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Random Measure Algebras Under Convolution

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Jason Hong Jae Park

June 2015

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The most valuable lesson that he taught me is “Never, ever, ever give up.”

To myself for going through hard work.

ABSTRACT OF THE DISSERTATION

Random Measure Algebras Under Convolution

by

Jason Hong Jae Park

Doctor of Philosophy, Graduate Program in Mathematics

University of California, Riverside, June 2015

Dr. M. M. Rao , Chairperson

In this thesis, we investigate a convolution product between random measures. The key importance of convolution is the convolution product of random measures is again the random measure. Since the operation is closed, a ring structure of random measures can be built. The covariance function of the random measure plays a key role in this research since each random measure has its corresponding covariance function.

First, we investigate the convolution and algebra structure of covariance functions. Let Z_1, Z_2 be the second order random measures. If β_1, β_2 are corresponding covariance functions of Z_1 and Z_2 , respectively, then define $Z = Z_1 * Z_2$, where Z is the corresponding random measure of the known convolution product $\beta_1 * \beta_2$. We construct the ring and algebra structure for the newly defined (random) convolution.

Second, we investigate the pointwise product of the covariance functions β_1, β_2 . If we take the diagonal of the domain of $\beta_1 \cdot \beta_2$, then we have a closed operation of covariance functions. We define odot product $Z = Z_1 \odot Z_2$, where Z is the corresponding random measure of $\beta_1 \cdot \beta_2$ with restricted diagonal domain, and we build algebra structures of random measures under $\odot$ operation. We also provide an example of the convolution of covariance functions that are related to Wiener processes.

Third, we define a convolution of covariance functions by Morse-Transue integral $\beta_1 * \beta_2(A, B) = \int_S \int_S \beta_1(A - x, B - y) d\beta_2(x, y)$. Then we define $Z = Z_1 * Z_2$ by letting Z to be the corresponding random measure of $\beta_1 * \beta_2$. Ring and algebra structure is also investigated under this definition. We also provide examples of convolution of covariance functions that are related to Wiener process as well.

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Chapter 1

Introduction

Convolution is a mathematical product operation between two measurable functions f and g in L^p space on an LCA group. The most interesting characteristic of the convolution product is that it often produces into the same class of functions, which enables to build the algebra structure of a function space. One may extend this idea to different types of objects in mathematics. Convolution of two scalar measures μ and ν on an locally compact abelian group is well defined. Algebra of such measures is well illustrated in [Rud62]. In this thesis, our main interest is to construct a convolution of random measures as an 'analog' for stochastic processes.

1.1 Pointwise Product

The pointwise product of random measures, or one may say the pointwise product of two stochastic processes, runs into an immediate problem since it does not guarantee σ -additive property. However, some studies have been made in special types of stochastic process. D. Dehay has showed the product of two L^p -harmonizable series is harmonizable.([Deh91]) His main results are as follows:

Definition 1.1.1. A time series $X = (X_n, n \in \mathbb{Z})$ is said to be L^p -harmonizable series whenever there exists an L^p stochastic measure $\mu : \mathcal{B}(I) \rightarrow L^p, 1 \leq p$, such that

$$X_n = \int_I \exp(inu) \mu(du), n \in \mathbb{Z}$$

The Fubini type of result is obtained.

Theorem 1.1.2. [Deh91] Assume the product measure $\nu_1 \times \nu_2$ is defined on $\mathcal{B}(I^2)$. All bounded measurable functions $f_i : I \rightarrow \mathbb{C}, i = 1, 2$, satisfy

$$\int_I f_1(u) \nu_1(du) \cdot \int_I f_2(u) \nu_2(du) = \int \int_{I^2} f_1(u_1) f_2(u_2) \nu_1 \times \nu_2(du_1, du_2)$$

Furthermore, whenever $g : I^2 \rightarrow \mathbb{C}$ denotes a bounded measurable function, the expression

$$\mu(A) = \int \int_{I \times A} g(u_1, u_2) \nu_1 \times \nu_2(du_1, du_2)$$

defines an L^p -stochastic measure on $\mathcal{B}(I)$ and every bounded measurable function $f : I \rightarrow \mathbb{C}$ satisfies

$$\int_I f(u) \mu(du) = \int \int_{I^2} f(u_2) g(u_1, u_2) \nu_1 \times \nu_2(du_1, du_2)$$

Dehay establishes the product of two L^p -harmonizable process is L^p -harmonizable.

Theorem 1.1.3. [Deh91] Let $X_j = (X_{j,n}, n \in \mathbb{Z})$ be an L^{p_j} -harmonizable series with spectral stochastic measure ν_j for $j = 1, 2$, such that $(1/p_1) + (1/p_2) = (1/p) \leq 1$ and the product measure $\nu_1 \times \nu_2$ is defined on Borel σ -algebra on $I^2 = I \times I, \mathcal{B}(I^2)$. Then the time series $X_1 \cdot X_2 = (X_{1,n} \cdot X_{2,n}, n \in \mathbb{Z})$ is L^p -harmonizable with spectral stochastic measure defined on I by $\mu(A) = \nu_1 \times \nu_2$, where $\tilde{A} = \{(u, v) : u + v \in A\}, A \in \mathcal{B}(I)$.

Dehay's result follows mainly from Fubini's Theorem. Since the product measure $\nu_1 \times \nu_2$ is defined, we have

$$X_{1,n} \cdot X_{2,n} = \int \int_{I^2} \exp(in(u_1 + u_2)) \nu_1 \times \nu_2(du_1, du_2),$$

and the change of measure yields $X_{1,n} \cdot X_{2,n} = \int_I \exp(inu) d\mu(du)$. Therefore the product is L^p -harmonizable.

The readers should keep in mind that the usual pointwise product of stochastic processes generally does not produce the product in the same class. The product of two gaussian processes is not gaussian, and the product of two stationary processes is not always defined, and when it is defined, it may not be a stationary process. The pointwise product creates too many other problems.

1.2 Itô Integral

Another way to define the convolution of two stochastic processes is through Stochastic Integrals. It seems natural to think that a convolution product should be defined through integration since the convolution in L^2 -space is defined through Lebesgue integration. A stochastic integral seems to be the natural choice when we try to define the convolution in stochastic processes.

A stochastic integral mimics the idea of the Riemann integral. Consider the interval $[0, t]$, and we partition it into n pieces. Let $0 = t_0 < t_1 < \dots < t_n = t$ and let the approximating sum S_n be

$$S_n = \sum_{i=0}^{n-1} C(s_i)(X(t_{i+1}) - X(t_i))$$

where C, X are stochastic processes and $s_i \in [t_i, t_{i+1}]$. We want to define an integral by letting $\max\{t_{i+1} - t_i\} \rightarrow 0$ as $n \rightarrow \infty$. For convenience, denote $C(s_i) = C_i$. We will get different types of integrals depending on how we let s_i be. For instance, if we let $s_i = t_i$, then it leads to the **Itô Integral**. If we let $s_i = (t_i + t_{i+1})/2$, then it leads to **Fisk-Stratonovich Integral**. If we let $s_i = \alpha t_i + (1 - \alpha)t_{i+1}$, then it will be some other type of integral.

Itô integral is the most successful integral in stochastic theory. It is most widely used and has most elegant structure. However, Itô integral requires the condition that integrand must be a martingale. Otherwise we encounter measurability(filtration) problems.

Definition 1.2.1. A stochastic process $Y : T \times \Omega \rightarrow S$ is a martingale with respect to a filtration $\mathcal{F}_*$ (defined below) and probability measure P if

1. $\mathcal{F}_*$ is a filtration of the underlying probability space $(\Omega, \mathcal{F}, P)$ i.e. $\mathcal{F}_* = \{\mathcal{F}_t\}_{t \in T}$ is an increasing sequence of σ -algebras such that $\mathcal{F}_t \subset \mathcal{F}$ for all $t \in T$ and $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s < t$, and $\mathcal{F}_s = \mathcal{F}_{s+0} = \bigcap_{t>s} \mathcal{F}_t$. (right continuity)
2. Y is adapted to the filtration $\mathcal{F}_*$. i.e. for each t in the index set T , the random variable Y_t is a $\mathcal{F}_t$ -measurable function.
3. For each t , $Y_t \in L^1(P)$, i.e. $E(|Y_t|) < \infty$.
4. For all s and t with $s < t$, $Y_s = E(Y_t | \mathcal{F}_s)$.

If $Y_s \leq E(Y_t | \mathcal{F}_s)$, then Y is a submartingale. If $Y_s \geq E(Y_t | \mathcal{F}_s)$, then Y is called supermartingale.

Definition 1.2.2. (Itô integral) Let $X = \{X_t\}_{t \in I}$ be a martingale, a submartingale or a supermartingale and let $C = \{C_t\}_{t \in I}$ be predictable random process(i.e. C is measurable with respect to the σ -algebra generated by all left-continuous adapted processes) on a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and X_t, C_t are $\mathcal{F}_t$ -measurable(or adapted). Let $0 = t_0 \leq t_1 < t_2 \cdots < t_n = t$. Denote $C(t_i) = C_i$, and $X(t_i) = X_i$ for convenience. By the martingale transform of X by C we mean the stochastic process

$Y_t = \int_0^t C_i dX_i = \int_0^t C dX$ defined by

$$\begin{cases} Y_0 = 0 \\ Y_n = \sum_{i=1}^n C_i \cdot (X_i - X_{i-1}) \text{ for } n \in \mathbb{N} \end{cases}$$

Denote $Y_t = \int_0^t C_i dX_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n C_i \cdot (X_i - X_{i-1}) = \lim Y_n$

$Y = \{Y_t\}_{t \in I}$ is then an adapted process. In other words, Y_t is $\mathcal{F}_t$ -measurable because C_i, X_i , and X_{i-1} for $i = 1, \dots, n$ are all $\mathcal{F}_t$ -measurable. If we don't have Itô integration restriction, then we immediately encounter the measurability problem. $\mathcal{F}_t$ -measurability of Y_t is not guaranteed. Since we are dealing with a stochastic process $X = \{X_t(\omega)\}_{t \in I}$, we have different ω to sit in there as t increases. Whenever we try to define a stochastic integral in Lebesgue sense, filtration problem intervenes. Therefore traditional way of integration is not suitable to define a convolution in stochastic theory.

1.3 The Results

In this thesis, we introduce a general definition of convolution of the second order random measures by using covariance functions, and construct an algebra of random measures. A random measure is an analog of a stochastic process. Here is an example of Wiener process and the random measure.

Definition 1.3.1. Let $\{W(t) \in \mathbb{R}^+\}$ be a Wiener Process. Let $A \in \mathcal{B}(\mathbb{R}^+)$, say $A = (t, s)$, and Z be a random measure such that $Z(A) = W(s) - W(t)$ and extend it onto $\mathcal{B}(\mathbb{R}^+)$. This random measure defined by the Wiener process is called a **Wiener random measure**.

The covariance functions play a key role in defining the convolution. We focus on covariance functions because the finite-dimensional distributions of a Gaussian process are fully determined by two functions - the **mean function** $\mu : [0, \infty) \rightarrow \mathbb{R}$ and

the **covariance function** $\Sigma : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$. If the Gaussian process has mean zero, then its behavior is completely determined by the covariance function. Also, if a covariance function is given, we can construct a Gaussian process corresponding to the covariance function. It is natural to look in the covariance functions when we make a convolution between to Gaussian Processes. Convolution of covariance functions thus will include the convolution of Gaussian processes. Therefore, analysis of covariance functions is the analysis of Gaussian measures.

Gaussian Process has its own importance in the stochastic analysis. Any two Gaussian measures are either equivalent or orthogonal by Feldman-Hajek Theorem. Also, Girsanov and Hitsuda [Gir60] [Hit68] have obtained a canonical representation of Gaussian process which are equivalent (or mutually absolutely continuous) to Wiener process.

We will investigate the algebra structure of covariance function space, and we introduce three different types of convolution products of random measures.

In chapter 3, we define the convolution of second order random measures $Z : \mathcal{B}(\mathbb{R}) \rightarrow L_0^2(P)$. To define the convolution, we look into the corresponding covariance functions of the given pair of random measures. Graham and Schreiber [GS84] have defined the convolution of bilinear forms and construct algebra of them. We translate bilinear forms in terms of covariance functions, and define the corresponding random measure of the convolution product of covariance functions by using Ylisen's result in [Yli88]. With a well defined convolution product, we show that the space of second order random measures forms a normed $\mathbb{C}$ -algebra.

In chapter 4, we introduce a new type of product which is called o-dot $\odot$ product, and present its properties. The bimeasure of product space is well defined. We take the diagonal of the product space $(\mathcal{S} \times \mathcal{S}) \times (\mathcal{S} \times \mathcal{S})$, and define a new bimeasure

convolution. With $\odot$ product, we will provide the ring and algebra structure of random measures.

In chapter 5, the convolution is defined through the Morse-Transue integral. This definition mimics the convolution of the scalar-valued measures. Since the MT integral is well-defined, we can find an explicit form of the convolution of two bimeasures which corresponds to the Wiener process. Also, we will provide the ring and algebra structures of random measures.

The convolution product is an interesting subject. It has the major difference from pointwise product which operation is closed to be in the same class. The most important thing is to construct an algebra structure by defining the operation of convolution. The pointwise product does not have the desired property. Mixture of analysis and algebra is essential to open a whole new research area.

Remark 1.3.2. H.S. Chung, D. Skoug and J. Chang [HCC14] introduced a convolution product on Wiener Space. Their convolution is the convolution of nonrandom functionals on a Wiener Space which is a vector space of nonrandom integrable functionals relative to the Wiener measure which is translation invariant in an infinite dimensional space $\mathbb{R}^\infty$. However, there is no randomness involved in the functions. It is an interesting functional analysis problem, but our interest is in the convolution of stochastic processes and measures of various types, and is distinct from their works.

Chapter 2

Background and Preliminaries

In this chapter we will present some of the required preliminary material needed to follow this work. All these results are standard and we will provide a motivation and references when necessary.

2.1 Random and Vector measures

In this section, we present an outline to random and vector measures.

Definition 2.1.1. 1. Let $(S, \mathcal{S})$ be a measurable space with $\mathcal{S}$ as a ring, and let $\mathfrak{X}$ be a topological vector space. A mapping $\mu : \mathcal{S} \rightarrow \mathfrak{X}$ is an *additive set function* if $A, B \in \mathcal{S}, A \cap B = \emptyset$, then

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

The additive set function μ is called a *vector measure* if it is a σ -additive in the topology of $\mathfrak{X}$.

2. If $\mathcal{S}$ is a σ -ring and $\mathfrak{X}$ is a vector space as above, then μ is termed σ -additive

whenever, $A_n \in \mathcal{S}$, $A_n \cap A_m = \emptyset$, $m \neq n$, and $\cup A_n \in \mathcal{S}$ implies

$$\mu \left(\bigcup_{i=1}^{\infty} A_n \right) = \sum_{i=1}^{\infty} \mu(A_n)$$

where the series on the right is convergent in the topology of $\mathfrak{X}$, and μ is then termed a vector measure.

