of p, since $pqs = bs = 1$, so C is projective.

Now assume C is projective. There is a deflation from the mapping cone of the identity on $C[-1]$ to C:

$$\begin{pmatrix} 0 & 1 \end{pmatrix} : \operatorname{cone}(id_{C[-1]}) = C[-1] \oplus C \to C.$$

Note that the differential on $\operatorname{cone}(id_{C[-1]})$, which we denote by d_i^* for ease of notation, takes the form

$$d_i^* = \begin{pmatrix} d_i^{C[-1]} & 1 \\ 0 & -d_{i-1}^{C[-1]} \end{pmatrix} = \begin{pmatrix} -d_{i+1}^C & 1 \\ 0 & d_i^C \end{pmatrix}.$$

Since C is projective, the deflation must split, giving $s \colon C \to \operatorname{cone}(id_{C[-1]})$ which must take the form

$$s_i = \begin{pmatrix} h_i \\ 1 \end{pmatrix} : C_i \to C_{i+1} \oplus C_i$$

at each degree. The splitting is a morphism of complexes, so $d_i^* s_i = s_{i-1} d_i^C$, hence

$$d_i^* s_i = \begin{pmatrix} -d_{i+1}^C & 1 \\ 0 & d_i^C \end{pmatrix} \begin{pmatrix} h_i \\ 1 \end{pmatrix} = \begin{pmatrix} -d_{i+1}^C h_i + 1 \\ d_i^C \end{pmatrix} = \begin{pmatrix} h_{i-1} d_i^C \\ d_i^C \end{pmatrix} = s_{i-1} d_i^C$$

Equality of the top entry means $h_{i-1} d_1^C + d_{i+1}^C h_i = 1$, so $\{h_i\}_{i \in \mathbf{Z}}$ is a homotopy between the identity on C and the zero map. Consequently, C is contractible.

We sketch the other biconditional: Suppose C is contractible, say with homotopy $\{h_i \colon C_i \to C_{i+1}\}_{i \in \mathbf{Z}}$. To show every inflation $j \colon C \hookrightarrow X$ splits, it suffices to show the inflation $a\colon C \hookrightarrow I_C$ given by

$$a_i := \begin{pmatrix} (-1)^{i+1} \\ d_i^C \end{pmatrix} : C_i \to C_i \oplus C_{i-1}$$

splits, since there exists $k\colon X \to I_C$ so that $kj = a$. The splitting $s\colon I_C \to C$ given by

$$s_i := \begin{pmatrix} (-1)^{i+1} d_{i+1}^C h_i & h_{i-1} \end{pmatrix} : C_i \oplus C_{i-1} \to C_i$$

does the trick. Now if C is injective, every inflation must split, so in particular inclusion into the first component

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} : C \to \mathrm{cone}(id_C) = C \oplus C[1]$$

must split. The splitting must take the form

$$s_i := \begin{pmatrix} 1 & h_{i-1} \end{pmatrix} : C_i \oplus C_{i-1} \to C_i$$

where the maps $h_{i-1}\colon C_{i-1} \to C_i$ are furnished by the splitting. Then $\{h_i\}_{i \in \mathbf{Z}}$ is the required homotopy, which can be verified as before. Therefore an object in $\mathsf{Ch}(\mathcal{X})$ is projective if and only if it is injective, hence $\mathsf{Ch}(\mathcal{X})$ is Frobenius. $\square$

Definition 3.1.10. The projective/injective stabilization of $\mathsf{Ch}(\mathcal{X})$ is called the **homotopy category of complexes** in $\mathcal{X}$, and is written $\mathsf{K}(\mathcal{X})$. Homotopy equivalent chain complexes are isomorphic in $\mathsf{K}(\mathcal{X})$, so contractible complexes—the projective/injective objects of $\mathsf{Ch}(\mathcal{X})$—are all sent to zero.

Corollary 3.1.11. *For any additive category $\mathcal{X}$, the homotopy category of complexes $\mathsf{K}(\mathcal{X})$ is triangulated.*

Proof. By Theorem 1.1.19 and Proposition 3.1.9. $\square$

Per Lemma 1.1.20, the pushout of $X \hookrightarrow I_X$ along some $f\colon X \to Y$ fits into a commutative diagram of the following form.

$$\begin{array}{ccccc} X & \hookrightarrow & I_X & \twoheadrightarrow & X[1] \\ \downarrow f & & \downarrow & & \| \\ Y & \hookrightarrow & Z & \twoheadrightarrow & X[1] \end{array}$$

Then the bottom row is degree-wise split exact by definition, so $Z_i \cong Y_i \oplus X_{i-1}$ for each i. In fact, there exists an isomorphism of complexes $Z \cong \operatorname{cone}(f)$, so that

$$Y \xrightarrow{\binom{1}{0}} Y \oplus X[1] \xrightarrow{(0\ 1)} X[1]$$

is isomorphic to $Y \hookrightarrow Z \twoheadrightarrow X[1]$. Following Theorem 1.1.19, distinguished triangles in $\mathsf{K}(\mathcal{X})$ arise from pushout diagrams like the one above, i.e.,

$$X \xrightarrow{f} Y \to Z \to X[1]$$

is distinguished. Triangles isomorphic to those excised from pushout diagrams are deemed distinguished, so the (standard) triangle

$$X \xrightarrow{f} Y \xrightarrow{\binom{1}{0}} \operatorname{cone}(f) \xrightarrow{(0\ 1)} X[1]$$

is distinguished. We may thus realize any distinguished triangle in $\mathsf{K}(\mathcal{X})$, up to a shift, as such a mapping cone sequence.

3.1.2. The subcategory $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ and equivalence. Throughout, let R be a noetherian ring, $\mathsf{mod}\,R$ the category of finitely generated right R-modules, and $\mathsf{proj}(R)$ the full subcategory of $\mathsf{mod}\,R$ consisting of projective R-modules. As $\mathsf{proj}(R)$ is an additive category, we can define the category of complexes $\mathsf{Ch}(\mathsf{proj}(R))$ which is Frobenius when equipped with the degree-wise split exact structure. Then passing to the quotient, we obtain the triangulated category $\mathsf{K}(\mathsf{proj}(R))$.

Remark 3.1.12. In an abundance of caution, we reconcile the notion of an admissible morphism with our current situation. Recall Definition 1.1.7 and the subsequent discussion, and observe that every morphism in $\mathsf{mod}\, R$ is admissible, regarding the abelian structure as an exact structure. Moreover, a sequence of composable morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is acyclic if $\operatorname{im}(f) \to B \to \operatorname{im}(g)$ is a short exact sequence (conflation), i.e., if $\operatorname{im}(f) \cong \ker(g)$. In this way, we can carry out the usual enterprise of homological algebra in an abelian category without fear of the subtlety brought on by consideration of exact structures.

Definition 3.1.13. The **homotopy category of acyclic complexes of projectives** $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R))$ is the full additive subcategory of $\mathsf{K}(\mathsf{mod}\, R)$ consisting of acyclic complexes of finitely generated projective right R-modules.

Example 3.1.14. Let k be a field and consider the hypersurface $\Lambda = k[x]/(x^n)$. Recall Example 1.1.6, where we showed that free modules are the only projective modules in $\mathsf{mod}\, \Lambda$. The most basic example of an object in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(\Lambda))$ is

$$\cdots \to \Lambda \xrightarrow{\cdot x^i} \Lambda \xrightarrow{\cdot x^{n-i}} \Lambda \xrightarrow{\cdot x^i} \Lambda \xrightarrow{\cdot x^{n-i}} \Lambda \xrightarrow{\cdot x^i} \Lambda \to \cdots$$

with $0 \leq i \leq n$, though the rest of the objects in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(\Lambda))$ are not much more complicated.

We may in fact say something stronger about $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R))$, in effect that it is a triangulated category unto itself. The triangulation on $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R))$ is inherited from $\mathsf{K}(\mathsf{proj}(R))$ in the following sense, as we shall verify shortly.

DEFINITION 3.1.15 ([12, Definition 1.5.1]). A full additive subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is a **triangulated subcategory** if it is closed under shifts, isomorphisms, and distinguished triangles in the following way: If

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle in $\mathcal{T}$ with X, Y in $\mathcal{S}$, then Z must be in $\mathcal{S}$.

REMARK 3.1.16. Before continuing, we discuss homology. Given $L \xrightarrow{f} M \xrightarrow{g} N$ with $gf = 0$, we can make sense of $\ker(g)/\operatorname{im}(f)$ since $\mathsf{mod}\, R$ is abelian. We extend this idea to **homology of complexes** in $\mathsf{mod}\, R$ by defining, for a complex X, the n^{th} homology of X as $H_n X := \ker(d_n^X)/\operatorname{im}(d_{n+1}^X)$. Homology is of particular importance when considering the homotopy category of complexes $\mathsf{K}(\mathsf{mod}\, R)$, as a distinguished triangle

$$X \to Y \to Z \to X[1]$$

gives rise to a long exact sequence

$$\cdots H_{n+1}Z \to H_n X \to H_n Y \to H_n Z \to H_{n-1} X \to \cdots$$

of homology modules—this follows from the Snake Lemma. Now, while $\mathsf{proj}(R)$ is not abelian (it lacks cokernels of monomorphisms in general), we can compute homology in $\mathsf{mod}\, R$, so we still discuss homology of complexes in $\mathsf{K}(\mathsf{proj}(R))$. Lastly, a morphism of complexes $f\colon X \to Y$ is a **quasi-isomorphism** if the induced map $f_n\colon H_n X \to H_n Y$ on homology is an isomorphism for all n. It is helpful to note that homotopy equivalence is a quasi-isomorphism.

