Reversibility of whole-plane SLE for $\kappa > 8$

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Abstract

Whole-plane Schramm-Loewner evolution (SLE_κ) is a random fractal curve between two points on the Riemann sphere. Zhan established for $\kappa \le 4$ that whole-plane SLE_κ is *reversible*, meaning invariant in law under conformal automorphisms swapping its endpoints. Miller and Sheffield extended this to $\kappa \le 8$. We prove whole-plane SLE_κ is reversible for $\kappa > 8$, resolving the final case and answering a conjecture of Viklund and Wang. Our argument depends on a novel mating-of-trees theorem of independent interest, where Liouville quantum gravity on the disk is decorated by an independent radial space-filling SLE curve.

Keywords Schramm-Loewner evolution, reversibility, Liouville quantum gravity, mating-of-trees
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1 Introduction

In the past two decades, Schramm-Loewner evolution (SLE) has emerged as a central object of study in probability theory. SLE is a random fractal curve in the plane [41, 39] describing the scaling limits of many statistical physics models at criticality [47, 29, 42, 48]. It has a parameter $\kappa > 0$: when $\kappa \in (0, 4]$ SLE is a simple curve, when $\kappa \in (4, 8)$ SLE is self-intersecting but not self-crossing, and when $\kappa \ge 8$ SLE is space-filling. See for instance [27, 10] for expository works on SLE.

For context, we first discuss chordal SLE, a random curve in a simply connected domain $D \subset \mathbb{C}$ from a boundary point x to another boundary point y. We say a random curve from x to y is *reversible* if it is invariant in law under conformal automorphisms of D switching x and y. More precisely, fixing such a conformal automorphism f, if η is a curve from x to y and $\widetilde{\eta}$ is the time-reversal of $f \circ \eta$, then reversibility means η and $\widetilde{\eta}$ agree in law up to monotone reparametrization of time.

The problem of SLE reversibility dates back to the very foundation of the subject. Schramm's definition of SLE was entirely motivated by the study of scaling limits of lattice models at criticality [41]: assuming a domain Markov property inherited from discrete models and the ansatz of conformal invariance, he deduced a stochastic differential equation encoding the growth of SLE. Inherent in his definition is a time-asymmetry where the starting and ending points of the curve are not interchangeable. On the other hand, many lattice models expected to converge to chordal SLE satisfy endpoint symmetry. In this way, the question of reversibility reflects a fundamental tension between the construction of SLE and its initial motivation.

The conjecture that chordal SLE is reversible for $\kappa \in (0, 8]$ was first recorded in [39]; at the time of that conjecture, reversibility was already known for $\kappa \in \{2, 8/3, 6, 8\}$ via scaling limits of lattice models. Reversibility of chordal SLE was proved by Zhan for $\kappa \in (0, 4]$ [52] and by Miller and Sheffield for $\kappa \in (4, 8)$ [33]. On the other hand, for $\kappa > 8$ chordal SLE is not reversible [39, 51].

We now turn to whole-plane SLE_κ, a random curve in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ from 0 to ∞. A random curve from 0 to ∞ is *reversible* if it is invariant in law under conformal automorphisms of $\hat{\mathbb{C}}$ switching 0 and ∞. Zhan proved that whole-plane SLE_κ is reversible for $\kappa \le 4$ [53], and Miller and Sheffield proved reversibility for $\kappa \in (4, 8]$ [35]. We resolve the final case of $\kappa > 8$.

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Theorem 1.1. *Whole-plane* SLE_κ *is reversible when* $\kappa > 8$.

Theorem 1.1 is surprising not only because of non-reversibility of chordal SLE_κ for $\kappa > 8$ and non-reversibility of a variant called whole-plane $\mathrm{SLE}_\kappa(\rho)$ for $\kappa > 8$ and $\rho > \frac{\kappa}{2} - 4$ [35, Remark 1.21], but also because it reveals a fundamental property of SLE not apparent through the lens of *imaginary geometry*. The imaginary geometry framework [31, 32, 35, 36] introduced by Miller and Sheffield studies SLE by coupling it with a Gaussian free field, and has proven an essential tool with wide-ranging applications such as [20, 14, 23]. The reversibility of chordal and whole-plane SLE_κ for $\kappa \le 8$ can be shown by imaginary geometry [33, 35] (in fact, for $\kappa \in (4, 8)$, this is the only known approach). However, the reversibility of whole-plane SLE_κ for $\kappa > 8$ seems unnatural from the perspective of imaginary geometry since the left and right boundaries of the curve interact in a complicated way [35, Remark 1.22]. The reversibility for whole-plane $\mathrm{SLE}_\kappa(\rho)$ with $\kappa > 8$ and $\rho \in (-2, \frac{\kappa}{2} - 4]\backslash\{0\}$ remains an open problem.

To our knowledge, apart from the illuminating work of Viklund and Wang [49], there had been no reason to expect the reversibility of whole plane SLE_κ for $\kappa > 8$. They proved the inversion invariance of the $\kappa \to \infty$ large deviation rate function of whole-plane SLE_κ, and consequently conjectured the reversibility of whole-plane SLE_κ for large κ. Theorem 1.1 confirms their conjecture.

