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Orthogonal Partial Conformal Change

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Orthogonal Partial Conformal Change

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Juan Manuel Zaragoza

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Dedication

For my wonderful and loving wife... Bibiana Lopez.

ABSTRACT OF THE DISSERTATION

Orthogonal Partial Conformal Change

by

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University of California, Riverside, June 2012
Dr. Fred Wilhelm, Chairperson

If we start with a generic nonnegatively curved metric on a Riemannian manifold and apply a deformation to the metric, then nonnegative curvature is usually not preserved. From the perspective of systems of equations, the problem of finding a metric with a prescribed curvature tensor is a highly overdetermined system. However, if every zero curvature plane is tangent to a totally geodesic flat that is fixed by a deformation, then the problem of finding the prescribed metric becomes more manageable. In 2008 Petersen and Wilhelm exploited this idea and used it to develop a deformation called Orthogonal Partial Conformal Change (OPCC). Then they successfully combined it with other deformations to put a positively curved metric on the Gromoll-Meyer sphere.

Here we introduce new examples of manifolds that admit an OPCC deformation. In particular, we generalize a result obtained by Walschap on the canonical line bundle over the complex projective space. One peculiar characteristic of one of our OPCC deformations is that they are not verifiable via the machinery developed by Petersen and Wilhelm.

Contents

Chapter 1

Introduction

1.1 Background

If we start with a generic nonnegatively curved metric on a Riemannian manifold and apply a deformation to the metric, then nonnegative curvature is usually not preserved. From the perspective of systems of equations, the problem of finding a metric with a prescribed curvature tensor is a highly overdetermined system. However, if every zero curvature plane is tangent to a totally geodesic flat that is fixed by the deformation, then the problem of finding the prescribed metric becomes more manageable. In 2008 Petersen and Wilhelm exploited this idea and used it to develop a deformation called Orthogonal Partial Conformal Change [PetWilh3]. Then they successfully combined this new deformation with other tools, such as Cheeger deformations, to put a positively curved metric on the Gromoll-Meyer sphere.

The above reasons serve as an indication of why the number of deformations that preserve nonnegative curvature is very limited. Until a couple of years ago the only deformation with this property was the deformation introduced by Cheeger in 1972 [Cheeg].[1] Then in 1988 Walschap obtained a new type of deformation which made use of the properties of souls of manifolds [Wals]. More recently, in 1998 Guijarro obtained a deformation through a gluing construction [Guija].

The thesis is structured as follows. In Chapter 1 we review some curvature formulas for the families of spaces found in this thesis. We also briefly describe Cheeger deformations and explain

[1] Cheeger deformations were a generalization of a construction obtained by Berger. However, in these days people simply refer to deformations of this type as Cheeger deformations.

how they can help us remove flatness. In Chapter 2 we introduce some background results obtained by Petersen and Wilhelm regarding deformations of the metric in the C^2 and C^1 topologies. The formulas presented in this chapter will serve as the foundation for the rest of the thesis. In Chapter 3 we combine all our results to perform an OPCC deformation on $\mathbb{C}P^n\# -\mathbb{C}P^n$ and other connected sums. We also generalize Walschap's construction to other vector bundles. One peculiar characteristic of one of our OPCC deformations is that they are not verifiable via the machinery developed by Petersen and Wilhelm. Finally, in Chapter 4 we provide a list of potential future research projects.

1.2 Some Curvature Formulas

Every time we perturb a Riemannian metric we will be interested in describing the curvature of the new manifold. We use this section to review some curvature formulas for the types of manifolds that will be encountered in this thesis.

If (M,g) is a Riemannian manifold with Riemannian connection ∇, then the curvature tensor is defined by

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

on vector fields X, Y, Z. Using the metric g we can change it to a $(4,0)$-tensor, which is defined by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

The curvature tensor satisfies these basic properties [cite Pet]:

1. $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(Y,X,W,Z)$
2. $R(X,Y,Z,W) = R(Z,W,X,Y)$
3. $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$
4. $(\nabla_Z R)(X,Y)W + (\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W = 0$

To get sectional curvature we restrict R to pairs of vectors $v, w \in T_pM$ and write

$$\begin{aligned}
\sec(v,w) &= \frac{g(R(v,w)w,v)}{g(v,v)\,g(w,w) - g(v,w)^2} \\
&= \frac{g(R(v,w)w,v)}{(\text{Area of trapezoid spanned by } v \text{ and } w)^2}
\end{aligned}$$

If $\sigma \subset T_pM$ is a 2-dimensional plane and $v, w \in \sigma$ are linearly independent, then sec(v, w) =sec(σ) is independent of the choices of v, w [DoCarmo]. If sec(σ)= 0 for every plane tangent to M then we say that M is flat.

In this thesis we will use the notation $\operatorname{curv}(B, C) = R(B, C, C, B)$. The properties of R can be used to derive the following formulas:

$$\begin{aligned}
\operatorname{curv}(A, C + D) &= \operatorname{curv}(A, C) + \operatorname{curv}(A, D) + 2R(A, C, D, A) \\
\operatorname{curv}(A + B, C) &= \operatorname{curv}(A, C) + \operatorname{curv}(B, C) + 2R(A, C, C, B) \\
R(A, C + D, C + D, B) &= R(A, C, C, B) + R(A, C, D, B) + R(A, D, C, B) + R(A, D, D, B)
\end{aligned}$$

If M has nonnegative sectional curvature and span$\{A, C\}$ is a zero curvature plane, then Proposition 1 in [ProWilh] implies that $R(C, A) A = 0 = R(A, C) C$. Using this we simplify the above formulas as follows:

$$\begin{aligned}
\operatorname{curv}(A, C + D) &= \operatorname{curv}(A, D) \\
\operatorname{curv}(A + B, C) &= \operatorname{curv}(B, C) \\
R(A, C + D, C + D, B) &= R(A, C, D, B) + R(A, D, C, B) + R(A, D, D, B)
\end{aligned}$$

Using these equations we derive a formula for curv$(A + B, C + D)$

$$\begin{aligned}
\operatorname{curv}(A + B, C + D) &= \operatorname{curv}(A, C + D) + \operatorname{curv}(B, C + D) + 2R(A, C + D, C + D, B) \\
&= \operatorname{curv}(A, D) + \operatorname{curv}(B, C) + \operatorname{curv}(B, D) + 2R(B, C, D, B) \\
&\quad +2R(A, C, D, B) + 2R(A, D, C, B) + 2R(A, D, D, B)
\end{aligned}$$

From [Lee] we know that any plane $\sigma = span\{X, W\} \subset T_pM$ has an open neighborhood (in the Grassmannian topology) given by $U_\sigma = \{span\{X + Z, W + V\} \mid Z, V \in T_pM\}$. With this in mind let us now define the curvature polynomial [PetWilh3] centered at $\sigma = span\{X, W\}$.

Definition 1. *Fix vectors X,$W \in T_pM$. The curvature polynomial $P_M : T_pM \times T_pM \to \mathbb{R}$ is defined by $P_M(Z, V) = \operatorname{curv}(X + Z, W + V)$.*

Suppose span$\{X, W\}$ is a zero curvature plane and curv≥ 0. Then $P_M(Z,V)$ is given by

$$\begin{aligned} P_M(Z,V) &= \operatorname{curv}(X,V) + \operatorname{curv}(Z,W) + 2\left[R(X,W,V,Z) + R(X,V,W,Z)\right] \\ &\quad +2R(X,V,V,Z) + 2R(Z,W,V,Z) \\ &\quad +\operatorname{curv}(Z,V) \end{aligned} \tag{1.1}$$

1.2.1 Fundamental Equations

In Chapter 2 we will require curvature calculations that involve distance functions, as well as their gradients and Hessians. Here we provide all the essential formulas which can be found in chapter 2 in [Pet].

Let $r: U \to \mathbb{R}$ be a distance function where $U \subset (M, g)$ is an open subset of a Riemannian manifold. Let the gradient be denoted by $\nabla r = \partial r$. The level sets for r are $U_t = \{x \in U : r(x) = t\}$ and the induced metric on U_t is g_t. In this spirit ∇^t,R^t are the Riemannian connection and curvature on (U_t, g_t). The $(1,1)$ version of the Hessian of r is denoted by $S(.) = \nabla_. \partial r$, i.e. $Hessr(X,Y) = g(S(X), Y)$. Here S stands for the shape operator and when $X, Y \in U_t$ we also have

$$Hessr(X,Y) = g(S(X), Y) = II(X,Y)$$

where II is the second fundamental form. We now state the equations without proof.

The first equation is called the Radial Curvature Equation

$$\nabla_{\partial r} S + S^2 = -R(., \partial r)\partial r \tag{1.2}$$

The next two equations are called the Tangential Curvature Equation (Gauss Equation) and the Mixed Curvature Equation (Codazzi-Minardi Equation), respectively,

$$g(R(X,Y)Z,W) = g_t\left(R^t(X,Y)Z,W\right) - II(Y,Z)II(X,W) + II(X,Z)II(Y,W) \tag{1.3}$$

$$\begin{aligned} g(R(X,Y)Z,\partial r) &= g(-(\nabla_X T)(Y) + (\nabla_Y T)(X), Z) \\ &= -(\nabla_X II)(Y,Z) + (\nabla_Y II)(X,Z) \end{aligned} \tag{1.4}$$

Here X,Y,Z,W are tangent to the level sets U_t.

1.2.2 Submersions

Our second tool is the horizontal curvature formula for submersions [On1]. Recall that if $\pi:(M,\widetilde{g})\rightarrow(B,g)$ is a Riemannian submersion and X,Y,Z,H are vector fields tangent to B with horizontal lifts $\widetilde{X}$,$\widetilde{Y}$, $\widetilde{Z}$, $\widetilde{H}$, then

$$\begin{aligned} R^{B}(X,Y,Z,H) &= R^{M}\left(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{H}\right)-2\widetilde{g}\left(A_{\widetilde{X}}\widetilde{Y},A_{\widetilde{Z}}\widetilde{H}\right)+\widetilde{g}\left(A_{\widetilde{Y}}\widetilde{Z},A_{\widetilde{X}}\widetilde{H}\right) \\ &\quad +\widetilde{g}\left(A_{\widetilde{Z}}\widetilde{X},A_{\widetilde{Y}}\widetilde{H}\right) \end{aligned} \tag{1.5}$$

Here R^B, R^M are the curvature tensors of B, M,respectively and $A_{\widetilde{X}}\widetilde{Y}=\frac{1}{2}\left[\widetilde{X},\widetilde{Y}\right]^{Vert}$ is the integrability tensor for the horizontal distribution. We will refer to equation 1.5 as the "full horizontal curvature equation" for submersions. When we restrict it to orthonormal vectors x,$y\in T_pB$, then we get the well-known O'Neill's Horizontal Curvature Equation,

$$\sec_B(x,y)=\sec_M(\widetilde{x},\widetilde{y})+3\left\|A_{\widetilde{x}}\widetilde{y}\right\|^2 \tag{1.6}$$

