Title

Mapping the QCD phase diagram with statistics-friendly distributions

Journal

Physical Review C, 100(5)

ISSN

2469-9985

Authors

Bzdak, Adam
Koch, Volker

Publication Date

2019-11-01

DOI

10.1103/physrevc.100.051902

Peer reviewed

Mapping the QCD phase diagram with statistics friendly distributions

Adam Bzdak[1,*] and Volker Koch[2,†]

[1] *AGH University of Science and Technology, Faculty of Physics and Applied Computer Science, 30-059 Kraków, Poland*
[2] *Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA, 94720, USA*

We demonstrate that the multiplicity distribution of a system located in the vicinity of a first-order phase transition can be successfully measured in terms of its factorial cumulants with a surprisingly small number of events. This finding has direct implications for the experimental search of a QCD phase transition conjectured to be located in the high baryon density region of the QCD phase diagram.

One of the key questions of the physics of strong interactions is the possible existence of a first-order phase transition accompanied by a critical point. While lattice QCD has established that the transition at vanishing net-baryon density is an analytic cross over [1], the presence of a first-order transition accompanied by a critical point has been conjectured based on many model calculations (see e.g. [2, 3] for a review). To search for such a possible transition in experiment, fluctuations of conserved charges in relativistic heavy ion collisions have been considered as promising probes [4–18]. Special attention has been paid to the cumulants of the net-baryon or net-proton[1] number distribution as they are particularly sensitive to the details of the transition from hadron gas to quark-gluon plasma in the cross-over region [7, 14] as well as near a potential critical point [6]. This sensitivity is expected to increase with the order of the cumulant [6], the measurement of which is commonly believed to require increasing statistics.

In this paper we show, quite generally, that it requires surprisingly few events to determine if a system is located close to a first-order phase transition. This finding has direct implications on the search for the QCD phase transition, but will also be relevant for any other (mesoscopic) systems where fluctuation measurements are meaningful. It is well known that the multiplicity distribution of a system close to a first-order phase transition is a two-component or bi-modal distribution reflecting the two (dense and dilute) phases. If the system is right at the transition it has two maxima of equal magnitude, reflecting the equal probability of the two phases. As one moves away from the transition, one of the maxima becomes smaller, reflecting the fact that away from the transition one phase is much more probable than the other. Thus, for small systems and not too far from the transition, the presence of the other phase still shows up in the multiplicity distribution (for a detailed discussion, see [24]). As discussed in [24], such a two-component multiplicity distribution, even in the case when one of the components is rather small, has a very characteristic behavior of its factorial cumulants: with increasing order they increase rapidly in magnitude with alternating sign (in contrast, ultrarelativistic quantum molecular dymamics (UrQMD) calculations give higher order factorial cumulants consistent with zero [25]). This characteristic may be used to establish the existence of a two-component multiplicity distribution, which in turn would provide strong evidence that the system is close to a first-order phase transition.[2]

Such a characterization requires factorial cumulants of many orders that are commonly believed to require large statistics. However, as we show, the two-component distributions relevant for a first-order phase transition are remarkably statistics friendly in the sense that for a given and rather limited number of events factorial cumulants can be reliably extracted to a surprisingly high order. Surprisingly, this is even the case if the second mode (component) is rather tiny and is difficult to see directly in the multiplicity distribution. This finding, therefore, demonstrates that a search for a first-order phase transition via fluctuation measurements is practically feasible and does not require unrealistic levels of statistics.

In the following we illustrate our findings in the context of preliminary results of the STAR Collaboration. However, our arguments are quite general and are not restricted to the QCD phase transition. The preliminary results from the STAR Collaboration for the ratio of fourth-order over second-order (net)-proton cumulants show an intriguing pattern [27]. It grows rapidly with decreasing beam energy from $\sqrt{s} = 19.6$ GeV reaching a large value at 7.7 GeV. It was argued [26] that this behavior is caused by a strong increase of multi-proton correlations with decreasing energy. In addition it was found [24], that at the lowest energy, $\sqrt{s} = 7.7\,\text{GeV}$, where the deviation of the cumulants from a Poisson baseline

[1] Experimentally, one is usually restricted to the measurement of cumulants of the net-proton distribution [19–21] since neutrons are difficult to measure. However, as shown in [22, 23] given fast isospin-exchange processes due to the abundance of pions the connection to the net-baryon number cumulants can be made.