Our arguments are substantially different from those of Zhan for $\kappa \le 4$, who applied commutation relations for SLE [52, 53], and Miller and Sheffield for $\kappa \le 8$, who used imaginary geometry. Rather, we employ the *mating-of-trees* approach [14] where a random planar surface called *Liouville quantum gravity (LQG)* is coupled with an independent SLE curve. All previously known mating-of-trees theorems [14, 36, 2] involved either chordal SLE or an SLE loop in $\hat{\mathbb{C}}$ or $\mathbb{D}$. We establish a mating-of-trees theorem for LQG on the disk coupled with radial SLE, and for LQG on $\hat{\mathbb{C}}$ coupled with whole-plane SLE, resolving another conjecture of [49]. These novel mating-of-trees theorems are noteworthy in their own right; see for instance the survey [18] for some applications of the mating-of-trees framework.

The starting point of the original mating-of-trees theorem is the *quantum zipper* coupling of reverse SLE with a certain LQG surface, from which "zooming in" on the base of the curve gives in the limit a forward SLE trace on a *scale-invariant* LQG surface [14]. All subsequent mating-of-trees theorems were derived from the original by limiting arguments. However, our radial setting is not scale-invariant, nor can it be derived from a scale-invariant picture. Our proof depends on two crucial insights. Firstly, as shown by the first author [1], the quantum zipper describes dynamics on LQG surfaces arising in *Liouville conformal field theory (LCFT)* [11, 22]. The LCFT perspective allows us to use the quantum zipper without zooming in on a boundary point, giving us access to non-scale-invariant LQG surfaces. See [3, 5, 7, 4, 6, 8] for other works that explore the interplay between LCFT and SLE. Secondly, to pass from reverse SLE to forward SLE, we work with the infinite measure $\int_0^\infty \mathrm{raSLE}_\kappa^t \, dt$ corresponding to "radial SLE run until a Lebesgue-typical capacity time". This allows us to exploit the fixed-time symmetry of forward and reverse radial SLE without fixing a capacity time, which is important since capacity time is unnatural for the quantum zipper.

To prove Theorem 1.1, we first derive a radial mating-of-trees theorem (Theorem 3.1) by building on the LCFT dynamics of [1]. Next, using a limiting argument pinching a disk into a sphere, we obtain a whole-plane mating-of-trees theorem (Theorem 4.1) identifying a two-pointed LQG sphere decorated by an independent whole-plane SLE curve with a 2D Brownian excursion. By the time-reversal symmetry of Brownian motion, the decorated quantum surface is invariant in law when the two points are interchanged and the curve is reversed. We conclude that whole-plane SLE is reversible. See Figure 7 for a proof summary. Our use of mating-of-trees to prove SLE reversibility is parallel to the arguments of [49] where a "mating-of-trees energy duality" is used to establish inversion invariance of the SLE large deviation functional as κ tends to infinity.

Outline. Section 2 gives preliminary background on LQG, Liouville conformal field theory, SLE, and mating-of-trees. In Section 3 we prove a radial mating-of-trees result (Theorem 3.1). In Section 4 we take a limit to obtain a whole-plane mating-of-trees (Theorem 4.1), then use it to prove Theorem 1.1. We mention related results in the literature and list some open questions in Section 5.

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2 Preliminaries

In this paper we work with non-probability measures and extend the terminology of ordinary probability to this setting. For a finite or σ-finite measure space $(\Omega, \mathcal{F}, M)$, we say X is a random variable if X is an $\mathcal{F}$-measurable function with its *law* defined via the push-forward measure $M_X = X_* M$. In this case, we say X is *sampled* from M_X and write $M_X[f]$ for $\int f(x) M_X(dx)$. *Weighting* the law of X by $f(X)$ corresponds to working with the measure $d\widetilde{M}_X$ with Radon-Nikodym derivative $\frac{d\widetilde{M}_X}{dM_X} = f$. *Conditioning* on some event $E \in \mathcal{F}$ (with $0 < M[E] < \infty$) refers to the probability measure $\frac{M[E \cap \cdot]}{M[E]}$ on the measurable space $(E, \mathcal{F}_E)$ with $\mathcal{F}_E = \{A \cap E : A \in \mathcal{F}\}$, while *restricting* to E refers to the measure $M[E \cap \cdot]$.

2.1 The Gaussian Free Field and Liouville quantum gravity

Let $m_{\mathbb{D}}$ (resp. $m_{\mathbb{H}}$) be the uniform measure on the unit circle $\partial\mathbb{D}$ (resp. half circle $\mathbb{H} \cap \partial\mathbb{D}$). For $X \in \{\mathbb{D}, \mathbb{H}\}$, define the Dirichlet inner product $\langle f, g \rangle_\nabla = (2\pi)^{-1} \int_X \nabla f \cdot \nabla g$ on the space $\{f \in C^\infty(X) : \int_X |\nabla f|^2 < \infty;\ \int f(z) m_X(dz) = 0\}$, and let $H(X)$ be the closure of this space w.r.t. the inner product $\langle f, g \rangle_\nabla$. Let $(f_n)_{n \geq 1}$ be an orthonormal basis of $H(X)$, and $(\alpha_n)_{n \geq 1}$ be a collection of independent standard Gaussian variables. Then the summation

$$h_X = \sum_{n=1}^{\infty} \alpha_n f_n$$

a.s. converges in the space of distributions on X, and h_X is the *Gaussian free field (GFF)* on X normalized such that $\int h_X(z) m_X(dz) = 0$. We denote its law by P_X. See [14, Section 4.1.4] for more details.