Let P_B and P_M be the curvature polynomials of B and M. Then equation 1.5 can be used to express P_B in terms of P_M and A as follows. First, notice that if span$\{X,W\}$ is zero curvature plane in B, then equation 1.6 combined with $\mathrm{curv}_M\geq 0$ imply that

$$\begin{aligned} P_B(Z,V) &= \mathrm{curv}_B(X+Z,W+V) \\ &= \mathrm{curv}_M\left(\widetilde{X},\widetilde{V}\right)+\mathrm{curv}_M\left(\widetilde{Z},\widetilde{W}\right)+2\left[R^M\left(\widetilde{X},\widetilde{W},\widetilde{V},\widetilde{Z}\right)+R^M\left(\widetilde{X},\widetilde{V},\widetilde{W},\widetilde{Z}\right)\right] \\ &\quad +3\left\|A_{\widetilde{V}}\widetilde{X}+A_{\widetilde{W}}\widetilde{Z}\right\|^2+2\left[R^M\left(\widetilde{X},\widetilde{V},\widetilde{V},\widetilde{Z}\right)+R^M\left(\widetilde{Z},\widetilde{W},\widetilde{V},\widetilde{Z}\right)\right] \\ &\quad +6\widetilde{g}\left(A_{\widetilde{V}}\widetilde{Z},A_{\widetilde{V}}\widetilde{X}+A_{\widetilde{W}}\widetilde{Z}\right)+\mathrm{curv}_M\left(\widetilde{Z},\widetilde{V}\right)\ +3\left\|A_{\widetilde{Z}}\widetilde{V}\right\|^2 \\ &= \mathrm{curv}_M\left(\widetilde{X}+\widetilde{Z},\widetilde{W}+\widetilde{V}\right)+3\left\|A_{\widetilde{V}}\widetilde{X}+A_{\widetilde{W}}\widetilde{Z}+A_{\widetilde{V}}\widetilde{Z}\right\|^2 \\ &= P_M\left(\widetilde{Z},\widetilde{V}\right)+3\left\|A_{\widetilde{V}}\widetilde{X}+A_{\widetilde{W}}\widetilde{Z}+A_{\widetilde{V}}\widetilde{Z}\right\|^2 \end{aligned} \tag{1.7}$$

1.2.3 Compact Lie Groups and Homogenous Spaces

The curvature polynomial takes a simpler form when the Riemannian manifold is a compact Lie group with a bi-invariant metric. In this case, if $\langle . \rangle$ denotes the bi-invariant metric, then the curvature tensor has the form $R(X,Y,Z,W) = \frac{1}{4}\langle [X,Y], [W,Z]\rangle$ [Pet]. It then follows that $\mathrm{curv}(X,Y) = \frac{1}{4}\|[X,Y]\|^2$. In particular, we see that compact Lie groups with bi-invariant metrics are nonnegatively curved. Using equation 1.1, we can write P_M as

$$\begin{aligned}
P_M(Z,V) &= \frac{1}{4}\|[X,V]+[Z,W]\|^2 && \text{(quadratic term)} \\
&+\frac{1}{2}\langle [X,V]+[Z,W],[Z,V]\rangle && \text{(cubic term)} \\
&+\frac{1}{4}\|[Z,V]\|^2 && \text{(quartic term)} \\
&= \frac{1}{4}\|[X,V]+[Z,W]+[Z,V]\|^2 && \text{(compact form)}
\end{aligned} \tag{1.8}$$

If $\pi : (M, \langle . \rangle) \to (B, g)$ is a submersion and M is a Lie group, then we can combine equations 1.7 and 1.8 to write P_B as

$$\begin{aligned}
P_B(Z,V) &= P_M\left(\widetilde{Z},\widetilde{V}\right) + 3\left\|A_{\widetilde{V}}\widetilde{X} + A_{\widetilde{W}}\widetilde{Z} + A_{\widetilde{V}}\widetilde{Z}\right\|^2 \\
&= \frac{1}{4}\left\|\left[\widetilde{X},\widetilde{V}\right]+\left[\widetilde{Z},\widetilde{W}\right]+\left[\widetilde{Z},\widetilde{V}\right]\right\|^2 + 3\left\|A_{\widetilde{V}}\widetilde{X} + A_{\widetilde{W}}\widetilde{Z} + A_{\widetilde{V}}\widetilde{Z}\right\|^2
\end{aligned}$$

In addition, if M happens to be a normal homogeneous space G/H where G is compact with a bi-invariant metric, then we can simplify P_M even further

$$P_B(Z,V) = \frac{1}{4}\left\|\left[\widetilde{X},\widetilde{V}\right]+\left[\widetilde{Z},\widetilde{W}\right]+\left[\widetilde{Z},\widetilde{V}\right]+\left[\widetilde{X},\widetilde{V}\right]^{\mathrm{vert}}+\left[\widetilde{Z},\widetilde{W}\right]^{\mathrm{vert}}+\left[\widetilde{Z},\widetilde{V}\right]^{\mathrm{vert}}\right\|^2 \tag{1.9}$$

This formula shows that if the quadratic term of P_M vanishes, then so does the quadratic term of P_B. Using the language from chapter 2, this basically says that the A-tensor cannot be used to remove degeneracy in homogeneous spaces. The interaction between the A-tensor and zero curvature planes has been studied by several people (see [Wilk], [Tapp], and [ProWilh]) and the overall impression is that, in general, the A-tensor is not very useful for removing flatness. Later we'll explain the complete definition of degeneracy, as used in this paper, and its consequences in more detail. For now we should simply think of M being degenerate at p as having a plane $\mathit{span}\{X+Z, W+V\} \subset T_pM$

where curv($span\left\{X,W\right\}$) $=0$ and such that the quadratic term of $P_M\left(Z,V\right)$ vanishes, but not its quartic term.

Proposition 2. *Let $M=G/H$ be a normal homogeneous space described as above. If $\pi:M\longrightarrow B$ is a submersion such that the horizontal distribution is degenerate, then B is also degenerate.*

Proof. This is a straightforward application of equation 1.9. ■

1.2.4 Product Planes

Definition 3. *Let M and N be Riemannian manifolds and consider $M\times N$ with the product metric and let $T_{(p,q)}\left(M\times N\right)=T_{(p,q)}M\oplus T_{(p,q)}N$. Then a product plane is a plane $P=span\left\{x,y\right\}\subset T_{(p,q)}\left(M\times N\right)$ where $x\in T_{(p,q)}M$ and $y\in T_{(p,q)}N$.*

Notation. *In the next Proposition we make the following conventions:*

a. ∇^M,∇^N,∇ will denote the connections on manifolds M,N, and $M\times N$, respectively.

b. R^M,R^N,R will denote the curvature tensors on M,N, and $M\times N$, respectively.

We have the following basic results [On2].

Proposition 4. *Let $X,Y,Z\in\mathfrak{X}\left(M\right)$ and $U,V,W\in\mathfrak{X}(N)$ and let $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U},\widetilde{V},\widetilde{W}$ denote their respective lifts to $\mathfrak{X}\left(M\times N\right)$. Then*

1. $\nabla_{\widetilde{X}}\widetilde{Y}$ is the lift of $\nabla^M_XY\in\mathfrak{X}\left(M\right)$

2. $\nabla_{\widetilde{V}}\widetilde{W}$ is the lift of $\nabla^N_VW\in\mathfrak{X}\left(N\right)$

3. $\nabla_{\widetilde{V}}\widetilde{X}=0=\nabla_{\widetilde{X}}\widetilde{V}$

4. $R_{\widetilde{X}\widetilde{Y}}\widetilde{Z}=$ is the lift of $R^M_{XY}Z$

5. $R_{\widetilde{V}\widetilde{W}}\widetilde{U}$ is the lift of $R^N_{VW}U$

6. $R\equiv0$ for any other combination of $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U},\widetilde{V},\widetilde{W}$.

Proof. For (1)-(3) we can use the Koszul's formula and the fact that $[\widetilde{X},\widetilde{Y}]=\widetilde{[X,Y]}$. For (4)-(6) we use (1)-(3) and the definition of curvature $R\left(X,Y\right)Z=\nabla_Y\nabla_XZ-\nabla_X\nabla_YZ-\nabla_{[X,Y]}Z$. ■

This Proposition can be used to derive the following congruence

Lemma 5. *Let M,N be compact Riemannian manifolds with $sec>0$. Equip $M\times N$ with the product metric. Then the family of product planes and the family of zero curvature planes on $M\times N$ are the same.*

Proof. Using the Proposition above we clearly have that product planes have zero curvature. For the other direction write the curvature tensor as

$$\begin{aligned} R(X,Y,Y,X) &= R^{M}(\pi_1(X),\pi_1(Y),\pi_1(Y),\pi_1(X)) \\ &\quad + R^{N}(\pi_2(X),\pi_2(Y),\pi_2(Y),\pi_2(X)) \end{aligned}$$

where π_1 and π_2 are the projection maps onto M and N, respectively. Notice that if a plane has zero curvature, then its projections onto either factor must also have zero curvature making them one dimensional and hence our plane on $M \times N$ must be a product plane. ■

1.3 Cheeger Deformations

A Berger sphere is constructed by taking a 3-dimensional sphere with the round metric and shrinking it along the Hopf-fibers. To accomplish the shrinking, Berger employed isometric actions to deform the Hopf-fibers (see chapter 3 in [Pet]). One characteristic of this deformation is that they preserve the curvature nature of planes in the original metric. In other words, a plane with sec≥ 0 in the original metric remains nonnegatively curved in the new metric. Similarly, a plane with sec > 0 in the original metric remains positively curved. Berger's construction was later generalized by Cheeger and nowadays we refer to it simply as a Cheeger deformation. The complete description of Cheeger deformations and some applications are given in [Cheeg] and [Muet]. However, for practical purposes, we offer only the following exposition, which along with Propositions 6 and 7 can be found in [PetWilh3].

Let G be a compact group of isometries of (M, g_M), g_{bi} a bi-invariant metric on G, and consider the one parameter family $l^2 g_{\mathrm{bi}} + g_M$ of metrics on $G \times M$. G acts on $\left(G \times M, l^2 g_{\mathrm{bi}} + g_M\right)$ via

$$g(p,m) = (pg^{-1}, gm). \tag{1.10}$$

Modding out by the action 1.10 we obtain a one parameter family g_l of metrics on M. As $l \to \infty$, (M, g_l) converges to g_M.

The quotient map for the action (1.10) is

$$q_{G\times M}:(g,m)\mapsto gm.$$

The vertical space for $q_{G\times M}$ at (g,m) is

$$\mathcal{V}_{q_{G\times M}}=\{(-k,k)\mid k\in\mathfrak{g}\}$$

where the $-k$ in the first factor stands for the value at g of the Killing field on G given by the circle action

$$(\exp(tk),g)\mapsto g\exp(-kt)$$

and the k in the second factor is the value of the Killing field

$$m\longmapsto\frac{d}{dt}\exp(tk)m$$

on M at m.

In [Cheeg], [PetWilh1] there is a reparametrization of the tangent space, that we will call the Cheeger reparametrization. It is denoted by

$$Ch_l:TM\rightarrow TM$$

and defined as

$$Ch_l\left(v\right)=Dq_{G\times M}\left(\hat{v}_l\right),$$

where $\hat{v}_l\in TG\times TM$ is the horizontal vector for $q_{G\times M}:\left(G\times M,l^2g_{\mathrm{bi}}+g_M\right)\longrightarrow(M,g_l)$ that projects to v under the projection $\pi_2:G\times M\longrightarrow M$.

The following result shows how Cheeger deformations preserve positive curvature.