[2] A two-component distribution could in principle also result from different effects, such as production vs. stopping of protons, problems with centrality determination, deuteron enhancement in some events, possible issues with a detector etc. However, it seems these effects should be also visible at, say, 19.6 GeV, where the higher order factorial cumulants are consistent with zero [26] and a two-component distribution is not visible.

(or rather binomial due to baryon conservation) are the largest, the first four (factorial) cumulants, so far measured by STAR, are consistent with a two-component proton multiplicity distribution, albeit with the second component being rather small. Of course the first four cumulants are not enough to sufficiently constrain the multiplicity distribution. Therefore, it is essential to measure (factorial) cumulants of higher order to either confirm or rule out that the underlying distribution is indeed a two-component one consistent with a first-order phase transition. As we show this is possible due to the "statistics friendly" properties of these two-component distributions even for the very limited statistics of the present STAR data set.

Specifically, in this paper we study various proton multiplicity distributions to evaluate the statistical errors of higher order factorial cumulants. In our studies we choose a rather small number of events, approximately 150000 (144393 to be more precise [28]), which is the statistics underlying the STAR measurement for the most central Au+Au collisions at $\sqrt{s} = 7.7$ GeV at RHIC [20, 27]. We will only consider multiplicity distributions of one species of particles, which are protons in our case.[3]

To evaluate the statistical errors numerically, we sample the number of protons, N, $n_{\text{events}} = 144393$ times from a given multiplicity distribution $P(N)$. We then calculate the cumulants, K_n and the factorial cumulants C_n for $n = 2, ..., 9$.[4] Next we repeat this sampling n_{run} times, where n_{run} is sufficiently large so that the results presented below do not depend on n_{run}. This procedure then gives us n_{run} "measurements" or samples of K_n and C_n, $\{K_n^{(1)}, \ldots, K_n^{(n_{\text{run}})}\}$ and $\{C_n^{(1)}, \ldots, C_n^{(n_{\text{run}})}\}$.

From these samples we calculate the variance, for example, in the case of the factorial cumulants C_n we have

$$\mathrm{Var}\left(C_n\right) = \frac{1}{n_{\text{run}}} \sum_{i=1}^{n_{\text{run}}} \left(C_n^{(i)}\right)^2 - \left(\frac{1}{n_{\text{run}}} \sum_{i=1}^{n_{\text{run}}} C_n^{(i)}\right)^2. \quad (1)$$

The expected *absolute* error, ΔC_n, is then given by $\Delta C_n = \sqrt{\mathrm{Var}\left(C_n\right)}$, whereas the *relative* error is $\Delta C_n / C_n$, where C_n denotes the true value directly calculated from the multiplicity distribution $P(N)$.

An alternative way to calculate the expected error is by means of the delta method (see, e.g., [10, 29, 30] for details). In the case at hand, where we want to calculate the errors for (factorial) cumulants, application of the delta method is straightforward. Let us discuss this in more detail for the case of the factorial cumulant. The random variables are the moments about zero, $\mu_k = \left\langle N^k \right\rangle$. Therefore, we express the factorial cumulant, C_n, in terms of the moments, $C_n = F(\mu_1, \ldots, \mu_n)$. Then according to the delta method the variance of C_n for a sample with n_{events} events is given by

$$\mathrm{Var}\left(C_n\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial F}{\partial \mu_i} \frac{\partial F}{\partial \mu_j} \mathrm{Cov}\left(\mu_i, \mu_j\right), \quad (2)$$

$$\mathrm{Cov}(\mu_i, \mu_j) = \frac{1}{n_{\text{events}}} \left(\mu_{i+j} - \mu_i \mu_j\right). \quad (3)$$

The *absolute* error is then again given by $\Delta C_n = \sqrt{\mathrm{Var}\left(C_n\right)}$. For example, we obtain the following for the variance of C_2 (after re-expressing the moments μ_i in terms of factorial cumulants)

$$\mathrm{Var}\left(C_2\right) = \frac{1}{n_{\text{events}}} \left[2\left(C_1 + C_2\right)^2 + 2C_2 + 4C_3 + C_4\right]. \quad (4)$$

We find that the so obtained errors are in perfect agreement with those determined via the aforementioned numerical sampling method.