Definition 2.1.2. Let $\mu : \mathcal{S} \rightarrow \mathfrak{X}$ (Banach space) be additive on a measurable space $(S, \mathcal{S})$ with $S \in \mathcal{S}$. Then its variation on each $A \in \mathcal{S}$, denoted $|\mu|(A)$ is defined by

$$|\mu|(A) = \sup \sum_{B \in \mathcal{S}} \|\mu(B)\|_{\mathfrak{X}}, \quad A \in \mathcal{S}, \quad (2.1)$$

where the supremum is over all finite partitions π of A into disjoint members of $\mathcal{S}$. If $|\mu|(S) < \infty$ then μ is said to be of *bounded variation* on the algebra $\mathcal{S}$.

The semi-variation $\|\mu\|$ of μ is defined by

$$\|\mu\|(S) = \sup \left\{ \left\| \sum_{i=1}^n a_i \mu(E_i) \right\| : |a_i| \leq 1, a_i \in \mathbb{C}, E_i \in \mathcal{S} \text{ disjoint} \right\}$$

$\|\mu\|$ is a monotone finitely-subadditive set function, and if $\|\mu\|(S) < \infty$, then μ is said to have *bounded semi-variation*, or simply *bounded*.

Proposition 2.1.3. Let $(S, \mathcal{S})$ be a measurable space and $\mu : \mathcal{S} \rightarrow \mathfrak{X}$ be σ - additive, where $\mathfrak{X}$ is a Banach space and $\mathcal{S}$ is a σ -algebra. Then μ is bounded in the sense that

$$\|\mu(S)\| = \sup\{\|\mu(A)\|_{\mathfrak{X}} : A \in \mathcal{S}\} < \infty$$

but not necessarily of finite variation and only $|\mu|(S) \leq \infty$ as in (2.1)

Theorem 2.1.4. Let $(S, \mathcal{S})$ be a measurable space and $\mu : \mathcal{S} \rightarrow \mathfrak{X}$, (a Banach space), be σ -additive. Then there exists a measure $\lambda : \mathcal{S} \rightarrow \mathbb{R}^+$ which controls μ in the sense

that for each $\varepsilon > 0$ there is $\delta > 0$ such that for $A \in \mathcal{S}$, $\lambda(A) < \varepsilon \Rightarrow \|\mu(A)\| < \delta$, i.e., μ is λ -continuous.

Recall the *Dunford-Schwarz (or D-S) integral*. Suppose $(S, \mathcal{S})$ is a measurable space with $\mathcal{S}$ as a σ -ring, $\mathfrak{X}$ is a Banach space, and $Z : \mathcal{S} \rightarrow \mathfrak{X}$ is a vector measure. If $f : S \rightarrow \mathbb{R}$ or $\mathbb{C}$, is an $\mathcal{S}$ -measurable function, then there exists a sequence of $\mathcal{S}$ -simple functions $f_n = \sum_{i=1}^n a_i \chi_{A_i}$, $a_i \in \mathbb{R}$ or $\mathbb{C}$, $A_i \in \mathcal{S}$, disjoint, such that $f_n \rightarrow f$ pointwise and $|f_n| \nearrow |f|$.

Then we may define

$$\int_A f dZ = \lim_{n \rightarrow \infty} \int_A f_n dZ, A \in \mathcal{S},$$

where

$$\int_A f_n dZ = \sum_{i=1}^n a_i Z(A_i \cap A) \in \mathfrak{X}, A \in \mathcal{S}$$

whenever the sequence $\{\int_A f_n dZ, n \geq 1\}$ converges in $\mathfrak{X}$ for each A .

This integral is uniquely defined and it does not depend on the sequence $\{f_n \geq 1\}$ approximating f . The detail is included in [DS58].

Definition 2.1.5. If $Z : \mathcal{S} \rightarrow \mathfrak{X}$ is a vector measure into a Banach space as above, then $T : f \rightarrow \int_S f dZ$ is called the Dunford-Schwarz (or D-S) integral, when it is defined as above.

It is proved in [DS58] that T is both linear and bounded and also the dominated convergence theorems hold for this integral.

2.2 Bimeasure

Note $(S, \mathcal{S})$ is a measurable space with $\mathcal{S}$ as a σ -algebra, $L^p(P)$ is the usual Lebesgue space of p^{th} order ($p \geq 0$) (real) random variables on a probability triple

(Ω, Σ, P) , and if $Z : \mathcal{S} \rightarrow L^p(P)$ is a σ -additive function in the metric of $L^p(P)$, it is called a random measure. Thus for any family $\{A_i \in \mathcal{S}, i \in I\}$, the collection $\{Z(A_i), i \in I\} \subset L^p(P)$ is a stochastic process indexed by the A_i , and its finite dimensional distributions for $i_1, \dots, i_n \in I, z_i \in \mathbb{R}$ are thus given by

$$P[Z(A_i) < z_j, j = 1, \dots, n] = F_{Z(A_{i_1}), \dots, Z(A_{i_n})}(z_1, \dots, z_n)$$

which satisfy the classical Kolmogorov consistency conditions and the $Z(A_i)$ is valued in the vector space $L^p(P)$.

Let $\mathbb{H}$ be a (complex) Hilbert space, and let $(\Omega, \mathcal{F})$ be a measurable space. Here $\mathcal{F}$ is a field of subsets of Ω . The terminology and notations are as introduced in Diestel and Uhl [DU77].

Definition 2.2.1. 1. A mapping $\beta : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{C}$ is called a *bimeasure*, if it is separately additive, i.e. if $\beta(E, \cdot)$ and $\beta(\cdot, F)$ are (scalar valued) additive measures for every $E, F \in \mathcal{F}$.

2. A bimeasure $\beta : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{C}$ is called *σ -additive*, if it is separately *σ -additive*.

3. A bimeasure $\beta : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{C}$ is *bounded*, if its semivariation is bounded, i.e. if

$$\sup \left| \sum_{i=1}^m \sum_{j=1}^n a_i b_j \beta(E_i, F_j) \right| < \infty$$

where the supremum is taken over all finite $\mathcal{F}$ -measurable partitions $\Omega = E_1 + \dots + E_m, \Omega = F_1 + \dots + F_n$ where E_i and F_i are disjoint, and $a_i, b_j \in \mathbb{C}, |a_i| \leq 1, |b_j| \leq 1; i = 1, \dots, m, j = 1, \dots, n; m, n \in \mathbb{N}$

4. A bimeasure $\beta : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{C}$ is *positive definite*, if

$$\sum_{i=1}^n \sum_{j=1}^n a_i \bar{a}_j \beta(E_i, E_j) \geq 0$$

for all $a_i \in \mathbb{C}, E_i, i = 1, \dots, n; n \in \mathbb{N}$.

Lemma 2.2.2. 1. Let $Z : \mathcal{F} \rightarrow \mathbb{H}$ (Hilbert Space) be a (bounded) vector measure.

Then, the mapping $\beta : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{C}$ defined by

$$\beta(E, F) = (Z(E), Z(F))_{\mathbb{H}}, E, F \in \mathcal{F}$$

is a (bounded) positive definite bimeasure and it is the bimeasure induced by Z .

2. Conversely, it follows from (basic) properties of the reproducing kernel Hilbert space [Aro50] that for a (bounded) positive (semi) definite bimeasure $\beta : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{C}$, that there exist $\mathbb{H}_\beta$ and a (bounded) vector measure $Z_\beta : \mathcal{F} \rightarrow \mathbb{H}_\beta$ such that

$$(Z_\beta(E), Z_\beta(F))_{\mathbb{H}_\beta} = \beta(E, F), E, F \in \mathcal{F}, \overline{\text{sp}(Z_\beta)} = \mathbb{H}_\beta$$

where $\text{sp}(Z_\beta)$ stands for the linear hull of $\{Z(E) | E \in \mathcal{F}\}$.

Remark 2.2.3. In the following work, we take the (Hilbert) space $\mathbb{H} = L_0^2(P)$ where P is a probability measure and all members of $\mathbb{H}$ being centered. Then, we have

$$\beta(E, F) = (Z(E), Z(F))_{\mathbb{H}} = E(Z(E)\overline{Z(F)}) = \int_{\Omega} Z(E)\overline{Z(F)}dP$$

Definition 2.2.4. If $(\Omega_i, \Sigma_i), i = 1, 2$ are measurable spaces and (Ω, Σ) is their product (so $\Omega = \Omega_1 \times \Omega_2$ and $\Sigma = \Sigma_1 \otimes \Sigma_2$), $f_i : \Omega_i \rightarrow \mathbb{C}$ (measurable relative to $\Sigma_i, i = 1, 2$) are given then the pair (f_1, f_2) is called *strictly* β -integrable where β is a bimeasure on $\Sigma_1 \times \Sigma_2$, provided the following two conditions hold:

1. f_1 is $\beta(\cdot, B)$ -integrable (L-S) for each $B \in \Sigma_2$ and f_2 is $\beta(A, \cdot)$ -integrable (L-S) for each $A \in \Sigma_1$ such that $\tilde{\beta}(A, F) = \int_F f_2(w_2)\beta(A, dw_2)$ is σ -additive in $A \in \Sigma_1$ for each $F \in \Sigma_2$ and $\tilde{\beta}_2(E, B) = \int_E f_1(w_1)\beta(dw_1, B)$ is σ -additive in $B \in \Sigma_2$ for each $E \in \Sigma_1$; [L-S for Lebesgue Stieltjes]

2. f_1 is $\tilde{\beta}_1(\cdot, F)$ -integrable (L-S), f_2 is $\tilde{\beta}_2(E, \cdot)$ -integrable (L-S) and

$$\int_E f_1(w_1)\tilde{\beta}_1(dw_1, F) = \int_F f_2(w_2)\tilde{\beta}_2(E, dw_2), E \in \Sigma_1, F \in \Sigma_2 \quad (2.2)$$

The common value in (2.2) is denoted $\int_E \int_F (f_1, f_2) d\beta$. It is called the *strict Morse-Transue integral*, if (2.2) holds each pair (E, F) as above.

The detail about Morse-Transue integral is also included in [CR86]. Now we recall the Dominated Covergence Thoerem of Morse-Transue integral from the same reference.

Theorem 2.2.5. *Let $(S_i, \mathcal{S}_i), i = 1, 2$ be measurable spaces, $f_{in} : S_i \rightarrow \mathbb{C}, n = 1, 2, \dots$ be measurable scalar functions and $\beta : S_1 \times S_2 \rightarrow \mathbb{C}$ be a bimeasure of finite Frechét variation (which is automatic if the $\mathcal{S}_i, i = 1, 2$ are σ -algebras). Let h_1, h_2 be β -integrable in the strict MT-sense, and let $|f_{in}| \leq |h_i|, i = 1, 2$.*

If $f_{in} \rightarrow f_i$ as $n \rightarrow \infty$ then the f_{in} as well as the limits $f_i, i = 1, 2$ are β -integrable in the same sense, and moreover for each $A \in \mathcal{S}_1$, and $B \in \mathcal{S}_2$,

$$\lim_{m, n \rightarrow \infty} \int_A \int_B (f_{1m}(s_1), f_{2n}(s_2)) \beta(ds_1, ds_2) = \int_A \int_B (f_1(s_1), f_2(s_2)) \beta(ds_1, ds_2)$$

holds, the order of the limits does not matter. In particular if the $\mathcal{S}_i$ are σ -alegebras (not δ -rings), then $h_i = \alpha_i > 0$ a constant is integrable and the resulting bounded convergence statement is also valid.

Definition 2.2.6. If $-\infty < \beta(A, B) < \infty$ for all compact sets A, B contained in the Hausdorff spaces S_1 and S_2 respectively then β is *regular from above(below)* if it can be approximated from above(below) in the following sense:

Given $A \in \mathcal{B}_0(S_1)$ and $B \in \mathcal{B}_0(S_2)$ and $\varepsilon > 0$, there exist open sets $O_1 \supset A_1, O_2 \supset B_1$ (compact sets $C_1 \subset A$ and $C_2 \subset B$) such that for each $A_2 \in \mathcal{B}_0(S_1), B_2 \in \mathcal{B}_0(S_2)$ satisfying $A \subset A_2 \subset A_1, B \subset B_2 \subset B_1$ (respectively $\tilde{C}_1 \in \mathcal{B}_0(S_1), \tilde{C}_2 \in \mathcal{B}_0(S_2)$ satisfying $C_1 \subset \tilde{C}_1 \subset A$ and $C_2 \subset \tilde{C}_2 \subset B$) for which one has

$$|\beta(A, B) - \beta(A_2, B_2)| < \varepsilon \quad (|\beta(A, B) - \beta(\tilde{C}_1, \tilde{C}_2)| < \varepsilon),$$

then β is termed a *regular bimeasure* (also called a *Radon bimeasure*) if it is simultaneously regular from above and below.

Definition 2.2.7. Suppose G is a locally compact abelian (or LCA) group, $\mathcal{S} = \mathcal{B}_0(G)$, the δ -ring of bounded Borel sets and μ as some regular measure.

Then a mapping $Z : \mathcal{B}_0(G) \rightarrow L^2(P)$ is *shift (or translation) invariant* if

1. $E(Z(A)) = E(Z(\tau_x A))$, $A \in \mathcal{B}_0(G)$, where $\tau_x A = x + A = \{x + y; y \in A\}$, $x \in G_1$, where '+' stands for the group operation.
2. $Z(\cdot)$ is σ -additive on $\mathcal{B}_0(G)$.
3. $E(Z(\tau_x A) \overline{Z(\tau_x B)}) = (Z(\tau_x A), Z(\tau_x B)) = (Z(A), Z(B))$, $A, B \in \mathcal{B}_0(G)$, $x \in G_1$.

Here $\tau_x : G \rightarrow G$ is called the *shift* or *translation* operator. Then the mapping $\beta : (A, B) \rightarrow (Z(A), Z(B))$ defined on $\mathcal{B}_0(G) \times \mathcal{B}_0(G) \rightarrow \mathbb{C}$ is called, a *shift invariant bimeasure*.

Theorem 2.2.8. Let $(S, \mathcal{B}_0(S))$ be a measurable space and $\beta : \mathcal{B}_0(S) \times \mathcal{B}_0(S) \rightarrow \mathbb{C}$ be a bounded bimeasure. Then it is positive definite iff there is a probability space (Ω, Σ, P) and a random measure $Z : \mathcal{B}_0(S) \rightarrow L^2(P)$ inducing the bimeasure in the sense that $\beta(A, B) = (Z(A), Z(B))_{L^2(P)}$, $A, B \in \mathcal{B}_0(S)$.