Let $|z|_+ = \max\{|z|, 1\}$. For $z, w \in \bar{\mathbb{H}}$, we define

$$G_{\mathbb{H}}(z, w) = -\log|z - w| - \log|z - \bar{w}| + 2\log|z|_+ + 2\log|w|_+; \quad G_{\mathbb{H}}(z, \infty) = 2\log|z|_+.$$

Similarly, for $z, w \in \bar{\mathbb{D}}$, set

$$G_{\mathbb{D}}(z, w) = -\log|z - w| - \log|1 - z\bar{w}|.$$

Then the GFF h_X is the centered Gaussian field on X with covariance structure $\mathbb{E}[h_X(z) h_X(w)] = G_X(z, w)$.

Now let $\gamma \in (0, 2)$ and $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$. For a conformal map $g : D \to \widetilde{D}$ and a generalized function h on D, define the generalized function $g \bullet_\gamma h$ on $\widetilde{D}$ by setting

$$g \bullet_\gamma h := h \circ g^{-1} + Q \log|(g^{-1})'| \tag{2.1}$$

A quantum surface is a $\sim_\gamma$-equivalence class of pairs (D, h) where $(D, h) \sim_\gamma (\widetilde{D}, \widetilde{h})$ if there is a conformal map g with $\widetilde{h} = g \bullet_\gamma h$. We call a representative (D, h) an *embedding* of the quantum surface. We will also consider quantum surfaces decorated by points and a curve; in this case we say $(D, h, \eta, (z_i)) \sim_\gamma (\widetilde{D}, \widetilde{h}, \widetilde{\eta}, (\widetilde{z}_i))$ if there is a conformal map $g : D \to \widetilde{D}$ such that $g \bullet_\gamma h = \widetilde{h}$, $g \circ \eta = \widetilde{\eta}$, and $g(z_i) = \widetilde{z}_i$ for all i. As before we call a representative $(D, h, \eta, (z_i))$ an embedding of the decorated quantum surface.

For a γ-quantum surface (D, h), its *quantum area measure* $\mathcal{A}_h(dz)$ is defined by taking the weak limit as $\varepsilon \to 0$ of $\mathcal{A}_{h_\varepsilon}(dz) := \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\varepsilon(z)} dz$, where $h_\varepsilon(z)$ is the circle average of h over $\partial B(z, \varepsilon)$. When $D = \mathbb{H}$, we can also define the *quantum boundary length measure* $\mathcal{L}_h(dx) := \lim_{\varepsilon \to 0} \varepsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} h_\varepsilon(x)} dx$ where $h_\varepsilon(x)$ is the average of h over the semicircle $\{x + \varepsilon e^{i\theta} : \theta \in (0, \pi)\}$. It has been shown in [15, 45] that all these weak limits are well-defined for the GFF and its variants we are considering in this paper, and if f is a conformal automorphism of $\mathbb{H}$ then $f_* \mathcal{A}_h = \mathcal{A}_{f \bullet_\gamma h}$ and $f_* \mathcal{L}_h = \mathcal{L}_{f \bullet_\gamma h}$. This latter point allows us to define $\mathcal{A}_h$ and $\mathcal{L}_h$ on other domains by conformally mapping to $\mathbb{H}$.

2.2 The Liouville field

Recall that $P_{\mathbb{D}}$ (resp. $P_{\mathbb{H}}$) is the law of the free boundary GFF on $\mathbb{D}$ (resp. $\mathbb{H}$) normalized to have average zero on $\partial\mathbb{D}$ (resp. $\partial\mathbb{D} \cap \mathbb{H}$). In the following definitions we use the shorthand $|z|_+ = \max\{|z|, 1\}$ for $z \in \mathbb{C}$.

Definition 2.1. *Let $(h, \mathbf{c})$ be sampled from $P_{\mathbb{D}} \times [e^{-Qc} dc]$ and $\phi = h + \mathbf{c}$. We call ϕ the Liouville field on $\mathbb{D}$, and we write $\mathrm{LF}_{\mathbb{D}}$ for the law of ϕ.*

Definition 2.2. *Let* $(h, \mathbf{c})$ *be sampled from* $P_{\mathbb{H}} \times [e^{-Qc}dc]$ *and* $\phi = h - 2Q\log|z|_+ + \mathbf{c}$. *We call* ϕ *the Liouville field on* $\mathbb{H}$, *and we write* $\mathrm{LF}_{\mathbb{H}}$ *for the law of* ϕ.

Definition 2.3. *Let* $(\alpha, w) \in \mathbb{R} \times \mathbb{H}$ *and* $(\beta, s) \in \mathbb{R} \times \partial\mathbb{H}$. *Let*

$$C_{\mathbb{H}}^{(\alpha,w),(\beta,s)} = (2\operatorname{Im} w)^{-\frac{\alpha^2}{2}} |w|_+^{-2\alpha(Q-\alpha)} |s|_+^{-\beta(Q-\frac{\beta}{2})} e^{\frac{\alpha\beta}{2}G_{\mathbb{H}}(w,s)}.$$

Let $(h, \mathbf{c})$ *be sampled from* $C_{\mathbb{H}}^{(\alpha,w),(\beta,s)} P_{\mathbb{H}} \times [e^{(\alpha+\frac{\beta}{2}-Q)c}dc]$, *and*

$$\phi(z) = h(z) - 2Q\log|z|_+ + \alpha G_{\mathbb{H}}(z,w) + \frac{\beta}{2}G_{\mathbb{H}}(z,s) + \mathbf{c}.$$

We write $\mathrm{LF}_{\mathbb{H}}^{(\alpha,w),(\beta,s)}$ *for the law of* ϕ *and call a sample from* $\mathrm{LF}_{\mathbb{H}}^{(\alpha,w),(\beta,s)}$ *the Liouville field on* $\mathbb{H}$ *with insertions* $(\alpha, w), (\beta, s)$.