After having presented the methods for error determination let us turn to the results. The essential point of the present paper is the observation that a small deviation from Poisson or binomial distributions can result in rather peculiar distributions, which we call statistics friendly distributions. From these distributions one may obtain factorial cumulants of high orders with a rather limited number of events. One example is a simple two-component distribution discussed recently in Ref. [24]

$$P(N) = (1 - \alpha) P_{(a)}(N) + \alpha P_{(b)}(N), \quad (5)$$

where both $P_{(a)}$ and $P_{(b)}$ are the proton multiplicity distributions characterized by small or even vanishing factorial cumulants.[5] This distribution not only serves as a nice example for a statistics friendly distribution, but also, as argued recently in Ref. [24], such a distribution would be consistent with a system with a finite number of particles being close to a first-order phase transition. The analysis in Ref. [24] found that $P(N)$ given by Eq. (5) with $\alpha \approx 0.0033$, $P_{(a)}(N)$ given by binomial ($B = 350$, $p \approx 0.1144$) and $P_{(b)}(N)$ given by Poisson ($\langle N_{(b)} \rangle = 25.3525$) is able to reproduce the preliminary results by the STAR Collaboration for the proton cumulants at $\sqrt{s} = 7.7\,\text{GeV}$ [27]. In addition it was found that the above distribution predicts factorial cumulants to roughly scale like $C_{n+1}/C_n \sim -15$, i.e., they alter in

[3] It would be interesting to explore if similar statistics friendly distributions also exist for more than one species, such as net-proton distributions which involve protons and anti-protons.

[4] As a reminder, the cumulants and factorial cumulants are obtained from the multiplicity distribution $P(N)$ as $K_n = \frac{d^n}{dt^n} \ln\left[\sum_N P(N) e^{Nt}\right]\Big|_{t=0}$, $C_n = \frac{d^n}{dz^n} \ln\left[\sum_N P(N) z^N\right]\Big|_{z=1}$.

[5] The simplest two-component distribution could result from two Poissons with different means. We take the main distribution to be binomial to conserve the baryon number, however, this is not important for our conclusions.

sign from order to order while increasing in absolute value by more than an order of magnitude.[6] In other words, the small admixture of a Poisson distribution changes the factorial cumulants dramatically, from being close to zero to almost exponentially increasing in magnitude. The same dramatic difference can also be seen in the expected error for a finite sampling. This is shown in Fig. 1, where in panel (a) we show the histogram of $C_n^{(i)}/C_n$ from our numerical sampling (based on $n_{\rm events} = 144393$ events) for the binomial distribution only, i.e., $P_{(a)}(N)$.[7] For completeness we note that the analytical values of C_n for binomial are given by $C_n = (-1)^{n+1}(n-1)!Bp^n$. The distribution gets very wide already for C_3. In contrast, in panel (b) of Fig. 1 we show equivalent histograms for the two-component distribution, Eq. (5). Again, the small admixture of a Poisson distribution changes the situation dramatically. In this case the distributions are so narrow that a measurement of even the 8-th order factorial cumulants may be feasible with as little as 150000 events.

This finding is quantified in Fig. 2, where we show the *relative* errors $\Delta C_n/C_n$ for various distributions again based on $n_{\rm events} = 144393$ event. The relative error for both the binomial distribution and the negative binomial distribution (NBD)[8] with $\langle N \rangle = 40$ and $k = 80$, increase essentially exponentially with increasing order of the factorial cumulant. Obviously all of these distributions are statistics hungry, and the measurement of higher order factorial cumulants with good accuracy requires very large statistics. For the two-component model, labeled "Binomial + Poisson", on the other hand the relative errors remain very small even for C_9. The actual values for the relative errors are (0.036, 0.16, 0.13, 0.14, 0.18, 0.26, 0.42, 0.91) for $(\Delta C_2/C_2, \ldots, \Delta C_9/C_9)$.

We also show as "Binomial + Poisson + effi" the result one would obtain, if one takes a finite detection efficiency of $\epsilon = 0.65$ into account, that is $\langle N_{(b)} \rangle = 25.3525 \times \epsilon$, $p = 0.1144 \times \epsilon$ so that $\langle N \rangle = 40 \times \epsilon$. Again, the relative error for the factorial cumulants remains small but larger than that in the case without efficiency. Here we have (0.056, 0.29, 0.27, 0.31, 0.41, 0.61, 1.06, 2.55) for $(\Delta C_2/C_2, \ldots, \Delta C_9/C_9)$. This also means that using the efficiency uncorrected STAR data one could try to measure the factorial cumulants up to the seventh order where $\Delta C_7/C_7 = 1 \pm 0.61$.