Proposition 6. *Let $\pi_1:G\times M\longrightarrow G$ be projection to the first factor.*

1 *If a plane P is positively curved with respect to g_M, then $Ch_l\left(P\right)$ is positively curved with respect to g_l.*

2 *If $curv_{g_{\mathrm{bi}}}\pi_1\left(\hat{P}_1\right)>0$, then $curv_{g_l}Ch_l\left(\hat{P}_l\right)>0$.*

Proof. If $\mathrm{curv}_{g_M}(P) > 0$, then so is $\mathrm{curv}_{l^2 g_{\mathrm{bi}}+g_M}\left(\hat{P}_l\right) > 0$. Since $Ch_l(P) = Dq_{G\times M}\left(\hat{P}_l\right)$, Part 1 follows from O'Neill's equation.

On the other hand, if

$$\begin{aligned}\hat{P}_1 &= \mathrm{span}\left\{\hat{v}, \hat{w}\right\}\\ &= \mathrm{span}\left\{(k_v, v), (k_w, w)\right\},\end{aligned}$$

then for arbitrary l,

$$\hat{P} = \mathrm{span}\left\{\left(\frac{k_v}{l^2}, v\right), \left(\frac{k_w}{l^2}, w\right)\right\}.$$

So

$$\begin{aligned}\mathrm{curv}_{(M,g_l)}\left(Ch_l(v), Ch_l(w)\right) &= \mathrm{curv}_{(M,g_l)}\left(Dq_{G\times M}\left(\frac{k_v}{l^2}, v\right), Dq_{G\times M}\left(\frac{k_w}{l^2}, w\right)\right)\\ &\geq \mathrm{curv}_{l^2 g_{\mathrm{bi}}}\left(\frac{k_v}{l^2}, \frac{k_w}{l^2}\right) + \mathrm{curv}_{g_M}(v, w)\\ &= \frac{1}{l^6}\mathrm{curv}_{g_{\mathrm{bi}}}(k_v, k_w) + \mathrm{curv}_{g_M}(v, w)\end{aligned}$$

Since both terms on the right are nonnegative we conclude that

$$\mathrm{curv}_{(M,g_l)}\left(Dq_{G\times M}\left(\frac{k_v}{l^2}, v\right), Dq_{G\times M}\left(\frac{k_w}{l^2}, w\right)\right) > 0.$$

■

The Cheeger reparametrization also preserves horizontal spaces of Riemannian submersions. This is a Corollary of the following (cf [PetWilh1]).

Proposition 7. *For all $u, w \in TM$*

$$g_M(u, w) = g_l(u, Ch_l(w)). \tag{1.11}$$

Proof. Starting with the left and side we take the horizontal lifts to $G \times M$

$$g_l(u, Ch_l(w)) = \left(l^2 g_{\mathrm{bi}} + g_M\right)\left((0, u) - (0, u)^{\mathcal{V}}, \hat{w}_l\right)$$

Since $\hat{w}_l$ is horizontal this becomes

$$\begin{aligned} g_l\left(u, Ch_l\left(w\right)\right) &= \left(l^2 g_{\mathrm{bi}} + g_M\right)\left(\left(0,u\right), \hat{w}_l\right) \\ &= g_M\left(u,w\right). \end{aligned}$$

■

Corollary 8. *Let $\pi : (M, g_M) \longrightarrow \Sigma$ be a Riemannian submersion of a compact manifold (M, g_M). Suppose that G acts isometrically on (M, g_M) and let $(M, g_l)_{l>0}$ be the Cheeger deformed metrics. Suppose that $\pi : (M, g_l) \longrightarrow \Sigma$ is a Riemannian submersion for all l. Let $\mathcal{H}_{\pi,g_M}$ and $\mathcal{H}_{\pi,g_l}$ be the horizontal spaces of π with respect to g_M and g_l. Then*

$$Ch_l\left(\mathcal{H}_{\pi,g_M}\right) = \mathcal{H}_{\pi,g_l}.$$

Proof. Just take u to be any vertical vector in equation 1.11. ■

Example 1. *Let $\left(\mathbb{R}^2, \overline{g}\right)$ be the Cheeger deformation of $\mathbb{R}^2$ obtained from the standard S^1 action on $\mathbb{R}^2$. Let s and g be the usual metrics on S^1 and $\mathbb{R}^2$, respectively. This new metric is positively curved and is a paraboloid asymptotic to a cylinder of radius 1.*

Example 2. *Let $\psi : [0, \pi] \longrightarrow [0, \infty)$ be a smooth, concave-down function. Let $g_\psi : dt^2 + \psi\left(t\right)^2 d\theta^2$ be a rotationally symmetric metric on $S^2 = [0, \pi] \times S^1$ (e.g. ψ is independent of θ). As before, S^1 acts isometrically on $\left(S^2, g_\psi\right)$, so we get a Riemannian submersion*

$$\left(S^2, g_\psi\right) \times S^1 \longrightarrow \left(S^2, \overline{g_\psi}\right)$$

where $\overline{g_\psi}$ is the metric induced by the submersion.

Chapter 2

Orthogonal Partial Conformal Change

In this chapter we will describe the deformation technique that is known as Orthogonal Partial Conformal Change (OPCC). Petersen and Wilhelm discovered it during a series of attempts to put a positively curved metric on the Gromoll-Meyer sphere. They eventually accomplished this breakthrough by combining several deformations, which included Cheeger deformations, OPCC deformations, and fiber scaling [PetWilh3]. The complete method used for the Gromoll-Meyer sphere is very long and this makes it hard to apply to an arbitrary manifold. A more practical approach is looking for applications of the individual deformation techniques. This observation became the inspiration for this paper.

This chapter is divided as follows. In the first section we introduce some results obtained by Petersen and Wilhelm that concern C^2-small deformations of a metric, and then we explain how they used them to generalize a deformation given in [Wals]. In the next section we present the OPCC in the context of C^1-small deformations. In this regard, Petersen and Wilhelm obtained some interesting results that allow the redistribution of curvatures while preserving the nonnegativity of the sectional curvature. In the last section, we describe the property of ε-nondegeneracy (which is required whenever we attempt to apply an OPCC). We will provide some methods for verifying this condition, and at the same time, we study two Lie groups that have a degenerate zero locus.

2.1 Main Definitions

Definition 9. *Let $\mathcal{D} \subset TM$ be a distribution and decompose the original metric as $g = g_{\mathcal{D}} + g_{\mathcal{D}^{\perp}}$.*
1. If we change $g_{\mathcal{D}}$ (by multiplying by a positive function) while fixing $g_{\mathcal{D}^{\perp}}$, then we have a ***partial conformal change****.*
2. An ***orthogonal partial conformal change*** *(OPCC) is obtained when all the zero curvature planes are orthogonal to the distribution $\mathcal{D}$.*

In [cite Walschap] Walschap applied a C^2-small deformation to the canonical line bundle over $\mathbb{C}P^1$ that preserved nonnegative curvature. However, the impact of this deformation on the curvature tensor is limited by the fact that the curvature tensor is continuous in the C^2-topology. This means that a C^2-small change in the metric produces a small change in the curvature tensor. On the other hand, the C^1-small deformations that we present in this paper can have a greater impact.

Example 3. *Let $\{g_t\}_{t\geq 0}$ be a family of complete, rotationally symmetric positively curved metrics on $\mathbb{R}^2$. Let (M, g) be nonnegatively curved. Then*

$$\left\{\left(M \times \mathbb{R}^2, g + g_t\right)\right\}_{t\geq 0}$$

is a deformation of $\left(M \times \mathbb{R}^2, g + g_0\right)$ preserving nonnegative curvature.

Example 4. *(Walschap) Taking $M = S^3$ with a round metric mod out by the diagonal S^1 action to obtain a family of nonnegatively curved metrics on the canonical line bundle over $\mathbb{C}P^1$. Notice that it is not necessary for the deformation to be small. In our language this is an OPCC on the canonical line bundle, and on $S^3 \times \mathbb{R}^2$ it is an OPCC with respect to the horizontal zero curvature planes of $S^3 \times \mathbb{R}^2 \longrightarrow \left(S^3 \times \mathbb{R}^2\right)/S^1$.*

Here we chose to present Walschap's deformation in a way that is different from the actual construction in [Wals]. This choice was made simply because it is easier to see.

2.2 C^2-Small Deformations

In this section, we prove Theorem 12 and Corollary 13, which provide an abstract framework for Walschap's example. In the even more narrowly constrained situation of the next section, we prove

Theorem 16 which gives a method to deform and preserve nonnegative curvature with a deformation that is only C^1-small.

For the remaining of this paper we assume that M is complete and nonnegatively curved. When Petersen and Wilhelm applied an OPCC to the Gromoll-Meyer sphere they introduced a list of eight axioms which we have labeled as OPC1, OPC2, ..., OPC8. To perform a C^2-small deformation we only require the first three axioms listed below:

OPC1 $O \subset M$ is a precompact open set,

OPC2 $\mathcal{S}$ is a family of totally geodesic flats with $\cup_{S \in \mathcal{S}} (S \cap O) = O$.

OPC3 assume that on O the tangent planes to the $S \in \mathcal{S}$ are *locally ε-nondegenerate* in the following sense.

Definition 10. *Let (M, g) be a nonnegatively curved manifold, $p \in M$, and $\mathcal{Z}_p$ be the family of zero curvature planes in T_pM.*

We say that M is nondegenerate at p if there is a number $\varepsilon > 0$ and an open neighborhood $\mathcal{N}_p$ of $\mathcal{Z}_p$ (in the Grassmannian at p) so that sec(σ)> ε for every plane σ in $T_pM \backslash \mathcal{N}_p$ and such that every plane in $\mathcal{N}_p \backslash\ \mathcal{Z}_p$ can be written as $\operatorname{span}\{X + Z, W + V\}$ *where* $|X| = |W| = 1$, $\operatorname{curv}(X, W) = 0$, $Z \perp \ker R(\cdot, W) W$, $V \perp \ker R(\cdot, X) X$, *and*

$$\operatorname{curv}(Z, W) + 2 (R(X, W, V, Z) + R(X, V, W, Z)) + \operatorname{curv}(X, V) > \varepsilon \left(|Z|^2 + |V|^2\right)$$

If M is nondegenerate at every point, then we say that M is nondegenerate. If there is an $\varepsilon > 0$ that works for all $p \in M$ and all $\mathcal{Z}_p$, then we say that M is ε-nondegenerate.

We start with the following lemma [PetWilh3]

Lemma 11. *(Converse Gauss Lemma) Let $\tilde{g}$ be another metric on M satisfying*

$$g(Z, \cdot) = \tilde{g}(Z, \cdot) : TO \longrightarrow \mathbb{R}$$

for all vectors Z tangent to a submanifold in $\mathcal{S}$. Then $S \cap O \subset (M, \tilde{g})$ is totally geodesic and flat for all $S \in \mathcal{S}$.

Proof. We set

$$T\mathcal{S} = \bigcup_{S \in \mathcal{S}} TS \text{ and } T_p\mathcal{S} = \bigcup_{S \in \mathcal{S}} T_pS.$$

Let Z be a unit vector field on O with values in $T\mathcal{S}$. Let V a field on O that is normal to $T\mathcal{S}$. Our assumptions imply that Z is also a unit field for $\tilde{g}$ and V is also normal to $T\mathcal{S}$ with respect to $\tilde{g}$. Using that all $S \in \mathcal{S}$ are totally geodesic for g, we have

$$\begin{aligned} 0 &= g\left(\nabla_Z V, Z\right) \\ &= g\left(\left[Z, V\right], Z\right) \\ &= \tilde{g}\left(\left[Z, V\right], Z\right) \\ &= \tilde{g}\left(\tilde{\nabla}_Z V, Z\right) \end{aligned}$$

showing that they are also totally geodesic for $\tilde{g}$.

