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Lu, Xiaochuan

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Aspects of Particle Physics Beyond the Standard Model

by

Xiaochuan Lu

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

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in the

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of the

University of California, Berkeley

Committee in charge:

Professor Hitoshi Murayama, Chair

Professor Lawrence J. Hall

Professor Nouredine El. Karoui

Spring 2015

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Abstract

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Professor Hitoshi Murayama, Chair

This dissertation describes a few aspects of particles beyond the Standard Model, with a focus on the remaining questions after the discovery of a Standard Model-like Higgs boson. In specific, three topics are discussed in sequence: neutrino mass and baryon asymmetry, naturalness problem of Higgs mass, and placing constraints on theoretical models from precision measurements.

First, the consequence of the neutrino mass anarchy on cosmology is studied. Attentions are paid in particular to the total mass of neutrinos and baryon asymmetry through leptogenesis. With the assumption of independence among mass matrix entries in addition to the basis independence, Gaussian measure is the only choice. On top of Gaussian measure, a simple approximate $U(1)$ flavor symmetry makes leptogenesis highly successful. Correlations between the baryon asymmetry and the light-neutrino quantities are investigated. Also discussed are possible implications of recently suggested large total mass of neutrinos by the SDSS/BOSS data.

Second, the Higgs mass implies fine-tuning for minimal theories of weak-scale supersymmetry (SUSY). Non-decoupling effects can boost the Higgs mass when new states interact with the Higgs, but new sources of SUSY breaking that accompany such extensions threaten naturalness. I will show that two singlets with a Dirac mass can increase the Higgs mass while maintaining naturalness in the presence of large SUSY breaking in the singlet sector. The modified Higgs phenomenology of this scenario, termed “Dirac NMSSM”, is also studied.

Finally, the sensitivities of future precision measurements in probing physics beyond the Standard Model are studied. A practical three-step procedure is presented for using the Standard Model effective field theory (SM EFT) to connect ultraviolet (UV) models of new physics with weak scale precision observables. With this procedure, one can interpret precision measurements as constraints on the UV model concerned. A detailed explanation is given for calculating the effective action up to one-loop order in a manifestly gauge covariant fashion. This covariant derivative expansion method dramatically simplifies the process of matching a UV model with the SM EFT, and also makes available a universal formalism that is easy to use for a variety of UV models. A

few general aspects of RG running effects and choosing operator bases are discussed. Mapping results are provided between the bosonic sector of the SM EFT and a complete set of precision electroweak and Higgs observables to which present and near future experiments are sensitive. Many results and tools which should prove useful to those wishing to use the SM EFT are detailed in several appendices.

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Chapter 1

Introduction

The discovery of a Standard Model (SM)-like Higgs boson [1] marks a great triumph of experimental and theoretical particle physics. But many long standing questions remain unexplained. Phenomenologically, there are five pieces of empirical evidence for physics beyond the Standard Model (see, *e.g.*, [2] for a detailed discussion):

- non-baryonic dark matter
- dark energy that causes the accelerated expansion of the Universe
- finite neutrino masses responsible for their flavor oscillations
- apparently acausal density fluctuations observed in the cosmic microwave background
- cosmic baryon asymmetry

Theoretically, there are also many unanswered questions related to this new boson, such as the naturalness problem of its mass: how to stabilize the big hierarchy between the weak scale and the Planck scale; and its mysterious features: why does it seem to be the only scalar in the theory and what is the dynamical cause of its condensation? There is a thriving area of research motivated by answering these questions, entitled particle physics beyond the Standard Model. In this dissertation, I will discuss a few aspects of this body of research and present partial solutions to some of the questions above. Specifically, I will discuss three important topics in sequence, (1) neutrino mass anarchy and the Universe¹, (2) Dirac NMSSM and Semi-soft Supersymmetry Breaking², and (3) Standard Model Effective Field Theory³.

First, neutrino physics is a unique area in particle physics that has many direct consequences on the evolution history and the current state of Universe. It was one of the first hypotheses for the non-baryonic dark matter. Excluding this possibility relied on rather surprising constraint that

¹This part of discussions is base on two published works [3, 4] with Hitoshi Murayama

²This part of discussion is base on a published work [5] with Hitoshi Murayama, Joshua T. Ruderman, and Kohsaku Tobioka

³This part of discussion is base on two published works [6, 7] with Brian Henning and Hitoshi Murayama

the density of neutrinos would exceed that allowed by Fermi degeneracy in the core of dwarf galaxies! [8] Because of the free streaming, massive neutrinos would also suppress the large-scale structure, which is still subject to active research. The explosion mechanism of supernova is tied to properties of neutrinos, and hence the chemical evolution of galaxies depend on neutrinos. The number of neutrinos is relevant to Big-Bang Nucleosynthesis. In addition, neutrinos may well have created the baryon asymmetry of the Universe [9] or create the Universe itself with scalar neutrino playing the role of the inflaton [10, 11].

Many of the consequences of neutrino properties on the Universe rely on the mass of neutrinos. After many decades of searches, neutrino mass was discovered in 1998 in disappearance of atmospheric neutrinos by the Super-Kamiokande collaboration [12]. Subsequently the SNO experiment demonstrated transmutation of solar electron neutrinos to other active neutrino species [13] corroborated by reactor neutrino data from KamLAND [14]. Most recently, the last remaining mixing angle was discovered by the Daya Bay reactor neutrino experiment [15]. Other experiments confirmed this discovery [16, 17]. Although a large amount of information about neutrino masses and mixings have been extracted from neutrino oscillation experiments, some properties remain unknown, such as the type of the mass hierarchy of the neutrino mass, and the CP phases of neutrino mixing.

In Chapter 2, I will discuss a statistical method of studying neutrinos and hence probing the history of the Universe. This method, entitled *neutrino anarchy* was first proposed in [18] and further developed in [19–21]. I will contribute to its development by closely investigating the mass distribution, and studying its consequences and correlation with cosmic baryon asymmetry.

Next, I turn attention to the naturalness problem of this newly discovered Higgs boson. Weak scale supersymmetry (SUSY) is arguably the best known mechanism to ameliorate the naturalness problem. It can also provide a natural candidate for the cosmological dark matter. However, to explain the observed value of Higgs mass, the minimal models of SUSY typically require the existence of a soft parameter much larger than the weak scale. And due to mixing among soft parameters radiative corrections, this will generically re-introduce the naturalness problem in part [22].

In Chapter 3, I will discuss a new mechanism, entitled *semi-soft SUSY breaking*, where contrary to the generic picture, one or more soft masses can be prevented from (powerly) feeding into the radiative corrections to the weak scale soft parameters. These soft masses can thus be arbitrarily large without doing any harm to the naturalness of the weak scale. The apparent hard SUSY breaking turns out to be secretly equivalent to a soft SUSY breaking to the weak scale sector. A simple incarnation of this mechanism, entitled *Dirac NMSSM*, will also be discussed, in which two singlet superfields with a Dirac mass are added to the Minimal Supersymmetric Standard Model (MSSM). In this model, the tree-level Higgs mass can be boosted by a very large soft mass of the singlet, while the naturalness is still maintained.

Finally, I will discuss a generic method of placing constraints on beyond SM theories from near future precision measurements. It is exciting that ongoing and possible near future experiments can achieve an estimated per mille sensitivity on precision Higgs and EW observables [23–28]. This level of precision provides a window to indirectly explore the theory space of beyond SM physics

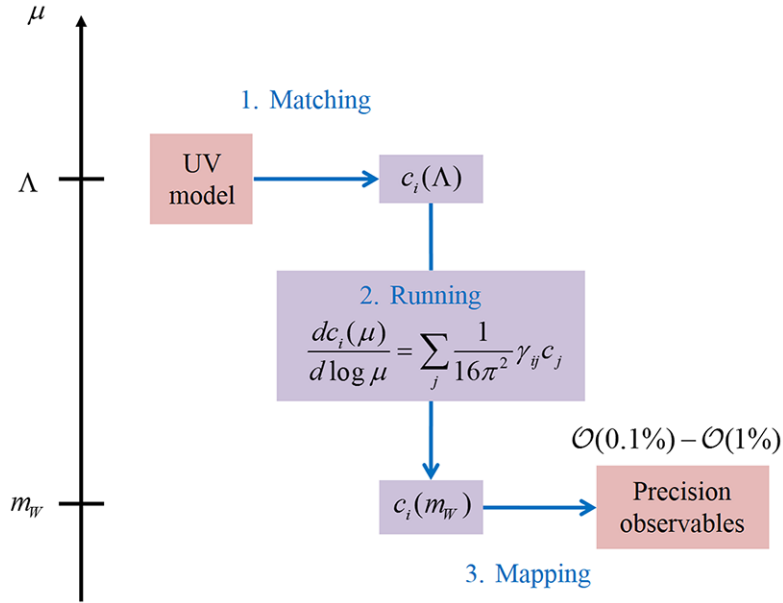


Figure 1.1: SM EFT as a bridge to connect UV models and weak scale precision observables.

and place constraints on specific ultraviolet (UV) models⁴. For this purpose, an efficient procedure of connecting new physics models with precision Higgs and EW observables is clearly desirable.

In Chapter 4, the Standard Model effective field theory (SM EFT) is used as a bridge to connect models of new physics with precision observables. The SM EFT consists of the renormalizable SM Lagrangian supplemented with higher-dimension interactions:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{1}{\Lambda^{d_i-4}} c_i \mathcal{O}_i. \quad (1.1)$$

In the above, Λ is the cutoff scale of the EFT, $\mathcal{O}_i$ are a set of dimension d_i operators that respect the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge invariance of $\mathcal{L}_{\text{SM}}$, and c_i are their Wilson coefficients that run as functions $c_i(\mu)$ of the renormalization group (RG) scale μ . The estimated per-mille sensitivity of future precision Higgs measurements justifies truncating the above expansion at dimension-six operators. The connection is then accomplished through a three-step procedure schematically described in Fig. 1.1. First, the UV model is matched onto the SM EFT at a high-energy scale Λ . This matching is performed order-by-order in a loop expansion. At each loop order, $c_i(\Lambda)$ is determined such that the S -matrix elements in the EFT and the UV model are the same at the RG scale $\mu = \Lambda$. Next, the $c_i(\Lambda)$ are run down to the weak scale $c_i(m_W)$ according to the RG equations of the SM EFT. The leading order solution to these RG equations is determined by the anomalous dimension matrix γ_{ij} . Finally, the effective Lagrangian at $\mu = m_W$ is used to compute weak scale

⁴In this dissertation, “UV model” is used to generically mean the SM supplemented with new states that couple to the SM. In particular, the UV model does not need to be UV complete; it may itself be an effective theory of some other, unknown description.

observables in terms of the $c_i(m_W)$ and SM parameters of $\mathcal{L}_{\text{SM}}$. In this dissertation, this third step is referred to as mapping the Wilson coefficients onto observables.

Chapter 2

Neutrino Mass Anarchy and the Universe

Neutrino physics is a unique area in particle physics that has many direct consequences on the evolution history and the current state of Universe. It was one of the first hypotheses for the non-baryonic dark matter. Excluding this possibility relied on rather surprising constraint that the density of neutrinos would exceed that allowed by Fermi degeneracy in the core of dwarf galaxies! [8] Because of the free streaming, massive neutrinos would also suppress the large-scale structure, which is still subject to active research. The explosion mechanism of supernova is tied to properties of neutrinos, and hence the chemical evolution of galaxies depend on neutrinos. The number of neutrinos is relevant to Big-Bang Nucleosynthesis. In addition, neutrinos may well have created the baryon asymmetry of the Universe [9] or create the Universe itself with scalar neutrino playing the role of the inflaton [10, 11].

Many of the consequences of neutrino properties on the Universe rely on the mass of neutrinos. After many decades of searches, neutrino mass was discovered in 1998 in disappearance of atmospheric neutrinos by the Super-Kamiokande collaboration [12]. Subsequently the SNO experiment demonstrated transmutation of solar electron neutrinos to other active neutrino species [13] corroborated by reactor neutrino data from KamLAND [14]. Most recently, the last remaining mixing angle was discovered by the Daya Bay reactor neutrino experiment [15]. Other experiments confirmed this discovery [16, 17].

On the other hand, fermion masses and mixings have been a great puzzle in particle physics ever since the discovery of muon. Through decades of intensive studies, we have discovered the existence of three generations and a bizarre mass spectrum and mixings among them. The underlying mechanism for this pattern is still not clear. But the hierarchical masses and small mixings exhibited by quarks and charged leptons seem to suggest that mass matrices are organized by some yet-unknown symmetry principles. The discovery of neutrino masses and mixings seem to even

complicate the puzzle. From the current data [29]

$$\Delta m_{12}^2 = (7.50 \pm 0.20) \times 10^{-5} \text{eV}^2, \quad (2.1)$$

$$|\Delta m_{23}^2| = (2.32^{+0.12}_{-0.08}) \times 10^{-3} \text{eV}^2, \quad (2.2)$$

$$\sin^2 2\theta_{23} > 0.95 \text{ (90\% C.L.)}, \quad (2.3)$$

$$\sin^2 2\theta_{12} = 0.857 \pm 0.024, \quad (2.4)$$

$$\sin^2 2\theta_{13} = 0.095 \pm 0.010, \quad (2.5)$$

the neutrinos also display a small hierarchy $\Delta m_{12}^2 / |\Delta m_{23}^2| \sim \frac{1}{30}$ ¹, which is quite mild compared to quarks and charged leptons. In addition, unlike quarks and charged leptons, the neutrinos have both large and small mixing angles. Many attempts were made to explain this picture using ordered, highly structured neutrino mass matrices, especially when θ_{13} was consistent with zero [30–34].

Quite counterintuitively, however, it was pointed out that the pattern of neutrino masses and mixings can also be well accounted for by structureless mass matrices [18]. Mass matrices with independently random entries can naturally produce the small hierarchical mass spectrum and the large mixing angles. This provides us with an alternative point of view: instead of contrived models for the mass matrix, one can simply take the random mass matrix as a low energy effective theory [19]. This anarchy approach is actually more naturally expected, because after all, the three generations possess the exact same gauge quantum numbers. From the viewpoint of anarchy, however, the mass spectrum with large hierarchy and small mixings exhibited by quarks and charged leptons need an explanation. It turns out that introducing an approximate $U(1)$ flavor symmetry [35, 36] can solve this problem well [19]. This combination of random mass matrix and approximate $U(1)$ flavor symmetry has formed a new anarchy-hierarchy approach to fermion masses and mixings [19].

To be consistent with the spirit of anarchy, the measure to generate the random mass matrices has to be basis independent [19]. This requires that the measure over the unitary matrices be Haar measure, which unambiguously determines the distributions of the mixing angles and CP phases. Consistency checks of the predicted probability distributions of neutrino mixing angles against the experimental data were also performed in great detail [20, 21]. Although quite successful already, this anarchy-hierarchy approach obviously has one missing brick: a choice of measure to generate the eigenvalues of the random mass matrices. With the only restriction being basis independence, one can still choose any measure for the diagonal matrices at will. However, as is shown in Appendix A, if in addition to basis independence, one also wants the entries of the matrix to be independently distributed, then the only choice is the Gaussian measure. Another interesting result is that under large dimension limit, random matrices with independent and identically distributed entries have a universal asymptotic eigenvalue distribution, regardless of the choice of the entry distribution. This result is known as the Marchenko-Pastur law [37]. In Appendix B, an alternative proof of it is presented—a direct diagrammatic method which is more familiar to the particle physics community.

¹Throughout this chapter, we use *small hierarchy* for a mass spectrum typically like $1 : 3 : 10$, and *large hierarchy* for a mass spectrum typically like $1 : 10^2 : 10^4$. So in our wording, the neutrinos exhibit a small hierarchy, while the quarks and leptons exhibit a large hierarchy

In this chapter, let us focus on the Gaussian measure to investigate the consequences. As pointed out in [19], the quantities most sensitive to this choice would be those closely related to neutrino masses. Let us study a few such quantities of general interest, including the effective mass of neutrinoless double beta decay m_{eff} , the neutrino total mass m_{total} , and the baryon asymmetry η_{B0} obtained through a standard leptogenesis [38, 39]. A correlation analysis between η_{B0} and light-neutrino parameters is also presented. Recently, the correlation between leptogenesis and light-neutrino quantities was also studied by taking linear measure in [40]. Their results are in broad qualitative agreement with ours in this chapter.

The rest of this chapter is organized as following. We first motivate our sampling model—Gaussian measure combined with approximate $U(1)$ flavor symmetry—in Section 2.1. Then the consequences of this sampling model is presented in Section 2.2, where Monte Carlo predictions are shown on light-neutrino mass hierarchy, effective mass of neutrinoless double beta decay, light-neutrino total mass, and baryon asymmetry through leptogenesis. In Section 2.3, the correlations between baryon asymmetry and light-neutrino quantities are investigated. A recent Baryon Oscillation Spectroscopic Survey (BOSS) analysis suggests that the total neutrino mass could be quite large [41]. Section 2.4 is devoted to discuss the consequence of this suggestion. We conclude in Section 2.5.

2.1 Sampling Model

2.1.1 Gaussian Measure

To accommodate the neutrino masses, let us consider the standard model with an addition of three generations of right-handed neutrinos ν_R , which are singlets under $SU(2)_L \times U(1)_Y$ gauge transformations. Then there are two neutrino mass matrices, the *Majorana mass matrix* m_R and the *Dirac mass matrix* m_D , allowed by gauge invariance

$$\Delta\mathcal{L} \supset -\epsilon^{ab}\bar{L}_a H_b^\dagger y_\nu \nu_R - \frac{1}{2}\bar{\nu}_R^c m_R \nu_R + h.c. \supset -\bar{\nu}_L m_D \nu_R - \frac{1}{2}\bar{\nu}_R^c m_R \nu_R + h.c., \quad (2.6)$$

where $y_\nu = \frac{\sqrt{2}}{v} m_D$, with $v = 246$ GeV. With the spirit of anarchy, we should not discard either of them without any special reason. Both should be considered as random inputs. Let us parameterize them as

$$m_R = \mathcal{M} \cdot U_R D_R U_R^T, \quad (2.7)$$

$$m_D = \mathcal{D} \cdot U_1 D_0 U_2^\dagger, \quad (2.8)$$

where D_R and D_0 are real diagonal matrices with non-negative elements; U_R , U_1 and U_2 are unitary matrices; $\mathcal{M}$ and $\mathcal{D}$ are overall scales.

At this point, the requirement of neutrino basis independence turns out to be very powerful. It requires that the whole measure of the mass matrix factorizes into that of the unitary matrices and

diagonal matrices [19]:

$$dm_R \sim dU_R dD_R, \quad (2.9)$$

$$dm_D \sim dU_1 dU_2 dD_0. \quad (2.10)$$

Furthermore, it also demands the measure of U_R , U_1 and U_2 to be the Haar measure of $U(3)$ group [19]:

$$U = e^{i\eta} e^{i\phi_1 \lambda_3 + i\phi_2 \lambda_8} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \\ \times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{i\chi_1 \lambda_3 + i\chi_2 \lambda_8}, \quad (2.11)$$

$$dU = ds_{23}^2 dc_{13}^4 ds_{12}^2 d\delta \cdot d\chi_1 d\chi_2 \cdot d\eta d\phi_1 d\phi_2, \quad (2.12)$$

where $\lambda_3 = \text{diag}(1, -1, 0)$, $\lambda_8 = \text{diag}(1, 1, -2)/\sqrt{3}$, and $c_{12} = \cos \theta_{12}$, etc.

Although basis independence is very constraining, it does not uniquely fix the measure choice of m_R or m_D , because arbitrary measure of the diagonal matrices D_R and D_0 is still allowed. Now let us look at the entries of m_R and m_D . There are 9 complex free entries for m_D and 6 complex free entries for the symmetric matrix $m_R = m_R^T$. Generating each free entry independently is probably the most intuitive way of getting a random matrix. However, if one combines this free entry independence with the basis independence requirement, then it turns out there is only one option left—the Gaussian measure:

$$dm_D \sim \prod_{ij} e^{-|m_{D,ij}|^2} dm_{D,ij} = \left(\prod_{ij} dm_{D,ij} \right) e^{-\text{tr}(m_D m_D^\dagger)}, \quad (2.13)$$

$$dm_R \sim \prod_i e^{-|m_{R,ii}|^2} dm_{R,ii} \prod_{i<j} e^{-2|m_{R,ij}|^2} dm_{R,ij} = \left(\prod_{i \leq j} dm_{R,ij} \right) e^{-\text{tr}(m_R m_R^\dagger)}. \quad (2.14)$$

Although this is a well known result in random matrix theory [42, 43], a proof is included in Appendix A.

On one hand, basis independence is necessary from the spirit of anarchy. On the other hand, free entry independence is also intuitively preferred. With these two conditions combined, we are led uniquely to the Gaussian measure. Now the only parameters left free to choose are the two units $\mathcal{M}$ and $\mathcal{D}$. Following the spirit of anarchy, $\mathcal{D}$ should be chosen to make the neutrino Yukawa coupling of order unity,

$$y_\nu = \frac{\sqrt{2}}{v} m_D \sim O(1). \quad (2.15)$$

Throughout this chapter, $\mathcal{D} = 30 \text{ GeV}$ is chosen, which yields $y_\nu \sim 0.6$. Then $\mathcal{M}$ is chosen to fix the next largest mass-square difference of light-neutrinos Δm_l^2 at $2.5 \times 10^{-3} \text{ eV}^2$ in accordance with the data.

right-handed neutrino	$U(1)$ flavor charge
$\nu_{R,1}$	2
$\nu_{R,2}$	1
$\nu_{R,3}$	0

Table 2.1: The $U(1)$ flavor charge assignments for right-handed neutrinos.

2.1.2 Approximate $U(1)$ Flavor Symmetry

Our model (Eq.(2.6)) is capable of generating a baryon asymmetry η_{B0} through thermal leptogenesis [9, 44]. For the simplicity of analysis, let us focus on the scenario with two conditions:

(1) There is a hierarchy among heavy-neutrino masses $M_1 \ll M_2, M_3$, so that the dominant contribution to η_{B0} is given by the decay of the lightest heavy neutrino N_1 [38].

(2) If we use h_{ij} to denote the Yukawa couplings of heavy-neutrino mass eigenstates

$$\Delta\mathcal{L} \supset -h_{ij}\epsilon^{ab}\bar{L}_{ai}H_b^\dagger N_j, \quad (2.16)$$

then the condition $h_{i1} \ll 1$ for all $i = 1, 2, 3$ would justify the neglect of annihilation process $N_1 N_1 \rightarrow l\bar{l}$, and also help driving the decay of N_1 out of equilibrium [38, 44]. This condition used to be taken as an assumption [44].

Both conditions above can be achieved by imposing a simple $U(1)$ flavor charge on the right-handed neutrinos. For example, one can make charge assignments as shown in Table. 2.1. Assuming a scalar field ϕ carries -1 of this $U(1)$ flavor charge, one can construct neutral combinations ν_ϕ as

$$\nu_\phi = \begin{pmatrix} \nu_{\phi,1} \\ \nu_{\phi,2} \\ \nu_{\phi,3} \end{pmatrix} = \begin{pmatrix} \nu_{R,1} \cdot \phi^2 \\ \nu_{R,2} \cdot \phi \\ \nu_{R,3} \cdot 1 \end{pmatrix}. \quad (2.17)$$

Now it only makes sense to place the random coupling matrices among these neutral combinations

$$\Delta\mathcal{L} \supset -\bar{\nu}_L m_{D0} \nu_\phi - \frac{1}{2} \bar{\nu}_\phi^c m_{R0} \nu_\phi + h.c., \quad (2.18)$$

where m_{R0} and m_{D0} should be generated according to Gaussian measure as in Eq. (2.13)-(2.14). Upon $U(1)$ flavor symmetry breaking $\langle\phi\rangle = \epsilon$ with $\epsilon \simeq 0.1$, this gives

$$\nu_\phi \supset \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \nu_R \equiv D_\epsilon \cdot \nu_R, \quad (2.19)$$

and hence the mass matrices

$$m_R = D_\epsilon m_{R0} D_\epsilon = \mathcal{M} \cdot D_\epsilon U_R D_R U_R^T D_\epsilon, \quad (2.20)$$

$$m_D = m_{D0} D_\epsilon = \mathcal{D} \cdot U_1 D_0 U_2^\dagger D_\epsilon. \quad (2.21)$$

Let us parameterize the heavy-neutrino mass matrix as $m_N = U_N M U_N^T$, then

$$m_N \approx m_R \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} \begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}, \quad (2.22)$$

where one can identify

$$M \sim \begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_N \sim \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}. \quad (2.23)$$

Clearly a hierarchy among heavy neutrino masses is achieved. Furthermore, the heavy neutrino mass eigenstates are $N = U_N^T \nu_R$. Since

$$\Delta \mathcal{L} \supset -\epsilon^{ab} \bar{L}_a H_b^\dagger y_\nu \nu_R \equiv -h_{ij} \epsilon^{ab} \bar{L}_{ai} H_b^\dagger N_j, \quad (2.24)$$

we obtain the Yukawa coupling h_{ij} as

$$\begin{aligned} h &= y_\nu U_N^* = \frac{\sqrt{2}}{v} m_D U_N^* \sim \frac{\sqrt{2}}{v} m_{D0} \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} \\ &\sim \frac{\sqrt{2}}{v} m_{D0} \begin{pmatrix} \epsilon^2 & \epsilon^3 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^2 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}. \end{aligned} \quad (2.25)$$

We see that $h_{i1} \sim \epsilon^2 \ll 1$ is guaranteed for all $i = 1, 2, 3$.

It is worth noting that the light-neutrino mass matrix m_ν is not affected by our U(1) flavor change assignment on right-handed neutrinos (Table. 2.1). The hierarchical matrix D_ϵ cancels due to the seesaw mechanism [45–50]:

$$m_\nu = m_D m_R^{-1} m_D^T = m_{D0} m_{R0}^{-1} m_{D0}^T. \quad (2.26)$$

Therefore all properties of light neutrinos do not depend on the particular U(1) flavor charge assignment nor the size of the breaking parameter ϵ . The leptogenesis aspect is the only discussion in the chapter where this flavor structure is relevant.

2.2 Consequences

In this section, Monte Carlo results based on the sampling measure described in the last section are presented.

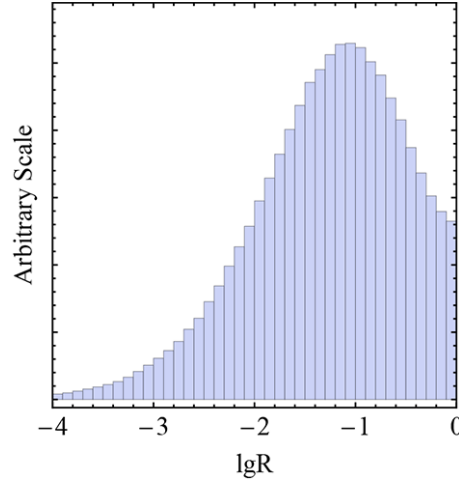


Figure 2.1: Histogram of $\lg R = \log_{10} R$ with 10^6 occurrences collected.

2.2.1 Light-neutrino Mixings and Mass Splitting Ratio

Let us parameterize the light neutrino mass matrix as

$$m_\nu = m_D m_R^{-1} m_D^T \equiv U_\nu m U_\nu^T, \quad (2.27)$$

with U_ν a unitary matrix and m a real diagonal matrix with non-negative elements. Let us also follow a conventional way to label the three masses of the light neutrinos: First sort them in the ascending order $m_{11} \leq m_{22} \leq m_{33}$. Then there are two mass-squared splittings $m_{22}^2 - m_{11}^2$ and $m_{33}^2 - m_{22}^2$. Δm_s^2 and Δm_l^2 are used to denote the smaller and larger one respectively. If Δm_s^2 is the first one, we call this scenario “normal” and take the definitions $m_1 \equiv m_{11}, m_2 \equiv m_{22}, m_3 \equiv m_{33}$. Otherwise, we call it “inverted” and take $m_1 \equiv m_{22}, m_2 \equiv m_{33}, m_3 \equiv m_{11}$. The columns of the unitary matrix U_ν should be arranged accordingly.

Predictions on light-neutrino mixings—the distributions of mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$, CP phase δ_{CP} , and other physical phases χ_1, χ_2 —are certainly the same as in general study of basis independent measures [19], since U_ν is totally governed by the Haar measure. A statistical Kolmogorov-Smirnov test shows that the predicted mixing angle distribution is completely consistent with the experimental data [20, 21].

The mass-squared splitting ratio $R \equiv \Delta m_s^2 / \Delta m_l^2$ is observed to be around [29]

$$R_{\text{exp}} \equiv \frac{7.50 \times 10^{-5}}{2.32 \times 10^{-3}}. \quad (2.28)$$

Fig. 2.1 shows our predicted distribution of this ratio. With a probability of $R < R_{\text{exp}}$ being 36.1%, the prediction is completely consistent with the data.

For the purpose of studying other quantities, let us introduce the following *Mixing-Split cuts* as suggested by the experimental data [29] on light-neutrino mixings and mass-squared splitting

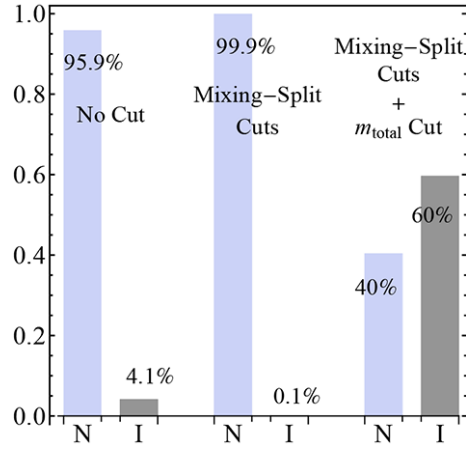


Figure 2.2: Fractions of normal and inverted mass hierarchy scenarios under different cuts selections, where “N” stands for normal hierarchy and “I” stands for inverted hierarchy. Each of the first two columns consists of 10^6 occurrences, while the last column “Mixing-Split Cuts + m_{total} Cut” contains only 10^4 occurrences.

ratio:

$$\sin^2 2\theta_{23} = 1.0 \quad (2.29)$$

$$\sin^2 2\theta_{12} = 0.857 \quad (2.30)$$

$$\sin^2 2\theta_{13} = 0.095 \quad (2.31)$$

$$R \in R_{\text{exp}} \times (1 - 0.05, 1 + 0.05) \quad (2.32)$$

2.2.2 Mass Hierarchy, m_{eff} and m_{total}

For the mass hierarchy of light-neutrino, our anarchy model predicts normal scenario being dominant. A 10^6 sample Monte Carlo finds the fraction of normal scenario being 95.9% before the Mixing-Split cuts (Eq. (2.29)-(2.32)), and 99.9% after applying the cuts. Fig. 2.2 shows the fractions of each scenario.

Neutrino anarchy allows for a random, nonzero Majorana mass matrix m_R . This means that the light neutrinos are Majorana and thus there can be neutrinoless double beta decay process. The

effective mass of this process $m_{\text{eff}} \equiv \left| \sum_i m_i U_{v,ei}^2 \right|$ is definitely a very broadly interested quantity.

Another quantity of general interest is the light-neutrino total mass $m_{\text{total}} \equiv m_1 + m_2 + m_3$, due to its presence in cosmological processes. Our predictions on m_{eff} and m_{total} are shown in Fig. 2.3 and Fig. 2.4 respectively. For each quantity, we plot both its whole distribution under Gaussian measure and its distribution after applying the Mixing-Split cuts. Distributions of m_{eff} and m_{total} under Gaussian measure were also studied recently in [51]. Their results are in agreement with ours. The small difference is due to the difference in taking cuts on neutrino mixing angles.

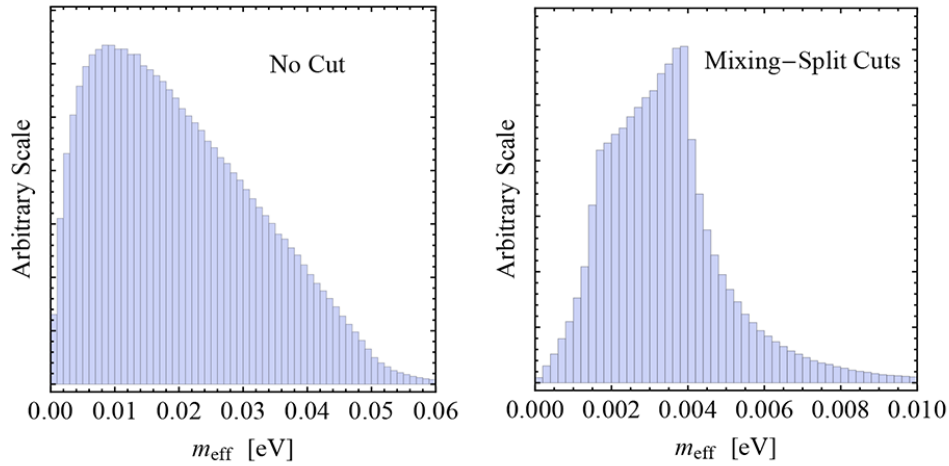


Figure 2.3: Histogram of m_{eff} with 10^6 occurrences collected. Left/Right panel shows distribution before/after applying the Mixing-Split cuts.

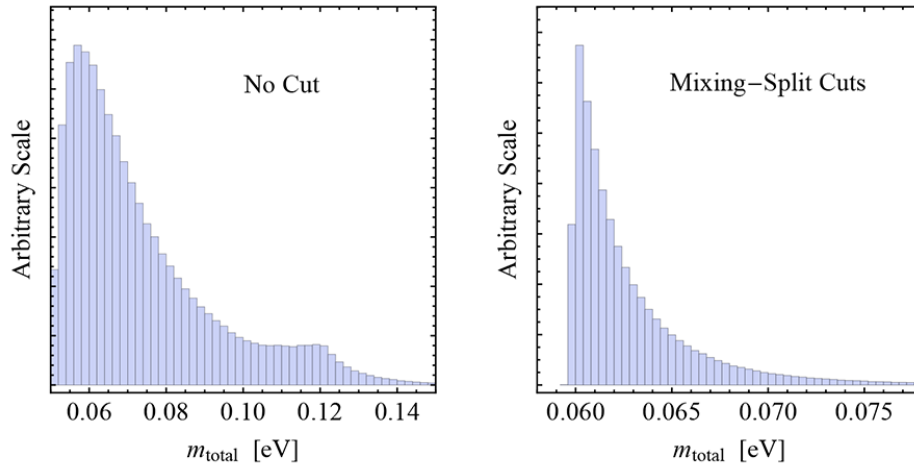


Figure 2.4: Histogram of m_{total} , with 10^6 occurrences collected. Left/Right panel shows distribution before/after applying the Mixing-Split cuts.

We see from Fig. 2.3 that most of the time $m_{\text{eff}} \lesssim 0.05$ eV. It becomes even smaller after we apply the Mixing-Split Cuts, mainly below $m_{\text{eff}} \lesssim 0.01$ eV. This is very challenging to experimental sensitivity. For m_{total} , Fig. 2.4 shows it being predicted to be very close to the current lower bound ~ 0.06 eV. The kink near 0.1 eV is due to the superposition of the two mass hierarchy scenarios.

To understand each component better, we show the distributions of m_{eff} and m_{total} in Fig. 2.5 and Fig. 2.6 for both before/after applying the cuts and normal/inverted hierarchy scenario. As Fig. 2.6 shows, under either hierarchy scenario, m_{total} is likely to reside very close to its cor-

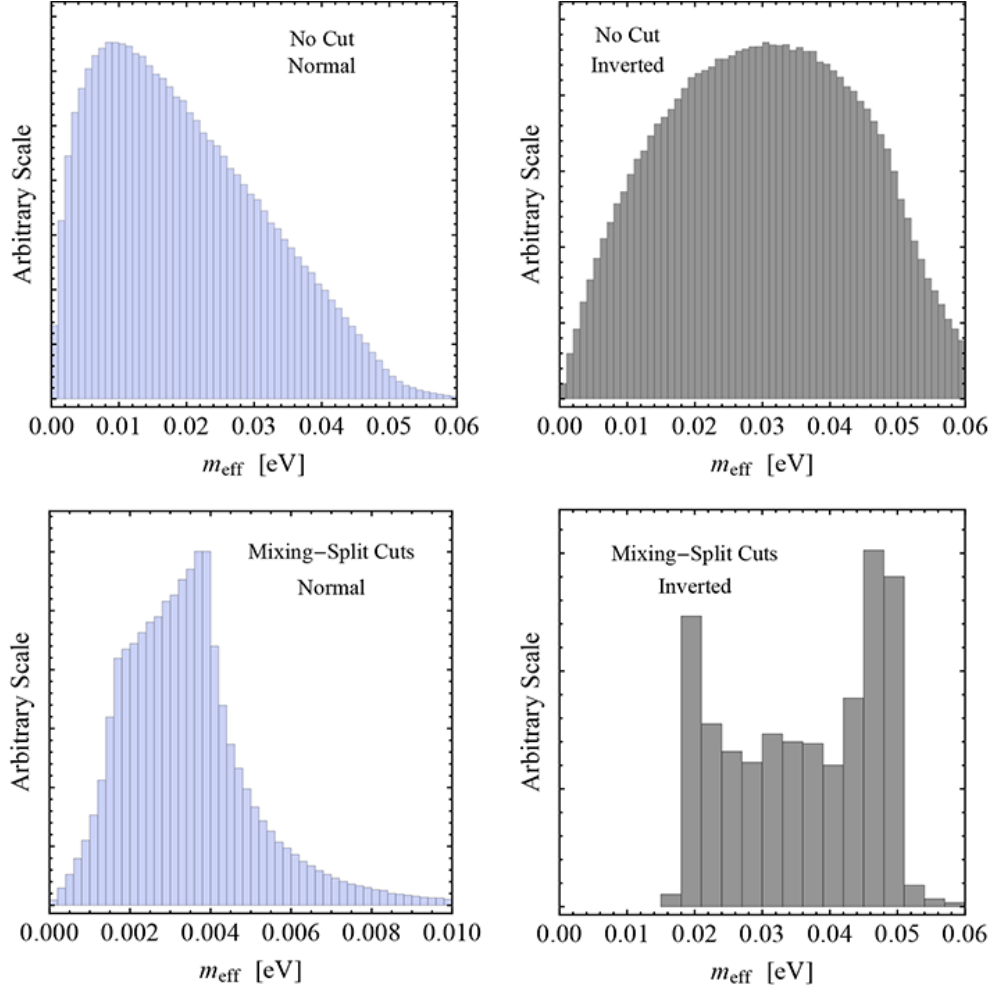


Figure 2.5: Histogram of m_{eff} with two mass hierarchy scenarios plotted separately. Left/Right column shows distribution under normal/inverted scenario. Upper/Lower row shows distribution before/after applying the Mixing-Split cuts. The plot of inverted scenario with Mixing-Split cuts applied (right bottom) contains 10^4 occurrences, while other plots contain 10^6 occurrences.

responding lower bound, especially after applying the cuts. Very interestingly, however, recent BOSS analysis suggests m_{total} could be quite large, with a center value ~ 0.36 eV [41]. As seen clearly from Fig. 2.6, a large value of m_{total} would favor inverted scenario. Some possible consequences of this suggestion will be discussed in Section 2.4.

2.2.3 Leptogenesis

As explained in Section 2.1, with our approximate $U(1)$ flavor symmetry, the baryon asymmetry $\eta_{B0} \equiv \frac{n_B}{n_\gamma}$ can be computed through the standard leptogenesis calculations [38, 39]. For each

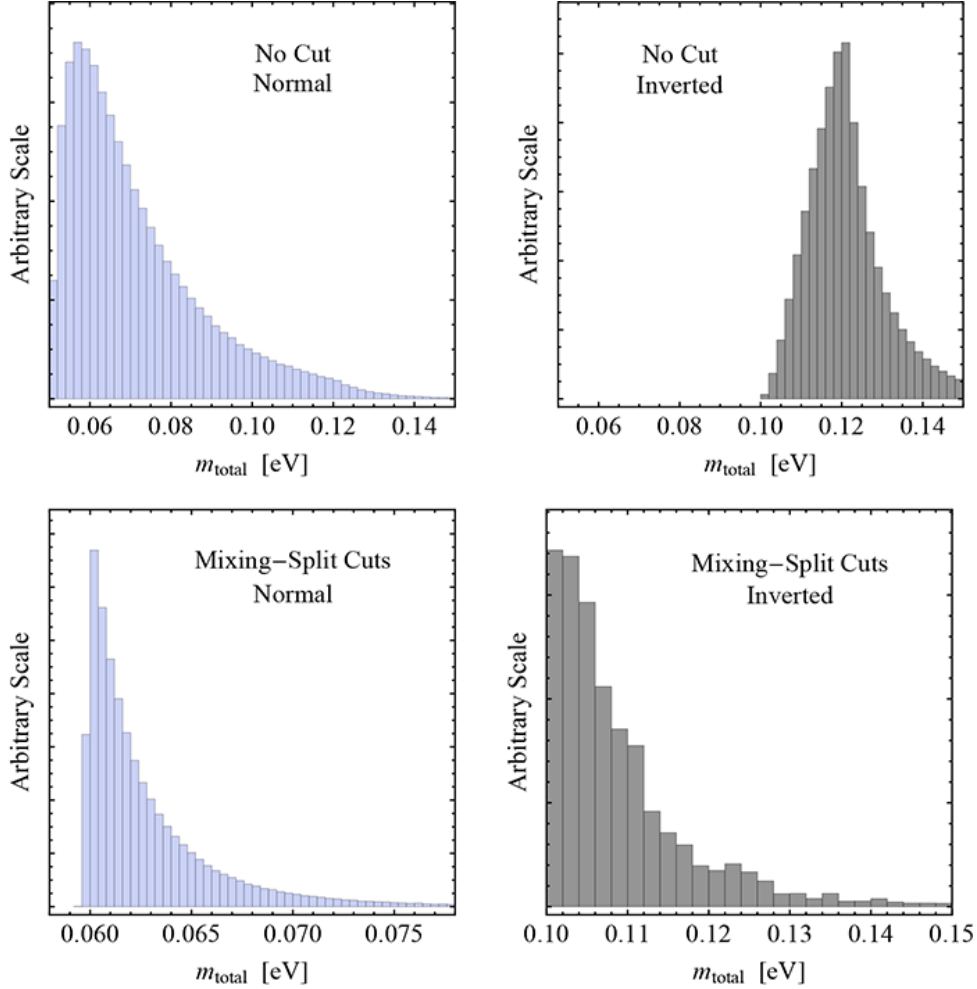


Figure 2.6: Histogram of m_{total} with two mass hierarchy scenarios plotted separately. Left/Right column shows distribution under normal/inverted scenario. Upper/Lower row shows distribution before/after applying the Mixing-Split cuts. The plot of inverted scenario with Mixing-Split cuts applied (right bottom) contains 10^4 occurrences, while other plots contain 10^6 occurrences.

event of m_R and m_D generated, we solve the following Boltzmann equations numerically.

$$\frac{dN_1}{dz} = -(N_1 - N_1^{\text{eq}})(D + S), \quad (2.33)$$

$$\frac{dN_{B-L}}{dz} = -(N_1 - N_1^{\text{eq}})\varepsilon D - N_{B-L}W, \quad (2.34)$$

where we have followed the notations in [38] and [39].

The argument $z \equiv M_1/T$ is evolved from $z_i = 0.001$ to $z_f = 20.0$. Evolving z further beyond 20.0 is not necessary, because the value of N_{B-L} becomes frozen shortly after $z > 10.0$. The baryon asymmetry is then given by $\eta_{B0} = 0.013N_{B-L}^0 \approx 0.013N_{B-L}(z_f)$ [38]. Due to randomly

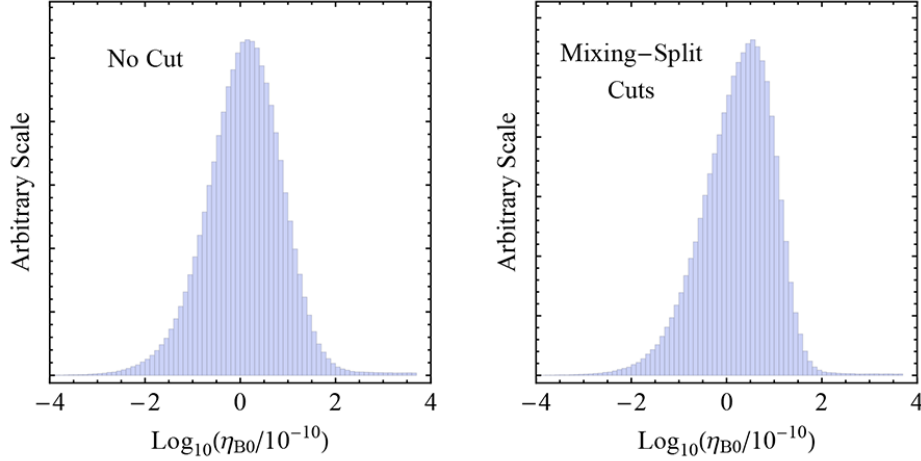


Figure 2.7: Histogram of η_{B0} with 10^6 occurrences collected. Left/Right panel shows distribution before/after applying Mixing-Split cuts.

generated m_R and m_D , we have equal chances for ε to be positive or negative. It is the absolute value that is meaningful. The initial condition $N_{B-L}(z_i) = 0$ is taken. Actually two typical initial conditions for N_1 are tried, $N_1(z_i) = 0$ and $N_1(z_i) = N_1^{\text{eq}}(z_i)$, and no recognizable differences is found. The distributions of η_{B0} , both before and after applying the Mixing-Split cuts, are shown in Fig. 2.7. We see from figure that our prediction on η_{B0} is about the correct order of magnitude as $\eta_{B0,\text{exp}} \approx 6 \times 10^{-10}$ [52].

Let us try to understand the results from some crude estimates. First, let us see why there is almost no difference between the two initial conditions $N_1(z_i) = 0$ and $N_1(z_i) = N_1^{\text{eq}}(z_i)$. The decay function $D(z)$ has the form [39]:

$$D(z) = \alpha_D \frac{K_1(z)}{K_2(z)} z, \quad (2.35)$$

where $K_1(z)$ and $K_2(z)$ are modified Bessel functions, and the constant α_D is proportional to the effective neutrino mass $\tilde{m}_1$:

$$\alpha_D \propto \tilde{m}_1 \equiv \frac{(m_D^\dagger m_D)_{11}}{M_1}. \quad (2.36)$$

So roughly speaking, after $z \gtrsim 1$, the modified Bessel functions become rather close and $D(z)$ increases linearly with z . But in our prescription, the value of $D(z) + S(z)$ becomes already quite large, typically around 50, at $z = 1.0$. So N_1 is forced to be very close to N_1^{eq} thereafter and the solution to the first differential equation (Eq. (2.33)) is approximately

$$N_1 - N_1^{\text{eq}} = -\frac{1}{D+S} \frac{dN_1}{dz} \approx -\frac{1}{D+S} \frac{dN_1^{\text{eq}}}{dz}. \quad (2.37)$$

Of course, this initial-condition-independent solution only holds when $D + S$ is large enough, typically for $z > 1.0$. The values of $N_1 - N_1^{\text{eq}}$ at $z < 1.0$ certainly have a considerable dependence on $N_1(z_i)$. However, the solution to the second differential equation (Eq.(2.34)) is

$$N_{B-L}^0 = \varepsilon \cdot \int_0^\infty dz \frac{D}{D+S} \frac{dN_1}{dz} e^{-\int_z^\infty W(z') dz'} . \quad (2.38)$$

And due to the shape of $W(z)$, yield of N_{B-L}^0 at $z < 1.0$ is significantly suppressed by a factor $e^{-\int_z^\infty W(z') dz'}$. Therefore η_{B0} turns out to be almost independent of $N_1(z_i)$.