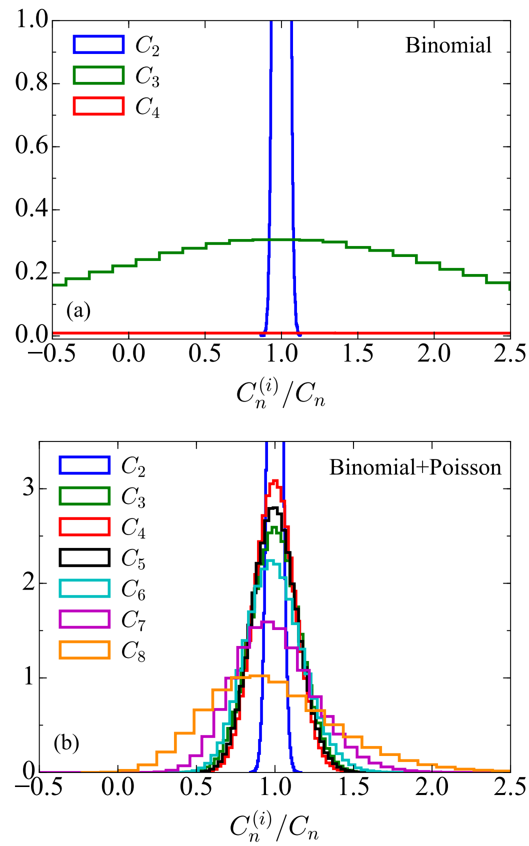

FIG. 1. Histogram (normalized to unity) of the factorial cumulant, $C_n^{(i)}$, fluctuating from experiment to experiment, divided by a known (evaluated analytically) value, C_n, based on 144393 events sampled from (a) the binomial distribution ($B = 350$, $p = 0.114$, $\langle N \rangle = pB \approx 40$) and (b) a distribution given by Eq. (5) (see text for details). The statistical errors are given by the widths of the corresponding histograms. In the panel (a), the histograms' order by the height at the maximum is (from largest to smallest) C_2, C_3, and C_4. In the panel (b) the order (from largest to smallest) is: C_2, C_4, C_5, C_3, C_6, C_7, and C_8.

The above results may be understood qualitatively in the following way. In general we have two types of multiplicity distributions, $P(N)$. One where the higher order factorial cumulants are driven by the tails (Poisson, binomial, NBD etc.) and the other one where the higher order factorial cumulants are driven by some structure away from the tails. This is exactly the case of our model.[9] To be a bit more precise the factorial cumulants of Eq. (5), assuming $\alpha \ll 1$ are given by[10]

$$C_n \approx C_n^{(a)} + (-1)^n \alpha \bar{N}^n, \tag{6}$$

[6] The actual ratios slightly decrease with increasing n: $C_4/C_3 = -17$, $C_5/C_4 = -15.56$, $C_6/C_5 = -15.46$, $C_7/C_6 = -15.04$, $C_8/C_7 = -13.85$, $C_9/C_8 = -10.66$, and for C_{10} the pattern breaks and $C_{10}/C_9 = 0.72$.

[7] We found that the absolute error of C_n for the binomial distribution is close to that of the Poisson distribution, which can be easily calculated using Eq. (2) and is given by $\sqrt{n!}\langle N \rangle^{n/2}/\sqrt{n_{\rm events}}$.

[8] For the NBD $C_n = (n-1)!\langle N \rangle^n/k^{n-1}$, where k measures the deviation from a Poisson distribution, e.g., $\langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle(1 + \langle N \rangle/k)$.

[9] Another example of a statistics friendly distribution is a uniform distribution. For example, taking $P(N) = \text{const}$ for $N \in [0, 80]$ we obtain (0.0026, 0.0309, 0.0071, 0.0447, 0.0114, 0.0477, 0.0157, 0.0487) for $(\Delta C_2/C_2, \ldots, \Delta C_9/C_9)$.