LEMMA 3.1.17. *$\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R))$ is a triangulated subcategory of $\mathsf{K}(\mathsf{proj}(R))$.*

Proof. All we really need to check is triangle closure. Suppose $X \to Y \to Z \to X[1]$ is a distinguished triangle in $\mathsf{K}(\mathsf{proj}(R))$ with X, Y in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R))$. Taking homology, we get a long exact sequence

$$\cdots \to H_nX \to H_nY \to H_nZ \to H_{n-1}X \to \cdots$$

of modules with $H_nY \cong 0 \cong H_{n-1}X$ for all n (because X and Y are assumed acyclic). Therefore H_nZ vanishes too, so Z is acyclic as required. □

Recall from Section 1.1.2 the syzygy functor Ω. We extend the notion of syzygies to $\mathsf{Ch}(\mathsf{proj}(R))$ by defining the i^{th} **syzygy of a complex** X to be $\Omega^i X := \operatorname{coker}(d^X_{i+1})$ for $i \in \mathbf{Z}$.

Proposition 3.1.18. *Taking the i^{th} syzygy defines an additive functor $\Omega^i\colon \mathsf{Ch}(\mathsf{proj}(R)) \to \mathsf{mod}\, R$. Moreover, we find the composition*

$$\mathsf{Ch}(\mathsf{proj}(R)) \xrightarrow{\Omega^i} \mathsf{mod}\, R \to \underline{\mathsf{mod}}\, R$$

factors uniquely through $\mathsf{K}(\mathsf{proj}(R))$.

Proof. Let $f\colon P \to Q$ be a morphism in $\mathsf{Ch}(\mathsf{proj}(R))$ and let $q\colon Q_i \twoheadrightarrow \Omega^i Q$ be the projection onto the cokernel of d^Q_{i+1}. As

$$q f_i d^P_{i+1} = q d^Q_{i+1} f_{i+1} = 0,$$

there exists a unique map $\Omega^i f\colon \Omega^i P \to \Omega^i Q$ induced by factoring through the cokernel.

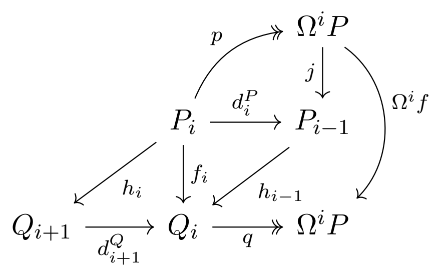
$$\begin{array}{ccccc} P_{i+1} & \xrightarrow{d^P_{i+1}} & P_i & \twoheadrightarrow & \Omega^i P \\ \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \exists ! \Omega^i f} \\ Q_{i+1} & \xrightarrow{d^Q_{i+1}} & Q_i & \overset{q}{\twoheadrightarrow} & \Omega^i Q \end{array}$$

It is straightforward to check (via the uniqueness of the induced maps) that $\Omega^i\colon \mathsf{Ch}(\mathsf{proj}(R)) \to \mathsf{mod}\, R$ is an additive functor.

To show the composition

$$\mathsf{Ch}(\mathsf{proj}(R)) \xrightarrow{\Omega^i} \mathsf{mod}\, R \to \underline{\mathsf{mod}}\, R$$

descends to an additive functor on $\mathsf{K}(\mathsf{proj}(R))$, it suffices to show null-homotopic maps in $\mathsf{Ch}(\mathsf{proj}(R))$ are sent to zero in $\underline{\mathsf{mod}}\, R$, equivalently, to maps in $\mathsf{mod}\, R$ that factor over a projective. Let $f\colon P \to Q$ be a null-homotopic morphism of complexes in $\mathsf{Ch}(\mathsf{proj}(R))$ with homotopy $\{h_i\colon P_i \to Q_{i+1}\}_{i\in\mathbf{Z}}$ such that $f_i = h_{i-1}d_i^P + d_{i+1}^Q h_i$. As before, let $q\colon Q_i \twoheadrightarrow \Omega^i Q$ and say $p\colon P_i \twoheadrightarrow \Omega^i P$ is the other epimorphism, so there exists a unique $\Omega^i f\colon \Omega^i P \to \Omega^i Q$ such that $(\Omega^i f)p = qf_i$. Additionally, there exists a map $j\colon \Omega^i P \to P_{i-1}$ (since P is a complex) such that $jp = d_i^P$.



From here, we compute

$$(\Omega^i f)p = qf_i = q(h_{i-1}d_i^P + d_{i+1}^Q h_i) = qh_{i-1}d_i^P = qh_{i-1}jp.$$

As p is an epimorphism, we conclude $\Omega^i f = qh_{i-1}j$, so $\Omega^i f$ factors over a projective—in fact two: both P_{i-1} and Q_i. Therefore the composition in question descends to an additive functor $\mathsf{K}(\mathsf{proj}(R)) \to \underline{\mathsf{mod}}\, R$. $\square$

REMARK 3.1.19. In the proof above, we conclude there exists an additive functor $\mathsf{K}(\mathsf{proj}(R)) \to \underline{\mathsf{mod}}\, R$. Upon restricting the source to $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R))$, we have a functor into $\underline{\mathsf{mod}}\, R$ which will be our primary charge henceforth. We call this new functor $\Omega^i\colon \mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R)) \to \underline{\mathsf{mod}}\, R$ as there is no opportunity for confusion.

In Chapter 2, projective co-resolutions of MCM modules were an essential tool in proofs. Recall that a projective co-resolution of an MCM module M is obtained by dualizing a projective resolution of the dual module M^*. This construction depends on MCMs being reflexive, and we should not expect that all modules possess projective co-resolutions. In fact, admitting a projective co-resolution is equivalent to being MCM, as was shown in Corollary 2.3.7. Following Remark 2.3.8, we now make formal the definition of a complete resolution.

Definition 3.1.20. A **complete resolution** of a module M is an acyclic complex P such that each P_i is projective and $\operatorname{coker}(d_1^P) \cong M$.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & P_1 & \xrightarrow{d_1^P} & P_0 & \longrightarrow & P_{-1} & \longrightarrow & P_{-2} & \longrightarrow \cdots \\ & & & & \searrow \;\; M \;\; \nearrow & & & & \end{array}$$

A module thus admits a complete resolution if and only if it admits a projective co-resolution.

Notice that a complete resolution P of a module M is a priori an acyclic complex in $\mathsf{Ch}(\mathsf{proj}(R))$, and M can be realized as the 0^{th} syzygy of P. There is a deficit to this perspective: When it exists, a complete resolution of a module need not be unique. However, just like projective resolutions, complete resolutions of the same module are homotopy equivalent. Looking instead at the homotopy category $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R))$, we can assign complete resolutions uniquely.

The cost of shifting perspective is that we can only investigate syzygy modules up to projectively stable equivalence, but this approach has been encoded in $\underline{\mathsf{mod}}\,R$ all along. Recall from Section 2.1 that taking syzygies of modules is unique up to projectively stable equivalence, and in much the same way, Proposition 3.1.18 says extracting syzygies of complexes in

$\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R))$ enjoys the same uniqueness. Beyond analogy, there is even a compatibility with the notion of syzygy from Section 2.1: If M is a module (in $\underline{\mathsf{mod}}\, R$) admitting a complete resolution P, then the 1^{st} syzygy of P is isomorphic to the syzygy of M, in symbols, $\Omega^1 P \cong \Omega M$. Higher syzygies of M correspond precisely to higher syzygies of P, that is, if we write

$$\Omega^i M := \underbrace{\Omega \cdots \Omega}_{i \text{ times}} M,$$

then $\Omega^i P \cong \Omega^i M$ $(i \geq 0)$.

Such an intimate connection between the 0^{th} syzygy module of an acyclic complex of projectives and its complete resolution encourages us to identify $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(R))$ with a subcategory of $\underline{\mathsf{mod}}\, R$, but there is hazard in being so hasty. While we can restrict the target to make Ω^0 essentially surjective, there is no indication that Ω^0 would be full or faithful. The way forward is to return to Gorenstein rings, where we can make precise the link between modules admitting complete resolutions and acyclic complexes of projectives.

Theorem 3.1.21. *Let S be a Gorenstein ring and $\underline{\mathsf{MCM}}(S)$ the projectively stable category of Maximal Cohen-Macaulay modules. The functor $\Omega^0 \colon \mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S)) \to \underline{\mathsf{MCM}}(S)$ is an equivalence of categories.*

To prove this result, we will need the Comparison Theorem.