Since the intrinsic metric on members of $\mathcal{S}$ does not change, each $S \in \mathcal{S}$ remains flat. ■

To explain the name of the lemma, let (M, g) be an arbitrary Riemannian manifold. Let $\mathcal{S}$ be the collection of minimal geodesics emanating from a fixed point $p \in M$, and let O be the open dense set on which $dist\,(p, \cdot) : M \longrightarrow \mathbb{R}$ is smooth. If $\tilde{g}$ agrees with g on the tangent field to the radial g-geodesics from p, then they are also $\tilde{g}$–geodesics.

As an application of the Converse Gauss Lemma we have [PetWilh3]

Theorem 12. *Let M, O, and $\mathcal{S}$ be as above, and assume that there is an $\varepsilon > 0$ so that $T_p\mathcal{S}$ satisfies the* local ε-nondegeneracy *condition for all $p \in O$.*

Let $\tilde{g}$ be another metric so that on O

$$g\left(Z, \cdot\right) = \tilde{g}\left(Z, \cdot\right) : TO \longrightarrow \mathbb{R}$$

for all $Z \in T\mathcal{S}$.

There is a neighborhood $\widetilde{\mathcal{N}}$ of $T\mathcal{S}|_O$ so that $(M, \tilde{g})$ is nonnegatively curved on $\widetilde{\mathcal{N}}$ with precisely the same zero curvature planes as g, provided $\tilde{g}$ is sufficiently close to g in the C^2–topology.

Proof. Since we have the local ε–nondegeneracy condition at each point $p \in O$, we have a neighborhood $\mathcal{N}_p$ of $T_p\mathcal{S}$ such that any plane in $\mathcal{N}_p$ has the form span$\left\{X + Z, W + V\right\}$ where

- span$\{X, W\} \in T_p\mathcal{S}$

- $|X| = |W| = 1$

$$\operatorname{curv}(Z,W) + 2\left(R(X,W,V,Z) + R(X,V,W,Z)\right) + \operatorname{curv}(X,V) > \varepsilon\left(|Z|^2 + |V|^2\right) \qquad (2.1)$$

Set $\mathcal{N} \equiv \cup_{p\in O}\mathcal{N}_p$. If necessary, we restrict $\mathcal{N}$ and get that there is a $\delta > 0$ so that

- all planes in $\mathcal{N}$ have the above form, $\operatorname{span}\{X+Z, W+V\}$, with $|Z|, |V| < \delta$.

For each $\{X,W\} \in T_p\mathcal{S}$ with $|X| = |W| = 1$ we let $\mathcal{ND}_{X,W} = \{(Z,V) \in T_pM \times T_pM \,|\, 2.1 \text{ holds}\}$.

- It follows that the functions

$$\begin{aligned} \operatorname{curv}_{X,W} \quad &: \quad \mathcal{ND}_{X,W} \longrightarrow \mathbb{R}, \\ (Z,V) \quad &\longmapsto \quad \operatorname{curv}(X+Z, W+V) \end{aligned}$$

are nonnegative and vanish only if $Z = V = 0$ or $\max\{|Z|, |V|\} > \delta$.

The Theorem will follow if we can show there is a $\tilde{\delta} > 0$ so that for all choices of $\{X,Z,W,V\}$ as above all of the functions

$$(Z,V) \longmapsto \widetilde{\operatorname{curv}}(X+Z, W+V)$$

are nonnegative for $|Z|, |V| < \tilde{\delta}$ and only vanishes when $Z = V = 0$, provided $\tilde{g}$ is sufficiently C^2 close to g.

Because X and W are tangent to a 0–curvature plane in a nonnegatively curved manifold

$$R(X,W)W = R(W,X)X = 0.$$

Since they are also tangent to a totally geodesic flat that is preserved under our deformation we have

$$\tilde{R}(X,W)W = \tilde{R}(W,X)X = 0.$$

So both g and $\tilde{g}$ satisfy

$$\operatorname{curv}(X,W) = R(X,W,W,Z) = R(W,X,X,Z) = 0$$

If $\tilde{g}$ is sufficiently close to g, then for $|Z|, |V| < \delta$ we have that

$$\widetilde{\mathrm{curv}}(Z,W) + 2\left(\tilde{R}(X,Z,V,W) + \tilde{R}(X,V,Z,W)\right) + \widetilde{\mathrm{curv}}(X,V) > \varepsilon\left(|Z|^2 + |V|^2\right).$$

If the C^2–distance between g and $\tilde{g}$ is σ, then

$$\left|\tilde{R}(A,B,C,D) - R(A,B,C,D)\right| \leq K\sigma |A| |B| |C| |D|,$$

for some $K > 0$.

So

$$\begin{aligned}\widetilde{\mathrm{curv}}(X+Z,W+V) \quad &\geq \quad \varepsilon\left(|Z|^2 + |V|^2\right) \\ &\quad -2K\sigma\left(|Z| |V|^2 + |Z|^2 |V| + |Z|^2 |V|^2\right)\end{aligned}$$

The right hand side is positive if $0 < |Z| + |V| < \tilde{\delta}$, and σ and $\tilde{\delta}$ are sufficiently small. ■

The previous result is local (within the Grassmannian). On the other hand, it does state that nonnegative curvature is globally preserved. So, for example, it is not an abstraction of Theorem 2.1 in [Wals]. To remedy this we offer the following Corollary, which will not be used in the sequel.

Corollary 13. *Let M, O, $\mathcal{S}$, and $\tilde{g}$ be as in Theorem 12. In addition assume that every zero plane in M is in $T\mathcal{S}$, and there is an $\varepsilon > 0$ so that on O, $T\mathcal{S}$ satisfies the ε-*nondegeneracy *condition. Then $(M, \tilde{g})$ is nonnegatively curved provided $\tilde{g}|_{M\setminus O} = g|_{M\setminus O}$ and $\tilde{g}$ is sufficiently close to g in the C^2-topology.*

Proof. In addition to the structure from the proof of Theorem 12 we have that $\sec_g > \varepsilon$ on the complement of $\mathcal{N}$. Thus $\sec_{\tilde{g}} > \varepsilon$ on the complement of $\widetilde{\mathcal{N}}$, provided the C^2–distance between g and $\tilde{g}$ is sufficiently small. Since we have all of the hypotheses of Theorem 12, we also have that $(M, \tilde{g})$ is nonnegatively curved on $\widetilde{\mathcal{N}}$ with precisely the same zero curvature planes as (M, g), provided $\tilde{g}$ is sufficiently close to g in the C^2-topology. ■

2.3 C^1-Small Deformations

Although Theorem 12 and Corollary 13 give us a means to deform nonnegative curvature, it is impossible for them to have a large quantitative effect on the curvature tensor. This is because the deformations discussed are C^2-small.

On the other hand, Theorem 16, which we prove in this section, holds for certain deformations that are only C^1-small.

In addition to the standing axioms of the previous section, for this section we assume the following.

OPC4 Let $f : O \longrightarrow \mathbb{R}$ be the distance function from a closed submanifold N defined on the open set O.

OPC5 Assume that O has compact closure and f is smooth on the closure of O. Define

$$\mathrm{II}\,(U,V) = -\frac{1}{2}\,(L_X g)\,(U,V) = -g\,(\nabla_U X, V) = -g\,(T\,(U)\,, V)$$

OPC6 Set $X \equiv \nabla f$. Suppose that everywhere on $O - N$, X is tangent to every $S \in \mathcal{S}$.

OPC7 Suppose there is a distribution $\mathcal{D}$ on $O - N$ that is perpendicular to each $S \in \mathcal{S}$ such that along each integral curve for X the distribution $\mathcal{D}$ becomes tangent to N as it approaches N

OPC8 Let

$$\varphi \equiv \varphi \circ f$$

where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_+$ is smooth and assume that $|\varphi - 1|$ is supported in $O - N$.

Let $\tilde{g}$ be obtained from g by multiplying the lengths of all vectors in $\mathcal{D}$ by φ, while keeping the orthogonal complement and the metric on the orthogonal complement of $\mathcal{D}$ fixed. That is, $\tilde{g}$ is obtained from g by doing a partial conformal change. The reason for the assumption about the behavior of $\mathcal{D}$ near N is that it keeps the Hessian of f bounded on $\mathcal{D}$.

We start by estimating how curvatures change on $O - N$. The following two lemmas will be crucial

Lemma 14. *If $U,V,W,Z\perp X$ are vectors in T_pM with $p\in O-N$, then*

$$\begin{aligned}\tilde{R}(U,V,W,Z)-R(U,V,W,Z) &= O\left(|\varphi-1|(1+|\mathrm{II}|)+|D_X\varphi|^2\right)|U|_g|V|_g|W|_g|Z|_g\\ &+\varphi(D_X\varphi)\left(g\left(V^{\mathcal{D}},W^{\mathcal{D}}\right)\mathrm{II}(U,Z)+g\left(U^{\mathcal{D}},Z^{\mathcal{D}}\right)\mathrm{II}(V,W)\right)\\ &-\varphi(D_X\varphi)\left(g\left(V^{\mathcal{D}},Z^{\mathcal{D}}\right)\mathrm{II}(U,W)+g\left(U^{\mathcal{D}},W^{\mathcal{D}}\right)\mathrm{II}(W,Z)\right)\end{aligned} \tag{2.2}$$

$$\tilde{R}(U,V,W,X)-R(U,V,W,X) = O(|D_X\varphi|+|\varphi-1|)|U|_g|V|_g|W|_g+O(|\varphi-1|)|\mathrm{II}||U|_g|V|_g|W|_g \tag{2.3}$$

$$\begin{aligned}\tilde{R}(U,X,X,U)-R(U,X,X,U) &= -\varphi(D_XD_X\varphi)g\left(U^{\mathcal{D}},U^{\mathcal{D}}\right)\\ &+O(|\varphi-1|+|D_X\varphi|)|U|_g^2+O(|\varphi-1|)|\mathrm{II}||U|_g^2\end{aligned} \tag{2.4}$$

Remark. *These equations show that the only curvatures that are possibly affected by second derivatives are of the form $R(U,X,X,U)$. Even for those the second derivative for φ will only appear if $U^{\mathcal{D}}$ doesn't vanish. These estimates can be made more explicit if we include bounds for $\nabla g|_{\mathcal{D}}$, $L_X(g|_{\mathcal{D}})$ and and $\nabla_X L_X(g|_{\mathcal{D}})$. Note that in any case these terms remain bounded as we have assumed that $\mathcal{D}$ stays tangent to N along all gradient curves for X.*

We also need the following basic property for the second fundamental form near N.