Theorem 2.2.9. [Yli88] A bounded bimeasure $\beta : \mathcal{B}_0(S) \times \mathcal{B}_0(S) \rightarrow \mathbb{C}$ admits a representation as:

$$\beta = \beta_1 - \beta_2 + i(\beta_3 - \beta_4)$$

where each β_j ($j = 1, \dots, 4$) is a (bounded) positive definite bimeasure on $\mathcal{B}_0(S) \times \mathcal{B}_0(S)$.

The decomposition is clearly not unique.

2.3 Bilinear Forms and Random Measures

Definition 2.3.1. Let $(\Omega_i, \Sigma_i), i = 1, 2$ be measurable spaces with Ω_i as locally compact Hausdorff and Σ_i as its Borel σ -algebra. Let $\beta : \Sigma_1 \times \Sigma_2 \rightarrow \mathbb{C}$ be a bimeasure, $C_c(\Omega_i)$ be the space of scalar continuous functions with compact supports. Then we define the corresponding **bilinear form** $B : C_c(\Omega_1) \times C_c(\Omega_2) \rightarrow \mathbb{C}$ as follows:

$$B(f, g) = \int_{\Omega_2} \int_{\Omega_1} (f_1, f_2)(w_1, w_2) \beta(dw_1, dw_2), f_i \in C_c(\Omega_i)$$

The integral here is the strict Morse-Transue integral.

Definition 2.3.2. A bilinear form B is *bounded* if there is a constant C (C depends on B) such that

$$|B(f_1, f_2)| \leq C \|f_1\| \|f_2\|$$

for all $f_i \in C_c(\Omega_i)$

From the previous section, we know that there exists a corresponding bilinear form for each bimeasure. Now it is natural to ask whether there exists a corresponding bimeasure for a given bilinear form. However, it may be seen that $B(\chi_A, \chi_B) = \beta(A, B)$ since

$$B(f, g) = \int_S \int_S (f, g) \beta(ds, dt),$$

This will imply that the bilinear form $B(\cdot, \cdot)$ on $B(S) \times B(S)$ is defined, where $B(S)$ is the space of scalar bounded $\mathcal{B}_0(S)$ -measurable functions, and the bimeasure β on $\mathcal{B}_0(S) \times \mathcal{B}_0(S)$ are in one-to-one relation on simple functions. By the Dominated Convergence Theorem for MT integrals (Theorem 2.2.5), we know $B(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ determines each other uniquely.

Definition 2.3.3. Let $(S, \mathcal{B}_0(S))$ be a measurable space where $\mathcal{B}_0(S)$ is a δ -ring, and consider $Z : \mathcal{B}_0(S) \rightarrow L_0^2(P)$, a random measure into a Hilbert space of centered (for

convenience) square integrable functions (or random variables) on a probability space. Then $Z(\cdot)$ is termed $L^{2,2}$ -bounded relative to a $(\sigma$ -finite) measure $\mu : \mathcal{B}_0(S) \rightarrow \overline{\mathbb{R}}^+$, if there is an absolute constant $C > 0$ such that for each $\mathcal{B}_0(S)$ -simple function $f : S \rightarrow \mathbb{C}$ one has

$$E \left(\left| \int_S f dZ \right|^2 \right) \leq C \int_S |f|^2 d\mu$$

where $\int_S f dZ = \sum_{i=1}^n a_i Z(A_i)$ for $f = \sum_{i=1}^n a_i \chi_{A_i}$, $A_i \in \mathcal{B}_0(S)$ disjoint. This integral is well defined since it clearly does not depend on the representation of f as a simple function.

The notion of $L^{2,2}$ -boundedness was first introduced by Bochner, [Boc55].

Theorem 2.3.4. *Let S_i be a compact Hausdorff space and $C(S_i)$ be the (Banach) space of continuous scalar functions and $B : C(S_1) \times C(S_2) \rightarrow \mathbb{C}$ be a bounded bilinear form. Then there exist regular measures $\mu_i : \mathcal{B}(S_i) \rightarrow \mathbb{R}^+(\mathcal{B}(S_i))$, being the Borel σ -algebra, $i = 1, 2$, such that*

$$|B(f_1, f_2)|^2 \leq K \int_{S_1} |f_1|^2 d\mu_1 \int_{S_2} |f_2|^2 d\mu_2, f_i \in C(S_i), i = 1, 2$$

where $K > 0$ is an absolute constant (which can be absorbed into μ_i or the μ_i can be taken as probability measure on $\mathcal{B}(S_i)$) and, the resulting $K = K_G > 0$ is termed the Grothendieck constant which for real spaces is known to satisfy $\frac{\pi}{2} < K_G \leq \pi(2 \log(1 + \sqrt{2}))^{-1} = 1.782 \dots$.

Definition 2.3.5. A bilinear form $B : B(S) \times B(S) \rightarrow \mathbb{C}$ is the (associated through MT-integration) bilinear form then it is *positive definite* if for each $f \in B(S)$,

$$B(f, f) = \int_S \int_S f(s) \bar{f}(s') d\beta(s, s') \geq 0$$

where the integral is in the MT-sense.

Remark 2.3.6. Suppose β is the corresponding bimeasure for the bilinear form B of Definition 2.3.5. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$, $a_i \in \mathbb{C}$, $A_i \in \mathcal{B}_0(S)$. Then we obtain the definition of positive definiteness of the bimeasure β , which is Definition 2.2.1.

Lemma 2.3.7. Assume the condition of Definition 2.3.1. If β and B are the corresponding bimeasure and bilinear form, then β is bounded if and only if B is bounded.

Definition 2.3.8. For a pair $\{\mathfrak{X}_i, \|\cdot\|_i\}$, $i = 1, 2$ of Banach spaces, the tensor product space $\mathfrak{X}_1 \otimes \mathfrak{X}_2$ is the linear span of formal expressions $\sum_{i=1}^n f_i \otimes g_i$, satisfying the following conditions for all $n \geq 1$ [$\sim$ for equivalence]:

1. $\sum_{i=1}^n f_i \otimes g_i \sim \sum_{j=1}^n f_{i_j} \otimes g_{i_j}$, $(i_1, \dots, i_n)$ is a permutation of $(1, \dots, n)$.
2. $\sum_{i=1}^n (a_i f_i) \otimes g_i \sim \sum_{i=1}^n f_i \otimes (a_i g_i)$, $a_i \in \mathbb{R}$ (or $\mathbb{Q}$)
3. $\sum_{i=1}^n (f'_i + f''_i) \otimes g_i = \sum_{i=1}^n f'_i \otimes g_i + \sum_{i=1}^n f''_i \otimes g_i$

and a norm $\alpha : \mathfrak{X}_1 \otimes \mathfrak{X}_2 \rightarrow \mathbb{R}^+$ is defined to satisfy the conditions

1. $\alpha(\sum_{i=1}^n f_i \otimes g_i) = 0$, if and only if $\sum_{i=1}^n f_i \otimes g_i \sim 0 \otimes 0$,

and α may be complex valued.

2. $\alpha(a \sum_{i=1}^n f_i \otimes g_i + \sum_{i=1}^n f'_i \otimes g'_i) \leq |a| \alpha(\sum_{i=1}^n f_i \otimes g_i) + \alpha(\sum_{i=1}^n f'_i \otimes g'_i)$.
3. $\alpha(\sum_{i=1}^n f_i \otimes g_i) = \alpha(\sum_{i=1}^n f'_i \otimes g'_i)$, if $\sum_{i=1}^n f'_i \otimes g'_i \sim \sum_{i=1}^n (f_i \otimes g_i)$.

Such an α is called a *cross-norm* if moreover we have

4. $\alpha(f \otimes g) = \|f\|_1 \|g\|_2$, for all $f \in \mathfrak{X}_1, g \in \mathfrak{X}_2$ (and $= 0$ if and only if $f = 0$ or $g = 0$).

In this work, we are specially interested in two cross norms. They are the least cross-norm (l.c.n) and the greatest cross-norm (g.c.n).

Definition 2.3.9. If $\mathfrak{X}_i^*$ denotes the dual of a Banach space $\mathfrak{X}_i, i = 1, 2$, and $f_i \in \mathfrak{X}_1, g_i \in \mathfrak{X}_2, x_i^* \in \mathfrak{X}_i^*$ are any elements, then the *least cross-norm*, denoted by $\lambda(\cdot)$, is defined as:

$$\lambda\left(\sum_{i=1}^n f_i \otimes g_i\right) = \sup\left\{\left|\sum x_1^*(f_i)x_2^*(g_i)\right| : \|x_1^*\| \leq 1, \|x_2^*\| \leq 1\right\}$$

Definition 2.3.10. If $f_i \in \mathfrak{X}_1, g_i \in \mathfrak{X}_2$ and $\sum_{i=1}^n f_i \otimes g_i \sim \sum_{i=1}^n f'_i \otimes g'_i$ are equivalent representations, then the *greatest cross-norm*, denoted $\gamma(\cdot)$, is defined as:

$$\gamma\left(\sum_{i=1}^n f_i \otimes g_i\right) = \inf\left\{\sum_{i=1}^n \|f'_i\|_1 \|g'_i\|_2 : \sum_{i=1}^n f_i \otimes g_i \sim \sum_{i=1}^n f'_i \otimes g'_i\right\}$$

Remark 2.3.11. It can be verified that if $\lambda(\cdot), \gamma(\cdot)$ above are the cross-norms, and if $\alpha(\cdot)$ is any cross-norm, then one has

$$\lambda\left(\sum_{i=1}^n f_i \otimes g_i\right) \leq \alpha\left(\sum_{i=1}^n f_i \otimes g_i\right) \leq \gamma\left(\sum_{i=1}^n f_i \otimes g_i\right)$$

and further

$$\mathfrak{X}_1 \otimes_\lambda \mathfrak{X}_2 \supset \mathfrak{X}_1 \otimes_\alpha \mathfrak{X}_2 \supset \mathfrak{X}_1 \otimes_\gamma \mathfrak{X}_2.$$

Alternative name for $\mathfrak{X}_1 \otimes_\lambda \mathfrak{X}_2$ is *injective* tensor product space and alternative name for $\mathfrak{X}_1 \otimes_\gamma \mathfrak{X}_2$ is *projective* tensor product space.

Now we need to introduce a norm for a bilinear form.

Definition 2.3.12. For a $B \in (\mathfrak{X}_1 \times \mathfrak{X}_2)^*$, we define a nonnegative function $\|\cdot\| : (\mathfrak{X}_1 \times \mathfrak{X}_2)^* \rightarrow \mathbb{R}$ by

$$\|B\| = \sup\{|B(x, y)| : \|x\|_1 \leq 1, \|y\|_2 \leq 1, x \in \mathfrak{X}_1, y \in \mathfrak{X}_2\}$$

and using Grothendieck's inequality one defines $|||B|||$ as:

$$|||B||| = \inf\{C > 0 : |B(f, g)| \leq C\|f\|_{\mathfrak{X}_1}\|g\|_{\mathfrak{X}_2}, f \in \mathfrak{X}_1, g \in \mathfrak{X}_2\}$$

Graham and Schreiber [GS84] described the above bimeasure algebras on LCA groups.

Lemma 2.3.13. *The function $\|\cdot\|$ is a norm in $(\mathfrak{X}_1 \times \mathfrak{X}_2)^*$.*

Proof. Here we provide the proof of the triangle inequality. The other properties are trivial. Note $\|B_1 + B_2\| = \sup\{|(B_1 + B_2)(x, y)| : \|x\| \leq 1, \|y\| \leq 1\}$. Obviously, $|(B_1 + B_2)(x, y)| = |B_1(x, y) + B_2(x, y)| \leq |B_1(x, y)| + |B_2(x, y)|$. Then $\sup\{|(B_1 + B_2)(x, y)| : \|x\| \leq 1, \|y\| \leq 1\} \leq \sup\{|B_1(x, y)| + |B_2(x, y)| : \|x\| \leq 1, \|y\| \leq 1\} \leq \sup\{|B_1(x, y)| : \|x\| \leq 1, \|y\| \leq 1\} + \sup\{|B_2(x, y)| : \|x\| \leq 1, \|y\| \leq 1\}$. Therefore, $\|B_1 + B_2\| \leq \|B_1\| + \|B_2\|$. $\square$

2.4 A Conjugate-closed Algebra of Functions in $\hat{G}_1 \times \hat{G}_2$

Definition 2.4.1. Let G_i be LCA groups whose character (or dual) groups are denoted by $\hat{G}_i, i = 1, 2$ so that for the product group, $G = G_1 \times G_2$ its dual is $\hat{G}_1 \times \hat{G}_2$. For each character $(\lambda_1, \lambda_2) \in \hat{G}_1 \times \hat{G}_2$ (so $\hat{g}_i, i = 1, 2$), consider

$$\hat{B}(\lambda_1, \lambda_2) = \langle \bar{\lambda}_1 \otimes \bar{\lambda}_2, B \rangle \quad (2.3)$$

where $\hat{B} : \hat{G}_1 \times \hat{G}_2 \rightarrow \mathbb{C}$ is a function, well-defined for each bilinear form B . If $\tilde{B}(G_1, G_2)$ stands for the set of bounded (complex) bilinear forms on $B_0(G_1) \times B_0(G_2)$, let $S(\hat{G}_1, \hat{G}_2)$ be the corresponding class of their Fourier transforms $\hat{B}$ defined by (2.3) so that each $B \in \tilde{B}(G_1, G_2)$ has its image $\hat{B} \in S(\hat{G}_1, \hat{G}_2)$.

Definition 2.4.2. If $\hat{G}_1, \hat{G}_2$ are LCA groups, $V_1(\cdot), V_2(\cdot)$ are strongly continuous unitary representatives of $\hat{G}_1, \hat{G}_2$ on a Hilbert space $\mathbb{H}$ (see e.g., [DU77]), let

$$S(\hat{G}_1, \hat{G}_2) = \{\alpha : \hat{G}_1 \times \hat{G}_2 \rightarrow \mathbb{C} \mid \langle V_1(\lambda_1)\xi, V_2(\lambda_2)\eta \rangle_{\mathbb{H}} = \alpha(\lambda_1, \lambda_2)\} \quad (2.4)$$

for some $\xi, \eta \in \mathbb{H}$. (Here ξ, η are arbitrarily fixed.)