Second, let us crudely estimate the order of magnitude of η_{B0} . In addition to $D(z)$, the scattering functions $S(z)$ and washout function $W(z)$ are also proportional to $\tilde{m}_1$. So $\tilde{m}_1$ is the key factor that significantly affects the evolution of Eq. (2.33) and (2.34) [39]. In our anarchy model, apart from the overall units, the mass matrix entries are $O(1)$ numbers, so we expect

$$\tilde{m}_1 = O(1) \cdot \frac{\mathcal{D}^2}{\mathcal{M}} . \quad (2.39)$$

This is just the light-neutrino mass scale. Because in our simulation, $\mathcal{M}$ is chosen such that $\Delta m_l^2 = 2.5 \times 10^3 \text{ eV}^2$, we have

$$\frac{\mathcal{D}^2}{\mathcal{M}} = O(1) \cdot \sqrt{\Delta m_l^2} = O(1) \cdot 0.05 \text{ eV} . \quad (2.40)$$

Therefore most of the time, our model is in the “strong washout regime” [39], since $\tilde{m}_1$ is much larger than the *equilibrium neutrino mass* m_* :

$$\tilde{m}_1 \sim 0.05 \text{ eV} \gg m_* \sim 10^{-3} \text{ eV} . \quad (2.41)$$

In this regime, the integral in Eq.(2.38), which is called *efficiency factor* κ_f , should be around [39]

$$\kappa_f = \int_0^\infty dz \frac{D}{D+S} \frac{dN_1}{dz} e^{-\int_z^\infty W(z') dz'} \sim 10^{-2} . \quad (2.42)$$

Thus our baryon asymmetry is

$$\eta_{B0} = 0.013 N_{B-L}^0 \sim 1.3 \times 10^{-4} \cdot \varepsilon . \quad (2.43)$$

To estimate the CP asymmetry ε , we notice that (following the notation of [39])

$$K \equiv h^\dagger h \sim \left(\frac{\sqrt{2}}{v} \mathcal{D} \right)^2 \begin{pmatrix} \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} , \quad (2.44)$$

and thus

$$\begin{aligned} \varepsilon &= \varepsilon_V + \varepsilon_M \sim \frac{3}{16\pi} \sum_{k=1}^3 \frac{\text{Im}(K_{1k}^2)}{K_{11}} \frac{M_1}{M_k} = \frac{3}{16\pi} \left[\frac{\text{Im}(K_{12}^2)}{K_{11}} \frac{M_1}{M_2} + \frac{\text{Im}(K_{13}^2)}{K_{11}} \frac{M_1}{M_3} \right] \\ &\sim \frac{3}{16\pi} \left(\frac{\sqrt{2}}{v} \mathcal{D} \right)^2 \left(\frac{\epsilon^6}{\epsilon^4} \epsilon^2 + \frac{\epsilon^4}{\epsilon^4} \epsilon^4 \right) \sim \frac{3}{4\pi} \left(\frac{\mathcal{D}}{v} \right)^2 \epsilon^4 \sim 3 \times 10^{-7} . \end{aligned} \quad (2.45)$$

So the baryon asymmetry is expected to be around $\eta_{B0} \sim 1.3 \times 10^{-4} \cdot \varepsilon \sim 4 \times 10^{-11}$, multiplied by some $O(1)$ factor. This is what we see from Fig. 2.7.

Our use of $U(1)$ flavor symmetry breaking plays an essential role to produce the correct order of ε ($\varepsilon \sim \epsilon^4$) and thus η_{B0} . It is thus interesting to investigate what would happen if we had a different $U(1)$ charge assignment. An arbitrary charge assignment would be, upon symmetry breaking, equivalent to an arbitrary choice of D_ϵ parameterized as

$$D_\epsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} = \epsilon_3 \begin{pmatrix} \epsilon_{r1}\epsilon_{r2} & 0 & 0 \\ 0 & \epsilon_{r2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.46)$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \lesssim 1$, and we have defined $\epsilon_{r2} \equiv \epsilon_2/\epsilon_3$ and $\epsilon_{r1} \equiv \epsilon_1/\epsilon_2$ for convenience. To make the simplest scenario of leptogenesis work, we need the hierarchy among the heavy-neutrino masses. So we restrict ourselves to the case $\epsilon_{r1}, \epsilon_{r2} \ll 1$.

The Majorana mass matrix now becomes

$$\begin{aligned} m_R &= D_\epsilon m_{R0} D_\epsilon \sim \epsilon_3^2 \begin{pmatrix} \epsilon_{r1}^2 \epsilon_{r2}^2 & \epsilon_{r1} \epsilon_{r2}^2 & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2}^2 & \epsilon_{r2}^2 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix} \\ &\sim \epsilon_3^2 \begin{pmatrix} 1 & \epsilon_{r1} & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} & 1 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix} \begin{pmatrix} \epsilon_{r1}^2 \epsilon_{r2}^2 & 0 & 0 \\ 0 & \epsilon_{r2}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon_{r1} & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} & 1 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix}, \end{aligned} \quad (2.47)$$

which gives

$$M \sim \epsilon_3^2 \begin{pmatrix} \epsilon_{r1}^2 \epsilon_{r2}^2 & 0 & 0 \\ 0 & \epsilon_{r2}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.48)$$

$$U_N \sim \begin{pmatrix} 1 & \epsilon_{r1} & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} & 1 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix}. \quad (2.49)$$

The Dirac mass matrix becomes

$$m_D = m_{D0} D_\epsilon, \quad (2.50)$$

which gives the Yukawa coupling h and $K \equiv h^\dagger h$ as

$$h = \frac{\sqrt{2}}{v} m_D U_N^* \sim \frac{\sqrt{2}}{v} m_{D0} \epsilon_3 \begin{pmatrix} \epsilon_{r1} \epsilon_{r2} & \epsilon_{r1}^2 \epsilon_{r2} & \epsilon_{r1}^2 \epsilon_{r2}^2 \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & \epsilon_{r2}^2 \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix} \quad (2.51)$$

$$K \sim \left(\frac{\sqrt{2}}{v} m_{D0}\right)^2 \epsilon_3^2 \begin{pmatrix} \epsilon_{r1}^2 \epsilon_{r2}^2 & \epsilon_{r1} \epsilon_{r2}^2 & \epsilon_{r1} \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2}^2 & \epsilon_{r2}^2 & \epsilon_{r2} \\ \epsilon_{r1} \epsilon_{r2} & \epsilon_{r2} & 1 \end{pmatrix}. \quad (2.52)$$

So our ε is given by

$$\varepsilon = \varepsilon_V + \varepsilon_M \sim \frac{3}{16\pi} \sum_{k=1}^3 \frac{\text{Im}(K_{1k}^2)}{K_{11}} \frac{M_1}{M_k} \sim \frac{3}{8\pi} \left(\frac{\sqrt{2}}{v}\mathcal{D}\right)^2 \epsilon_3^2 \epsilon_{r1}^2 \epsilon_{r2}^2 \sim \frac{3}{4\pi} \left(\frac{\mathcal{D}}{v}\right)^2 \epsilon_1^2. \quad (2.53)$$

We see that under the condition $\epsilon_{r1}, \epsilon_{r2} \ll 1$, ε is only sensitive to the value of ϵ_1 .

On the other hand, the value of $\tilde{m}_1$ is not affected by changing $U(1)$ flavor charge assignments:

$$\tilde{m}_1 \equiv \frac{(m_D^\dagger m_D)_{11}}{M_1} \sim \frac{(m_{D0}^\dagger m_{D0})_{11}}{(M_1)_0} \frac{\epsilon_1^2}{\epsilon_1^2} = \frac{(m_{D0}^\dagger m_{D0})_{11}}{(M_1)_0} = (\tilde{m}_1)_0. \quad (2.54)$$

Here a subscript “0” is used to denote the value when there is no $U(1)$ flavor charge assignment, as we did in Eq. (2.18). So the strong washout condition (Eq. (2.41)) still holds, and we are again led to Eq. (2.43). Therefore, the baryon asymmetry η_{B0} can only be affected through ε , which in turn is only sensitive to ϵ_1 , under the condition $\epsilon_{r1}, \epsilon_{r2} \ll 1$.

2.3 Correlations between η_{B0} and Light-neutrino Parameters

As we can see from Fig. 2.7, the baryon asymmetry is slightly enhanced after applying the Mixing-Split cuts Eq. (2.29)-(2.32). This indicates some correlation between η_{B0} and light-neutrino parameters. To understand this better, let us systematically investigate the correlations between η_{B0} and the light-neutrino mass matrix $m_\nu = U_\nu m U_\nu^T$.²

Although both of η_{B0} and m_ν seem to depend on the random inputs m_R and m_D in a complicated way, it is not hard to see that there should be no correlation between η_{B0} and U_ν (This was also pointed out in [40]). Recall that we parameterize m_R and m_D as in Eq. (2.7)-(2.8). And due to the decomposition Eq. (2.9)-(2.10), there are five independent random matrices: U_1, U_2, U_R, D_0 and D_R . The first thing to observe is that changing U_1 with the other four matrices fixed will not affect η_{B0} . This is because:

1. The baryon asymmetry η_{B0} we have been computing is the total baryon asymmetry, including all the three generations. So m_D enters the calculation of leptogenesis only through the form of the matrix

$$K \equiv h^\dagger h = \left(\frac{\sqrt{2}}{v}\right)^2 U_N^T m_D^\dagger m_D U_N^*, \quad (2.55)$$

with $m_D = \mathcal{D} \cdot U_1 D_0 U_2^\dagger$. Obviously U_1 cancels in K .

2. Throughout the simulation, we are also applying a built-in cut $\Delta m_l^2 = 2.5 \times 10^{-3} eV^2$ by choosing the value of $\mathcal{M}$ to force it. Due to this cut, m_D can potentially affect η_{B0} through the value of $\mathcal{M}$. However, since the actual relation is

$$m_\nu = m_D m_R^{-1} m_D^T = \frac{\mathcal{D}^2}{\mathcal{M}} U_1 D_0 U_2^\dagger m_R^{-1} U_2^* D_0 U_1^T = U_\nu m U_\nu^T, \quad (2.56)$$

²The correlation between leptogenesis and light-neutrino parameters has been recently studied in [40]. They have some overlap with our results.

we see that changing U_1 would only affect U_ν , not m . So no further adjustment of $\mathcal{M}$ is needed when we change U_1 .

The second point to observe is that any change in U_ν can be achieved by a left translation

$$U_{\nu a} \rightarrow U_{\nu b} = (U_{\nu b} U_{\nu a}^{-1}) U_{\nu a} \equiv U_L U_{\nu a} \quad (2.57)$$

This in turn, can be accounted for by just a left translation in U_1 : $U_{1a} \rightarrow U_{1b} = U_L U_{1a}$, with the other four random matrices unchanged (see Eq. (2.56)). This left translation in U_1 is thus a one-to-one mapping between the sub-sample generating $U_{\nu a}$ and the sub-sample generating $U_{\nu b} = U_L U_{\nu a}$. Any two events connected through this one-to-one mapping generate the same value of η_{B0} , because changing U_1 does not change η_{B0} . In addition, the two events have the same chance to appear, because the measure over U_1 is the Haar measure, which is invariant under the left translation. Thus the sub-sample with $U_\nu = U_{\nu a}$ and $U_\nu = U_{\nu b}$, for any arbitrary $U_{\nu a}$ and $U_{\nu b}$, will give the same distribution of η_{B0} , namely that η_{B0} is independent of U_ν . So immediately we conclude that η_{B0} cannot be correlated with the three mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$, the CP phase δ_{CP} , or the phases χ_1, χ_2 .

Three of the Mixing-Split cuts applied to η_{B0} are cuts on mixing angles which we just showed not correlated with η_{B0} . So clearly, the enhancement of η_{B0} is due to its non-zero correlation with R . To study more detail about the correlation between η_{B0} and the light-neutrino masses m , we apply a χ^2 test of independence numerically to the joint distribution between η_{B0} and quantities related to m , including $\lg R$, m_{eff} and m_{total} . For each quantity with η_{B0} , we construct a discrete joint distribution by counting the number of occurrences O_{ij} ($i, j = 1, \dots, 10$) in an appropriate 10×10 partitioning grid. Then we obtain the expected number of occurrences E_{ij} as

$$E_{ij} = \frac{1}{n} \left(\sum_{c=1}^{10} O_{ic} \right) \left(\sum_{r=1}^{10} O_{rj} \right), \quad (2.58)$$

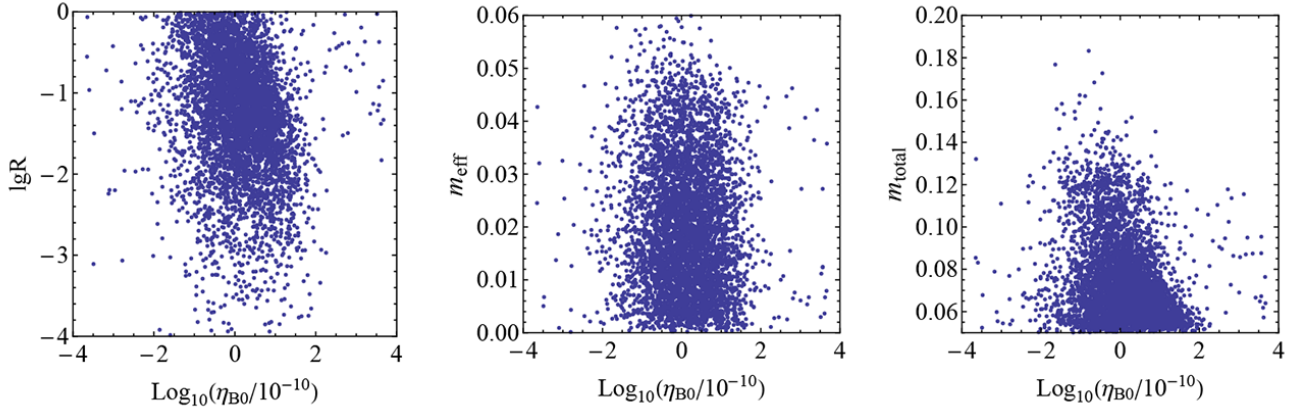
where n is the total number of occurrences in all 10×10 partitions. If the two random variables in question were independent of each other, we would have the test statistic

$$X = \sum_{i,j=1}^{10} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}, \quad (2.59)$$

satisfying the χ^2 distribution with degrees of freedom $(10-1) \times (10-1) = 81$. We then compute the probability $P(\chi^2 > X)$ for the hypothesis distribution $\chi^2(81)$ to see if the independence hypothesis is likely. Our results from a $n = 3,000,000$ sample Monte Carlo are shown in Table. 2.2.

Unambiguously, η_{B0} has nonzero correlations with $\lg R$, m_{eff} , and m_{total} . To see the tendency of the correlations, we draw scatter plots with 5,000 occurrences (Fig. 2.8). The plots show that all the three quantities are negatively correlated with η_{B0} . For example the left panel of Fig. 2.8 tells us that a smaller $\lg R$ would favor a larger η_{B0} . This explains the slight enhancement of η_{B0} after applying Mixing-Split cuts. But as the scatter plots show, the correlations are rather weak.

	X	$P(\chi^2 > X)$
$\lg R$	1.43×10^5	3.81×10^{-30908}
m_{eff}	8.06×10^3	1.24×10^{-1655}
m_{total}	1.04×10^5	7.24×10^{-22445}

Table 2.2: χ^2 test of independence between η_{B0} and $\lg R$, m_{eff} , m_{total} .Figure 2.8: Scatter plots for η_{B0} with $\lg R$, m_{eff} , and m_{total} . Each plot shows a sample of 5,000 occurrences.

2.4 Possible Consequences of a Large m_{total}

As mentioned previously, a recent BOSS analysis suggests m_{total} possibly quite large, $m_{\text{total}} = 0.36 \pm 0.10$ eV [41]. Currently their uncertainty is still large, and thus no conclusive argument can be made. If in future the uncertainty pins down near its current central value, anarchy prediction (Fig. 2.4) would be obviously inconsistent with it and becomes ruled out. On the other hand, if the central value also comes down significantly, it could still be well consistent with anarchy prediction.

Without the knowledge of future data, we would like to answer the following question: Assuming the future data be consistent with anarchy, could a relatively large m_{total} dramatically change anarchy's predictions on other quantities? For this purpose, we introduce a heuristic m_{total} cut:

$$m_{\text{total}} > 0.1 \text{ eV}, \quad (2.60)$$

just to get a sense of how much our predictions could be changed if there turns out to be a large but still consistent m_{total} .

We collect 10^4 occurrences that pass both the Mixing-Split cuts and the m_{total} cut. It turns out that the predictions change quite significantly. We see from Fig. 2.2 that the mass hierarchy

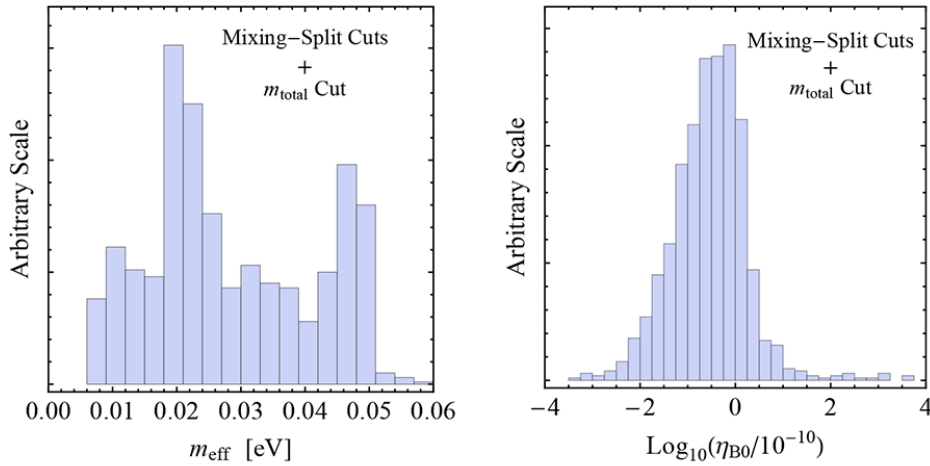


Figure 2.9: Histograms of m_{eff} (left) and η_{B0} (right) with 10^4 occurrences passed both Mixing-Split cuts and the m_{total} cut.

prediction is overturned, with normal hierarchy only 40% and inverted scenario more likely. This can be expected from Fig. 2.6. The predictions of m_{eff} and η_{B0} are shown in Fig. 2.9. We see that m_{eff} exhibits a very interesting bipolar distribution. Its overall expectation value also becomes about an order of magnitude larger than before and thus much less challenging to the neutrinoless double beta decay experiments. The prediction on η_{B0} drops by about an order of magnitude, but the observed baryon asymmetry is still very likely to be achieved.

2.5 Discussions

We have shown that basis independence and free entry independence lead uniquely to Gaussian measure for m_R and m_D . We also showed that an approximate $U(1)$ flavor symmetry can make leptogenesis feasible for neutrino anarchy. Combining the two, we find anarchy model successfully generate the observed amount of baryon asymmetry. Same sampling model is used to study other quantities related to neutrino masses. We found the chance of normal mass hierarchy is as high as 99.9%. The effective mass of neutrinoless double beta decay m_{eff} would probably be well beyond the current experimental sensitivity. The neutrino total mass m_{total} is a little more optimistic. Correlations between baryon asymmetry and light-neutrino quantities were also investigated. We found η_{B0} not correlated with light-neutrino mixings or phases, but weakly correlated with R , m_{eff} , and m_{total} , all with negative correlation. Possible implications of recent BOSS analysis result have been discussed.

Chapter 3

Dirac NMSSM and Semi-soft Supersymmetry Breaking

The discovery of a new resonance at 125 GeV [1], that appears to be the long-sought Higgs boson, marks a great triumph of experimental and theoretical physics. On the other hand, the presence of this light scalar forces us to face the naturalness problem of its mass. Arguably, the best known mechanism to ease the naturalness problem is weak-scale supersymmetry (SUSY), but the lack of experimental signatures is pushing SUSY into a tight corner. In addition, the observed mass of the Higgs boson is higher than what was expected in the Minimal Supersymmetric Standard Model (MSSM), requiring fine-tuning of parameters at the 1% level or worse [22].

If SUSY is realized in nature, one possibility is to give up on naturalness [53]. Alternatively, theories that retain naturalness must address two problems, (I) the missing superpartners and (II) the Higgs mass. The collider limits on superpartners are highly model-dependent and can be relaxed when superpartners unnecessary for naturalness are taken to be heavy [54, 55], when less missing energy is produced due to a compressed mass spectrum [56] or due to decays to new states [57], and when R -parity is violated [58]. Even if superpartners have evaded detection for one of these reasons, we must address the surprisingly heavy Higgs mass.

There have been many attempts to extend the MSSM to accommodate the Higgs mass. In such extensions, new states interact with the Higgs, raising its mass by increasing the strength of the quartic interaction of the scalar potential. If the new states are integrated out supersymmetrically, their effects decouple and the Higgs mass is not increased. On the other hand, SUSY breaking can lead to non-decoupling effects that increase the Higgs mass. One possibility is a non-decoupling F -term, as in the NMSSM (MSSM plus a singlet) [59, 60] or λ SUSY (allowing for a Landau pole) [61–63]. A second possibility is a non-decoupling D -term that results if the Higgs is charged under a new gauge group [64]. In general, these extensions require new states at the few hundred GeV scale, so that the new sources of SUSY breaking do not spoil naturalness.

For example, consider the NMSSM, where a singlet superfield, S , interacts with the MSSM Higgses, $H_{u,d}$, through the superpotential,

$$W \supset \lambda S H_u H_d + \frac{M}{2} S^2 + \mu H_u H_d. \quad (3.1)$$

The Higgs mass is increased by,

$$\Delta m_h^2 = \lambda^2 v^2 \sin^2 2\beta \left(\frac{m_S^2}{M^2 + m_S^2} \right), \quad (3.2)$$

where m_S^2 is the SUSY breaking soft mass $m_S^2 |S|^2$, $\tan \beta = v_u/v_d$ is the ratio of the VEVs of the up and down-type Higgses, and $v = \sqrt{v_u^2 + v_d^2} = 174$ GeV. Notice that this term decouples in the supersymmetric limit, $M \gg m_S$, which means m_S should not be too small. On the other hand, m_S feeds into the Higgs soft masses, $m_{H_{u,d}}^2$ at one-loop, requiring fine-tuning if $m_S \gg m_h$. Therefore, with M at the weak-scale, there is tension between raising the Higgs mass, which requires large m_S relative to M , and naturalness, which demands small m_S .

In this chapter, we will see that, contrary to the above example, a *lack of light scalars* can help raise the Higgs mass without a cost to naturalness, if the singlet has a Dirac mass. We begin by introducing the model and discussing the Higgs mass and naturalness properties. Then, we discuss the phenomenology of the Higgs sector, which can be discovered or constrained with future collider data. We finish with our conclusions.

3.1 Dirac NMSSM

To illustrate this possibility, we consider a modification of Eq. (3.1) where S receives a Dirac mass with another singlet, $\bar{S}$,

$$W = \lambda S H_u H_d + M S \bar{S} + \mu H_u H_d. \quad (3.3)$$

We call this model the *Dirac NMSSM*. The absence of various dangerous operators (such as large tadpoles for the singlets) follows from a $U(1)_{PQ} \times U(1)_{\bar{S}}$ Peccei-Quinn-like symmetry,

	H_u	H_d	S	$\bar{S}$	μ	M
$U(1)_{PQ}$	1	1	-2	-2	-2	4
$U(1)_{\bar{S}}$	0	0	0	1	0	-1

Here, $U(1)_{\bar{S}}$ has the effect of differentiating S and $\bar{S}$ and forbidding the operator $\bar{S} H_u H_d$. Because μ and M explicitly break the $U(1)_{PQ} \times U(1)_{\bar{S}}$ symmetry, we regard them to be spurions originating from chiral superfields (“flavons” [65]) so that holomorphy is used to avoid certain unwanted terms (“SUSY zeros” [66]). By classifying all possible operators induced by these spurions, we see that a tadpole for $\bar{S}$ is suppressed adequately,

$$W \supset c_{\bar{S}} \mu M \bar{S}, \quad (3.4)$$

where $c_{\bar{S}}$ is a $\mathcal{O}(1)$ coefficient. Other terms involving only singlets are forbidden by the symmetries or suppressed by the cutoff.

The following soft supersymmetry breaking terms are allowed by the symmetries,

$$\begin{aligned}\Delta V_{\text{soft}} = & m_{H_u}^2 |H_u|^2 + m_{H_d}^2 |H_d|^2 + m_S^2 |S|^2 + m_{\bar{S}}^2 |\bar{S}|^2 \\ & + \lambda A_\lambda S H_u H_d + M B_S S \bar{S} + \mu B H_u H_d + c.c. \\ & + t_{\bar{S}} \bar{S} + t_S S + c.c.\end{aligned}\quad (3.5)$$

The last tadpole arises from a non-holomorphic term $\mu^\dagger S$. The small hierarchy, which we consider later, between the soft masses of $\bar{S}$ and others can be naturally obtained in gauge mediation models if $\bar{S}$ couples to the messengers in the superpotential. It is easy to write down a model where $m_{\bar{S}}^2$ is positive at the one-loop level, while the soft masses for squarks and sleptons arise at the two-loop level via gauge mediation [67]. The tadpole for $\bar{S}$ is generated at one-loop and hence $t_{\bar{S}} \simeq \mu M m_{\bar{S}} / 4\pi$, while soft masses and the tadpole for S are generated at higher order. We checked these singlet tadpoles do not introduce extra tuning even with large M and $m_{\bar{S}}$.

We would like to understand whether the new quartic term, $|\lambda H_u H_d|^2$, can naturally raise the Higgs mass. Integrating out S and $\bar{S}$ we find the following potential for the doublet-like Higgses,

$$V_{\text{eff}} = |\lambda H_u H_d|^2 \left(1 - \frac{M^2}{M^2 + m_{\bar{S}}^2} \right) - \frac{\lambda^2}{M^2 + m_{\bar{S}}^2} |A_\lambda H_u H_d + \mu^* (|H_u|^2 + |H_d|^2)|^2. \quad (3.6)$$

where we keep leading $(M^2 + m_{\bar{S}}^2)^{-1}$ terms and neglect the tadpole terms for simplicity. The additional Higgs quartic term does not decouple when $m_{\bar{S}}^2$ is large, as in the NMSSM at large $m_{\bar{S}}^2$. The SM-like Higgs mass becomes,

$$m_h^2 = m_{h,\text{MSSM}}^2(m_{\bar{t}}) + \lambda^2 v^2 \sin^2 2\beta \left(\frac{m_{\bar{S}}^2}{M^2 + m_{\bar{S}}^2} \right) - \frac{\lambda^2 v^2}{M^2 + m_{\bar{S}}^2} |A_\lambda \sin 2\beta - 2\mu^*|^2, \quad (3.7)$$

in the limit where the VEVs and mass-eigenstates are aligned.

The Higgs sector is natural when there are no large radiative corrections to $m_{H_{u,d}}^2$. The renormalization group (RG) of the up-type Higgs contains the terms,

$$\mu \frac{d}{d\mu} m_{H_u}^2 = \frac{1}{8\pi^2} \left(3y_t^2 [m_{\tilde{Q}_3}^2 + m_{\tilde{t}_R}^2] + \lambda^2 m_S^2 \right) + \dots \quad (3.8)$$

While heavy stops or m_S^2 lead to fine-tuning, we find that $m_{\bar{S}}^2$ does not appear. In fact, the RGs for $m_{H_{u,d}}^2$ are independent of $m_{\bar{S}}^2$ to all orders in mass-independent schemes, because $\bar{S}$ couples to the MSSM+ S sector only through the dimensionful coupling M . There is logarithmic sensitivity to $m_{\bar{S}}^2$ from the one-loop finite threshold correction,

$$\delta m_H^2 \equiv \delta m_{H_{u,d}}^2 = \frac{(\lambda M)^2}{(4\pi)^2} \log \frac{M^2 + m_{\bar{S}}^2}{M^2}. \quad (3.9)$$

which still allows for very heavy $m_{\bar{S}}^2$ without fine-tuning.

One may wonder if there are dangerous finite threshold corrections to $m_{H_u}^2$ at higher order, after integrating out $\bar{S}$. In fact, there is no quadratic sensitivity to $m_{\bar{S}}^2$ to all orders. This follows

because any dependence on $m_{\bar{S}}^2$ must be proportional to $|M|^2$ (since $\bar{S}$ becomes free when $M \rightarrow 0$ and by conservation of $U(1)_{\bar{S}}$), but $|M|^2 m_{\bar{S}}^2$ has too high mass dimension. The mass dimension cannot be reduced from other mass parameters appearing in the denominator because threshold corrections are always analytic functions of IR mass parameters [68].

It may seem contradictory that naturalness is maintained in the limit of very heavy $m_{\bar{S}}$, since removing the $\bar{S}$ scalar from the spectrum constitutes a hard breaking of SUSY. The reason is that the effective theory, with the $\bar{S}$ fermion but no scalar present at low energies is actually equivalent to a theory with only softly broken supersymmetry, where the MSSM is augmented by the Kähler operators,

$$\mathcal{K}_{\text{eff}} = \bar{S}^\dagger \bar{S} - \theta^2 \bar{\theta}^2 (M \mathcal{D}^\alpha S \mathcal{D}_\alpha \bar{S} + c.c. + M^2 |S + c_{\bar{S}} \mu|^2), \quad (3.10)$$

and where the scalar and F -term of $\bar{S}$ are reintroduced at low-energy but completely decoupled from the other states. We call this mechanism *semi-soft supersymmetry breaking*. It is crucial that $\bar{S}$ couples to the other fields only through dimensionful couplings. Note that Dirac gauginos are a different example where adding new fields can lead to improved naturalness properties [69].

The most natural region of parameter space, summarized in Fig. 3.1, has m_S and M at the hundreds of GeV scale, to avoid large corrections to m_{H_u} , and large $m_{\bar{S}} \gtrsim 10$ TeV, to maximize the second term of Eq. (3.7). The tree-level contribution to the Higgs mass can be large enough such that $m_{\tilde{t}}$ takes a natural value at the hundreds of GeV scale.

We have performed a quantitative study of the fine-tuning in the Dirac NMSSM, shown to the left of Fig. 3.2 as a function of $(M, m_{\bar{S}})$. We computed the radiative corrections from the top sector to the Higgs mass at RG-improved Leading-Log order, analogous to [70]. We have confirmed that our results match the FeynHiggs software [71], for the MSSM, within $\Delta m_h \simeq 1$ GeV in the parameter regime of interest. We fix $A_t = 0$ for simplicity, and other parameters are fixed according to the table, shown below. Here, we adopt a parameter $\mu_{\text{eff}} \equiv \mu + \lambda \langle S \rangle$ for convenience. We have chosen λ to saturate the upper-limit such that it does not reach a Landau pole below the

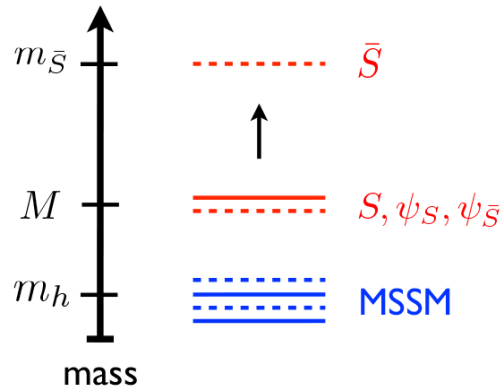


Figure 3.1: A typical spectrum of the Dirac NMSSM, which allows for large $m_{\bar{S}}$ without spoiling naturalness.

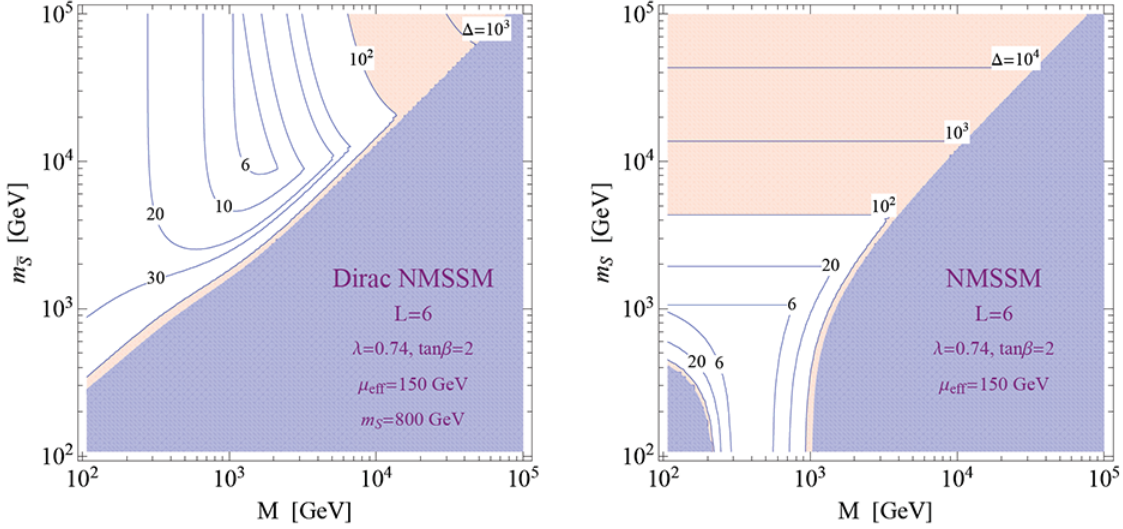


Figure 3.2: The tuning Δ , defined in Eq. (3.11), for the Dirac NMSSM is shown on the *left* as a function of M and $m_{\bar{S}}$. For comparison, the tuning of the NMSSM is shown on the *right*, as a function of M and m_S . The red region has high fine-tuning, $\Delta > 100$, and the purple region requires $m_{\tilde{t}} > 2$ TeV, signaling severe fine-tuning $\gtrsim \mathcal{O}(10^3)$.

unification scale [72]. For each value of $(M, m_{\bar{S}})$, the stop soft masses, $m_{\tilde{t}} = m_{\tilde{t}_R} = m_{\tilde{Q}_3}$, are chosen to maintain the lightest scalar mass at 125 GeV.

The degree of fine-tuning is estimated by

$$\Delta = \frac{2}{m_h^2} \max \left(m_{H_u}^2, m_{H_d}^2, \frac{dm_{H_u}^2}{d \ln \mu} L, \frac{dm_{H_d}^2}{d \ln \mu} L, \delta m_H^2, b_{\text{eff}} \right), \quad (3.11)$$

where $b_{\text{eff}} = \mu B + \lambda(A_\lambda \langle S \rangle + M \langle \bar{S} \rangle)$ and we take $L \equiv \log(\Lambda/m_{\tilde{t}}) = 6$, corresponding to low-scale SUSY breaking. We assume that contributions through gauge couplings to the RGs for $m_{H_{u,d}}^2$ are subdominant.

For comparison, the right of Fig. 3.2 shows the tuning in the NMSSM, which corresponds to the Dirac NMSSM replacing $\bar{S} \rightarrow S$ (which removes the $U(1)_{\bar{S}}$ symmetry). The superpotential of the NMSSM corresponds to Eq.(3.1) plus the tadpole $c_S \mu M S$. We treat m_S as a free parameter instead of $m_{\bar{S}}$ and use the same fine-tuning measure of Eq.(3.11), except the threshold correction δm_H^2 is absent and $b_{\text{eff}} = \mu B + \lambda \langle S \rangle (A_\lambda + M)$.

benchmark parameters		
$\lambda = 0.74$	$\tan \beta = 2$	$\mu_{\text{eff}} = 150 \text{ GeV}$
$b_{\text{eff}} = (190 \text{ GeV})^2$	$A_\lambda = 0$	$B_s = 100 \text{ GeV}$
$M = 1 \text{ TeV}$	$m_{\bar{S}} = 10 \text{ TeV}$	$m_S = 800 \text{ GeV}$

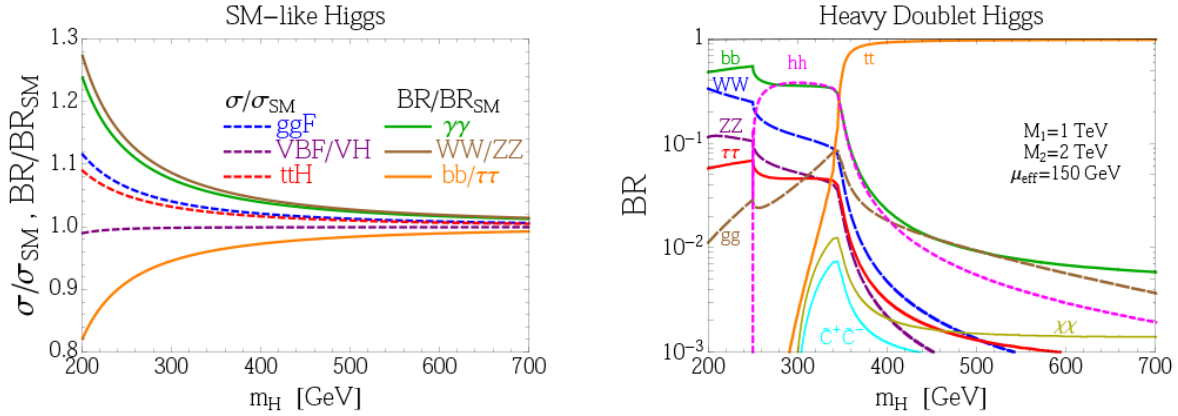


Figure 3.3: The branching ratios and production cross sections of the SM-like Higgs are shown, normalized to the SM values [73, 74], on the *left* as a function of the heavy doublet-like Higgs mass, m_H . On the *right*, we show several branching ratios of the heavy doublet-like Higgs as a function of its mass. Note that the location of the chargino/neutralino thresholds depend on the -ino spectrum. Here we take heavy gauginos and $\mu_{\text{eff}} = 150$ GeV.

We see that the least-tuned region of the Dirac NMSSM corresponds to $M \sim 2$ TeV and $m_{\tilde{S}} \gtrsim 10$ TeV, where the tree-level correction to the Higgs mass is maximized. The fine-tuning is dominated by δm_H^2 in the large M region, and by the contribution of $m_{\tilde{t}}$ to m_{H_u} in the rest of the plane. On the other-hand, the NMSSM becomes highly tuned when m_S is large (since it radiatively corrects $m_{H_{u,d}}$), and then $m_S \lesssim 1$ TeV is favored. Note that region of low-tuning in the NMSSM extends to the supersymmetric limit, $m_S \rightarrow 0$. In this region the Higgs mass is increased by a new contribution to the quartic coupling proportional to $\lambda^2(M\mu \sin 2\beta - \mu^2)/M^2$ (see Ref.[60] for more details).

3.2 Higgs Phenomenology

We now discuss the experimental signatures of the Dirac NMSSM. The phenomenology of the NMSSM is well-studied [22, 83]. The natural region of the Dirac NMSSM differs from the NMSSM in that the singlet states are too heavy to be produced at the LHC. The low-energy Higgs phenomenology is that of a two Higgs doublet model, and we focus here on the nature of the SM-like Higgs, h , and the heavier doublet-like Higgs, H [84]. The properties of the two doublets differ from the MSSM due to the presence of the non-decoupling quartic coupling $|\lambda H_u H_d|^2$, which raises the Higgs mass by the semi-soft SUSY breaking, described above.

We consider the potential with radiative corrections from the stop sector, and we find that the couplings of the SM-like Higgs to leptons and down-type quarks are lowered, while couplings to the up-type quarks are slightly increased compared to those in the SM, which results in the deviations to the cross sections and decay patterns shown to the left of Fig. 3.3. These effects

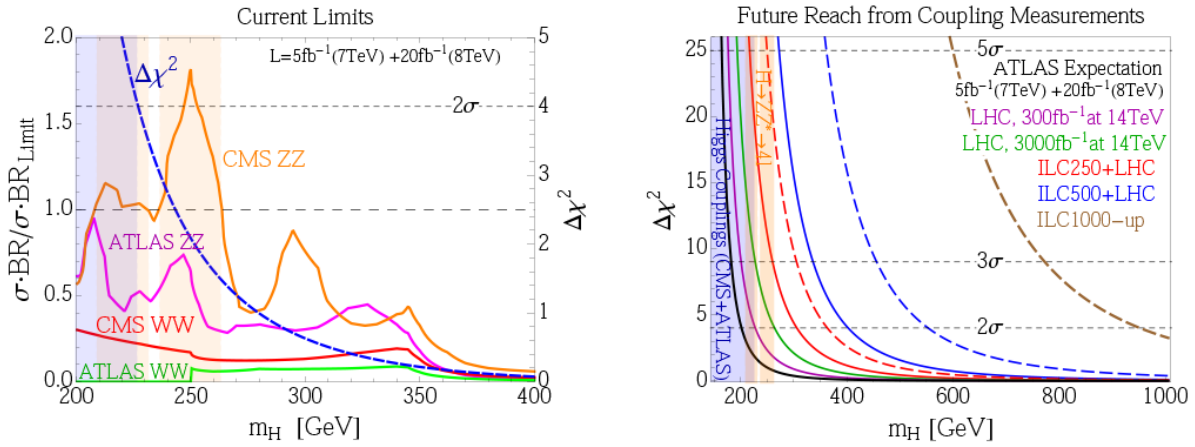


Figure 3.4: The *left* plot shows current constraints on our model. The right axis corresponds to $\Delta\chi^2$ for the SM-like Higgs couplings with the 7 and 8 TeV datasets [75–78] neglecting correlations between the measurements of different couplings. The left axis shows $\sigma/\sigma_{\text{lim}}$ for direct searches, $H \rightarrow ZZ, WW$ [76, 78]. The *right* plot shows the expected $\Delta\chi^2$ from combined measurements of the Higgs-like couplings at the high-luminosity LHC at $\sqrt{s} = 14$ TeV [79] and the ILC at $\sqrt{s} = 250, 500, 1000$ GeV. The optimistic [80, 81] (conservative [82]) ILC reach curves are solid (dashed) and neglect (include) theoretical uncertainties in the Higgs branching ratios. The ILC analyses include the expected LHC measurements. For comparison we show the present limits and also the expected limit of the current ATLAS measurements (solid, black).

decouple in limit $m_H \gg m_h$, which corresponds to large b_{eff} . We also show, to the right of Fig. 3.3, the decay branching ratios of H . Due to the non-decoupling term, di-Higgs decay, $H \rightarrow 2h$, becomes the dominant decay once its threshold is opened, $m_H \gtrsim 250$ GeV.

There are now two relevant constraints on the Higgs sector of the Dirac NMSSM. The first comes from measurements of the couplings of the SM-like Higgs from ATLAS [75, 76] and CMS [77, 78]. The second comes from direct searches for the heavier state decaying to dibosons, $H \rightarrow ZZ, WW$ [76, 78]. The former excludes $m_H \lesssim 220$ GeV at 95%, while the latter extends this limit to $m_H \sim 260$ GeV by the CMS search for $H \rightarrow ZZ$ (except for a small gap near $m_H \approx 235$ GeV), as can be seen to the left of Fig. 3.4. We also estimate the future reach to probe m_H with future Higgs coupling measurements [79–82], shown to the right of Fig. 3.4. The 2σ exclusion reach is $m_H \simeq 280$ GeV at the high-luminosity LHC [79], $m_H \simeq 400$ GeV with theoretical uncertainty at ILC500 [82], and $m_H \simeq 950$ GeV without theoretical uncertainty at upgraded ILC1000 [81]. The increased sensitivity at the ILC is dominated by the improved measurements projected for the $b\bar{b}$ and $\tau^+\tau^-$ couplings [80, 81].

3.3 Discussions

The LHC has discovered a new particle, consistent with the Higgs boson, with a mass near 125 GeV. Weak-scale SUSY must be reevaluated in light of this discovery. Naturalness demands new dynamics beyond the minimal theory, such as a non-decoupling F -term, but this implies new sources of SUSY breaking that themselves threaten naturalness. In this chapter, we have identified a new model where the Higgs couples to a singlet field with a Dirac mass. The non-decoupling F -term is naturally realized through semi-soft SUSY breaking, because large $m_{\tilde{S}}$ helps raise the Higgs mass but does not threaten naturalness. The first collider signatures of the Dirac NMSSM are expected to be those of the MSSM fields, with the singlet sector naturally heavier than 1 TeV.

The key feature of semi-soft SUSY breaking in the Dirac NMSSM is that $\tilde{S}$ couples to the MSSM only through the dimensionful Dirac mass M . We note that interactions between $\tilde{S}$ and other new states are not constrained by naturalness, even if these states experience SUSY breaking. Therefore, the Dirac NMSSM represents a new type of portal, whereby our sector can interact with new sectors, with large SUSY breaking, without spoiling naturalness in our sector.

Chapter 4

Standard Model Effective Field Theory

The discovery of a Standard Model (SM)-like Higgs boson [1] is a milestone in particle physics. Direct study of this boson *will* shed light on the mysteries surrounding the origin of the Higgs boson and the electroweak (EW) scale. Additionally, it will potentially provide insight into some of the many long standing experimental observations that remain unexplained (see, *e.g.*, [2]) by the SM. In attempting to answer questions raised by the EW sector and these presently unexplained observations, a variety of new physics models have been proposed, with little clue which—if any—Nature actually picks.

It is exciting that ongoing and possible near future experiments can achieve an estimated per mille sensitivity on precision Higgs and EW observables [23–28]. This level of precision provides a window to indirectly explore the theory space of BSM physics and place constraints on specific UV models. For this purpose, an efficient procedure of connecting new physics models with precision Higgs and EW observables is clearly desirable.

In this chapter, the Standard Model effective field theory (SM EFT) is used as a bridge to connect models of new physics with experimental observables. The SM EFT consists of the renormalizable SM Lagrangian supplemented with higher-dimension interactions:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{1}{\Lambda^{d_i-4}} c_i \mathcal{O}_i. \quad (4.1)$$

In the above, Λ is the cutoff scale of the EFT, $\mathcal{O}_i$ are a set of dimension d_i operators that respect the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge invariance of $\mathcal{L}_{\text{SM}}$, and c_i are their Wilson coefficients that run as functions $c_i(\mu)$ of the renormalization group (RG) scale μ . The estimated per-mille sensitivity of future precision Higgs measurements justifies truncating the above expansion at dimension-six operators.

It is worth noting that the SM EFT parameterized by the c_i of Eq. (4.1) is totally different from the widely used seven- κ parametrization (*e.g.*, [85]), which captures only a change in size of each of the SM-type Higgs couplings. In fact, the seven κ 's parameterize models that do not respect the electroweak gauge symmetry, and hence, violate unitarity. As a result, future precision programs can show spuriously high sensitivity to the κ . The SM EFT of Eq. (4.1), on the other hand, param-

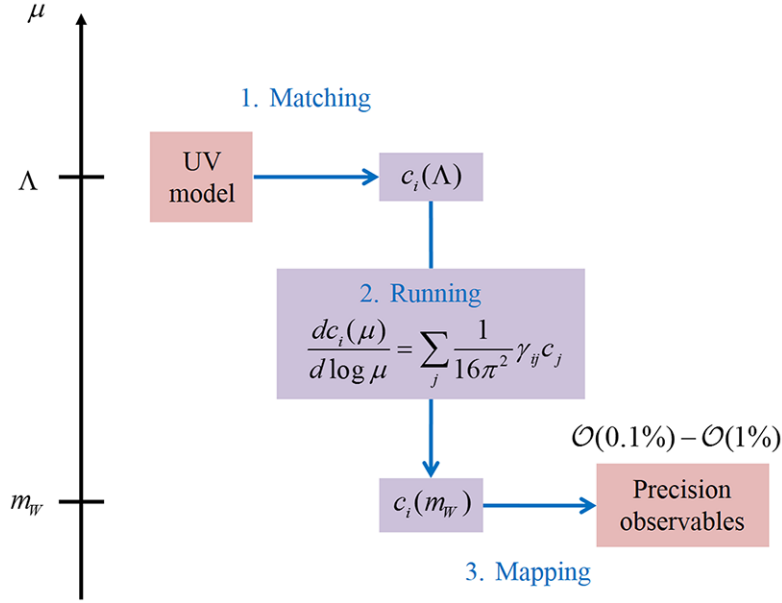


Figure 4.1: SM EFT as a bridge to connect UV models and weak scale precision observables.

eterizes new physics in directions that respect the SM gauge invariance and are therefore free from unitarity violations.¹

In an EFT framework, the connection of UV models² with low-energy observables is accomplished through a three-step procedure schematically described in Fig. 4.1.³ First, the UV model is matched onto the SM EFT at a high-energy scale Λ . This matching is performed order-by-order in a loop expansion. At each loop order, $c_i(\Lambda)$ is determined such that the S -matrix elements in the EFT and the UV model are the same at the RG scale $\mu = \Lambda$. Next, the $c_i(\Lambda)$ are run down to the weak scale $c_i(m_W)$ according to the RG equations of the SM EFT. The leading order solution to these RG equations is determined by the anomalous dimension matrix γ_{ij} . Finally, the effective Lagrangian at $\mu = m_W$ is used to compute weak scale observables in terms of the $c_i(m_W)$ and SM parameters of $\mathcal{L}_{\text{SM}}$. In this chapter, this third step is referred to as mapping the Wilson coefficients onto observables.