[10] Again, we assume that both $P_{(a)}$ and $P_{(b)}$ are the proton multiplicity distributions characterized by small (or even vanishing)

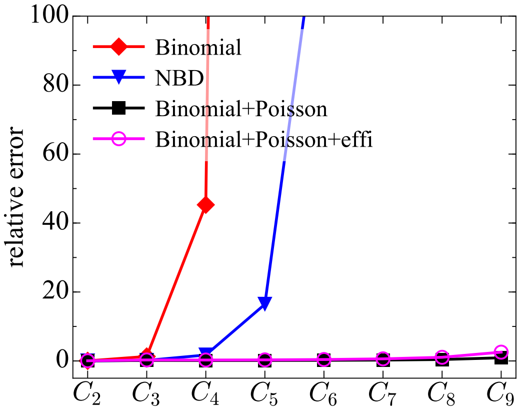

FIG. 2. The relative error, $\Delta C_n/C_n$, of factorial cumulants for various proton multiplicity distributions based on 144393 events, as present in the most central $Au + Au$ collisions at RHIC. The binomial and negative binomial distributions presented here are statistically very demanding, whereas the distribution given by. Eq. (5) (Binomial+Poisson) with $\langle N\rangle = 40$, allows to successfully measure higher order factorial cumulants with a relatively small number of events. This feature is also present for the efficiency uncorrected distribution (Binomial+Poisson+effi) where $\langle N\rangle = 40 \times 0.65$.

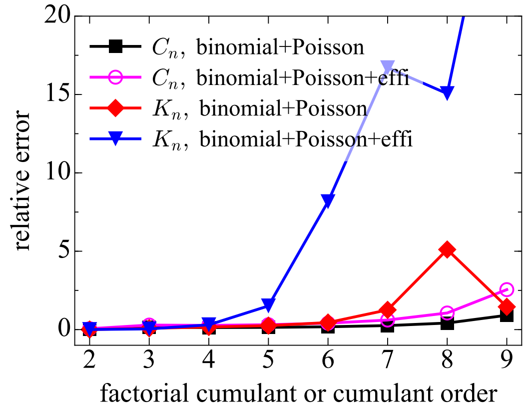

FIG. 3. The relative errors of the factorial cumulants, $\Delta C_n/C_n$, and the regular cumulants, $\Delta K_n/K_n$, based on 144393 events sampled from a distribution given by Eq. (5).

where $C_n^{(a)}$ is a factorial cumulant characterizing $P_{(a)}(N)$ and $\overline{N} = \langle N_{(a)}\rangle - \langle N_{(b)}\rangle$. For $C_n^{(a)}$ being a Poisson or binomial the values of C_n are completely dominated by the term $\alpha\overline{N}^n$, which results in very large factorial cumulants. The error, ΔC_n, on the other hand, is of the same magnitude as that of the first term, $\Delta C_n^{(a)}$ (in practice $\Delta C_n^{(a)}/\Delta C_n$ ranges from ~ 0.95 for $n = 2$ to ~ 0.2 for $n = 9$). Thus we have a situation, where the error of the factorial cumulant is of the same magnitude as that of a binomial distribution, but the factorial cumulant is orders of magnitude larger. Consequently, and not surprisingly, the *relative* error is much smaller for the two-component distribution than for the binomial distribution. It is worth noting that C_n scales linearly with α and the two-component distribution is statistics friendly even if the second mode is tiny, i.e., α is small (provided $\overline{N}^n$ is large enough).

Finally, we note that in the case of Eq. (5), the regular cumulnats are less statistics friendly. This is presented in Fig. 3. The reason for this is the same as just stated. The absolute errors for both cumulants and factorial cumulants are of the same magnitude, $\Delta K_n \sim \Delta C_n$. On the other hand, for the two-component model, the factorial cumulants are very large while the regular cumulants are only modestly larger than that of a simple binomial distribution. This is a result of the alternating signs of the factorial cumulants. For example, the sixth order cumulant, K_6, is given in terms of the factorial cumulants as $K_6 = \langle N\rangle + 31C_2 + 90C_3 + 65C_4 + 15C_5 + C_6$ (see e.g., Ref. [24]). For our example of "binomial+Poisson+effi", where we see a rapid increase in the relative error, we have $C_6 \approx 3080$, $15C_5 \approx -4600$ and $65C_4 \approx 1970$. As a result, $K_6 \approx 180 \ll C_6$, and consequently the relative error is much larger for K_6 as compared to C_6.