Theorem 2.4.3. [GS84] Let G_1, G_2 be LCA groups with $\hat{G}_1, \hat{G}_2$ as their dual groups. Let $\tilde{B}(G_1, G_2)$ be the space of all bounded biinear forms [obtained from bounded bimeasures through the MT-integration as before] and $S(\hat{G}_1, \hat{G}_2)$ be the corresponding function space of Definition 2.4.2. Then the following conclusions obtain:

1. For each $B \in \tilde{B}(G_1, G_2)$, its transform $\hat{B}$ exists, and satisfies $\hat{B} \in S(\hat{G}_1, \hat{G}_2)$, and for each $\alpha \in S(\hat{G}_1, \hat{G}_2)$ there is a unique $B \in \tilde{B}(G_1, G_2)$ satisfying $\alpha = \hat{B}$.
2. For each $\alpha \in S(\hat{G}_1, \hat{G}_2)$ and $B \in \tilde{B}(G_1, G_2)$ of (1) so that $\alpha = \hat{B}$, we have

$$\|B\| \leq \|\xi\| \|\eta\|, \text{ where } (\xi, \eta) \text{ defines } \alpha \text{ as in (2.4)}$$

Definition 2.4.4. Let B_1, B_2 be a pair of bounded bilinear functionals on LCA groups G_1, G_2 and $\hat{B}_1, \hat{B}_2$ be their Fourier transforms as given in Theorem 2.4.3, which are elements of $S(\hat{G}_1, \hat{G}_2)$. We define the *convolution* $B_1 * B_2$ as an element of $\tilde{B}(G_1, G_2)$ by the equation $(B_1 * B_2)^\wedge = \hat{B}_1 \cdot \hat{B}_2$ which is unambiguously defined in the (complex-valued) space $S(\hat{G}_1, \hat{G}_2)$.

Proposition 2.4.5. If B_1, B_2 are bounded bilinear forms on $C_0(G_1) \times C_0(G_2)$, then their composition $B_1 * B_2$, of Definition 2.4.4 is continuous and satisfies

$$\|B_1 * B_2\| \leq K_G^2 \|B_1\| \|B_2\|, \tag{2.5}$$

where $K_G > 0$ is the Grothendieck constant which is known to satisfy $\frac{\pi}{2} < K_G \leq \pi(2 \log(1 + \sqrt{2}))^{-1} = 1.782 \dots$.

Lemma 2.4.6. The set $S(\hat{G}_1, \hat{G}_2)$ of Definition 2.4.2 is closed under pointwise products, sums and complex conjugation so that it is an algebra.

This follows from definition.

2.5 Wiener Measure

Definition 2.5.1. A process $\{W(t), t \in \mathbb{R}^+\}$ is called a Wiener process with a positive diffusion coefficient σ if

1. Each increment $W(s+t) - W(s)$ is $N(0, \sigma^2 t)$
2. For every pair of disjoint time intervals $(t_1, t_2], (t_3, t_4]$ with $0 \leq t_1 < t_2 \leq t_3 < t_4$, the increments $W(t_4) - W(t_3)$ and $W(t_2) - W(t_1)$ are independent random variables, and similarly for n disjoint time intervals, where n is an arbitrary positive integer.
3. $W(0) = 0$ and $W(t)$ is continuous as a function of t .

By the above definition, $W(s+t) - W(s)$ is independent of the past, in other words, if we know $W(s) = x$, then no further knowledge of $W(\tau)$ for $\tau < s$ is needed for learning about $W(s+t) - W(s)$. Often we fix $W(0) = x$ for some arbitrary number x . Note that $\text{Var}(W(t) - W(0)) = \sigma^2 t$, but we usually write $\text{Var}(W(t)) = \sigma^2 t$ by assuming the Wiener process to start at $W(0)$. Therefore, $\tilde{W}(t) = W(t)\sigma$ has unit variance, i.e. $\text{Var}(\tilde{W}(t)) = t$ and is called a standard Wiener process. The computations of probability concerning the Wiener process are as follows with $\Phi(\cdot)$ being the standard Gaussian distribution function:

$$\begin{aligned} P(W(s+t) \leq y | W(s) = x) &= P(W(s+t) - W(s) \leq y - x | W(s) = x) \\ &= P(W(s+t) - W(s) \leq y - x) = \Phi\left(\frac{y-x}{\sigma\sqrt{t}}\right) \end{aligned}$$

Definition 2.5.2. Given a centered $L^2 P$ -stochastic process $\{Z(t) : t \in \mathbb{R}^+\}$, its **covariance function**, or **kernel** is given by $C(t, s) = \text{Cov}(Z(t), Z(s))$.

Lemma 2.5.3. For the Wiener Process $\{W(t) : t \in \mathbb{R}^+\}$, the covariance function is $\text{Cov}(W(s), W(t)) = \sigma^2 \min\{s, t\}$ for $s, t \geq 0$.

Proof. Recall $E(W(t)) = 0$, and $E[W(t)]^2 = \sigma^2 t$. Then for $0 \leq s < t$,

$$\begin{aligned}
Cov(W(s), W(t)) &= E[W(s)W(t)] \\
&= E(W(s)\{W(t) - W(s) + W(s)\}) \\
&= E[W^2(s)] + E[W(s)]E[W(t) - W(s)] \\
&= \sigma^2 s + 0 = \sigma^2 s.
\end{aligned}$$

Similarly, for $0 \leq t < s$, we obtain $Cov[W(s), W(t)] = \sigma^2 t$, which leads to the formula

$$Cov(W(s), W(t)) = \sigma^2 \min\{s, t\} \text{ for } s, t \geq 0$$

□

Definition 2.5.4. Let $\{W(t) \in \mathbb{R}^+\}$ be a Wiener Process. Let $A \in \mathcal{B}(\mathbb{R}^+)$, say $A = (t, s)$, and Z be a random measure such that $Z(A) = W(s) - W(t)$ and extend it onto $\mathcal{B}(\mathbb{R}^+)$. This random measure defined by the Wiener process is called a **Wiener random measure**. The bimeasure β and the bilinear form B , which are induced from Wiener random measure are called **Wiener bimeasure** and **Wiener bilinear form**, respectively.

Lemma 2.5.5. *If Z is a Wiener random measure and $A, B \in \mathcal{B}(\mathbb{R}^+)$, then its bimeasure β is given by $\beta(A, B) = E(Z(A)Z(B)) = \sigma^2 \mu(A \cap B)$, where μ is the Lebesgue measure.*

Proof. If A, B are disjoint, $Z(A), Z(B)$ are independent. Therefore, $\beta(A, B) = E[Z(A)Z(B)] = E[Z(A)]E[Z(B)] = 0$.

Let's consider $A \cap B \neq \emptyset$. If $s_1 < t_1 < s_2 < t_2$ and $A = (s_1, s_2), B = (t_1, t_2)$,

then

$$\begin{aligned}\beta(A, B) &= E[Z(A)Z(B)] \\ &= E[(W(s_2) - W(s_1))(W(t_2) - W(t_1))] \\ &= E[W(s_2)W(t_2) - W(s_2)W(t_1) - W(s_1)W(t_2) + W(s_1)W(t_1)] \\ &= E[W(s_2)W(t_2)] - E[W(s_2)W(t_1)] - E[W(s_1)W(t_2)] + E[W(s_1)W(t_1)] \\ &= \sigma^2 s_2 - \sigma^2 t_1 - \sigma^2 s_1 + \sigma^2 s_1 \\ &= \sigma^2(s_2 - t_1) \\ &= \sigma^2 \mu(A \cap B)\end{aligned}$$

□

Chapter 3

Convolution of Random Measures valued in $L^2(P)$ by the Covariation Method

Let $(S, \mathcal{S})$ be a measurable space. Consider $Z : \mathcal{S} \rightarrow L^2(\Omega, \Sigma, P)$ a second order random measure. Let $\beta : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$ be the corresponding bimeasure defined by $\beta(A, B) = E(Z(A)\overline{Z(B)}) = \int_{\Omega} Z(A)\overline{Z(B)}dP$. Let $B : C_0(S) \times C_0(S) \rightarrow \mathbb{C}$ be the bilinear form of β , defined by $B(f, g) = \int_S \int_S (f, g)(s_1, s_2)d\beta(s_1, s_2)$. Let $RM(S), BM(S), BL(S)$ be the spaces of random measures, bimeasures and bilinear forms, respectively. If a bimeasure β corresponds to the random measure Z , and a bilinear form B corresponds to the bimeasure β , then we denote $Z \sim \beta$ and $B \sim \beta$, respectively.

3.1 Some Basic Structure

The following elementary results will be used later, and so each is called a theorem.

Theorem 3.1.1. *$BM(\mathbb{R}, +)$ is an abelian group.*

Proof. Let $\beta_1, \beta_2, \beta_3 \in BM(\mathbb{R})$. Then for associativity, $(\beta_1 + (\beta_2 + \beta_3))(A, B) = \beta_1(A, B) + (\beta_2(A, B) + \beta_3(A, B)) = \beta_1(A, B) + \beta_2(A, B) + \beta_3(A, B) = (\beta_1(A, B) + \beta_2(A, B)) + \beta_3(A, B) = ((\beta_1 + \beta_2) + \beta_3)(A, B)$. Let $\vec{0}$ be a zero bimeasure, so that $\vec{0}(A, B) = 0$ for all $A, B \in \mathcal{S}$. Then $(\beta_1 + \vec{0})(A, B) = \beta_1(A, B) = (\vec{0} + \beta_1)(A, B)$. $-\beta_1$ is the additive inverse of β_1 . Also $BM(\mathbb{R})$ is commutative. $\beta_1(A, B) + \beta_2(A, B) = \beta_2(A, B) + \beta_1(A, B)$. $\square$

Theorem 3.1.2. *$BM(\mathbb{R}, +)$ is a unitary module over $\mathbb{C}$.*

Proof. Let $r, s \in \mathbb{C}$, and $\beta_1, \beta_2 \in BM(\mathbb{R})$. i) For the distributive property, $r(\beta_1 + \beta_2)(A, B) = r[E(Z_1(A)\overline{Z_1(B)}) + E(Z_2(A)\overline{Z_2(B)})] = rE(Z_1(A)\overline{Z_1(B)}) + rE(Z_2(A)\overline{Z_2(B)}) = r\beta_1(A, B) + r\beta_2(A, B)$, and $(r+s)\beta_1(A, B) = (r+s)E(Z_1(A)\overline{Z_1(B)}) = rE(Z_1(A)\overline{Z_1(B)}) + sE(Z_1(A)\overline{Z_1(B)}) = r\beta_1(A, B) + s\beta_1(A, B)$. ii) For associativity, $r(s\beta_1(A, B)) = r(sE(Z_1(A)\overline{Z_1(B)})) = rsE(Z_1(A)\overline{Z_1(B)}) = (rs)\beta_1(A, B)$. iii) 1 is the multiplicative element in $\mathbb{C}$, and $1 \cdot \beta_1(A, B) = \beta_1(A, B) = \beta_1(A, B) \cdot 1$, so $BM(\mathbb{R}, +)$ is unitary. $\square$

Theorem 3.1.3. *$BL(\mathbb{R}, +)$ is an abelian group algebraically under addition.*

Proof. Let $B_1, B_2, B_3 \in BL(\mathbb{R})$. i) $[B_1 + (B_2 + B_3)](f, g) = [(B_1 + B_2) + B_3](f, g)$. ii) Let $\vec{0} \in BL(\mathbb{R})$ be a zero bilinear form, in other words, $\vec{0}(f, g) = 0$ for all $f, g \in C_0(\mathbb{R})$. Then $(\vec{0} + B_1)(f, g) = B_1(f, g) = (B_1 + \vec{0})(f, g)$. iii) $-B_1$ is usual additive inverse of B_1 . iv) $(B_1 + B_2)(f, g) = (B_2 + B_1)(f, g)$, so it is commutative. $\square$

Theorem 3.1.4. $BL(\mathbb{R}, +)$ is a unitary module over $\mathbb{C}$.

Proof. Let $r, s \in \mathbb{C}$, and $B_1, B_2 \in BL(\mathbb{R})$. i) $r(B_1 + B_2) = rB_1 + rB_2$. ii) $(r + s)B_1 = rB_1 + sB_1$. iii) $r(sB_1) = (rs)B_1$. iv) $1 \in \mathbb{C}$ is the multiplicative identity, then $1 \cdot B_1 = B_1 = B_1 \cdot 1$. $\square$

Theorem 3.1.5. $RM(\mathbb{R}, +)$ is an abelian group under addition.

Proof. Let $Z_1, Z_2, Z_3 \in RM(\mathbb{R})$. i) $[Z_1 + (Z_2 + Z_3)](A) = [(Z_1 + Z_2) + Z_3](A)$. ii) Let $\vec{0} \in RM(\mathbb{R})$ be a zero random measure.(i.e. $\vec{0}(A) = 0$) for all $A \in \mathcal{B}(\mathbb{R})$. Then $Z_1(A) + \vec{0}(A) = Z_1(A) = \vec{0}(A) + Z_1(A)$. iii) $(Z_1 - Z_2)(A) = (-Z_2 + Z_1)(A)$. iv) $(Z_1 + Z_2)(A) = (Z_2 + Z_1)(A)$. $\square$

Lemma 3.1.6. $RM(\mathbb{R}, +)$ is a module over $\mathbb{C}$.

Proof. It is clear that (i) $\alpha(Z_1 + Z_2) = \alpha Z_1 + \alpha Z_2$. (ii) $(\alpha + \beta)Z = \alpha Z + \beta Z$, (iii) $\alpha(\beta Z) = (\alpha\beta)Z$. And 1 is the usual identity for $\mathbb{C}$. $\square$

3.2 Defining Convolution

Now the random convolution concept is introduced as follows:

Theorem 3.2.1. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be the measurable space, where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra of $\mathbb{R}$. Suppose $\tilde{Z}_i : \mathcal{B}(\mathbb{R}) \rightarrow L_0^2(P), i = 1, 2$ be σ -additive random measures, and B_1, B_2 be the corresponding bounded bilinear forms on $C_0(\mathbb{R})$ so that their composition $B_1 * B_2$ is well-defined (whence (2.5) holds). Then there exists a unique random measure Z on $\mathcal{B}(\mathbb{R})$ with values in $L_0^2(P)$ such that its covariance bimeasure determines a bounded bilinear form B which is precisely $B_1 * B_2$ and if Z is defined as $Z = \tilde{Z}_1 * \tilde{Z}_2 : \mathcal{B}(\mathbb{R}) \rightarrow L_0^2(P)$ where the probability space (Ω, Σ, P) can be taken rich enough to support all this structure, then Z has its covariance bimeasure form as B .

Proof. Let $\tilde{Z}_1, \tilde{Z}_2$ be σ -additive random measures. Then the $\tilde{Z}_i$'s are bounded by Proposition 2.1.3. Suppose $\tilde{\beta}_1, \tilde{\beta}_2 : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{C}$ are their corresponding bimeasures. Then they are bounded and positive definite by Lemma 2.2.2. We define their bounded bilinear forms as B_1, B_2 , which are bounded by Lemma 2.3.7 and are positive definite by being represented with the corresponding bimeasures. Denote their convolution by $B = B_1 * B_2$ as it is defined by Definition 2.4.4. Also B is bounded by Proposition 2.4.5. It is trivial that B induces the corresponding bimeasure β , which is bounded. By Theorem 2.2.9, the bounded bimeasure β can be decomposed into $\beta = \beta_1 - \beta_2 + i(\beta_3 - \beta_4)$, where each $\beta_1, \beta_2, \beta_3, \beta_4$ is a bounded positive definite bimeasure. For each bimeasure β_i , there exist a (corresponding) random measure Z_{β_i} and a Hilbert space $\mathbb{H}_{\beta_i}$ such that $(Z_{\beta_i}(E), Z_{\beta_i}(F))_{\mathbb{H}_{\beta_i}} = \beta_i(E, F)$. Next define $Z = \tilde{Z}_1 * \tilde{Z}_2 = Z_1 - Z_2 + i(Z_3 - Z_4) : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{H}$ which is a random measure, where $\mathbb{H}$ is a Hilbert space which can be taken large enough, so that it contains isomorphic copies of all the above $\mathbb{H}_i$'s.