Definition 2.4. *Let* $\alpha, \alpha_1, \beta \in \mathbb{R}$, $w \in \mathbb{D}$ *and* $s \in \partial\mathbb{D}$. *Let*

$$C_{\mathbb{D}}^{(\alpha,0),(\alpha_1,w),(\beta,s)} = (1 - |w|^2)^{-\frac{\alpha_1^2}{2}} e^{\alpha_1\alpha G_{\mathbb{D}}(0,w) + \frac{\alpha_1\beta}{2}G_{\mathbb{D}}(s,w)}.$$

Let $(h, \mathbf{c})$ *be sampled from* $C_{\mathbb{D}}^{(\alpha,0),(\alpha_1,w),(\beta,s)} P_{\mathbb{D}} \times [e^{(\alpha+\alpha_1+\frac{\beta}{2}-Q)c}dc]$ *and*

$$\phi(z) = h(z) + \alpha G_{\mathbb{D}}(z,0) + \alpha_1 G_{\mathbb{D}}(z,w) + \frac{\beta}{2}G_{\mathbb{D}}(z,s) + \mathbf{c}.$$

We call ϕ *the Liouville field on* $\mathbb{D}$ *with insertions* $(\alpha, 0), (\alpha_1, w), (\beta, s)$ *and write* $\mathrm{LF}_{\mathbb{D}}^{(\alpha,0),(\alpha_1,w),(\beta,s)}$ *for the law of* ϕ.

As we will see later in Lemma 2.9, the Liouville fields introduced for $\mathbb{H}$ and $\mathbb{D}$ agree up to conformal coordinate change.

We now state the conformal covariance in $\mathbb{H}$. For a conformal map $f : D \to \widetilde{D}$ and a measure M on $H^{-1}(D)$, let f_*M be the pushforward of M under the LQG coordinate change map $\phi \mapsto f \bullet_\gamma \phi$. For $\alpha \in \mathbb{R}$, we set $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

Lemma 2.5. *Let* $(\alpha, w) \in \mathbb{R} \times \mathbb{H}$ *and* $(\beta, s) \in \mathbb{R} \times \mathbb{R}$. *Suppose* $f : \mathbb{H} \to \mathbb{H}$ *is a conformal map, such that* $f(s) \neq \infty$. *Then*

$$\mathrm{LF}_{\mathbb{H}}^{(\alpha,f(w)),(\beta,f(s))} = |f'(w)|^{-2\Delta_\alpha} |f'(s)|^{-\Delta_\beta} f_*\mathrm{LF}_{\mathbb{H}}^{(\alpha,w),(\beta,s)}.$$

In particular, when $f(s) = s = 0$, $f(w) = i$, *we have*

$$\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)} = (\operatorname{Im} w)^{2\Delta_\alpha - \Delta_\beta} |w|^{2\Delta_\beta} f_*\mathrm{LF}_{\mathbb{H}}^{(\alpha,w),(\beta,s)}. \tag{2.2}$$

Proof. This statement is proved in [22, Theorem 3.5]; see [5, Lemma 2.4] for an explanation. □

Now, we define the LCFT measure $\mathrm{LF}_{\mathbb{D},\ell}^{(\alpha,0),(\beta,1)}$ having fixed boundary length ℓ.

Definition 2.6. *Let* $\alpha \in \mathbb{R}, \beta < Q$. *Let* h *be a sample from* $P_{\mathbb{D}}$ *and set*

$$\widetilde{h}(z) = h + \alpha G_{\mathbb{D}}(z,0) + \frac{\beta}{2}G_{\mathbb{D}}(z,1).$$

Fix $\ell > 0$, *and let* $L = \mathcal{L}_{\widetilde{h}}(\partial\mathbb{D})$. *Define the measure* $\mathrm{LF}_{\mathbb{D},\ell}^{(\alpha,0),(\beta,1)}$ *to be the law of* $\widetilde{h} + \frac{2}{\gamma}\log\frac{\ell}{L}$ *under the reweighted measure* $\frac{2}{\gamma}\frac{\ell^{\frac{2\alpha+\beta-2Q}{\gamma}-1}}{L^{\frac{2\alpha+\beta-2Q}{\gamma}}}P_{\mathbb{D}}(dh)$.

Lemma 2.7. *In the setting of Definition 2.6,* $\{\mathrm{LF}_{\mathbb{D},\ell}^{(\alpha,0),(\beta,1)}\}_{\ell>0}$ *is a* disintegration *of* $\mathrm{LF}_{\mathbb{D}}^{(\alpha,0),(\beta,1)}$ *over its boundary length. That is, any sample* ϕ *from* $\mathrm{LF}_{\mathbb{D},\ell}^{(\alpha,0),(\beta,1)}$ *has* $\mathcal{L}_\phi(\partial\mathbb{D}) = \ell$, *and*

$$\mathrm{LF}_{\mathbb{D}}^{(\alpha,0),(\beta,1)} = \int_0^\infty \mathrm{LF}_{\mathbb{D},\ell}^{(\alpha,0),(\beta,1)} d\ell. \tag{2.3}$$

Moreover, if $\alpha + \frac{\beta}{2} > Q$, *we have* $|\mathrm{LF}_{\mathbb{D},\ell}^{(\alpha,0),(\beta,1)}| = C\ell^{\frac{2\alpha+\beta-2Q}{\gamma}-1}$ *for some finite constant* C.