Lemma 3.1.22 ([5, Proposition V.1.1]). *Assume the following takes place in a category with a suitable notion of acyclicity, i.e., an abelian category or more generally an exact category.*

(1) *Given the solid diagram*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \twoheadrightarrow & M \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ \cdots & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & \twoheadrightarrow & N \end{array}$$

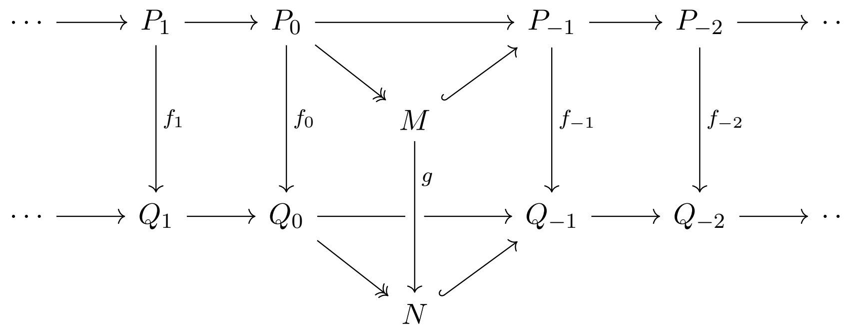
with the bottom row acyclic and the P_i projective objects, there exists a lift of f to a morphism of complexes $\{f_i\}$ that is unique up to homotopy.

(2) *Given the solid diagram*

$$\begin{array}{ccccccccc} M & \hookrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & E^2 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \\ N & \hookrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

with the top row acyclic and the I_i injective objects, there exists a lift of f to a morphism of complexes $\{f^i\}$ that is unique up to homotopy.

Proof of Theorem 3.1.21. By Corollary 2.3.7, $\Omega^0 X$ is MCM for all X in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$. What's more, the assignment of complete resolutions is functorial. Let $g\colon M \to N$ be a morphism of MCM modules. Then by Lemma 3.1.22, g lifts to a morphism between projective resolutions and a morphism between injective resolutions of M and N, both unique up to homotopy. This gives a lift f of g between complete resolutions, as injective resolutions in $\mathsf{MCM}(S)$ are projective co-resolutions by Lemma 2.4.5.



If g is the zero map, then Lemma 3.1.22 guarantees that each half of the lift is null-homotopic. We can complete the homotopies on each side to one for the entire complex by filling in the zero map for $P_{-1} \to Q_0$, so f is null-homotopic. Thus formation of complete resolutions yields a functor $\mathrm{CRes}^\sharp\colon \mathsf{MCM}(S) \to \mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$, and we conclude in particular that complete resolutions are unique up to isomorphism. Moreover, if g factors over a projective P, i.e., if the residue of g in $\underline{\mathsf{MCM}}(S)$ is 0, then $\mathrm{CRes}^\sharp g$ factors over $\mathrm{CRes}^\sharp P$, which is contractible: A complete resolution of P is given by $0 \to P = P \to 0$, which is a projective object in $\mathsf{Ch}(\mathsf{proj}(S))$ by Lemma 3.1.3, so contractible by Proposition 3.1.9. Therefore $\mathrm{CRes}^\sharp$ descends to a unique additive functor $\mathrm{CRes}\colon \underline{\mathsf{MCM}}(S) \to \mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$.

Maintaining notation, we have shown there exists a *unique* lift $f\colon P \to Q$ of $g\colon M \to N$ for any choice of complete resolutions P for M and Q for N. In particular, we can lift id_M to a map $\eta_M\colon P \to \mathrm{CRes}\, M$ and id_N to $\eta_N\colon Q \to \mathrm{CRes}\, N$. Since $id_N \cdot g = g \cdot id_M = g$, we find that

$$\eta_N f = (\mathrm{CRes}\, g)\eta_M$$

as both extend g to a map $P \to \mathrm{CRes}\, N$. In addition, η_M and η_N must be isomorphisms because complete resolutions are unique up to isomorphism. Now suppose we are given a map $f\colon P \to Q$ in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ with $\Omega^0 P =: M$

and $\Omega^0 Q =: N$, and set $g := \Omega^0 f$. We conclude from the equality above that

$$\eta_N f = (\operatorname{CRes}\Omega^0 f)\eta_M.$$

Therefore η describes a natural isomorphism from the identity on $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ to $\operatorname{CRes}\Omega^0$, proving Ω^0 is an equivalence of categories. $\square$

Example 3.1.23. If S is a semisimple ring (Example 2.2.7), then by definition $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ is just $\mathsf{K}_{\mathrm{ac}}(\mathsf{mod}\, S)$. However, there is nothing to be gained here: $\underline{\mathsf{MCM}}(S)$ is trivial, so by Theorem 3.1.21, $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ must be trivial too. More generally, if S is a Gorenstein ring of finite global dimension, then $\underline{\mathsf{MCM}}(S)$ and thus $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ are trivial; see Example 2.4.2.

As both $\underline{\mathsf{MCM}}(S)$ and $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ are triangulated categories, one may wonder whether Ω^0 is compatible with both triangulations. To state our next result, we need the following definition.

Definition 3.1.24 ([12, Definition 2.1.1]). A **triangulated functor** $F\colon \mathcal{T} \to \mathcal{D}$ is an additive functor between triangulated categories with natural isomorphisms

$$\xi_X\colon F(\Sigma X) \xrightarrow{\sim} \Sigma(FX)$$

such that for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in $\mathcal{T}$, the candidate triangle

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{\xi_X \cdot Fw} \Sigma(FX)$$

is distinguished in $\mathcal{D}$.

Proposition 3.1.25. *Ω^0 is a triangulated functor.*

PROOF. Suppose

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is a distinguished triangle in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$, so it comes from a pushout diagram

$$\begin{array}{ccc} X & \xhookrightarrow{\;i\;} & I_X \\ \downarrow{\scriptstyle \hat{u}} & & \downarrow{\scriptstyle g} \\ Y & \xhookrightarrow{\;\hat{v}\;} & Z \end{array}$$

in $\mathsf{Ch}(\mathsf{proj}(S))$. Let M be an MCM module and fix a complete resolution P for M. If we have maps $h\colon I_X \to P$ and $k\colon Y \to P$ such that $hi = k\hat{u}$, then there exists a unique map $\ell\colon Z \to P$ with $h = \ell g$ and $k = \ell\hat{v}$. By the uniqueness property of factoring through the cokernel, we get the following commutative diagram.

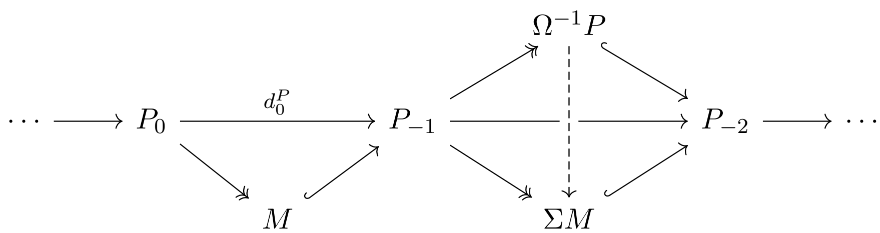
$$\begin{array}{ccccc} \operatorname{coker}(d_1^X) & \longrightarrow & \operatorname{coker}(d_1^{I_X}) & & \\ \downarrow & & \downarrow & \searrow & \\ \operatorname{coker}(d_1^Y) & \longrightarrow & \operatorname{coker}(d_1^Z) & & \\ & \searrow & & \overset{\exists !}{\searrow} & \\ & & & & M \end{array}$$

Note that both $\operatorname{coker}(d_1^X) \to \operatorname{coker}(d_1^{I_X})$ and $\operatorname{coker}(d_1^Y) \to \operatorname{coker}(d_1^Z)$ are inflations: If $A \hookrightarrow B$ is an inflation of acyclic complexes of projectives, then there is a conflation $A \hookrightarrow B \twoheadrightarrow C$, which is by definition a degree-wise split exact sequence. Thus C is an acyclic complex of projectives and $\operatorname{coker}(d_1^C)$ is MCM. It follows that $\operatorname{coker}(d_1^A) \to \operatorname{coker}(d_1^B)$ is an inflation in $\mathsf{MCM}(S)$. Therefore the diagram above is a pushout diagram. Noting $\Omega^0 A = \operatorname{coker}(d_1^A)$, we have that $\Omega^0 X \to \Omega^0 Y \to \Omega^0 Z \to \Sigma(\Omega^0 X)$ is a distinguished triangle in $\underline{\mathsf{MCM}}(S)$. If we can exhibit a natural isomorphism $\xi_X\colon \Omega^0(X[1]) \to \Sigma(\Omega^0 X)$, we will be done.

Notice first that Ω^0 is compatible with shifts:

$$\Omega^0(X[j]) = \operatorname{coker}(d_1^{X[j]}) = \operatorname{coker}((-1)^j d_{-j+1}^X) \cong \Omega^{i-j}X;$$

this isomorphism is non-canonical since, to describe the rightmost isomorphism, we have to choose to put the $-1\colon X_n \to X_n$ in either even or odd degrees when j is odd. Let M be an MCM module with complete resolution P. The idea with the natural isomorphism $\xi_P\colon \Omega^0(P[1]) \to \Sigma(\Omega^0 P)$ can be illustrated by the following diagram.