Also, $Z = Z_1 * Z_2$ is σ -additive. Let E_k 's be disjoint measurable sets and consider,

$$\begin{aligned}
Z(\bigcup_{k=1}^{\infty} E_k) &= \tilde{Z}_1 * \tilde{Z}_2(\bigcup_{k=1}^{\infty} E_k) \\
&= Z_1(\bigcup_{k=1}^{\infty} E_k) - Z_2(\bigcup_{k=1}^{\infty} E_k) + i(Z_3(\bigcup_{k=1}^{\infty} E_k) - Z_4(\bigcup_{k=1}^{\infty} E_k)) \\
&= \sum_{k=1}^{\infty} Z_1(E_k) - \sum_{k=1}^{\infty} Z_2(E_k) + i(\sum_{k=1}^{\infty} Z_3(E_k) - \sum_{k=1}^{\infty} Z_4(E_k)) \\
&= \sum_{k=1}^{\infty} [Z_1 - Z_2 + i(Z_3 - Z_4)](E_k) \\
&= \sum_{k=1}^{\infty} Z(E_k),
\end{aligned}$$

where E_k 's are disjoint sets. Also Z is bounded since $Z = Z_1 * Z_2$ and

$$\begin{aligned}
\|\tilde{Z}_1 * \tilde{Z}_2(A)\| &= \|(Z_1 - Z_2 + i(Z_3 - Z_4))(A)\| \\
&\leq \|Z_1(A)\| + \|Z_2(A)\| + \|Z_3(A)\| + \|Z_4(A)\| < \infty
\end{aligned}$$

Therefore, Z is a well-defined random measure. □

Let $RM(\mathbb{R})$ be a set of bounded σ -additive random measures. Therefore, an element $Z \in RM(\mathbb{R})$ is a random measure $Z : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{H} = L_0^2(P)$, where $\mathbb{H}$ can be assumed to be a Hilbert space large enough to support all its convolution products in its range. We next analyze various structural properties of the *random measures space* $RM(\mathbb{R})$.

3.3 Bimeasures and Bilinear Forms

First, we will verify some of the relationships between bimeasures and the corresponding bilinear forms.

Definition 3.3.1. Suppose B_1, B_2 are bounded bilinear forms and β_1, β_2 are their bimeasures as defined in Theorem 3.2.1. Then the convolution product $B_1 * B_2$ is well defined. Denote $\beta_1 * \beta_2$, the corresponding bimeasure of $B_1 * B_2$, and denote by $Z_1 * Z_2$ the induced random measure of the bimeasure $\beta_1 * \beta_2$, as given by Theorem 3.2.1.

Lemma 3.3.2. Let $\beta_1, \beta_2 : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{C}$ be (bounded) bimeasures and B_1, B_2 be their corresponding (bounded) bilinear forms, respectively. Then $\beta_1 + \beta_2$ induces the bilinear form $B_1 + B_2$.

Proof.

$$\begin{aligned} (B_1 + B_2)(f, g) &= B_1(f, g) + B_2(f, g) \\ &= \int_{S_1} \int_{S_2} fg \, d\beta_1(s_1, s_2) + \int_{S_1} \int_{S_2} fg \, d\beta_2(s_1, s_2) \\ &= \int_{S_1} \int_{S_2} fg \, d(\beta_1 + \beta_2), f, g \in L^2(P) \end{aligned}$$

Therefore, $\beta_1 + \beta_2$ has the induced bilinear form $B_1 + B_2$. $\square$

Lemma 3.3.3. Suppose β is a bimeasure and B is its corresponding bilinear form. If $\alpha \in \mathbb{C}$ then $\alpha\beta$ has its bilinear form αB . (i.e. If $\beta \sim B$, then $\alpha\beta \sim \alpha B$.)

Proof. Let $\beta' = \alpha\beta$ such that B' is the corresponding bilinear form. Then $B'(f, g) = \int f g d\beta' = \alpha \int f g d\beta = \alpha B(f, g)$. Hence, $B' = \alpha B$. $\square$

Lemma 3.3.4. *Let B_1, B_2 be bounded bilinear forms and $\hat{B}_1, \hat{B}_2$ be their Fourier transforms. Then $\widehat{B_1 + B_2} = \hat{B}_1 + \hat{B}_2$*

Proof.

$$\begin{aligned} (\hat{B}_1 + \hat{B}_2)(\gamma, \delta) &= \hat{B}_1(\gamma, \delta) + \hat{B}_2(\gamma, \delta) \\ &= \langle \pi_1(\gamma)\xi, \pi_2(\delta)\eta \rangle_{\mathbb{H}} + \langle \pi'_1(\gamma)\xi', \pi'_2(\delta)\eta' \rangle_{\mathbb{K}} \text{ for some } \xi, \eta \in \mathbb{H} \text{ and } \xi', \eta' \in \mathbb{K} \\ &= \langle (\pi_1 \oplus \pi'_1)(\gamma)(\xi \oplus \xi'), (\pi_2 \oplus \pi'_2)(\delta)(\eta \oplus \eta') \rangle_{\mathbb{H} \oplus \mathbb{K}} \\ &= \widehat{B_1 + B_2}(\gamma, \delta) \end{aligned}$$

$\square$

Lemma 3.3.5. *Suppose G is an LCA group. Then the algebra $S(\hat{G}, \hat{G})$ has an identity element (under pointwise multiplication).*

Proof. We need to find an element $\alpha \in S(\hat{G}, \hat{G})$ such that $\alpha(\lambda_1, \lambda_2) = 1$ for all $\lambda_1, \lambda_2 \in \hat{G}$. Recall $\alpha(\lambda_1, \lambda_2) = \langle V_1(\lambda_1)\xi, V_2(\lambda_2)\eta \rangle_{\mathbb{H}}$. Let $V_1 : L^1(\hat{G}) \rightarrow U(\mathbb{H})$ be a strongly continuous unitary representation in $\hat{G}_1$ such that $V_1(\lambda_1) = I$ identity operator for all $\lambda_1 \in \hat{G}$. Assume $V_1 = V_2$ and also $\xi = \eta$. Then $\alpha(\lambda_1, \lambda_2) = \langle V_1(\lambda_1)\xi, V_2(\lambda_2)\eta \rangle_{\mathbb{H}} = \langle \xi, \xi \rangle_{\mathbb{H}} = 1$. Therefore $\alpha(\lambda_1, \lambda_2)\beta(\lambda_1, \lambda_2) = \beta(\lambda_1, \lambda_2) = \beta(\lambda_1, \lambda_2)\alpha(\lambda_1, \lambda_2)$ which implies α is an identity element. $\square$

Lemma 3.3.6. *Suppose B is a bounded bilinear form and $\hat{B}$ is the corresponding Fourier transform. If $c \in \mathbb{C}$, then the bounded bilinear form cB has the corresponding Fourier transform $c\hat{B}$. (i.e If $B \sim \hat{B}$, then $cB \sim c\hat{B}$.)*

Proof. $\widehat{cB}(\lambda_1, \lambda_2) = \langle \bar{\lambda}_1 \otimes \bar{\lambda}_2, cB \rangle = cB(\bar{\lambda}_1, \bar{\lambda}_2) = c\langle (\bar{\lambda}_1 \otimes \bar{\lambda}_2, B) = c\hat{B}(\lambda_1, \lambda_2)$. Therefore $\widehat{cB} = c\hat{B}$. $\square$

Lemma 3.3.7. *Let $\alpha \in \mathbb{C}$, $Z : G \rightarrow L^2(\Omega, \Sigma, P)$ be a random measure, and β be the corresponding bimeasure of Z (i.e. $\beta(A, B) = E(Z(A)\overline{Z(B)})$). Then αZ has the bimeasure $|\alpha|^2\beta$ (i.e. If $Z \sim \beta$ then $\alpha Z \sim |\alpha|^2\beta$)*

Proof. Let β' be the corresponding bimeasure of αZ . Then $\beta'(A, B) = E[\alpha Z(A)\overline{\alpha Z(B)}] = |\alpha|^2 E[Z(A)\overline{Z(B)}] = |\alpha|^2 \beta(A, B)$. $\square$

3.4 Structure of $BL(\mathbb{R})$ under $*$

Graham and Schreiber have detailed the algebra structure of bounded bilinear forms in [GS84]. But Graham and Schreiber stop their work at convolution of bounded bilinear forms and constructing an algebra of bounded bilinear forms. We extend it to the bimeasure and random measure cases, and construct algebra of bimeasures and random measures under operation $*$.

Theorem 3.4.1. *[GS84] $BL(\mathbb{R})$ is a ring.*

Here we provide the commutativity and identity element of $BL(\mathbb{R}, *)$ which is not illustrated in [GS84]. Commutativity can be easily shown by $B_1 * B_2 \sim \hat{B}_1 \cdot \hat{B}_2 = \hat{B}_2 \cdot \hat{B}_1 \sim B_2 * B_1$. There exist the identity element $I \in S(\hat{\mathbb{R}}, \hat{\mathbb{R}})$ by Theorem 3.3.5. Then let B_I be the element corresponding to the element I by Theorem 2.4.3. $B_1 * B_I \sim \hat{B}_1 \cdot I = \hat{B}_1 \sim B_1$. Similary, $B_I * B_1 = B_1$. Therefore B_I is the identity element.

Theorem 3.4.2. *[GS84] $BL(\mathbb{R})$ is a $\mathbb{C}$ -algebra.*

Refer Graham and Schreiber [GS84] for detailed work.

3.5 Structure of $BM(\mathbb{R})$

Theorem 3.5.1. *$BM(\mathbb{R})$ is a commutative ring with identity.*

Proof. Let $\beta_1, \beta_2, \beta_3 \in BM(\mathbb{R})$, and B_1, B_2, B_3 be the corresponding bilinear forms of each of the bimeasures respectively. (i.e. $\beta_1 \sim B_1, \beta_2 \sim B_2, \beta_3 \sim B_3$). Observe the fact that space of bilinear forms is an algebra by Graham and Schreiber [GS84]. i) $BM(\mathbb{R}, +)$ is an abelian group by Theorem 3.1.3. ii) Since $\beta_2 * \beta_3 \sim B_2 * B_3, \beta_1 * (\beta_2 * \beta_3) \sim B_1 * (B_2 * B_3) = (B_1 * B_2) * B_3 \sim (\beta_1 * \beta_2) * \beta_3$. iii) $\beta_1 * (\beta_2 + \beta_3) \sim B_1 * (B_2 + B_3) = B_1 * B_2 + B_1 * B_3 \sim \beta_1 * \beta_2 + \beta_1 * \beta_3$. iv) $\beta_1 * \beta_2 \sim B_1 * B_2 = B_2 * B_1 \sim \beta_2 * \beta_1$. v) B_I is the identity element in $BL(\mathbb{R}, *)$. If $\beta_I \sim B_I$, then $\beta_I * \beta_1 \sim B_I * B_1 = B_1 \sim \beta_1$. Similarly, $\beta_1 * \beta_I = \beta_1$. □

Theorem 3.5.2. *$BM(\mathbb{R})$ is a $\mathbb{C}$ -algebra.*

Proof. Let $r \in \mathbb{C}$. i) $BM(\mathbb{R}, +)$ is a $\mathbb{C}$ -module by Theorem 3.1.2. ii) $r(\beta_1 * \beta_2) \sim r(B_1 * B_2) = rB_1 * B_2 = B_1 * rB_2 \sim r\beta_1 * \beta_2$ and $\beta_1 * r\beta_2$, respectively. □

3.6 Structure of $RM(\mathbb{R})$

Theorem 3.6.1. *$RM(\mathbb{R})$ is a ring with its binary operations as convolution and addition $(*, +)$.*

Proof. $RM(\mathbb{R})$ is an abelian group under addition by Lemma 3.1.6. Here we show associative and distributive properties under $*$.

To verify associativity, we must show the bilinear form of $Z_1 * (Z_2 * Z_3)$ is equal to the bilinear form of $(Z_1 * Z_2) * Z_3$. Let B_1, B_2, B_3 be bilinear forms associated with Z_1, Z_2, Z_3 respectively. Then $Z_2 * Z_3$ has the bilinear form $B_2 * B_3$ and $Z_1 * Z_2$ has the bilinear form $B_1 * B_2$. Also $\hat{B}_1 \cdot (B_2 * B_3)^\wedge$ has the bilinear form of $B_1 * (B_2 * B_3)$, which

has the induced random measures $Z_1 * (Z_2 * Z_3)$. Now $\hat{B}_1 \cdot (B_2 * B_3)^\wedge = \hat{B}_1 \cdot (\hat{B}_2 \cdot \hat{B}_3) = \hat{B}_1 \cdot \hat{B}_2 \cdot \hat{B}_3 = (B_1 * B_2)^\wedge \cdot \hat{B}_3$ has a bilinear form $(B_1 * B_2) * B_3$, which has the induced random measure $(Z_1 * Z_2) * Z_3$. Therefore $Z_1 * (Z_2 * Z_3) = (Z_1 * Z_2) * Z_3$.

Now we can establish the distributive property. Let Z_1, Z_2, Z_3 be random measures. If B_1, B_2, B_3 are their corresponding bounded bilinear forms then denote $\hat{B}_1, \hat{B}_2, \hat{B}_3$ as the Fourier transforms of their bilinear forms. It is trivial that $\hat{B}_1 \cdot (\hat{B}_2 + \hat{B}_3) = \hat{B}_1 \cdot \hat{B}_2 + \hat{B}_1 \cdot \hat{B}_3$. On the left side, we have $\hat{B}_1 \cdot (\hat{B}_2 + \hat{B}_3) = \hat{B}_1 \cdot (\widehat{B_2 + B_3})$, which has its corresponding bounded bilinear form $B_1 * (B_2 + B_3)$. Therefore its random measure is $Z_1 * (Z_2 + Z_3)$. On the right side, $\hat{B}_1 \cdot \hat{B}_2 + \hat{B}_1 \cdot \hat{B}_3$ has its corresponding bounded bilinear form $B_1 * B_2 + B_1 * B_3$, and its random measure is $Z_1 * Z_2 + Z_1 * Z_3$.