In summary, we demonstrated that for the multiplicity distribution given by Eq. (5), which is relevant in the context of searching for structures in the QCD phase diagram, factorial cumulants of high orders can be determined with a relatively small number of events. This is in contrast to various statistics hungry distributions (Poisson, binomial, NBD, etc.), for which the error increases nearly exponentially with increasing order. As shown in Ref. [24], the distribution, Eq. (5), describes the preliminary STAR data for proton cumulants (up to the forth order) in central Au+Au collisions at $\sqrt{s} = 7.7\,\mathrm{GeV}$. Because this distribution is statistics friendly, it can be further tested by evaluating the higher order factorial cumulants even with the presently available STAR data set of 144393 events for the most central collisions. We also pointed out that factorial cumulants are more statistics friendly when compared to regular cumulants, which, in the case of Eq. (5), results from a delicate cancellation of large factorial cumulants. Assuming that $C_4 = 170$ (as extracted from preliminary STAR data) we predict:

$$C_5 = -307\,(1 \pm 0.31), \quad C_6 = 3085\,(1 \pm 0.41),$$
$$C_7 = -30155\,(1 \pm 0.61), \quad C_8 = 271492\,(1 \pm 1.06),$$

for efficiency uncorrected data and

$$C_5 = -2645\,(1 \pm 0.14), \quad C_6 = 40900\,(1 \pm 0.18),$$
$$C_7 = -615135\,(1 \pm 0.26), \quad C_8 = 8520220\,(1 \pm 0.42),$$

for $\langle N\rangle = 40$, corresponding to the efficiency corrected

factorial cumulants. The whole idea is to obtain large factorial cumulants from two rather standard distributions.

data.[11] In the next phase of the RHIC beam energy scan the statistics is expected to increase by roughly a factor of ~ 25 [31] reducing the above errors by about a factor of 5. It would be desirable to also analyze C_5 and higher order proton (not net-proton) factorial cumulants at much higher energies, say, $\sqrt{s} = 200$ GeV, where a first-order phase transition is not anticipated. Thus the factorial cumulants are not expected to alter in sign while increasing in absolute value. It was checked in Ref. [26] that C_3 and C_4 alter in sign but their magnitudes are very small.

Our message does not rely on the ability to estimate the errors of C_n in an experiment. The reason is the following. We conjecture that the multiplicity distribution at 7.7 GeV is a two-component one and describe the preliminary data up to the fourth order. Next, we run a sufficient number of independent experiments with each experiment resulting in one measured number C_n. The histogram of the measured values, as shown in Fig. 1(b), is narrow if the distribution is given by our conjectured one. Now STAR makes one measurement only and obtains C_5, C_6, C_7, C_8. If our conjecture is correct, that is, the distribution is a two-component one, the numbers measured by STAR should be consistent with our predictions. If the numbers are significantly off our predictions, then our conjecture is falsified. We also note that this procedure is quite general and not restricted to the STAR data discussed here: Measure the first four factorial cumulants then see if they are consistent with a two-component distribution. If so, test this distribution by comparing the measured higher factorial cumulants with the prediction of the two-component model.

In conclusion, we have shown that two-component multiplicity distributions as expected in the vicinity of a first-order phase transition are "statistics-friendly". This allows for the determination of factorial cumulants of high order even with limited statistics, and opens a novel way to search for the phase structure of mesoscopic systems.

Acknowledgments: We thank Andrzej Bialas and Jan Steinheimer for useful comments. A.B. is partially supported by the Ministry of Science and Higher Education, and by the National Science Centre Grant No. 2018/30/Q/ST2/00101. V.K. is supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, under contract number DE-AC02-05CH11231. This work also received support within the framework of the Beam Energy Scan Theory (BEST) Topical Collaboration.

[11] We note that the errors quoted here are only due to the sample size and do not account for additional uncertainties due to the efficiency correction [30].