Proof. First, $\mathcal{L}_\phi(\partial\mathbb{D}) = \mathcal{L}_{\widetilde{h}+\frac{2}{\gamma}\log\frac{\ell}{L}}(\partial\mathbb{D}) = \frac{\ell}{L}\mathcal{L}_{\widetilde{h}}(\partial\mathbb{D}) = \ell$. Next, for any nonnegative measurable function F on $H^{-1}(\mathbb{D})$,

$$\int_0^\infty \int F(\widetilde{h} + \frac{2}{\gamma}\log\frac{\ell}{L})\frac{2}{\gamma}\frac{\ell^{\frac{2\alpha+\beta-2Q}{\gamma}-1}}{L^{\frac{2\alpha+\beta-2Q}{\gamma}}} P_\mathbb{D}(dh)d\ell = \int_\mathbb{R}\int F(\widetilde{h}+c)e^{(\alpha+\frac{\beta}{2}-Q)c}P_\mathbb{D}(dh)dc$$

using Fubini's theorem and the change of variables $c = \frac{2}{\gamma}\log\frac{\ell}{L}$. This justifies (2.3). For the last claim,

$$\begin{aligned}\mathrm{LF}_\mathbb{D}^{(\alpha,0),(\beta,1)}[\{\mathcal{L}_\phi(\partial\mathbb{D})\in[a,b]\}] &= \int\int \mathbb{1}_{e^{\frac{\gamma}{2}c}L\in[a,b]}e^{(\alpha+\frac{\beta}{2}-Q)c}P_\mathbb{D}(dh)dc \\ &= \frac{2}{\gamma}\int L^{-\frac{2\alpha+\beta-2Q}{\gamma}}P_\mathbb{D}(dh)\cdot\int_a^b \ell^{\frac{2\alpha+\beta-2Q}{\gamma}-1}d\ell\end{aligned} \tag{2.4}$$

where we used the change of variables $\ell = e^{\frac{\gamma}{2}c}L$. Since $\alpha+\frac{\beta}{2} > Q$, the integral $\int L^{-\frac{2\alpha+\beta-2Q}{\gamma}}P_\mathbb{D}(dh)$ is finite (see e.g. [22, 37]) and the claim then follows. □

As we see next, sampling a point from the LQG area measure corresponds to adding an LCFT insertion of size γ. Recall $\mathcal{A}_\phi(dz)$ denotes the quantum area measure.

Lemma 2.8. *Let $w\in\mathbb{D}$, $\alpha,\beta\in\mathbb{R}$ and $s\in\partial\mathbb{D}$. Then we have*

$$\mathcal{A}_\phi(dz)\mathrm{LF}_\mathbb{D}^{(\alpha,0),(\beta,s)}(d\phi) = \mathrm{LF}_\mathbb{D}^{(\alpha,0),(\beta,s),(\gamma,z)}(d\phi)dz.$$

Proof. The proof is identical to that of [1, Proposition 2.5]. □

Finally, the Liouville fields on $\mathbb{H}$ and $\mathbb{D}$ agree up to coordinate change; we now verify the case that we need for this paper.

Lemma 2.9. *Let $\alpha,\beta\in\mathbb{R}$ with $\alpha+\frac{\beta}{2}=Q$. For $w\in\mathbb{H}$, let $g:\mathbb{H}\to\mathbb{D}$ be a conformal map with $g(w)=0$ and $g(0)=1$. Then*

$$\mathrm{LF}_\mathbb{D}^{(\alpha,0),(\beta,1)} = 2^{\frac{\alpha^2}{2}}(\mathrm{Im}\,w)^{2\Delta_\alpha-\Delta_\beta}|w|^{2\Delta_\beta}g_*\mathrm{LF}_\mathbb{H}^{(\alpha,w),(\beta,0)}. \tag{2.5}$$

Proof. We will show the claim for $w=i$, then the general case follows by using Lemma 2.5.

Let $g:\mathbb{H}\to\mathbb{D}$ be the conformal map such that $g(i)=0, g(0)=1$. Explicitly, it is given by $g(z)=\frac{i-z}{i+z}$. By the conformal invariance of the free boundary GFF viewed as a distribution modulo additive constant, if $(h_\mathbb{H},\mathbf{c}_\mathbb{H})\sim P_\mathbb{H}\times dc$ and $(h_\mathbb{D},\mathbf{c}_\mathbb{D})\sim P_\mathbb{D}\times dc$, then $h_\mathbb{H}+\mathbf{c}_\mathbb{H}\overset{d}{=}(h_\mathbb{D}+\mathbf{c}_\mathbb{D})\circ g$. Next, using the formulas for $G_\mathbb{H}$ and $G_\mathbb{D}$ in Section 2.1, one can directly check that for some constant C, we have

$$\alpha G_\mathbb{H}(\cdot,i)+\frac{\beta}{2}G_\mathbb{H}(\cdot,0)-2Q\log|\cdot|_+ = (\alpha G_\mathbb{D}(\cdot,0)+\frac{\beta}{2}G_\mathbb{D}(\cdot,1))\circ g + Q\log|g'|+C \quad \text{for all } z\in\mathbb{H}.$$