Therefore one can conclude $Z_1 * (Z_2 + Z_3) = Z_1 * Z_2 + Z_1 * Z_3$. $\square$

Theorem 3.6.2. *$RM(\mathbb{R})$ is a commutative ring.*

Proof. Suppose $Z_1, Z_2 \in RM(\mathbb{R})$. $Z_1 * Z_2 = Z$ is a random measure which has a bounded bilinear form B such that $\hat{B} = \hat{B}_1 \cdot \hat{B}_2$. Suppose $Z_2 * Z_1 = Z'$ is the random measure with the bounded bilinear form B' such that $\hat{B}' = \hat{B}_2 \cdot \hat{B}_1$. Since multiplication here is commutative, $\hat{B} = \hat{B}'$ implies $B = B'$. Therefore, $Z_1 * Z_2 = Z_2 * Z_1$. $\square$

Theorem 3.6.3. *$RM(\mathbb{R})$ is a ring with identity.*

Proof. Here we will set $\mathbb{R} = G$. By Theorem 2.4.3, for each element $\alpha \in S(\hat{G}, \hat{G})$ there exists a corresponding bilinear form B . Let I be the identity element in $S(\hat{G}, \hat{G})$. Such an element I exist by Lemma 3.3.5. Then by definition $I(\lambda_1, \lambda_2) = \langle \bar{\lambda}_1 \otimes \bar{\lambda}_2, B_I \rangle = B_I(\bar{\lambda}_1, \bar{\lambda}_2) = \int_G \int_G \bar{\lambda}_1 \bar{\lambda}_2 d\beta_I(\lambda_1, \lambda_2) = 1$ for all $\lambda_1, \lambda_2 \in \hat{G}$. Suppose B' is some other element. Then $I \cdot \hat{B}' = \hat{B}'$. Hence, $B_I * B' = B'$. Now $B' * B_I = B'$ by commutativity. Let Z_I and Z' be the corresponding random measures of B_I and B' respectively. Then

$Z * Z'$ is the random measure of $B_I * B'$ and so $Z_I * Z' = Z'$. Also $Z' * Z_I = Z'$ by commutativity. $\square$

Theorem 3.6.4. *$RM(\mathbb{R})$ is a normed-ring, whose norm $\|\cdot\|$ is the semi-variation.*

Proof. Here, we provide the proof of triangle inequality. Other properties of the norm are trivial.

$$\begin{aligned} \|\sum_{i=1}^n a_i(Z_1 + Z_2)(E_i)\| &= \|a_1(Z_1 + Z_2)(E_1) + a_2(Z_1 + Z_2)(E_2) + \cdots + a_n(Z_1 + Z_2)(E_n)\| \\ &= \|[a_1Z_1(E_1) + a_2Z_1(E_2) + \cdots + a_nZ_1(E_n)] \\ &\quad + [a_1Z_2(E_1) + a_2Z_2(E_2) + \cdots + a_nZ_2(E_n)]\| \\ &\leq \|a_1Z_1(E_1) + \cdots + a_nZ_1(E_n)\| + \|a_1Z_2(E_1) + \cdots + a_nZ_2(E_n)\| \end{aligned}$$

Therefore, $\|Z_1 + Z_2\| \leq \|Z_1\| + \|Z_2\|$. $\square$

Theorem 3.6.5. *$RM(\mathbb{R})$ is a $\mathbb{C}$ -Algebra.*

Proof. By Lemma 3.1.6, $RM(\mathbb{R})$ is a unitary $\mathbb{C}$ -module. It suffices to show the property $\alpha(Z_1 * Z_2) = (\alpha Z_1) * Z_2 = Z_1 * (\alpha Z_2)$ for $\alpha \in \mathbb{C}$ and $Z_1, Z_2 \in RM(\mathbb{R})$.

Suppose Z_1, Z_2 are random measures, β_1, β_2 the corresponding bimeasures, B_1, B_2 bilinear forms, and $\hat{B}_1, \hat{B}_2$ Fourier transforms of bilinear forms. Note that αZ_i induces bimeasure $\alpha^2 \beta_i$, bilinear forms $\alpha^2 B_i$, and Fourier transform $\alpha^2 \hat{B}_i$. $\alpha(Z_1 * Z_2)$ induce bimeasure $\alpha^2(\beta_1 * \beta_2)$, bilinear form $\alpha^2 B_1 * B_2$ and its Fourier transform $\alpha^2(\hat{B}_1 \cdot \hat{B}_2)$. Since multiplication here is commutative, $\alpha^2(\hat{B}_1 \cdot \hat{B}_2) = (\alpha \hat{B}_1) \cdot \hat{B}_2 = \hat{B}_1 \cdot (\alpha \hat{B}_2)$. Each of the expressions, $(\alpha \hat{B}_1) \cdot \hat{B}_2, \hat{B}_1 \cdot (\alpha \hat{B}_2)$ induces equivalent random measures $(\alpha Z_1) * Z_2, Z_1 * (\alpha Z_2)$, respectively. This completes the proof that $RM(\mathbb{R})$ is a $\mathbb{C}$ -algebra. $\square$

Theorem 3.6.6. *$RM(\mathbb{R})$ is a normed algebra.*

Proof. Proof is immediate by combining Theorem 3.6.4 and 3.6.5 $\square$

Theorem 3.6.7. $RM(\mathbb{R})$ is a normed linear space, where its norm $\|\cdot\|$ is the semi-variation.

Proof. It is trivial that $RM(\mathbb{R})$ is a linear space. The norm is the semi-variation by Theorem 3.6.4. □

3.7 Wiener Processes

Here we provide the explicit form of the Wiener process bilinear form as an illustration of the above work.

Theorem 3.7.1. If $f_N = \sum_{i=1}^N a_i \chi_{A_i}$, $g_M = \sum_{j=1}^M b_j \chi_{B_j}$, then the Wiener bilinear form is

$$B(f_N, g_M) = \int_{\mathbb{R}} \int_{\mathbb{R}} f_N g_M d\beta = \sigma^2 \sum_{i=1}^N \sum_{j=1}^M a_i b_j \mu(A_i \cap B_j).$$

Proof.

$$\begin{aligned} B(f_N, g_M) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f_N g_M d\beta \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\sum_{i=1}^N a_i \chi_{A_i} \right) \left(\sum_{j=1}^M b_j \chi_{B_j} \right) d\beta \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{i=1}^N \sum_{j=1}^M a_i b_j \chi_{A_i} \chi_{B_j} d\beta \\ &= \sum_{i=1}^N \sum_{j=1}^M a_i b_j \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{A_i} \chi_{B_j} d\beta \\ &= \sum_{i=1}^N \sum_{j=1}^M a_i b_j \sigma^2 \mu(A_i \cap B_j) \end{aligned}$$

□

Theorem 3.7.2. Suppose f, g are measurable functions and $f_N = \sum_{i=1}^N a_i \chi_{A_i} \nearrow f$, $g_M = \sum_{j=1}^M b_j \chi_{B_j} \nearrow g$. Then the Wiener bilinear form

$$B(f, g) = \int_{\mathbb{R}} \int_{\mathbb{R}} f g d\beta = \sigma^2 \lim_{N, M \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^M a_i b_j \mu(A_i \cap B_j).$$

Proof.

$$\begin{aligned}
B(f, g) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f g d\beta \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\lim_{N \rightarrow \infty} \sum_{i=1}^N a_i \chi_{A_i} \right) \left(\lim_{N \rightarrow \infty} \sum_{j=1}^M b_j \chi_{B_j} \right) d\beta \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \lim_{N, M \rightarrow \infty} \sum_{i=1}^N a_i \chi_{A_i} \sum_{j=1}^M b_j \chi_{B_j} d\beta \\
&= \lim_{N, M \rightarrow \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{i=1}^N a_i \chi_{A_i} \sum_{j=1}^M b_j \chi_{B_j} d\beta \\
&= \lim_{N, M \rightarrow \infty} B(f_N, g_M) \\
&= \lim_{N, M \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^M a_i b_j \sigma^2 \mu(A_i \cap B_j)
\end{aligned}$$

□

Explicit form of $B_1 * B_2$ is not simple. The definition of convolution $*$ is defined in an abstract way. The exact formula of the form of $Z_1 * Z_2$ still requires much further research.

Chapter 4

Odot Product and Convolution of Bimeasures

4.1 O-dot Product

The following proposition motivates the concept.

Proposition 4.1.1. *Let $\beta_i : \mathcal{S}_i \times \mathcal{S}_i \rightarrow \mathbb{C}, i = 1, 2$ be a pair of positive definite kernels and $\beta = \beta_1 \cdot \beta_2 : (\mathcal{S}_1 \times \mathcal{S}_1) \times (\mathcal{S}_2 \times \mathcal{S}_2) \rightarrow \mathbb{C}$ as their pointwise product. Then β is positive definite. If we let $\mathcal{H}_\beta, \mathcal{H}_{\beta_1}, \mathcal{H}_{\beta_2}$ the corresponding reproducing kernel Hilbert (or Aronszajn) spaces, then $\mathcal{H}_\beta = \mathcal{H}_{\beta_1} \otimes \mathcal{H}_{\beta_2}$, so that $\mathcal{H}_\beta$ is a tensor product of $\mathcal{H}_{\beta_1}$ and $\mathcal{H}_{\beta_2}$.*

The problem with pointwise product occurs immediately. In fact the product $\beta_1 \cdot \beta_2$ does not produce the bimeasure of the same class. The dimension of domain gets larger as we multiply. The interesting part of this proposition is since the product is positive definite, it induces a corresponding random measure. The proof of the Proposition 4.1.1 is detailed in [Rao12], p.172.

Definition 4.1.2. Let $(S, \mathcal{S})$ be a measurable space and $Z_i : \mathcal{S} \rightarrow L_0^2(P)$ be a pair ($i = 1, 2$) of random measures into $L_0^2(P)$ the Hilbert space of (equivalence classes of) centered (complex) random variables on a probability space (Ω, Σ, P) with covariance bimeasures $\beta_i : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$ given by $\beta_i(A, B) = (Z_i(A), Z_i(B))$ using the inner product notation. Let $\beta = \beta_1 \cdot \beta_2 : (\mathcal{S} \times \mathcal{S}) \times (\mathcal{S} \times \mathcal{S}) \rightarrow \mathbb{C}$ be the product, pointwise as in Proposition 4.1.1.

The product β in Definition 4.1.2 does not have the same domain as β_1, β_2 . β is not in the same class as β_1, β_2 . However, we can take a diagonal of $(\mathcal{S} \times \mathcal{S}) \times (\mathcal{S} \times \mathcal{S})$ which is isomorphic to $\mathcal{S} \times \mathcal{S}$. The reproducing kernel Hilbert space $H(S, \beta)$ of β is isometric with the subspace of $H(S \times S, \beta_1 \otimes \beta_2)$. Also $H(S \times S, \beta_1 \otimes \beta_2)$ is the reproducing kernel Hilbert space of $\beta_1 \otimes \beta_2$. Consider the following definition. Let $\tilde{\beta} : (\mathcal{S} \times \mathcal{S}) \times (\mathcal{S} \times \mathcal{S}) \rightarrow \mathbb{C}$ be such that

$$\tilde{\beta} = \begin{cases} \beta & \text{on diagonal of } (\mathcal{S} \times \mathcal{S}) \times (\mathcal{S} \times \mathcal{S}) \\ 0 & \text{otherwise} \end{cases}$$

$\tilde{\beta}$ is positive definite since β is positive definite by Theorem 4.1.1. Since domain of $\tilde{\beta}$ is "diagonal", denote $\tilde{\beta}(A \times B, A \times B) = \tilde{\beta}(A, B)$ without ambiguity. Next lemma shows that $\tilde{\beta}$ is σ -additive as well.

Lemma 4.1.3. $\tilde{\beta} : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$ is separately σ -additive.

Proof. First, we consider the disjoint set (A, C) and (B, D) . Let $\tilde{\beta}$ be the pointwise product of $\beta_1, \beta_2 \in BM(\mathbb{R})$, and $A, B, C, D \in \mathcal{B}(\mathbb{R})$, where $A \cap C = \emptyset, B \cap D = \emptyset$.

Then

$$\begin{aligned}
& \tilde{\beta}((A \cup C), (B \cup D)) \\
&= \tilde{\beta}((A \cup C) \times (B \cup D), (A \cup C) \times (B \cup D)) \\
&= \tilde{\beta}((A \times B) \cup (A \times D) \cup (C \times B) \cup (C \times D), \\
&\quad (A \times B) \cup (A \times D) \cup (C \times B) \cup (C \times D)) \\
&= \tilde{\beta}(A \times B, A \times B) + \tilde{\beta}(A \times B, A \times D) + \tilde{\beta}(A \times B, C \times B) + \tilde{\beta}(A \times B, C \times D) \\
&\quad + \tilde{\beta}(A \times D, A \times B) + \tilde{\beta}(A \times D, A \times D) + \tilde{\beta}(A \times D, C \times B) + \tilde{\beta}(A \times D, C \times D) \\
&\quad + \tilde{\beta}(C \times B, A \times B) + \tilde{\beta}(C \times B, A \times D) + \tilde{\beta}(C \times B, C \times B) + \tilde{\beta}(C \times B, C \times D) \\
&\quad + \tilde{\beta}(C \times D, A \times B) + \tilde{\beta}(C \times D, A \times D) + \tilde{\beta}(C \times D, C \times B) + \tilde{\beta}(C \times D, C \times D) \\
&\quad \text{Note that } \tilde{\beta}(A \times B, A \times D) = 0 \text{ by the definition.} \\
&= \tilde{\beta}(A \times B, A \times B) + \tilde{\beta}(A \times D, A \times D) + \tilde{\beta}(C \times B, C \times B) + \tilde{\beta}(C \times D, C \times D) \\
&= \tilde{\beta}(A, B) + \tilde{\beta}(A, D) + \tilde{\beta}(C, B) + \tilde{\beta}(C, D)
\end{aligned}$$

Since β is σ -additive, $\tilde{\beta}$ is also σ -additive. $\square$

We have now well-defined a product of two positive definite bimeasures.

Definition 4.1.4. Suppose the $\beta = \beta_1 \cdot \beta_2 : (\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})) \times (\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})) \rightarrow \mathbb{C}$ as in Definition 4.1.2. Let $\tilde{\beta} = \beta$ on the diagonal of $(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})) \times (\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}))$, and 0 otherwise. Then $\tilde{\beta}$ is positive definite and is a separately σ -additive bimeasure, so $\tilde{\beta} \in BM(\mathbb{R})$.

Moreover, there exist a reproducing kernel Hilbert space, $\mathcal{H}$ of $\tilde{\beta}$ and a random measure Z such that $\tilde{\beta}(A, B) = E(Z(A)\overline{Z(B)})$. If Z_1, Z_2 and $\mathcal{H}_1, \mathcal{H}_2$ are the corresponding random measures and reproducing kernel Hilbert spaces for bimeasures β_1, β_2 , then define the odot product of Z_1 and Z_2 as $Z = Z_1 \odot Z_2$, whose bimeasure is $\tilde{\beta}$.