In the rest of this chapter, let us consider each of these three steps—matching, running, and mapping—in detail for the SM EFT. In the SM EFT, the main challenge presented at each step is complexity: truncating the expansion in (4.1) at dimension-six operators leaves us with $\mathcal{O}(10^2)$ independent deformations of the Standard Model.⁴ This large number of degrees of freedom can

¹Equation (4.1) is a linear-realization of EW gauge symmetry. An EFT constructed as a non-linear realization of EW gauge symmetry is, of course, perfectly acceptable.

²In this chapter, “UV model” is used to generically mean the SM supplemented with new states that couple to the SM. In particular, the UV model does not need to be UV complete; it may itself be an effective theory of some other, unknown description.

³For an introduction to the basic techniques of effective field theories see, for example, [86].

⁴This counting excludes flavor. With flavor, this number jumps to $\mathcal{O}(10^3)$.

obscure the incredible simplicity and utility that the SM EFT has to offer. One of the main purposes of the present chapter is to provide tools and results to help a user employ the SM EFT and take advantage of the many benefits it can offer.

A typical scenario one can imagine is that a person has some UV model containing massive BSM states and she wishes to understand how these states affect Higgs and EW observables. With a UV model in hand she can, of course, compute these effects using the UV model itself. This option sounds more direct and can, in principle, be more accurate since it does not require an expansion in powers of Λ^{-1} . However, performing a full computation with the UV model is typically quite involved, especially at loop-order and beyond, and needs to be done on a case-by-case basis for each UV model. Among the great advantages of using an EFT is that the computations related to running and mapping, being intrinsic to the EFT, only need to be done once; in other words, once the RG evolution and physical effects of the $\mathcal{O}_i$ are known (to a given order), the results can be tabulated for general use.

Moreover, for many practical purposes, a full computation in the UV model does not offer considerable improvement in accuracy over the EFT approach when one considers future experimental resolution. The difference between an observable computed using the UV theory versus the (truncated) EFT will scale in powers of E_{obs}/Λ , typically beginning at $(E_{\text{obs}}/\Lambda)^2$, where $E_{\text{obs}} \sim m_W$ is the energy scale at which the observable is measured. The present lack of evidence for BSM physics coupled to the SM requires in many cases Λ to be at least a factor of a few above the weak scale. With an estimated per mille precision of future Higgs and EW observables, this means that the leading order calculation in the EFT will rapidly converge with the calculation from the UV model, providing essentially the same result for $\Lambda \gtrsim (\text{several} \times E_{\text{obs}})$.⁵ For the purpose of determining the physics reach of future experiments on specific UV models—*i.e.* estimating the largest values of Λ in a given model that experiments can probe—the EFT calculation is sufficiently accurate in almost all cases.

As mentioned above, the steps of RG running the $\mathcal{O}_i$ and mapping these operators to observables are done within the EFT; once these results are known they can be applied to any set of $\{c_i(\Lambda)\}$ obtained from matching a given UV model onto the SM EFT. Therefore, an individual wishing to study the impact of some UV model on weak scale observables “only” needs to obtain the $c_i(\Lambda)$ at the matching scale Λ . “only” is put in quotes because this step, while straightforward, can also be computationally complex owing to the large number of operators in the SM EFT.

A large amount of literature pertaining to the SM EFT already exists, some of which dates back a few decades, and is rapidly growing and evolving. Owing to the complexity of the SM EFT, many results are scattered throughout the literature at varying levels of completeness. This body of research can be difficult to wade through for a newcomer (or expert) wishing to use the SM EFT to study the impact of BSM physics on Higgs and EW observables. An explication from a UV perspective, oriented to consider how one uses the SM EFT as a bridge to connect UV models with weak-scale precision observables, should be warranted. Such a perspective is shown by providing

⁵For example, in considering the impact of scalar tops on the associated Zh production cross-section at an e^+e^- collider, Craig *et. al.* recently compared [87] the result of a full NLO calculation versus the SM EFT calculation. They found that the results were virtually indistinguishable for stop masses above 500 GeV.

new results and tools with the full picture of matching, running, and mapping in mind. Moreover, the results are aimed to be complete and systematic—especially in regards to the mapping onto observables—as well as usable and self-contained. These goals have obviously contributed to the considerable length of this chapter. In the rest of this introduction, the results are summarized more explicitly in order to provide an overview for what is contained where in this chapter.

In section 4.1, a method (covariant derivative expansion) is presented to considerably ease the matching of a UV model onto the SM EFT. The SM EFT is obtained by taking a given UV model and integrating out the massive BSM states. The resultant effective action is given by (4.1), where the higher dimension operators are suppressed by powers of $\Lambda = m$, the mass of the heavy BSM states. Although every $\mathcal{O}_i$ respects SM gauge invariance, traditional methods of evaluating the effective action, such as Feynman diagrams, require working with gauge non-invariant pieces at intermediate steps, so that the process of arranging an answer back into the gauge invariant $\mathcal{O}_i$ can be quite tedious. Utilizing techniques introduced in [88, 89] and termed the covariant derivative expansion (CDE), we will discuss a method of computing the effective action through one-loop order in a manifestly gauge-invariant manner. By working solely with gauge-covariant quantities, an expansion of the effective action is obtained that immediately produces the gauge-invariant operators $\mathcal{O}_i$ of the EFT and their associated Wilson coefficients.

At one-loop order, the effective action that results when integrating out a heavy field Φ of mass m is generally of the form

$$\Delta S_{\text{eff,1-loop}} \propto i \text{Tr} \log \left[D^2 + m^2 + U(x) \right], \quad (4.2)$$

where $D^2 = D_\mu D^\mu$ with D_μ a gauge covariant derivative and $U(x)$ depends on the light, SM fields. The typical method for evaluating the functional trace relies on splitting the covariant derivative into its component parts, $D_\mu = \partial_\mu - iA_\mu$ with A_μ a gauge field, and performing a derivative expansion in $\partial^2 - m^2$. This splitting clearly causes intermediate steps of the calculation to be gauge non-covariant. Many years ago, Gaillard found a transformation [88] that allows the functional trace to be evaluated while keeping gauge covariance manifest at every step of the calculation, which will be derived and explained in detail in section 4.1. In essence, the argument of the logarithm in Eq. (4.2) is transformed such that the covariant derivative only appears in a series of commutators with itself and $U(x)$. The effective action is then evaluated in a series of “free propagators” of the form $(q^2 - m^2)^{-1}$ with q_μ a momentum parameter that is integrated over. The coefficients of this expansion are the commutators of D_μ with itself and $U(x)$ and correspond to the $\mathcal{O}_i$ of the EFT. Thus, one immediately obtains the gauge-invariant $\mathcal{O}_i$ of the effective action.

In this chapter, we will clarify and streamline certain aspects of the derivation and use of the covariant derivative expansion of [88, 89]. Moreover, we will generalize the results of [88, 89] and provide explicit formulas for scalars, fermions, and massless as well as massive vector bosons. For the massive vector boson case, we will show an algebraic proof (in Appendix C) that the magnetic dipole coefficient is universal. Additionally, a method presented for obtaining the tree-level effective action in a covariant derivative expansion. While this tree-level evaluation is very straightforward, it has not appeared elsewhere in the literature.

In section 4.2, we will consider the step of running Wilson coefficients from the matching scale Λ to the electroweak scale m_W where measurements are made. Over the past few years, the RG evolution of the SM EFT has been investigated quite intensively [90–100]. It is a great accomplishment that the entire one-loop anomalous dimension matrix within a complete operator basis has been obtained [92–95],⁶ as well as components of γ_{ij} in other operator bases [96, 97]. As the literature has been quite thorough on the subject, there is little new to discuss in terms of calculations; instead, our discussion on RG running will primarily concern determining when this step is important to use and how to use it. Since future precision observables have a sensitivity of $\mathcal{O}(0.1\%)$ – $\mathcal{O}(1\%)$, they will generically be able to probe new physics at one-loop order. RG evolution introduces a loop factor; therefore, as a rule of thumb, RG running of the $c_i(\Lambda)$ to $c_i(m_W)$ is usually only important if the $c_i(\Lambda)$ are tree-level generated. As a relevant topic, all the possible UV models⁷ that could generate tree-level effective action are enumerated. RG evolution includes a logarithm which may serve to counter its loop suppression; however, from $v^2/\Lambda^2 \sim 0.1\%$, we see that Λ can be probed at most to a few TeV, so that the logarithm is not large, $\log(\Lambda/m_W) \sim 3$. Note that this estimate also means that in a perturbative expansion, a truncation by loop-order counting is more appropriate than by logarithm power counting.

A common theme in the SM EFT is the choice of operator basis. We will discuss this in detail in section 4.2, but let us comment here on relevance of choosing an operator basis to each of the matching, running, and mapping steps. One does not need to choose an operator basis at the stage of matching a UV model onto the effective theory. The effective action obtained by integrating out some massive modes will simply produce a set of higher-dimension operators. One can then decide to continue to work with this UV generated operator set as it is, or to switch to a different set due to some other considerations. An operator basis needs to be picked once one RG evolves the Wilson coefficients using the anomalous dimension matrix γ_{ij} , as the anomalous dimension matrix is obviously basis dependent. When the running is relevant, one can choose any operator set compatible with the RG analysis to proceed with it. In section 4.2, we will discuss more about what kind of operator set is compatible with the RG analysis. In the last step, mapping, one studies the effects on the precision observables of each dimension-six operators. The result is obviously uncorrelated among different operators. Therefore, unlike the anomalous dimension matrix γ_{ij} in RG analysis, the mapping result for each Wilson coefficient is actually operator set independent. Of course, the larger the operator set one chooses to work with, the more complete is the mapping result. But except for the purpose of completeness, one does not need to worry too much about what operator set to choose, as the mapping result of each individual dimension-six operator stands by itself.

In section 4.3 we will consider the mapping step, *i.e.* obtaining Higgs and EW precision observables as functions of the Wilson coefficients at the weak scale, $c_i(m_W)$. While there have been a variety of studies concerning the mapping of operators onto weak-scale observables in the literature [87, 94, 96, 97, 102–115], a complete and systematic list does not exist yet. In Sec-

⁶Not only is the computation of γ_{ij} practically useful, its structure may be hinting at something deep in regards to renormalization and effective actions [101].

⁷under the restriction that the UV model is renormalizable and the heavy field being integrated out *only couples with the bosonic fields of the SM*

tion 4.3, we will study a complete set of the Higgs and EW precision observables that present and possible near future experiments can have a decent (1% or better) sensitivity on. These include the seven Electroweak precision observables (EWPO) S, T, U, W, Y, X, V up to p^4 order in the vacuum polarization functions, the three independent triple gauge couplings (TGC), the deviation in Higgs decay widths $\{\Gamma_{h \rightarrow f\bar{f}}, \Gamma_{h \rightarrow gg}, \Gamma_{h \rightarrow \gamma\gamma}, \Gamma_{h \rightarrow \gamma Z}, \Gamma_{h \rightarrow WW^*}, \Gamma_{h \rightarrow ZZ^*}\}$, and the deviation in Higgs production cross sections at both lepton and hadron colliders $\{\sigma_{ggF}, \sigma_{WWh}, \sigma_{Wh}, \sigma_{Zh}\}$. These precision observables are written up to linear power and tree-level order in the Wilson coefficients $c_i(m_W)$ of a complete set of dimension-six CP-conserving bosonic operators⁸ shown in Table 4.1. Quite a bit calculation steps are also listed in Appendix E. These include a list of two-point and three-point Feynman rules (Appendix E.1) from operators in Table 4.1, interference corrections to Higgs decay widths (Appendix E.2) and production cross sections (Appendix E.3), and general analysis on residue modifications (Appendix E.4) and Lagrangian parameter modifications (Appendix E.5). Since the primary interest is new physics that only couples with bosons in the SM, the Wilson coefficients of all the fermionic operators have been taken to be zero while calculating the mapping results. However, the general analysis presented for calculating the Higgs decay widths and production cross sections completely apply to fermionic operators.

In Section 4.4, two examples will be given to demonstrate the process of using the SM EFT as a bridge to connect UV models with weak-scale precision observables, and thus placing constraints on the UV models. One example is a singlet scalar coupled to the Higgs boson, where impacts arise at the tree level. It can achieve first-order electroweak phase transition (EWPT) which would allow electroweak baryogenesis. The other is the scalar top in the Minimal Supersymmetric Standard Model (MSSM), where impacts arise at the one-loop level. It will help minimize the fine-tuning in the Higgs mass-squared. In both cases, future precision Higgs and precision electroweak measurements are found to be sensitive probes.

Finally, the chapter is concluded in Section 4.5.

4.1 Covariant Derivative Expansion

The point of this section is to present a method for computing the one-loop effective action that leaves gauge invariance manifest *at every step of the calculation*. By this we mean that one only works with gauge covariant quantities, such as the covariant derivative. It is somewhat surprising that this method—developed in the 80s by Gaillard [88] (see also her summer school lectures [116] and the work by Cheyette [89])—is not widely known considering the incredible simplifications it provides. Therefore, in order to spread the good word so to speak, we will explain the method of the covariant derivative expansion (CDE) as developed in [88, 89]. Along the way, we will make more rigorous and clear a few steps in the derivation, present a more transparent expansion method to evaluate the CDE, and provide generalized results for scalars, fermions, and massless as well as massive gauge bosons. We will also discuss how to evaluate the tree-level effective action in manifestly gauge-covariant manner.

⁸In this chapter, the term “bosonic operators” is used to refer to the operators that contain only bosonic fields, *i.e.* Higgs and gauge bosons. Other operators will be referred to as “fermionic operators”.

Besides providing an easier computational framework, the CDE illuminates a certain universality in computing Wilson coefficients from different UV theories. This occurs because individual terms in the expansion split into a trace over internal indices (gauge, flavor, *etc.*) involving covariant derivatives times low energy fields—these are the operators in the EFT—times a simple momentum integral whose value corresponds to the Wilson coefficient of the operator. The UV physics is contained in the specific form of the covariant derivatives and low energy fields, but the momentum integral is independent of these details and therefore can be considered universal.

So far our discussion has been centered around the idea of integrating out some heavy mode to get an effective action, to which the CDE is claimed a useful tool. But more precisely, the CDE is a technique for evaluating functional determinants of a generalized Laplacian operator, $\det[D^2 + U(x)]$, where D is some covariant derivative. Since functional determinants are prolific in the computation of the (1PI or Wilsonian) effective action to one-loop order, the use of the CDE extends far beyond integrating out some heavy field and can be used as a tool to, for example, renormalize a (effective) field theory or compute thermal effects.

The 1980s saw considerable effort in developing methods to compute the effective action with arbitrary background fields. While we cannot expect to do justice to this literature, let us provide a brief outline of some relevant works. The CDE developed in [88, 89] built upon the derivative expansion technique of [117, 118]. A few techniques for covariant calculation of the one-loop effective action were developed somewhat earlier in [119]. While these techniques do afford considerable simplification over traditional methods, they are less systematic and more cumbersome than the CDE presented here [88]. In using a heat kernel to evaluate the effective action, a covariant derivative expansion has also been developed, see, *e.g.*, [120]. This method utilizes a position space representation and is significantly more involved than the approach presented here, where we work in Fourier space.

An outline for this section is as follows. In Section 4.1.1, we will consider the tree and one-loop contributions to the effective action in turn and discuss how to evaluate each using a covariant derivative expansion. The explicit extension to fermions and gauge bosons is provided in Section 4.1.2 together with summary formulas of the CDE for different spin particles. In Section 4.1.3, we will discuss how to explicitly evaluate terms in the CDE. Following this, universal formulas for terms in the expansion are presented, which can be used to immediately obtain the one-loop effective action for a wide variety of theories.

4.1.1 CDE Method at Tree-level and One-loop

Setting up the problem

Consider Φ to be a heavy, real scalar field of mass m that we wish to integrate out. Let $S[\phi, \Phi]$ denote the piece of the action in the full theory consisting of Φ and its interactions with Standard Model fields ϕ . The effective action resultant from integrating out Φ is given by

$$e^{iS_{\text{eff}}[\phi](\mu)} = \int \mathcal{D}\Phi e^{iS[\phi, \Phi](\mu)}. \quad (4.3)$$

The above defines the effective action at the scale $\mu \sim m$, where we have matched the UV theory onto the effective theory. In the following we will not write the explicit μ dependence and it is to be implicitly understood that the effective action is being computed at $\mu \sim m$.

Following standard techniques, S_{eff} can be computed to one-loop order by a saddle point approximation to the above integral. To do this, expand Φ around its minimum value, $\Phi = \Phi_c + \eta$, where Φ_c is determined by

$$\frac{\delta S[\phi, \Phi]}{\delta \Phi} = 0 \Rightarrow \Phi_c[\phi]. \quad (4.4)$$

Expanding the action around this minimum,

$$S[\phi, \Phi_c + \eta] = S[\Phi_c] + \frac{1}{2} \frac{\delta^2 S}{\delta \Phi^2} \Big|_{\Phi_c} \eta^2 + \mathcal{O}(\eta^3), \quad (4.5)$$

the integral is computed as⁹

$$e^{iS_{\text{eff}}[\phi]} = \int \mathcal{D}\eta e^{iS[\phi, \Phi_c + \eta]} \approx e^{iS[\Phi_c]} \left[\det \left(- \frac{\delta^2 S}{\delta \Phi^2} \Big|_{\Phi_c} \right) \right]^{-1/2}, \quad (4.6)$$

so that the effective action is given by

$$S_{\text{eff}} \approx S[\Phi_c] + \frac{i}{2} \text{Tr} \log \left(- \frac{\delta^2 S}{\delta \Phi^2} \Big|_{\Phi_c} \right). \quad (4.7)$$

The first term in the above is the tree-level piece when integrating out a field, *i.e.* solving for a field's equation of motion and plugging it back into the action, while the second term is the one-loop piece.

As is clear in the defining equation of the effective action, Eq. (4.3), the light fields ϕ are held fixed while the path integral over Φ is computed. The $\phi(x)$ fields are therefore referred to as background fields. The fact that the background fields are held fixed while only Φ varies in Eq. (4.3) leads to an obvious diagrammatic interpretation of the effective action: the effective action is the set of all Feynman diagrams with ϕ as external legs and only Φ fields as internal legs. The number of loops in these diagrams correspond to a loop expansion of the effective action.

CDE at tree level

First, let us see how to evaluate the tree-level piece to the effective action in a covariant fashion. The most naïve guess of how to do this turns out to be correct: in the exact same way one would do a derivative expansion, one can do a covariant derivative expansion.

To have a tree-level contribution to the effective action there needs to be a term in the UV Lagrangian that is linear in the heavy field Φ . Let us take a Lagrangian of a complex scalar Φ ,

$$\mathcal{L}[\Phi, \phi] \supset (\Phi^\dagger B(x) + \text{h.c.}) + \Phi^\dagger (-D^2 - m^2 - U(x))\Phi + \mathcal{O}(\Phi^3), \quad (4.8)$$

⁹The minus sign inside the logarithm comes from Wick rotating to Euclidean space, computing the path integral using the method of steepest descent, and then Wick rotating back to Minkowski space.

where $B(x)$ and $U(x)$ are generically functions of the light fields $\phi(x)$ and we have not specified the interaction terms that are cubic or higher in Φ . To get the tree-level effective action, one simply solves the equation of motion for Φ , and plugs it back into the action. The equation of motion for Φ is

$$(P^2 - m^2 - U(x))\Phi = -B(x) + \mathcal{O}(\Phi^2), \quad (4.9)$$

where $P_\mu \equiv iD_\mu = i\partial_\mu + A_\mu(x)$ is the covariant derivative¹⁰ that acts on Φ . The solution of this gives $\Phi_c[\phi]$ denoted in Eq. (4.4). To leading approximation, we can linearize the above equation to solve for Φ_c ,

$$\Phi_c = -\frac{1}{P^2 - m^2 - U(x)}B(x). \quad (4.10)$$

If the covariant derivative were replaced with the partial derivative, $P^2 = -\partial^2$, one would evaluate the above in an inverse-mass expansion producing a series in ∂^2/m^2 . The exact same inverse-mass expansion can be used with the covariant derivative as well to obtain¹¹

$$\begin{aligned} \Phi_c &= \left[1 - \frac{1}{m^2}(P^2 - U)\right]^{-1} \frac{1}{m^2}B \\ &= \frac{1}{m^2}B + \frac{1}{m^2}(P^2 - U)\frac{1}{m^2}B + \frac{1}{m^2}(P^2 - U)\frac{1}{m^2}(P^2 - U)\frac{1}{m^2}B + \dots \end{aligned} \quad (4.11)$$

In general, the mass-squared matrix need not be proportional to the identity, so that $1/m^2$ should be understood as the inverse of the matrix m^2 . In this case, $1/m^2$ would not necessarily commute with U and hence we used the matrix expansion from Eq. (4.24) in the above equation.

Plugging Φ_c back into the Lagrangian gives the tree-level effective action. Using the linearized solution to the equation of motion, Eq. (4.10), we have

$$\mathcal{L}_{\text{eff,tree}} = -B^\dagger \frac{1}{P^2 - m^2 - U(x)}B + \mathcal{O}(\Phi_c^3). \quad (4.12)$$

Although we have not specified the interactions in Eq. (4.8) that are cubic or higher in Φ , one needs to also substitute Φ_c for these pieces as well, as indicated in the above equation. The first few terms in the inverse mass expansion are

$$\mathcal{L}_{\text{eff,tree}} = B^\dagger \frac{1}{m^2}B + B^\dagger \frac{1}{m^2}(P^2 - U)\frac{1}{m^2}B + \dots + \mathcal{O}(\Phi_c^3). \quad (4.13)$$

¹⁰ $A_\mu = A_\mu^a T^a$ with T^a in the representation of Φ . We will not specify the coupling constant in the covariant derivative. Of course, the coupling constant can be absorbed into the gauge field; however, unless otherwise stated, for calculations in this chapter it is implicitly assumed that the coupling constant is in the covariant derivative. The primary reason we will not explicitly written the coupling constant is because Φ may carry multiple gauge quantum numbers. For example, if Φ is charged under $SU(2)_L \times U(1)_Y$ then we will take $D_\mu = \partial_\mu - igW_\mu - ig'YB_\mu$.

¹¹This is trivially true. In the case of a partial derivative, $-\partial^2 - m^2 - U(x)$, the validity of the expansion relies not only on $\partial^2/m^2 \ll 1$ but also on $U(x)/m^2 \ll 1$, *i.e.* momenta in the EFT need to be less than m which also means the fields in the EFT need to be slowly varying on distance scales of order m^{-1} . Obviously, the same conditions can be imposed on the covariant derivative as a whole.

CDE at one-loop level

Now let us discuss the one-loop piece of the effective action. Let Φ be field of mass m that we wish to integrate out to obtain a low-energy effective action in terms of light fields. Assume that Φ has quantum numbers under the low-energy gauge groups. The one-loop contribution to the effective action that results from integrating out Φ is

$$\Delta S_{\text{eff}} = ic_s \text{Tr} \log \left(-P^2 + m^2 + U(x) \right), \quad (4.14)$$

where $c_s = +1/2, +1$, or $-1/2$ for Φ a real scalar, complex scalar, or fermion, respectively.¹²

We evaluate the trace in the usual fashion by inserting a complete set of momentum and spatial states to arrive at

$$\Delta S_{\text{eff}} = ic_s \int d^4x \int \frac{d^4q}{(2\pi)^4} \text{tr} e^{iq \cdot x} \log \left(-P^2 + m^2 + U(x) \right) e^{-iq \cdot x}, \quad (4.15)$$

where the lower case “tr” denotes a trace on internal indices, *e.g.* gauge, spin, flavor, *etc.* For future shorthand let us define $dx \equiv d^4x$ and $dq \equiv d^4q/(2\pi)^4$. Using the Baker-Campbell-Hausdorff (BCH) formula,

$$e^B A e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} L_B^n A, \quad L_B A = [B, A], \quad (4.16)$$

together with the fact that we can bring the $e^{\pm iq \cdot x}$ into the logarithm, we see that the $P_\mu \rightarrow P_\mu + q_\mu$. Then, after changing variables $q \rightarrow -q$, the one-loop effective action is given by

$$\Delta S_{\text{eff}} = ic_s \int dx dq \text{tr} \log \left[- (P_\mu - q_\mu)^2 + m^2 + U(x) \right]. \quad (4.17)$$

Following [88, 89], we sandwich the above by $e^{\pm P^\mu \partial / \partial q^\mu}$

$$\Delta S_{\text{eff}} = ic_s \int dx dq \text{tr} e^{P \cdot \frac{\partial}{\partial q}} \log \left[- (P_\mu - q_\mu)^2 + m^2 + U(x) \right] e^{-P \cdot \frac{\partial}{\partial q}}. \quad (4.18)$$

In the above it is to be understood that the derivatives $\partial/\partial q$ and $\partial/\partial x \subset P$ act on unity to the right (for $e^{-P \cdot \partial/\partial q}$) and, by integration by parts, can be made to act on unity to the left (for $e^{P \cdot \partial/\partial q}$). Since the derivative of one is zero, the above insertion is allowed. Let us emphasize that the ability to insert $e^{\pm P \cdot \partial/\partial q}$ in Eq. (4.18) *does not* rely on cyclic property of the trace: the “tr” trace in Eq. (4.18) is over internal indices only and we therefore cannot cyclically permute the infinite dimensional matrices in Eq. (4.18).

One advantage of this choice of insertion is that it makes the linear term in P_μ vanish when transforming the combination $(P_\mu - q_\mu)$, and so the expansion starts from a commutator $[P_\mu, P_\nu]$

¹²The reason fermions have $c_s = -1/2$ instead of the usual -1 is because we have squared the usual argument of the logarithm, $\Delta S_{\text{eff}} = -\frac{i}{2} \text{Tr} \log(i \not{D} + \dots)^2$, to bring it to the form in Eq. (4.14). A vector boson has the same c_s as a real scalar. It just has more spin degrees of freedom. A Fadeev-Popov ghost, relevant when integrating out gauge bosons, has $c_s = -1$. See Appendix D for details.

which is the field strength. Indeed, by making use of the BCH formula and the fact $(L_{P \cdot \partial / \partial q}) q_\mu = [P \cdot \partial / \partial q, q_\mu] = P_\mu$, we get

$$\begin{aligned}
e^{P \cdot \frac{\partial}{\partial q}} (P_\mu - q_\mu) e^{-P \cdot \frac{\partial}{\partial q}} &= \sum_{n=0}^{\infty} \frac{1}{n!} (L_{P \cdot \partial / \partial q})^n P_\mu - \sum_{n=0}^{\infty} \frac{1}{n!} (L_{P \cdot \partial / \partial q})^n q_\mu \\
&= -q_\mu + \sum_{n=1}^{\infty} \frac{n}{(n+1)!} (L_{P \cdot \partial / \partial q})^n P_\mu \\
&= -q_\mu - \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!} \left[P_{\alpha_1}, \left[\dots [P_{\alpha_n}, [D_\nu, D_\mu]] \right] \right] \frac{\partial^n}{\partial q_{\alpha_1} \dots \partial q_{\alpha_n}} \frac{\partial}{\partial q_\nu} \\
&\equiv - \left(q_\mu + \tilde{G}_{\nu\mu} \frac{\partial}{\partial q_\nu} \right), \tag{4.19}
\end{aligned}$$

and similarly

$$e^{P \cdot \frac{\partial}{\partial q}} U e^{-P \cdot \frac{\partial}{\partial q}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[P_{\alpha_1}, \left[P_{\alpha_2}, \left[\dots [P_{\alpha_n}, U] \right] \right] \right] \frac{\partial^n}{\partial q_{\alpha_1} \dots \partial q_{\alpha_n}} \equiv \tilde{U}. \tag{4.20}$$

Bringing the $e^{\pm P \cdot \partial / \partial q}$ into the logarithm to compute the transformation of the integrand in Eq. (4.18), one gets the results obtained in [88, 89]

$$\Delta S_{\text{eff}} = \int dx \Delta \mathcal{L}_{\text{eff}} = i c_s \int dx \int dq \text{tr} \log \left[- \left(q_\mu + \tilde{G}_{\nu\mu} \frac{\partial}{\partial q_\nu} \right)^2 + m^2 + \tilde{U} \right], \tag{4.21}$$

where we have defined

$$\tilde{G}_{\nu\mu} = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!} \left[P_{\alpha_1}, \left[P_{\alpha_2}, \left[\dots [P_{\alpha_n}, [D_\nu, D_\mu]] \right] \right] \right] \frac{\partial^n}{\partial q_{\alpha_1} \partial q_{\alpha_2} \dots \partial q_{\alpha_n}}, \tag{4.22a}$$

$$\tilde{U} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[P_{\alpha_1}, \left[P_{\alpha_2}, \left[\dots [P_{\alpha_n}, U] \right] \right] \right] \frac{\partial^n}{\partial q_{\alpha_1} \partial q_{\alpha_2} \dots \partial q_{\alpha_n}}. \tag{4.22b}$$

The commutators in the above correspond to manifestly gauge invariant higher dimension operators: In Eq. (4.22a) the commutators of P 's with $[D_\nu, D_\mu] = -i G_{\nu\mu}$, where $G_{\nu\mu}$ is the gauge field strength, correspond to higher dimension operators of the field strength and its derivatives. In Eq. (4.22b), the commutators will generate higher dimension derivative operators on the fields inside $U(x)$.

While it should be clear, it is worth emphasizing that x and $\partial / \partial x$ commute with q and $\partial / \partial q$, *i.e.* $P = i \partial / \partial x + A(x)$ and $U(x)$ commute with q and $\partial / \partial q$. This, together with the fact that the commutators in Eq. (4.22) correspond to higher dimension operators, allows us to develop a simple expansion of Eq. (4.21) in terms of higher dimension operators whose coefficients are determined from easy to compute momentum integrals, which let us now describe.

Instead of working with the logarithm, let us work with its derivative with respect to m^2 . Using ∂_μ to denote the derivative *with respect to* q , $\partial_\mu \equiv \partial/\partial q^\mu$, and defining $\Delta \equiv (q^2 - m^2)^{-1}$, the effective Lagrangian is

$$\Delta\mathcal{L}_{\text{eff}} = -ic_s \int dq \int dm^2 \text{tr} \frac{1}{\Delta^{-1} \left[1 + \Delta \left(\{q_\mu, \tilde{G}_{\nu\mu} \partial^\nu\} + \tilde{G}_{\sigma\mu} \tilde{G}_\nu^\sigma \partial^\mu \partial^\nu - \tilde{U} \right) \right]}. \quad (4.23)$$

In the above, Δ is a free propagator for a massive particle; we can develop an expansion of powers of Δ and its derivatives (from the q derivatives inside $\tilde{G}$ and $\tilde{U}$) where the coefficients are the higher dimension operators. The derivatives and integrals in q are then simple, albeit tedious, to compute and correspond to the Wilson coefficient of the higher dimension operator. Explicitly, using

$$[A^{-1}(1 + AB)]^{-1} = A - ABA + ABABA - \dots, \quad (4.24)$$

we have (using obvious shorthand notation)

$$\begin{aligned} \Delta\mathcal{L}_{\text{eff}} = -ic_s \int dq dm^2 \text{tr} & \left[\Delta - \Delta \left(\{q, \tilde{G}\} + \tilde{G}^2 - \tilde{U} \right) \Delta \right. \\ & \left. + \Delta \left(\{q, \tilde{G}\} + \tilde{G}^2 - \tilde{U} \right) \Delta \left(\{q, \tilde{G}\} + \tilde{G}^2 - \tilde{U} \right) \Delta + \dots \right]. \end{aligned} \quad (4.25)$$

There are two points worth noting:

Power counting Power counting is very transparent in the expansion in Eq. (4.25). This makes it simple to identify the dimension of the operators in the resultant EFT and to truncate the expansion at the desired order. For example, the lowest dimension operator in $\tilde{G}_{\mu\nu}$ is the field strength $[D_\mu, D_\nu] = -iG_{\mu\nu}$; each successive term in $\tilde{G}$ increases the EFT operator dimension by one through an additional P_α . The dimension increase from additional P 's is compensated by additional q derivatives which, by acting on Δ , increase the numbers of propagators.

Universality When the mass squared matrix m^2 is proportional to the identity then Δ commutes with the matrices in $\tilde{G}$ and $\tilde{U}$. In this case, for any given term in the expansion in Eq. (4.25), the q integral trivially factorizes out of the trace and can be calculated separately. Because of this, there is a certain universality of the expansion in Eq. (4.25): specifics of a given UV theory are contained in P_μ and $U(x)$, but the coefficients of EFT operators are determined by the q integrals and can be calculated without any reference to the UV model.

Before ending this section, let us introduce a more tractable notation that we use in later calculations and results. As we already have used, $\partial_\mu \equiv \partial/\partial q^\mu$. The action of the covariant derivative on matrix is defined as a commutator and let us use as shorthand $P_\mu A \equiv [P_\mu, A]$. Let us also define

$G'_{\mu\nu} \equiv [D_\mu, D_\nu]$.¹³ To summarize and repeat ourselves:

$$\partial_\mu \equiv \frac{\partial}{\partial q^\mu}, \quad P_\mu A \equiv [P_\mu, A], \quad G'_{\mu\nu} \equiv [D_\mu, D_\nu]. \quad (4.26)$$

Finally, as everything is explicitly Lorentz invariant, we will typically not bother with raised and lowered indices. With this notation, $\tilde{G}$ and $\tilde{U}$ as defined in Eq. (4.22) are given by

$$\tilde{G}_{\nu\mu} = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!} (P_{\alpha_1} \dots P_{\alpha_n} G'_{\nu\mu}) \partial_{\alpha_1 \dots \alpha_n}^n, \quad (4.27a)$$

$$\tilde{U} = \sum_{n=0}^{\infty} \frac{1}{n!} (P_{\alpha_1} \dots P_{\alpha_n} U) \partial_{\alpha_1 \dots \alpha_n}^n. \quad (4.27b)$$

4.1.2 CDE Summary Formulas for Scalars, Fermions, and Gauge Bosons

The CDE as presented in Section 4.1.1 is for evaluating functional determinants of the form

$$\log \det (-P^2 + W(x)) = \text{Tr} \log (-P^2 + W(x)), \quad (4.28)$$

where $P_\mu = iD_\mu$ is a covariant derivative. As such, the results of section 4.1.1 apply for any generalized Laplacian operator of the form $-P^2 + W(x)$.¹⁴ The lightning summary is

$$\begin{aligned} \text{Tr} \log (-P^2 + W) &= \int dx dq \text{tr} e^{P \cdot \partial_q} e^{iq \cdot x} \log (-P^2 + W) e^{-iq \cdot x} e^{-P \cdot \partial_q} \\ &= \int dx dq \text{tr} \log \left[- \left(q_\mu + \tilde{G}_{\nu\mu} \partial_\mu \right)^2 + \tilde{W} \right], \end{aligned} \quad (4.29)$$

where $\tilde{G}$ and $\tilde{W}$ are given in Eq. (4.27) with U replaced by W and we are using the notation defined in Eq. (4.26). In section 4.1.1 we took $W(x) = m^2 + U(x)$ for its obvious connection to massive scalar fields.

When we integrate out fermions and gauge bosons, at one-loop they also give functional determinants of generalized Laplacian operators of the form $-P^2 + W(x)$. It is straightforward to apply the steps of section 4.1.1 to these cases. Nevertheless, it is useful to tabulate these results for easy reference. Therefore, let us summarize in this subsection the results for integrating out massive scalars, fermions, and gauge bosons. We also include the result of integrating out the high energy modes of a massless gauge field to compute its RG beta function. Detailed derivations of the fermion and gauge boson results are shown in Appendix D. The results for fermions were first obtained in [88]¹⁵ and for gauge bosons in [89].

¹³If $D_\mu = \partial/\partial x^\mu - igA_\mu$, then $G'_{\mu\nu}$ is related to the usual field strength as $G'_{\mu\nu} = [D_\mu, D_\nu] = -igG_{\mu\nu}$. In the case where we have integrated out multiple fields with possibly multiple and different gauge numbers, it is easier to just work with D_μ , hence the definition of $G'_{\mu\nu}$.

¹⁴This is loosely speaking, but applies to many of the cases physicists encounter. More correctly, the functional determinant should exist and so we actually work in Euclidean space and consider elliptic operators of the form $+P^2 + W(x)$ with W hermitian, positive-definite. The transformations leading to the CDE in section 4.1.1 then apply to these elliptic operators as well. In the cases we commonly encounter in physics, these properties are satisfied by the fact that operator is the second variation of the Euclidean action which is typically taken to Hermitian, positive-definite.

¹⁵There is an error in the results for fermions in [88] (see Appendix D).

Let us state the general result and then specify how it specializes to the various cases under consideration. The one-loop effective action is given by

$$\Delta S_{\text{eff},1\text{-loop}} = ic_s \text{Tr} \log \left(-P^2 + m^2 + U(x) \right), \quad (4.30)$$

where the constant c_s and the form of U depend on the species we integrate out, as we will discuss below. After evaluating the trace and using the transformations introduced in [88] and explained in section 4.1.1, the one-loop effective Lagrangian is given by

$$\Delta \mathcal{L}_{\text{eff},1\text{-loop}} = ic_s \int dq \text{tr} \log \left[- \left(q_\mu + \tilde{G}_{\nu\mu} \partial_\mu \right)^2 + m^2 + \tilde{U} \right], \quad (4.31)$$

where the lower case trace, “tr”, is over internal indices and

$$\tilde{G}_{\nu\mu} = \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!} (P_{\alpha_1} \dots P_{\alpha_n} G'_{\nu\mu}) \partial_{\alpha_1 \dots \alpha_n}^n, \quad (4.32a)$$

$$\tilde{U} = \sum_{n=0}^{\infty} \frac{1}{n!} (P_{\alpha_1} \dots P_{\alpha_n} U) \partial_{\alpha_1 \dots \alpha_n}^n, \quad (4.32b)$$

$$P_\mu = iD_\mu, \quad \partial_\mu \equiv \frac{\partial}{\partial q^\mu}, \quad G'_{\nu\mu} \equiv [D_\nu, D_\mu]. \quad (4.32c)$$

Real scalars The effective action originates from the Gaussian integral

$$\exp(i\Delta S_{\text{eff},1\text{-loop}}) = \int \mathcal{D}\Phi \exp \left[i \int dx \frac{1}{2} \Phi^T (P^2 - m^2 - M^2(x)) \Phi \right].$$

For this case, in Eqs. (4.30) and (4.31) we have

$$c_s = 1/2, \quad U(x) = M^2(x). \quad (4.33)$$

Complex scalars The effective action originates from the Gaussian integral

$$\exp(i\Delta S_{\text{eff},1\text{-loop}}) = \int \mathcal{D}\Phi \mathcal{D}\Phi^* \exp \left[i \int dx \Phi^\dagger (P^2 - m^2 - M^2(x)) \Phi \right].$$

For this case, in Eqs. (4.30) and (4.31) we have

$$c_s = 1, \quad U(x) = M^2(x). \quad (4.34)$$

Massive fermions Let us work with Dirac fermions. The effective action originates from the Gaussian integral

$$\exp(i\Delta S_{\text{eff},1\text{-loop}}) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[i \int dx \bar{\psi} (\not{P} - m - M(x)) \psi \right],$$

where $\not{P} = \gamma^\mu P_\mu$ with γ^μ the usual gamma matrices. As shown in Appendix D, in Eqs. (4.30) and (4.31) we have

$$c_s = -1/2, \quad U = U_{\text{ferm}} \equiv -\frac{i}{2}\sigma^{\mu\nu}G'_{\mu\nu} + 2mM + M^2 + \not{P}M, \quad (4.35)$$

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$ and, by definition, $\not{P}M = [\not{P}, M]$. Note that the trace in (4.31) includes tracing over the spinor indices. The $2mM$ and M^2 terms in U_{ferm} and the $-P^2$ term are proportional to the identity matrix in the spinor indices which, since we use the 4×4 gamma matrices, is the 4×4 identity matrix $\mathbf{1}_4$.

Massless gauge fields Let us take pure Yang-Mills theory for non-abelian gauge group G ,

$$\mathcal{L}_{YM} = -\frac{1}{2g^2\mu(G)}\text{tr} F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = F_{\mu\nu}^a t_G^a,$$

where t_G^a are generators in the adjoint representation and $\mu(G)$ is the Dynkin index for the adjoint representation.¹⁶ We are considering the 1PI effective action, $\Gamma[A]$, of the gauge field A_μ .

Let us explain the essential details here and explicate them in full in Appendix D. The 1PI effective action is evaluated using the background field method: the gauge field is expanded around a background piece and a fluctuating piece, $A_\mu(x) = A_{B,\mu}(x) + Q_\mu$, and we integrate out Q_μ . The field Q_μ is gauge-fixed in such a way as to preserve the background field gauge invariance. The gauge-fixed functional integral we evaluate is,

$$\begin{aligned} \exp(i\Gamma_{\text{1-loop}}[A_B]) &= \int \mathcal{D}Q_\mu^a \mathcal{D}c^a \mathcal{D}\bar{c}^a \\ &\times \exp\left[i \int dx -\frac{1}{2g^2}Q_\rho^a(P^2 + i\mathcal{J}^{\mu\nu}G'_{\mu\nu})_\sigma^{\rho,ab}Q^{\sigma,b} + \bar{c}^a(P^2)^{ab}c^b\right], \end{aligned}$$

where c^a are Fadeev-Popov ghosts. In the above, $G'_{\mu\nu} = [D_\mu, D_\nu]$ where $D_\mu = \partial_\mu - iA_{B,\mu}$ is the covariant derivative with respect to the background field, $\mathcal{J}^{\mu\nu}$ is the generator of Lorentz transformations on four-vectors,¹⁷ and we have taken Feynman gauge ($\xi = 1$).

The effective Lagrangian is composed of two-pieces of the form in Eqs. (4.30) and (4.31) with $m^2 = 0$. The first is the ghost piece, for which $c_s = -1$ since the ghost fields are anti-commuting and $m^2 = U = 0$:

$$\text{Ghost piece:} \quad c_s = -1, \quad m^2 = U = 0. \quad (4.36)$$

¹⁶For representation R , the Dynkin index is given by $\text{tr} T_R^a T_R^b = \mu(R)\delta^{ab}$. For $SU(N)$, $\mu(G) = N$ while the fundamental representation has $\mu(\square) = 1/2$. In the adjoint representation $(t_G^b)_{ac} = if^{abc}$ where f^{abc} are the structure constants, $[T^a, T^b] = if^{abc}T^c$.

¹⁷Note the similarity with the fermion case, where $\sigma^{\mu\nu}/2$ is the generator of Lorentz transformations on spinors. Explicitly, the components of $\mathcal{J}^{\mu\nu}$ are given by $(\mathcal{J}^{\mu\nu})_{\rho\sigma} = i(\delta_\rho^\mu\delta_\sigma^\nu - \delta_\rho^\nu\delta_\sigma^\mu)$.

The second piece is the from the gauge field Q_μ^a which gives Eqs. (4.30) and (4.31) with $m^2 = 0$, $c_s = 1/2$ since each component of Q_μ^a is a real boson, and $U = -i\mathcal{J} \cdot G'$

$$\text{Gauge piece: } c_s = 1/2, \quad U = U_{\text{gauge}} \equiv -i\mathcal{J}^{\mu\nu} G'_{\mu\nu}, \quad m^2 = 0. \quad (4.37)$$

With $m^2 = 0$, Eqs. (4.30) and (4.31) contain IR divergences. These IR divergences can be regulated by adding a mass term for Q_μ^a and c^a (essentially keeping m^2 in Eqs. (4.30) and (4.31)).

Massive gauge bosons Let us consider a UV model with gauge group G that is spontaneously broken into H . A set of gauge bosons Q_μ^i , $i = 1, 2, \dots, \dim(G) - \dim(H)$ that correspond to the broken generators obtain mass m_Q by “eating” the Nambu-Goldstone bosons χ^i . Here, we restrict ourselves to the degenerate mass spectrum of all Q_μ^i for simplicity. These heavy gauge bosons form a representation of the unbroken gauge group. As is shown in Appendix C, the general gauge-kinetic piece of the Lagrangian up to quadratic term in Q_μ^i is

$$\mathcal{L}_{\text{g.k.}} \supset \frac{1}{2} Q_\mu^i \{ -P^2 g^{\mu\nu} + P^\nu P^\mu - [P^\mu, P^\nu] \}^{ij} Q_\nu^j, \quad (4.38)$$

where $P_\mu = iD_\mu$, with D_μ denotes the covariant derivative that contains only the unbroken gauge fields. One remarkable feature of this general gauge-kinetic term is that the coefficient of the “magnetic dipole term” $\frac{1}{2} Q_\mu^i \{ -[P^\mu, P^\nu] \}^{ij} Q_\nu^j$ is universal, namely that its coefficient is fixed to 1 relative to the “curl” terms $\frac{1}{2} Q_\mu^i \{ -P^2 g^{\mu\nu} + P^\nu P^\mu \}^{ij} Q_\nu^j$, regardless of the details of the symmetry breaking. In Appendix C, we will give both an algebraic derivation and a physical argument to prove Eq. (4.38).

The piece shown in Eq. (4.38) is to be combined with a gauge boson mass term due to the symmetry breaking, a generalized R_ξ gauge fixing term which preserves the unbroken gauge symmetry, an appropriate ghost term, and a possible generic interaction term. More details about all these terms are in appendix D. The resultant one-loop effective action is given by computing

$$\begin{aligned} \exp(i\Delta S_{\text{eff,1-loop}}) &= \int \mathcal{D}Q_\mu^i \mathcal{D}\chi^i \mathcal{D}c^i \mathcal{D}\bar{c}^i \\ &\times \exp \left\{ i \int dx \left[\frac{1}{2} Q_\mu^i (-P^2 g^{\mu\nu} + m_Q^2 g^{\mu\nu} - 2[P^\mu, P^\nu] + M^{\mu\nu})^{ij} Q_\nu^j \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \chi^i (P^2 - m_Q^2)^{ij} \chi^j + \bar{c}^i (P^2 - m_Q^2)^{ij} c^j \right] \right\}, \end{aligned} \quad (4.39)$$

where $c^i, \bar{c}^i$ denote the ghosts, $M^{\mu\nu}$ parameterizes the possible generic interaction term, and we have taken Feynman gauge $\xi = 1$. Clearly, the effective Lagrangian is composed of

three-pieces of the form in Eqs. (4.30) and (4.31)

$$\text{Gauge piece: } c_s = 1/2, \quad U = -i\mathcal{J}^{\mu\nu} \left(G'_{\mu\nu} + \frac{1}{2}M_{\mu\nu} \right), \quad m^2 = m_Q^2. \quad (4.40)$$

$$\text{Goldstone piece: } c_s = 1/2, \quad U = 0, \quad m^2 = m_Q^2. \quad (4.41)$$

$$\text{Ghost piece: } c_s = -1, \quad U = 0, \quad m^2 = m_Q^2. \quad (4.42)$$

4.1.3 Evaluating the CDE and Universal Results

In the present subsection, we will explicitly discuss how to evaluate terms in covariant derivative expansion of the one-loop effective action in Eqs. (4.23) and (4.25). Following this, provided are the results of the expansion through a given order in covariant derivatives. Specifically, for an effective action of the form $S_{\text{eff}} \propto \text{Tr} \log(-P^2 + m^2 + U)$ the results of the CDE are provided through dimension-six operators assuming U is at least linear in background fields. These results make no explicit reference to a specific UV model, and therefore they are, in a sense, universal. This universal result is tabulated in Eq. (4.59) and can be immediately used to compute the effective action of a given UV model.

Evaluating terms in CDE

Let us consider how to evaluate expansion terms from the effective Lagrangian of Eq. (4.23), which we reproduce here for convenience

$$\Delta\mathcal{L}_{\text{eff,1-loop}} = -ic_s \int dq \int dm^2 \text{tr} \frac{1}{\Delta^{-1} \left[1 - \Delta \left(-\{q_\mu, \tilde{G}_{\nu\mu}\} \partial_\nu - \tilde{G}_{\mu\sigma} \tilde{G}_{\nu\sigma} \partial_\mu \partial_\nu + \tilde{U} \right) \right]}.$$

In the above, $\tilde{G}$ and $\tilde{U}$ are as defined in Eq. (4.27), $dq \equiv d^4q/(2\pi)^4$, $\Delta \equiv 1/(q^2 - m^2)$, and we employ the shorthand notation defined in Eq. (4.26). We also used the fact that $\{q_\mu, \tilde{G}_{\nu\mu}\} \partial_\nu = \{q_\mu, \tilde{G}_{\nu\mu}\} \partial_\nu$ which follows from $\{A, BC\} = \{A, B\}C + B\{C, A\}$ and the antisymmetry of $\tilde{G}_{\nu\mu}$, $\tilde{G}_{\nu\mu} = -\tilde{G}_{\mu\nu}$. Using the matrix expansion

$$\frac{1}{A^{-1}(1 - AB)} = \sum_{n=0}^{\infty} (AB)^n A, \quad (4.43)$$

we define the integrals

$$\mathcal{I}_n \equiv \text{tr} \int dq dm^2 \left[\Delta \left(-\{q, \tilde{G}\} \partial - \tilde{G}^2 \partial^2 + \tilde{U} \right) \right]^n \Delta. \quad (4.44)$$

The effective action from a given $\mathcal{I}_n$ integral is given by $\Delta\mathcal{L}_{\mathcal{I}_n} = -ic_s \mathcal{I}_n$.