Combining this with the translation invariance of Lebesgue measure, we conclude that

$$g\bullet_\gamma(h_\mathbb{H}+\mathbf{c}_\mathbb{H}+\alpha G_\mathbb{H}(\cdot,i)+\frac{\beta}{2}G_\mathbb{H}(\cdot,0)-2Q\log|\cdot|_+)\overset{d}{=}h_\mathbb{D}+\mathbf{c}_\mathbb{D}+\alpha G_\mathbb{D}(\cdot,0)+\frac{\beta}{2}G_\mathbb{D}(\cdot,1).$$

Thus (2.5) holds for $w=i$, as needed. □

2.3 Forward and reverse SLE

In this section we briefly recall the forward and reverse radial SLE processes, and whole-plane SLE. We will not give precise definitions since they will not be used later, but curious readers can refer to [27].

Forward radial SLE$_\kappa$ in $\mathbb{D}$ from 1 to 0 is a random non-self-crosing curve $\eta:[0,\infty)\to\overline{\mathbb{D}}$ with $\eta(0)=1$ and $\lim_{t\to\infty}\eta(t)=0$. Let K_t be the compact subset of $\overline{\mathbb{D}}$ such that $\overline{\mathbb{D}}\backslash K_t$ is the connected component of $\mathbb{D}\backslash\eta([0,t])$ containing 0, and let $g_t:\mathbb{D}\backslash K_t\to\mathbb{D}$ be the conformal map with $g_t(0)=0$ and $g_t'(0)>0$. The curve η is parametrized by log conformal radius, meaning that for each t we have $g_t'(0)=e^t$. It turns out that there is a random process $U_t\overset{d}{=}e^{i\sqrt{\kappa}B_t}$ (where B_t is standard Brownian motion) such that

$$dg_t(z)=\Phi(U_t,g_t(z))\,dt \quad \text{for } z\in\mathbb{D}\backslash K_t \text{ and } \Phi(u,z):=z\frac{u+z}{u-z}. \tag{2.6}$$

In fact, (2.6) and the initial condition $g_0(z) = z$ define the family of conformal maps $(g_t)_{t\geq 0}$ and hence radial SLE_κ, see [27] for details.

Similarly, whole-plane SLE_κ is a random non-self-crossing curve $\eta : (-\infty, \infty) \to \mathbb{C}$ from 0 to ∞, such that if K_t is the compact set such that $\mathbb{C}\backslash K_t$ is the unbounded connected component of $\mathbb{C}\backslash\eta((-\infty, t))$, and $g_t : \mathbb{C}\backslash K_t \to \mathbb{C}\backslash\mathbb{D}$ is the conformal map such that $g_t(\infty) = \infty$ and $g_t'(\infty) > 0$, then

$$dg_t(z) = \Phi(U_t, g_t(z))\, dt \quad \text{ for } z \in \mathbb{C}\backslash K_t$$

where $U_t \stackrel{d}{=} e^{i\sqrt{\kappa}B_t}$ and $(B_t)_{t\in\mathbb{R}}$ is two-sided standard Brownian motion. This curve extends continuously to its starting and ending points, i.e. $\lim_{t\to-\infty} \eta(t) = 0$ and $\lim_{t\to\infty} \eta(t) = \infty$ [28, 35].

Now we discuss *centered reverse* radial SLE. Unlike the forward case where we have a single random curve, centered reverse radial SLE is a random process of curves $(\eta_t)_{t\geq 0}$. Each curve $\eta_t : [0, t] \to \overline{\mathbb{D}}$ is parametrized by log conformal radius and has starting point $\eta_t(0) = 1$, and $(\eta_t)_{t\geq 0}$ satisfies the compatibility relation that for $s < t$, if $\widetilde{f}_{s,t}$ is the conformal map from $\mathbb{D}$ to the connected component of $\mathbb{D}\backslash\eta_t([0, t - s])$ containing 0 such that $\widetilde{f}_{s,t}(1) = \eta_t(t - s)$ and $\widetilde{f}_{s,t}(0) = 0$, then $\eta_s = \widetilde{f}_{s,t}^{-1} \circ \eta_t(\cdot + t - s)|_{[0,s]}$. Informally, this compatibility relation means that the process $(\eta_t)_{t\geq 0}$ grows from the base of the curve. We call $\widetilde{f}_{0,t}$ the centered reverse Loewner map. The process $(\eta_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$d\widetilde{f}_{0,t}(z) = -i\sqrt{\kappa}\widetilde{f}_{0,t}(z)dB_t - \Phi(1, \widetilde{f}_{0,t}(z))\, dt \text{ for } z \in \overline{\mathbb{D}}. \tag{2.7}$$

One can show via the time-reversal symmetry of Brownian motion that for each fixed t, the curve η_t has the law of forward radial SLE run for time t.