Remark 4.1.5. $Z = Z_1 \odot Z_2$ is well defined in the Definition 4.1.4. Since $\tilde{\beta}$ is positive definite bimeasure, there exist the random measure Z and Hilbert space $\mathcal{H}$ such that

$\tilde{\beta}(A, B) = (Z(A), Z(B))_{\mathcal{H}}$ by Theorem 2.2.2

4.2 Structure of $BM(\mathbb{R}, \cdot)$

Theorem 4.2.1. $BM(\mathbb{R}, \cdot)$ is a ring.

Proof. $BM(\mathbb{R}, +)$ is an abelian group by Theorem 3.1.1. The distributive property as well as commutativity is trivial since it is mere multiplication. $\square$

Remark 4.2.2. The multiplicative identity of $BM(\mathbb{R})$ is not trivial. One can think of a bimeasure $\delta(A, B) = 1$ for all $A, B \in \mathcal{B}(\mathbb{R})$. However, this δ will not have the additive property of bimeasure.

Theorem 4.2.3. $BM(\mathbb{R}, \cdot)$ is an algebra over $\mathbb{C}$.

Proof. i) $BM(\mathbb{R})$ is a unitary $\mathbb{C}$ -module by Theorem 3.1.2. ii) Let $a \in \mathbb{C}$. Then $a(\beta_1 \cdot \beta_2)(A, B) = a(\beta_1(A, B) \cdot \beta_2(A, B)) = a\beta_1(A, B) \cdot \beta_2(A, B) = \beta_1(A, B) \cdot a\beta_2(A, B)$.

$\square$

4.3 Structure of $RM(\mathbb{R}, \odot)$

Theorem 4.3.1. $RM(\mathbb{R}, \odot)$ is a ring.

Proof. Let $Z_1, Z_2, Z_3 \in RM(\mathbb{R})$, and $\beta_1, \beta_2, \beta_3$ be their corresponding bimeasures. (i.e. $Z_1 \sim \beta_1, Z_2 \sim \beta_2, Z_3 \sim \beta_3$). i) $RM(\mathbb{R}, \odot)$ is an abelian group by Theorem 3.1.3. ii) Note $Z_1 \odot Z_2 \sim \beta_1 \cdot \beta_2$ by definition. $(Z_1 \odot Z_2) \odot Z_3 \sim (\beta_1 \cdot \beta_2) \cdot \beta_3 = \beta_1 \cdot (\beta_2 \cdot \beta_3) \sim Z_1 \odot (Z_2 \odot Z_3)$. iii) $Z_1 \odot (Z_2 + Z_3) \sim \beta_1 \cdot (\beta_2 + \beta_3) = \beta_1 \cdot \beta_2 + \beta_1 \cdot \beta_3 \sim Z_1 \odot Z_2 + Z_1 \odot Z_3$. iv) $Z_1 \odot Z_2 \sim \beta_1 \cdot \beta_2 = \beta_2 \cdot \beta_1 \sim Z_2 \odot Z_1$. $\square$

Theorem 4.3.2. $RM(\mathbb{C}, \odot)$ is an algebra over $\mathbb{C}$.

Proof. Let $Z_1, Z_2 \in RM(\mathbb{R})$ and $k \in \mathbb{C}$. Then i) $RM(\mathbb{R}, +)$ is a unitary $\mathbb{C}$ -module by Theorem 3.1.6. ii) Recall $kZ \sim |k|^2\beta$ by Lemma 3.3.7. $k(Z_1 \odot Z_2) \sim |k|^2(\beta_1 \cdot \beta_2) = |k|^2\beta_1 \cdot \beta_2 = \beta_1 \cdot |k|^2\beta_2 \sim kZ_1 \odot Z_2$ and $\sim Z_1 \odot kZ_2$, respectively.

□

4.4 Wiener Process

Lemma 4.4.1. *Suppose $Z_1 : \mathcal{B}_0(\mathbb{R}) \rightarrow L^2(P)$ is a Wiener random measure, and β_1 is its corresponding bimeasure (i.e. β_1 is a scalar bimeasure induced from a Wiener process). If $Z = Z_1 \odot Z_1 : \mathcal{B}_0(\mathbb{R}) \times \mathcal{B}_0(\mathbb{R}) \rightarrow L^2(P)$, then Z has the bimeasure $\beta : (\mathcal{B}_0(\mathbb{R}) \times \mathcal{B}_0(\mathbb{R})) \times (\mathcal{B}_0(\mathbb{R}) \times \mathcal{B}_0(\mathbb{R})) \rightarrow \mathbb{C}$ such that $\beta(A \times B, \tilde{A} \times \tilde{B}) = \sigma^4 \mu(A \cap B) \mu(\tilde{A} \cap \tilde{B})$*

Proof. By the definition of β ,

$$\begin{aligned} \beta(A \times B, \tilde{A} \times \tilde{B}) &= \beta_1(A, B) \cdot \beta_1(\tilde{A}, \tilde{B}) \\ &= \sigma^2 \mu(A \cap B) \cdot \sigma^2 \mu(\tilde{A} \cap \tilde{B}) \\ &= \sigma^4 \mu(A \cap B) \mu(\tilde{A} \cap \tilde{B}) \end{aligned}$$

This is a bimeasure since it is σ -additive on $(\mathcal{B}_0(\mathbb{R}) \times \mathcal{B}_0(\mathbb{R})) \times (\mathcal{B}_0(\mathbb{R}) \times \mathcal{B}_0(\mathbb{R}))$. □

Lemma 4.4.2. *Suppose $\tilde{\beta} = \beta|_{\text{diagonal of } S \times S}$. Then $\tilde{\beta}(A, B) = \sigma^4 (\mu(A \cap B))^2$.*

Proof. Proof is immediate from previous Lemma. □

$Z \odot Z$ has the covariance bimeasure $\sigma^4 (\mu(A \cap B))^2$, where Z is a Wiener random measure. There exist a unique Gaussian Process corresponding to a given bimeasure. However, $Z \odot Z$ itself will not be a Wiener measure in the classical sense of Wiener's.

Chapter 5

Convolution by strict Morse-Transue integral

5.1 Convolution of Bimeasures

The following propostion also can be considered as a definition.

Proposition 5.1.1. *If $Z_i : \mathcal{B}_0([0, 1]) \rightarrow L^2(P), i = 1, 2$ are a pair of random measures, then it is possible to obtain a convolution product giving a new random measure, using their induced bimeasures β_i , that have finite Vitali variations, as follows: a pair of sets A, B employing the Lebesgue-integration, consider*

$$(\beta_1 * \beta_2)(A, B) = \int_0^1 \int_0^1 \beta_1(A - x, B - y) \beta_2(dx, dy), A, B \in \mathcal{B}_0([0, 1]) \quad (5.1)$$

*Then $(\beta_1 * \beta_2)(\cdot, \cdot)$ is a well-defined positive definite bimeasure on $\mathcal{B}_0([0, 1]) \times \mathcal{B}_0([0, 1])$ and there is a random measure $Z : \mathcal{B}_0([0, 1]) \rightarrow L^2(P)$ whose bimeasure is $(\beta_1 * \beta_2)(\cdot, \cdot)$.*

Proof. We will first show that $\beta_1 * \beta_2(A, \cdot)$ of (5.1), for each A , is additive. Suppose

E, F are disjoint. Then

$$\begin{aligned}
(\beta_1 * \beta_2)(A, E \cup F) &= \int_0^1 \int_0^1 \beta_1(A - x, E \cup F - y) \beta_2(dx, dy) \\
&= \int_0^1 \int_0^1 [\beta_1(A - x, E - y) + \beta_1(A - x, F - y)] \beta_2(dx, dy) \\
&\quad \text{since } \beta_1 \text{ is } \sigma\text{-additive in each variable,} \\
&= \int_0^1 \int_0^1 \beta_1(A - x, E - y) \beta_2(dx, dy) \\
&\quad + \int_0^1 \int_0^1 \beta_1(A - x, F - y) \beta_2(dx, dy) \\
&= (\beta_1 * \beta_2)(A, E) + (\beta_1 * \beta_2)(A, F),
\end{aligned}$$

so that $\beta_1 * \beta_2(A, \cdot)$ is σ -additive since $\beta_1(A, \cdot)$ is. Similarly, $\beta_1 * \beta_2(\cdot, B)$ is σ -additive and so $\beta_1 * \beta_2$ is well-defined and is a bimeasure.

Next, we claim $\beta_1 * \beta_2$ is positive definite. For $a_i \in \mathbb{C}$,

$$\begin{aligned}
\sum_{i,j=1}^n a_i \bar{a}_j \beta_1 * \beta_2(A_i, A_j) &= \sum_{i,j=1}^n a_i \bar{a}_j \int_0^1 \int_0^1 \beta_1(A_i - x, A_j - y) \beta_2(dx, dy) \\
&= \int_0^1 \int_0^1 \sum_{i,j=1}^n a_i \bar{a}_j \beta_1(A_i - x, A_j - y) \beta_2(dx, dy) \\
&\geq 0 \text{ since } \beta_1 \text{ is positive definite} \\
&\text{and } \beta_2 \text{ is a positive definite bimeasure.}
\end{aligned}$$

Therefore, $\beta_1 * \beta_2$ is positive definite.

But then there exists a random measure Z , whose induced bimeasure is $\beta_1 * \beta_2$ by Theorem 2.2.8. □

Lemma 5.1.2. *The convolution products of positive definite bimeasures are commutative.*

Proof.

$$\begin{aligned}
\beta_1 * \beta_2(A, B) &= \int_0^1 \int_0^1 \beta_1(A - x, B - y) d\beta_2(x, y) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \chi_{A,B}(a + x, b + y) d\beta_1(a, b) d\beta_2(x, y) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \chi_{A,B}(a + x, b + y) d\beta_2(x, y) d\beta_1(a, b) \\
&= \int_0^1 \int_0^1 \beta_2(A - a, B - b) d\beta_1(a, b) \\
&= \beta_2 * \beta_1(A, B)
\end{aligned}$$

□

5.2 Structure of $BM([0, 1])$

Definition 5.2.1. Let $BM([0, 1])$ be the space of positive definite bimeasures $\beta : \mathcal{B}_0([0, 1]) \times \mathcal{B}_0([0, 1]) \rightarrow \mathbb{C}$. We call it a bimeasure space.

Theorem 5.2.2. *The bimeasure space $BM([0, 1], *)$ is a ring with identity.*

Proof. It is trivial to show that $BM([0, 1])$ is a group under addition. We show that $BM([0, 1])$ is a monoid space under convolution $*$ and has the distributive property.

First, we want to show $(\beta_1 * \beta_2) * \beta_3 = \beta_1 * (\beta_2 * \beta_3)$. We use Fubini's Theorem for bimeasures and commutative properties.

$$\begin{aligned}
((\beta_1 * \beta_2) * \beta_3)(A, B) &= \int_0^1 \int_0^1 \beta_1 * \beta_2(A - x, B - y) \beta_3(dx, dy) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \beta_1(A - x - a, B - y - b) d\beta_2(a, b) d\beta_3(x, y) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \beta_1(A - x - a, B - y - b) d\beta_3(x, y) d\beta_2(a, b) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \beta_3(A - x - a, B - y - b) d\beta_1(x, y) d\beta_2(a, b) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \beta_3(A - x - a, B - y - b) d\beta_2(a, b) d\beta_1(x, y) \\
&= \int_0^1 \int_0^1 \beta_3 * \beta_2(A - x, B - y) d\beta_1(x, y) \\
&= ((\beta_3 * \beta_2) * \beta_1)(A, B) \\
&= (\beta_1 * (\beta_3 * \beta_2))(A, B) = (\beta_1 * (\beta_2 * \beta_3))(A, B)
\end{aligned}$$

Similiarly, one can show $(\beta_1 * (\beta_3 * \beta_2))(A, B) = (\beta_1 * (\beta_2 * \beta_3))(A, B)$.

$BM([0, 1])$ has a unit $\delta_0(\cdot, \cdot)$, where $\delta_0(\cdot, \cdot)$ is defined by

$$\delta_0(A, B) = \begin{cases} 1 & \text{if } 0 \in A \text{ and } 0 \in B \\ 0 & \text{otherwise} \end{cases}$$

Observe that δ_0 is a bimeasure since it has the σ -additive property $\delta_0(A, \cup_{n \in I} B_n) = \sum_{n \in I} \delta_0(A, B_n)$. Also $\beta * \delta_0(A, B) = \int_0^1 \int_0^1 \beta(A - x, B - y) d\delta_0(dx, dy) = \beta(A, B)$, and $\delta_0 * \beta(A, B) = \int_0^1 \int_0^1 \delta_0(A - a, B - b) d\beta(a, b) = \beta(A, B)$.

For the multiplicative distributive property, we have

$$\begin{aligned} \beta_1 * (\beta_2 + \beta_3)(A, B) &= \int_0^1 \int_0^1 \beta_1(A - x, B - y) d(\beta_2 + \beta_3)(x, y) \\ &= \int_0^1 \int_0^1 \beta_1(A - x, B - y) d\beta_2 + \int_0^1 \int_0^1 \beta_1(A - x, B - y) d\beta_3 \\ &= (\beta_1 * \beta_2 + \beta_1 * \beta_3)(A, B) \end{aligned}$$

Therefore, $BM[0, 1]$ is a ring. □

Theorem 5.2.3. $BM([0, 1])$ is a $\mathbb{C}$ -algebra.

Proof. $BM([0, 1])$ is a $\mathbb{C}$ -module by Theorem 3.1.2. We have the compatibility with scalars, that is $(a\beta_1) * (b\beta_2)(A, B) = \int_0^1 \int_0^1 a\beta_1(A - x, B - y) d(b\beta_2)(x, y) = ab \int_0^1 \int_0^1 \beta_1(A - x, B - y) d\beta_2(x, y) = ab\beta_1 * \beta_2(A, B)$. □

5.3 Structure of $RM([0, 1], *)$

Definition 5.3.1. Given the Borel measurable space $([0, 1], \mathcal{B}([0, 1]))$, let $Z_i : \mathcal{B}([0, 1]) \rightarrow L_0^2(P)$ be a pair ($i = 1, 2$) of random measures into the Hilbert space of (equivalence classes of) centered (complex) random variables on a probability space (Ω, Σ, P) with covariance bimeasures $\beta_i : \mathcal{B}([0, 1]) \times \mathcal{B}([0, 1]) \rightarrow \mathbb{C}$ given by $\beta_i(A, B) = (Z_i(A), Z_i(B))$

using the inner product notation. Let $\beta = \beta_1 * \beta_2 : \mathcal{B}([0, 1]) \times \mathcal{B}([0, 1]) \rightarrow \mathbb{C}$ be the convolution product given in Proposition 4.1.1. If $Z : \mathcal{B}([0, 1]) \rightarrow L_0^2(P)$ is the induced random measure by β , then denote by $Z = Z_1 * Z_2 : \mathcal{B}([0, 1]) \rightarrow L_0^2(P)$, the random measure. It is well defined as convolution of Z_1 and Z_2 .