As projective co-resolutions are injective resolutions in $\mathsf{MCM}(S)$, both $\Omega^{-1}P$ and ΣM are defined as $\operatorname{coker}(d_0^P)$. Therefore there exists an isomorphism $\Omega^{-1}P \xrightarrow{\sim} \Sigma M$, which is natural by the unique factorization property of the cokernel. Then with $\Omega^0(P[1]) \cong \Omega^{-1}P$ and $M \cong \Omega^0 P$, we find $\Omega^0(P[1]) \cong \Sigma(\Omega^0 P)$. Continuing on from the last paragraph, we conclude

$$\Omega^0 X \xrightarrow{\Omega^0 u} \Omega^0 Y \xrightarrow{\Omega^0 v} \Omega^0 Z \xrightarrow{\xi_X \cdot \Omega^0 w} \Sigma(\Omega^0 X)$$

is a distinguished triangle in $\underline{\mathsf{MCM}}(S)$. $\square$

Example 3.1.26. Let k be a field. For the hypersurface $\Lambda = k[x]/(x^n)$, consider, as in Example 3.1.14, an object

$$\cdots \to \Lambda \xrightarrow{\cdot x^i} \Lambda \xrightarrow{\cdot x^{n-i}} \Lambda \xrightarrow{\cdot x^i} \Lambda \xrightarrow{\cdot x^{n-i}} \Lambda \xrightarrow{\cdot x^i} \Lambda \to \cdots$$

in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(\Lambda))$, $0 \leq i \leq n$. Compute syzygies of this object and compare with the formulas derived in Examples 1.1.11 and 1.1.14. Conclude that objects of this form are complete resolutions of indecomposable Λ-modules.

EXAMPLE 3.1.27. Let k be a field and G a finite group. The equivalence of $\underline{\mathsf{MCM}}(kG)$ and $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(kG))$ yields another way to calculate the Tate cohomology groups of G introduced in Example 2.5.3. We can now take a complete resolution of k and consider the complex $\mathrm{Hom}(\mathrm{CRes}\,k, k)$, whereby

$$\widehat{H}^n(G) \cong H^n \,\mathrm{Hom}(\mathrm{CRes}\,k, k).$$

This isomorphism is demonstrated in [3, Lemma 6.1.2.ii] by different means than Theorem 3.1.21.

3.2. The singularity category

It is at this juncture that historical motivation leans decidedly geometric. Verdier's effort to axiomatize triangulated categories (see Remark 1.1.16) was in service of describing the derived category, a notion proposed by his doctoral advisor Alexander Grothendieck. The advent of the derived category, which isolates homological data of a given abelian category, marked a technological leap in algebraic geometry. Formal construction of the derived category evaded Grothendieck, perhaps by want of time over anything else. He insisted that results—dependent on the yet formalized derived category—were within his grasp, if only the mathematics could be described to state these insights. Grothendieck, in a tradition as old as mathematics, subcontracted the problem to his student, Verdier.

Verdier localization allows us to define the derived category of $\mathsf{mod}\,S$. We describe subcategories of the derived category that restrict where complexes have nonzero homology, including the bounded derived category, where a complex has nonzero homology in only finitely many degrees. With the bounded derived category as our starting point, we localize yet again to define the singularity category $\mathsf{D}_{\mathrm{sg}}(S)$, encoding MCM approximation through the canonical inclusion of $\mathsf{mod}\, S$. By studying isomorphism classes in the singularity category, along with connections between $\mathsf{D}_{\mathrm{sg}}(S)$ and $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$, we will show that $\mathsf{D}_{\mathrm{sg}}(S)$ and $\underline{\mathsf{MCM}}(S)$ are equivalent as triangulated categories.

3.2.1. Verdier localization. Verdier localization is a procedure by which a class of morphisms is formally inverted. While not intractably complicated, the construction is technical, and a thorough treatment requires careful lemmas and lengthy argument. We present here an abridged version, so as to provide enough flavor to whet the appetite, and refer the intent reader to [12, Chapter 2] for the buffet.

We paraphrase [9, Section 1.3] for an overview of localization. The idea with localization is to invert a class $\mathcal{S}$ of morphisms in a category $\mathcal{T}$ so that the canonical localization functor $q : \mathcal{T} \to \mathcal{T}[\mathcal{S}^{-1}]$ enjoys two properties:

- qs is an isomorphism for all $s \in \mathcal{S}$.
- If $F\colon \mathcal{T} \to \mathcal{D}$ is a functor such that Fs is an isomorphism for all $s \in \mathcal{S}$, then F factors uniquely through q. Therefore q is universal with respect to this factoring property.

We impose several axioms on $\mathcal{S}$ to ensure the resulting $\mathcal{T}[\mathcal{S}^{-1}]$ is a category and that q possesses the universal factoring property. For our purposes, we concern ourselves only with localizations of triangulated categories, whereby we require $\mathcal{S}$ is compatible with the triangulation. The next definition loosely follows [9].

Definition 3.2.1. A **multiplicative system** $\mathcal{S}$ in a category $\mathcal{T}$ is a class of morphisms subject to the following constraints.

MS0 For every object X in $\mathcal{T}$, $id_X \in \mathcal{S}$.

MS1 If $s, t \in \mathcal{S}$ are composable, then $ts \in \mathcal{S}$.

MS2 Suppose $s\colon X \to Y$ is a morphism in $\mathcal{S}$. Any map $Y' \to Y$ can be completed to a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\ Y' & \longrightarrow & Y \end{array}$$

with $s' \in \mathcal{S}$, and symmetrically any map $X \to X''$ can be completed to a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X'' \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s''} \\ Y & \longrightarrow & Y'' \end{array}$$

with $s'' \in \mathcal{S}$.

MS3 For two maps $f, g\colon X \to Y$ in $\mathcal{T}$, there exists $s\colon X' \to X$ in $\mathcal{S}$ with $fs = gs$ if and only if there exists $t\colon Y \to Y'$ in $\mathcal{S}$ such that $tf = tg$.

When $\mathcal{T}$ is a triangulated category, we require that the multiplicative system be compatible with the triangulation, i.e., that $\mathcal{S}$ meets the following additional criteria.

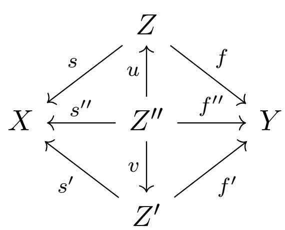
MS4 For all $n \in \mathbf{Z}$ and $s \in \mathcal{S}$, $\Sigma^n s \in \mathcal{S}$.

MS5 If (s, t, u) is a morphism of distinguished triangles in $\mathcal{T}$ with $s, t \in \mathcal{S}$, then there exists $u' \in \mathcal{S}$ making (s, t, u') a morphism of distinguished triangles.

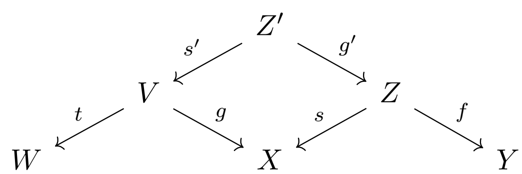
Objects in the localization $\mathcal{T}[\mathcal{S}^{-1}]$ are precisely those in $\mathcal{T}$. A morphism $X \to Y$ in $\mathcal{T}[\mathcal{S}^{-1}]$ is an equivalence class of diagrams of the form

$$X \xleftarrow{\;s\;} Z \xrightarrow{\;f\;} Y$$

with $s \in \mathcal{S}$ and $f \in \mathrm{Hom}_{\mathcal{T}}(Z, Y)$, which we will abbreviate as $(Z, s, f)\colon X \to Y$. These diagrams should be thought of as representing fs^{-1} and are often called **fractions** accordingly. Morphisms in $\mathcal{T}[\mathcal{S}^{-1}]$ are subject to an equivalence relation: Diagrams (Z, s, f) and (Z', s', f') are equivalent if there exists a diagram $(Z'', s'', f'')\colon X \to Y$ and morphisms $u\colon Z'' \to Z$ and $v\colon Z'' \to Z'$ in $\mathcal{T}$ such that the following diagram commutes.



Composition of (equivalence classes of) morphisms $(V, t, g)\colon W \to X$ and $(Z, s, f)\colon X \to Y$ is accomplished by the following commutative diagram.



By MS2, the commutative rhombus at center exists and $s' \in \mathcal{S}$. Then $ts' \in \mathcal{S}$ by MS1, hence the composition is given by $(Z', ts', fg')\colon W \to Y$. The localization functor $q\colon \mathcal{T} \to \mathcal{T}[\mathcal{S}^{-1}]$ sends a morphism $f\colon X \to Y$ to the triple (X, id_X, f).