Lemma 15. *Let*

$$T(O-N)=\operatorname{span}\{X\}\oplus\mathcal{T}\oplus\mathcal{N}$$

be an orthogonal splitting on $O-N$ with the property that for any geodesic $\gamma:[0,b]\longrightarrow O$ with $\gamma(0)\in N$, $\dot{\gamma}(t)=X_{\gamma(t)}$, $t>0$ we have

$$\lim_{t\to 0}\mathcal{T}_{\gamma(t)}=T_{\gamma(0)}N$$

Then there is a constant $C>0$ so that on $O-N$ we have

$$\left|\mathrm{Hess}f-\frac{1}{f}g|_{\mathcal{N}}\right|<C$$

Proof. The function $\frac{1}{2}f^2$ is smooth on O with N being the minimum set. Furthermore at any $p \in N$

$$\mathrm{Hess}\left(\frac{1}{2}f^2\right)_p = g|_{\nu_p N}$$

where $T_pO = T_pN \oplus \nu_p N$ is the splitting into tangent and normal spaces for N. Our assumptions guarantee that

$$\lim_{t \to 0} \mathrm{span}\left\{\dot{\gamma}(t)\right\} \oplus \mathcal{N}|_{\gamma(t)} = \nu_{\gamma(0)} N$$

Thus

$$\frac{1}{f}\left(\mathrm{Hess}\left(\frac{1}{2}f^2\right) - df^2 - g|_{\mathcal{N}}\right) = \mathrm{Hess} f - \frac{1}{f} g|_{\mathcal{N}}$$

is bounded on O. ■

Using the two previous lemmas Petersen and Wilhelm obtained the following result

Theorem 16. *Assume $X, W \in S \subset \mathcal{S}$, $V \perp W$ and $Z \perp W + V$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if*

$$\begin{aligned} & \mathrm{curv}(X+Z, W+V) - \varphi D_X D_X \varphi \left|V^{\mathcal{D}}\right|^2 + \varphi D_X \varphi \mathrm{II}(Z,Z)\left|V^{\mathcal{D}}\right|^2 + \varphi D_X \varphi \mathrm{II}(V,V)\left|Z^{\mathcal{D}}\right|^2 \\ \geq\ & \varepsilon\left(|Z|^2 + |V|^2 + |Z|^2|V|^2\right), \end{aligned}$$

and

$$\begin{aligned} |\varphi - 1| + |D_X\varphi| &< \delta, \\ |1-\varphi|\,|\mathrm{II}| &< \delta \end{aligned}$$

then

$$\widetilde{\mathrm{curv}}(X+Z, W+V) > \frac{\varepsilon}{4}\left(|Z|^2 + |V|^2 + |Z|^2|V|^2\right).$$

Proof. By the Converse Gauss Lemma, $\tilde{R}(X,W)W = \tilde{R}(W,X)X = 0$. Therefore

$$\begin{aligned} \widetilde{\mathrm{curv}}(X+Z, W+V) &= \widetilde{\mathrm{curv}}(X,V) + 2\left(\tilde{R}(X,W,V,Z) + \tilde{R}(X,V,W,Z)\right) + \widetilde{\mathrm{curv}}(Z,W) \\ &\quad +2\left(\tilde{R}(Z,W,V,Z) + \tilde{R}(X,V,V,Z)\right) + \widetilde{\mathrm{curv}}(Z,V) \end{aligned}$$

Lemma 14 together with $W^{\mathcal{D}} = 0$ and $\nabla_W X = 0$ implies

$$\begin{aligned}\widetilde{\operatorname{curv}}\left(X+Z,W+V\right) &= \operatorname{curv}\left(X+Z,W+V\right)-\varphi D_X D_X \varphi \left|V^{\mathcal{D}}\right|^2 \\ &\quad +\varphi D_X \varphi \left|V^{\mathcal{D}}\right|^2 \mathrm{II}\left(Z,Z\right)+\varphi D_X \varphi \left|Z^{\mathcal{D}}\right|^2 \mathrm{II}\left(V,V\right) \\ &\quad -2\varphi D_X \varphi g\left(Z^{\mathcal{D}},V^{\mathcal{D}}\right)\mathrm{II}\left(Z,V\right) \\ &\quad -O\left(\left|D_X\varphi\right|+\left|\varphi-1\right|\right)\left(\left|Z\right|^2+\left|V\right|^2\right) \\ &\quad -O\left(\left|\varphi-1\right|\right)\left|\mathrm{II}\right|\left(\left|Z\right|^2+\left|V\right|^2\right)\end{aligned}$$

Now using that $\mathrm{II}\left(Z,V\right)=\mathrm{II}\left(Z,V+W\right)$ the condition $Z\perp W+V$ shows, by the previous lemma, that this second fundamental form is uniformly bounded on O. Thus we obtain

$$\widetilde{\operatorname{curv}}\left(X+Z,W+V\right)\geq\frac{\varepsilon}{2}\left(\left|Z\right|^2+\left|V\right|^2+\left|Z\right|^2\left|V\right|^2\right)-O\left(\delta\right)\left(\left|Z\right|^2+\left|V\right|^2\right)$$

■

The deformations we introduced in this chapter depend on whether the approximations for the metric take place in the C^1- and C^2-topologies. We saw that a C^1-small deformation is preferred because it may have a greater impact on the curvature tensor. All this work culminated in Theorem 16 which provides a method to deform the metric globally while preserving nonnegative curvature. The importance of this Theorem is that it can be used to perform both types of deformations. To show how this is done we offer the following overview:

Type I If we have ε-*nondegeneracy*, $1-\varphi$ has compact support in O, and $\left|1-\varphi\right|$ is sufficiently small in the C^2-topology, then we preserve nonnegative curvature in compact subsets. This follows from Theorem 16 because on compact subsets of O the second fundamental form is bounded. Thus, we can make $\varphi D_X D_X \varphi, \varphi D_X \varphi \mathrm{II}\left(Z,Z\right),$ and $\varphi D_X \varphi \mathrm{II}\left(V,V\right)$ sufficiently small by choosing $\left|1-\varphi\right|$ sufficiently small in the C^2-topology.

Type II If $1-\varphi$ has compact support in O, $\left|1-\varphi\right|$ is sufficiently small in the C^1-topology, and we replace ε-*nondegeneracy* with the inequality

$$\begin{aligned}&\operatorname{curv}\left(X+Z,W+V\right)-\varphi D_X D_X \varphi\left|V^{\mathcal{D}}\right|^2 \\ \geq\ &\varepsilon\left(\left|Z\right|^2+\left|V\right|^2+\left|Z\right|^2\left|V\right|^2\right),\end{aligned}$$

then we preserve nonnegative curvature in compact subsets of O.

Type III For the points near the singularity (e.g. where we give up compactness of the support of $1-\varphi$) we assume that $|1-\varphi|$ is sufficiently small in the C^1-topology, $\varphi' < 0$ near the singularity, and $|1-\varphi|\,|II|$ is C^0-small. Then we preserve nonnegative curvature. The hypothesis $\varphi' < 0$ is necessary to make $\varphi D_X \varphi \mathrm{II}\,(Z, Z)$ and $\varphi D_X \varphi \mathrm{II}\,(V, V)$ positive.

Notice that deformations of types I and II are local while those of type III are global. Theorem 16 is the result of combining all the conditions from types II and III. In chapter 3 we will perform deformations of types I and II on some connected sums of projective spaces. We won't be able to apply a deformation of type III in our examples because the axiom OPC7 is not applicable. Nevertheless, we can still apply a global OPCC by means of O'Neill's horizontal curvature equation.

2.4 Epsilon Nondegenerate Riemannian Manifolds

In this section we study two methods for constructing *ε-nondegenerate* manifolds. We then apply these methods to find examples of spaces that exhibit this property, and at the same time, we also study two Lie groups that fail to satisfy this condition. The ability to apply an OPCC depends on our ability to check that conditions OPC1-OPC8 hold and from this list ε-nondegeneracy is one of the hardest to satisfy. This is why we have reserved this entire section to understand this property better.

First we show how to write open neighborhoods in the Grassmannian topology in a "nicer" way. By this we mean a form that allows us to make curvature calculations more manageable. In the argument below we will use the notation $A_{B^\perp}$ to denote the orthogonal projection of A onto the complement of B.

Proposition 17. *Let σ =span$\{X, W\}$ where X and W are orthogonal and let U_σ be a neighborhood of σ in the Grassmannian topology. Then we can "orthogonalize" U_σ and write each element $\sigma' \in U_\sigma$ as σ' =span$\{X + Z, W + V\}$ where V and Z are orthogonal to X and W.*

Proof. We show this in two steps. First, suppose σ' is a plane close to σ that contains W. Then we can write it as σ' =span$\{X + Z, W\}$. We claim that we can choose Z such that $Z \perp X, W$. To see this write $Z = \lambda W + Z_{W^\perp} = \lambda W + \lambda' X + (Z_{W^\perp})_{X^\perp}$. Then we get

$$\begin{aligned}
\sigma' &= \operatorname{span}\{X + Z, W\} \\
&= \operatorname{span}\left\{X + \lambda W + \lambda' X + (Z_{W^\perp})_{X^\perp}, W\right\} \\
&= \operatorname{span}\left\{\left(1 + \lambda'\right) X + (Z_{W^\perp})_{X^\perp}, W\right\} \\
&= \operatorname{span}\left\{X + \overline{Z}, W\right\}
\end{aligned}$$

where $\overline{Z} = \frac{1}{1+\lambda'} (Z_{W^\perp})_{X^\perp}$ is orthogonal to both X and W. Now, any plane near span$\{X, W\}$ is near a plane that contains W and hence has the form span$\{X + Z, W + V\}$ where $Z \perp X, W$. Write $V = \delta W + V^\perp + \delta' X$ where $V^\perp \perp \operatorname{span}\{X, W\}$. Then

$$\operatorname{span}\{X + Z, W + V\} = \operatorname{span}\left\{X + Z, (1 + \delta) W + V^\perp + \delta' X\right\}$$

Dividing the second vector by $(1 + \delta)$ and redefining $V^\perp + \delta' X$ we get

$$\operatorname{span}\{X + Z, W + V\} = \operatorname{span}\left\{X + Z, W + V^\perp + \delta' X\right\}.$$

Subtracting the correct multiple of $X + Z$ from $W + V^\perp + \delta' X$ gives

$$\operatorname{span}\{X + Z, W + V\} = \operatorname{span}\left\{X + Z, W + V^\perp + \delta'' Z\right\}$$

Since $Z \perp X, W$ our plane obtains the desired form by setting $V = V^\perp + \delta'' Z$. ■

The first method for constructing ε-*nondegeneracy* involves Riemannian products.

Lemma 18. *Let M,N be compact Riemannian manifolds with sec > 0. Then the family of zero curvature planes on $M \times N$ with the product metric is ε-nondegenerate for some $\varepsilon > 0$.*

Proof. Let σ =span$\{X, W\} \subset T_{(p,q)}(M \times N)$ be a product plane where $|X| = |W| = 1$. Let σ' =span$\{X + Z, W + V\}$ be a plane close to σ. From Proposition 17 we can assume V and Z are orthogonal to σ. Write $V = V_1 + V_2$ and $Z = Z_1 + Z_2$ where V_1,$Z_1 \in T_p(M)$ and $V_2, Z_2 \in T_q(N)$. Then

$$\sigma' = \operatorname{span}\{X + Z_1 + Z_2, W + V_1 + V_2\}$$

has curvature

$$
\begin{aligned}
\operatorname{curv}(\sigma') &= \operatorname{curv}(X, V_1) + 2R(X, V_1, V_1, Z_1) + \operatorname{curv}(Z_1, V_1) \\
&\quad + curv(Z_2, W) + 2R(Z_2, W, V_2, Z_2) + curv(V_2, Z_2) \\
&= \|V_1\|^2 \sec\{X, V_1\} + \|Z_2\|^2 \sec\{Z_2, W\} + O\left(\|V_1\|^2 \|Z_1\| + \|V_2\| \|Z_2\|^2\right) \\
&\geq \|V_1\|^2 \min\sec_M + \|Z_2\|^2 \min\sec_N
\end{aligned}
$$

where $\min\sec_M$, $\min\sec_N$ are the minimum sectional curvatures of M,N respectively. To complete the proof we let $\varepsilon = \min\{\min\sec_M,\ \min\sec_N\}$. ■

Next we look at some submersions that preserve nondegeneracy. In the following Theorem R^B and R^M denote the curvature tensors of B and M, respectively, and if X is a vector tangent to B then we let $\widetilde{X}$. We'll use the same notation for liftings of planes. We also use the notation $\mathcal{H}$ and $\mathcal{V}$ to denote the horizontal and vertical components of X so that $X = X^{\mathcal{V}} + X^{\mathcal{H}}$.