For $z_0 \in \mathbb{H}$ and $\rho \in \mathbb{R}$, there is also a random process $(\eta_t)_{t\geq 0}$ called *centered reverse chordal* $\mathrm{SLE}_\kappa(\rho)$ *with force point at* z_0 (see e.g. [40, Section 4.3], [14, Section 3.3.1]). Each $\eta_t : [0, t] \to \overline{\mathbb{H}}$ is parametrized by half-plane capacity, has $\eta_t(0) = 0$, and satisfies a compatibility relation analogous to that of the radial case. It is defined by a stochastic differential equation similar to (2.7) which we omit here. For each $t > 0$ let $\widetilde{f}_{\mathbb{H},t} : \mathbb{H} \to \mathbb{H}\backslash\eta_t$ be the conformal map with $\widetilde{f}_{\mathbb{H},t}(0) = \eta_t(t)$ and $\widetilde{f}_{\mathbb{H},t}(z) = z + O(1)$ as $z \to \infty$; we call $\widetilde{f}_{\mathbb{H},t}$ the centered reverse Loewner map.

Finally, [40, Theorem 4.6][1] gives a change of coordinates result for reverse chordal SLE:

Lemma 2.10. *Fix $\kappa > 0$. Let $(\eta_t)_{t\geq 0}$ be a centered reverse chordal $\mathrm{SLE}_\kappa(\kappa + 6)$ process with force point at $\widetilde{z}_0 \in \mathbb{H}$. Let $\widetilde{f}_t$ be its associated reverse centered Loewner map. Let $\varphi_0 : \mathbb{H} \to \mathbb{D}$ be the conformal map with $\varphi_0(\widetilde{z}_0) = 0$ and $\varphi_0(0) = 1$, and $\varphi_t : \mathbb{H} \to \mathbb{D}$ the conformal map such that $\varphi_t(\widetilde{f}_t(\widetilde{z}_0)) = 0$ and $\varphi_t(0) = 1$. Let η_t' be $\varphi_t \circ \eta_t$ parametrized by log conformal radius. Then up to a time change, $(\eta_t')_{t\geq 0}$ has the law of centered reverse radial SLE_κ stopped at the time $\varphi_0(\infty)$ hits the driving function, i.e. the first time s when $\widetilde{f}_{0,s}(\varphi_0(\infty)) = 1$ where $\widetilde{f}_{0,s}$ is the centered reverse Loewner map of the reverse radial SLE_κ.*

2.4 Chordal mating-of-trees and special quantum surfaces

In this section we state the chordal mating-of-trees theorem of [14], and recall the definition of the *quantum cone* from [44, 14] and the *quantum cell* from [1].

Let $\mathcal{C} = (\mathbb{R} \times [0, 2\pi])/\sim$ be the horizontal cylinder obtained by gluing the upper and lower boundaries of the strip via the identification $x \sim x + 2\pi i$. We define the GFF on $\mathcal{C}$ as in Section 2.1, with $m_\mathcal{C}$ the uniform measure on $(\{0\} \times [0, 2\pi])/\sim$, and likewise define the Hilbert space $H(\mathcal{C})$. As explained in, e.g., [14, Section 4.1.7], we may decompose $H(\mathcal{C}) = H_{\mathrm{av}}(\mathcal{C}) \oplus H_{\mathrm{lat}}(\mathcal{C})$, where $H_{\mathrm{av}}(\mathcal{C})$ (resp. $H_{\mathrm{lat}}(\mathcal{C})$) is the subspace of functions which are constant (resp. have mean 0) on $\{t\} \times [0, 2\pi]$ for each $t \in \mathbb{R}$. This gives a decomposition $h_\mathcal{C} = h_{\mathrm{av}} + h_{\mathrm{lat}}$ of $h_\mathcal{C}$ into two independent components.

Now we introduce the γ-LQG surfaces called quantum cones via an embedding in $(\mathcal{C}, -\infty, +\infty)$. Near $-\infty$ it has finite quantum area, but every neighborhood of $+\infty$ has infinite quantum area.

Definition 2.11 (α-quantum cone)**.** *Fix $\alpha < Q$. Suppose ψ_{av} and ψ_{lat} are independent distributions on $\mathcal{S}$ such that:*

- *We have $\psi_{\mathrm{av}}(z) = X_{\mathrm{Re}\, z}$ for $z \in \mathcal{C}$, where*

$$X_t := \begin{cases} B_t - (Q - \alpha)t & \text{for } \quad t \geq 0 \\ \widetilde{B}_{-t} + (Q - \alpha)t & \text{for } \quad t < 0 \end{cases} \tag{2.8}$$

[1]They use a different notation for weights of force points, see Remark 2 immediately after [40, Corollary 4.8].

and $(B_t)_{t\geq 0}$ and $(\widetilde{B}_t)_{t\geq 0}$ are independent standard Brownian motions conditioned on $\widetilde{B}_t-(Q-\alpha)t<0$ for all $t>0$[2]*;*

- *ψ_{lat} has the same law as h_{lat}.*

*Set $\psi=\psi_{\mathrm{av}}+\psi_{\mathrm{lat}}$. We call $(\mathcal{C},\psi,-\infty,+\infty)/{\sim_\gamma}$ an α-*quantum cone.

For $\kappa>4$, there is a random curve in $\mathbb{C}$ called *space-filling* SLE_κ *from* ∞ *to* ∞. It is defined via the imaginary geometry flow lines of a whole-plane GFF. Space-filling SLE_κ from ∞ to ∞ is reversible since its construction is symmetric. Moreover, if $\kappa\geq 8$, for each $z\in\mathbb{C}$ the regions covered by the curve before and after hitting z are simply connected, and conditioned on the curve up until it hits z, it subsequently evolves as chordal SLE_κ from z to ∞ in the complementary domain. This follows from the flow line construction of space-filling SLE_κ, see [35, Section 1.2.3] for more details.