Let $\mathcal{T}$ be a triangulated category and $\mathcal{U}$ a triangulated subcategory. Define $\mathcal{S}(\mathcal{U})$ to be the class of maps $f\colon X \to Y$ in $\mathcal{T}$ such that there exists a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to \Sigma X$$

with $Z \in \mathcal{U}$. By [14, Proposition II.2.1.8], $\mathcal{S}(\mathcal{U})$ is a multiplicative system compatible with the triangulation.

DEFINITION 3.2.2. The **Verdier quotient** of $\mathcal{T}$ by $\mathcal{U}$ is the localization

$$\mathcal{T}/\mathcal{U} := \mathcal{T}[\mathcal{S}(\mathcal{U})^{-1}].$$

As always, there is a canonical functor $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{U}$. The formation of a Verdier quotient is a special case of localization, so it is often attributively referred to as **Verdier localization**.

PROPOSITION 3.2.3 ([14, Théorème II.2.2.6]). *There exists a unique triangulated structure on the Verdier quotient $\mathcal{T}/\mathcal{U}$ such that q is a triangulated functor. Distinguished triangles in $\mathcal{T}/\mathcal{U}$ are isomorphic to images via q of distinguished triangles in $\mathcal{T}$. If $F\colon \mathcal{T} \to \mathcal{D}$ is a triangulated functor with the property that Fs is an isomorphism for all $s \in \mathcal{S}(\mathcal{U})$, then F factors uniquely through q.*

There is a sense in which the Verdier quotient $\mathcal{T}/\mathcal{U}$ can be thought of as modding out by $\mathcal{U}$. Let

$$X \xrightarrow{f} Y \to Z \to \Sigma X$$

be a distinguished triangle in $\mathcal{T}$ with $Z \in \mathcal{U}$. Localizing, we find that

$$qX \xrightarrow{qf} qY \to qZ \to \Sigma(qX)$$

is a distinguished triangle in $\mathcal{T}/\mathcal{U}$ and qf is invertible. Axioms TR1 and TR3 guarantee that we can construct a morphism of distinguished triangles

$$\begin{array}{ccccccc}
qX & \xrightarrow{qf} & qY & \longrightarrow & qZ & \longrightarrow & \Sigma(qX) \\
\| & & \downarrow{\scriptstyle (qf)^{-1}} & & \downarrow{\scriptstyle w} & & \| \\
qX & = & qX & \longrightarrow & 0 & \longrightarrow & \Sigma(qX)
\end{array}$$

where w is an isomorphism by Lemma 1.1.18 (Triangulated 5 Lemma). Therefore q sends objects in $\mathcal{U}$ to zero. We can in fact characterize the kernel of q.

DEFINITION 3.2.4. The **kernel** of a triangulated functor $F\colon \mathcal{T} \to \mathcal{D}$ is the full additive subcategory $\ker F$ of $\mathcal{T}$ consisting of objects sent to zero by F, i.e., $X \in \ker F$ if and only if $FX \cong 0$.

It is clear that $\ker F$ is isomorphism closed. As F is a triangulated functor,

$$F(\Sigma X) \cong \Sigma(FX) \cong 0$$

whenever $X \in \ker q$, so the kernel is closed under shifts. Lastly, suppose $X \to Y \to Z \to \Sigma X$ is a distinguished triangle in $\mathcal{T}$ with $X, Y \in \ker F$. By TR1, $0 \to 0 \to 0 \to 0$ is a distinguished triangle in $\mathcal{D}$, and TR3 says there exists a morphism of distinguished triangles

$$\begin{array}{ccccccc} FX & \longrightarrow & FY & \longrightarrow & FZ & \longrightarrow & \Sigma(FX) \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle w} & & \big\downarrow{\scriptstyle \Sigma u} \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

in $\mathcal{D}$. But u and v are isomorphisms by assumption, so Lemma 1.1.18 (Triangulated 5 Lemma) implies w is an isomorphism, hence $Z \in \ker F$. Therefore $\ker F$ is a triangulated subcategory of $\mathcal{T}$. Notice moreover that if $X \oplus X' \in \ker F$, then $X \in \ker F$:

$$0 \cong F(X \oplus X') \cong FX \oplus FX',$$

so $FX \cong 0$. We conclude that the kernel of a triangulated functor is closed under direct summands.

DEFINITION 3.2.5. A triangulated subcategory is called **thick** if it is closed under direct summands.

Remark 3.2.6. We remark that thick subcategories have utility beyond kernels. Let k be a field, and recall from Example 2.5.3 the cohomology ring $H^\star(G)$ for a finite group G. Understanding one invariant of $H^\star(G)$, namely the associated projective variety, amounts to classifying thick subcategories of the triangulated category $\underline{\mathsf{mod}}\, kG$. In fact, there is a more general program that studies topological invariants of algebraic objects by classifying thick subcategories of an associated triangulated category.

Returning to the localization $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{U}$, we have that $\mathcal{U}$ is certainly contained in $\ker q$. To characterize the latter, note that $\ker q$ is the smallest thick triangulated subcategory containing $\mathcal{U}$. We can in fact construct $\ker q$ as the full additive subcategory containing all direct summands in $\mathcal{T}$ of objects in $\mathcal{U}$.

To see Verdier localization in action, we close this section with an example. Let $\mathcal{A}$ be an abelian category. Perhaps *the* canonical example of Verdier localization is the derived category $\mathsf{D}(\mathcal{A})$. Recall homology of complexes and quasi-isomorphisms from Remark 3.1.16 and notice that it immediately generalizes from $\mathsf{mod}\, R$ to $\mathcal{A}$. The derived category of $\mathcal{A}$ consists of complexes of objects in $\mathcal{A}$ with the property that all quasi-isomorphisms are invertible. Start with the homotopy category of complexes $\mathsf{K}(\mathcal{A})$. Let $\mathsf{K}_{\mathrm{ac}}(\mathcal{A})$ be the full additive subcategory of $\mathsf{K}(\mathcal{A})$ comprised of acyclic complexes. It is quick to see that $\mathsf{K}_{\mathrm{ac}}(\mathcal{A})$ is a triangulated subcategory, as the proof of Lemma 3.1.17 generalizes immediately. Furthermore we remark that $\mathsf{K}_{\mathrm{ac}}(\mathcal{A})$ is thick. Define the **derived category** as the Verdier quotient

$$\mathsf{D}(\mathcal{A}) := \mathsf{K}(\mathcal{A}) / \mathsf{K}_{\mathrm{ac}}(\mathcal{A}).$$

Note that a morphism $f\colon X \to Y$ in $\mathsf{K}(\mathcal{A})$ is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is acyclic, so quasi-isomorphisms are invertible in $\mathsf{D}(\mathcal{A})$, and acyclic complexes are sent to zero. We will build on the derived category in the next section to introduce the third actor in our analysis.

3.2.2. The quotient $\mathsf{D}_{\mathrm{sg}}(S)$ and equivalences. Let $\mathcal{A}$ be an abelian category. So far in Chapter 3, we have considered only unbounded chain complexes, but now that we have defined $\mathsf{Ch}(\mathcal{A})$, $\mathsf{K}(\mathcal{A})$, and $\mathsf{D}(\mathcal{A})$, we mention boundedness conditions for all three. Define the category of chain complexes in $\mathcal{A}$ with bounded above homology

$$\mathsf{Ch}^{-}(\mathcal{A}) := \{X \in \mathsf{Ch}(\mathcal{A}) : H_n X = 0 \text{ for all } n \ll 0\},$$

the category of chain complexes with bounded below homology

$$\mathsf{Ch}^{+}(\mathcal{A}) := \{X \in \mathsf{Ch}(\mathcal{A}) : H_n X = 0 \text{ for all } n \gg 0\},$$

and the category of chain complexes with bounded homology

$$\mathsf{Ch}^{\mathrm{b}}(\mathcal{A}) := \{X \in \mathsf{Ch}(\mathcal{A}) : H_n X = 0 \text{ for all } |n| \gg 0\}.$$

As with $\mathsf{Ch}(\mathcal{A})$, we can stabilize $\mathsf{Ch}^{\star}(\mathcal{A})$ for $\star \in \{\mathrm{b}, -, +\}$ with respect to contractible complexes, giving the homotopy category $\mathsf{K}^{\star}(\mathcal{A})$ of complexes with bounded (above/below) homology. Then taking the Verdier quotient of $\mathsf{K}^{\star}(\mathcal{A})$ by the triangulated subcategory of acyclic complexes, we get the bounded (above/below) derived category $\mathsf{D}^{\star}(\mathcal{A})$. By [14, Théorème III.1.2.3], we can think of $\mathsf{K}^{\star}(\mathcal{A})$ and $\mathsf{D}^{\star}(\mathcal{A})$ as triangulated subcategories of $\mathsf{K}(\mathcal{A})$ and $\mathsf{D}(\mathcal{A})$ respectively, as we may induce full embeddings from the inclusion functor $\mathsf{Ch}^{\star}(\mathcal{A}) \to \mathsf{Ch}(\mathcal{A})$.