$\tilde{G}_{\nu\mu}$ and $\tilde{U}$ are an infinite expansions in covariant derivatives of $G'_{\nu\mu}$ and U , and thus contain higher-dimension operators (HDOs). Therefore, each $\mathcal{I}_n$ is an infinite expansion containing these HDOs. For this work, motivated by present and future precision measurements, we are interested

in corrections up to dimension-six operators. This dictates how many $\mathcal{I}_n$ we have to calculate as well as what order in $\tilde{G}_{\nu\mu}$ and $\tilde{U}$ we need to expand within a given $\mathcal{I}_n$.

As a typical example to demonstrate how to evaluate the $\mathcal{I}_n$, let us consider $\mathcal{I}_1$

$$\mathcal{I}_1 = \text{tr} \int dq dm^2 \Delta \left(-\{q, \tilde{G}\} \partial - \tilde{G}^2 \partial^2 + \tilde{U} \right) \Delta. \quad (4.45)$$

The fact that q_μ and ∂_μ commute with P_μ and U makes the $\mathcal{I}_n$ very simple to compute. We also assume that the mass-squared matrix m^2 commutes with $G'_{\mu\nu}$ and $\tilde{U}$. In this case, Δ commutes with the HDOs in $\tilde{G}$ and $\tilde{U}$, *i.e.* $[\Delta, P_{\alpha_1} \dots P_{\alpha_n} G'_{\mu\nu}] = 0$ and similarly for the HDOs in $\tilde{U}$. This allows us to separate the q -integral from the trace over the HDOs. Let us consider the $\tilde{U}$ term first,

$$\mathcal{I}_1 \supset \text{tr} \int dq dm^2 \Delta \tilde{U} \Delta = \sum_{n=0}^{\infty} \frac{1}{n!} \text{tr}(P_{\alpha_1} \dots P_{\alpha_n} U) \times \int dq \Delta \partial_{\alpha_1 \dots \alpha_n}^n \Delta. \quad (4.46)$$

Recall that the covariant derivative action on a matrix is defined as the commutator, *e.g.* $P_\alpha U = [P_\alpha, U]$. Since the trace of a commutator vanishes, all the $n \geq 1$ terms become total derivatives after the evaluation of the trace, and therefore do not contribute to the effective action. Thus

$$\text{tr} \int dq dm^2 \Delta \tilde{U} \Delta = \text{tr} U \times \int dq dm^2 \Delta^2. \quad (4.47)$$

The above term is divergent. In this chapter, dimensional regularization with $\overline{\text{MS}}$ is used as our renormalization scheme, in which case

$$\text{tr} U \int dq dm^2 \Delta^2 = \text{tr} U \int dq \Delta = -\frac{i}{(4\pi)^2} m^2 \left(\log \frac{m^2}{\mu^2} - 1 \right) \text{tr} U, \quad (4.48)$$

where μ is the renormalization scale.

Let us now turn our attention to the pieces in $\mathcal{I}_1$ involving $\tilde{G}_{\mu\nu}$. The term linear in $\tilde{G}$ in $\mathcal{I}_1$ vanishes since it is the trace of a commutator, as was the case for the higher derivative terms in $\tilde{U}$ discussed above. Thus, only the $\tilde{G}^2$ term is non-zero and we seek to evaluate

$$\mathcal{I}_1 \supset -\text{tr} \int dq dm^2 \Delta \tilde{G}_{\mu\sigma} \tilde{G}_{\nu\sigma} \partial_{\mu\nu}^2 \Delta. \quad (4.49)$$

We evaluate the above up to dimension-six operators. Since $G'_{\mu\nu} = -[P_\mu, P_\nu]$ is $\mathcal{O}(P^2)$, we need the expansion of $\tilde{G}\tilde{G}$ to $\mathcal{O}(P^6)$:

$$\begin{aligned} \tilde{G}_{\mu\sigma} \tilde{G}_{\nu\sigma} \partial_{\mu\nu}^2 &= \frac{1}{4} G'_{\mu\sigma} G'_{\nu\sigma} \partial_{\mu\nu}^2 + \frac{1}{9} (P_\alpha G'_{\mu\sigma}) (P_\beta G'_{\nu\sigma}) \partial_{\alpha\beta\mu\nu}^4 \\ &+ \frac{1}{16} \left[G'_{\mu\sigma} (P_{\beta_1} P_{\beta_2} G'_{\nu\sigma}) \partial_{\beta_1\beta_2\mu\nu}^4 + (P_{\alpha_1} P_{\alpha_2} G'_{\mu\sigma}) G'_{\nu\sigma} \partial_{\alpha_1\alpha_2\mu\nu}^4 \right], \end{aligned} \quad (4.50)$$

where we have dropped the $\mathcal{O}(P^5)$ terms since they vanish as required by Lorentz invariance. It is straightforward to plug the above back into $\mathcal{I}_1$ and compute the q -derivatives and integrals. For

example, the $G'^2 \partial^2$ term requires computing

$$\begin{aligned}
 \int dq dm^2 \Delta \partial_{\mu\nu}^2 \Delta &= \int dq dm^2 \Delta (-2g_{\mu\nu} \Delta^2 + 8q_\mu q_\nu \Delta^3) \\
 &= 2g_{\mu\nu} \int dq dm^2 (-\Delta^3 + q^2 \Delta^4) \\
 &= 2g_{\mu\nu} \int dq \left(-\frac{1}{2} \Delta^2 + \frac{1}{3} q^2 \Delta^4 \right) \\
 &= 2g_{\mu\nu} \cdot \frac{i}{(4\pi)^2} \cdot \frac{1}{6} \cdot \left(\log \frac{m^2}{\mu^2} - 1 \right), \tag{4.51}
 \end{aligned}$$

where we used the fact

$$\partial_{\mu\nu}^2 \Delta = -2g_{\mu\nu} \Delta^2 + 8q_\mu q_\nu \Delta^3 \Rightarrow \text{under } q\text{-integral: } \partial_{\mu\nu}^2 \Delta = 2g_{\mu\nu} (-\Delta^2 + q^2 \Delta^3). \tag{4.52}$$

and dimensional regularization with $\overline{\text{MS}}$ as our renormalization scheme. Thus, we see that

$$\mathcal{I}_1 \supset -\frac{1}{4} \text{tr}(G'_{\mu\sigma} G'_{\nu\sigma}) \int dq dm^2 \Delta \partial_{\mu\nu}^2 \Delta = -\frac{i}{(4\pi)^2} \cdot \left(\log \frac{m^2}{\mu^2} - 1 \right) \cdot \frac{1}{12} \cdot \text{tr}(G'_{\mu\nu} G'_{\mu\nu}), \tag{4.53}$$

which we clearly recognize as a contribution to the β function of the gauge coupling constant.

The other $\mathcal{O}(P^6)$ terms in the expansion of $\tilde{G}^2$ are computed similarly. The end result of computing the q -integrals for the $\mathcal{O}(P^6)$ terms in $\mathcal{I}_1$ gives

$$\begin{aligned}
 \mathcal{I}_1 \supset -\frac{i}{(4\pi)^2} \frac{1}{30} \frac{1}{m^2} \text{tr} \left\{ \frac{4}{9} \left[(P_\mu G'_{\mu\nu})^2 + (P_\mu G'_{\nu\sigma})(P_\mu G'_{\nu\sigma}) + (P_\mu G'_{\nu\sigma})(P_\nu G'_{\mu\sigma}) \right] \right. \\
 \left. + \frac{1}{2} \left[G'_{\mu\nu} (P^2 G_{\mu\nu} + P_\mu P_\sigma G'_{\sigma\nu} + P_\sigma P_\mu G'_{\sigma\nu}) \right] \right\}. \tag{4.54}
 \end{aligned}$$

There are only two possible dimension-six operators involving just P_μ and $G'_{\mu\nu}$, namely $\text{tr}(P_\mu G'_{\mu\nu})^2$ and $\text{tr}(G'_{\mu\nu} G'_{\nu\sigma} G'_{\sigma\mu})$. Using the Bianchi identity and integration by parts

$$\text{tr}[A(P_\mu B)] = -\text{tr}[(P_\mu A)B] + \text{total derivative terms}, \tag{4.55}$$

the above can be arranged into just these two dimension-six operators:

$$-\frac{i}{(4\pi)^2} \frac{1}{m^2} \left[\frac{1}{135} \text{tr}(P_\mu G'_{\mu\nu})^2 + \frac{1}{90} \text{tr}(G'_{\mu\nu} G'_{\nu\sigma} G'_{\sigma\mu}) \right]. \tag{4.56}$$

Combining all these terms together, we find the contribution to the effective Lagrangian from $\mathcal{I}_1$ is

$$\begin{aligned}
 \Delta \mathcal{L}_{\mathcal{I}_1} = -ic_s \mathcal{I}_1 = -\frac{c_s}{(4\pi)^2} \left[\left(\log \frac{m^2}{\mu^2} - 1 \right) \frac{1}{12} \text{tr}(G'_{\mu\nu} G'_{\mu\nu}) + \frac{1}{m^2} \frac{1}{135} \text{tr}(P_\mu G'_{\mu\nu})^2 \right. \\
 \left. + \frac{1}{m^2} \frac{1}{90} \text{tr}(G'_{\mu\nu} G'_{\nu\sigma} G'_{\sigma\mu}) \right] + \text{dim-8 operators}. \tag{4.57}
 \end{aligned}$$

In a similar fashion, one can compute the other $\mathcal{I}_n$. In the next subsection we tabulate the result of all possible contributions to dimension-six operators from the $\mathcal{I}_n$.

Universal results

We just discussed how to evaluate terms in the CDE to a given order. Here we will tabulate the results that allow one to compute the one-loop effective action through dimension six operators.

The one-loop effective action is given by

$$\Delta S_{\text{eff,1-loop}} = ic_s \text{Tr} \log \left(-P^2 + m^2 + U(x) \right), \quad (4.58)$$

where, as discussed these in Section 4.1.2, c_s and $U(x)$ depend on the species we integrate out. Let us assume that the mass-squared matrix m^2 commutes with U and $G'_{\mu\nu}$. Under this assumption, we can tabulate results of the CDE through dimension-six operators. In general, U may have terms which are linear in the background fields.¹⁸ In this case, although the scaling dimension of U is two, its operator dimension may be one. Simple power counting tells us that we will have to evaluate terms in the $\mathcal{I}_n$ integrals of Eq. (4.44) through $\mathcal{I}_6$.¹⁹ Gathering all of the terms together, the one-loop effective action is:

¹⁸For example, a Yukawa interaction $y\phi\bar{\psi}\psi$ for massive fermions leads to a term linear in the light field ϕ : from Eq. (4.35), $U_{\text{ferm}} \supset 2mM(x) = ym\phi$.

¹⁹While this is tedious, it isn't too hard. Moreover, there are many terms within each $\mathcal{I}_n$ that we don't need to compute since they lead to too large of an operator dimension. For example, the only term in $\mathcal{I}_6$ that we need to compute is

$$\mathcal{I}_6 = \text{tr} \int dq dm^2 \left[\Delta(-\{q, \tilde{G}\}\partial - \tilde{G}^2\partial^2 + \tilde{U}) \right]^6 \Delta \supset \text{tr} U^6 \int dq dm^2 \Delta^7 = \text{tr} U^6 \cdot \frac{i}{(4\pi)^2} \cdot \frac{1}{120} \cdot \frac{1}{m^8}.$$

All other terms in $\mathcal{I}_6$ have too large of operator dimension and can be dropped.

$$\begin{aligned}
\Delta\mathcal{L}_{\text{eff},1\text{-loop}} = \frac{c_s}{(4\pi)^2} \text{tr} \Bigg\{ & \\
& + m^4 \left[-\frac{1}{2} \left(\log \frac{m^2}{\mu^2} - \frac{3}{2} \right) \right] \\
& + m^2 \left[- \left(\log \frac{m^2}{\mu^2} - 1 \right) U \right] \\
& + m^0 \left[-\frac{1}{12} \left(\log \frac{m^2}{\mu^2} - 1 \right) G_{\mu\nu}^{\prime 2} - \frac{1}{2} \log \frac{m^2}{\mu^2} U^2 \right] \\
& + \frac{1}{m^2} \left[-\frac{1}{60} (P_\mu G'_{\mu\nu})^2 - \frac{1}{90} G'_{\mu\nu} G'_{\nu\sigma} G'_{\sigma\mu} - \frac{1}{12} (P_\mu U)^2 - \frac{1}{6} U^3 - \frac{1}{12} U G'_{\mu\nu} G'_{\mu\nu} \right] \\
& + \frac{1}{m^4} \left[\frac{1}{24} U^4 + \frac{1}{12} U (P_\mu U)^2 + \frac{1}{120} (P^2 U)^2 + \frac{1}{24} (U^2 G'_{\mu\nu} G'_{\mu\nu}) \right. \\
& \quad \left. - \frac{1}{120} [(P_\mu U), (P_\nu U)] G'_{\mu\nu} - \frac{1}{120} [U[U, G'_{\mu\nu}]] G'_{\mu\nu} \right] \\
& + \frac{1}{m^6} \left[-\frac{1}{60} U^5 - \frac{1}{20} U^2 (P_\mu U)^2 - \frac{1}{30} (U P_\mu U)^2 \right] \\
& \left. + \frac{1}{m^8} \left[\frac{1}{120} U^6 \right] \right\}. \tag{4.59}
\end{aligned}$$

Equation (4.59) is one of the central results, so let us make a few comments about it:

- This formula is the expansion of a functional trace of the form $ic_s \text{Tr} \log [-P^2 + m^2 + U(x)]$ where $P_\mu = iD_\mu$ is a covariant derivative and $U(x)$ is an arbitrary function of spacetime. We have worked in Minkowski space and defined the one-loop action and Lagrangian from $ic_s \text{Tr} \log [-P^2 + m^2 + U] = \Delta S_{\text{eff},1\text{-loop}} = \int d^4x \Delta\mathcal{L}_{\text{eff},1\text{-loop}}$.
- The results of Eq. (4.59) are valid when the mass-squared matrix m^2 commutes with $U(x)$ and $G'_{\mu\nu} = [D_\mu, D_\nu]$.
- The lower case “tr” in (4.59) is over internal indices. These indices may include gauge indices, Lorentz indices (spinor, vector, *etc.*), flavor indices, *etc.*.
- c_s is a constant which relates the functional trace to the effective action, á la the first bullet point above. For example, for real scalars, complex scalars, Dirac fermions, gauge bosons, and Fadeev-Popov ghosts $c_s = 1/2, 1, -1/2, 1/2$, and -1 , respectively. $U(x)$ is a function of the background fields. In Section 4.1.2 we discussed the form of $U(x)$ for various particle species, namely scalars, fermions, and gauge bosons.

- Given the above statements, it is clear that (4.59) is universal in the sense that it applies to any effective action of the form $\text{Tr} \log (-P^2 + m^2 + U)$.²⁰ For any specific theory, one only needs to determine the form of the covariant derivative P_μ and the matrix $U(x)$ and then (4.59) may be used.
- Equation (4.59) is an expansion of the effective Lagrangian through dimension-six operators. U has scaling dimension two, but its operator dimension may be one or greater. In the case U contains a term with unit operator dimension, one needs all the terms in (4.59) to capture all dimension-six operators.
- The lines proportional to m^4 , m^2 , and m^0 in (4.59) come from UV divergences in the evaluation of the trace; μ is a renormalization scale and we used dimensional regularization and $\overline{\text{MS}}$ scheme.
- The lines proportional to m^2 and m^0 can always be absorbed by renormalization. They can also be used to find the contribution of the particles we integrate out to the β -functions of operators.

4.2 Running of Wilson Coefficients and Choosing Operator Set

In the last section, a detailed description was given on the covariant derivative expansion method, a technique to efficiently achieve the first step in Fig. 4.1 — matching a UV model with a SM EFT. In this section, let us move on to the second step — connecting $c_i(\Lambda)$ with $c_i(m_W)$. This step is nothing but evolving the Wilson coefficients $c_i(\mu)$ from the matching scale $\mu = \Lambda$ down to the weak scale $\mu = m_W$, according to their renormalization group (RG) equations. At the leading order, these RG equations are governed by the anomalous dimension matrix γ_{ij} :

$$\frac{dc_i(\mu)}{d \log \mu} = \sum_j \frac{1}{16\pi^2} \gamma_{ij} c_j. \quad (4.60)$$

The solution at leading order is

$$c_i(\mu) = c_i(\Lambda) + \sum_j \frac{1}{16\pi^2} \gamma_{ij} c_j(\Lambda) \log \frac{\mu}{\Lambda}, \quad (4.61)$$

which gives

$$c_i(m_W) = c_i(\Lambda) - \sum_j \frac{1}{16\pi^2} \gamma_{ij} c_j(\Lambda) \log \frac{\Lambda}{m_W}. \quad (4.62)$$

So this RG running step seems really straightforward once the anomalous dimension matrix γ_{ij} is known. However, the calculation of γ_{ij} is far from easy. It is quite tedious and depends on the

²⁰Under the assumption m^2 commutes with U and $G'_{\mu\nu}$; see the second bullet point.

choice of the operator set $\{\mathcal{O}_i\}$. At least, one needs to choose a RG closed set of operators to make Eq. (4.60) valid [92]. Also, there are generically indirect contributions to γ_{ij} in addition to the direct contribution. Here direct contribution refers to the case when $\mathcal{O}_j$ generates $\mathcal{O}_i$ directly through a loop Feynman diagram, while indirect contribution means that $\mathcal{O}_j$ generates some $\mathcal{O}_k$ outside the operator set chosen, whose elimination (through equation of motion or identity) in turn gives $\mathcal{O}_i$.

4.2.1 Practical Relevance of RG Running Effects

It is worth pointing out that although the running of Wilson coefficients is conceptually an important step, there are stringent requirements on the UV models for it to be practically relevant. This is because the near future measurements can at best achieve a precision of per mille level. From $\frac{v^2}{\Lambda^2} \sim 0.1\%$, we see that Λ can be probed at most up to a few TeV. So the logarithm is not large $\log \frac{\Lambda}{m_W} \sim 3$. Therefore in the perturbative calculation, a truncation by loop order counting is more appropriate than by logarithm power counting. Then the per mille level precision means that we can truncate our perturbative calculations at the one-loop level. It is clear from Eq. (4.62) that the RG running effect is one loop smaller than $c_j(\Lambda)$. So first, if $c_j(\Lambda)$ itself is already of one-loop size from the UV model, then its running effects are of two-loop size and hence practically negligible. Second, even in the case that $c_j(\Lambda)$ is generated at tree level, its one-loop size RG running contribution to $c_i(m_W)$ would be subdominant if $c_i(\Lambda)$ is also generated at tree level together with $c_j(\Lambda)$. In summary, practically one needs to take into account of γ_{ij} only when both of the following conditions are satisfied:

1. $c_j(\Lambda)$ is generated at tree level from the UV model.
2. $c_i(\Lambda)$ is not generated at tree level from the UV model.

However, being able to generate Wilson coefficients at tree level turns out to be a stringent requirement for UV models. If we require the UV model to be renormalizable and the heavy field being integrated out *only couples with the bosonic fields of the SM*, it turns out that there are only five of them:

1. A real singlet scalar Φ

$$\Delta\mathcal{L} \supset \Phi |H|^2. \quad (4.63)$$

2. A real (complex) $SU(2)_L$ triplet scalar $\Phi_0 = \Phi_0^a \tau^a$ ($\Phi_1 = \Phi_1^a \tau^a$) with hypercharge $Y_\Phi = 0$ ($Y_\Phi = 1$)

$$\Delta\mathcal{L} \supset H^\dagger \Phi_0 H, \quad (4.64)$$

$$\Delta\mathcal{L} \supset H^\dagger \Phi_1 \tilde{H} + c.c., \quad (4.65)$$

where $\tilde{H} = i\sigma^2 H^*$.

3. A complex $SU(2)_L$ doublet scalar Φ with $U(1)_Y$ hypercharge $Y_\Phi = \frac{1}{2}$

$$\Delta\mathcal{L} \supset |H|^2 (\Phi^\dagger H + c.c.). \quad (4.66)$$

4. A complex $SU(2)_L$ quartet scalar $\Phi_{3/2}$ ($\Phi_{1/2}$) with hypercharge $Y_\Phi = \frac{3}{2}$ ($Y_\Phi = \frac{1}{2}$)

$$\Delta\mathcal{L} \supset \Phi^\dagger H^3 + c.c., \quad (4.67)$$

5. A heavy $U(1)$ gauge boson K_μ

$$\Delta\mathcal{L} \supset B^{\mu\nu} K_{\mu\nu}, \quad (4.68)$$

where $K_{\mu\nu} = \partial_\mu K_\nu - \partial_\nu K_\mu$ and B_μ denotes the SM $U(1)_Y$ gauge boson.

The above list exhausts the possibilities of tree-level Wilson coefficients from renormalizable UV models in which heavy states only couple with the bosonic fields of the SM. First, in order to have tree-level generated Wilson coefficients, the UV Lagrangian must contain a term that is linear in the heavy field being integrated out. Any heavy fermionic field Ψ would need a fermionic field from the SM to form a Lagrangian term that is linear in it, so it immediately goes out of the category. Second, a heavy vector boson has to be gauged in order for the UV theory to be renormalizable. And by requirement of gauge invariance and dimension counting, one can easily see that the No.5 above is the only possibility. Now the only case left is a heavy scalar Φ that couples to the SM Higgs H and gauge bosons. But a renormalizable interaction term between Φ and the SM gauge bosons cannot be linear in Φ due to gauge invariance. Therefore what is left for us is to count all the possible Lagrangian terms formed by Φ and H that is linear in Φ . After appropriate diagonalization of Φ and H , we do not need to consider the quadratic terms. Then there are only two types of renormalizable interactions $H^a H^b \Phi^{ab}$ and $H^a H^b H^c \Phi^{abc}$, where we have written the SM Higgs field H in terms of its four real components H^a with $a = 1, 2, 3, 4$. Because only symmetric combinations are non-vanishing, it is clear that there are in total 10 real components Φ^{ab} that are enumerated by No.1 and No.2 in the above list, and 20 real components Φ^{abc} that are enumerated by No.3 and No.4. Of course, there are more UV models with tree-level generated Wilson coefficients, if one allows the new physics to couple with SM fermions directly.

Among the five models above, No.3 and No.4 can generate only $\mathcal{O}_6$ at tree-level. Since $\mathcal{O}_6$ does not run into other dimension-six operators, clearly the condition (2) above cannot be satisfied by No.3 or No.4. So RG analysis is needed for only the rest three on the list. Of course, if one allows the new physics to couple with SM fermions directly, then more UV models can be found in which the RG analysis is relevant.

4.2.2 Popular Operator Bases in the Literature

A few popular choices of dim-6 operator set are commonly used in the literature (see [109] for a recent review). They have been developed with two different types of motivations: (1) completeness, and (2) phenomenological relevance. In spite of that, however, they are actually not very

different from each other. In this subsection, we will briefly describe each basis and then discuss the relation among them.

With a motivation of completeness, one starts with enumerating all the possible dim-6 operators that respect the Standard Model gauge symmetry. A systematic classification of the dim-6 operators would be very helpful in order not to miss any of them. Then many combinations of them are found to be zero due to simple identities. One can remove these redundances and shrink the operator set. In addition, many other combinations are zero upon using equation of motions, and hence would not contribute to physical observables which are on-shell quantities. These combinations can also be removed because they are redundant in respect of describing physics. After all of these reductions, one arrives at an operator set that is non redundant but still complete, in a sense that it has the full capability of describing the physical effects of any dim-6 operators. Clearly, the non redundant complete set of operators form an “operator basis”. There are, of course, multiple choices of operator bases, all related by usual basis transformations.

The first attempt of this completeness motivated construction dates back to [121], where 80 dim-6 operators were claimed to be independent. However, it was later discovered that there were still some redundant combinations within the set of 80. The non-redundant basis was eventually found to contain only 59 dim-6 operators [122]. To respect this first success, let us call the 59 dim-6 operators listed in [122] the “standard basis”. During the past year, the full anomalous dimension matrix γ_{ij} has been calculated in the standard basis [92–95].

The second type of motivation in choosing operator set is the relevance to phenomenology. With this kind of motivation, one usually starts with a quite small set of operators that are immediately relevant to the physics concerned. But if the RG running effects turns out to be important, a complete operator set is required for the analysis. In that case, one can extend the initial operator set into a complete operator basis by adding enough non redundant operators to it. Popular operator bases constructed along this line include the “EGGM basis” [97], the “HISZ basis” [123], and the “SILH basis” [96, 107, 124]. These three bases are all motivated by studying physics relevant to the Higgs boson and the electroweak bosons. As a result, they all maximize the use of bosonic operators. And in fact, they are very close to each other. Consider the following seven operators $\{\mathcal{O}_W, \mathcal{O}_B, \mathcal{O}_{WW}, \mathcal{O}_{WB}, \mathcal{O}_{BB}, \mathcal{O}_{HW}, \mathcal{O}_{HB}\}$, where $\mathcal{O}_{HW}$ and $\mathcal{O}_{HB}$ are defined as

$$\mathcal{O}_{HW} \equiv 2ig(D^\mu H)^\dagger \tau^a (D^\nu H) W_{\mu\nu}^a, \quad (4.69)$$

$$\mathcal{O}_{HB} \equiv ig'(D^\mu H)^\dagger (D^\nu H) B_{\mu\nu}, \quad (4.70)$$

and the other five are defined in Table 4.1. There are two identities among them as following

$$\mathcal{O}_W = \mathcal{O}_{HW} + \frac{1}{4}(\mathcal{O}_{WW} + \mathcal{O}_{WB}), \quad (4.71)$$

$$\mathcal{O}_B = \mathcal{O}_{HB} + \frac{1}{4}(\mathcal{O}_{BB} + \mathcal{O}_{WB}). \quad (4.72)$$

So only five out of the seven are non redundant. The difference among “EGGM basis”, “HISZ basis”, and “SILH basis” just lies in different ways of choosing five operators out of these seven: “EGGM basis” drops $\{\mathcal{O}_{HW}, \mathcal{O}_{HB}\}$, “HISZ basis” drops $\{\mathcal{O}_W, \mathcal{O}_B\}$, and “SILH basis” drops $\{\mathcal{O}_{WW}, \mathcal{O}_{WB}\}$.

The three phenomenological relevance motivated bases are not that different from the standard basis either. As mentioned before, due to motivation difference, the second type maximizes the use of bosonic operators. It turns out that to obtain the “EGGM basis” from the standard basis, one only needs to do the following basis transformation (trading five fermionic operators into five bosonic operators using equation of motion):

$$(H^\dagger \tau^a \overleftrightarrow{D}^\mu H)(\bar{L}_1 \gamma_\mu \tau^a L_1) \rightarrow O_W = ig(H^\dagger \tau^a \overleftrightarrow{D}^\mu H)(D^\nu W_{\mu\nu}^a), \quad (4.73)$$

$$(H^\dagger \overleftrightarrow{D}^\mu H)(\bar{e} \gamma_\mu e) \rightarrow O_B = ig' Y_H (H^\dagger \overleftrightarrow{D}^\mu H)(\partial^\nu B_{\mu\nu}), \quad (4.74)$$

$$(\bar{u} \gamma^\mu t_s^A u)(\bar{d} \gamma_\mu t_s^A d) \rightarrow O_{2G} = -\frac{1}{2}(D^\mu G_{\mu\nu}^a)^2, \quad (4.75)$$

$$(\bar{L}_1 \gamma^\mu \tau^a L_1)(\bar{L}_1 \gamma_\mu \tau^a L_1) \rightarrow O_{2W} = -\frac{1}{2}(D^\mu W_{\mu\nu}^a)^2, \quad (4.76)$$

$$(\bar{e} \gamma^\mu e)(\bar{e} \gamma_\mu e) \rightarrow O_{2B} = -\frac{1}{2}(\partial^\mu B_{\mu\nu})^2. \quad (4.77)$$

4.2.3 Choosing Operator Set in Light of RG Running Analysis

There are obviously three types of operator sets in general: (1) an exact complete set which is just an operator basis that related to the standard basis by a basis transformation, (2) an over complete set that has some redundant operators, and (3) an incomplete set that lacks of some components compared to a complete operator basis. Let us comment on these three choices one by one.

- Working with a Complete Operator Basis

This case is fairly straightforward. Since the full anomalous dimension matrix γ_{ij} in the standard basis has been computed [92–94], one just needs to carry out the basis transformation to obtain the γ_{ij} in the new basis.

- Working with an Over Complete Operator Set

Sometimes it is helpful to use a redundant operator set because it can make the physics more transparent. For example, the matching from UV model may just generate an over complete set of effective operators. An obvious drawback of working with an over complete set of operators is that the size of γ_{ij} would be larger than necessary, and that the value of γ_{ij} would not be unique. However, this does not necessarily mean that γ_{ij} is harder to calculate. For example, consider the extreme case of using all the dim-6 operators, before removing any redundant combination. This is a super over complete set, and as a result the size of γ_{ij} would be way larger than that in standard basis. But with this choice of operator set, all the contributions to γ_{ij} are direct contributions by definition. Some of these direct contributions would become indirect in a smaller operator set, and one has to accommodate them by using equation of motion or identity, which is a further step of calculation. Therefore, in some cases, it is the reduction from an over complete set to an exact complete set that requires

more work. The ambiguity in γ_{ij} would not cause any problem, because one can pick any of them to work with. And they will eventually all lead to the same result.

- Working with an Incomplete Operator Set

An operator basis contains 59 operators, which has 76 (2499) real Wilson coefficients for the number of generation being one (three) [94]. So that is practically a very large basis to work with. In some cases, only a small number of operators are relevant to the physics considered, then it is very tempting to just focus on this small incomplete set, for purpose of simplification. However, while a complete or over operator basis is obviously guaranteed to be RG closed, an incomplete operator set is typically not. When the incomplete operator set is not RG closed, Eq. (4.60) no longer holds. To fix this problem, one can view the incomplete operator set $\{\mathcal{O}_i\}$ as a subset of a certain complete operator basis $\{\mathcal{O}_i, \mathcal{O}_a\}$. Once this full operator basis is specified, one has a clear definition of the sub matrix γ_{ij} to compute the RG induced effects. Obviously, the off diagonal block γ_{ia} is generically nonzero, which means some operator $\mathcal{O}_a$ outside the chosen operator set $\{\mathcal{O}_i\}$ can also be RG induced, which could bring additional constraints on the UV model. The ignorance of this makes the constraints over conservative. (see also the discussion in section 2 of [97]).

4.3 Mapping Wilson Coefficients onto Precision Observables

So far we discussed how to compute the Wilson coefficients $c_i(\Lambda)$ from a given UV model, and how to run them down to the weak scale $c_i(m_W)$ with the appropriate anomalous dimension matrix γ_{ij} . This section then is devoted to the last step in Fig. 4.1 — mapping c_i ²¹ onto the weak scale precision observables. The Wilson coefficients c_i will bring various corrections to the precision observables at the weak scale. The goal of this section is to study the deviation of each weak scale precision observable as a function of c_i .

It is worth noting that our SM EFT parameterized by Eq. (4.1) and c_i is totally different from the widely used seven- κ parametrization (for example see [85]), which parameterizes only a size change in each of the SM type Higgs couplings. The seven- κ actually parameterize models that do not respect the electroweak gauge symmetry and hence violates unitarity. As a result, future precision programs show spuriously high sensitivity on them. Our SM EFT on the other hand, parameterize new physics in the direction that respects the SM gauge invariance and free from unitarity violation.

In order to provide a concrete mapping result, let us specify a set of operators to work with. With in mind a special interest in UV models in which new physics is CP preserving and couples with the SM only through the Higgs and gauge bosons, let us choose the set of dim-6 operators that are purely bosonic and CP conserving. All the dim-6 operators satisfying these conditions are listed in Table 4.1. This set of effective operators coincides with the set chosen in [97], supplemented

²¹Throughout this section, all the Wilson coefficients mentioned will be at the weak scale $\mu = m_W$. In order to reduce the clutter, we hence suppress this specification of the RG scale and use c_i as a shorthand for $c_i(m_W)$.

$\mathcal{O}_{GG} = g_s^2 H ^2 G_{\mu\nu}^a G^{a,\mu\nu}$	$\mathcal{O}_H = \frac{1}{2} (\partial_\mu H ^2)^2$
$\mathcal{O}_{WW} = g^2 H ^2 W_{\mu\nu}^a W^{a,\mu\nu}$	$\mathcal{O}_T = \frac{1}{2} (H^\dagger \overleftrightarrow{D}_\mu H)^2$
$\mathcal{O}_{BB} = g'^2 H ^2 B_{\mu\nu} B^{\mu\nu}$	$\mathcal{O}_R = H ^2 D_\mu H ^2$
$\mathcal{O}_{WB} = 2gg' H^\dagger \tau^a H W_{\mu\nu}^a B^{\mu\nu}$	$\mathcal{O}_D = D^2 H ^2$
$\mathcal{O}_W = ig (H^\dagger \tau^a \overleftrightarrow{D}^\mu H) D^\nu W_{\mu\nu}^a$	$\mathcal{O}_6 = H ^6$
$\mathcal{O}_B = ig' Y_H (H^\dagger \overleftrightarrow{D}^\mu H) \partial^\nu B_{\mu\nu}$	$\mathcal{O}_{2G} = -\frac{1}{2} (D^\mu G_{\mu\nu}^a)^2$
$\mathcal{O}_{3G} = \frac{1}{3!} g_s f^{abc} G_\rho^{a\mu} G_\mu^{b\nu} G_\nu^{c\rho}$	$\mathcal{O}_{2W} = -\frac{1}{2} (D^\mu W_{\mu\nu}^a)^2$
$\mathcal{O}_{3W} = \frac{1}{3!} g \epsilon^{abc} W_\rho^{a\mu} W_\mu^{b\nu} W_\nu^{c\rho}$	$\mathcal{O}_{2B} = -\frac{1}{2} (\partial^\mu B_{\mu\nu})^2$

Table 4.1: Dimension-six bosonic operators for our mapping analysis.

by the operators $\mathcal{O}_D$ and $\mathcal{O}_R$. Wilson coefficients of all the fermionic operators are assumed to be zero.

There are four categories of precision observables on which present and near future precision programs will be able to reach a per mille level sensitivity: (1) Electroweak Precision Observables (EWPO), (2) Triple Gauge Couplings (TGC), (3) Higgs decay widths, and (4) Higgs production cross sections. In the mapping calculation, we can keep only up to linear order of Wilson coefficients and we can include only tree-level diagrams of the Wilson coefficients. This is because the near future precision experiments would only be sensitive to one-loop physics, and we practically consider each power of $\frac{1}{\Lambda^2} c_i$ as one-loop size, since it is already known that the SM is a very good theoretical description and the deviations should be small. Although in some UV models, Wilson coefficients can arise at tree-level, the corresponding $\frac{1}{\Lambda^2}$ must be small enough to be consistent with the current constraints. So considering $\frac{1}{\Lambda^2} c_i$ as one-loop size is practically appropriate.

4.3.1 Electroweak Precision Observables

Electroweak precision observables represent the oblique corrections to the propagators of electroweak gauge bosons. Specifically, there are four transverse vacuum polarization functions: $\Pi_{WW}(p^2)$, $\Pi_{ZZ}(p^2)$, $\Pi_{\gamma\gamma}(p^2)$, and $\Pi_{\gamma Z}(p^2)$ ²², each of which can be expanded in p^2

$$\Pi(p^2) = a_0 + a_2 p^2 + a_4 p^4 + \mathcal{O}(p^6). \quad (4.78)$$

Two out of these expansion coefficients are fixed to zero by the masslessness of the photon $\Pi_{\gamma\gamma}(0) = \Pi_{\gamma Z}(0) = 0$. Another three combinations are fixed (absorbed) by the definition of the three free parameters g , g' , and v in electroweak theory. So up to p^2 order, there are three left-over parameters

²²Throughout this chapter, $\Pi(p^2)$ is used to denote the additional part of the transverse vacuum polarization function due to the Wilson coefficients. In a more precise notation, one should use $\Pi^{\text{new}}(p^2)$ as in [29] or $\delta\Pi(p^2)$ as in [125] for it, but let us simply use $\Pi(p^2)$ to reduce the clutter. That being said, our $\Pi(p^2)$ at leading order is linear in c_i .

$S = -\frac{4c_Z s_Z}{\alpha} \Pi'_{3B}(0)$
$X = -\frac{1}{2} m_W^2 \Pi''_{3B}(0)$
$T = \frac{1}{\alpha} \frac{1}{m_W^2} [\Pi_{WW}(0) - \Pi_{33}(0)]$
$U = \frac{4s_Z^2}{\alpha} [\Pi'_{WW}(0) - \Pi'_{33}(0)]$
$V = \frac{1}{2} m_W^2 [\Pi''_{WW}(0) - \Pi''_{33}(0)]$
$W = -\frac{1}{2} m_W^2 \Pi''_{33}(0)$
$Y = -\frac{1}{2} m_W^2 \Pi''_{BB}(0)$

Table 4.2: Definitions of the EWPO parameters, where the single/double prime denotes the first/second derivative of the transverse vacuum polarization functions.

that can be used to test the predictions of the model. These are the Peskin-Takeuchi parameters S , T , and U defined as in [29, 126], which captures all the possible non-decoupling electroweak oblique corrections. As higher energy scales were probed at LEP II, it was proposed to include also the coefficients of p^4 terms, which brings us four additional parameters W , Y , X , V [125, 127, 128].

So in total, there are seven EWPO parameters in consideration, S, T, U, W, Y, X, V . In this chapter, let us take the definitions of them as listed in Table 4.2,²³ where for the purpose of conciseness, we use the alternative set $\{\Pi_{33}, \Pi_{BB}, \Pi_{3B}\}$ instead of $\{\Pi_{ZZ}, \Pi_{\gamma\gamma}, \Pi_{\gamma Z}\}$.²⁴ And due to the relation $W^3 = c_Z Z + s_Z A$ and $B = -s_Z Z + c_Z A$ ²⁵, the two set are simply related by the

²³Our definitions in Table 4.2 agree with [126] and [128]. Many other popular definitions are in common use as well (e.g. see [26, 29, 125]). The main differences lie in the choice of using derivatives of $\Pi(p^2)$ evaluated at $p^2 = 0$, such as $\Pi'_{WW}(0)$ etc., versus using some form of finite distance subtraction, such as $\frac{\Pi_{WW}(m_W^2) - \Pi_{WW}(0)}{m_W^2}$ etc. Up to p^4 order in $\Pi(p^2)$, this discrepancy would only cause a disagreement in the result of U . For example, the definition in [29] would result in nonzero U parameter from the custodial preserving operator $\mathcal{O}_{2W}$: $U = \frac{s_Z^4}{\alpha} \frac{4m_Z^2}{\Lambda^2} c_{2W} \neq 0$. In this chapter, let us stick to the definition in [126] to make U a purely custodial violating parameter. Under our definition, $U = 0$ at dim-6 level.

²⁴One may also have a concern that these definitions through the transverse polarization functions $\Pi(p^2)$ are not generically gauge invariant. In principle, these $\Pi(p^2)$ functions can be promoted to gauge invariant ones $\bar{\Pi}(p^2)$ by a “pinch technique” prescription. (For examples, see discussions in [108, 129, 130].)

²⁵Throughout this chapter, we adopt the notation $c_Z \equiv \cos \theta_Z$ etc., with θ_Z denoting the weak mixing angle. We do not use θ_W in order to avoid clash with the Wilson coefficient for the operator $\mathcal{O}_W$.

$S = \frac{c_Z^2 s_Z^2}{\alpha} \frac{4m_Z^2}{\Lambda^2} (4c_{WB} + c_W + c_B)$	$W = \frac{m_W^2}{\Lambda^2} c_{2W}$
$T = \frac{1}{\alpha} \frac{2v^2}{\Lambda^2} c_T$	$Y = \frac{m_W^2}{\Lambda^2} c_{2B}$
$U = 0$	$X = V = 0$

Table 4.3: EWPO parameters in terms of Wilson coefficients, where $v = 174\text{GeV}$.

transformation

$$\Pi_{33} = c_Z^2 \Pi_{ZZ} + s_Z^2 \Pi_{\gamma\gamma} + 2c_Z s_Z \Pi_{\gamma Z}, \quad (4.79)$$

$$\Pi_{BB} = s_Z^2 \Pi_{ZZ} + c_Z^2 \Pi_{\gamma\gamma} - 2c_Z s_Z \Pi_{\gamma Z}, \quad (4.80)$$

$$\Pi_{3B} = -c_Z s_Z \Pi_{ZZ} + c_Z s_Z \Pi_{\gamma\gamma} + (c_Z^2 - s_Z^2) \Pi_{\gamma Z}. \quad (4.81)$$

Table 4.3 summarizes the mapping results of the seven EWPO parameters, *i.e.* each of them as a linear function (to leading order) of the Wilson coefficients c_i . These results are straightforward to calculate. First, we calculate $\Pi_{WW}(p^2)$, $\Pi_{ZZ}(p^2)$, $\Pi_{\gamma\gamma}(p^2)$, and $\Pi_{\gamma Z}(p^2)$ in terms of c_i . This can be done by expanding out the dim-6 operators in Table 4.1, identifying the relevant Lagrangian terms, and reading off the two-point Feynman rules. The details of these steps together with the results of $\Pi_{WW}(p^2)$, $\Pi_{ZZ}(p^2)$, $\Pi_{\gamma\gamma}(p^2)$, and $\Pi_{\gamma Z}(p^2)$ (Table E.1) are shown in Appendix E.1. Next, we compute the alternative combinations $\Pi_{WW}(p^2) - \Pi_{33}(p^2)$, $\Pi_{33}(p^2)$, $\Pi_{BB}(p^2)$, $\Pi_{3B}(p^2)$ using the transformation relation Eq. (4.79)-Eq. (4.81), the results of which are also summarized in Appendix E.1 (Table E.2). Finally, we combine Table E.2 with the definitions of EWPO parameters (Table 4.2) to obtain the results in Table 4.3.

The importance of W and Y parameters should be emphasized. It should be clear from the definitions Table 4.2 that the seven EWPO parameters fall into four different classes: $\{S, X\}$, $\{T, U, V\}$, $\{W\}$, and $\{Y\}$. Therefore W and Y out of the four p^4 order EWPO parameters supplement the classes formed by S, T, U (see also the discussions in [128]). Our mapping results (Table 4.3) also show that W and Y are practically more important compared to X and V , for W and Y are nonzero while X and V vanish at dim-6 level.

4.3.2 Triple Gauge Couplings

The TGC parameters can be described by the a phenomenological Lagrangian [131–133]

$$\begin{aligned} \mathcal{L}_{\text{TGC}} = & igc_Z Z^\mu \cdot g_1^Z (\hat{W}_{\mu\nu}^- W^{+\nu} - \hat{W}_{\mu\nu}^+ W^{-\nu}) + igW_\mu^+ W_\nu^- (\kappa_Z \cdot c_Z \hat{Z}^{\mu\nu} + \kappa_\gamma \cdot s_Z \hat{A}^{\mu\nu}) \\ & + \frac{ig}{m_W^2} \hat{W}_\mu^{-\rho} \hat{W}_{\rho\nu}^+ (\lambda_Z \cdot c_Z \hat{Z}^{\mu\nu} + \lambda_\gamma \cdot s_Z \hat{A}^{\mu\nu}), \end{aligned} \quad (4.82)$$

where $\hat{V}_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$. Among the five parameters above, there are two relations due to an accidental custodial symmetry. Let us take g_1^Z , κ_γ , and λ_γ as the three independent parameters.

$$\begin{aligned} \delta g_1^Z &= -\frac{m_Z^2}{\Lambda^2} c_W \\ \delta \kappa_\gamma &= \frac{4m_W^2}{\Lambda^2} c_{WB} \\ \lambda_\gamma &= -\frac{m_W^2}{\Lambda^2} c_{3W} \end{aligned}$$

Table 4.4: TGC parameters in terms of Wilson coefficients.

The other two can be expressed as [133]

$$\kappa_Z = g_1^Z - \frac{s_Z^2}{c_Z^2} (\kappa_\gamma - 1), \quad (4.83)$$

$$\lambda_Z = \lambda_\gamma. \quad (4.84)$$

The SM values of TGC parameters are $g_{1,\text{SM}}^Z = \kappa_{\gamma,\text{SM}} = 1$, $\lambda_{\gamma,\text{SM}} = 0$. Their deviations from SM are currently constrained at percent level [134], and will be improved to 10^{-4} level at ILC500 (see No.2 in [23]). Their mapping results are summarized in Table 4.4.²⁶

4.3.3 Deviations in Higgs Decay Widths

The dim-6 operators bring deviations in the Higgs decay widths from the Standard Model. In this chapter, we will study all the SM Higgs decay modes that near future linear colliders can have sub-percent sensitivity on, *i.e.* $\Gamma \in \{\Gamma_{h \rightarrow f\bar{f}}, \Gamma_{h \rightarrow gg}, \Gamma_{h \rightarrow \gamma\gamma}, \Gamma_{h \rightarrow \gamma Z}, \Gamma_{h \rightarrow WW^*}, \Gamma_{h \rightarrow ZZ^*}\}$. Our analysis for the decay modes through off-shell vector gauge bosons $h \rightarrow WW^*$ and $h \rightarrow ZZ^*$ apply to all their fermionic modes, namely that $h \rightarrow WW^* \rightarrow Wl\bar{\nu}/Wd\bar{u}$ and $h \rightarrow ZZ^* \rightarrow Zf\bar{f}$.

For each decay width Γ above, let us define its deviation from the SM

$$\epsilon \equiv \frac{\Gamma}{\Gamma_{\text{SM}}} - 1. \quad (4.85)$$

It turns out that at leading order (linear power) in c_i , this deviation is generically a sum of three parts, (1) the “interference correction” ϵ_I , (2) the “residue correction” ϵ_R , and (3) the “parametric correction” ϵ_P :

$$\epsilon = \epsilon_I + \epsilon_R + \epsilon_P. \quad (4.86)$$

In the following, we will first briefly discuss the meaning and the mapping results of each part, and then discuss in detail how to derive these results.

Brief description of the results

- “Interference Correction” ϵ_I

²⁶These results are also obtained in [97].

$$\begin{aligned}
\epsilon_{h\bar{f}f,I} &= 0 \\
\epsilon_{hgg,I} &= \frac{(4\pi)^2}{\text{Re}A_{hgg}^{\text{SM}}} \frac{16v^2}{\Lambda^2} c_{GG} \\
\epsilon_{h\gamma\gamma,I} &= \frac{(4\pi)^2}{\text{Re}A_{h\gamma\gamma}^{\text{SM}}} \frac{8v^2}{\Lambda^2} (c_{WW} + c_{BB} - c_{WB}) \\
\epsilon_{h\gamma Z} &= \frac{(4\pi)^2}{\text{Re}A_{h\gamma Z}^{\text{SM}}} \frac{4v^2}{\Lambda^2} \frac{1}{c_Z} \left[2(c_Z^2 c_{WW} - s_Z^2 c_{BB}) - (c_Z^2 - s_Z^2) c_{WB} \right] \\
\epsilon_{hWW^*,I} &= \left[2I_a(\beta_W) - I_b(\beta_W) \right] \frac{m_W^2}{\Lambda^2} c_{2W} - \left[2I_b(\beta_W) - I_c(\beta_W) \right] \frac{4m_W^2}{\Lambda^2} c_{WW} \\
&\quad - I_a(\beta_W) \frac{2m_W^2}{\Lambda^2} c_W - I_b(\beta_W) \frac{v^2}{\Lambda^2} c_R + \frac{2m_h^2}{\Lambda^2} c_D \\
\epsilon_{hZZ^*,I} &= + \left[2I_a(\beta_Z) - I_b(\beta_Z) \right] \frac{m_Z^2}{\Lambda^2} (c_Z^2 c_{2W} + s_Z^2 c_{2B}) \\
&\quad - \left[2I_b(\beta_Z) - I_c(\beta_Z) \right] \frac{4m_Z^2}{\Lambda^2} (c_Z^4 c_{WW} + s_Z^4 c_{BB} + c_Z^2 s_Z^2 c_{WB}) \\
&\quad - I_a(\beta_Z) \frac{2m_Z^2}{\Lambda^2} (c_Z^2 c_W + s_Z^2 c_B) + I_b(\beta_Z) \frac{v^2}{\Lambda^2} (2c_T - c_R) + \frac{2m_h^2}{\Lambda^2} c_D \\
&\quad + \frac{2Q_f c_Z^2 s_Z^2}{T_f^3 - s_Z^2 Q_f} \left\{ \begin{aligned} &\left[I_a(\beta_Z) - I_b(\beta_Z) - 1 \right] \frac{m_Z^2}{\Lambda^2} (c_{2W} - c_{2B} - c_W + c_B) \\ &+ I_d(\beta_Z) \frac{m_Z^2}{\Lambda^2} \left[2c_Z^2 c_{WW} - 2s_Z^2 c_{BB} - (c_Z^2 - s_Z^2) c_{WB} \right] \end{aligned} \right\}
\end{aligned}$$

Table 4.5: Interference corrections ϵ_I to Higgs decay widths, with $\beta_W \equiv \frac{m_W}{m_h}$, $\beta_Z \equiv \frac{m_Z}{m_h}$, and the auxiliary integrals $I_a(\beta)$, $I_b(\beta)$, $I_c(\beta)$, $I_d(\beta)$ listed in the appendix (Eq. (E.29)-(E.32)). In the table, $v = 174\text{GeV}$. The A_{hgg}^{SM} , $A_{h\gamma\gamma}^{\text{SM}}$, and $A_{h\gamma Z}^{\text{SM}}$ are the standard form factors, whose expressions are listed in Appendix E.2 (Eq. (E.33)-(E.35)).