* bzdak@fis.agh.edu.pl
† vkoch@lbl.gov

[1] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, Nature **443**, 675 (2006), arXiv:hep-lat/0611014.
[2] M. A. Stephanov, Prog. Theor. Phys. Suppl. **153**, 139 (2004), arXiv:hep-ph/0402115.
[3] A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov, and N. Xu, (2019), arXiv:1906.00936 [nucl-th].
[4] S. Jeon and V. Koch, Phys. Rev. Lett. **85**, 2076 (2000), arXiv:hep-ph/0003168 [hep-ph].
[5] M. Asakawa, U. W. Heinz, and B. Muller, Phys. Rev. Lett. **85**, 2072 (2000), hep-ph/0003169.
[6] M. Stephanov, Phys. Rev. Lett. **102**, 032301 (2009), arXiv:0809.3450 [hep-ph].
[7] V. Skokov, B. Friman, and K. Redlich, Phys. Rev. **C83**, 054904 (2011), arXiv:1008.4570 [hep-ph].
[8] M. Stephanov, Phys. Rev. Lett. **107**, 052301 (2011), arXiv:1104.1627 [hep-ph].
[9] X.-F. Luo, B. Mohanty, H. G. Ritter, and N. Xu, Phys. Atom. Nucl. **75**, 676 (2012), arXiv:1105.5049 [nucl-ex].
[10] X. Luo and N. Xu, Nucl. Sci. Tech. **28**, 112 (2017), arXiv:1701.02105 [nucl-ex].
[11] C. Herold, M. Nahrgang, Y. Yan, and C. Kobdaj, Phys. Rev. **C93**, 021902 (2016), arXiv:1601.04839 [hep-ph].
[12] D.-M. Zhou, A. Limphirat, Y.-l. Yan, C. Yun, Y.-p. Yan, X. Cai, L. P. Csernai, and B.-H. Sa, Phys. Rev. **C85**, 064916 (2012), arXiv:1205.5634 [nucl-th].
[13] X. Wang and C. B. Yang, Phys. Rev. **C85**, 044905 (2012), arXiv:1202.4857 [nucl-th].
[14] F. Karsch and K. Redlich, Phys. Rev. **D84**, 051504 (2011), arXiv:1107.1412 [hep-ph].
[15] B. J. Schaefer and M. Wagner, Phys. Rev. **D85**, 034027 (2012), arXiv:1111.6871 [hep-ph].
[16] L. Chen, X. Pan, F.-B. Xiong, L. Li, N. Li, Z. Li, G. Wang, and Y. Wu, J. Phys. **G38**, 115004 (2011).
[17] W.-j. Fu, Y.-x. Liu, and Y.-L. Wu, Phys. Rev. **D81**, 014028 (2010), arXiv:0910.5783 [hep-ph].
[18] M. Cheng *et al.*, Phys. Rev. **D79**, 074505 (2009), arXiv:0811.1006 [hep-lat].
[19] M. M. Aggarwal *et al.* (STAR), Phys. Rev. Lett. **105**, 022302 (2010), arXiv:1004.4959 [nucl-ex].
[20] L. Adamczyk *et al.* (STAR), Phys. Rev. Lett. **112**, 032302 (2014), arXiv:1309.5681 [nucl-ex].
[21] A. Rustamov (ALICE), *Proceedings, 26th International Conference on Ultra-relativistic Nucleus-Nucleus Collisions (Quark Matter 2017): Chicago, Illinois, USA, February 5-11, 2017*, Nucl. Phys. **A967**, 453 (2017), arXiv:1704.05329 [nucl-ex].
[22] M. Kitazawa and M. Asakawa, Phys. Rev. **C85**, 021901 (2012), arXiv:1107.2755 [nucl-th].
[23] M. Kitazawa and M. Asakawa, Phys. Rev. **C86**, 024904 (2012), [Erratum: Phys. Rev. C86, 069902 (2012)], arXiv:1205.3292 [nucl-th].
[24] A. Bzdak, V. Koch, D. Oliinychenko, and J. Steinheimer, Phys. Rev. **C98**, 054901 (2018), arXiv:1804.04463 [nucl-th].
[25] S. He and X. Luo, Phys. Lett. **B774**, 623 (2017), arXiv:1704.00423 [nucl-ex].
[26] A. Bzdak, V. Koch, and N. Strodthoff, Phys. Rev. **C95**, 054906 (2017), arXiv:1607.07375 [nucl-th].
[27] X. Luo (STAR), *Proceedings, 9th International Workshop*

on Critical Point and Onset of Deconfinement (CPOD 2014): Bielefeld, Germany, November 17-21, 2014, PoS **CPOD2014**, 019 (2015), arXiv:1503.02558 [nucl-ex].
[28] X. Luo (STAR), private communication (2018).
[29] A. Davison, *Statistical Models*, Cambridge Series in Statistical and Probabilistic Mathematics (Cambridge University Press, 2003).
[30] X. Luo, Phys. Rev. **C91**, 034907 (2015), arXiv:1410.3914 [physics.data-an].
[31] STAR Note 598, https://drupal.star.bnl.gov/STAR/starnotes/public/sn0598 (2014).