Let S be a Gorenstein ring and write $\mathsf{D}(S)$ for $\mathsf{D}(\mathsf{Mod}\, S)$. Additionally, write $\mathsf{D}^{\mathrm{b}}(S)$ for $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, S)$, the bounded derived category of complexes of finitely generated S-modules. A **perfect complex** is a complex X that is isomorphic in $\mathsf{D}(S)$ to a finite complex of finitely generated projective

S-modules; by finite complex we mean that $X_i \neq 0$ for only finitely many i. Clearly if X is perfect, then X has bounded homology, whereby $X \in \mathsf{D}^{\mathrm{b}}(S)$, and we denote the full additive subcategory of perfect complexes by $\mathsf{perf}(S)$. Suppose $\mathcal{T}$ is another triangulated category and $F\colon \mathsf{D}^{\mathrm{b}}(S) \to \mathcal{T}$ is a triangulated functor such that $FS \cong 0$. The kernel of any such functor is easily characterized.

LEMMA 3.2.7. *For any triangulated functor $F\colon \mathsf{D}^{\mathrm{b}}(S) \to \mathcal{T}$ such that $FS \cong 0$, $\ker F = \mathsf{perf}(S)$.*

PROOF. Suppose that X is a perfect complex, so without loss of generality we can write X as a finite complex of finitely generated projective modules concentrated in nonnegative degrees:

$$0 \to X_i \to \cdots \to X_0 \to 0.$$

For each $0 \leq j \leq i$, there exists a module Y_j and an integer n_j such that $X_j \oplus Y_j \cong S^{n_j}$, and the complex Y given by

$$0 \to Y_i \xrightarrow{0} Y_{i-1} \xrightarrow{0} \cdots \xrightarrow{0} Y_1 \xrightarrow{0} Y_0 \to 0$$

must be perfect. Then $F(X_j \oplus Y_j) \cong F(S^{n_j}) \cong 0$ for each j since F is additive, and we must have $0 \cong F(X \oplus Y) \cong FX \oplus FY$. Hence $X \in \ker F$, and it follows that $\mathsf{perf}(S) \subset \ker F$.

Note that $\mathsf{perf}(S)$ is a triangulated subcategory: The only thing we possibly need to check is triangle closure, but this is immediate upon applying the triangulated functor F. We claim $\mathsf{perf}(S)$ is thick. For this, it suffices to notice that a summand of a perfect complex is (isomorphic to) a summand of a finite complex of projectives. As summands of projectives are themselves projective, we conclude $\mathsf{perf}(S)$ is closed under summands. Therefore $\mathsf{perf}(S)$ is a thick subcategory containing S and contained in $\ker F$, but $\ker F$ is the smallest thick subcategory containing S, so we must have $\mathsf{perf}(S) = \ker F$. □

It follows in turn that every such functor F factors through the *same* Verdier quotient, a somewhat remarkable result considering our assumptions on F are minimal. This Verdier quotient is deserving of examination in its own right, comprising the remainder of our efforts.

DEFINITION 3.2.8. The **singularity category** is the Verdier quotient

$$\mathsf{D}_{\mathrm{sg}}(S) := \mathsf{D}^{\mathrm{b}}(S)/\,\mathsf{perf}(S).$$

There is a unique triangulated functor $q\colon \mathsf{D}^{\mathrm{b}}(S) \to \mathsf{D}_{\mathrm{sg}}(S)$ with the property that any triangulated functor $\mathsf{D}^{\mathrm{b}}(S) \to \mathcal{T}$ sending S to 0 factors uniquely through q.

REMARK 3.2.9. An object C in a triangulated category $\mathcal{T}$ is called **compact** if the functor $\mathrm{Hom}_{\mathcal{T}}(C, -)$ preserves all set-indexed coproducts that exists in $\mathcal{T}$. It is interesting to remark that the compact objects in $\mathsf{D}(S)$ are in fact the perfect complexes. As $\mathsf{perf}(S) \subset \mathsf{D}^{\mathrm{b}}(S)$, the singularity category may alternatively be described as the Verdier quotient of $\mathsf{D}^{\mathrm{b}}(S)$ with respect to the compact objects in $\mathsf{D}(S)$.

Yet another way to characterize $\mathsf{perf}(S)$ is the isomorphism closed full subcategory generated by shifts of projective S-modules. This essentially follows from the fact that $\mathsf{perf}(S)$ is the smallest thick subcategory containing S, i.e., the thick subcategory generated by S. Closing under summands of finite sums of S gives all finitely generated projective modules, and taking mapping cones of shifts of projectives gives any perfect complex, up to isomorphism.

We can regard any module as a complex concentrated in degree 0, which we may formalize as a composition of functors

$$\mathsf{mod}\, S \to \mathsf{Ch}^{\mathrm{b}}(\mathsf{mod}\, S) \to \mathsf{K}^{\mathrm{b}}(\mathsf{mod}\, S) \to \mathsf{D}^{\mathrm{b}}(S) \to \mathsf{D}_{\mathrm{sg}}(S).$$

Any projective module is clearly sent to 0 in the composition, so any map in $\mathsf{mod}\, S$ that factors over a projective is consequently sent to the zero map. We conclude the following.

LEMMA 3.2.10 ([3, Lemma 2.2.2]). *The functor* $\mathsf{mod}\, S \to \mathsf{D}_{\mathrm{sg}}(S)$ *factors uniquely through the canonical reduction functor* $\mathsf{mod}\, S \to \underline{\mathsf{mod}}\, S$*, giving a functor* $\iota\colon \underline{\mathsf{mod}}\, S \to \mathsf{D}_{\mathrm{sg}}(S)$*. Moreover,* ι *takes the syzygy functor* Ω *to the inverse of the translation functor.*

PROOF. The first assertion is demonstrated in the paragraph before the lemma. For the second, remark that the functor $\mathsf{Ch}(\mathsf{mod}\, S) \to \mathsf{D}(S)$ is a δ-functor, in the sense of [14, Section III.1.3], so any short exact sequence of complexes in $\mathsf{Ch}(\mathsf{mod}\, S)$ gets sent to a distinguished triangle in $\mathsf{D}(S)$. Start with a morphism $f\colon M \to N$ in $\underline{\mathsf{mod}}\, S$. Extend f to a commutative diagram (in $\mathsf{mod}\, S$)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega M & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \Omega f} & & \big\downarrow & & \big\downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \Omega N & \longrightarrow & Q & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with short exact rows, choosing lifts $\Omega M, \Omega N$ appropriately. Viewing each module as a complex concentrated in degree 0, we can think of the above commutative diagram as one in $\mathsf{Ch}(\mathsf{mod}\, S)$, and we get a morphism of distinguished triangles in $\mathsf{D}(S)$.

$$\begin{array}{ccccccc} \iota\Omega M & \longrightarrow & \iota P & \longrightarrow & \iota M & \xrightarrow{\delta} & (\iota\Omega M)[1] \\ \big\downarrow{\scriptstyle \iota\Omega f} & & \big\downarrow & & \big\downarrow{\scriptstyle \iota f} & & \big\downarrow{\scriptstyle (\iota\Omega f)[1]} \\ \iota\Omega N & \longrightarrow & \iota Q & \longrightarrow & \iota N & \xrightarrow{\delta'} & (\iota\Omega N)[1] \end{array}$$

As ιP and ιQ are perfect, δ and δ' are isomorphisms, with naturality following from commutativity of the rightmost square. $\square$

Consider again the inclusion of $\mathsf{mod}\, S$ into $\mathsf{D}^{\mathrm{b}}(S)$, and note that fpd modules are sent to perfect complexes. Upon localizing, we find that every fpd module is isomorphic to 0 in $\mathsf{D}_{\mathrm{sg}}(S)$, so $\iota\colon \underline{\mathsf{mod}}\, S \to \mathsf{D}_{\mathrm{sg}}(S)$ cuts out the torsion-free class in $\underline{\mathsf{mod}}\, S$. Recall from Proposition 2.6.2 and the discussion following that the inclusion $\mathsf{i}\colon \underline{\mathsf{MCM}}(S) \to \underline{\mathsf{mod}}\, S$ admits a right adjoint c given by MCM-approximation. With $\iota(\underline{\mathsf{fpd}}(S))$ vanishing, we can draw conclusions about the interaction of ι and MCM-approximation, so that up to natural isomorphism, ι is just a functor from $\underline{\mathsf{MCM}}(S)$.