ϵ_I captures the effects of new amputated Feynman diagrams $iM_{\text{AD,new}}(c_i)$ introduced by the dim-6 effective operators. This modifies the value of the total amputated diagram

$$iM_{\text{AD}} = iM_{\text{AD,SM}} + iM_{\text{AD,new}}(c_i). \quad (4.87)$$

Upon modulus square, the cross term, namely the interference between the new amplitude and the SM amplitude, gives the leading order contribution to the deviation:

$$\epsilon_I = \frac{\int d\Pi_f \overline{M_{\text{AD,SM}}^* M_{\text{AD,new}}(c_i)} + c.c.}{\int d\Pi_f |M_{\text{AD,SM}}|^2}, \quad (4.88)$$

where $\int d\Pi_f$ denotes the phase space integral, and the overscore denotes any step needed for getting the unpolarized result, namely that a sum of final spins and an average over the initial

	ϵ_R	ϵ_P
$\Gamma_{hf\bar{f}}$	Δr_h	$\Delta w_{y_f^2}$
Γ_{hgg}	0	0
$\Gamma_{h\gamma\gamma}$	0	0
$\Gamma_{h\gamma Z}$	0	0
Γ_{hWW^*}	$\Delta r_h + \Delta r_W$	$3\Delta w_{g^2} + \Delta w_{v^2}$
Γ_{hZZ^*}	$\Delta r_h + \Delta r_Z$	$3\Delta w_{g^2} + \Delta w_{v^2} + \left(3\frac{s_Z^2}{c_Z^2} - \frac{2s_Z^2 Q_f}{T_f^3 - s_Z^2 Q_f}\right) \Delta w_{s_Z^2}$

Table 4.6: Residue corrections ϵ_R and parametric corrections ϵ_P to Higgs decay widths. The results of residue modifications $\Delta r_h, \Delta r_W, \Delta r_Z$ and parameter modifications $\Delta w_{g^2}, \Delta w_{v^2}, \Delta w_{s_Z^2}, \Delta w_{y_f^2}$ are listed in the appendix.

spins, if any. The results of ϵ_I are summarized in Table 4.5. More details of the calculation are in the appendix. Specifically, Appendix E.1 contains a list of the new set of Feynman rules generated by the dim-6 operators. And then Appendix E.2 collects all the relevant new amputated diagrams involved in each ϵ_I . Due to the phase space integral, there are some complicated auxiliary integrals involved in the results. The definitions and values of these auxiliary integrals are also given in Appendix E.2 (Eq. (E.29)-(E.32)). The $A_{hgg}^{\text{SM}}, A_{h\gamma\gamma}^{\text{SM}}$, and $A_{h\gamma Z}^{\text{SM}}$ in Table 4.5 are the standard form factors, detailed expressions of which are shown in Appendix E.2 (Eq. (E.33)-(E.35)).

- “Residue Correction” ϵ_R

ϵ_R captures the effects of mass pole residue modifications by the dim-6 effective operators. We know from LSZ reduction formula that the invariant amplitude iM should equal to the value of amputated diagram iM_{AD} multiplied by the square root of the mass pole residue r_k of each external leg particle k

$$iM = \left(\prod_{k \in \{\text{external legs}\}} r_k^{1/2} \right) \cdot iM_{\text{AD}}. \quad (4.89)$$

So besides the corrections to iM_{AD} discussed before, a mass pole residue modification Δr_k of an external leg particle k would also feed into the decay width deviation. Upon modulus

square, this part of deviation is

$$\epsilon_R = \sum_{k \in \{\text{external legs}\}} \Delta r_k \quad (4.90)$$

The results of ϵ_R for each decay width are summarized in the second column of Table 4.6. The values of the relevant residue modifications Δr_k are listed in Appendix E.4 (Table E.3). Note that unlike the interference correction ϵ_I , the residue correction ϵ_R corresponds to a contribution with the size of $\Gamma_{\text{SM}} \times c_i$. But for Γ_{hgg} , $\Gamma_{h\gamma\gamma}$, and $\Gamma_{h\gamma Z}$, the SM value Γ_{SM} is already of one-loop size. So $\epsilon_{hgg,R}$, $\epsilon_{h\gamma\gamma,R}$, and $\epsilon_{h\gamma Z,R}$ should be one-loop size in Wilson coefficients, namely that $\frac{1}{16\pi^2} \times c_i$. In our order of approximation, this size should be neglected for consistency.

- “Parametric Correction” ϵ_P

ϵ_P captures the effects of Lagrangian parameter modifications by the dim-6 effective operators. When computing the decay width Γ , one usually writes it in terms of a set of Lagrangian parameters $\{\rho\}$, which in our case are $\{\rho\} = \{g^2, v^2, s_Z^2, y_f^2\}$. So $\Gamma = \Gamma(\rho, c_i)$ is what one usually calculates. However, the deviation ϵ is supposed to be a physical observable that describes the change of the relation between Γ and other physical observables $\{obs\}$, which in our case can be taken as $\{obs\} = \{\hat{\alpha}, \hat{G}_F, \hat{m}_Z^2, \hat{m}_f^2\}$. So one should eliminate $\{\rho\}$ in terms of $\{obs\}$. This elimination brings additional dependence on $\{c_i\}$, because the Wilson coefficients also modify the relation between $\{\rho\}$ and $\{obs\}$ through $\rho = \rho(obs, c_i)$. Therefore, to include the full dependence on c_i , one should write the decay width as

$$\Gamma = \Gamma(\rho(obs, c_i), c_i). \quad (4.91)$$

The ϵ_I and ϵ_R discussed previously only take into account of the explicit dependence on c_i , with $\{\rho\}$ held fixed. The implicit dependence on c_i through modifying the Lagrangian parameter ρ is what we will call “parametric correction”:

$$\epsilon_P = \sum_{\rho \in \{g^2, v^2, s_Z^2, y_f^2\}} \frac{\partial \ln \Gamma(\rho, c_i)}{\partial \ln \rho} \Delta \ln \rho = \sum_{\rho \in \{g^2, v^2, s_Z^2, y_f^2\}} \frac{\partial \ln \Gamma(\rho, c_i)}{\partial \ln \rho} \Delta w_\rho, \quad (4.92)$$

where Δw_ρ denotes the Lagrangian parameter modification

$$\Delta w_\rho = \Delta \ln \rho = \frac{\Delta \rho}{\rho}. \quad (4.93)$$

The parametric correction ϵ_P in terms of Δw_ρ are summarized in the third column of Table 4.6. And a detailed calculation of Δw_ρ is in Appendix E.5, with the results summarized in Table E.4. As with the residue correction case, $\epsilon_{hgg,P}$, $\epsilon_{h\gamma\gamma,P}$, and $\epsilon_{h\gamma Z,P}$ are one-loop size in Wilson coefficients and hence neglected for consistency.

Detailed derivation

Clearly from Eq. (4.1), our SM EFT goes back to the SM when all $c_i = 0$. Thus up to linear power of c_i , the deviation defined in Eq. (4.85) is

$$\epsilon \equiv \frac{\Gamma}{\Gamma_{\text{SM}}} - 1 = \frac{\Gamma(c_i)}{\Gamma(c_i = 0)} - 1 = \left. \frac{d \ln \Gamma}{dc_i} \right|_{c_i=0} c_i. \quad (4.94)$$

As explained before, this function $\Gamma(c_i)$ in Eq. (4.94) should be understood as the dependence of Γ on $\{c_i\}$ with the values of $\{obs\}$ held fixed. Practically, it is most convenient to first compute both Γ and $\{obs\}$ in terms of the Lagrangian parameters $\{\rho\}$:

$$\Gamma = \Gamma(\rho, c_i), \quad (4.95)$$

$$obs = obs(\rho, c_i), \quad (4.96)$$

One can then plug the inverse of the second function $\rho = \rho(obs, c_i)$ into the first to get

$$\Gamma(c_i) = \Gamma(\rho(obs, c_i), c_i). \quad (4.97)$$

Clearly, in addition to the explicit dependence on c_i , Γ also has an implicit dependence on c_i through the Lagrangian parameters $\rho(obs, c_i)$:

$$\frac{d \ln \Gamma}{dc_i} = \frac{\partial \ln \Gamma(\rho, c_i)}{\partial c_i} + \sum_{\rho} \frac{\partial \ln \Gamma(\rho, c_i)}{\partial \ln \rho} \frac{\partial \ln \rho(obs, c_i)}{\partial c_i}. \quad (4.98)$$

Putting it another way, the first term in the above shows the deviation when ρ are fixed numbers. But ρ are not fixed numbers. They are a set of Lagrangian parameters determined by a set of experimental measurements obs through relations that get modified by c_i as well. So the truly fixed numbers are the experimental inputs obs . By adding the second piece in Eq. (4.98), we get the full amount of deviation with obs as fixed input numbers. By putting obs in the place of $\ln \Gamma$, one can also explicitly check that Eq. (4.98) keeps obs fixed. Making use of the fact

$$\frac{\partial \ln \rho(obs, c_i)}{\partial c_i} = \left. \frac{d \ln \rho}{dc_i} \right|_{obs=\text{const}} = - \frac{\left. \frac{\partial(obs)}{\partial c_i} \right|_{\rho}}{\left. \frac{\partial(obs)}{\partial \ln \rho} \right|_{c_i}}, \quad (4.99)$$

we clearly see that

$$\frac{d(obs)}{dc_i} = \frac{\partial(obs)}{\partial c_i} + \sum_{\rho} \frac{\partial(obs)}{\partial \ln \rho} \left. \frac{\partial \ln \rho}{\partial c_i} \right|_{c_i} = 0. \quad (4.100)$$

Because of Eq. (4.98), the deviation Eq. (4.94) is split into two parts

$$\begin{aligned} \epsilon &= \left. \frac{\partial \ln \Gamma(\rho, c_i)}{\partial c_i} \right|_{c_i=0} c_i + \sum_{\rho} \left[\left. \frac{\partial \ln \Gamma(\rho, c_i)}{\partial \ln \rho} \right|_{c_i=0} \left(\left. \frac{\partial \ln \rho(obs, c_i)}{\partial c_i} \right|_{c_i=0} c_i \right) \right] \\ &= \left. \frac{\partial \ln \Gamma(\rho, c_i)}{\partial c_i} \right|_{c_i=0} c_i + \epsilon_P, \end{aligned} \quad (4.101)$$

where the implicit dependence part is defined as the parametric correction ϵ_P

$$\epsilon_P \equiv \sum_{\rho} \frac{\partial \ln \Gamma(\rho, c_i)}{\partial \ln \rho} \Big|_{c_i=0} \Delta w_{\rho}, \quad (4.102)$$

with the parameter modifications Δw_{ρ} defined as

$$\Delta w_{\rho} \equiv \frac{\partial \ln \rho(obs, c_i)}{\partial c_i} \Big|_{c_i=0} c_i = \Delta \ln \rho = \frac{\Delta \rho}{\rho}. \quad (4.103)$$

The explicit dependence part can be further split by noting that

$$iM_{AD} = iM_{AD,SM} + iM_{AD,new}(c_i), \quad (4.104)$$

$$iM = \left(\prod_{k \in \{\text{external legs}\}} r_k^{1/2} \right) \cdot iM_{AD}, \quad (4.105)$$

$$\Gamma(\rho, c_i) = \frac{1}{2m_h} \int d\Pi_f \overline{|M|^2} = \frac{1}{2m_h} \left(\prod_{k \in \{\text{external legs}\}} r_k \right) \cdot \int d\Pi_f \overline{|M_{AD}|^2}. \quad (4.106)$$

Therefore we have

$$\begin{aligned} \frac{\partial \ln \Gamma(\rho, c_i)}{\partial c_i} \Big|_{c_i=0} c_i &= \frac{\partial \ln \left[\int d\Pi_f \overline{|M_{AD}|^2} \right]}{\partial c_i} \Big|_{c_i=0} c_i + \sum_{k \in \{\text{external legs}\}} \frac{\partial \ln r_k}{\partial c_i} \Big|_{c_i=0} c_i \\ &= \frac{\Delta \left(\int d\Pi_f \overline{|M_{AD}|^2} \right)}{\int d\Pi_f \overline{|M_{AD}|^2}} \Big|_{c_i=0} + \sum_{k \in \{\text{external legs}\}} \frac{\Delta r_k}{r_k} \Big|_{c_i=0} \\ &= \frac{\int d\Pi_f \overline{M_{AD,SM}^* M_{AD,new}(c_i) + c.c.}}{\int d\Pi_f \overline{|M_{AD,SM}|^2}} + \sum_{k \in \{\text{external legs}\}} \Delta r_k \\ &= \epsilon_I + \epsilon_R, \end{aligned} \quad (4.107)$$

with ϵ_I and ϵ_R defined as

$$\epsilon_I \equiv \frac{\int d\Pi_f \overline{M_{AD,SM}^* M_{AD,new}(c_i) + c.c.}}{\int d\Pi_f \overline{|M_{AD,SM}|^2}}, \quad (4.108)$$

$$\epsilon_R \equiv \sum_{i \in \{\text{external legs}\}} \Delta r_i. \quad (4.109)$$

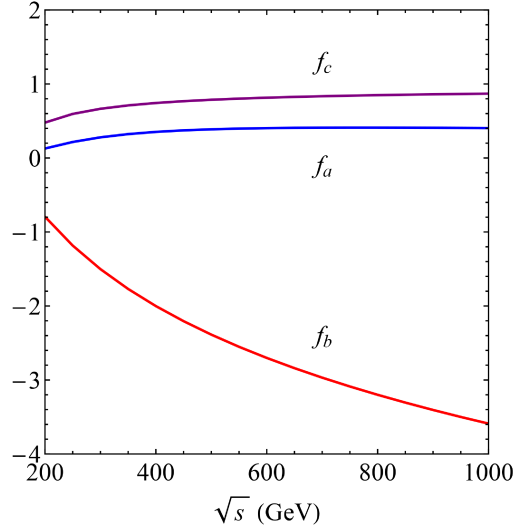


Figure 4.2: Numerical results of auxiliary functions $f_a(s)$, $f_b(s)$, and $f_c(s)$ in $\epsilon_{WW_h,I}(s)$.

So in summary, the total deviation in decay width has three parts $\epsilon = \epsilon_I + \epsilon_R + \epsilon_P$, with

$$\epsilon_I = \frac{\int d\Pi_f \overline{M_{\text{AD,SM}}^* M_{\text{AD,new}}(c_i) + c.c.}}{\int d\Pi_f |M_{\text{AD,SM}}|^2}, \quad (4.110)$$

$$\epsilon_R = \sum_{i \in \{\text{external legs}\}} \Delta r_i, \quad (4.111)$$

$$\epsilon_P = \sum_{\rho \in \{g^2, v^2, s_Z^2, y_f^2\}} \left. \frac{\partial \ln \Gamma(\rho, c_i)}{\partial \ln \rho} \right|_{c_i=0} \Delta w_\rho, \quad (4.112)$$

where

$$\Delta w_\rho \equiv \left. \frac{\partial \ln \rho(\text{obs}, c_i)}{\partial c_i} \right|_{c_i=0} c_i = \Delta \ln \rho = \frac{\Delta \rho}{\rho}. \quad (4.113)$$

For each decay width in consideration, we computed these three parts of deviation. The results are summarized in Table 4.5 and Table 4.6. It is worth noting that this splitting is a convenient intermediate treatment of the calculation, but each of ϵ_I , ϵ_R , ϵ_P alone would not be physical, because it depends on the renormalization scheme as well as the choice of operator basis. It is the total sum of the three that reflects the physical deviation in the decay widths.

4.3.4 Deviations in Higgs Production Cross Sections

The dim-6 operators also induce deviations in the Higgs production cross sections. In this chapter, we will focus on the production modes $\sigma \in \{\sigma_{ggF}, \sigma_{WW_h}, \sigma_{Wh}, \sigma_{Zh}\}$, which are the most

$$\begin{aligned}
\epsilon_{ggF,I} &= \frac{(4\pi)^2}{\text{Re}(A_{hgg}^{\text{SM}})} \frac{16v^2}{\Lambda^2} c_{GG} \\
\epsilon_{WWh,I}(s) &= \left[-f_b(s) - f_c(s) \right] \frac{2m_W^2}{\Lambda^2} c_{2W} + \left[-f_a(s) + 2f_c(s) \right] \frac{8m_W^2}{\Lambda^2} c_{WW} \\
&\quad + \left[f_b(s) + 2f_c(s) \right] \frac{2m_W^2}{\Lambda^2} c_W + f_c(s) \frac{2v^2}{\Lambda^2} c_R + \frac{2m_h^2}{\Lambda^2} c_D \\
\epsilon_{Wh,I} &= \frac{1}{1 - \eta_W^2} \left[-\frac{2s}{\Lambda^2} c_{2W} + I_{VH}(\eta_h, \eta_W) \frac{16m_W^2}{\Lambda^2} c_{WW} \right. \\
&\quad \left. + (1 + 2\eta_W^2 - \eta_W^4) \frac{2s}{\Lambda^2} c_W + (2 - \eta_W^2) \frac{2v^2}{\Lambda^2} c_R \right] + \frac{2m_h^2}{\Lambda^2} c_D \\
\epsilon_{Zh,I} &= \frac{1}{1 - \eta_Z^2} \left[-\frac{2s}{\Lambda^2} (c_Z^2 c_{2W} + s_Z^2 c_{2B}) \right. \\
&\quad + I_{VH}(\eta_h, \eta_Z) \frac{16m_Z^2}{\Lambda^2} (c_Z^4 c_{WW} + s_Z^4 c_{BB} + c_Z^2 s_Z^2 c_{WB}) \\
&\quad + (1 + 2\eta_Z^2 - \eta_Z^4) \frac{2s}{\Lambda^2} (c_Z^2 c_W + s_Z^2 c_B) \\
&\quad \left. + (2 - \eta_Z^2) \frac{2v^2}{\Lambda^2} (-2c_T + c_R) \right] + \frac{2m_h^2}{\Lambda^2} c_D \\
&\quad + \frac{2Q_f c_Z^2 s_Z^2}{T_f^3 - s_Z^2 Q_f} \left\{ -\frac{s}{\Lambda^2} (c_{2W} - c_{2B} - c_W + c_B) \right. \\
&\quad \left. + I_{VH}(\eta_h, \eta_Z) \frac{4m_Z^2}{\Lambda^2} [2c_Z^2 c_{WW} - 2s_Z^2 c_{BB} - (c_Z^2 - s_Z^2) c_{WB}] \right\}
\end{aligned}$$

Table 4.7: Interference corrections ϵ_I to Higgs production cross sections, with $\eta_h \equiv \frac{m_h}{\sqrt{s}}$, $\eta_Z \equiv \frac{m_Z}{\sqrt{s}}$, and the auxiliary function defined as $I_{VH}(\eta_h, \eta_V) \equiv 1 + \frac{6(1-\eta_h^2+\eta_V^2)(1-\eta_V^2)}{(1-\eta_h^2+\eta_V^2)^2+8\eta_V^2}$. In the table, $v = 174\text{GeV}$. The numerical results of the auxiliary functions $f_a(s)$, $f_b(s)$, and $f_c(s)$ in $\epsilon_{WWh,I}(s)$ are shown in Fig. 4.2.

important ones for both hadron colliders such as the LHC and future lepton colliders such as the ILC. As with the decay width case, let us define the cross section deviation

$$\epsilon \equiv \frac{\sigma}{\sigma_{\text{SM}}} - 1. \quad (4.114)$$

There are again three types of corrections

$$\epsilon = \epsilon_I + \epsilon_R + \epsilon_P. \quad (4.115)$$

The mapping results are summarized in Table 4.7 and Table 4.8. Relevant new amputated Feynman diagrams for ϵ_I are listed in Appendix E.3. The calculation of the interference correction to σ_{WWh} turns out to be very involved. Its lengthy analytical expression $\epsilon_{WWh,I}(s)$ does not help much, so let us instead show its numerical results in Table 4.7. The auxiliary functions $f_a(s)$, $f_b(s)$, $f_c(s)$ in $\epsilon_{WWh,I}(s)$ are defined in Appendix E.3 (Eq. (E.52)-(E.54)), where more details of the phase space

	ϵ_R	ϵ_P
σ_{ggF}	0	0
σ_{WW_h}	Δr_h	$4\Delta w_{g^2} + \Delta w_{v^2}$
σ_{Wh}	$\Delta r_h + \Delta r_W$	$3\Delta w_{g^2} + \Delta w_{v^2}$
σ_{Zh}	$\Delta r_h + \Delta r_Z$	$3\Delta w_{g^2} + \Delta w_{v^2} + \left(3\frac{s_Z^2}{c_Z^2} - \frac{2s_Z^2 Q_f}{T_f^3 - s_Z^2 Q_f}\right) \Delta w_{s_Z^2}$

Table 4.8: Residue corrections ϵ_R and parametric corrections ϵ_P to Higgs production cross sections. The results of residue modifications and parameter modifications are listed in the appendix.

integral are also shown. The numerical values of $f_a(s)$, $f_b(s)$, $f_c(s)$ are plotted in Fig. 4.2. The [mathematica code](#) to obtain Fig. 4.2 is also provided.

4.4 Example Applications of Standard Model Effective Field Theory

In this section, two examples are given to demonstrate the process of using the SM EFT as a bridge to connect UV models with weak-scale precision observables, and thus placing constraints on the UV models. One example is a singlet scalar coupled to the Higgs boson, where impacts arise at the tree level. It can achieve first-order electroweak phase transition (EWPT) which would allow electroweak baryogenesis. The other is the scalar top in the Minimal Supersymmetric Standard Model (MSSM), where impacts arise at the one-loop level. It will help minimize the fine-tuning in the Higgs mass-squared. In both cases, future precision Higgs and precision electroweak measurements are found to be sensitive probes.

4.4.1 A Heavy Real Singlet Scalar

In this example, let us consider a heavy real gauge singlet scalar field Φ with mass m_S that couples to the SM via a Higgs portal

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{1}{2} (\partial_\mu \Phi)^2 - \frac{1}{2} m_S^2 \Phi^2 - A |H|^2 \Phi - \frac{1}{2} k |H|^2 \Phi^2 - \frac{1}{3!} \mu \Phi^3 - \frac{1}{4!} \lambda_S \Phi^4. \quad (4.116)$$

This example is one of the five UV models (as listed in Section 4.2.1) that can generate Wilson coefficients at tree-level. It is also among the three surviving candidates (among the listed five) to which RG analysis may be relevant. So it serves as a good example of demonstrating our matching and running steps. Phenomenologically, there are also several motivations for studying

this singlet model. This single additional degree of freedom can successfully achieve a strongly first-order electroweak phase transition (EWPT) [135]. Additionally, singlet sectors of the above form—with particular relations among the couplings—arise in the NMSSM [136] and its variants, *e.g.* [62]. Finally, the effects of Higgs portal operators are captured through the trilinear and quartic interactions $\Phi |H|^2$ and $\Phi^2 |H|^2$, respectively.

Integrating out this heavy scalar Φ generates a SM EFT that matches the above UV theory at the scale $\Lambda = m_S$. Let us first calculate the tree-level contribution to the effective Lagrangian. Following the procedure described in Section 4.1.1, we first obtain the equation of motion

$$(\partial^2 + m_S^2 + k |H|^2) \Phi + \frac{1}{2} \mu \Phi^2 + \frac{1}{3!} \lambda_S \Phi^3 = -A |H|^2, \quad (4.117)$$

which gives the solution

$$\Phi_c = -\frac{1}{\partial^2 + m_S^2 + k |H|^2} A |H|^2 = -\frac{1}{m_S^2} A |H|^2 + \frac{1}{m_S^4} (\partial^2 + k |H|^2) A |H|^2. \quad (4.118)$$

Plugging this solution back to Eq. (4.116), we get the tree-level effective Lagrangian

$$\begin{aligned} \Delta \mathcal{L}_{\text{eff,tree}} &= -A |H|^2 \Phi_c + \frac{1}{2} \Phi_c (-\partial^2 - m_S^2 - k |H|^2) \Phi_c - \frac{1}{3!} \mu \Phi_c^3 - \frac{1}{4!} \lambda_S \Phi_c^4 \\ &= -\frac{1}{2} A |H|^2 \Phi_c - \frac{1}{3!} \mu \Phi_c^3 - \frac{1}{4!} \lambda_S \Phi_c^4 \\ &= \frac{1}{2m_S^2} A^2 |H|^4 + \frac{A^2}{m_S^4} \mathcal{O}_H + \left(-\frac{k A^2}{2m_S^4} + \frac{1}{3!} \frac{\mu A^3}{m_S^6} \right) \mathcal{O}_6. \end{aligned} \quad (4.119)$$

Next let us compute the 1-loop piece of the effective Lagrangian, which according to Eq. (4.7), is

$$\begin{aligned} \Delta S_{\text{eff,1-loop}} &= \frac{i}{2} \text{Tr} \log \left(-\frac{\delta^2 S}{\delta \Phi^2} \Big|_{\Phi_c} \right) = \frac{i}{2} \text{Tr} \log \left(\partial^2 + m_S^2 + k |H|^2 + \mu \Phi_c + \frac{1}{2} \lambda_S \Phi_c^2 \right) \\ &= \frac{i}{2} \text{Tr} \log (\partial^2 + m_S^2 + k |H|^2) = \frac{i}{2} \text{Tr} \log (-P^2 + m_S^2 + U). \end{aligned} \quad (4.120)$$

This is clearly in a form of Eq. (4.14), with $U = k |H|^2$ and $G'_{\mu\nu} = [D_\mu, D_\nu] = [\partial_\mu, \partial_\nu] = 0$. Plugging these specific values of U and $G'_{\mu\nu}$ into Eq. (4.59), we obtain

$$\begin{aligned} \Delta \mathcal{L}_{\text{eff,1-loop}} &= \frac{1}{2(4\pi)^2} \frac{1}{m_S^2} \left[-\frac{1}{12} (P_\mu U)^2 - \frac{1}{6} U^3 \right] = \frac{1}{2(4\pi)^2} \frac{1}{m_S^2} \left[\frac{k^2}{12} (\partial_\mu |H|^2)^2 - \frac{k^3}{6} |H|^6 \right] \\ &= \frac{1}{(4\pi)^2} \frac{1}{m_S^2} \left(\frac{k^2}{12} \mathcal{O}_H - \frac{k^3}{12} \mathcal{O}_6 \right). \end{aligned} \quad (4.121)$$

Putting together Eqs. (4.119) and (4.121), we see that two Wilson coefficients c_H and c_6 are both generated at tree-level and one-loop level. Practically, one can neglect the one loop contribu-

tions

$$c_H = \frac{A^2}{m_S^2} + \frac{1}{(4\pi)^2} \frac{k^2}{12} \approx \frac{A^2}{m_S^2}, \quad (4.122)$$

$$c_6 = -\frac{kA^2}{2m_S^2} + \frac{1}{3!} \frac{\mu A^3}{m_S^4} - \frac{1}{(4\pi)^2} \frac{k^3}{12} \approx -\frac{kA^2}{2m_S^2} + \frac{1}{3!} \frac{\mu A^3}{m_S^4}. \quad (4.123)$$

Several consequences follow from the generated $\mathcal{O}_H$ and $\mathcal{O}_6$. First, upon electroweak symmetry breaking, $\mathcal{O}_H$ modifies the wavefunction of the physical Higgs h and therefore universally modifies all the Higgs couplings, through the residue correction Δr_h (Table E.3 in Appendix E.4)

$$\Delta r_h = -\frac{2v^2}{\Lambda^2} c_H = -\frac{2v^2}{m_S^2} c_H, \quad (4.124)$$

where $c_H = A^2/m_S^2$. This universal *Higgs oblique correction* Δr_h can be quite sensitive to new physics [137–139] since future lepton colliders, such as the ILC, can probe it at the per mille level [25]. In Fig. 4.3, the 2σ contour of this oblique correction is shown. The contour is obtained by combining the future expected sensitivities of Higgs couplings across all 7 channels in Table 1-20 of [25] for an ILC 500up program, except for the $h\gamma\gamma$ channel where the updated value provided by the second column in Table 6 of [140] is used. As shown, the ILC is quite sensitive to this oblique correction, exploring masses up to several TeV and much of the parameter space of the singlet's couplings to the SM.

In addition to the *Higgs oblique correction* Δr_h , $\mathcal{O}_H$ will also generate, through RG running effect discussed in Section 4.2, the operators $\mathcal{O}_W$, $\mathcal{O}_B$, $\mathcal{O}_T$, and hence measurable contributions to EWPO. Let us take the results of the anomalous dimension matrix calculated in [97].²⁷

$$\gamma_{c_H \rightarrow c_W} = \gamma_{c_H \rightarrow c_B} = -\frac{1}{3}, \quad \gamma_{c_H \rightarrow c_T} = \frac{3}{2} g'^2. \quad (4.125)$$

Combining it with our mapping results (Table 4.3 in Section 4.3)

$$S = \frac{c_Z^2 s_Z^2}{\alpha} \frac{4m_Z^2}{\Lambda^2} [4c_{WB}(m_W) + c_W(m_W) + c_B(m_W)], \quad (4.126)$$

$$T = \frac{1}{\alpha} \frac{2v^2}{\Lambda^2} c_T(m_W), \quad (4.127)$$

we find for the singlet model at hand

$$S = \frac{1}{6\pi} \left[\frac{2v^2}{m_S^2} c_H(m_S) \right] \log \frac{m_S}{m_W}, \quad (4.128)$$

$$T = -\frac{3}{8\pi c_Z^2} \left[\frac{2v^2}{m_S^2} c_H(m_S) \right] \log \frac{m_S}{m_W}. \quad (4.129)$$

²⁷The work [97] calculates γ_{ij} within a complete operator basis even though they provide only a subset of the full anomalous dimension matrix. Further, upon changing bases, the results of [97] agree with another recent computation of the full anomalous dimension matrix [92–94].

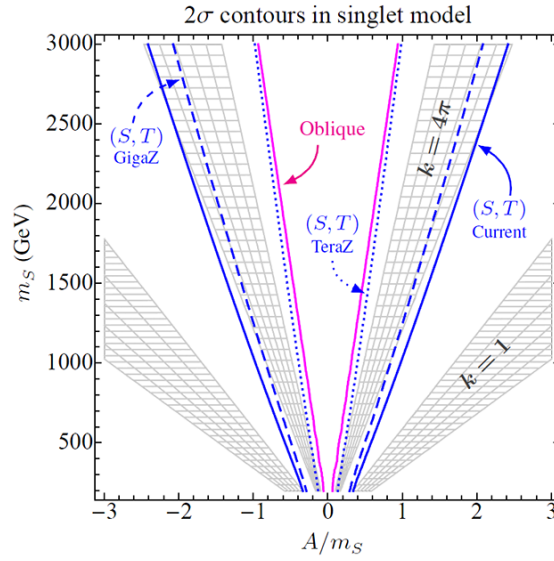


Figure 4.3: 2σ contours of future precision measurements on the singlet model in Eq. (4.116). Regions below the contours will be probed. The magenta contour is the 2σ sensitivity to the universal Higgs oblique correction in Eq. (4.124) at ILC 500up. Blue contours show the 2σ RG-induced constraints from the S and T parameters in Eqs. (4.128)-(4.129) from current measurements (solid) [26] and future sensitivities at ILC GigaZ (dashed) [27] and TLEP TeraZ (dotted) [141]. Regions of a viable first order EW phase transition, from Eq. (4.132), are shown in the gray, hatched regions for $k = 1$ and 4π .

It is worth noting that S and T are highly correlated—current fits find a correlation coefficient of $+0.91$ [26]—while the RG evolution of c_H generates S and T in the orthogonal direction of this correlation, as depicted in Fig. 4.4. This orthogonality feature enhances the sensitivity of EWPO to oblique Higgs corrections, even when the new physics does not directly couple to the EW sector.

The current best fit of the S and T parameters are [26]

$$S = 0.05 \pm 0.09, \quad T = 0.08 \pm 0.07. \quad (4.130)$$

This precision is already sensitive to potential next-to-leading order physics which typically comes with a loop suppression, as in our singlet model. Future lepton colliders will significantly increase the precision measurements of S and T ; a GigaZ program at the ILC would increase precision to $\Delta S = \Delta T = 0.02$ [23, 27] while a TeraZ program at TLEP estimates precision of $\Delta S = 0.007$, $\Delta T = 0.004$ [24, 141]. Constraints on our singlet model from current and prospective future lepton collider measurements of S and T are shown in Fig. 4.3. As seen in the figure, the combination of increased precision measurements together with the fact that the singlet generates S and T in the anti-correlated direction, makes these EWPO a particularly sensitive probe of the singlet. Note that the apparent lack of improvement by GigaZ is an artifact of current non-zero central values in S and T .

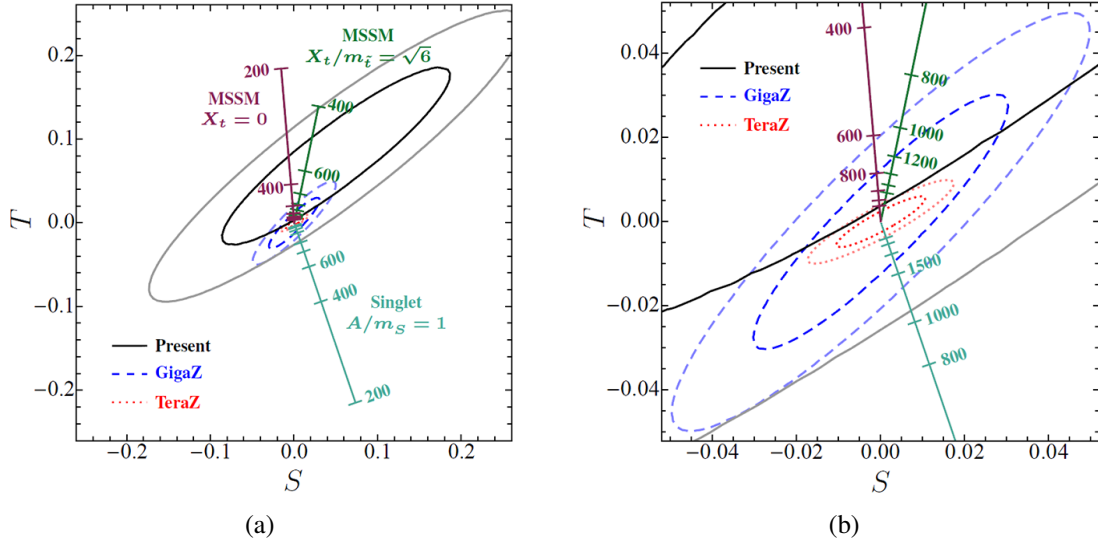


Figure 4.4: The 1σ (darker) and 2σ (lighter) ellipses of precision EW parameters S and T . Current fits (solid, black) is shown together with projected sensitivities at ILC GigaZ (dashed, blue) and TLEP TeraZ (dotted, red). The lines show the size of S and T parameters in our singlet model with $A = m_S$ (teal) and in the MSSM with $\tan\beta = 30$ for $X_t = \sqrt{6}m_{\tilde{t}}$ (green) and $X_t = 0$ (purple). The tick marks show specific mass values in each model; $m_{\tilde{t}}$ values in 200 GeV increments starting from $m_{\tilde{t}} = 200$ GeV for $X_t = 0$ and $m_{\tilde{t}} = 400$ GeV for $X_t = \sqrt{6}m_{\tilde{t}}$ in the MSSM; m_S values in 200 GeV increments between 200-1000 GeV and 500 GeV increments between 1000-3000 GeV in the singlet model.

Finally, this simple singlet model can achieve a strongly first-order EWPT, as previously mentioned,. Essentially, this occurs by having a negative quartic Higgs coupling while stabilizing the potential with $\mathcal{O}_6$,

$$V_H \sim a_2 |H|^2 - a_4 |H|^4 + a_6 |H|^6, \quad (4.131)$$

for positive coefficients $a_{4,6}$. Within a thermal mass approximation,²⁸ a first-order EWPT occurs when [135]

$$\frac{4v^4}{m_H^2} < \frac{2m_s^4}{kA^2} < \frac{12v^4}{m_H^2}, \quad (4.132)$$

where we have set $\mu = 0$ for simplicity. The lower bound comes from requiring EW symmetry breaking at zero temperature, while the upper bound comes from requiring $a_4 > 0$, which guarantees the phase transition is first order. The region of viability for a strongly first-order EWPT within the singlet model is shown in Fig. 4.3, for nominal values of the coupling k (note that k has an upper limit of $k \lesssim 4\pi$ from perturbativity and lower limit $k > 0$ from stability). Current EWPO

²⁸ A full one-loop calculation at finite temperature does not drastically alter the bounds in Eq. (4.132); the lower bound remains the same, while the upper bound is numerically raised by about 25% [142]. This region is still well probed by future lepton colliders.

already constrain a substantial fraction of the viable parameter space, while future lepton colliders will probe the entire parameter space.

4.4.2 Light Scalar Tops in MSSM

As a second benchmark scenario, let us consider the MSSM with light scalar tops (stops) and examine the low energy EFT resultant from integrating out these states. Stops hold a privileged position in alleviating the naturalness problem, *e.g.* [55]. This motivates us to consider a spectrum with light stops while other supersymmetric partners are decoupled. Since the stops carry all SM gauge quantum numbers, all of the dimension-six operators in Table 4.1 are generated at leading order (1-loop). Therefore, they also serve as an excellent computational example to estimate the parametric size of Wilson coefficients of the operators in Table 4.1 resultant from heavy scalar particles with SM quantum numbers. Since the Wilson coefficients are generated at 1-loop leading order, their relatively smaller RG running effects (2-loop) can be neglected.

More specifically, let us integrate out the multiplet $\Phi = (\tilde{Q}_3, \tilde{t}_R)^T$, the Lagrangian of which up to quadratic order is given by

$$\mathcal{L} = \Phi^\dagger (-D^2 - m^2 - U) \Phi, \quad (4.133)$$

where degenerate soft masses $m_{\tilde{Q}_3}^2 = m_{\tilde{t}_R}^2 \equiv m_t^2$ are taken for simplicity

$$m^2 = \begin{pmatrix} m_{\tilde{Q}_3}^2 & 0 \\ 0 & m_{\tilde{t}_R}^2 \end{pmatrix} = \begin{pmatrix} m_t^2 & 0 \\ 0 & m_t^2 \end{pmatrix}, \quad (4.134)$$

and the matrix U is

$$\begin{pmatrix} (y_t^2 s_\beta^2 + \frac{1}{2} g^2 c_\beta^2) \tilde{H} \tilde{H}^\dagger + \frac{1}{2} g^2 s_\beta^2 H H^\dagger - \frac{1}{2} (g'^2 Y_{Qc_{2\beta}} + \frac{1}{2} g^2) |H|^2 & y_t s_\beta X_t \tilde{H} \\ y_t s_\beta X_t \tilde{H}^\dagger & (y_t^2 s_\beta^2 - \frac{1}{2} g'^2 Y_{t_R c_{2\beta}}) |H|^2 \end{pmatrix}$$

Now with both the representation and the interaction matrix U at hand, we are ready to make use of Eq. (4.59) to compute the Wilson coefficients. It is worth noting that due to the appearance of X_t , U is no longer quadratic in H , but also contains linear term of H . This means that one has to keep up to $\mathcal{O}(U^6)$ in computing the Wilson coefficients of dimension-six operators. Through a straightforward, albeit tedious use of Eq. (4.59), we obtain the final result of Wilson coefficients listed in Table 4.9, where $h_t \equiv m_t/v$ and $X_t = A_t - \mu \cot \beta$.

Similar with the singlet model, these Wilson coefficients will correct Higgs widths universally through Eq. (4.124), as well as contribute to S and T parameters through Eq. (4.126)-(4.127). In contrast to the singlet case, the stops contribute to both the Higgs oblique correction (via $\mathcal{O}_H$) and EWPOs (via $\mathcal{O}_{WB}$, $\mathcal{O}_W$, $\mathcal{O}_B$ and $\mathcal{O}_T$) at leading order (1-loop). Additionally, vertex corrections to $h \rightarrow gg$ and $h \rightarrow \gamma\gamma$ decay widths—arising from $\mathcal{O}_{GG}$, $\mathcal{O}_{WW}$, $\mathcal{O}_{BB}$, and $\mathcal{O}_{WB}$ —are sensitive probes since these are already at loop-level in the SM. Combining the results in Table 4.5 and

$c_{GG} = \frac{h_t^2}{(4\pi)^2} \frac{1}{12} \left[\left(1 + \frac{1}{12} \frac{g'^2 c_{2\beta}}{h_t^2} \right) - \frac{1}{2} \frac{X_t^2}{m_{\tilde{t}}^2} \right]$ $c_{WW} = \frac{h_t^2}{(4\pi)^2} \frac{1}{16} \left[\left(1 - \frac{1}{6} \frac{g'^2 c_{2\beta}}{h_t^2} \right) - \frac{2}{5} \frac{X_t^2}{m_{\tilde{t}}^2} \right]$ $c_{BB} = \frac{h_t^2}{(4\pi)^2} \frac{17}{144} \left[\left(1 + \frac{31}{102} \frac{g'^2 c_{2\beta}}{h_t^2} \right) - \frac{38}{85} \frac{X_t^2}{m_{\tilde{t}}^2} \right]$	$c_{WB} = -\frac{h_t^2}{(4\pi)^2} \frac{1}{24} \left[\left(1 + \frac{1}{2} \frac{g'^2 c_{2\beta}}{h_t^2} \right) - \frac{4}{5} \frac{X_t^2}{m_{\tilde{t}}^2} \right]$ $c_W = \frac{h_t^2}{(4\pi)^2} \frac{1}{40} \frac{X_t^2}{m_{\tilde{t}}^2}$ $c_B = \frac{h_t^2}{(4\pi)^2} \frac{1}{40} \frac{X_t^2}{m_{\tilde{t}}^2}$
$c_{3G} = \frac{g_s^2}{(4\pi)^2} \frac{1}{20}$ $c_{3W} = \frac{g^2}{(4\pi)^2} \frac{1}{20}$ $c_{2G} = \frac{g_s^2}{(4\pi)^2} \frac{1}{20}$ $c_{2W} = \frac{g^2}{(4\pi)^2} \frac{1}{20}$ $c_{2B} = \frac{g'^2}{(4\pi)^2} \frac{1}{20}$	$c_H = \frac{h_t^4}{(4\pi)^2} \frac{3}{4} \left[\left(1 + \frac{1}{3} \frac{g'^2 c_{2\beta}}{h_t^2} + \frac{g'^4 c_{2\beta}^2}{12 h_t^4} \right) - \frac{7}{6} \frac{X_t^2}{m_{\tilde{t}}^2} \left(1 + \frac{(g^2 + 2g'^2) c_{2\beta}}{14 h_t^2} \right) + \frac{7}{30} \frac{X_t^4}{m_{\tilde{t}}^4} \right]$ $c_T = \frac{h_t^4}{(4\pi)^2} \frac{1}{4} \left[\left(1 + \frac{1}{2} \frac{g^2 c_{2\beta}}{h_t^2} \right)^2 - \frac{1}{2} \frac{X_t^2}{m_{\tilde{t}}^2} \left(1 + \frac{1}{2} \frac{g^2 c_{2\beta}}{h_t^2} \right) + \frac{1}{10} \frac{X_t^4}{m_{\tilde{t}}^4} \right]$ $c_R = \frac{h_t^4}{(4\pi)^2} \frac{1}{2} \left[\left(1 + \frac{1}{2} \frac{g^2 c_{2\beta}}{h_t^2} \right)^2 - \frac{3}{2} \frac{X_t^2}{m_{\tilde{t}}^2} \left(1 + \frac{1}{12} \frac{(3g^2 + g'^2) c_{2\beta}}{h_t^2} \right) + \frac{3}{10} \frac{X_t^4}{m_{\tilde{t}}^4} \right]$ $c_D = \frac{h_t^2}{(4\pi)^2} \frac{1}{20} \frac{X_t^2}{m_{\tilde{t}}^2}$
$c_6 = -\frac{h_t^6}{(4\pi)^2} \frac{1}{2} \left\{ \begin{aligned} & \left[1 + \frac{1}{12} \frac{(3g^2 - g'^2) c_{2\beta}}{h_t^2} \right]^3 + \left[-\frac{1}{12} \frac{(3g^2 + g'^2) c_{2\beta}}{h_t^2} \right]^3 + \left(1 + \frac{1}{3} \frac{g'^2 c_{2\beta}}{h_t^2} \right)^3 \\ & - \frac{X_t^2}{m_{\tilde{t}}^2} \left[2 \left(1 + \frac{1}{12} \frac{(3g^2 - g'^2) c_{2\beta}}{h_t^2} \right) \left(1 + \frac{1}{8} \frac{(g^2 + g'^2) c_{2\beta}}{h_t^2} \right) + \left(1 + \frac{1}{3} \frac{g'^2 c_{2\beta}}{h_t^2} \right)^2 \right] \\ & + \frac{X_t^4}{m_{\tilde{t}}^4} \left[1 + \frac{1}{8} \frac{(g^2 + g'^2) c_{2\beta}}{h_t^2} \right] - \frac{X_t^6}{m_{\tilde{t}}^6} \frac{1}{10} \end{aligned} \right\}$	

Table 4.9: Wilson coefficients c_i for the operators $\mathcal{O}_i$ in Table 4.1 generated from integrating out MSSM stops with degenerate soft mass $m_{\tilde{t}}$. g_s, g , and g' denote the gauge couplings of $SU(3)$, $SU(2)_L$, and $U(1)_Y$, respectively, $h_t = m_t/v$ with $v = 174 \text{ GeV}$, and $\tan \beta = \langle H_u \rangle / \langle H_d \rangle$ in the MSSM.

Table 4.6, we find the deviations of these two decay widths are

$$\epsilon_{hgg} = \frac{(4\pi)^2}{\text{Re} A_{hgg}^{\text{SM}}} \frac{16v^2}{m_{\tilde{t}}^2} c_{GG}, \quad (4.135)$$

$$\epsilon_{h\gamma\gamma} = \frac{(4\pi)^2}{\text{Re} A_{h\gamma\gamma}^{\text{SM}}} \frac{8v^2}{m_{\tilde{t}}^2} (c_{WW} + c_{BB} - c_{WB}), \quad (4.136)$$

where A_{hgg}^{SM} and $A_{h\gamma\gamma}^{\text{SM}}$ are the standard form factors, whose expressions are listed in Appendix E.2 (Eq. (E.33) and (E.34)).

2σ sensitivity contours are shown in Fig. 4.5. It should be stressed that here we are focused on the experimental sensitivities on the scalar top mass, while assuming improvements on relevant theoretical uncertainties will catch up in time. As seen in Fig. 4.5, future precision Higgs and EW measurements from the ILC offer comparable sensitivities while a TeraZ program significantly increases sensitivity. Moreover, the most natural region of the MSSM—where $X_t \sim \sqrt{6} m_{\tilde{t}}$ and $m_{\tilde{t}} \sim 1 \text{ TeV}$ (e.g. [22])—can be well probed by future precision measurements.