Theorem 19. *Suppose that $\pi : M \to B$ is a Riemannian submersion such that horizontal zero curvature planes are projected to zero curvature planes. If the horizontal distribution is ε-nondegenerate at p, then B is ε-nondegenerate at $\pi(p)$.*

Proof. Let Z_M and Z_B denote the zero loci of $T_p(M)$ and $T_{\pi(p)}(B)$, respectively, and let $Z_M^{\mathcal{H}} = Z_M \cap \mathcal{H}$. Since $Z_M^{\mathcal{H}}$ is *ε-nondegenerate* there is a $\varepsilon > 0$ and a open neighborhood $\mathcal{N}_p \supset Z_M^{\mathcal{H}}$ (in the Grassmannian at p) satisfying the ε-nondegenerate conditions. We claim that $d\pi(\mathcal{N}_p) \supset Z_B$ is the neighborhood that induces *ε-nondegeneracy* on B. Indeed, for any plane $P \notin d\pi(\mathcal{N}_p)$ its lift $\widetilde{P} \notin \mathcal{N}_p$ satisfies $\sec_M\left(\widetilde{P}\right) > \varepsilon$ and by O'Neill's formula we must have $\sec_B(P) > \varepsilon$. Since we assumed that zero curvature planes project to zero curvature planes we have $d\pi\left(Z_M^{\mathcal{H}}\right) \subset Z_B$. Now let $P = \operatorname{span}\{X + Z, W + V\} \in d\pi(\mathcal{N}_p) \setminus Z_B$ where $\operatorname{curv}_B(X, W) = 0$, $Z \perp \ker R(\cdot, W) W$, and $V \perp \ker R(\cdot, X) X$. Using O'Neill's horizontal curvature equation we lift these conditions to $\operatorname{curv}_M\left(\widetilde{X}, \widetilde{W}\right) = 0$, $\widetilde{Z} \perp \ker R^M\left(\cdot, \widetilde{W}\right)\widetilde{W}$, $\widetilde{V} \perp \ker R^M\left(\cdot, \widetilde{X}\right)\widetilde{X}$. Now using O'Neill's full horizontal curvature equation we can write the quadratic term of $\operatorname{curv}_B(X + Z, W + V)$ as

$$
\operatorname{curv}_M\left(\widetilde{X}, \widetilde{V}\right) + 2\left(R^M\left(\widetilde{X}, \widetilde{W}, \widetilde{V}, \widetilde{Z}\right) + R^M\left(\widetilde{X}, \widetilde{V}, \widetilde{W}, \widetilde{Z}\right)\right) + \operatorname{curv}_M\left(\widetilde{Z}, \widetilde{W}\right) + 3\|A_V X + A_W Z\|^2
$$

Since $Z_M^{\mathcal{H}}$ is ε*-nondegenerate* we have

$$\operatorname{curv}_M\left(\widetilde{X},\widetilde{V}\right)+2\left(R^M\left(\widetilde{X},\widetilde{W},\widetilde{V},\widetilde{Z}\right)+R^M\left(\widetilde{X},\widetilde{V},\widetilde{W},\widetilde{Z}\right)\right)+\operatorname{curv}_M\left(\widetilde{Z},\widetilde{W}\right)>\varepsilon\left(\left|\widetilde{Z}\right|^2+\left|\widetilde{V}\right|^2\right)$$

and this implies

$$\operatorname{curv}_B(Z,W)+2\left(R^B(X,W,V,Z)+R^B(X,V,W,Z)\right)+\operatorname{curv}_B(X,V)>\varepsilon\left(|Z|^2+|V|^2\right)$$

■

Remark. *A surprising feature of ε-nondegeneracy is that even if M is ε-nondegenerate, then B is not necessarily ε-nondegenerate. We will see examples of this in chapter 3 where we will explain how this limits the type of OPCC deformations we can apply. On the other hand, we will show that if M is nondegenerate, then $\pi(M)$ must be nondegenerate.*

Corollary 20. *If $\pi : M \to B$ is a Riemannian submersion that projects zero curvature planes to zero curvature planes, then π preserves nondegeneracy, i.e. B is nondegenerate whenever M is nondegenerate.*

Proof. First we notice that for any subspace $\mathcal{Y} \subset T_pM$ there is $\varepsilon(\mathcal{Y})>0$ such $\mathcal{Y}$ is $\varepsilon(\mathcal{Y})$-nondegenerate. To finish we simply choose $\mathcal{Y}$ to be the horizontal distribution and repeat the argument of Theorem 19. ■

Corollary 21. *Let M, N be compact positively curved manifolds and let G act principally on $M\times N$. Then the zero locus of the quotient $(M\times N)/G$ is nondegenerate.*

Proof. Since M and N have positive curvature then under the product metric every horizontal zero curvature plane is tangent to a flat. Since the action on $M\times N$ is principal we can apply Theorem 4 in [ProWilh] to conclude that horizontal zero curvature planes are projected to zero curvature planes in the quotient. To finish we apply Corollary 20. ■

We can actually obtain a more general result, one that applies to products of nonnegatively curved manifolds. More precisely we have,

Corollary 22. *Let M,N be compact nonnegatively curved manifolds. Let $\pi : M\times N \to B$ be a Riemannian submersion such that the horizontal zero curvature planes are product planes and* *they're projected to zero curvature planes. Then the zero locus of the horizontal distribution is nondegenerate.*

Proof. The proof follows along the same lines as in Lemma 18. If $span\left\{X,W\right\}$ is a zero curvature plane, then nearby planes can be written $P' = span\left\{X+Z_1+Z_2, W+V_1+V_2\right\}$ where V_1,$Z_1 \in \mathfrak{X}\left(M\right)$ and $V_2, Z_2 \in \mathfrak{X}\left(N\right)$. Its curvature is given by

$$\begin{aligned} \operatorname{curv}\left(P'\right) &= \operatorname{curv}\left(X,V_1\right)+2R\left(X,V_1,V_1,Z_1\right)+\operatorname{curv}\left(Z_1,V_1\right) \\ &\quad +curv\left(Z_2,W\right)+2R\left(Z_2,W,V_2,Z_2\right)+curv\left(V_2,Z_2\right) \end{aligned}$$

From the hypothesis we have $\operatorname{curv}\left(X,V_1\right)>0$ and $curv(Z_2,W)>0$. We finish the proof by repeating the argument of Lemma 18. ■

Remark. *In all of the above results we rely on the fact that the submersion $\pi: M \to B$ sends horizontal zero curvature planes to zero curvature planes. Some methods to verify this condition are discussed in [ProWilh].*

2.4.1 Examples

Now that we have some tools for checking ε-nondegeneracy, we are going to apply them to a family of connected sums. All of these manifolds will exhibit ε-nondegeneracy, which as we know depends on the footpoint. In fact, in chapter 3 we will discover that $\varepsilon \to 0$ as we approach a certain subset N of the connected sum. After doing this we will revisit Walschap's example.

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Example 5. *(Complex) Here we view $\mathbb{C}P^n\#-\mathbb{C}P^n$ as $\left(S^{2n-1}\times S^2\right)/S^1$, where S^1 acts freely on S^{2n-1} (the Hopf action) and by rotations on S^2 [Tot]. We equip $S^{2n-1}\times S^2$ with the product metric and $\mathbb{C}P^n\#-\mathbb{C}P^n$ with the quotient metric induced by the submersion $\pi: S^{2n-1}\times S^2 \to \mathbb{C}P^n\#-\mathbb{C}P^n$. Under this action all the horizontal zero curvature planes are tangent to flat tori $S^1\times S^1$ (e.g. they exponentiate to flats) and therefore we can apply Corollary 21.*

Example 6. *(Real and Quaternionic) A similar argument yields* nondegeneracy *for $\mathbb{R}P^n\#-\mathbb{R}P^n = \left(S^{n-1}\times S^1\right)/S^0$ and $\mathbb{H}P^n\#-\mathbb{H}P^n = \left(S^{4n-1}\times S^4\right)/S^3$.*

Example 7. *(Octonionic) Now consider* $\mathbf{Ca}P^2\# - \mathbf{Ca}P^2$ *which according to [Tot] can be written as the quotient* $\left(Spin\left(9\right) \times S^8\right)/Spin\left(8\right)$. *Here* $Spin\left(8\right) \rightharpoonup Spin\left(9\right)$ *is the standard inclusion and* $Spin\left(8\right)$ *acts on* $S^8 \subset \mathbb{R}^9$ *by the direct sum of an 8-dimensional real spin representation and the trivial representation. By Corollary 22 it suffices to show that the horizontal zero curvature planes are product planes. Let* $Spin\left(8\right)$ *act on* $Spin\left(9\right)$ *and* S^8 *as described above. Let* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *be the decomposition of the Lie algebra of* $Spin\left(9\right)$ *where* $\mathfrak{h}$ *denotes the subalgebra generated by* $Spin\left(8\right)$ *and* $\mathfrak{m}$ *is its orthogonal complement. Using that* $Spin\left(7\right) \subset Spin\left(8\right)$ *we can write* $\mathfrak{h} = \mathfrak{h}_7 \oplus \mathfrak{m}_7$ *where* $\mathfrak{h}_7$ *is the Lie algebra generated by* $Spin\left(7\right)$ *and* $\mathfrak{m}_7$ *is its orthogonal complement in* $\mathfrak{h}$. *We know* $Spin\left(9\right)/Spin\left(8\right) = S^8$ *and therefore we can write* $\mathfrak{m} = span\left\{m_1, m_2, m_3, ..., m_8\right\}$. *Now let* $\left\{\partial r, \partial\theta_1, \partial\theta_2, ..., \partial\theta_7\right\}$ *be a basis for* $T\left(S^8\right)$ *where the* $\partial\theta_i$*'s are tangent to the orbits of the* $Spin\left(8\right)$ *action and* ∂r *is orthogonal to the orbit. The horizontal distribution of the submersion* $\pi : Spin\left(9\right) \times S^8 \to Spin\left(9\right) \times S^8/Spin\left(8\right)$ *is given by*

$$\left\{\left(0, \partial r\right), \left(m_1, 0\right), \left(m_2, 0\right), .., \left(m_8, 0\right), \left(a_1\partial\overline{\theta}_1, b_1\partial\theta_1\right), \left(a_2\partial\overline{\theta}_2, b_2\partial\theta_2\right), .., \left(a_7\partial\overline{\theta}_7, b_7\partial\theta_7\right)\right\}$$

where $\partial\overline{\theta}_i \in \mathfrak{m}_7$ *for all* i *and* a_i, b_i *are numbers that depend on the footpoint. First, let* P *be a plane with a 2-dimensional projection onto the space*

$$span\left\{\left(m_1, 0\right), ..., \left(m_8, 0\right), \left(a_1\partial\overline{\theta}_1, b_1\partial\theta_1\right), ..., \left(a_7\partial\overline{\theta}_7, b_7\partial\theta_7\right)\right\}$$