Proposition 3.2.11. *The counit of the adjunction* $\mathsf{i} \dashv \mathsf{c}$ *induces a natural isomorphism of functors* $\iota\varepsilon\colon \iota\mathsf{ic} \Rightarrow \iota$. *The functor* $\iota\mathsf{i}\colon \underline{\mathsf{MCM}}(S) \to \mathsf{D}_{\mathrm{sg}}(S)$ *is triangulated.*

Proof. The second claim is a corollary to Lemma 3.2.10, so we need only demonstrate the first. Let N be a finitely generated S-module. As $(\mathsf{MCM}(S), \mathsf{fpd}(S))$ is a complete cotorsion pair in $\mathsf{mod}\, S$ (Proposition 2.6.1), we may exhibit a short exact sequence

$$0 \to U \to M \to N \to 0$$

with M a MCM module and U a fpd module. The inclusion of $\mathsf{mod}\, S$ into $\mathsf{D}_{\mathrm{sg}}(S)$ is a δ-functor, thus the above short exact sequence is sent to a distinguished triangle

$$U \to M \to N \to U[1].$$

Noting that U is perfect, we conclude the images of M and N are isomorphic in $\mathsf{D}_{\mathrm{sg}}(S)$. What's more, $M \cong \mathsf{c}N$ in $\underline{\mathsf{MCM}}(S)$—recall from the discussion following Proposition 2.6.2 that M is a MCM-approximation for N. To proceed, let $f\colon N \to L$ be a map in $\underline{\mathsf{mod}}\, S$. The counit ε of the adjunction $\mathsf{i} \dashv \mathsf{c}$ induces a commutative diagram

$$\begin{array}{ccc} \mathsf{ic}N & \xrightarrow{\mathsf{ic}f} & \mathsf{ic}L \\ \big\downarrow{\scriptstyle \varepsilon_N} & & \big\downarrow{\scriptstyle \varepsilon_L} \\ N & \xrightarrow{\quad f \quad} & L \end{array}$$

in $\underline{\mathsf{mod}}\, S$, which upon applying ι yields a commutative diagram

$$\begin{array}{ccc} \iota\mathsf{ic}N & \xrightarrow{\iota\mathsf{ic}f} & \iota\mathsf{ic}L \\ \big\downarrow{\scriptstyle \iota\varepsilon_N} & & \big\downarrow{\scriptstyle \iota\varepsilon_L} \\ \iota N & \xrightarrow{\quad \iota f \quad} & \iota L \end{array}$$

in $\mathsf{D}_{\mathrm{sg}}(S)$. As was just shown, the image (via ι) of any module N is isomorphic in $\mathsf{D}_{\mathrm{sg}}(S)$ to the image of its MCM-approximation $\mathsf{c}N$, so $\iota\varepsilon_N$ is an isomorphism for all N. Therefore $\iota\varepsilon$ is a natural isomorphism. $\square$

This natural isomorphism of functors hints at a deeper connection between maximal Cohen-Macaulay modules and objects in the singularity category, in the sense that including any module into $\mathsf{D}_{\mathrm{sg}}(S)$ automatically encodes its MCM approximation. Strengthening this connection further, we now show that *every* object in the singularity category arises as the image of an MCM module, up to isomorphism. For ease of notation and in light of the last proposition, we write simply ι for $\iota\mathsf{i}$.

PROPOSITION 3.2.12. *The triangulated functor* $\iota\colon \underline{\mathsf{MCM}}(S) \to \mathsf{D}_{\mathrm{sg}}(S)$ *is essentially surjective.*

PROOF. Start by taking a projective resolution of a complex X in $\mathsf{D}_{\mathrm{sg}}(S)$, that is, take P to be a bounded above (vanishing in degrees $i \ll 0$) complex of finitely generated projective S-modules such that P is quasi-isomorphic to X. Without loss of generality assume that $P_i = 0$ for all $i < 0$. Since X has bounded homology, there exists an integer m such that $H_m X \neq 0$ and

$H_nX = 0$ for all $n > m$. Let $r := m + \mathrm{idim}(S) + 1$, and break up P into a short exact sequence of complexes in $\mathsf{Ch}^{\mathrm{b}}(\mathsf{mod}\, S)$.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & P_{r+1} & \longrightarrow & P_{r+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & P_r & \longrightarrow & P_r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{r-1} & \longrightarrow & P_{r-1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

We write $0 \to A \to P \to B \to 0$ for the above short exact sequence of complexes, and note that P is quasi-isomorphic to X. Then the δ-functor $\mathsf{Ch}^{\mathrm{b}}(\mathsf{mod}\, S) \to \mathsf{D}_{\mathrm{sg}}(S)$ sends the above short exact sequence to a distinguished triangle $A \to P \to B \to A[1]$ with the outer terms—perfect complexes—vanishing. Therefore X is isomorphic in $\mathsf{D}_{\mathrm{sg}}(S)$ to the truncated complex B. As B has homology only in degree r, we can further post-compose with the quasi-isomorphism of complexes

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_{r+1} & \xrightarrow{d^P_{r+1}} & P_r & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{coker}(d^P_{r+1}) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

to conclude X is isomorphic to the complex $\mathrm{coker}(d^P_{r+1})[r]$.

Let $M := \mathrm{coker}(d^P_{r+1})$. Then $P[-r]$ is almost a complete resolution of M, the deficit being the nonzero homology off to the right, but this is no issue—enough of the complex to the right is acyclic to show that M is MCM.

Consider taking cosyzygies of M, and let $j := \operatorname{idim}(S)$.

$$\begin{array}{ccccccccccc} \cdots \longrightarrow P_r & & \longrightarrow & & P_{r-1} & & \longrightarrow & & P_{r-2} \longrightarrow \cdots \longrightarrow P_{m+1} & & \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ & & M & & & & \Sigma M & & & & \Sigma^j M \end{array}$$

By dimension shifting (Lemma 2.3.1), we find that M is MCM:

$$\operatorname{Ext}^i(M, S) \cong \operatorname{Ext}^{i+j}(\Sigma^j M, S) = 0$$

for all $i > 0$. Therefore $\Sigma^r M$ exists (by taking a complete resolution of M) and is MCM, so $\iota(\Sigma^r M) \cong (\iota M)[r] \cong X$ since ι is triangulated, completing the proof. $\square$

Breaking up a complex into a short exact sequence with perfect kernel is a handy trick—one that we employ again soon. Of particular utility was the cokernel of the short exact sequence (see proof above), a truncated complex, which was enough, in $\mathsf{D}_{\mathrm{sg}}(S)$, to recover the complex with which we started. In the ways they simplify our investigation, these truncated complexes are the key to better understanding the singularity category and its connection to maximal Cohen-Macaulay modules. Toward that end, recall naïve right truncation of a complex: For a complex A and an integer k, the **naïve right truncation** $\sigma_{\geq k} A$ is the complex with $(\sigma_{\geq k} A)_i = A_i$ if $i \geq k$ and 0 otherwise, i.e.,

$$\cdots \to A_{k+2} \to A_{k+1} \to A_k \to 0.$$

There is a canonical surjective map $\pi_k^j \colon \sigma_{\geq j} A \to \sigma_{\geq k} A$ for all $j \leq k$ given by 0 in degrees less than k and identity elsewhere. It is immediate that naïve truncation defines an endofunctor on $\mathsf{K}(\operatorname{\mathsf{mod}} S)$ and that

$$\sigma_{\geq k}(A[i]) = (\sigma_{\geq k-i} A)[i] \tag{3.2.1}$$

for all $i, k \in \mathbf{Z}$.

To complete our description of ι, we appeal to complete resolutions of MCM modules; our approach here roughly follows [3]. Notice that if P is a complex in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$, then $\sigma_{\geq k}P$ is a projective resolution of $\Omega^k P$. Put another way, the complexes $\sigma_{\geq k}P$ and $\iota(\Omega^k P)[k]$ are isomorphic in $\mathsf{D}^{\mathrm{b}}(S)$.

At the cost of slightly more notation, we introduce the homotopy category of bounded above complexes of projective modules with bounded homology

$$\mathsf{K}^{-,b}(\mathsf{proj}(S)) := \{P \in \mathsf{K}(\mathsf{proj}(S)) : P_i = 0 \text{ for } i \ll 0,\, H_n P = 0 \text{ for } |n| \gg 0\}.$$

It is well known that $\mathsf{K}^{-,b}(\mathsf{proj}(S))$ is equivalent to $\mathsf{D}^{\mathrm{b}}(S)$—the functor into $\mathsf{D}^{\mathrm{b}}(S)$ is just inclusion and the quasi-inverse is given by taking a projective resolution. Now for each $k \in \mathbf{Z}$, we can regard truncation as an additive functor from $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ to $\mathsf{D}^{\mathrm{b}}(S)$:

$$\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S)) \xrightarrow{\sigma_{\geq k}} \mathsf{K}^{-,b}(\mathsf{proj}(S)) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(S);$$

we maintain the notation $\sigma_{\geq k}$ for this composition. To check that this assignment is functorial, suppose $f\colon P \to Q$ in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ is null-homotopic. Then $\Omega^k f$ factors over a projective, i.e., $\iota\Omega^k f$ factors over a perfect complex, and the commutative diagram

$$\begin{array}{ccc} \sigma_{\geq k}P & \xrightarrow{\sim} & \iota(\Omega^k P)[k] \\ \downarrow{\scriptstyle \sigma_{\geq k} f} & & \downarrow{\scriptstyle \iota(\Omega^k f)[k]} \\ \sigma_{\geq k}Q & \xrightarrow{\sim} & \iota(\Omega^k Q)[k] \end{array} \tag{3.2.2}$$

in $\mathsf{D}^{\mathrm{b}}(S)$ assures $\sigma_{\geq k}f$ vanishes. Notice moreover that $\sigma_{\geq k}$ is fully faithful for all $k \in \mathbf{Z}$: This follows essentially from Lemma 3.1.22 and the subsequent description of maps lifted from $\underline{\mathsf{MCM}}(S)$ to $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$. To show $\sigma_{\geq k}$ is fully faithful, if we are given $f\colon \sigma_{\geq k}P \to \sigma_{\geq k}Q$, then there exists $\bar{f}\colon \Omega^k P \to \Omega^k Q$ that is unique in $\underline{\mathsf{MCM}}(S)$, so functoriality of CRes implies there is a unique lift $\operatorname{CRes} \bar{f}$ such that $\sigma_{\geq k} \operatorname{CRes} \bar{f} = f$. Thus $\sigma_{\geq k}$ is a full embedding, or put another way, $\sigma_{\geq k} \mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ is a full subcategory of $\mathsf{D}^{\mathrm{b}}(S)$.