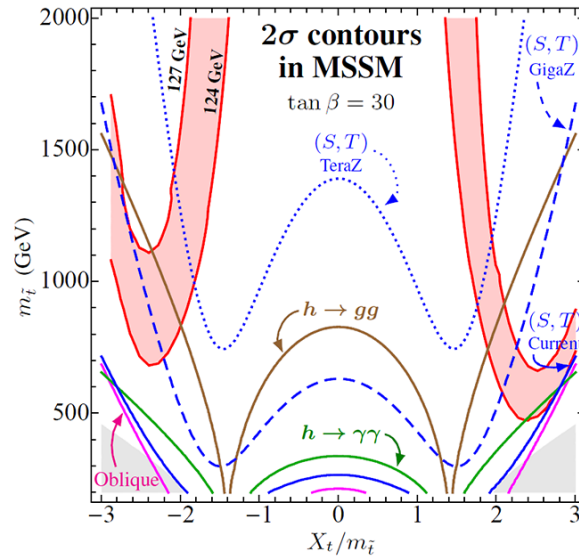


Figure 4.5: 2σ contours of precision Higgs and EW observables as a function of $m_{\tilde{t}}$ and X_t in the MSSM. The contours show 2σ sensitivity of ILC 500up to the universal Higgs oblique correction (magenta) and modifications of $h \rightarrow gg$ (brown) and $h \rightarrow \gamma\gamma$ (green). Constraints from S and T parameters are shown in blue for current measurements (solid), ILC GigaZ (dashed), and TLEP TeraZ (dotted). The shaded red region shows contours of Higgs mass between 124-127 GeV calculated using FeynHiggs [71, 143]. The shaded gray regions are unphysical because one of the stop mass eigenvalues becomes negative.

4.5 Discussions

Advocated in this chapter is a practical three-step procedure of using the SM EFT to connect UV models with weak scale precision observables. This procedure helps translating precision measurements into constraints on the UV models in concern. For the first step, a detailed explanation is given on the matching technique between UV models and the SM EFT — the covariant derivative expansion method. With this technique, one can compute the effective action up to one-loop order in a manifestly gauge covariant fashion, and also obtain a universal formalism that is easy to use for different UV models. We discussed the rigorous derivation of its formalism, its application to integrating out heavy scalars, fermions, and gauge bosons, and a universal formalism for all types of UV models. Then about the second step, we discussed how to run the Wilson coefficients down from Λ to the weak scale m_W , and a closely related topic—choosing the operator set. Finally for the third step, mapping results are provided, up to one-loop order, between the bosonic sector of the SM EFT and a complete set of precision electroweak and Higgs observables to which present and near future experiments are sensitive. Many results and tools which should prove useful to those wishing to use the SM EFT are detailed in several appendices. We also discussed two explicit examples of using this advocated framework to place constraints on UV models.

Chapter 5

Conclusions

In this dissertation, I have discussed a few aspects of the research in particle physics beyond the Standard Model.

In Chapter 2, I discussed neutrino mass anarchy and the baryon asymmetry. I showed that an approximate $U(1)$ flavor symmetry can make leptogenesis feasible for neutrino anarchy. Combining the two, we find anarchy model successfully generate the observed amount of baryon asymmetry. Same sampling model is used to study other quantities related to neutrino masses. We found the chance of normal mass hierarchy is as high as 99.9%. The effective mass of neutrinoless double beta decay m_{eff} would probably be well beyond the current experimental sensitivity. The neutrino total mass m_{total} is a little more optimistic. Correlations between baryon asymmetry and light-neutrino quantities were also investigated. We found η_{B0} not correlated with light-neutrino mixings or phases, but weakly correlated with R , m_{eff} , and m_{total} , all with negative correlation. Possible implications of recent BOSS analysis result have been discussed.

In Chapter 3, I discussed the naturalness problem of the Higgs mass. We have identified a new model where the Higgs couples to a singlet field with a Dirac mass. The non-decoupling F -term is naturally realized through semi-soft SUSY breaking, because large $m_{\tilde{S}}$ helps raise the Higgs mass but does not threaten naturalness. The first collider signatures of the Dirac NMSSM are expected to be those of the MSSM fields, with the singlet sector naturally heavier than 1 TeV. The key feature of semi-soft SUSY breaking in the Dirac NMSSM is that $\tilde{S}$ couples to the MSSM only through the dimensionful Dirac mass M . We note that interactions between $\tilde{S}$ and other new states are not constrained by naturalness, even if these states experience SUSY breaking. Therefore, the Dirac NMSSM represents a new type of portal, whereby our sector can interact with new sectors, with large SUSY breaking, without spoiling naturalness in our sector.

In Chapter 4, I advocated a practical three-step procedure of using the SM EFT to connect UV models with weak scale precision observables. This procedure helps translating precision measurements into constraints on the UV models in concern. For the first step, a detailed explanation is give on the matching technique between UV models and the SM EFT — the covariant derivative expansion method. With this technique, one can compute the effective action up to one-loop order in a manifestly gauge covariant fashion, and also obtain a universal formalism that is easy to use for different UV models. We discussed the rigorous derivation of its formalism, its application to

integrating out heavy scalars, fermions, and gauge bosons, and a universal formalism for all types of UV models. Then about the second step, we discussed how to run the Wilson coefficients down from Λ to the weak scale m_W , and a closely related topic—choosing the operator set. Finally for the third step, mapping results are provided, up to one-loop order, between the bosonic sector of the SM EFT and a complete set of precision electroweak and Higgs observables to which present and near future experiments are sensitive. Many results and tools which should prove useful to those wishing to use the SM EFT are detailed in several appendices. We also discussed two explicit examples of using this advocated framework to place constraints on UV models.

Appendix A

Basis and Entry Independence Uniquely Lead to Gaussian Measure

In this appendix, let us show that Gaussian measure is the only measure that satisfies both requirements of basis independence and entry independence. Let us abstractly write all choices of measure in the form

$$dm = \left(\prod_{ij} dm_{ij} \right) \cdot e^{-f(\{m_{ij}\})}, \quad (\text{A.1})$$

where m stands for m_R or m_D , $\prod_{ij}$ and $\{m_{ij}\}$ run over all the free entries of m . We want the form of $f(\{m_{ij}\})$ so that the measure above has both basis independence and independence among m_{ij} .

Let us first consider m_D . For $N \times N$ m_D , there are N^2 free entries: $m_{11}, m_{12}, \dots, m_{NN}$. For convenience, let us rename them as $x_1, x_2, \dots, x_n$, where $n = N^2$. Then $\{x_i\}$ forms an irreducible unitary representation of the basis transformation group $U(3)_L \times U(3)_R$ in flavor space:

$$m_D \rightarrow m'_D = U_L m_D U_R^\dagger, \quad (\text{A.2})$$

$$x \rightarrow x' = \Lambda x, \quad (\text{A.3})$$

where $\Lambda = U_L \otimes U_R^*$ is obviously unitary.

Independence of $\{m_{ij}\}$, namely $\{x_i\}$, requires $f(\{m_{ij}\})$ having the form

$$f(\{x_i\}) = f_1(x_1) + f_2(x_2) + \dots + f_n(x_n). \quad (\text{A.4})$$

Here since x_i are complex arguments, all the functions $f_i(x_i)$ are actually abbreviations of $f_i(x_i, x_i^*)$. Under the transformation of Eq.(A.3), basis independence requires

$$f_1(x'_1) + \dots + f_n(x'_n) = f_1(x_1) + \dots + f_n(x_n). \quad (\text{A.5})$$

Taking a derivative with respect to x_i^* yields

$$\sum_j \Lambda_{ji}^* \frac{\partial f_j(x'_j)}{\partial x_j'^*} = \frac{\partial f_i(x_i)}{\partial x_i^*}. \quad (\text{A.6})$$

Since Λ is unitary, this is the same as

$$\frac{\partial f_i(x'_i)}{\partial x'^*_i} = \sum_j \Lambda_{ij} \frac{\partial f_j(x_j)}{\partial x^*_j}. \quad (\text{A.7})$$

Thus $\frac{\partial f_i(x_i)}{\partial x^*_i}$ transform in the same way as x_i . Because x_i forms an irreducible representation of the transformation Eq.(A.3), the only possibility for $\frac{\partial f_i(x_i)}{\partial x^*_i}$ is

$$\begin{pmatrix} \frac{\partial f_1(x_1)}{\partial x^*_1} \\ \vdots \\ \frac{\partial f_n(x_n)}{\partial x^*_n} \end{pmatrix} = c_a \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad (\text{A.8})$$

with c_a an arbitrary constant. Similarly, taking a derivative of Eq.(A.5) with respect to x_i will give us

$$\begin{pmatrix} \frac{\partial f_1(x_1)}{\partial x_1} \\ \vdots \\ \frac{\partial f_n(x_n)}{\partial x_n} \end{pmatrix} = c_b \begin{pmatrix} x^*_1 \\ \vdots \\ x^*_n \end{pmatrix}. \quad (\text{A.9})$$

Combining Eq.(A.8) and (A.9) we get

$$f(\{m_{D,ij}\}) = c_1(x_1x^*_1 + \cdots + x_nx^*_n) + c_2 = c_1 \left(\sum_{ij} |m_{D,ij}|^2 \right) + c_2. \quad (\text{A.10})$$

An important condition in this proof is that x forms an irreducible unitary representation of the basis transformation group, $U(3)_L \times U(3)_R$ in the case of m_D . For the case of m_R , this condition still holds. The relevant basis transformation group for m_R is just $U(3)_R$

$$m_R \rightarrow m'_R = U_R m_R U_R^T, \quad (\text{A.11})$$

$$x \rightarrow x' = \Lambda x. \quad (\text{A.12})$$

$\Lambda = U_R \otimes U_R$ is reducible in general: $3 \otimes 3 = 6 \oplus 3$, but our m_R is symmetric by definition, which only forms the irreducible subspace “6” (Note that if m_R were real symmetric, this symmetric subspace “6” would be further reducible.). So same as in Eq. (A.10), we get

$$f(\{m_{R,ij}\}) = c_1(x_1x^*_1 + \cdots + x_nx^*_n) + c_2 = c_1 \left(\sum_i |m_{R,ii}|^2 + 2 \sum_{i < j} |m_{R,ij}|^2 \right) + c_2. \quad (\text{A.13})$$

In Eq. (A.10) and (A.13), c_1 corresponds to the freedom of adjusting $\mathcal{D}$ and $\mathcal{M}$ in Eqs. (2.7)-(2.8), while c_2 is just an overall normalization factor. Plugging them back into Eq. (A.1), we get the Gaussian measure of m_D and m_R as in Eq. (2.13) and (2.14).

Appendix B

Universal Eigenvalue Distribution of Large Random Matrices

Random matrix theory is widely used in the study of theoretical physics. And the large N limit has been a very useful tool, both at a qualitative and a quantitative level [144]. It works as a great approximation, even when the actual dimension of the matrix is not a very large number, such as $N_c = 3$ in QCD [145, 146] and $N_f = 3$ in neutrino anarchy [18, 19, 43]. Therefore, the behavior of random matrices under large N limit can provide insight to many theoretical studies in physics.

Often in this kind of studies, the behavior of eigenvalues are of special interests to us, as they usually represent crucial quantities of the model, such as masses in cases of particle physics. But of course the eigenvalue distribution generically depends on the prior of the random matrices, and there is no privileged choice. However, under large dimension limit, a powerful theorem—Marchenko-Pastur (MP) law—states that there is a universal asymptotic eigenvalue distribution as long as all the entries are independent and identically distributed (i.i.d), regardless of the choice of the entry distribution [37]. To be more concrete (still a sketch here, see Section B.1 for the precise statement), let X be a random $M \times N$ matrix with $M \times N$ i.i.d. complex/real entries, then under large N limit, the asymptotic eigenvalue distribution of the matrix $A = \frac{1}{N} X X^\dagger$ is universal and called Marchenko-Pastur distribution. The theorem is named after Ukrainian mathematicians Vladimir Marchenko and Leonid Pastur who proved this result in 1967 [37]. After the original paper, a lot of works followed and the theorem is sharpened into a few different versions, each of which has a different set of premises assumed (see e.g. [147] for a brief summary of the story).

The key point of the MP law is the universality, namely that any i.i.d. entry distribution (within some restrictions, see Section B.1 for details) will yield the same asymptotic eigenvalue distribution. This law got proven almost 50 years ago, but an alternative proof is provided in this appendix—a direct diagrammatic method which is more familiar to the particle physics community, so that this universality can be better understood. In addition, the original MP law only covers the case of X being an arbitrary complex/real matrix with $M \times N$ i.i.d. complex/real entries. But in many physics models, cases of restricted X are of interests, such as symmetric, antisymmetric, Hermitian, *etc.* For these types of restricted X , is the large N eigenvalue distribution of $A = \frac{1}{N} X X^\dagger$ still MP distribution? With the direct diagrammatic method presented in

	i.i.d. Gaussian							i.i.d non-Gaussian						
cases of X	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
our method	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[37]	(✓)			(✓)				(✓)			(✓)			
[148, 149]					(✓)							(✓)		
[150–152]					✓		✓					✓		✓
[153]	✓			(✓)			(✓)	(✓)			(✓)			(✓)

Table B.1: Comparison between the method presented in this appendix and some of the closely related works in the literature. For each work, we put a “✓” if a case is proved by a diagrammatic method, a “(✓)” if the case is proved by a non-diagrammatic method, and leave it blank if the case is not discussed.

this appendix, one can answer this question easily. As will be shown in Section B.3, the method generalizes to six types of restricted X with little effort, making it very transparent that the MP universality should hold for all of the following seven cases of X :

- (1) Complex arbitrary
- (2) Complex symmetric
- (3) Complex antisymmetric
- (4) Real arbitrary
- (5) Real symmetric
- (6) Real antisymmetric
- (7) Hermitian

with case (1) and (4) being the original MP law.

Both asymptotic eigenvalue distribution and diagrammatic methods had been discussed extensively in the literature. Table B.1 summarizes a comparison between the method presented in this appendix and several closely related literatures, from which one can see the novel aspects of this new method and how it serves as a complementary approach to many other works.

First, a very well known result is the Wigner’s semicircle law [148, 149]. It states that if X is a real symmetric matrix, then for any i.i.d.¹ entry distribution, the asymptotic eigenvalue density of $\frac{1}{\sqrt{N}}X$ is given by a semicircle curve. With Wigner’s semicircle law, it follows trivially that the asymptotic eigenvalue distribution of $A = \frac{1}{N}XX^\dagger$ is the MP distribution. However, this derivation of MP law from Wigner’s semicircle law only applies to real symmetric X , *i.e.* our case (5) in the list. Also, the original proof of Wigner’s semicircle law [148, 149] did not use a diagrammatic method.

¹The i.i.d. ensembles are also known as “Wigner Class” in random matrix theory literatures.

Diagrammatic proof of Wigner’s semicircle law also exists in the literature [150–152]. This diagrammatic method also applies to Hermitian X (*i.e.* our case (7)). When X is real symmetric or Hermitian, one needs not to distinguish between X and $X^\dagger$, hence only one kind of vertex is needed in the diagrams. Our new proof here treats more general X by using two types of vertices in the diagrams, and thus serves as a generalization of the diagrammatic method in [150–152] to all the seven cases of X listed.

Another paper particularly close is [153], where a diagrammatic method was presented among some other methods to calculate the large N asymptotic eigenvalue distribution of the matrix

$$H = \begin{pmatrix} 0 & X^\dagger \\ X & 0 \end{pmatrix}.$$

and of the matrix A closely related to it, with X being an arbitrary complex matrix (*i.e.* our case (1)). However, an important difference is that the diagrammatic method presented in [153] is limited to entry distribution being Gaussian. As to the proof of the universality for generic i.i.d. entry distributions, a large N RG method was resorted. Remarkably, this RG picture can help better understanding the universality. But this approach is clearly not a *direct diagrammatic method*. Actually, it was claimed in [153] (the first paragraph of section 3) that it is rather difficult to develop a direct diagrammatic method for entry distributions beyond Gaussian. We believe the main difficulty is that a crucial prerequisite for a diagrammatic calculation is the notion of “propagator”, which comes trivially under Gaussian ensemble, but not otherwise. In this appendix, three subsections (Section B.2.2, B.2.3, and B.2.4) are devoted to prove that the notion and use of “propagator” is legitimate because of a *grouping by pairs* requirement (see Section B.2.4 for details) under large N limit. Therefore, this new method has overcome the difficulty claimed in [153] and led us to a direct diagrammatic method applicable to any i.i.d. entry distribution and any of the seven cases of X listed.

The rest of this appendix is organized as follows. First, a precise statement of (a selected version of) the Marchenko-Pastur law is given in Section B.1. Then in Section B.2, a detailed proof of MP law using Feynman diagrams is shown. The method is generalized to six types of restricted X in Section B.3.

B.1 Marchenko-Pastur Law

As mentioned in the introduction section, there are more than one versions of the MP law with the differences residing in how strong the premises are assumed. Let us focus on its following version, which is sufficient for most large N models in physics.

Let X be a $M \times N$ random complex matrix, whose entries X_{ij} are generated according to the following conditions:

$$(1) \text{ independent, identical distribution (i.i.d.),} \tag{B.1}$$

$$(2) \langle X_{ij} \rangle_X = 0, \langle X_{ij}^2 \rangle_X = 0, \text{ and } \langle |X_{ij}|^2 \rangle_X = 1, \tag{B.2}$$

$$(3) \langle |X_{ij}|^{2+\varepsilon} \rangle_X < \infty \text{ for any } \varepsilon > 0, \tag{B.3}$$

where and throughout this appendix, $\langle \mathcal{O} \rangle_X$ is used to denote the expectation value of a random variable $\mathcal{O}$ under the ensemble of X . Then construct an $M \times M$ hermitian matrix $A = \frac{1}{N}XX^\dagger$, whose eigenvalues are denoted by λ_k , with $k = 1, 2, \dots, M$. Then the empirical distribution of these eigenvalues is defined as

$$F_M(x) \equiv \frac{1}{M} \sum_{k=1}^M I_{\{\lambda_k \leq x\}}, \quad (\text{B.4})$$

where I_B denotes the indicator of an event B :

$$I_{\{B\}} = \begin{cases} 1 & \text{if } B \text{ is true} \\ 0 & \text{if } B \text{ is false} \end{cases} \quad (\text{B.5})$$

Consider the limit $N \rightarrow \infty$. If the limit of the ratio M/N is finite

$$b \equiv \lim_{N \rightarrow \infty} M/N \in (0, \infty), \quad (\text{B.6})$$

then $\langle F_M(x) \rangle_X \rightarrow F(x)$ ², where $F(x)$ denotes the cumulative distribution function of the Marchenko-Pastur distribution whose density function is

$$f(x) = \frac{1}{2\pi} \frac{\sqrt{(x_2 - x)(x - x_1)}}{x} \frac{1}{b} \cdot I_{\{x \in (x_1, x_2)\}} + (1 - \frac{1}{b})\delta(x) \cdot I_{\{b \in [1, \infty)\}}, \quad (\text{B.7})$$

with $x_1 = (1 - \sqrt{b})^2$ and $x_2 = (1 + \sqrt{b})^2$. In the special case of a square matrix X , namely $b = 1$, this becomes

$$f(x) = \frac{1}{2\pi} \sqrt{\frac{4}{x} - 1} \cdot I_{x \in (0, 4)}. \quad (\text{B.8})$$

B.2 Proof of Marchenko-Pastur Law with Feynman Diagrams

B.2.1 Stieltjes Transformation

For a single matrix X generated, the distribution density of eigenvalues is

$$\rho_X(E) = \frac{1}{M} \sum_{k=1}^M \delta(E - \lambda_k). \quad (\text{B.9})$$

Our goal is to compute its expectation

$$\rho(E) \equiv \langle \rho_X(E) \rangle_X = \int dX \cdot \rho_X(E), \quad (\text{B.10})$$

²The actual Marchenko-Pastur law states that $F_M(x)$ will converge to $F(x)$ in probability, namely that $\lim_{N \rightarrow \infty} \Pr(|F_M(x) - F(x)| > \epsilon) = 0$ irrespective of $\epsilon > 0$, where ‘‘Pr’’ stands for ‘‘Probability’’. This is a much stronger statement than $\langle F_M(x) \rangle_X \rightarrow F(x)$. But the method discussed in this appendix only proves the weaker version of the MP law.

and prove that $\rho(E)$ approaches the MP density function (Eq. B.7) as $N \rightarrow \infty$. Here dX denotes the normalized measure of X :

$$dX = \prod_{ij} g(X_{ij}) dX_{ij}, \quad \int dX = 1, \quad (\text{B.11})$$

with $g(X_{ij})$ denoting the normalized distribution density of each X_{ij} .

Since $\rho(E)$ is not easy to compute directly, let us make use of a method known in mathematics as ‘‘Stieltjes transformation’’. That is, from the identity, where x is a real variable

$$\delta(x) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \text{Im} \frac{1}{x + i\varepsilon}, \quad (\text{B.12})$$

we get

$$\rho(E) = \int \rho(x) \delta(E - x) dx = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \text{Im} \int \frac{\rho(x)}{E + i\varepsilon - x} dx. \quad (\text{B.13})$$

Thus for any distribution density function $\rho(x)$, we can define its Stieltjes transformation $G(z)$, a complex function as an integral over the support of $\rho(x)$

$$G(z) \equiv \int \frac{\rho(x)}{z - x} dx. \quad (\text{B.14})$$

Then according to Eq. (B.13), $\rho(x)$ can be obtained from the inverse formula

$$\rho(E) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \text{Im} G(E + i\varepsilon). \quad (\text{B.15})$$

For our case, $G(z)$ can be computed as following

$$\begin{aligned} G(z) &= \int \frac{\rho(x)}{z - x} dx = \int \frac{1}{z - x} \langle \rho_X(x) \rangle_X dx = \left\langle \frac{1}{M} \sum_{k=1}^M \int \frac{1}{z - x} \delta(x - \lambda_k) dx \right\rangle_X \\ &= \left\langle \frac{1}{M} \sum_{k=1}^M \frac{1}{z - \lambda_k} \right\rangle_X = \left\langle \frac{1}{M} \text{tr} \left(\frac{1}{z - A} \right) \right\rangle_X = \frac{1}{M} \frac{1}{z} \text{tr} [B(z)], \end{aligned} \quad (\text{B.16})$$

where a matrix $B(z)$ has been defined as

$$B(z) \equiv \left\langle \frac{z}{z - A} \right\rangle_X \quad (\text{B.17})$$

$$= \left\langle \sum_{n=0}^{\infty} \left(\frac{A}{z} \right)^n \right\rangle_X = \left\langle \sum_{n=0}^{\infty} \left(\frac{1}{zN} X X^\dagger \right)^n \right\rangle_X. \quad (\text{B.18})$$

This expansion is a valid analytical form of $B(z)$ in the vicinity of $z = \infty$. We will compute $B(z)$ in this vicinity first, and then analytically continue it to the whole complex plane. Once $B(z)$ is obtained, $G(z)$ and $\rho(E)$ would follow immediately. The following several subsections are devoted to calculate this *target function* $B(z)$ under the limit $N \rightarrow \infty$.

B.2.2 Group into “Boxes”

Our target function is a sum of various terms, in which a typical n -term looks like

$$B_{ij}(z) \supset \left(\frac{1}{zN} \right)^n \left\langle \prod_{p=1}^n X_{\alpha_p \beta_p} X_{\beta_p \alpha_{p+1}}^\dagger \right\rangle_X = \left(\frac{1}{zN} \right)^n \left\langle X_{i\beta_1} X_{\beta_1 \alpha_2}^\dagger \cdots X_{\alpha_n \beta_n} X_{\beta_n j}^\dagger \right\rangle_X, \quad (\text{B.19})$$

with an identification $\alpha_1 \equiv i, \alpha_{n+1} \equiv j$ and a sum over all the dummy indices $\alpha_2, \alpha_3, \dots, \alpha_n$ from 1 to M , and $\beta_1, \beta_2, \dots, \beta_n$ from 1 to N .

As stated in the condition Eq. (B.1), different elements X_{ij} are independent. Therefore any such n -term expectation can be factorized

$$\left\langle X_{i\beta_1} X_{\beta_1 \alpha_2}^\dagger \cdots X_{\alpha_n \beta_n} X_{\beta_n j}^\dagger \right\rangle_X = \langle f(X_{k_1 l_1}) \rangle_X \langle f(X_{k_2 l_2}) \rangle_X \cdots, \quad (\text{B.20})$$

where each individual expectation $\langle f(X_{kl}) \rangle_X$ contains only X_{kl} and its complex conjugate X_{kl}^*

$$\langle f(X_{kl}) \rangle_X = \langle (X_{kl})^{m_1} (X_{kl}^*)^{m_2} \rangle_X. \quad (\text{B.21})$$

Namely that the independence among elements allows us to group same X_{kl} (and the complex conjugate) together into one factor. For future convenience, let us call each such factor a “box”.

Now in evaluating the n -term (Eq. B.19), the sum over the dummy indices α 's and β 's can be decomposed into two steps. (1) There are many ways to group the $2n$ elements into boxes. We need to sum over all possible grouping configurations. (2) Under each grouping configuration, all α 's within the same box are forced equal, so are all β 's, thus some dummy indices are tied to others and hence no longer free to sum. But generically there are still some *free dummy indices* remaining, which needs to be summed. In summary, this decomposition of sum can be expressed as

$$\sum_{\alpha, \beta} = \sum_{\text{grouping configurations}} \sum_{\text{free dummy indices}}. \quad (\text{B.22})$$

B.2.3 Key Statement in Power Counting of N

To evaluate an n -term (Eq. B.19) under $N \rightarrow \infty$, an efficient way to count the power of N is definitely crucial. The suppression factor $\frac{1}{N^n}$ in front of Eq. (B.19) contributes a factor N^{-n} . On the other hand, with the sum decomposition Eq. B.22, under each grouping configuration, summing

over each free dummy index will give us one power of N : $\sum_{\alpha=1}^M \delta_{\alpha\alpha} = M = bN$, $\sum_{\beta=1}^N \delta_{\beta\beta} = N$. From

this competition, we end up with a factor $N^{-(n-n_f)}$, where n_f denotes the total number of free dummy indices under a given grouping configuration. There are $2n - 1$ dummy indices in total: $\alpha_2, \dots, \alpha_n, \beta_1, \dots, \beta_n$, but not all of them are free, because within each box, the different α 's and β 's are forced into same value respectively. Apparently, n_f largely depends on the grouping configuration.

To study how n_f depends on the grouping configuration, we resort to some graphical analysis. First, let us draw a map to represent each grouping configuration, where each grouping box is drawn as an isolated island. We notice that in the sequence of Eq. (B.19)

$$X_{i\beta_1} X_{\beta_1\alpha_2}^\dagger \cdots X_{\alpha_n\beta_n} X_{\beta_n j}^\dagger, \quad (\text{B.23})$$

every dummy index appears twice, i.e. in pair. Each such dummy index pair could be grouped into the same box, or two different boxes. If any two boxes share a dummy index pair, let us connect those two islands by a “bridge” on the map. So every grouping configuration is described by a map of boxes (or islands) and a number of bridges connecting them.

Suppose that there are m boxes fully connected by bridges. Within each box, after all α ’s and β ’s are identified respectively, we are left with at most 2 free dummy indices. Then collecting all the m boxes, we get at most $2m$ free indices. But to connect all these m boxes, we need at least $m - 1$ bridges. If we remove all the redundant bridges and thus chop the map into a tree map, then each of the remaining $m - 1$ bridges would effectively reduce one free index out of the $2m$. Thus we have arrived at our *key statement*:

If m boxes are fully connected by bridges, then we can get at most $m + 1$ free dummy indices out of them.

Clearly, the whole sequence Eq. (B.23) is fully connected by dummy-index bridges, so we can apply this key statement to it. Suppose there are n_b boxes in total for a grouping configuration, then we get $n_f \leq n_b + 1$. However, this upper bound is obtained by counting i and j also as dummy indices. But they are not. So we need to further subtract 1, if i and j are already identified by the grouping; or subtract 2 if they are not. To sum up, we get

$$n_f \leq \begin{cases} n_b & i, j \text{ identified by the grouping} \\ n_b - 1 & i, j \text{ not identified by the grouping} \end{cases} \quad (\text{B.24})$$

B.2.4 Association with Feynman Diagrams

With the result Eq. (B.24), we are ready to study what kinds of grouping configurations can give nonzero contribution. Due to the condition Eq. (B.2), each box needs to contain at least two elements in order not to vanish. But there are in total only $2n$ elements in an n -term. So if any box has more than two elements, then the total number of boxes n_b must be less than n . Consequently, $n_f \leq n_b < n$ and the grouping configuration is suppressed by a factor $N^{-(n-n_f)}$. Due to condition Eq. (B.3), the coefficient multiplying this factor must be finite. Thus the contribution from this kind of grouping configuration vanishes under $N \rightarrow \infty$. To sum up, only grouping the elements by pairs can give nonzero contributions. We can call each such grouped pair a “contraction”. It is this *grouping by pairs* requirement that justifies the notion and the use of “contraction”, which in turn makes a diagrammatic approach possible.

$$X_{\alpha_p \beta_p} \begin{array}{c} \xleftarrow{\alpha_p} \\ \xleftarrow{\alpha_{q+1}} \\ \xrightarrow{\beta_p} \\ \xrightarrow{\beta_q} \end{array} X_{\beta_q \alpha_{q+1}}^\dagger \quad \delta_{\alpha_p \alpha_{q+1}} \delta_{\beta_p \beta_q}$$

Figure B.1: Feynman rule propagator

$$\begin{array}{c} \xleftarrow{\alpha_k} \downarrow \downarrow \xrightarrow{\beta_k} \\ X_{\alpha_k \beta_k} \end{array} \quad \frac{1}{\sqrt{N}} \frac{1}{\sqrt{z}} \quad \begin{array}{c} \downarrow \downarrow \xleftarrow{\beta_k} \xleftarrow{\alpha_{k+1}} \\ X_{\beta_k \alpha_{k+1}}^\dagger \end{array} \quad \frac{1}{\sqrt{N}} \frac{1}{\sqrt{z}}$$

Figure B.2: Feynman rule vertices

$$\begin{array}{ccccccc} i & \downarrow \downarrow & \beta_1 & \uparrow \uparrow & \alpha_2 & \downarrow \downarrow & \beta_2 & \uparrow \uparrow & \alpha_3 & \dots & \alpha_{n-1} & \downarrow \downarrow & \beta_{n-1} & \uparrow \uparrow & \alpha_n & \downarrow \downarrow & \beta_n & \uparrow \uparrow & j \\ X_{i\beta_1} & & X_{\beta_1\alpha_2}^\dagger & & X_{\alpha_2\beta_2} & & X_{\beta_2\alpha_3}^\dagger & & X_{\alpha_{n-1}\beta_{n-1}} & & X_{\beta_{n-1}\alpha_n}^\dagger & & X_{\alpha_n\beta_n} & & X_{\beta_n j}^\dagger \end{array}$$

 Figure B.3: Feynman diagrams of n -term (before specifying contraction structure).

Here is also a bonus result: Even under pair grouping configurations where $n_b = n$, i and j must be identified by the grouping in order to get large enough n_f (see Eq. B.24). So i and j must be equal, which means that the matrix $B(z)$ must be diagonal under $N \rightarrow \infty$.

According to the condition Eq. (B.2), only contracting X and $X^\dagger$ can be nonzero

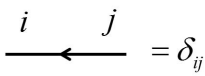
$$\langle X_{ij} X_{kl}^\dagger \rangle_X = \langle X_{ij} X_{lk}^* \rangle_X = \delta_{il} \delta_{jk}. \quad (\text{B.25})$$

Let us call this contraction ‘‘propagator’’, which corresponds to the Feynman rule shown in Fig. B.1.

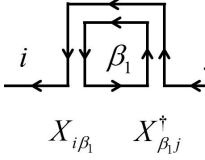
Our goal is to evaluate the target function $B(z)$ (Eq. B.18). Since each matrix element has two indices, and each pair of $XX^\dagger$ always comes with a factor $\frac{1}{zN}$, we are naturally led to the Feynman rules of vertices shown in Fig. B.2. The arrow flow is used to distinguish X from $X^\dagger$. This is necessary because our X is generically not hermitian. Then a typical n -term (Eq. B.19) can be calculated by summing over Feynman diagrams corresponding to all the possible contraction structures of Fig. B.3. To see a few examples, let us enumerate all nonzero diagrams contributing to $n = 0$, $n = 1$ and $n = 2$ terms in Fig. B.4.

B.2.5 Simplification: Planar Diagrams only for $N \rightarrow \infty$

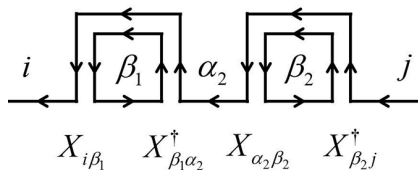
Now we have developed a diagrammatic way of evaluating $B_{ij}(z)$ as described by Fig. B.3, which is well organized and quite routine. But the actual calculation is still rather complicated, because there are so many ways of contracting the vertices. Large N limit, however, brings us



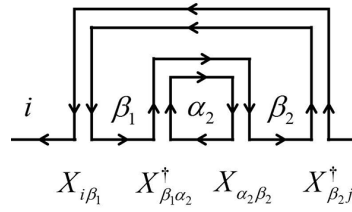
(a) $n = 0$ term



(b) $n = 1$ term



(c) $n = 2$ case 1



(d) $n = 2$ case 2

Figure B.4: Feynman diagram examples.

another great simplification: Any diagram with crossed contractions will vanish under $N \rightarrow \infty$. This means that we only need to consider the type of contraction shown in the first line of the following, but not that kind shown in the second line.

$$\begin{aligned}
 & \cdots X_{\alpha_2\beta_2} X_{\beta_2\alpha_3}^\dagger X_{\alpha_3\beta_3} X_{\beta_3\alpha_4}^\dagger X_{\alpha_4\beta_4} X_{\beta_4\alpha_5}^\dagger \cdots \\
 & \cdots X_{\alpha_2\beta_2} X_{\beta_2\alpha_3}^\dagger X_{\alpha_3\beta_3} X_{\beta_3\alpha_4}^\dagger X_{\alpha_4\beta_4} X_{\beta_4\alpha_5}^\dagger \cdots
 \end{aligned}$$

Once crossed contractions are forbidden, all the propagators can only form two types of structures: “side by side” as in the example of Fig. B.4(c) or “nesting” as in Fig. B.4(d). A combination of these two types gives us a general “planar” diagram. Only planar diagrams have nonzero contributions under $N \rightarrow \infty$.

This requirement also follows from our key statement. Assume that we have a contraction jumping k couples of elements:

$$\cdots X_{\beta_{p-1}\alpha_p}^\dagger X_{\alpha_p\beta_p} X_{\beta_p\alpha_{p+1}}^\dagger \cdots X_{\alpha_q\beta_q} X_{\beta_q\alpha_{q+1}}^\dagger X_{\alpha_{q+1}\beta_{q+1}} \cdots$$

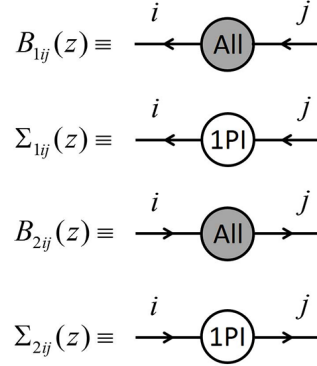


Figure B.5: Auxiliary functions defined in terms of planar Feynman diagrams.

with $q = p + k$. This contraction identifies α_{q+1} with α_p and β_q with β_p . After summing over these non-free dummy indices α_{q+1} and β_q , we get the result proportional to (with a finite coefficient)

$$\cdots X_{\beta_{p-1}\alpha_p}^\dagger \cdot X_{\beta_p\alpha_{p+1}}^\dagger \cdots X_{\alpha_q\beta_p} \cdot X_{\alpha_p\beta_{q+1}} \cdots \quad (\text{B.26})$$

Now we are only left with $n - 1$ couples of X and $X^\dagger$. With the number of boxes $n_b = n - 1$ then, it seems impossible to make $n_f = n$, according to our previous result Eq. (B.24). However, by a careful look at the new sequence Eq. (B.26), we realize that it is no longer guaranteed to be fully connected by bridges. Instead, it consists of two parts: inside the contraction and outside the contraction, each part fully connected. So as long as we do not group any element inside with any element outside into one box (i.e. no crossed contraction!), we can only apply our key statement to each part separately. In this case, we are just lucky enough to save it: we can get up to $k + 1$ free dummy indices from the inside part and $n - 1 - k$ from the outside part. Together, we can still make $n_f = n$. On the other hand, if we do make a contraction crossed with the first one, then the divided two parts are reconnected through this contraction box and Eq. (B.24) can be applied to the whole sequence Eq. (B.26): $n_f \leq n_b = n - 1 < n$. Therefore diagram with crossed contractions will vanish under $N \rightarrow \infty$.

B.2.6 Diagrammatic Calculation for $N \rightarrow \infty$

Now we are finally ready to calculate our target function $B_{ij}(z)$ by summing over all the planar Feynman diagrams formed from Fig. B.3. For convenience, let us define four functions as shown in Fig. B.5, two 1PI (1 Particle Irreducible) functions $\Sigma_{1ij}(z)$, $\Sigma_{2ij}(z)$, and two two-point functions $B_{1ij}(z)$, $B_{2ij}(z)$. Here B_{1ij} is nothing but our target function $B_{ij}(z) = B_{1ij}(z)$.

First, let us study the 1PI functions. $\Sigma_{1ij}(z)$ sums over all the 1PI planar diagrams with external single arrows pointing to the left. Each 1PI planar diagram must have a double-line contraction coating it at the most outside, with nested inside anything. Clearly, the sum of the nested part gives

nothing but $B_2(z)$. So we get a relation as shown in Fig. B.6(a):

$$\Sigma_{1ij} = \sum_{\beta_p, \beta_q=1}^N \frac{1}{zN} \delta_{ij} \delta_{\beta_p \beta_q} B_{2\beta_p \beta_q} = \left(\frac{1}{zN} \sum_{\beta_p=1}^N B_{2\beta_p \beta_p} \right) \delta_{ij} \equiv \Sigma_1 \delta_{ij}. \quad (\text{B.27})$$

We see that $\Sigma_{1ij}(z)$ is proportional to the identity matrix. There is a similar relation for $\Sigma_{2ij}(z)$ (as shown in Fig. B.6(b)), which is also proportional to identity matrix:

$$\Sigma_{2ij} = \sum_{\alpha_p, \alpha_q=1}^M \frac{1}{zN} \delta_{ij} \delta_{\alpha_p \alpha_q} B_{1\alpha_p \alpha_q} = \left(\frac{1}{zN} \sum_{\alpha_p=1}^M B_{1\alpha_p \alpha_p} \right) \delta_{ij} \equiv \Sigma_2 \delta_{ij}. \quad (\text{B.28})$$

Now let us turn to the two point functions $B_{1ij}(z)$ and $B_{2ij}(z)$. Same as in computing a two-point correlation function in QFT, all the diagrams contributing to $B_{1ij}(z)$ ($B_{2ij}(z)$) can be organized into a geometric series of the 1PI functions $\Sigma_{1ij}(z)$ ($\Sigma_{2ij}(z)$). Since both $\Sigma_{1ij}(z)$ and $\Sigma_{2ij}(z)$ are proportional to identity matrix, $B_{1ij}(z)$ and $B_{2ij}(z)$ are also proportional to identity matrix:

$$\begin{aligned} B_{1ij} &= \delta_{ij} + \Sigma_{1ij} + \Sigma_{1i\alpha} \Sigma_{1\alpha j} + \dots \\ &= (1 + \Sigma_1 + \Sigma_1^2 + \dots) \delta_{ij} = \frac{1}{1 - \Sigma_1} \delta_{ij} \\ &\equiv B_1 \delta_{ij}, \end{aligned} \quad (\text{B.29})$$

$$\begin{aligned} B_{2ij} &= \delta_{ij} + \Sigma_{2ij} + \Sigma_{2i\beta} \Sigma_{2\beta j} + \dots \\ &= (1 + \Sigma_2 + \Sigma_2^2 + \dots) \delta_{ij} = \frac{1}{1 - \Sigma_2} \delta_{ij} \\ &\equiv B_2 \delta_{ij}. \end{aligned} \quad (\text{B.30})$$

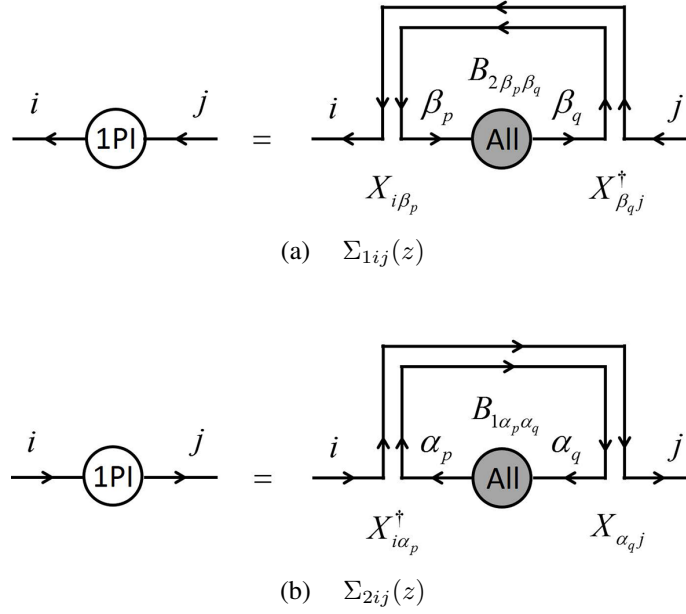
This confirms our bonus result in subsection B.2.4 that B_{1ij} and B_{2ij} have to be diagonal. Going back to Eq. (B.27) and Eq. (B.28), we get

$$\Sigma_1 = \frac{1}{zN} \sum_{\beta_p=1}^N B_{2\beta_p \beta_p} = \frac{B_2}{zN} \sum_{\beta_p=1}^N \delta_{\beta_p \beta_p} = \frac{1}{z} B_2, \quad (\text{B.31})$$

$$\Sigma_2 = \frac{1}{zN} \sum_{\alpha_p=1}^M B_{1\alpha_p \alpha_p} = \frac{B_1}{zN} \sum_{\alpha_p=1}^M \delta_{\alpha_p \alpha_p} = \frac{b}{z} B_1. \quad (\text{B.32})$$

Combining the work above, we get the following equation set

$$\left\{ \begin{aligned} B_1 &= \frac{1}{1 - \Sigma_1} \\ B_2 &= \frac{1}{1 - \Sigma_2} \\ \Sigma_1 &= \frac{1}{z} B_2 \\ \Sigma_2 &= \frac{b}{z} B_1 \end{aligned} \right. \quad (\text{B.33})$$


 Figure B.6: 1PI diagrams (a) $\Sigma_{1ij}(z)$, and (b) $\Sigma_{2ij}(z)$, in terms of two point full diagrams

Because we are eventually interested in $B_{ij}(z) = B_{1ij}(z) = B_1(z)\delta_{ij}$, we eliminate the other three variables and get the equation of $B_1(z)$:

$$bB_1^2 - [z - (1 - b)] B_1 + z = 0. \quad (\text{B.34})$$

Solving this and plugging it into Eq. (B.16) and Eq. (B.15), we get the result of $\rho(E)$

$$\begin{aligned} \rho(E) &= -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \text{Im} G(E + i\varepsilon) \\ &= \frac{1}{2\pi} \frac{1}{b} \frac{\sqrt{(x_2 - E)(E - x_1)}}{E} I_{\{E \in (x_1, x_2)\}} + \left(1 - \frac{1}{b}\right) \delta(E) I_{\{b \geq 1\}}, \end{aligned} \quad (\text{B.35})$$

which is exactly what we want to prove (Eq. B.7). Note that there are two solutions for Eq. (B.34), and one needs be cautious while choosing the root and taking the limit. It is a straight forward but slightly tedious procedure.

B.3 Generalizing to Cases of Restricted X

As mentioned in the introduction section, our direct diagrammatic approach can be readily generalized to six types of restricted X . This makes a total list of seven cases of X which let us

reproduce here for convenience:

- (1) Complex arbitrary
- (2) Complex symmetric
- (3) Complex antisymmetric
- (4) Real arbitrary
- (5) Real symmetric
- (6) Real antisymmetric
- (7) Hermitian

Many of these cases are interesting in physics models. For example, in the case of large N analysis of neutrino anarchy, the Majorana mass matrix is complex symmetric (case (2) above).

It is understood that for the cases (2)-(7) above, the conditions (Eq. B.1-Eq. B.3) on the random entries of X should be modified accordingly. First, in the condition Eq. (B.1), “independent” should be understood as only among the *free entries* of X . For cases (1) and (4), X has $M \times N$ free entries. Other cases require that $M = N$ ($b = 1$). For cases (2), (5), and (7), X has $N(N+1)/2$ free entries.³ For cases (3) and (6), X has $N(N-1)/2$ free entries. Second, the condition Eq. (B.2) is specifically for complex valued entries of X . In certain cases above, all or part of the entries of X are required to be real valued. For real valued X_{ij} , the condition Eq. (B.2) should be replaced by the following

$$\langle X_{ij} \rangle_X = 0, \quad \langle X_{ij}^2 \rangle_X = 1.$$

To see that our whole analysis through Section B.2 still works for the other six cases of X , we need a few observations. First, it is clear that the *grouping by pairs* requirement (see the first paragraph of Section B.2.4) holds for all the seven cases above. As is emphasized before, this requirement justifies the notion and the use of “contraction” and makes a diagrammatic approach possible. Second, one may worry that for cases (4)-(7), $\langle X_{ij} X_{kl} \rangle_X$ can be nonzero, namely that X can contract with not only $X^\dagger$ but also X . This will not be a problem, because when an X contracts with another X (or an $X^\dagger$ contracts with another $X^\dagger$) in Eq. (B.23), an odd number of entries of X are left inside this contraction, which makes a crossed contraction inevitable. And from Section B.2.5, we know that terms with crossed contractions vanish under $N \rightarrow \infty$. So we still only need to consider the type of contraction $\langle X_{ij} X_{kl}^\dagger \rangle_X$, namely the type of propagator shown in Fig. B.1, even in cases (4)-(7). However, for cases (2)-(7), the value of the propagator could be different from Eq. (B.25) or that shown in Fig. B.1. Therefore, a final observation needed is how Eq. (B.25) is changed and how that affects the calculations through Section B.2. It is easy

³For case (2), all of these free entries are complex valued. For case (5), all of these free entries are real valued. For case (7), $N(N-1)/2$ of these free entries are complex valued, and N of these free entries are real valued.

to see that for the seven cases of X above, Eq. (B.25) should be modified into three values

$$\left\langle X_{ij} X_{kl}^\dagger \right\rangle_{X, \text{arbitrary}} = \delta_{il} \delta_{jk}, \quad (\text{B.36})$$

$$\left\langle X_{ij} X_{kl}^\dagger \right\rangle_{X, \text{symmetric}} = \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl} (1 - \delta_{ij}), \quad (\text{B.37})$$

$$\left\langle X_{ij} X_{kl}^\dagger \right\rangle_{X, \text{antisymmetric}} = \delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}. \quad (\text{B.38})$$

In the above, Eq. (B.36) applies to cases (1), (4), and (7); Eq. (B.37) applies to cases (2) and (5); and Eq. (B.38) applies to cases (3) and (6). Clearly, the modifications of Eq. (B.25) are additional terms proportional to $\delta_{ik} \delta_{jl}$. This modification could affect our calculations in Section B.2 only through Eq. (B.27) and (B.28). However, we can easily see that the additional terms in these two equations vanish under $N \rightarrow \infty$

$$\begin{aligned} \Sigma_{1ij, \text{additional}}(z) &\propto \sum_{\beta_p, \beta_q=1}^N \frac{1}{zN} \delta_{i\beta_q} \delta_{\beta_p j} B_{2\beta_p \beta_q} = \frac{1}{zN} B_{2ji} \rightarrow 0, \\ \Sigma_{2ij, \text{additional}}(z) &\propto \sum_{\alpha_p, \alpha_q=1}^M \frac{1}{zN} \delta_{i\alpha_q} \delta_{\alpha_p j} B_{1\alpha_p \alpha_q} = \frac{1}{zN} B_{1ji} \rightarrow 0. \end{aligned}$$

Therefore, the diagrammatic proof presented in Section B.2 holds for all the seven cases of X .