By construction $m_i \in \mathfrak{m}$ *and* $a_i\partial\overline{\theta}_i \in \mathfrak{m}_7$ *are orthogonal to the* $Spin\left(7\right)$*-orbits and in [GrovZil2] it was proved that* $Spin\left(9\right)/Spin\left(7\right) = S^{15}$. *This implies that (under the product metric)* $curv\left(P\right) > 0$, *and therefore every horizontal zero-curvature plane must have* $\left(0, \partial r\right)$ *as one of its spanning vectors. Now if* P *has a two-dimensional projection onto*

$$span\left\{\left(0, \partial r\right), \left(a_1\partial\overline{\theta}_1, b_1\partial\theta_1\right), \left(a_2\partial\overline{\theta}_2, b_2\partial\theta_2\right), .., \left(a_7\partial\overline{\theta}_7, b_7\partial\theta_7\right)\right\}$$

then we can use $Spin\left(9\right)/Spin\left(8\right) = S^8$ *to conclude that* $curv\left(P\right) > 0$. *In conclusion, every horizontal zero-curvature plane must be of the form* $span\left\{\left(m_i, 0\right), \left(0, \partial r\right)\right\}$ *and hence is a product plane.*

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Example 8. *(Generalized Walschap) Proceeding as in example 4 we let* S^1 *be the usual action on* $(\mathbb{R}^2, g_t)$ *where* g_t *is a rotationally symmetric positively curved metric. Let* S^1 *act on* S^{2n+1} *by the Hopf-action so that* $S^{2n+1}/S^1 = \mathbb{C}P^n$. *Next we mod out* $S^{2n+1} \times \mathbb{R}^2$ *by the diagonal* S^1*-action to obtain a family of nonnegatively curved metrics on the canonical line bundle over* $\mathbb{C}P^n$. *Then nondegeneracy follows immediately from Corollary 21.*

Improving Nondegeneracy

In order to apply an OPCC deformation we need ε-nondegeneracy. Here we show how we can verify this condition for the connected sums discussed above. Let k =dim($\mathbb{K}$) and let $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$, **Ca** for $k = 2, 4, 8$, respectively. Let $M = \mathbb{K}P^n \# - \mathbb{K}P^n$ for $k = 2, 4$ and let $M = \mathbf{Ca}P^2 \# - \mathbf{Ca}P^2$ for $k = 8$. Recall that $\mathbb{K}P^n \# - \mathbb{K}P^n = \left(S^{kn-1} \times S^k\right)/S^{k-1}$ for $k = 2, 4$ and $\mathbb{K}P^2 \# - \mathbb{K}P^2 = \left(Spin\,(9) \times S^8\right)/Spin\,(8)$ for $k = 8$. Under this correspondence the set $\mathcal{Z}$ of zero curvature planes of M are projections of horizontal zero curvature planes of the form span$\{(0, \partial r), (W, 0)\}$. If $k = 2, 4$ then W denotes a vector that is orthogonal to the S^{k-1}-action on S^{kn-1} and when $k = 8$ then W denotes a vector orthogonal to the $Spin\,(8)$-action on $Spin\,(9)$. For $k = 2, 4$ let $\{*, -*\}$ denote the two fixed points in S^k under the S^{k-1}-action. Finally, let $\overline{O} = S^{kn-1} \times S^k - S^{kn-1} \times \{*, -*\}$ and $O = \pi\left(\overline{O}\right)$ where $\pi : S^{kn-1} \times S^k \to \mathbb{K}P^n \# - \mathbb{K}P^n$ is the canonical submersion. We construct an open set $O \subset \mathbf{Ca}P^2 \# - \mathbf{Ca}P^2$ in a similar way.

Claim 23. *Let* M,$\mathcal{Z}$, *and* $O \subset M$ *be as above. Then for any compact subset* $C \subset O$ *there is an* $\varepsilon\,(C) > 0$ *so that* $\mathcal{Z}|_C$ *is* $\varepsilon\,(C)$*–nondegenerate.*

Proof. Suppose this is not the case. Then there is a zero curvature plane spanned by $X = \partial r$ and W (with W as described above) and sequences $\{p_i\}_{i=1}^{\infty} \subset C, \{X_i\}_{i=1}^{\infty}, \{W_i\}_{i=1}^{\infty}, \{Z_i\}_{i=1}^{\infty}$, and $\{V_i\}_{i=1}^{\infty}$ such that

$$\begin{aligned} p_i &\to p \\ X_i &\to X, \\ W_i &\to W, \\ Z_i &\to Z, \\ V_i &\to V \end{aligned}$$

with $\text{curv}(X_i, W_i) = 0$, $X_i \perp Z_i$, $V_i \perp W_i$, $Z_i \perp \ker R(\cdot, W_i) W_i$, $V_i \perp \ker R(\cdot, X_i) X_i$ and

$$\text{curv}(Z_i, W_i) + 2(R(X_i, Z_i, V_i, W_i) + R(X_i, V_i, Z_i, W_i)) + \text{curv}(X_i, V_i) < \frac{1}{i}\left(|Z_i|^2 + |V_i|^2\right).$$

Since $\ker R(\cdot, W_i) W_i \to \ker R(\cdot, W) W$ and $\ker R(\cdot, X_i) X_i \to \ker R(\cdot, X) X$ then we have $Z \perp \ker R(\cdot, W) W$ and $V \perp \ker R(\cdot, X) X$. If we combine this with the equation

$$\text{curv}(Z, W) + 2(R(X, Z, V, W) + R(X, V, Z, W)) + \text{curv}(X, V) = 0$$

then this contradicts the nondegeneracy of $\mathcal{Z}|_O$. ■

2.4.2 Some Riemannian Manifolds with Degenerate Zero Loci

The previous subsection was devoted to Riemannian manifolds that have an ε-*nondegenerate* zero locus. Here we provide two manifolds that fail to satisfy this condition, and at the same time, we try to describe possible ways to fix their degeneracy.

Let M be a compact Lie group with bi-invariant metric $\langle , \rangle$ and let $\mathfrak{g}$ be its Lie algebra. Let $ad_X : \mathfrak{g} \to \mathfrak{g}$ be the linear map given by $ad_X(Y) = [X, Y]$. Recall that the curvature polynomial on M is given by

$$\begin{aligned} P_M(Z, V) &= \frac{1}{4}\|[X, V] + [Z, W]\|^2 && \text{(quadratic term)} \\ &\quad + \frac{1}{2}\langle [X, V] + [Z, W], [Z, V]\rangle && \text{(cubic term)} \\ &\quad + \frac{1}{4}\|[Z, V]\|^2 && \text{(quartic term)} \\ &= \frac{1}{4}\|[X, V] + [Z, W] + [Z, V]\|^2 && \text{(compact form)} \end{aligned}$$

For planes that are very close to $\text{span}\{X, W\}$ the quadratic term of P_M dominates the rest of the terms and this is the term we track when we apply an OPCC. When this quadratic term vanishes, then we lose control of the deformation. Manifolds for which this situation occurs will be called degenerate. We can use the formula $R(X, W, Z) = \frac{1}{4}\langle [X, Y], [W, Z]\rangle$ to translate the conditions

$$\text{curv}(X, W) = Z \perp \ker R(\cdot, W) W, and V \perp \ker R(\cdot, X) X$$

to

$$[X,W]=0, Z\perp\ker(ad_W), V\perp\ker(ad_X)$$

Thus, for compact Lie groups the conditions of degeneracy can be expressed in the following way.

Proposition 24. *A compact Lie group with a bi-invariant metric is* degenerate *if we can find elements V,W,X,Z in the Lie algebra of M satisfying*

D1. $[X,W]=0$

D2. $[X,V]=-[Z,W]$ *with* $V\perp\ker(ad_X)$ *and* $Z\perp\ker(ad_W)$

D3. $[Z,V]\neq 0$

In the examples that follow we will equip every Lie algebra with the bi-invariant metric given by $\langle A,B\rangle = trace\left(AB^T\right).$

Example 9. *The Special Unitary Group $SU(3)$.*

We show that the zero locus of $SU(3)$ is degenerate *by finding V,W,X,Z satisfying conditions (D1)-(D3). Recall that the Lie algebra of $SU(3)$ is given by*

$$\mathfrak{su}(3)=\{A\in\mathfrak{gl}(3,\mathbb{C}) : A^*=-A \textit{ and } trace(A)=0\}$$

where A^* is the conjugate transpose of A. In [Cahn] a basis for $\mathfrak{su}(3)$ is given by

$$\begin{bmatrix}0&i&0\\i&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}i&0&0\\0&-i&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&i\\0&0&0\\i&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&i\\0&i&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix},$$
$$\begin{bmatrix}i&0&0\\0&i&0\\0&0&-2i\end{bmatrix}.$$

Let $X=\begin{bmatrix}0&i&0\\i&0&0\\0&0&0\end{bmatrix}$, $W=\begin{bmatrix}i&0&0\\0&i&0\\0&0&-2i\end{bmatrix}$, $V=\begin{bmatrix}0&0&i\\0&0&0\\i&0&0\end{bmatrix}$, $Z=\frac{1}{3}\begin{bmatrix}0&0&0\\0&0&i\\0&i&0\end{bmatrix}_x$. *Then we ob-*

tain $[X,W]=0,\ [X,V]=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}=-\ [Z,W],$ *and* $[Z,V]=\frac{1}{3}\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}$. *We have*

$$\begin{aligned}\ker(ad_W) &= span\left\{W,\begin{bmatrix}0&i&0\\i&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}i&0&0\\0&-i&0\\0&0&0\end{bmatrix}\right\},\\ \ker(ad_X) &= span\left\{\begin{bmatrix}i&0&0\\0&i&0\\0&0&-2i\end{bmatrix}\right\}\end{aligned}$$

Using the metric $\langle A,B\rangle = trace\left(AB^T\right)$ *we see that* $V\perp\ \ker ad_X$ *and* $Z\perp\ \ker ad_W$.

Example 10. *The Symplectic Group* $Sp(2)$.