Example 3.2.13. Consider again the case of the hypersurface $\Lambda = k[x]/(x^n)$ for k a field. In Example 3.1.26 we classified complete resolutions of indecomposable Λ-modules. Let M be an indecomposable Λ-module, i.e., $M \cong k[x]/(x^i)$ for $1 \leq i \leq n$ or $M \cong 0$. Show for any integer m that the $2m^{th}$ naïve right truncation $\sigma_{\geq 2m} \operatorname{CRes} M$ of a complete resolution of M is isomorphic to M in $\mathsf{D}^{\mathrm{b}}(\Lambda)$ up to a shift.

Passing to the singularity category via the localization $q\colon \mathsf{D}^{\mathrm{b}}(S) \to \mathsf{D}_{\mathrm{sg}}(S)$, we have that the complexes $\sigma_{\geq j} P$ and $\sigma_{\geq k} P$ are isomorphic in $\mathsf{D}_{\mathrm{sg}}(S)$ for integers $j \leq k$: The kernel of the natural map $\pi^j_k\colon \sigma_{\geq j} P \to \sigma_{\geq k} P$ is a finite complex of projectives, so the short exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & P_{k+1} & \longrightarrow & P_{k+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & P_{k} & \longrightarrow & P_{k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{k-1} & \longrightarrow & P_{k-1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{j+1} & \longrightarrow & P_{j+1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{j} & \longrightarrow & P_{j} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is sent to a distinguished triangle

$$\ker \pi^j_k \to q\sigma_{\geq j} P \to q\sigma_{\geq k} P \to (\ker \pi^j_k)[1]$$

in $\mathsf{D}_{\mathrm{sg}}(S)$, so $q\sigma_{\geq j}P \cong q\sigma_{\geq k}P$ in $\mathsf{D}_{\mathrm{sg}}(S)$ since $\ker \pi_k^j$ is perfect. Moreover, the map $f\colon P \to Q$ yields a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker & \longrightarrow & \sigma_{\geq j}P & \longrightarrow & \sigma_{\geq k}P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \sigma_{\geq j}f} & & \downarrow{\scriptstyle \sigma_{\geq k}f} & & \\
0 & \longrightarrow & \ker' & \longrightarrow & \sigma_{\geq j}Q & \longrightarrow & \sigma_{\geq k}Q & \longrightarrow & 0
\end{array}$$

with short exact rows, which is sent to a morphism of distinguished triangles

$$\begin{array}{ccccccc}
\ker & \longrightarrow & q\sigma_{\geq j}P & \xrightarrow{\sim} & q\sigma_{\geq k}P & \longrightarrow & \ker[1] \\
\downarrow & & \downarrow{\scriptstyle q\sigma_{\geq j}f} & & \downarrow{\scriptstyle q\sigma_{\geq k}f} & & \downarrow \\
\ker' & \longrightarrow & q\sigma_{\geq j}Q & \xrightarrow{\sim} & q\sigma_{\geq k}Q & \longrightarrow & \ker'[1]
\end{array}$$

in $\mathsf{D}_{\mathrm{sg}}(S)$, whereby naturality of the isomorphism follows from the commutativity of the middle square. In summary, we have the following.

PROPOSITION 3.2.14. *The collection* $\{q\sigma_{\geq k} : k \in \mathbf{Z}\}$ *is a directed system of functors from* $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$ *to* $\mathsf{D}_{\mathrm{sg}}(S)$ *with natural isomorphisms for transition maps. For each* $k \in \mathbf{Z}$*, there is a natural isomorphism of functors* $q\sigma_{\geq k}\,\mathrm{CRes} \cong \iota$ *from* $\underline{\mathsf{MCM}}(S)$ *to* $\mathsf{D}_{\mathrm{sg}}(S)$*. The limit*

$$\sigma_{\geq} := \varprojlim_{k \in \mathbf{Z}} q\sigma_{\geq k}$$

is a triangulated functor with $\sigma_{\geq}\,\mathrm{CRes} \cong \iota$*.*

PROOF. The first claim is immediate from the preceding discussion and the second follows from Theorem 3.1.21 and (3.2.2). For the last claim, we note first that the limit exists, since $q\sigma_{\geq k}$ satisfies the universal property for any $k \in \mathbf{Z}$. In particular, $\sigma_{\geq}$ is naturally isomorphic to $q\sigma_{\geq k}$ for all $k \in \mathbf{Z}$. Then (3.2.1) implies that $\sigma_{\geq}$ is triangulated and the second claim implies $\sigma_{\geq}\,\mathrm{CRes} \cong \iota$. $\square$

THEOREM 3.2.15. *The functor* $\iota\colon \underline{\mathsf{MCM}}(S) \to \mathsf{D}_{\mathrm{sg}}(S)$ *is an equivalence of triangulated categories.*

PROOF. By Propositions 3.2.11, 3.2.12, and 3.2.14, all we need to show is that $\sigma_{\geq}$ is fully faithful. Considering the natural isomorphism of functors $\sigma_{\geq} \cong q\sigma_{\geq k}$ for all $k \in \mathbf{Z}$, it suffices to find an integer k such that $q\sigma_{\geq k}$ is fully faithful. Moreover, as $\sigma_{\geq k}$ is fully faithful for all $k \in \mathbf{Z}$, it only remains to show that the localization q is fully faithful when restricted to the full subcategory $\sigma_{\geq k}\,\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$. We take an approach similar to [3, Proof of Theorem 4.4.1], whereby we first reduce the problem: By [14, Proposition II.2.3.3], it suffices to show, for any perfect complex X and any complex Q in $\mathsf{K}_{\mathrm{ac}}(\mathsf{proj}(S))$, that there exists an integer k such that

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(S)}(X, \sigma_{\geq k}Q) = 0.$$

Since both X and $\sigma_{\geq k}Q$ are bounded above complexes of projectives, the equivalence $\mathsf{K}^{-,b}(\mathsf{proj}(S)) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(S)$ implies

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(S)}(X, \sigma_{\geq k}Q) \cong \mathrm{Hom}_{\mathsf{K}^{-,b}(\mathsf{proj}(S))}(X, \sigma_{\geq k}Q).$$

By Remark 3.2.9, we can assume $X = P[i]$ for P some projective module and $i \in \mathbf{Z}$. Thus all we need to find is some $k \in \mathbf{Z}$ such that f is null-homotopic, and $k = i + 2$ works, so we are done. $\square$

REMARK 3.2.16. We can now describe a quasi-inverse of ι. The data of a quasi-inverse is twofold: a module and a shift. Assigning a shifted MCM module for each object in $\mathsf{D}_{\mathrm{sg}}(S)$—as is detailed in the proof of Proposition 3.2.12—is now seen to be functorial. To check this is well-defined, note that we need not truncate exactly at the degree given in the proof, as any degree greater will also yield a MCM module at the cost of a greater shift. Maintaining the notation from the proof of Proposition 3.2.12, suppose X is a complex in $\mathsf{D}_{\mathrm{sg}}(S)$ and let P be a projective resolution of X. Now choose two different MCM modules.

$$\begin{array}{ccccccccc} P_t & & \longrightarrow & & P_{t-1} \longrightarrow \cdots \longrightarrow P_r & & \longrightarrow & & P_{r-1} \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & \operatorname{coker}(d^P_{t+1}) & & & & \operatorname{coker}(d^P_{r+1}) & & \end{array}$$

Then the assignment $X \mapsto \Sigma^r \operatorname{coker}(d^P_{r+1})$ is well-defined, since

$$\Sigma^r \operatorname{coker}(d^P_{r+1}) \cong \Sigma^r(\Sigma^{t-r} \operatorname{coker}(d^P_{t+1})) \cong \Sigma^t \operatorname{coker}(d^P_{t+1})$$

in $\underline{\mathsf{MCM}}(S)$. For a morphism $f\colon X \to Y$, we can truncate far enough to the left so that both X and Y can be replaced by shifted MCM modules concentrated in the same degree. The resulting morphism between these modules must be the image of a unique morphism in $\underline{\mathsf{MCM}}(S)$ as ι is fully faithful, so send f to the corresponding morphism in $\underline{\mathsf{MCM}}(S)$ between the shifted MCM modules. It is quick to check that this assignment is a quasi-inverse of ι.

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