Appendix C

Proof of the Universality of Magnetic Dipole Term

Assuming that there is a weakly coupled renormalizable UV model,^{1,2,3} let us consider a general picture that the full gauge symmetry group G of the UV model is spontaneously broken into a subgroup H . A set of gauge bosons Q_μ^i have “eaten” the Nambu-Goldstone bosons χ^i and obtained mass m_Q . For this setup, it turns out that Q_μ^i form a certain representation of the unbroken gauge group H , and under this representation, the general form of the gauge-kinetic piece of the Lagrangian up to quadratic term in Q_μ^i is given by Eq. (4.38), which we reproduce here for convenience

$$\mathcal{L}_{\text{g.k.}} \supset \frac{1}{2} Q_\mu^i (D^2 g^{\mu\nu} - D^\nu D^\mu + [D^\mu, D^\nu])^{ij} Q_\nu^j, \quad (\text{C.1})$$

with D_μ denoting the covariant derivative that contains only the massless gauge bosons. One remarkable feature of this general gauge-kinetic term is that the coefficient of the “magnetic dipole term” $\frac{1}{2} Q_\mu^i \{[D^\mu, D^\nu]\}^{ij} Q_\nu^j$ is universal, namely that its coefficient is fixed to 1 relative to the “curl” terms $\frac{1}{2} Q_\mu^i \{D^2 g^{\mu\nu} - D^\nu D^\mu\}^{ij} Q_\nu^j$, regardless of the details of the symmetry breaking. In this appendix, two ways of proving Eq. (C.1) will be presented.

¹In general, this need not be the case. For example, the Q_μ could be composite particles in the low-energy effective description of some strongly interacting theory. Another example is when additional massive vector bosons are needed to UV complete the theory. For example, an effective theory with a massive vector transforming as a doublet under a $SU(2)$ gauge symmetry is non-renormalizable—a valid UV completion could be an $SU(3)$ gauge symmetry broken to $SU(2)$, but this requires an additional doublet and singlet vector. For our present purposes, relaxing this assumption only alters one result, which is discussed below.

²As in all the other cases considered in this work, although never explicitly stated, we are also assuming the fields we integrate out are weakly coupled amongst themselves and the low-energy fields, so that it makes sense to integrate them out.

³ G itself may be contained in some larger group $\mathcal{G}$ which also contains exact and approximate global symmetries and the same mechanism responsible for breaking $G \rightarrow H$ may also break some of these global symmetries. These generalities do not affect our results below, which concern the transformation of Q_μ and its associated NGBs under H . We therefore stick to our simplified picture for clarity.

C.1 Algebraic Proof

Let us first give an algebraic derivation of Eq. (C.1). Let G have a general structure of product group

$$G = G_1 \times G_2 \times \cdots \times G_n. \quad (\text{C.2})$$

Let T^A be the set of generators of G , with $A = 1, 2, \dots, \dim(G)$. Due to Eq. (C.2), the set of generators T^A are composed by a number of subsets

$$\{T^A\} = \{T_1^{A_1}\} \cup \{T_2^{A_2}\} \cup \cdots \cup \{T_n^{A_n}\}, \quad (\text{C.3})$$

with $A_i = 1, 2, \dots, \dim(G_i)$. Let f_G^{ABC} denote the structure constant of G :

$$[T^A, T^B] = if_G^{ABC} T^C. \quad (\text{C.4})$$

Obviously $f_G^{ABC} = 0$ if any two indices belong to different subsets in Eq. (C.3).

The full covariant derivative $\bar{D}$ of the UV model and its commutator is

$$\bar{D}_\mu = \partial_\mu - ig^A G_\mu^A T^A, \quad (\text{C.5})$$

$$[\bar{D}_\mu, \bar{D}_\nu] = -ig^A G_{\mu\nu}^A T^A, \quad (\text{C.6})$$

where G_μ^A denote the gauge fields, $G_{\mu\nu}^A$ the field strengths, and g^A the gauge couplings that could be arbitrarily different for T^A of different subsets in Eq. (C.3). Note that the above expression of the full covariant derivative holds for any representation of G .

Because we have put the arbitrary gauge couplings into the covariant derivative, the gauge boson kinetic term of the UV Lagrangian is simply

$$\mathcal{L}_{\text{g.k.}} = -\frac{1}{4}(G_{\mu\nu}^{A_1})^2 - \frac{1}{4}(G_{\mu\nu}^{A_2})^2 - \cdots - \frac{1}{4}(G_{\mu\nu}^{A_n})^2. \quad (\text{C.7})$$

In order to write this kinetic term in terms of the full covariant derivative $\bar{D}_\mu$, let us define an inner product in the generator space $\{T^A\}$:

$$\langle T^A, T^B \rangle \equiv \frac{1}{2(g^A)^2} \delta^{AB}, \quad (\text{C.8})$$

which just looks like a scaled version of trace. However, let us emphasize that, although it should be quite clear from definition, this inner product is essentially very different from the trace. The inner product can only be taken over two vectors in the generator space, while a trace action can be taken over arbitrary powers of generators. Nevertheless, the inner product defined in Eq. (C.8) has many similar properties as the trace action. For example, if one of the two vectors is given in a form of a commutator of two other generators, a cyclic permutation is allowed

$$\begin{aligned} \langle T^A, [T^B, T^C] \rangle &= \langle T^A, if_G^{BCD} T^D \rangle = if_G^{BCD} \frac{1}{2(g^A)^2} \delta^{AD} \\ &= if_G^{ABC} \frac{1}{2(g^A)^2} = if_G^{ABC} \frac{1}{2(g^C)^2} \\ &= if_G^{CAB} \frac{1}{2(g^C)^2} = \langle T^C, [T^A, T^B] \rangle. \end{aligned} \quad (\text{C.9})$$

Note that the second line above is true because for the case $g^A \neq g^C$, $f^{ABC} = 0$. As we shall see shortly, this cyclic permutation property will play a very important role in our derivation. With the inner product defined in Eq. (C.8), the gauge boson kinetic term Eq. (C.7) can be very conveniently written as

$$\mathcal{L}_{\text{g.k.}} = \frac{1}{2} \langle [\bar{D}_\mu, \bar{D}_\nu], [\bar{D}^\mu, \bar{D}^\nu] \rangle. \quad (\text{C.10})$$

Now let us consider the subgroup H of G . Let t^a be the generators of H , which span a subspace of the full group generator space, and have closed algebra

$$[t^a, t^b] = i f_H^{abc} t^c, \quad (\text{C.11})$$

with f_H^{abc} denotes the structure constant of H , and $a = 1, 2, \dots, \dim(H)$. Once the full group G is spontaneously broken into H , it is obviously convenient to divide the full generator space into the unbroken generators t^a and the broken generators X^i , $i = 1, 2, \dots, \dim(G) - \dim(H)$, with the corresponding massless gauge fields A_μ^a and massive gauge bosons Q_μ^i

$$(t^A) = \begin{pmatrix} g_H^a t^a \\ X^i \end{pmatrix}, \quad (W_\mu^A) = \begin{pmatrix} A_\mu^a \\ Q_\mu^i \end{pmatrix}. \quad (\text{C.12})$$

In the above, we write t^A instead of T^A , and W_μ^A instead of G_μ^A , because t^a is generically a linear combination of T^A , and there is a linear transformation between t^A and T^A , as well as between W_μ^A and G_μ^A in accordance. This linear transformation is typically chosen to be orthogonal between gauge field⁴, in order to preserve the universal coefficients structure in Eq. (C.7). Then we have

$$W_\mu^A = O^{AB} G_\mu^B, \quad \text{with } O^T O = 1. \quad (\text{C.13})$$

The full covariant derivative Eq. (C.5) can be rewritten as

$$\bar{D}_\mu = \partial_\mu - i W_\mu^A t^A = \partial_\mu - i g_H^a A_\mu^a t^a - i Q_\mu^i X^i = D_\mu - i Q_\mu^i X^i, \quad (\text{C.14})$$

$$t^A = O^{AB} g^B T^B, \quad (\text{C.15})$$

where the second line serves as the definition of t^A in terms of T^A . Note that a factor g_H^a is needed in Eq. (C.12) to make Eqs (C.4), (C.11) and (C.15) consistent. This is how one determines the gauge coupling constant g_H^a of the unbroken gauge group. We have also used D_μ to denote the covariant derivative that contains only the massless gauge bosons A_μ^a . The above definition of t^A preserves the orthogonality of them under the inner product defined in Eq. (C.8)

$$\langle t^A, t^B \rangle = \langle O^{AC} g^C T^C, O^{BD} g^D T^D \rangle = \frac{1}{2(g^C)^2} O^{AC} O^{BD} g^C g^D \delta^{CD} = \frac{1}{2} \delta^{AB}, \quad (\text{C.16})$$

which specifically means that

$$\langle t^a, t^b \rangle = \frac{1}{2(g_H^a)^2} \delta^{ab}, \quad \langle X^i, X^j \rangle = \frac{1}{2} \delta^{ij}, \quad \langle t^a, X^i \rangle = 0. \quad (\text{C.17})$$

⁴Other linear transformations will lead to equivalent theories upon field redefinition.

Let us first prove that Q_μ^i defined through Eq. (C.12) and Eq. (C.13) form a representation under the unbroken gauge group H . This is essentially to prove that the commutator between t^a and X^i is only a linear combination of X^i

$$[t^a, X^i] = -(t_Q^a)^{ij} X^j, \quad (\text{C.18})$$

with a certain set of matrices $(t_Q^a)^{ij}$ that also need to be antisymmetric between i, j . Both points can be easily proven by making use of our inner product defined in Eq. (C.8) and its cyclic permutation property Eq. (C.9). Eq. (C.18) is obvious from

$$\langle t^b, [t^a, X^i] \rangle = \langle X^i, [t^b, t^a] \rangle = 0, \quad (\text{C.19})$$

and the antisymmetry is clear from

$$(t_Q^a)^{ij} = -2 \langle X^j, [t^a, X^i] \rangle = -2 \langle t^a, [X^i, X^j] \rangle. \quad (\text{C.20})$$

Once Eq. (C.18) is proven, it follows that

$$[t^a, Q_\mu^i X^i] = -Q_\mu^i (t_Q^a)^{ij} X^j = (t_Q^a)^{ij} Q_\mu^j X^i, \quad (\text{C.21})$$

where we see that t_Q^a serves as the generator matrix or “charge” of Q_μ^i . And therefore

$$\begin{aligned} [D_\mu, Q_\nu^i X^i] &= (\partial_\mu Q_\nu^i) X^i - i g_H^a A_\mu^a [t^a, Q_\nu^i X^i] = (\partial_\mu Q_\nu^i) X^i - i g_H^a A_\mu^a (t_Q^a)^{ij} Q_\nu^j X^i \\ &= \left[(\partial_\mu Q_\nu^i) - i g_H^a A_\mu^a (t_Q^a)^{ij} Q_\nu^j \right] X^i = (D_\mu Q_\nu^i) X^i. \end{aligned} \quad (\text{C.22})$$

With all the above preparations, we are eventually ready to decompose the full gauge boson kinetic term in Eq. (C.7). First, the commutator of the full covariant derivative is

$$\begin{aligned} [\bar{D}_\mu, \bar{D}_\nu] &= [D_\mu - i Q_\mu^i X^i, D_\nu - i Q_\nu^j X^j] \\ &= [D_\mu, D_\nu] - i \{ [\bar{D}_\mu, Q_\nu^j X^j] - [\bar{D}_\nu, Q_\mu^i X^i] \} - [Q_\mu^i X^i, Q_\nu^j X^j] \\ &= [D_\mu, D_\nu] - i [(D_\mu Q_\nu^i) - (D_\nu Q_\mu^i)] X^i - [Q_\mu^i X^i, Q_\nu^j X^j]. \end{aligned} \quad (\text{C.23})$$

Keeping only terms relevant and up to quadratic power for Q_μ^i , it follows from Eq. (C.10) that

$$\begin{aligned} \mathcal{L}_{\text{g.k.}} &= \frac{1}{2} \langle [\bar{D}_\mu, \bar{D}_\nu], [\bar{D}^\mu, \bar{D}^\nu] \rangle \\ &\supset -\frac{1}{4} [(D_\mu Q_\nu^i) - (D_\nu Q_\mu^i)]^2 - \langle [D^\mu, D^\nu], [Q_\mu^i X^i, Q_\nu^j X^j] \rangle \\ &= \frac{1}{2} Q_\mu^i (D^2 g^{\mu\nu} - D^\nu D^\mu)^{ij} Q_\nu^j - \langle Q_\mu^i X^i, [Q_\nu^j X^j, [D^\mu, D^\nu]] \rangle \\ &= \frac{1}{2} Q_\mu^i (D^2 g^{\mu\nu} - D^\nu D^\mu)^{ij} Q_\nu^j + \langle Q_\mu^i X^i, [D^\mu, [D^\nu, Q_\nu^j X^j]] \rangle - \langle Q_\mu^i X^i, [D^\nu, [D^\mu, Q_\nu^j X^j]] \rangle \\ &= \frac{1}{2} Q_\mu^i (D^2 g^{\mu\nu} - D^\nu D^\mu)^{ij} Q_\nu^j + Q_\mu^i \langle X^i, (D^\mu D^\nu Q_\nu^j) X^j \rangle - Q_\mu^i \langle X^i, (D^\nu D^\mu Q_\nu^j) X^j \rangle \\ &= \frac{1}{2} Q_\mu^i (D^2 g^{\mu\nu} - D^\nu D^\mu)^{ij} Q_\nu^j + \frac{1}{2} Q_\mu^i (D^\mu D^\nu - D^\nu D^\mu)^{ij} Q_\nu^j \\ &= \frac{1}{2} Q_\mu^i \{ D^2 g^{\mu\nu} - D^\nu D^\mu + [D^\mu, D^\nu] \}^{ij} Q_\nu^j, \end{aligned} \quad (\text{C.24})$$

where from the second line to the third line, we have used the cyclic permutation property of the inner product, and the fourth line follows from the third line due to Jacobi identity. This finishes our algebraic derivation of Eq. (C.1).

It should be stressed that in spite of the allowance of arbitrary gauge couplings for each simple group G_i , the end gauge-interaction piece of the Lagrangian of the heavy vector boson Q_μ^i has the above universal form, especially that the coefficient of the “magnetic dipole term” $\frac{1}{2}Q_\mu^i [D^\mu, D^\nu]^{ij} Q_\nu^j$ is fixed at 1 relative the the “curl” terms $\frac{1}{2}Q_\mu^i \{D^2 g^{\mu\nu} - D^\nu D^\mu\}^{ij} Q_\nu^j$.

C.2 Physical Proof

Now let us give a physical argument to explain this universality, which is from the tree-level unitarity. Let us consider one component of the massless background gauge boson and call it a “photon” A_μ with its coupling constant e and generator Q . It is helpful to use a complex linear combination of generators X^i to form X^α and $X^{\alpha\dagger}$ that are “eigenstates” of the generator Q , $[Q, X^\alpha] = q^\alpha X^\alpha$ and $[Q, X^{\alpha\dagger}] = -q^\alpha X^{\alpha\dagger}$. Let us also define Q_μ^α and $Q_\mu^{\alpha\dagger}$ to keep $Q_\mu^i X^i = Q_\mu^\alpha X^\alpha + Q_\mu^{\alpha\dagger} X^{\alpha\dagger}$. Note that Q_μ^i are real, but Q_μ^α are complex fields. The normalization of Q_μ^α is chosen such that $\frac{1}{2}Q_\mu^i Q^{\mu i} = Q_\mu^{\alpha\dagger} Q^{\mu\alpha}$. It should be clear that in this part of the appendix where we discuss integrating out a heavy gauge boson, indices α, β are used to denote the complex generators $X^\alpha, X^{\alpha\dagger}$, and their accordingly defined complex gauge fields $Q^\alpha, Q^{\alpha\dagger}$. Lorentz indices are denoted by μ, ν, ρ , etc.

First, one can check that the “curl” terms in Eq. (C.1) written in terms of Q_μ^i gives the correct kinetic term for Q_μ^α coupled to photon according to its charge q^α , because from Eq. (C.22) we have

$$\begin{aligned} (D_\mu Q_\nu^i) X^i &= [D_\mu, Q_\nu^i X^i] = [D_\mu, Q_\nu^\alpha X^\alpha + Q_\nu^{\alpha\dagger} X^{\alpha\dagger}] \\ &\supset (\partial_\mu Q_\nu^\alpha) X^\alpha + (\partial_\mu Q_\nu^{\alpha\dagger}) X^{\alpha\dagger} - ie A_\mu [Q, Q_\nu^\alpha X^\alpha + Q_\nu^{\alpha\dagger} X^{\alpha\dagger}] \\ &= (\partial_\mu Q_\nu^\alpha) X^\alpha + (\partial_\mu Q_\nu^{\alpha\dagger}) X^{\alpha\dagger} - ie q^\alpha A_\mu (Q_\nu^\alpha X^\alpha - Q_\nu^{\alpha\dagger} X^{\alpha\dagger}) \\ &= (\partial_\mu Q_\nu^\alpha - ie q^\alpha A_\mu Q_\nu^\alpha) X^\alpha + (\partial_\mu Q_\nu^{\alpha\dagger} + ie q^\alpha A_\mu Q_\nu^{\alpha\dagger}) X^{\alpha\dagger}, \end{aligned} \quad (C.25)$$

and the “curl” form derives from the original Yang-Mills Lagrangian in the UV theory

$$\mathcal{L}_{\text{Yang-Mills}} \supset -\frac{1}{4} (D_\mu Q_\nu^i - D_\nu Q_\mu^i)^2 = \frac{1}{2} Q_\mu^i \{D^2 g^{\mu\nu} - D^\nu D^\mu\}^{ij} Q_\nu^j. \quad (C.26)$$

What is the least obvious is the universal coefficient for the “magnetic dipole term”

$$\frac{1}{2} Q_\mu^i [D^\mu, D^\nu]^{ij} Q_\nu^j = -\frac{1}{2\mu(R)} \text{tr}([Q_\mu, Q_\nu][D^\mu, D^\nu]), \quad (C.27)$$

where $Q_\mu \equiv Q_\mu^i X^i = Q_\mu^\alpha X^\alpha + Q_\mu^{\alpha\dagger} X^{\alpha\dagger}$, and $\text{tr}(X^i X^j) = \mu(R) \delta^{ij}$. This term is gauge invariant under the unbroken gauge symmetry and one may wonder whether the coefficient can be arbitrary

and model dependent. Focusing on the “photon” coupling piece, this term contains

$$\begin{aligned}
-\frac{1}{2\mu(R)}\text{tr}([Q_\mu, Q_\nu][D^\mu, D^\nu]) &\supset \frac{ie\hat{A}^{\mu\nu}}{2\mu(R)}\text{tr}([Q_\mu, Q_\nu]Q) \\
&= \frac{ie\hat{A}^{\mu\nu}}{2\mu(R)}\text{tr}([Q, Q_\mu]Q_\nu) \\
&= \frac{ie\hat{A}^{\mu\nu}}{2\mu(R)}\text{tr}\{(q^\alpha Q_\mu^\alpha X^\alpha - q^\alpha Q_\mu^{\alpha\dagger} X^{\alpha\dagger})(Q_\nu^\beta X^\beta + Q_\nu^{\beta\dagger} X^{\beta\dagger})\} \\
&= \frac{-ie\hat{A}^{\mu\nu}}{2}q^\alpha(Q_\mu^{\alpha\dagger}Q_\nu^\alpha - Q_\nu^{\alpha\dagger}Q_\mu^\alpha) \\
&= -ie\hat{A}^{\mu\nu}q^\alpha Q_\mu^{\alpha\dagger}Q_\nu^\alpha,
\end{aligned} \tag{C.28}$$

where $\hat{A}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, and we have used the property $\text{tr}(X^\alpha X^{\beta\dagger}) = \mu(R)\delta^{\alpha\beta}$, $\text{tr}(X^\alpha X^\beta) = \text{tr}(X^{\alpha\dagger} X^{\beta\dagger}) = 0$. So it is clear that the coefficient of this “magnetic dipole term” is exactly the “triple gauge coupling” between the heavy gauge boson Q_μ^α and the massless gauge bosons A_μ . One can make it more transparent by taking the SM analog of Eq. (C.28). In the case of SM electroweak symmetry breaking, one recognizes $q^\alpha = -1$, $Q_\nu^\alpha = W_\nu^-$, and $Q_\mu^{\alpha\dagger} = W_\mu^+$, then Eq. (C.28) is nothing but the κ_γ term in Eq. (4.82). It is well known that the amplitude for $\gamma\gamma \rightarrow W^+W^-$ would grow as E_W^2 in the Standard Model if the magnetic dipole moment $\kappa_\gamma \neq 1$. The quadratic part of the Lagrangian (*i.e.* Eq. (C.1)) is sufficient to determine the tree-level amplitude, and the diagrams are exactly the same as those in the Standard Model. Unless $\kappa_\gamma = 1$, it violates perturbative unitarity at high energies. Because the amplitude does not involve the Higgs or other heavy vector bosons, the amplitude is exactly the same as that in the UV theory, which is unitary. Therefore, the perturbative unitarity for this amplitude needs to be satisfied with the quadratic Lagrangian, which requires the dipole moment to have this value.

Appendix D

Derivation of CDE for Fermions and Gauge Bosons

In this appendix, some details are shown for the derivation of CDE for fermions and gauge bosons.

D.1 Fermions

Let us now consider the functional determinant for massive fermion fields and provide the formulas for the covariant derivative expansion for them. Let us work in the notation of Dirac fermions, denoting the gamma matrices by γ^μ and employing slashed notation, *e.g.* $\not{D} = \gamma^\mu D_\mu$. This discussion is easily modified if one wants to consider Weyl fermions and use two-component notation.

Consider the Lagrangian containing the fermions to be

$$\mathcal{L}[\psi, \phi] = \bar{\psi}(i\not{D} - m - M(x))\psi, \quad (\text{D.1})$$

where m is the fermion mass and $M(x)$ is in general dependent on the light fields $\phi(x)$. Upon integrating over the Grassman valued fields in the path integral, the one-loop contribution to the effective action is given by

$$S_{\text{eff},1\text{-loop}} \equiv \Delta S_{\text{eff}} = -i\text{Tr} \log (\not{D} - m - M), \quad (\text{D.2})$$

where, as before, $P_\mu \equiv iD_\mu$. Using $\text{Tr} \log AB = \text{Tr} \log A + \text{Tr} \log B$ and the fact that the trace is invariant under changing signs of gamma matrices we have

$$\begin{aligned} \Delta S_{\text{eff}} &= -\frac{i}{2} \left[\text{Tr} \log (-\not{D} - m - M) + \text{Tr} \log (\not{D} - m - M) \right] \\ &= -\frac{i}{2} \text{Tr} \log \left(-\not{D}^2 + m^2 + 2mM + M^2 + \not{D}M \right). \end{aligned} \quad (\text{D.3})$$

where $\not{P}M \equiv [\not{P}, M]$, as defined in Eq. (4.26). With $\gamma^\mu \gamma^\nu = (\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu])/2 = g^{\mu\nu} - i\sigma^{\mu\nu}$,

$$\not{P}^2 = P^2 + \frac{i}{2}\sigma^{\mu\nu}[D_\mu, D_\nu] = P^2 + \frac{i}{2}\sigma \cdot G', \quad (\text{D.4})$$

where $G'_{\mu\nu} \equiv [D_\mu, D_\nu]$, as defined in Eq. (4.26).

We thus see that the trace for fermions,

$$\text{Tr} \log \left(-P^2 + m^2 - \frac{i}{2}\sigma^{\mu\nu}G'_{\mu\nu} + 2mM + M^2 + \not{P}M \right), \quad (\text{D.5})$$

is of the form $\text{Tr} \log(-P^2 + m^2 + U)$. Therefore, all the steps in evaluating the trace and shifting by the covariant derivative using $e^{\pm P \cdot \partial / \partial q}$ are the same as previously considered and we can immediately write down the answer from Eq. (4.21). Defining

$$U_{\text{ferm}} \equiv -\frac{i}{2}\sigma^{\mu\nu}G'_{\mu\nu} + 2mM + M^2 + \not{P}M, \quad (\text{D.6})$$

the one-loop effective Lagrangian for fermions is then given by

$$\Delta\mathcal{L}_{\text{eff,ferm}} = -\frac{i}{2} \int dq \text{tr} \log \left[-\left(q_\mu + \tilde{G}_{\nu\mu} \partial_\nu \right)^2 + \tilde{U}_{\text{ferm}} \right], \quad (\text{D.7})$$

where $\tilde{G}$ and $\tilde{U}_{\text{ferm}}$ are defined as in Eq. (4.27) with $U \rightarrow U_{\text{ferm}}$.

The result originally obtained in [88] contains an error (see Eq. (4.21) therein compared to our result above in Eq. (D.7)). This mistake originates from an error in Eq. (4.17) of [88] where a term proportional to $[\tilde{G}_{\mu\nu}\partial_\nu, \tilde{G}_{\rho\sigma}\partial_\sigma] \neq 0$ was missing.

D.2 Massless Gauge Bosons

Here let us consider the one-loop contribution to the 1PI effective action from massless gauge fields. The spirit here is slightly different from our previous discussions involving massive scalars and fermions; we are not integrating the gauge bosons out of the theory but instead are evaluating the 1PI effective action. Nevertheless, the manipulations are exactly the same since the one-loop contribution to the 1PI effective action is still a functional trace of the form $\text{Tr} \log(D^2 + U)$.

In evaluating the 1PI effective action, we split the gauge boson into a background piece plus fluctuations around this background, $A_\mu = A_{B,\mu} + Q_\mu$, and perform the path integral over the fluctuations Q_μ while holding the background $A_{B,\mu}$ fixed. In order to do the path integral, one must gauge fix the Q_μ fields. At first glance, one might think that gauge fixing destroys the possibility of keeping gauge invariance manifest while evaluating the one-loop effective action. However, this turns out not to be the case. It is well known that there is a convenient gauge fixing condition that leaves the gauge symmetry of the background $A_{B,\mu}$ field manifest, *i.e.* it only gauge fixes Q_μ and not $A_{B,\mu}$. This technique is known as the background field method (for example, see [154])

and references therein).¹ Because the gauge symmetry of the background $A_{B,\mu}$ field is not fixed, we will still be able to employ the techniques of the covariant derivative expansion, allowing a manifestly gauge invariant computation of the one-loop effective action.

The issues around gauge symmetry are actually quite distinct for the background field method versus the CDE. However, because similar words are used in both discussions, it is worth clarifying what aspects of gauge symmetry are handled in each case. The background field method makes it manifestly clear that the effective action of $A_{B,\mu}$ possesses a gauge symmetry by only gauge fixing the fluctuating field Q_μ . This is an all orders statement. However, when evaluating the effective action order-by-order, one still works with the non-covariant quantities $A_{B,\mu}$, Q_μ , and $\partial/\partial x^\mu$ at intermediate steps.² The covariant derivative expansion, on the other hand, is a technique for evaluating the *one-loop* effective action that keeps gauge invariance manifest at all stages of the computation by working with gauge covariant quantities such as D_μ . To understand this point more explicitly, one can compare the method of the CDE presented in this dissertation and in [89] with the evaluation of the functional determinant using the component fields as presented in detail in Peskin and Schroeder [155].

Now onto the calculation, we take pure $SU(N)$ gauge theory,

$$\mathcal{L}[A_\mu] = -\frac{1}{2Ng^2} \text{tr} F_{\mu\nu}^2 = -\frac{1}{4g^2} (F_{\mu\nu}^a)^2, \quad (\text{D.8})$$

where $F_{\mu\nu} = F_{\mu\nu}^a t^a$ and we take the t^a in the adjoint representation, $\text{tr} t^a t^b = N\delta^{ab}$, $(t^b)_{ac} = if^{abc}$. We denote the covariant derivative as $\mathcal{D}_\mu = \partial_\mu - iA_\mu$ with the field strength defined as usual, $F_{\mu\nu} = i[\mathcal{D}_\mu, \mathcal{D}_\nu]$. Note that we have normalized the gauge field such that the coupling constant does not appear in the covariant derivative.

Let $\Gamma[A_B]$ be the 1PI effective action. To find $\Gamma[A_B]$, we split the gauge field into a background piece and a fluctuating piece, $A_\mu = A_{B,\mu} + Q_\mu$, and integrate out the Q_μ fields.³ The one-loop contribution to Γ comes from the quadratic terms in Q_μ . We have

$$\mathcal{D}_\mu = \partial_\mu - i(A_{B,\mu} + Q_\mu) \equiv D_\mu - iQ_\mu, \quad (\text{D.9a})$$

$$F_{\mu\nu} = i[D_\mu, D_\nu] + D_\mu Q_\nu - D_\nu Q_\mu - i[Q_\mu, Q_\nu] \equiv G_{\mu\nu} + Q_{\mu\nu} - i[Q_\mu, Q_\nu], \quad (\text{D.9b})$$

$$\mathcal{L} = -\frac{1}{2Ng^2} \text{Tr}(G_{\mu\nu} + Q_{\mu\nu} - i[Q_\mu, Q_\nu])^2. \quad (\text{D.9c})$$

Note that $D_\mu = \partial_\mu - iA_{B,\mu}$ and $G_{\mu\nu} = i[D_\mu, D_\nu]$ are the covariant derivative and field strength of the background field alone.

¹All techniques of evaluating effective actions are, by the definition of holding fields fixed while doing a path integral, background field methods. Nevertheless, the term “background field method” is usually taken to refer to employing this special gauge fixing condition while evaluating the 1PI effective action.

²To one-loop order, one only deals with $A_{B,\mu}$ and ∂_μ .

³To keep the discussion short, we are being slightly loose here. In particular, a source term J for the fluctuating fields needs to be introduced. After integrating out the fluctuating field, we obtain an effective action which is a functional of J and the background fields, $W[J, A_B]$. The 1PI effective action, $\Gamma[A_B]$, is obtained by a Legendre transform of W . For more details see, for example, [154].

In order to get sensible results out of the path integral, we need to gauge fix. As in the background field method, let us employ a gauge fixing condition which is covariant with respect to the background field $A_{B,\mu}$. Namely, the gauge-fixing condition G^a is taken to be $G^a = D^\mu Q_\mu^a$. The resultant gauge-fixed Lagrangian—including ghosts to implement the Fadeev-Popov determinant—is, *e.g.* [154, 155],

$$\mathcal{L}_{g.f.} + \mathcal{L}_{gh} = -\frac{1}{2g^2\xi} (D^\mu Q_\mu^a)^2 + D^\mu \bar{c}^a (D_\mu c^a + f^{abc} Q_\mu^b c^c), \quad (\text{D.10})$$

where ξ is the gauge-fixing parameter. The utility of this gauge fixing condition is that the fluctuating Q_μ^a is gauge fixed while the Lagrangian (D.9c) together with $\mathcal{L}_{g.f.} + \mathcal{L}_{gh}$ possesses a manifest gauge symmetry with gauge field $A_{B,\mu}$ that is not gauge fixed. Thus we can perform the path integral over Q_μ^a while leaving the gauge invariance of the effective action of $A_{B,\mu}$ manifest. Under a background gauge symmetry transformation, $A_{B,\mu}$ transforms as a gauge field, $A_{B,\mu} \rightarrow V(A_{B,\mu} + i\partial_\mu)V^\dagger$ while Q_μ (and the ghosts c and $\bar{c}$) transforms simply as a field in the adjoint representation, $Q_\mu \rightarrow VQ_\mu V^\dagger$. Procedurally, when performing the path integral over Q and c , one can simply think about these fields as regular scalar and fermion⁴ fields in the adjoint of some gauge symmetry and with interactions dictated by the Lagrangians in (D.9c) and (D.10).

The quadratic piece of the combined Yang-Mills, gauge-fixing, and ghost Lagrangian is

$$\mathcal{L} = -\frac{1}{2g^2} Q_\mu^a \left[-g^{\mu\nu} (D^2)^{ac} - \frac{1-\xi}{\xi} (D^\mu D^\nu)^{ac} - 2f^{abc} G^{b\mu\nu} \right] Q_\nu^c + \bar{c}^a \left[- (D^2)^{ac} \right] c^c. \quad (\text{D.11})$$

Let us work in Feynman gauge with $\xi = 1$ so that we can drop the $D^\mu D^\nu$ term. Note that everything inside the square brackets in the above is in the adjoint representation (recall, $f^{abc} = -i(t^b)_{ac}$). Using the generator for Lorentz transformations on four-vectors, $(\mathcal{J}_{\rho\sigma})^{\mu\nu} = i(\delta_\rho^\mu \delta_\sigma^\nu - \delta_\rho^\nu \delta_\sigma^\mu)$, we can write

$$G^{\mu\nu} = -\frac{i}{2} (G^{\rho\sigma} \mathcal{J}_{\rho\sigma})^{\mu\nu}.$$

The quadratic piece of the Lagrangian is then given by

$$\mathcal{L} = -\frac{1}{2g^2} Q_\mu^a \left[-D^2 \mathbf{1}_4 + G \cdot \mathcal{J} \right]_\nu^{\mu,ac} Q_\nu^c + \bar{c}^a \left[-D^2 \right]^{ac} c^c, \quad (\text{D.12})$$

where $\mathbf{1}_4$ is the 4×4 identity matrix for the Lorentz indices, *i.e.* $(\mathbf{1}_4)_\nu^\mu = \delta_\nu^\mu$. Performing the path-integral over the gauge and ghost fields we obtain

$$\Gamma_{1\text{-loop}}[A_B] = \frac{i}{2} \text{Tr} \log (D^2 \mathbf{1}_4 - G \cdot \mathcal{J}) - i \text{Tr} \log (D^2), \quad (\text{D.13})$$

where the factor of $1/2$ in the first term is because the Q_μ^a are real bosons, while the factor of -1 in the second term is because the c^a are anti-commuting. Note that the functional traces makes

⁴Of course ghosts aren't fermions; they are anti-commuting scalars. We are speaking very loosely and by fermion we are referring to their anti-commuting properties.

totally transparent the role of the ghosts. The trace of the gauge boson term containing D^2 picks up a factor of 4 from the trace over Lorentz indices, one for each Q_μ $\mu = 0, 1, 2, 3$. Of course, the gauge boson only has two physical degrees of freedom; we see explicitly above that the ghost piece cancels the contribution of two of the degrees of freedom.

Each of the traces in the above are of the form $\text{Tr}(-P^2 + U)$, and thus we can immediately apply the transformations leading to the covariant derivative expansion. Switching to our notation $G'_{\mu\nu} = [D_\mu, D_\nu] = -iG_{\mu\nu}$ and defining

$$U_{\text{gauge}} = -i\mathcal{J}^{\mu\nu}G'_{\mu\nu}, \quad (\text{D.14})$$

we have

$$\begin{aligned} \Gamma_{\text{1-loop}}[A_B] = & \frac{i}{2} \int dx dq \text{tr} \log \left[- \left(q_\mu + \tilde{G}_{\nu\mu} \partial_\nu \right)^2 + \tilde{U}_{\text{gauge}} \right] \\ & - i \int dx dq \text{tr} \log \left[- \left(q_\mu + \tilde{G}_{\nu\mu} \partial_\nu \right)^2 \right], \end{aligned} \quad (\text{D.15})$$

where $\tilde{G}$ and $\tilde{U}_{\text{gauge}}$ are defined as in Eq. (4.27) with $U \rightarrow U_{\text{ferm}}$. The first term in the above is from the fluctuating gauge fields, while the second is from the ghosts. Note also that the trace “tr” in the first term includes over the Lorentz indices, just as the trace for fermions in Eq. (D.7) is over the Lorentz (spinor) indices. In fact, it should be clear that U_{gauge} is very similar to the first term in U_{ferm} (Eq. (D.6)): $U_{\text{ferm}} \supset -i(\sigma^{\mu\nu}/2)G'_{\mu\nu}$ where $\sigma^{\mu\nu}/2$ is the generator for Lorentz transformations on spinors.

Note that the effective action (D.15) contains infrared divergences from the massless gauge and ghost fields that we integrated out. These divergences can be regulated by adding a mass term for Q_μ^a and c^a because these mass terms respect the gauge invariance of the background field $A_{B,\mu}$.⁵

D.3 Massive Gauge Bosons

With our understanding of the story for massless gauge bosons, it turns out to be simple to obtain the result for massive gauge bosons. Let us consider massive vector bosons Q_μ transforming under an unbroken, low-energy gauge group. As is well known, beyond tree-level perturbation theory, the Nambu-Goldstone bosons (NGBs) “eaten” by the massive vector boson must be included, *i.e.* we cannot work in unitary gauge. By working in a generalized R_ξ gauge, we will be able to maintain manifest covariance of the low-energy gauge group. As we will see, mathematically, the results are essentially the same as the massless case in Eqs. (D.12) and (D.13), modified by the presence of mass terms for the Q_μ and ghosts as well as an additional term for the NGBs.

First, as is mentioned in the main text, the gauge-kinetic piece of the Lagrangian up to quadratic term in Q_μ^i is

$$\mathcal{L}_{\text{g.k.}} \supset \frac{1}{2} Q_\mu^i \left(D^2 g^{\mu\nu} - D^\nu D^\mu + [D^\mu, D^\nu] \right)^{ij} Q_\nu^j, \quad (\text{D.16})$$

⁵As stated previously, procedurally one can just think of Q_μ and c as scalars and fermions transforming in the adjoint of some gauge symmetry whose gauge field is $A_{B,\mu}$. Just as scalars and fermions can have mass terms without disturbing gauge-invariance, Q_μ and c can have mass terms without disturbing the background gauge-invariance.

where D_μ denotes the covariant derivative that contains only the unbroken gauge fields. In Appendix C, both an algebraic derivation and a physical argument to prove Eq. (D.16) are shown.

Second, because we are integrating out the heavy gauge bosons Q_μ^i perturbatively, we need to fix the part of gauge transformation corresponding to Q_μ^i . But we would also like to preserve the unbroken gauge symmetry. To achieve this, let us adopt a generalized R_ξ gauge fixing term as following

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{2\xi}(\xi m_Q \chi^i + D^\mu Q_\mu^i)^2, \quad (\text{D.17})$$

where $\partial^\mu Q_\mu^i$ in a usual R_ξ gauge fixing is promoted to $D^\mu Q_\mu^i$ to preserve the unbroken gauge symmetry.

Now combining Eq. (D.16) and (D.17) with the appropriate ghost term

$$\mathcal{L}_{\text{ghost}} = \bar{c}^i (-D^2 - \xi m_Q^2)^{ij} c^j, \quad (\text{D.18})$$

the mass term of Q_μ^i due to the symmetry breaking

$$\mathcal{L}_{\text{mass}} \supset \frac{1}{2}(D_\mu \chi^i - m_Q Q_\mu^i)^2, \quad (\text{D.19})$$

and a generic interaction term quadratic in Q_μ^i

$$\mathcal{L}_1 = \frac{1}{2} Q_\mu^i (M^{\mu\nu})^{ij} Q_\nu^j \quad (\text{D.20})$$

we get the full Lagrangian up to quadratic power in Q_μ^i as

$$\begin{aligned} \Delta\mathcal{L} = & \frac{1}{2} Q_\mu^i \left(D^2 g^{\mu\nu} - D^\nu D^\mu + m_Q^2 g^{\mu\nu} + [D^\mu, D^\nu] + \frac{1}{\xi} D^\mu D^\nu + M^{\mu\nu} \right)^{ij} Q_\nu^j \\ & + \frac{1}{2} \chi^i (-D^2 - \xi m_Q^2)^{ij} \chi^j + \bar{c}^i (-D^2 - \xi m_Q^2)^{ij} c^j \end{aligned} \quad (\text{D.21})$$

Taking Feynman gauge $\xi = 1$, we get

$$\begin{aligned} \Delta\mathcal{L} = & \frac{1}{2} Q_\mu^i (D^2 g^{\mu\nu} + m_Q^2 g^{\mu\nu} + 2[D^\mu, D^\nu] + M^{\mu\nu})^{ij} Q_\nu^j \\ & + \frac{1}{2} \chi^i (-D^2 - m_Q^2)^{ij} \chi^j + \bar{c}^i (-D^2 - m_Q^2)^{ij} c^j \end{aligned} \quad (\text{D.22})$$

This is the result (Eq. (4.39)) presented in the main text.

Appendix E

Detailed Steps of Mapping Wilson Coefficients on to Physical Observables

In this appendix, some details are shown about the calculations of the mapping step described in Section 4.3. First, all the relevant two-point and three-point Feynman rules from the set of dimension-six operator in Table 4.1 are list out in Appendix E.1. Transverse vacuum polarization functions, that can be readily read off from the two-point Feynman rules, are also tabulated. Then in Appendix E.2 and E.3 details are presented in calculating the “interference correction” ϵ_I for Higgs decay widths and Higgs production cross sections respectively. Relevant Feynman diagrams, definitions of auxiliary functions, and conventional form factors are listed out. Finally, in Appendix E.4 and E.5, detailed steps are shown for calculating the residue modifications and the parameter modifications, which are related to the “residue correction” ϵ_R and the “parametric correction” ϵ_P respectively.

E.1 Additional Feynman Rules from Dim-6 Effective Operators

E.1.1 Feynman Rules for Vacuum Polarization Functions

Throughout the calculations in Section 4.3, the relevant vacuum polarization functions are those of the vector bosons $i\Pi_{VV}^{\mu\nu}(p^2) \in \{i\Pi_{WW}^{\mu\nu}(p^2), i\Pi_{ZZ}^{\mu\nu}(p^2), i\Pi_{\gamma\gamma}^{\mu\nu}(p^2), i\Pi_{\gamma Z}^{\mu\nu}(p^2)\}$ and that of the Higgs boson $-i\Sigma(p^2)$. It is straightforward to expand out the dim-6 effective operators listed in Table 4.1, identify the relevant Lagrangian pieces, and obtain the Feynman rules. The relevant Lagrangian pieces are shown in Eq. (E.5)-Eq. (E.9). The resulting Feynman rules of the vacuum polarization functions are drawn in Fig. E.1, with the detailed values listed in Eq. (E.10)-Eq. (E.14). In the diagrams, a big solid dot is used to denote the interactions due to the dim-6 effective operators (*i.e.* due to Wilson coefficients c_i), while a simple direct connecting would represent the SM interaction.

$$\begin{aligned}
 \Pi_{WW}(p^2) &= p^4 \left(-\frac{1}{\Lambda^2} c_{2W} \right) + p^2 \frac{2m_W^2}{\Lambda^2} (4c_{WW} + c_W) + m_W^2 \frac{v^2}{\Lambda^2} c_R \\
 \Pi_{ZZ}(p^2) &= p^4 \left[-\frac{1}{\Lambda^2} (c_Z^2 c_{2W} + s_Z^2 c_{2B}) \right] + p^2 \frac{2m_Z^2}{\Lambda^2} \left[4(c_Z^4 c_{WW} + s_Z^4 c_{BB} + c_Z^2 s_Z^2 c_{WB}) \right. \\
 &\quad \left. + (c_Z^2 c_W + s_Z^2 c_B) \right] + m_Z^2 \frac{v^2}{\Lambda^2} (-2c_T + c_R) \\
 \Pi_{\gamma\gamma}(p^2) &= p^4 \left[-\frac{1}{\Lambda^2} (s_Z^2 c_{2W} + c_Z^2 c_{2B}) \right] + p^2 \frac{8m_Z^2}{\Lambda^2} c_Z^2 s_Z^2 (c_{WW} + c_{BB} - c_{WB}) \\
 \Pi_{\gamma Z}(p^2) &= p^4 \left[-\frac{1}{\Lambda^2} c_Z s_Z (c_{2W} - c_{2B}) \right] + p^2 \frac{m_Z^2}{\Lambda^2} c_Z s_Z \left[8(c_Z^2 c_{WW} - s_Z^2 c_{BB}) \right. \\
 &\quad \left. - 4(c_Z^2 - s_Z^2) c_{WB} + (c_W - c_B) \right] \\
 \Sigma(p^2) &= p^4 \left(-\frac{1}{\Lambda^2} c_D \right) + p^2 \left[-\frac{v^2}{\Lambda^2} (2c_H + c_R) \right]
 \end{aligned}$$

Table E.1: Transverse Vacuum polarization functions in terms of Wilson coefficients.

$$\begin{aligned}
 \Pi_{WW}(p^2) - \Pi_{33}(p^2) &= m_W^2 \frac{2v^2}{\Lambda^2} c_T \\
 \Pi_{33}(p^2) &= p^4 \left(-\frac{1}{\Lambda^2} c_{2W} \right) + p^2 \frac{2m_W^2}{\Lambda^2} (4c_{WW} + c_W) + m_W^2 \frac{v^2}{\Lambda^2} (-2c_T + c_R) \\
 \Pi_{BB}(p^2) &= p^4 \left(-\frac{1}{\Lambda^2} c_{2B} \right) + p^2 \frac{2m_Z^2 s_Z^2}{\Lambda^2} (4c_{BB} + c_B) + m_Z^2 s_Z^2 \frac{v^2}{\Lambda^2} (-2c_T + c_R) \\
 \Pi_{3B}(p^2) &= p^2 \left(-\frac{m_Z^2}{\Lambda^2} c_Z s_Z \right) (4c_{WB} + c_W + c_B) + m_Z^2 \frac{v^2}{\Lambda^2} c_Z s_Z (2c_T - c_R)
 \end{aligned}$$

Table E.2: Alternative set of transverse vacuum polarization functions that are used in the definitions of EWPO parameters Table 4.2.