The Lie algebra of $Sp(2)$ *can be described as*

$$\mathfrak{sp}(2)=\left\{\begin{bmatrix}a&b\\-\bar{b}&d\end{bmatrix}\in\mathfrak{gl}(2,\mathbb{H}):\mathrm{Re}(a)=\mathrm{Re}(d)=0\right\}$$

A basis for $\mathfrak{sp}(2)$ *is given by*

$$\begin{bmatrix}i&0\\0&0\end{bmatrix},\begin{bmatrix}j&0\\0&0\end{bmatrix},\begin{bmatrix}k&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&i\end{bmatrix},\begin{bmatrix}0&0\\0&j\end{bmatrix},\begin{bmatrix}0&0\\0&k\end{bmatrix},\begin{bmatrix}0&1\\-1&0\end{bmatrix},\begin{bmatrix}0&i\\i&0\end{bmatrix},\begin{bmatrix}0&j\\j&0\end{bmatrix},\begin{bmatrix}0&k\\k&0\end{bmatrix}.$$

Let $X=\begin{bmatrix}j&0\\0&0\end{bmatrix},\ W=\begin{bmatrix}0&0\\0&i\end{bmatrix},\ Z=\begin{bmatrix}0&i\\i&0\end{bmatrix},\ V=\begin{bmatrix}0&j\\j&0\end{bmatrix}$. *Then we have* $[X,W]=0$,

$[X,V]=[Z,W]$, *and* $[Z,V]=2\begin{bmatrix}k&0\\0&k\end{bmatrix}$. *An easy calculation shows that*

$$\begin{aligned}ker(ad_X) &= span\left\{X,\begin{bmatrix}0&0\\0&\alpha\end{bmatrix}\right\},\\ ker(ad_W) &= span\left\{W,\begin{bmatrix}\alpha&0\\0&0\end{bmatrix}\right\}\end{aligned}$$

where $\alpha\in\{i,j,k\}$. *Therefore, we have* $V\perp ker(ad_X)$ *and* $Z\perp ker(ad_W)$.

Cheeger deformations have the effect of removing zero curvature planes and this makes them very useful for removing degeneracy. In the case of $Sp(2)$ we can use a Cheeger deformation to make the plane span$\left\{\begin{bmatrix} j & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}\right\}$ be positively curved.

Example 11. *Cheeger Deformation on $Sp\left(2\right)$.*
Let $X = \begin{bmatrix} j & 0 \\ 0 & 0 \end{bmatrix}$, $W = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$. Let $Sp\left(1\right) = S^3$ act on $S^3 \times Sp\left(2\right)$ by the action given by

$$\begin{array}{rcl} S^3 \times S^3 \times Sp\left(2\right) & \longrightarrow & S^3 \times Sp\left(2\right) \\ \left(p, q, B\right) & \longmapsto & \left(p\overline{q}, B \begin{bmatrix} \overline{q} & 0 \\ 0 & \overline{q} \end{bmatrix}\right) \end{array}$$

Let g_{bi}, g be the metrics on S^3 and $Sp\left(2\right)$, respectively, and let $Q : \left(S^3 \times Sp\left(2\right), g_{bi} + g\right) \longrightarrow \left(Sp\left(2\right), g_{cheeg}\right)$ be the corresponding submersion where g_{cheeg} denotes the Cheeger deformed metric. The vertical space of Q is spanned by $\left\{\left(y_\alpha, \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\right) : \alpha \in \{i, j, k\}\right\}$ where y_α is the killing field on S^3 associated to $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$. Let $Ch(\sigma)$ =span$\left\{DQ\left(\widehat{X}\right), DQ\left(\widehat{W}\right)\right\}$ be the Cheeger reparametrization of σ =span$\{X, W\}$. Here $\widehat{X} = \left(k_X, X\right) \in T_*\left(S^3 \times Sp\left(2\right)\right)$ is horizontal with respect to Q and hence $\left(k_X, X\right) \perp \left(y_\alpha, \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}\right)$. If we let $\alpha = j$, then a simple calculation shows that $k_X = -j$. Similarly we have that $\widehat{W} = \left(k_W, W\right) = \left(-i, W\right)$ and hence the projection of $\widehat{\sigma}$ =span$\left\{\widehat{X}, \widehat{W}\right\}$ onto S^3 is a positively curved plane. Therefore, by Proposition 6 we have that σ is positively curved in $\left(Sp\left(2\right), g_{cheeg}\right)$.

Remark. *Since $\left(Sp\left(2\right), g_{cheeg}\right)$ is a normal homogeneous space then Proposition 2 tells us that the only possible way to remove degeneracy via a Cheeger deformation is by removing zero curvatures.*

Chapter 3

Examples of Orthogonal Partial Conformal Changes

This chapter is devoted to provide examples of OPCC deformations. We recall that there are three types of deformations that can be applied to a manifold (see the comments after Theorem 16) and these can be local or global. For convenience we list here all the conditions that are required to apply an OPCC as employed by Peterson and Wilhelm:

OPC1 $O \subset M$ is a precompact open set.

OPC2 $\mathcal{S}$ is a family of totally geodesic flats with $\cup_{S\in\mathcal{S}} (S \cap O) = O$.

OPC3 We assume that on O the tangent planes to the $S \in \mathcal{S}$ are *ε-nondegenerate*.

OPC4 Let $f : O \longrightarrow \mathbb{R}$ be the distance function from a closed submanifold N defined on O.

OPC5 f is smooth on the closure of O. Define

$$\mathrm{II}\,(U, V) = -\frac{1}{2}\,(L_X g)\,(U, V) = -g\,(\nabla_U X, V) = -g\,(T\,(U)\,, V)$$

OPC6 Set $X \equiv \nabla f$. Suppose that everywhere on $O - N$, X is tangent to every $S \in \mathcal{S}$.

OPC7 Suppose there is a distribution $\mathcal{D}$ on $O - N$ that is perpendicular to each $S \in \mathcal{S}$ such that along each integral curve for X the distribution $\mathcal{D}$ becomes tangent to N as it approaches N

OPC8 Let

$$\varphi \equiv \varphi \circ f$$

where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_+$ is smooth and assume that $|\varphi - 1|$ is supported in $O - N$.

In the next section we will apply deformations of type I and II to some connected sums of projective spaces. We won't be able to use the full strength of Theorem 16 since axiom OPC7 does not hold for these spaces. However, we will still be able to apply an OPCC via the O'Neill' horizontal curvature equation.

3.1 Connected Sums of Projective Spaces

Example 12. *Let k =dim($\mathbb{K}$) and let $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$,* **Ca** *for $k = 2, 4, 8$, respectively. Recall that for $k = 2, 4$ we can view $\mathbb{K}P^n \# - \mathbb{K}P^n$ as the image of the submersion $\pi : S^{kn-1} \times S^k \longrightarrow \left(S^{kn-1} \times S^k\right) / S^{k-1}$. Similarly for $k = 8$ we view $\mathbb{K}P^2 \# {-}\mathbb{K}P^2$ as $\left(Spin\,(9) \times S^8\right) / Spin\,(8)$. Let $k \in \{2, 4\}$ and let $\{*, -*\}$ denote the two fixed points in S^k under the S^{k-1} action. Let $\overline{N} = S^{kn-1} \times \{*\}$ and $\overline{O} = S^{kn-1} \times S^k - S^{kn-1} \times \{*, -*\}$. Consider the distance function dist:$\left(\overline{N}, .\right) : \overline{O} \longrightarrow \mathbb{R}$ and take $N = \pi\left(\overline{N}\right) = \mathbb{K}P^{n-1}$ and $O = \pi\left(\overline{O}\right)$. Then the required distance function on O becomes f =dist$(N, .)$ (e.g. f is the distance from one copy of $\mathbb{K}P^{n-1}$ in $\mathbb{K}P^n \# - \mathbb{K}P^n$). A similar construction can be obtained for the case $k = 8$ where f is the distance from $N = \mathbb{K}P^1 = S^8$.*

Now let $X = \nabla f$. By an argument somewhat similar to example 6 it can be proved that all zero curvature planes in $\mathbb{K}P^n \# - \mathbb{K}P^n$ are projections of product planes (in the total space of the corresponding submersion). In fact they all contain X and by Theorem 2 in [ProWilh] exponentiating these planes will yield a family of flats that cover $\mathbb{K}P^n \# - \mathbb{K}P^n$.

The level sets $U_r = \{p \in \mathbb{K}P^n \# - \mathbb{K}P^n : f\,(N, p) = r\}$ are Berger spheres where the vector fields spanning the Hopf-fibers are determined by the vector fields on $S^{kn-1} \times S^k$ that are tangent to the S^{k-1}-orbits. Thus, if $\{\partial\theta_1, .., \partial\theta_{k-1}\}$ span the Hopf-fibers of U_r and we take a sub-distribution $\mathcal{D} \subset$span$\,\{\partial\theta_1, .., \partial\theta_{k-1}\}$, then $\mathcal{D}$ will be orthogonal to each zero curvature plane. With all these conditions in place now we are ready to apply an OPCC.

*From Claim 23 in chapter 2 we know that these connected sums are ε-*nondegenerate *in compact subsets of O. Thus, on these sets, we obtain a uniform ε that can used to apply OPCC deformations of types I and II.*

However, condition OPC7 does not hold; the distribution $\mathcal{D}$ doesn't become tangent to N. This means that we cannot apply the full power of the machinery developed by Petersen and Wilhelm. Fortunately, we can still perform a global OPCC deformation. To do this we first equip the second factor S^k in the total space with a positively curved rotationally symmetric metric. Then O'Neill's horizontal curvature equation can be used to show that the deformed metric remains nonnegatively curved.

3.2 Vector Bundles

Example 13. *If we replace S^k with $\mathbb{R}^k$ in the connected sums above, then we obtain more examples of manifolds that admit an OPCC. We do this by equipping $\mathbb{R}^k$ with a rotationally symmetric metric and letting S^{k-1} act on $\mathbb{R}^k$ by rotations. Indeed, the case $k = 2 = n$ gives us the canonical line bundle over $\mathbb{C}P^1$ that we introduced in Example 4.*

Example 14. *Using Theorem 2.4 from [cite Walschap] we can extend our work from the canonical line bundle over $\mathbb{C}P^1$ to arbitrary 2-dimensional bundles. Let $\pi : E \longrightarrow \mathbb{C}P^1$ be any 2-dimensional vector bundle over $\mathbb{C}P^1 = S^2$ where $\mathbb{C}P^1$ has a positive metric. Then there exist a family of metrics of nonnegative curvature on E, each of which has a soul isometric to $\mathbb{C}P^1$, with totally geodesic fibers. Furthermore, the zero curvature planes have the form span$\{\partial r, W\}$ where W is horizontal.*

Chapter 4

Future Research

Petersen and Wilhelm applied an OPCC deformation to the Gromoll-Meyer sphere in the presence of submersions [PetWilh3]. So it is natural to ask how all our work can be generalized to this setting. In particular, we would like to find out how an OPCC can be used to deform biquotients and homogenous spaces. All of the examples we introduced above fall into this category. Another group of potential candidates is the family of cohomogeneity-one manifolds. A cohomogeneity-one manifold is a connected manifold M that admits an action by a compact Lie group G such that M/G is one-dimensional. In [GrovZil1] we already found one of these manifolds that seems like a potential candidate for an OPCC. The space is S^4 and the Lie group is $SO(3)$ which acts by conjugate multiplication.

In the examples of the vector bundles, all the zero-curvature planes were tangent to Perelman flat submanifolds. A Perelman flat is obtained when we take a vector tangent to the soul of the manifold and a vector in its normal bundle, and then exponentiate the plane spanned by these two vectors. Therefore, the next obvious step is to consider manifolds whose zero curvature planes satisfy this property.

All the examples we presented in this thesis admit a local OPCC of types I and II. Although we couldn't apply a type III OPCC using the machinery, we were still able to apply a global OPCC by means of the O'Neill's horizontal curvature equation. This immediately raised the question of how can we adapt axioms OPC1-OPC8 to cover a bigger class of manifolds.

As another area for further research we would like to find conditions on the distribution $\mathcal{D}$ such that a constant Partial Conformal Change preserves sec≥ 0. Any advance made in this direction might shed some light about constructions of collapsing sequences of Riemannian manifolds that have sec≥ 0.

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