The convolution product ' $*$ ' is different from $*$ in Chapter 3. Introducing a new notation would be required.

Theorem 5.3.2. *$RM([0, 1], *)$ is a ring with identity.*

Proof. i) $RM([0, 1], *)$ is an abelian group by Theorem 3.1.3. ii) $Z_1 * (Z_2 + Z_3) \sim \beta_1 * (\beta_2 + \beta_3) = \beta_1 * \beta_2 + \beta_1 * \beta_3 \sim Z_1 * Z_2 + Z_1 * Z_3$. iii) $(Z_1 + Z_2) * Z_3 \sim (\beta_1 + \beta_2) * \beta_3 = \beta_1 * \beta_3 + \beta_2 * \beta_3 \sim Z_1 * Z_3 + Z_2 * Z_3$. iv) $(Z_1 * Z_2) * Z_3 \sim (\beta_1 * \beta_2) * \beta_3 = \beta_1 * (\beta_2 * \beta_3) \sim Z_1 * (Z_2 * Z_3)$. v) $Z_1 * Z_2 \sim Z_2 * Z_1$ for commutativity. vi) Define a random measure $Z_0(E) = 1$ if $0 \in E$, 0 otherwise. Z_0 is finitely additive since $Z_0(\cup E_n) = \sum Z_0(E_n) = 1$ if $0 \in E_i$ for any i , 0 otherwise. Let δ_0 be corresponding bimeasure of Z_0 . Then $\delta_0(A, B) = E(Z(A)Z(B)) = 1$ if $0 \in A$ and $0 \in B$, 0 otherwise, which is the exact definition of Dirac δ -function. (i.e $Z_0 \sim \delta_0$. $Z_1 * Z_0 \sim \beta_1 * \delta_0 = \beta_1 \sim Z_1$. Similarly, $Z_0 * Z_1 \sim Z_1$.

□

Theorem 5.3.3. *$RM([0, 1], *)$ is a $\mathbb{C}$ -algebra.*

Proof. i) $RM([0, 1])$ is a $\mathbb{C}$ -module by Lemma 3.1.6. ii) $k(Z_1 * Z_2) \sim |k|^2(\beta_1 * \beta_2) = |k|^2\beta_1 * \beta_2 = \beta_1 * |k|^2\beta_2 \sim kZ_1 * Z_2$ and $\sim Z_1 * kZ_2$. □

5.4 Wiener Process

One should recall that Wiener bilinear form is $B(\chi_A, \chi_B) = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(x) \chi_B(y) d\beta(x, y) = \beta(A, B) = \sigma^2 \mu(A \cap B)$. Now, let us investigate the explicit form of convolution of

two Wiener bimeasures using the above definition of $L^2(P)$ -valued random measures. Suppose β_1, β_2 are two Wiener bimeasures. In other words $I = [0, 1]$ and $\beta_1, \beta_2 : \mathcal{B}(I) \times \mathcal{B}(I) \rightarrow \mathbb{C}$. Each bimeasure's corresponding random measure is $Z : \mathcal{B}(I) \rightarrow L^2(P)$ such that $Z([t_i, t_{i+1}]) = X_{t_{i+1}} - X_{t_i}$, where $\{X_{t_j}\}$ is the Wiener Process with $t_j \in I$. Note that $\beta_1(A, B) = E[Z(A)Z(B)]$. Let $A = [t_1, t_2], B = [s_1, s_2]$ with $s_1 < t_1 < s_2 < t_2$.

$$\begin{aligned}
\beta(A - x, B - y) &= E[Z(A - x)Z(B - y)] \text{ where } Z(\cdot) \text{ is as above} \\
&= E[(X(t_2 - x) - X(t_1 - x)) \cdot (X(s_2 - y) - X(s_1 - y))] \\
&= E[X(t_2 - x)X(s_2 - y)] - E[X(t_2 - x)X(s_1 - y)] \\
&\quad - E[X(t_1 - x)X(s_2 - y)] + E[X(t_1 - x)X(s_1 - y)] \\
&= \min\{t_2 - x, s_2 - y\} - \min\{t_2 - x, s_1 - y\} \\
&\quad - \min\{t_1 - x, s_2 - y\} + \min\{t_1 - x, s_1 - y\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\beta_1 * \beta_2(A, B) &= \int_0^1 \int_0^1 \beta_1(A - x, B - y) \beta_2(dx, dy) \\
&= \int_0^1 \int_0^1 \min\{t_2 - x, s_2 - y\} - \min\{t_2 - x, s_1 - y\} \\
&\quad - \min\{t_1 - x, s_2 - y\} + \min\{t_1 - x, s_1 - y\} d\beta_2(x, y) \\
&= \int_0^1 \int_0^1 \min\{t_2 - x, s_2 - y\} d\beta_2(x, y) \\
&\quad - \int_0^1 \int_0^1 \min\{t_2 - x, s_1 - y\} d\beta_2(x, y) \\
&\quad - \int_0^1 \int_0^1 \min\{t_1 - x, s_2 - y\} d\beta_2(x, y) \\
&\quad + \int_0^1 \int_0^1 \min\{t_1 - x, s_1 - y\} d\beta_2(x, y)
\end{aligned}$$

Observe that

$$\min\{t_2 - x, s_2 - y\} = \begin{cases} t_2 - x & \text{if } x - (t_2 - s_2) \geq y \\ s_2 - y & \text{if } x - (t_2 - s_2) < y \end{cases}$$

Also $d\beta(x, y) = d(\min(x, y)) = x dx dy$ if $x \leq y$, and $= y dx dy$ if $x > y$. We will now

calculate each of the integrals explicitly.

$$\begin{aligned}
& \int_0^1 \int_0^1 \min\{t_2 - x, s_2 - y\} d\beta_2(x, y) \\
&= \int_0^1 \int_0^y x(s_2 - y) dx dy + \int_0^{t_2-s_2} \int_y^{t_2-s_2} y(s_2 - y) dx dy \\
&+ \int_{t_2-s_2}^1 \int_{x-(t_2-s_2)}^x y(s_2 - y) dy dx + \int_{t_2-s_2}^1 \int_0^{x-(t_2-s_2)} y(t_2 - x) dy dx \\
&= -\frac{1}{8} + \frac{s_2}{6} \\
&- \frac{1}{12}(s_2 - t_2)^3(3s_2 - t_2) \\
&+ \frac{1}{6}(s_2^4 - s_2^3(4t_2 + 1) + 6s_2^2t_2^2 + s_2(-4t_2^3 + 3t_2^2 - 3t_2 + 2) + t_2(t_2^3 - 2t_2^2 + 3t_2 - 2)) \\
&+ \frac{1}{24}(s_2 - t_2 + 1)^3(s_2 + 3t_2 - 3)
\end{aligned}$$

Similarly, other integrals can be obtained. Thus,

$$\begin{aligned}
& \int_0^1 \int_0^1 \min\{t_2 - x, s_1 - y\} d\beta_2(x, y) \\
&= \int_0^1 \int_0^y x(s_1 - y) dx dy + \int_0^{t_2-s_1} \int_y^{t_2-s_1} y(s_1 - y) dx dy \\
&+ \int_{t_2-s_1}^1 \int_{x-(t_2-s_1)}^x y(s_1 - y) dy dx + \int_{t_2-s_1}^1 \int_0^{x-(t_2-s_1)} y(t_2 - x) dy dx \\
&= -\frac{1}{8} + \frac{s_1}{6} \\
&- \frac{1}{12}(s_1 - t_2)^3(3s_1 - t_2) \\
&+ \frac{1}{6}(s_1^4 - s_1^3(4t_2 + 1) + 6s_1^2t_2^2 + s_1(-4t_2^3 + 3t_2^2 - 3t_2 + 2) + t_2(t_2^3 - 2t_2^2 + 3t_2 - 2)) \\
&+ \frac{1}{24}(s_1 + 3t_2 - 3)(s_1 - t_2 + 1)^3,
\end{aligned}$$

And

$$\begin{aligned}
& \int_0^1 \int_0^1 \min\{t_1 - x, s_2 - y\} d\beta_2(x, y) \\
&= \int_{s_2-t_1}^1 \int_0^{y-s_2+t_1} x(s_2 - y) dx dy + \int_0^{1-s_2+t_1} \int_x^{x+s_2-t_1} x(t_1 - x) dy dx \\
&+ \int_{1-s_2+t_1}^1 \int_x^1 x(t_1 - x) dy dx + \int_0^1 \int_0^x y(t_1 - x) dy dx \\
&= \frac{1}{24}(1 + t_1 - s_2)^3(-3 + t_1 + 3s_2) \\
&+ \frac{1}{6}(1 + t_1 - s_2)^2(-t_1 + s_2)(-2 + t_1 + 2s_2) \\
&+ \frac{1}{12}(t_1 - s_2)^2(-6 + t_1^2 + 2t_1(-1 + s_2) + 8s_2 - 3s_2^2) \\
&+ \frac{t_1}{6} - \frac{1}{8}.
\end{aligned}$$

The last integral is

$$\begin{aligned}
& \int_0^1 \int_0^1 \min\{t_1 - x, s_1 - y\} d\beta_2(x, y) \\
&= \int_0^1 \int_0^y x(s_1 - y) dx dy + \int_0^{t_1-s_1} \int_y^{t_1-s_1} y(s_1 - y) dx dy \\
&+ \int_{t_1-s_1}^1 \int_{x-(t_1-s_1)}^x y(s_1 - y) dy dx + \int_{t_1-s_1}^1 \int_0^{x-(t_1-s_1)} y(t_1 - x) dy dx \\
&= -\frac{1}{8} + \frac{s_1}{6} \\
&- \frac{1}{12}(s_1 - t_1)^3(3s_1 - t_1) \\
&+ \frac{1}{6}(s_1^4 - s_1^3(4t_1 + 1) + 6s_1^2t_1^2 + s_1(-4t_1^3 + 3t_1^2 - 3t_1 + 2) + t_1(t_1^3 - 2t_1^2 + 3t_1 - 2)) \\
&+ \frac{1}{24}(s_1 + 3t_1 - 3)(s_1 - t_1 + 1)^3
\end{aligned}$$

We put these calculated integrals together.

$$\begin{aligned}
\beta_1 * \beta_2(A, B) &= \int_0^1 \int_0^1 \beta_1(A - x, B - y) \beta_2(dx, dy) \\
&= \int_0^1 \int_0^1 \min\{t_2 - x, s_2 - y\} d\beta_2(x, y) - \int_0^1 \int_0^1 \min\{t_2 - x, s_1 - y\} d\beta_2(x, y) \\
&- \int_0^1 \int_0^1 \min\{t_1 - x, s_2 - y\} d\beta_2(x, y) + \int_0^1 \int_0^1 \min\{t_1 - x, s_1 - y\} d\beta_2(x, y) \\
&= -1 + \frac{s_1}{3} - \frac{s_1^2}{2} - \frac{s_1^3}{3} - \frac{s_1^4}{12} + \frac{t_1}{3} + \frac{s_1 t_1}{2} + \frac{s_1^2 t_1}{2} + \frac{s_1^3 t_1}{6} - \frac{t_1^2}{2} - \frac{s_1 t_1^2}{2} - \frac{s_1^2 t_1^2}{4} \\
&+ \frac{s_2 t_1^3}{4} - \frac{t_1^4}{12} + \frac{s_2}{3} + \frac{t_1 s_2}{2} + \frac{t_1^2 s_2}{2} + \frac{t_1^3 s_2}{6} - \frac{s_2^2}{2} - \frac{t_1 s_2^2}{2} - \frac{t_1^2 s_2^2}{4} + \frac{t_1 s_2^3}{6} - \frac{s_2^4}{12} + \frac{t_2}{3} \\
&+ \frac{s_1 t_2}{2} + \frac{s_1^2 t_2}{2} + \frac{s_1^3 t_2}{6} + \frac{s_2 t_2}{2} + \frac{s_2^2 t_2}{2} + \frac{s_2^3 t_2}{6} - \frac{t_2^2}{2} - \frac{s_1 t_2^2}{2} - \frac{s_1^2 t_2^2}{4} - \frac{s_2 t_2^2}{2} - \frac{s_2^2 t_2^2}{4} \\
&+ \frac{t_2^3}{3} + \frac{s_1 t_2^3}{6} + \frac{s_2 t_2^3}{6} - \frac{t_2^4}{12}
\end{aligned}$$

There exists a unique (centered) Gaussian Process corresponding to a given covariance function and hence for that we just computed, since it is well known that a Gaussian process is determined by its mean and covariance functions. However, with such a complex representation, it is not trivial to express the exact representation of the Gaussian Process related to the covariance function above, although there exists a unique (Gaussian) process, which will not be the (traditional) Wiener process.

Chapter 6

Furture Work and Conclusion

Problem 6.0.1. Suppose $M(G)$ is a space of scalar valued measures on Borel sets of G , where G is a locally compact abelian group. The Fourier transform of $\mu \in M(G)$ is defined by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x), (\gamma \in \Gamma)$$

If $\sigma = \mu * \nu$, then $\hat{\sigma} = \hat{\mu} \cdot \hat{\nu}$. The set of all such functions $\hat{\mu}$ is denoted by $B(\Gamma)$. We may extend this idea to the random measures case. Niemi, H. [Nie75] has investigated the Fourier transform of stochastic processes. It is interesting to study when $Z_1 * Z_2$ implies $\hat{Z}_1 \cdot \hat{Z}_2$.

Problem 6.0.2. Let $M_c(G)$ denote the set of all continuous members of $M(G)$, and $M_d(G)$ denote the set of all discrete member of $M(G)$. $M_c(G)$ is a closed ideal of $M(G)$, and $M_d(G)$ is a closed subalgebra of $M(G)$. Detailed work for scalar valued measures are well-illustrated in Rudin [Rud62]. One may ask if there exist such a ideal and subalgebra for the algebra of random measures.

Problem 6.0.3. The algebra of random measures is a normed algebra. The Frechét variation is the norm in the random measure algebra. One can ask whether the algebra of random measures is complete.

Remark 6.0.4. In the literature, several authors considered convolution of functionals on infinite dimensional spaces equipped with Wiener measure which is translation invariant on an infinite dimensional "Wiener measure" spaces as an extension of the Lebesgue measure on $\mathbb{R}^n, 1 \leq n < \infty$. Thus the convolution defined for nonrandom functions with the method is structurally different from that used in the present work which involves a direct definition through covariance bimeasures. This important distinction should be kept in mind, and the above proposition is derived to clarify these differences. In what follows this procedure is to be kept the primary in all the work when the integrand and the integrator (i.e. measure) are both random.

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