For vector bosons, one can easily identify the transverse part of the vacuum polarization functions $\Pi_{VV}(p^2)$ from

$$i\Pi_{VV}^{\mu\nu}(p^2) = i \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \Pi_{VV}(p^2) + \left(i \frac{p^\mu p^\nu}{p^2} \text{ term} \right). \quad (\text{E.1})$$

These transverse vacuum polarization functions $\{\Pi_{WW}(p^2), \Pi_{ZZ}(p^2), \Pi_{\gamma\gamma}(p^2), \Pi_{\gamma Z}(p^2)\}$ together with $-i\Sigma(p^2)$ are summarized in Table E.1. In some occasions, such as defining the EWPO parameters, it is more concise to use an alternative set $\{\Pi_{33}, \Pi_{BB}, \Pi_{3B}\}$ instead of $\{\Pi_{ZZ}, \Pi_{\gamma\gamma}, \Pi_{\gamma Z}\}$. Due to the relation $W^3 = c_Z Z + s_Z A$ and $B = -s_Z Z + c_Z A$, there is a simple transformation

between these two sets

$$\Pi_{33} = c_Z^2 \Pi_{ZZ} + s_Z^2 \Pi_{\gamma\gamma} + 2c_Z s_Z \Pi_{\gamma Z}, \quad (\text{E.2})$$

$$\Pi_{BB} = s_Z^2 \Pi_{ZZ} + c_Z^2 \Pi_{\gamma\gamma} - 2c_Z s_Z \Pi_{\gamma Z}, \quad (\text{E.3})$$

$$\Pi_{3B} = -c_Z s_Z \Pi_{ZZ} + c_Z s_Z \Pi_{\gamma\gamma} + (c_Z^2 - s_Z^2) \Pi_{\gamma Z}, \quad (\text{E.4})$$

where we have adopted the notation $c_Z \equiv \cos \theta_Z$ *etc.*, with θ_Z denoting the weak mixing angle. This alternative set of vector boson transverse vacuum polarization functions are summarized in Table E.2.

$$\begin{aligned} \mathcal{L}_{WW} = & W_\mu^+ (\partial^4 g^{\mu\nu} - \partial^2 \partial^\mu \partial^\nu) W_\nu^- \cdot \left(-\frac{1}{\Lambda^2} c_{2W} \right) \\ & + W_\mu^+ (-\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu) W_\nu^- \cdot \frac{2m_W^2}{\Lambda^2} (4c_{WW} + c_W) \\ & + m_W^2 W_\mu^+ W^{-\mu} \cdot \frac{v^2}{\Lambda^2} c_R - W^{-\mu} (\partial_\mu \partial_\nu) W^{+\nu} \cdot \frac{m_W^2}{\Lambda^2} c_D, \end{aligned} \quad (\text{E.5})$$

$$\begin{aligned} \mathcal{L}_{ZZ} = & \frac{1}{2} Z_\mu (\partial^4 g^{\mu\nu} - \partial^2 \partial^\mu \partial^\nu) Z_\nu \cdot \left[-\frac{1}{\Lambda^2} (c_Z^2 c_{2W} + s_Z^2 c_{2B}) \right] \\ & + \frac{1}{2} Z_\mu (-\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu) Z_\nu \cdot \frac{2m_Z^2}{\Lambda^2} \left[\frac{4(c_Z^4 c_{WW} + s_Z^4 c_{BB} + c_Z^2 s_Z^2 c_{WB})}{(c_Z^2 c_W + s_Z^2 c_B)} \right] \\ & + \frac{1}{2} m_Z^2 Z_\mu Z^\mu \cdot \frac{v^2}{\Lambda^2} (-2c_T + c_R), \end{aligned} \quad (\text{E.6})$$

$$\begin{aligned} \mathcal{L}_{\gamma\gamma} = & \frac{1}{2} A_\mu (\partial^4 g^{\mu\nu} - \partial^2 \partial^\mu \partial^\nu) A_\nu \cdot \left[-\frac{1}{\Lambda^2} (s_Z^2 c_{2W} + c_Z^2 c_{2B}) \right] \\ & + \frac{1}{2} A_\mu (-\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu) A_\nu \cdot \frac{8m_Z^2}{\Lambda^2} c_Z^2 s_Z^2 (c_{WW} + c_{BB} - c_{WB}), \end{aligned} \quad (\text{E.7})$$

$$\begin{aligned} \mathcal{L}_{\gamma Z} = & A_\mu (\partial^4 g^{\mu\nu} - \partial^2 \partial^\mu \partial^\nu) Z_\nu \cdot \left[-\frac{1}{\Lambda^2} c_Z s_Z (c_{2W} - c_{2B}) \right] \\ & + A_\mu (-\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu) Z_\nu \cdot \frac{m_Z^2}{\Lambda^2} c_Z s_Z \left[\frac{8(c_Z^2 c_{WW} - s_Z^2 c_{BB})}{-4(c_Z^2 - s_Z^2) c_{WB} + (c_W - c_B)} \right], \end{aligned} \quad (\text{E.8})$$

$$\mathcal{L}_{hh} = \frac{1}{2} h (\partial^4) h \cdot \frac{1}{\Lambda^2} c_D + \frac{1}{2} h (-\partial^2) h \cdot \frac{v^2}{\Lambda^2} (2c_H + c_R). \quad (\text{E.9})$$

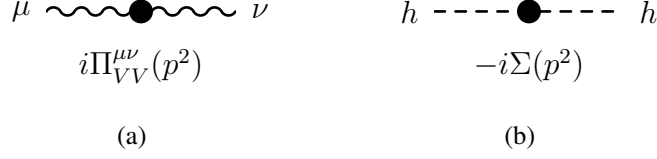


Figure E.1: Feynman rules for vacuum polarization functions.

$$\begin{aligned}
 i\Pi_{WW}^{\mu\nu}(p^2) = & i \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left[p^4 \left(-\frac{1}{\Lambda^2} c_{2W} \right) + p^2 \frac{2m_W^2}{\Lambda^2} (4c_{WW} + c_W) \right. \\
 & \left. + m_W^2 \frac{v^2}{\Lambda^2} c_R \right] + i \frac{p^\mu p^\nu}{p^2} \cdot \left(p^2 \frac{m_W^2}{\Lambda^2} c_D + m_W^2 \frac{v^2}{\Lambda^2} c_R \right), \tag{E.10}
 \end{aligned}$$

$$\begin{aligned}
 i\Pi_{ZZ}^{\mu\nu}(p^2) = & i \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left\{ p^4 \left[-\frac{1}{\Lambda^2} (c_Z^2 c_{2W} + c_{2B} s_Z^2) \right] \right. \\
 & + p^2 \frac{2m_Z^2}{\Lambda^2} [4(c_Z^4 c_{WW} + s_Z^4 c_{BB} + c_Z^2 s_Z^2 c_{WB}) + (c_Z^2 c_W + s_Z^2 c_B)] \\
 & \left. + m_Z^2 \frac{v^2}{\Lambda^2} (-2c_T + c_R) \right\} + i \frac{p^\mu p^\nu}{p^2} \cdot m_Z^2 \frac{v^2}{\Lambda^2} (-2c_T + c_R), \tag{E.11}
 \end{aligned}$$

$$\begin{aligned}
 i\Pi_{\gamma\gamma}^{\mu\nu}(p^2) = & i \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left\{ p^4 \left[-\frac{1}{\Lambda^2} (s_Z^2 c_{2W} + c_Z^2 c_{2B}) \right] \right. \\
 & \left. + p^2 \frac{8m_Z^2}{\Lambda^2} c_Z^2 s_Z^2 (c_{WW} + c_{BB} - c_{WB}) \right\}, \tag{E.12}
 \end{aligned}$$

$$\begin{aligned}
 i\Pi_{\gamma Z}^{\mu\nu}(p^2) = & i \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \cdot \left\{ p^4 \left[-\frac{1}{\Lambda^2} c_Z s_Z (c_{2W} - c_{2B}) \right] \right. \\
 & \left. + p^2 \frac{m_Z^2}{\Lambda^2} c_Z s_Z [8(c_Z^2 c_{WW} - s_Z^2 c_{BB}) - 4(c_Z^2 - s_Z^2) c_{WB} + c_W - c_B] \right\}, \tag{E.13}
 \end{aligned}$$

$$-i\Sigma(p^2) = i p^4 \frac{1}{\Lambda^2} c_D + i p^2 \frac{v^2}{\Lambda^2} (2c_H + c_R). \tag{E.14}$$

E.1.2 Feynman Rules for Three-point Vertices

Throughout Section 4.3, the relevant three-point vertices are hWW , hZZ , $h\gamma Z$, $h\gamma\gamma$, and hgg vertices. As with the vacuum polarization functions case, we can expand out the dim-6 effective

operators in Table 4.1 and identify the relevant Lagrangian pieces (Eq. (E.15)-Eq. (E.19)). These Lagrangian pieces generate the Feynman rules shown in Fig. E.2, with detailed values listed in Eq. (E.20)-Eq. (E.24).

$$\mathcal{L}_{hWW} = \frac{\sqrt{2}m_W^2}{v} \left\{ \frac{1}{2} h \hat{W}_{\mu\nu}^+ \hat{W}^{-\mu\nu} \cdot \frac{1}{\Lambda^2} 8c_{WW} + h \left[\begin{array}{c} W_\mu^+ (-\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu) W_\nu^- \\ + W_\mu^- (-\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu) W_\nu^+ \end{array} \right] \cdot \frac{1}{\Lambda^2} c_W \right. \\ \left. - \left[\begin{array}{c} (\partial^2 h) W_\mu^- W^{+\mu} + h (\partial_\mu W^{-\mu}) (\partial_\nu W^{+\nu}) \\ h W^{+\mu} (\partial_\mu \partial_\nu) W^{-\nu} + h W^{-\mu} (\partial_\mu \partial_\nu) W^{+\nu} \end{array} \right] \cdot \frac{1}{\Lambda^2} c_D + h W_\mu^+ W^{-\mu} \cdot \frac{2v^2}{\Lambda^2} c_R \right\}, \quad (\text{E.15})$$

$$\mathcal{L}_{hZZ} = \frac{\sqrt{2}m_Z^2}{v} \left\{ \frac{1}{4} h \hat{Z}_{\mu\nu} \hat{Z}^{\mu\nu} \cdot \frac{1}{\Lambda^2} 8 (c_Z^4 c_{WW} + s_Z^4 c_{BB} + c_Z^2 s_Z^2 c_{WB}) \right. \\ + \frac{1}{2} h Z_\mu (-\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu) Z_\nu \cdot \frac{1}{\Lambda^2} 2 (c_Z^2 c_W + s_Z^2 c_B) \\ + \left[-\frac{1}{2} (\partial^2 h) (Z_\mu Z^\mu) - \frac{1}{2} h (\partial_\mu Z^\mu) (\partial_\nu Z^\nu) - h Z^\mu (\partial_\mu \partial_\nu) Z^\nu \right] \cdot \frac{1}{\Lambda^2} c_D \\ \left. + \frac{1}{2} h Z_\mu Z^\mu \cdot \frac{2v^2}{\Lambda^2} (-2c_T + c_R) \right\}, \quad (\text{E.16})$$

$$\mathcal{L}_{h\gamma Z} = \frac{\sqrt{2}m_Z^2}{v} \left\{ \frac{1}{2} h \hat{Z}_{\mu\nu} A^{\mu\nu} \cdot \frac{1}{\Lambda^2} 4c_Z s_Z \left[2 (c_Z^2 c_{WW} - s_Z^2 c_{BB}) - (c_Z^2 - s_Z^2) c_{WB} \right] \right. \\ \left. + h Z_\mu (-\partial^2 g^{\mu\nu} + \partial^\mu \partial^\nu) A_\nu \cdot \frac{1}{\Lambda^2} c_Z s_Z (c_W - c_B) \right\}, \quad (\text{E.17})$$

$$\mathcal{L}_{h\gamma\gamma} = \frac{\sqrt{2}m_Z^2}{v} \frac{1}{4} h A_{\mu\nu} A^{\mu\nu} \cdot \frac{1}{\Lambda^2} 8c_Z^2 s_Z^2 (c_{WW} + c_{BB} - c_{WB}), \quad (\text{E.18})$$

$$\mathcal{L}_{hgg} = \frac{\sqrt{2}g_s^2 v^2}{2v} \frac{1}{4} h G_{\mu\nu}^a G^{a,\mu\nu} \cdot \frac{1}{\Lambda^2} 8c_{GG}. \quad (\text{E.19})$$

$$\begin{aligned}
 iM_{hWW}^{\mu\nu}(p_1, p_2) = & i \frac{\sqrt{2}m_W^2}{v} \left\{ - (p_1 p_2 g^{\mu\nu} - p_1^\nu p_2^\mu) \frac{1}{\Lambda^2} 8c_{WW} \right. \\
 & + \left[(p_1^2 g^{\mu\nu} - p_1^\mu p_1^\nu) + (p_2^2 g^{\mu\nu} - p_2^\mu p_2^\nu) \right] \frac{1}{\Lambda^2} c_W \\
 & \left. + \left[(p_1 + p_2)^2 g^{\mu\nu} + p_1^\mu p_2^\nu + p_1^\mu p_1^\nu + p_2^\mu p_2^\nu \right] \frac{1}{\Lambda^2} c_D + g^{\mu\nu} \frac{2v^2}{\Lambda^2} c_R \right\}, \quad (E.20)
 \end{aligned}$$

$$\begin{aligned}
 iM_{hZZ}^{\mu\nu}(p_1, p_2) = & i \frac{\sqrt{2}m_Z^2}{v} \left\{ - (p_1 p_2 g^{\mu\nu} - p_1^\nu p_2^\mu) \frac{1}{\Lambda^2} 8(c_Z^4 c_{WW} + s_Z^4 c_{BB} + c_Z^2 s_Z^2 c_{WB}) \right. \\
 & + \left[(p_1^2 g^{\mu\nu} - p_1^\mu p_1^\nu) + (p_2^2 g^{\mu\nu} - p_2^\mu p_2^\nu) \right] \frac{1}{\Lambda^2} (c_Z^2 c_W + s_Z^2 c_B) \\
 & + \left[(p_1 + p_2)^2 g^{\mu\nu} + p_1^\mu p_2^\nu + p_1^\mu p_1^\nu + p_2^\mu p_2^\nu \right] \frac{1}{\Lambda^2} c_D \\
 & \left. + g^{\mu\nu} \frac{2v^2}{\Lambda^2} (-2c_T + c_R) \right\}, \quad (E.21)
 \end{aligned}$$

$$\begin{aligned}
 iM_{h\gamma Z}^{\mu\nu}(p_1, p_2) = & i \frac{\sqrt{2}m_Z^2}{v} \left\{ - (p_1 p_2 g^{\mu\nu} - p_1^\nu p_2^\mu) \frac{4c_Z s_Z}{\Lambda^2} \left[2(c_Z^2 c_{WW} - s_Z^2 c_{BB}) - (c_Z^2 - s_Z^2) c_{WB} \right] \right. \\
 & \left. + (p_1^2 g^{\mu\nu} - p_1^\mu p_1^\nu) \frac{1}{\Lambda^2} c_Z s_Z (c_W - c_B) \right\}, \quad (E.22)
 \end{aligned}$$

$$iM_{h\gamma\gamma}^{\mu\nu}(p_1, p_2) = -i \frac{\sqrt{2}m_Z^2}{v} (p_1 p_2 g^{\mu\nu} - p_1^\nu p_2^\mu) \frac{1}{\Lambda^2} 8c_Z^2 s_Z^2 (c_{WW} + c_{BB} - c_{WB}), \quad (E.23)$$

$$iM_{hgg}^{\mu\nu}(p_1, p_2) = -i \frac{\sqrt{2}g_s^2 v^2}{2v} (p_1 p_2 g^{\mu\nu} - p_1^\nu p_2^\mu) \frac{1}{\Lambda^2} 8c_{GG}. \quad (E.24)$$

E.2 Details on Interference Corrections to the Higgs Decay Widths

There is no new amputated diagrams for $h \rightarrow f \bar{f}$ decay modes up to leading order (linear power and tree level) in Wilson coefficients, because we are considering only the bosonic dim-6 effective operators (Table 4.1). The $h \rightarrow gg$, $h \rightarrow \gamma\gamma$, and $h \rightarrow \gamma Z$ decay widths are already at one-loop order in the SM, so the only new amputated diagram up to leading order in Wilson coefficients is given by the new three-point vertices $iM_{hgg}^{\mu\nu}(p_1, p_2)$, $iM_{h\gamma\gamma}^{\mu\nu}(p_1, p_2)$, and $iM_{h\gamma Z}^{\mu\nu}(p_1, p_2)$ (Fig. E.2(d),

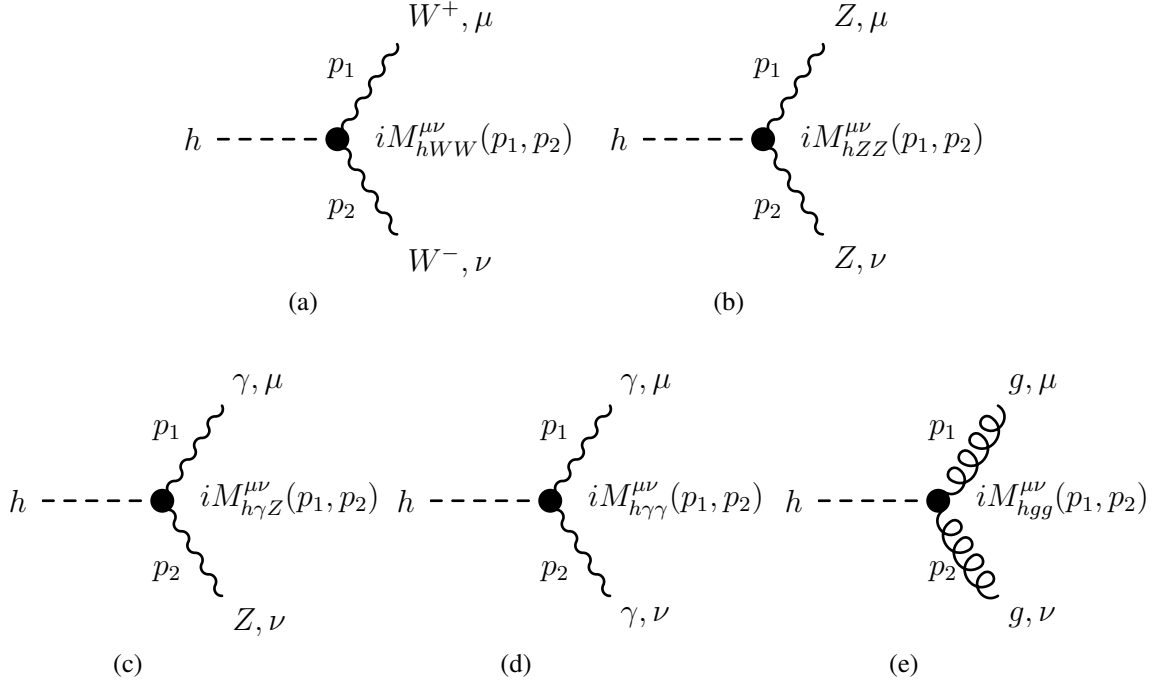


Figure E.2: Feynman rules for three-point vertices.

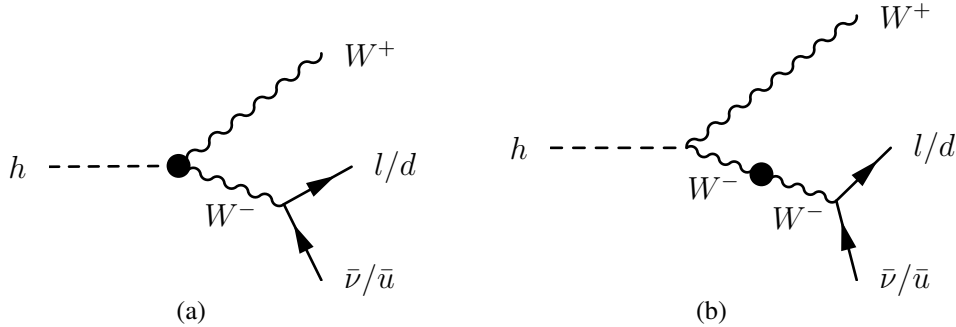

 Figure E.3: New amputated Feynman diagrams for Γ_{hWW^*} .

Fig. E.2(e), and Fig. E.2(c)) multiplied by appropriate polarization vectors

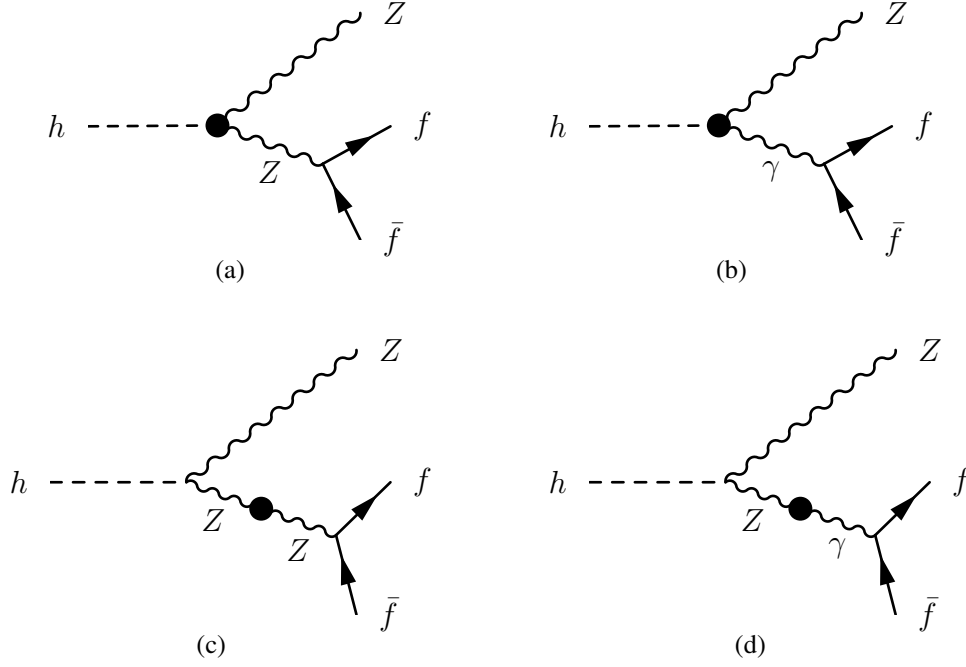
$$iM_{hgg, \text{AD,new}} = iM_{hgg}^{\mu\nu}(p_1, p_2)\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2), \quad (\text{E.25})$$

$$iM_{h\gamma\gamma, \text{AD,new}} = iM_{h\gamma\gamma}^{\mu\nu}(p_1, p_2)\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2), \quad (\text{E.26})$$

$$iM_{h\gamma Z, \text{AD,new}} = iM_{h\gamma Z}^{\mu\nu}(p_1, p_2)\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2). \quad (\text{E.27})$$

The $h \rightarrow WW^*$ and $h \rightarrow ZZ^*$ modes are a little more complicated, because they are at tree level in the SM. It turns out that there are two new amputated diagrams for $h \rightarrow WW^*$ mode as shown in Fig. E.3, and four new amputated diagrams for $h \rightarrow ZZ^*$ mode as shown in Fig. E.4.

It is straightforward to evaluate these relevant new diagrams using the new Feynman rules listed in Section E.1 (together with the SM Feynman rules). One can then compute the interference


 Figure E.4: New amputated Feynman diagrams for Γ_{hZZ^*} .

correction ϵ_I for each decay mode from its definition (Eq. (4.88)). The three-body phase space integrals are analytically manageable, albeit a little bit tedious. The final results of ϵ_I is summarized in Table 4.5, where the auxiliary integrals $I_a(\beta)$, $I_b(\beta)$, $I_c(\beta)$, and $I_d(\beta)$ are defined as

$$I_{\text{SM}}(\beta) \equiv \frac{1}{8\beta^2} [I_2(\beta) + 2(1 - 6\beta^2)I_1(\beta) + (1 - 4\beta^2 + 12\beta^4)I_0(\beta)], \quad (\text{E.28})$$

$$I_a(\beta) \equiv \frac{1}{8\beta^4 I_{\text{SM}}(\beta)} \left[\begin{aligned} &I_3(\beta) + (1 - 16\beta^2)I_2(\beta) + (1 - 12\beta^2 + 62\beta^4)I_1(\beta) \\ &- 4(\beta^2 - 5\beta^4 + 18\beta^6)I_0(\beta) + 2(\beta^4 - 4\beta^6 + 12\beta^8)I_{-1}(\beta) \end{aligned} \right], \quad (\text{E.29})$$

$$I_b(\beta) \equiv \frac{1}{4\beta^2 I_{\text{SM}}(\beta)} \left[\begin{aligned} &-2I_2(\beta) - (4 - 25\beta^2)I_1(\beta) - 2(1 - 5\beta^2 + 18\beta^4)I_0(\beta) \\ &+ \beta^2(1 - 4\beta^2 + 12\beta^4)I_{-1}(\beta) \end{aligned} \right], \quad (\text{E.30})$$

$$I_c(\beta) \equiv \frac{5I_2(\beta) + 2(2 - 3\beta^2)I_1(\beta) - (1 + 2\beta^2)I_0(\beta)}{2\beta^2 I_{\text{SM}}(\beta)}, \quad (\text{E.31})$$

$$I_d(\beta) \equiv \frac{7I_2(\beta) + 8(1 - 3\beta^2)I_1(\beta) + (1 - 4\beta^2 + 12\beta^4)I_0(\beta)}{2\beta^2 I_{\text{SM}}(\beta)}, \quad (\text{E.32})$$

where another set of auxiliary integrals $I_0(\beta)$, $I_1(\beta)$, $I_2(\beta)$, $I_3(\beta)$, $I_{-1}(\beta)$ are defined as follows,

with $\beta \in (\frac{1}{2}, 1)$

$$\begin{aligned}
 I_0(\beta) &\equiv \int_{2\beta-1}^{\beta^2} \frac{dy \sqrt{(y+1)^2 - 4\beta^2}}{y^2} = 1 - \frac{1}{\beta^2} - \ln \beta + \frac{\frac{\pi}{2} - \arcsin \frac{3\beta^2-1}{2\beta^3}}{\sqrt{4\beta^2-1}}, \\
 I_1(\beta) &\equiv \int_{2\beta-1}^{\beta^2} \frac{dy \sqrt{(y+1)^2 - 4\beta^2}}{y^2} y = 1 - \beta^2 - \ln \beta - \frac{\frac{\pi}{2} - \arcsin \frac{3\beta^2-1}{2\beta^3}}{\sqrt{4\beta^2-1}} (4\beta^2 - 1), \\
 I_2(\beta) &\equiv \int_{2\beta-1}^{\beta^2} \frac{dy \sqrt{(y+1)^2 - 4\beta^2}}{y^2} y^2 = \frac{1}{2} (1 - \beta^4) + 2\beta^2 \ln \beta, \\
 I_3(\beta) &\equiv \int_{2\beta-1}^{\beta^2} \frac{dy \sqrt{(y+1)^2 - 4\beta^2}}{y^2} (y^3 + y^2) = \frac{1}{3} (1 - \beta^2)^3, \\
 I_{-1}(\beta) &\equiv \int_{2\beta-1}^{\beta^2} \frac{dy \sqrt{(y+1)^2 - 4\beta^2}}{y^3} = \frac{2\beta^2 \left(\frac{\pi}{2} - \arcsin \frac{3\beta^2-1}{2\beta^3} \right)}{(4\beta^2 - 1)^{\frac{3}{2}}} - \frac{(1 - \beta^2)(3\beta^2 - 1)}{2\beta^4(4\beta^2 - 1)}.
 \end{aligned}$$

The A_{hgg}^{SM} , $A_{h\gamma\gamma}^{\text{SM}}$, and $A_{h\gamma Z}^{\text{SM}}$ in Table 4.5 are the standard form factors

$$A_{hgg}^{\text{SM}} = \sum_Q A_{1/2}(\tau_Q), \quad (\text{E.33})$$

$$A_{h\gamma\gamma}^{\text{SM}} = A_1(\tau_W) + \sum_f N_C Q_f^2 A_{1/2}(\tau_f), \quad (\text{E.34})$$

$$A_{h\gamma Z}^{\text{SM}} = A_1(\tau_W, \lambda_W) + \sum_f N_C \frac{2Q_f}{c_Z} (T_f^3 - 2s_Z^2 Q_f) A_{1/2}(\tau_f, \lambda_f), \quad (\text{E.35})$$

with $\tau_i \equiv \frac{4m_i^2}{m_h^2}$, $\lambda_i \equiv \frac{4m_i^2}{m_Z^2}$, and $A_{1/2}(\tau)$, $A_1(\tau)$, $A_{1/2}(\tau, \lambda)$, $A_1(\tau, \lambda)$ being the conventional form factors (for example see [156])

$$A_{1/2}(\tau) = 2\tau^{-2} \left[\tau + (\tau - 1) f(\tau) \right], \quad (\text{E.36})$$

$$A_1(\tau) = -\tau^{-2} \left[2\tau^2 + 3\tau + 3(2\tau - 1) f(\tau) \right], \quad (\text{E.37})$$

$$A_{1/2}(\tau, \lambda) = B_1(\tau, \lambda) - B_2(\tau, \lambda), \quad (\text{E.38})$$

$$A_1(\tau, \lambda) = c_Z \left\{ 4 \left(3 - \frac{s_Z^2}{c_Z^2} \right) B_2(\tau, \lambda) + \left[\left(1 + \frac{2}{\tau} \right) \frac{s_Z^2}{c_Z^2} - \left(5 + \frac{2}{\tau} \right) \right] B_1(\tau, \lambda) \right\}, \quad (\text{E.39})$$

with

$$B_1(\tau, \lambda) \equiv \frac{\tau\lambda}{2(\tau - \lambda)} + \frac{\tau^2\lambda^2}{2(\tau - \lambda)^2} \left[f\left(\frac{1}{\tau}\right) - f\left(\frac{1}{\lambda}\right) \right] + \frac{\tau^2\lambda}{(\tau - \lambda)^2} \left[g\left(\frac{1}{\tau}\right) - g\left(\frac{1}{\lambda}\right) \right],$$

$$B_2(\tau, \lambda) \equiv -\frac{\tau\lambda}{2(\tau - \lambda)} \left[f\left(\frac{1}{\tau}\right) - f\left(\frac{1}{\lambda}\right) \right],$$

and

$$f(\tau) = \begin{cases} \arcsin^2 \sqrt{\tau} & \tau \leq 1 \\ -\frac{1}{4} \left[\log \frac{1 + \sqrt{1 - \tau^{-1}}}{1 - \sqrt{1 - \tau^{-1}}} - i\pi \right]^2 & \tau > 1 \end{cases}, \quad (\text{E.40})$$

$$g(\tau) = \begin{cases} \sqrt{\tau^{-1} - 1} \arcsin \sqrt{\tau} & \tau \leq 1 \\ \frac{\sqrt{1 - \tau^{-1}}}{2} \left[\log \frac{1 + \sqrt{1 - \tau^{-1}}}{1 - \sqrt{1 - \tau^{-1}}} - i\pi \right] & \tau > 1 \end{cases}. \quad (\text{E.41})$$

E.3 Details on Interference Corrections to Higgs Production Cross Section

The ggF Higgs production mode is just the time reversal of the $h \rightarrow gg$ decay. Again as it is already at one-loop order in the SM, the only new amputated diagram up to leading order in Wilson coefficients is given by the new three-point vertex $iM_{hgg}^{\mu\nu}(p_1, p_2)$ (Fig. E.2(e)) multiplied by the polarization vectors

$$iM_{ggF, \text{AD, new}} = iM_{hgg}^{\mu\nu}(p_1, p_2)\epsilon_\mu(p_1)\epsilon_\nu(p_2). \quad (\text{E.42})$$

Obviously, the interference correction to ggF production cross section is the same as that to $h \rightarrow gg$ decay width

$$\epsilon_{ggF, I} = \epsilon_{hgg, I} = \frac{(4\pi)^2}{\text{Re}(A_{hgg}^{\text{SM}})} \frac{16v^2}{\Lambda^2} c_{GG}. \quad (\text{E.43})$$

The vector boson fusion production mode σ_{WW_h} has three new amputated diagrams as shown in Fig. E.7 (in which one of the fermion lines can be inverted to take account of production mode in lepton colliders such as the ILC). For the vector boson associate production modes, there are two new diagrams for σ_{Wh} (Fig. E.5) and four for σ_{Zh} (Fig. E.6).

Again from the definition (Eq. (4.88)), we can compute the interference correction ϵ_I for each Higgs production mode. The final results are summarized in Table 4.7. For σ_{Wh} and σ_{Zh} , the final states phase space integral is only two-body and quite simple. On the other hand, σ_{WW_h} requires to integrate over a three-body phase space, which turns out to be quite involved. The analytical result $\epsilon_{WW_h, I}(s)$ is several pages long and hence would not be that useful. Instead, numerical results of it are provided in Table 4.7, where three auxiliary functions $f_a(s)$, $f_b(s)$, and $f_c(s)$ are defined. Numerical results of these auxiliary functions are shown in Fig. 4.2.

To show the definition of $f_a(s)$, $f_b(s)$, and $f_c(s)$, we need to describe the three-body phase space integral of σ_{WW_h} . Let us take the center of mass frame of the colliding fermions and setup the spherical coordinates with the positive z -axis being the direction of $\vec{p}_a$. Then the various

momenta labeled in Fig. E.7 can be expressed as

$$p_a = \frac{\sqrt{s}}{2}(1, 0, 0, 1), \quad (\text{E.44})$$

$$p_b = \frac{\sqrt{s}}{2}(1, 0, 0, -1), \quad (\text{E.45})$$

$$p_3 = \frac{\sqrt{s}}{2}x_3(1, s_3, 0, c_3), \quad (\text{E.46})$$

$$p_4 = \frac{\sqrt{s}}{2}x_4(1, s_4 \cos \phi, s_4 \sin \phi, c_4). \quad (\text{E.47})$$

where we have defined $x_3 \equiv \frac{2E_3}{\sqrt{s}}$, $x_4 \equiv \frac{2E_4}{\sqrt{s}}$, and adopted the notation $c_3 \equiv \cos \theta_3$ etc. Due to the axial symmetry around the z -axis, we have also taken the parametrization $\phi_3 = 0$ and $\phi_4 = \phi$ without loss of generality. For further convenience, let us also define $\eta_h \equiv \frac{m_h}{\sqrt{s}}$, $\eta_W \equiv \frac{m_W}{\sqrt{s}}$, and $\alpha_\phi \equiv \frac{1}{2}(1 - c_3 c_4 - s_3 s_4 \cos \phi)$. The three-body phase space has nine variables to integrate over. But the axial symmetry and the δ -function of 4-momentum make five of them trivial, leaving us with four nontrivial ones, which we choose to be x_3 , c_3 , c_4 , and ϕ . Sometimes, we will still use the quantity x_4 to make the expression short, but it has been fixed by the energy δ -function and should be understood as a function of the other four

$$x_4(x_3, c_3, c_4, \phi) = \frac{1 - \eta_h^2 - x_3}{1 - \alpha_\phi x_3}. \quad (\text{E.48})$$

Now the phase space integral can be written as

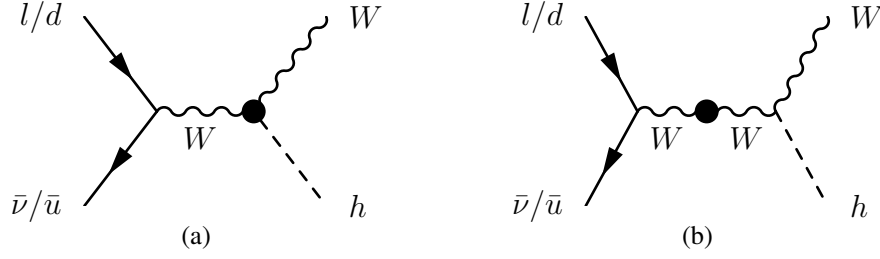
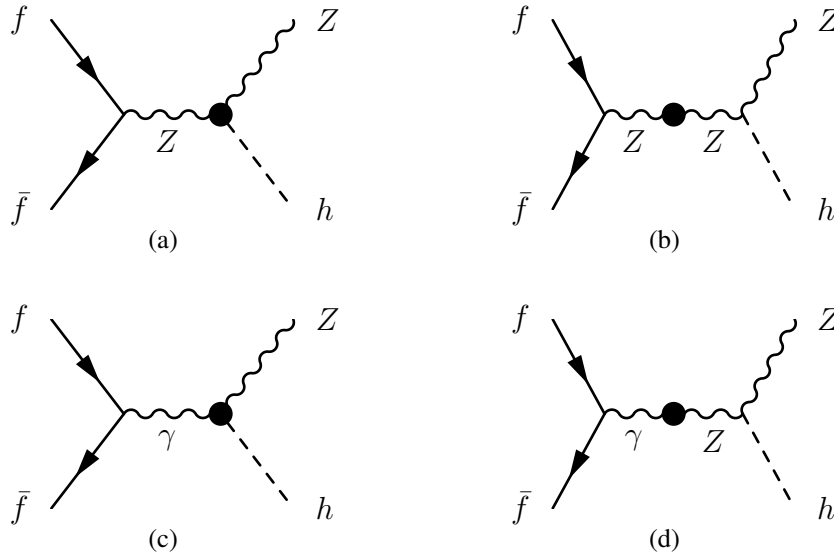
$$\begin{aligned} \frac{1}{2s} \int d\Pi_3(1, 3, 4) &= \frac{1}{2s} \int \frac{d^3\vec{p}_3}{(2\pi)^3} \frac{1}{2E_3} \frac{d^3\vec{p}_4}{(2\pi)^3} \frac{1}{2E_4} \frac{d^3\vec{p}_1}{(2\pi)^3} \frac{1}{2E_1} (2\pi)^4 \delta^4(p_1 + p_3 + p_4 - p) \\ &= \frac{1}{2048\pi^4} \int_0^{1-\eta_h^2} dx_3 \int_{-1}^1 dc_3 dc_4 \int_0^{2\pi} d\phi \frac{(1 - \eta_h^2 - x_3)x_3}{(1 - \alpha_\phi x_3)^2}. \end{aligned} \quad (\text{E.49})$$

The modulus square of the SM invariant amplitude is

$$\begin{aligned} \overline{|M_{WW h, SM}|^2} &= \left(\frac{g}{\sqrt{2}}\right)^4 \frac{2m_W^4}{v^2} \frac{g^{\mu\nu} g^{\alpha\beta} \frac{1}{4} \text{tr}(\not{p}_a \gamma_\alpha \not{p}_3 \gamma_\mu P_L) \text{tr}(\not{p}_b \gamma_\nu \not{p}_4 \gamma_\beta P_L)}{(k_1^2 - m_W^2)^2 (k_2^2 - m_W^2)^2} \\ &= \frac{m_W^4}{v^6} 2\eta_W^4 \frac{4x_3 x_4 (1 + c_3)(1 - c_4)}{[x_3(1 - c_3) + 2\eta_W^2]^2 [x_4(1 + c_4) + 2\eta_W^2]^2}. \end{aligned} \quad (\text{E.50})$$

Now we are about ready to show the definition of $f_a(s)$, $f_b(s)$, and $f_c(s)$. Let us introduce an “average” definition of A as

$$\langle A \rangle \equiv \frac{\frac{1}{2s} \int d\Pi_3(1, 3, 4) \overline{|M_{WW h, SM}|^2} A}{\frac{1}{2s} \int d\Pi_3(1, 3, 4) \overline{|M_{WW h, SM}|^2}}. \quad (\text{E.51})$$


 Figure E.5: New amputated Feynman diagrams for σ_{Wh} .

 Figure E.6: New amputated Feynman diagrams for σ_{Zh} .

Then $f_a(s)$, $f_b(s)$, and $f_c(s)$ are defined as

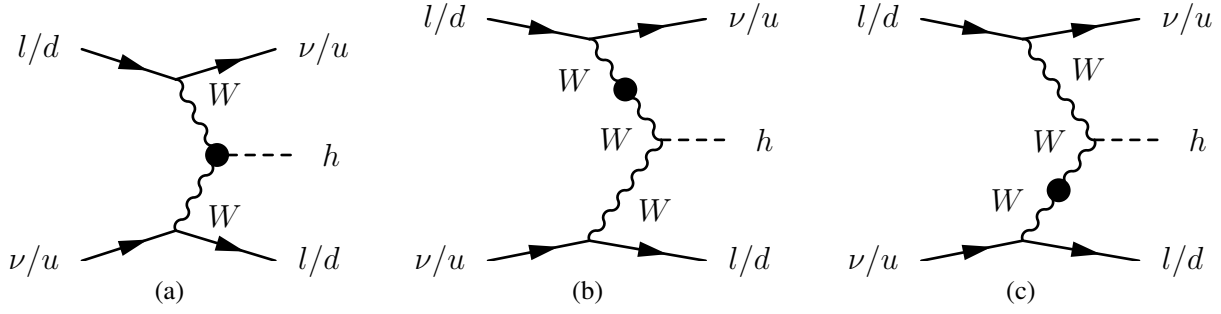
$$f_a(s) \equiv \left\langle \frac{(k_1 k_2 g^{\mu\nu} - k_1^\nu k_2^\mu) g^{\alpha\beta} \frac{1}{4} \text{tr}(\not{p}_a \gamma_\alpha \not{p}_3 \gamma_\mu P_L) \text{tr}(\not{p}_b \gamma_\nu \not{p}_4 \gamma_\beta P_L) + c.c.}{m_W^2 g^{\mu\nu} g^{\alpha\beta} \frac{1}{4} \text{tr}(\not{p}_a \gamma_\alpha \not{p}_3 \gamma_\mu P_L) \text{tr}(\not{p}_b \gamma_\nu \not{p}_4 \gamma_\beta P_L)} \right\rangle$$

$$= \left\langle -\frac{1}{2\eta_W^2} \left(\frac{x_4}{1+c_3} + \frac{x_3}{1-c_4} \right) s_3 s_4 \cos \phi \right\rangle, \quad (\text{E.52})$$

$$f_b(s) \equiv \left\langle \frac{k_1^2 + k_2^2}{m_W^2} \right\rangle = \left\langle -\frac{1}{2\eta_W^2} [x_3(1-c_3) + x_4(1+c_4)] \right\rangle, \quad (\text{E.53})$$

$$f_c(s) \equiv \left\langle \frac{k_1^2}{k_1^2 - m_W^2} + \frac{k_2^2}{k_2^2 - m_W^2} \right\rangle = \left\langle \frac{x_3(1-c_3)}{x_3(1-c_3) + 2\eta_W^2} + \frac{x_4(1+c_4)}{x_4(1+c_4) + 2\eta_W^2} \right\rangle, \quad (\text{E.54})$$

where various momenta are as labeled in Fig. E.7, and $P_L = \frac{1-\gamma^5}{2}$, with the γ matrices defined as usual.


 Figure E.7: New amputated Feynman diagrams for σ_{WWh} .

$\Delta r_h = -\frac{v^2}{\Lambda^2} (2c_H + c_R) - \frac{2m_h^2}{\Lambda^2} c_D$ $\Delta r_Z = \frac{2m_Z^2}{\Lambda^2} \left[-c_Z^2 c_{2W} - s_Z^2 c_{2B} + 4(c_Z^4 c_{WW} + s_Z^4 c_{BB} + c_Z^2 s_Z^2 c_{WB}) + c_Z^2 c_W + s_Z^2 c_B \right]$ $\Delta r_W = \frac{2m_W^2}{\Lambda^2} (-c_{2W} + 4c_{WW} + c_W)$

 Table E.3: Residue modifications Δr in terms of Wilson coefficients.

E.4 Calculation of Residue Modifications

The mass pole residue modification Δr_k of each external leg k can be computed using the corresponding vacuum polarization function. Throughout Section 4.3, the relevant mass pole residue modifications are

$$\Delta r_h = \left. \frac{d\Sigma(p^2)}{dp^2} \right|_{p^2=m_h^2}, \quad (\text{E.55})$$

$$\Delta r_W = \left. \frac{d\Pi_{WW}(p^2)}{dp^2} \right|_{p^2=m_W^2}, \quad (\text{E.56})$$

$$\Delta r_Z = \left. \frac{d\Pi_{ZZ}(p^2)}{dp^2} \right|_{p^2=m_Z^2}, \quad (\text{E.57})$$

where $-i\Sigma(p^2)$ denotes the vacuum polarization function of the physical Higgs field h . With all the vacuum polarization functions listed in Table E.1, it is straightforward to calculate Δr . The results are summarized in Table E.3.

E.5 Calculation of Lagrangian Parameter Modifications

The set of Lagrangian parameters relevant for us are $\{\rho\} = \{g^2, v^2, s_Z^2, y_f^2\}$. We would like to compute them in terms of the physical observables and the Wilson coefficients $\rho = \rho(obs, c_i)$, where the set of observables relevant to us can be taken as $\{obs\} = \{\hat{\alpha}, \hat{G}_F, \hat{m}_Z^2, \hat{m}_f^2\}$. A hat is

put on the quantities to denote that it is a physical observable measured from the experiments. On the other hand, for notation convenience, let us also define the following auxiliary Lagrangian parameters that are related to the basic ones $\{\rho\} = \{g^2, v^2, s_Z^2, y_f^2\}$:

$$m_W^2 \equiv \frac{1}{2}g^2v^2, \quad (\text{E.58})$$

$$m_Z^2 \equiv \frac{1}{2}g^2v^2 \frac{1}{1 - s_Z^2}. \quad (\text{E.59})$$

These auxiliary Lagrangian parameters are not hatted.

As explained in Section 4.3, in order to obtain $\rho = \rho(obs, c_i)$, we first need to compute the function $obs = obs(\rho, c_i)$, which up to linear order in c_i are

$$\hat{\alpha} = \frac{g^2 s_Z^2}{4\pi} \frac{p^2}{p^2 - \Pi_{\gamma\gamma}(p^2)} \Big|_{p^2 \rightarrow 0} = \frac{g^2 s_Z^2}{4\pi} [1 + \Pi'_{\gamma\gamma}(0)], \quad (\text{E.60})$$

$$\hat{G}_F = \frac{\sqrt{2}g^2}{8} \frac{-1}{p^2 - m_W^2 - \Pi_{WW}(p^2)} \Big|_{p^2=0} = \frac{1}{2\sqrt{2}v^2} \left[1 - \frac{1}{m_W^2} \Pi_{WW}(0) \right], \quad (\text{E.61})$$

$$\hat{m}_Z^2 = m_Z^2 + \Pi_{ZZ}(m_Z^2) = \frac{1}{2}g^2v^2 \frac{1}{1 - s_Z^2} \left[1 + \frac{1}{m_Z^2} \Pi_{ZZ}(m_Z^2) \right], \quad (\text{E.62})$$

$$\hat{m}_f^2 = y_f^2 v^2. \quad (\text{E.63})$$

Note that the vacuum polarization functions are linear in c_i and hence only kept up to first order. Next we need to take the inverse of these to get the function $\rho = \rho(obs, c_i)$. Again, because the vacuum polarization functions are already linear in c_i , one can neglect the modification of the Lagrangian parameters multiplying them when taking the inverse at the leading order. This gives

$$g^2 s_Z^2 = 4\pi \hat{\alpha} [1 - \Pi'_{\gamma\gamma}(0)], \quad (\text{E.64})$$

$$v^2 = \frac{1}{2\sqrt{2}\hat{G}_F} \left[1 - \frac{1}{m_W^2} \Pi_{WW}(0) \right], \quad (\text{E.65})$$

$$s_Z^2(1 - s_Z^2) = \frac{\pi \hat{\alpha}}{\sqrt{2}\hat{G}_F \hat{m}_Z^2} [1 - \Pi'_{\gamma\gamma}(0)] \left[1 - \frac{1}{m_W^2} \Pi_{WW}(0) \right] \left[1 + \frac{1}{m_Z^2} \Pi_{ZZ}(m_Z^2) \right], \quad (\text{E.66})$$

$$y_f^2 = 2\sqrt{2}\hat{G}_F \hat{m}_f^2 \left[1 + \frac{1}{m_W^2} \Pi_{WW}(0) \right]. \quad (\text{E.67})$$

Then taking log and derivative on both sides, we obtain

$$\Delta w_{g^2} + \Delta w_{s_Z^2} = -\Pi'_{\gamma\gamma}(0), \quad (\text{E.68})$$

$$\Delta w_{v^2} = -\frac{1}{m_W^2} \Pi_{WW}(0), \quad (\text{E.69})$$

$$\frac{c_Z^2 - s_Z^2}{c_Z^2} \Delta w_{s_Z^2} = -\Pi'_{\gamma\gamma}(0) - \frac{1}{m_W^2} \Pi_{WW}(0) + \frac{1}{m_Z^2} \Pi_{ZZ}(m_Z^2), \quad (\text{E.70})$$

$$\Delta w_{y_f^2} = \frac{1}{m_W^2} \Pi_{WW}(0), \quad (\text{E.71})$$

$$\begin{aligned}
 \Delta w_{g^2} &= \frac{m_W^2}{\Lambda^2} \frac{1}{c_Z^2 - s_Z^2} \left\{ (c_Z^2 c_{2W} + s_Z^2 c_{2B}) - 8 \left[(c_Z^2 - s_Z^2) c_{WW} + s_Z^2 c_{WB} \right] \right. \\
 &\quad \left. - 2(c_Z^2 c_W + s_Z^2 c_B) \right\} + \frac{c_Z^2}{c_Z^2 - s_Z^2} \frac{2v^2}{\Lambda^2} c_T \\
 \Delta w_{v^2} &= -\frac{v^2}{\Lambda^2} c_R \\
 \Delta w_{s_Z^2} &= \frac{m_W^2}{\Lambda^2} \frac{1}{c_Z^2 - s_Z^2} \left\{ - (c_Z^2 c_{2W} + s_Z^2 c_{2B}) \right. \\
 &\quad \left. + 8 \left[(c_Z^2 - s_Z^2) (c_Z^2 c_{WW} - s_Z^2 c_{BB}) + 2c_Z^2 s_Z^2 c_{WB} \right] \right. \\
 &\quad \left. + 2(c_Z^2 c_W + s_Z^2 c_B) \right\} - \frac{c_Z^2}{c_Z^2 - s_Z^2} \frac{2v^2}{\Lambda^2} c_T \\
 \Delta w_{y_f^2} &= \frac{v^2}{\Lambda^2} c_R
 \end{aligned}$$

 Table E.4: Parameter modifications Δw_ρ in terms of Wilson coefficients.

which leads us to the final results

$$\Delta w_{g^2} = -\Pi'_{\gamma\gamma}(0) - \frac{c_Z^2}{c_Z^2 - s_Z^2} \left[-\Pi'_{\gamma\gamma}(0) - \frac{1}{m_W^2} \Pi_{WW}(0) + \frac{1}{m_Z^2} \Pi_{ZZ}(m_Z^2) \right], \quad (\text{E.72})$$

$$\Delta w_{v^2} = -\frac{1}{m_W^2} \Pi_{WW}(0), \quad (\text{E.73})$$

$$\Delta w_{s_Z^2} = \frac{c_Z^2}{c_Z^2 - s_Z^2} \left[-\Pi'_{\gamma\gamma}(0) - \frac{1}{m_W^2} \Pi_{WW}(0) + \frac{1}{m_Z^2} \Pi_{ZZ}(m_Z^2) \right], \quad (\text{E.74})$$

$$\Delta w_{y_f^2} = \frac{1}{m_W^2} \Pi_{WW}(0). \quad (\text{E.75})$$

Plugging in the vacuum polarization functions listed in Table E.1, one can get the Lagrangian parameter modifications Δw_ρ summarized in Table E.4.

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