We are ready to state the mating-of-trees theorem [14, Theorem 1.9, Theorem 1.11]. We shall focus on the $\kappa>8$ regime.

Theorem 2.12. *Let $\kappa>8$ and $\gamma=\frac{4}{\sqrt{\kappa}}$. Let $(\mathbb{C},\phi,0,\infty)$ be an embedding of a γ-quantum cone and η an independent space-filling SLE_κ curve from ∞ to ∞, and we reparameterize η by the γ-LQG measure, in the sense that $\eta(0)=0$ and $\mathcal{A}_\phi(\eta([s,t]))=t-s$ for $-\infty<s<t<\infty$. Define X_t^-,X_t^+,Y_t^-,Y_t^+ as in Figure 1 (left, middle) and let $X_t:=X_t^+-X_t^-$ and $Y_t:=Y_t^+-Y_t^-$. Then $(X_t,Y_t)_{t\in\mathbb{R}}$ is a correlated two-sided two-dimensional Brownian motion with $X_0=Y_0=0$, with covariance*

$$\mathrm{var}(X_t)=\mathrm{var}(Y_t)=\mathrm{a}^2|t|;\quad \mathrm{cov}(X_t,Y_t)=-\cos(\frac{4\pi}{\kappa})\mathrm{a}^2|t|\quad \textit{where } \mathrm{a}^2:=\frac{2}{\sin(\frac{4\pi}{\kappa})}. \tag{2.9}$$

Moreover, the pair (X,Y) a.s. determines the decorated quantum surface $(\mathbb{C},\phi,\eta,0,\infty)/{\sim_\gamma}$.

We can interpret X_t (resp. Y_t) as the change in the quantum length of the left (resp. right) boundary of η relative to time 0. The covariance in (2.9) was computed in [17] while the constant a was obtained in [5].

Let (ϕ,η) and (X,Y) be as in the statement of Theorem 2.12. For each $a>0$, let $D_a=\eta([0,a])$, $p=\eta(0)=0$ and $q=\eta(a)$. Let x_L (resp. x_R) be the last point on the left (resp. right) boundary arc of $\eta((-\infty,0])$ hit by η before time a. See Figure 1 (right).

Definition 2.13. *We call the SLE_κ-decorated quantum surface $\mathcal{C}_a:=(D_a,h,\eta|_{[0,a]};p,q,x_L,x_R)/{\sim_\gamma}$ an area a quantum cell, and denote its law by P_a. We call $(X_t,Y_t)_{[0,a]}$ its boundary length process, and $X_a^-=-\inf_{0<t<a}X_t$, $X_a^+=X_a+X_a^-$, $Y_a^-=-\inf_{0<t<a}Y_t$, $Y_a^+=Y_a+Y_a^-$ its boundary lengths.*

Note that the quantum length of the arc between p and x_L (resp. x_R) is X_a^- (resp. Y_a^-), and the quantum length of the arc between q and x_L (resp. x_R) is X_a^+ (resp. Y_a^+). [1] gives a different but equivalent definition of the quantum cell in terms of the so-called weight $2-\frac{\gamma^2}{2}$ quantum wedge; the equivalence follows from the fact that in the setting of Theorem 2.12, the quantum surface $(\eta((0,\infty)),\phi,0,\infty)/{\sim}$ has the law of the weight $2-\frac{\gamma^2}{2}$ quantum wedge [14, Theorem 1.9].

By [1, Remark 2.9], $\mathcal{C}_a$ is measurable with respect to $(D_a,h,\eta|_{[0,a]})/{\sim_\gamma}$ since $\kappa>8$, and therefore we will often omit the marked points of $\mathcal{C}_a$ for notational simplicity. The quantum surface $(D_a,h,\eta|_{[0,a]})/{\sim_\gamma}$ is measurable with respect to $(X_t,Y_t)_{0\leq t\leq a}$ [2, Lemma 2.17], and we denote the map sending $(X_t,Y_t)_{0\leq t\leq a}$ to $(D_a,h,\eta|_{[0,a]})/{\sim_\gamma}$ by F. We now give two properties of F.

Lemma 2.14 (Reversibility of F)**.** *Fix $a>0$, sample $\mathcal{C}_a=(D,h,\eta)/{\sim_\gamma}$ from P_a, and let $(X_t,Y_t)_{[0,a]}$ be its boundary length process, so $F((X_t,Y_t)_{[0,a]})=\mathcal{C}_a$ a.s.. Let $\widetilde{\mathcal{C}}_a=(D,h,\widetilde{\eta})/{\sim_\gamma}$ where $\widetilde{\eta}$ is the time-reversal of η, and let $(\widetilde{X}_t,\widetilde{Y}_t)_{[0,a]}=(Y_{a-t},X_{a-t})_{[0,a]}$ be the time-reversal of $(X_t,Y_t)_{[0,a]}$. Then $F((\widetilde{X}_t,\widetilde{Y}_t)_{[0,a]})=\widetilde{\mathcal{C}}_a$ a.s..*

Proof. Let $(\mathbb{C},h,0,\infty)$ be an embedding of a γ-quantum cone and let η be an independent SLE from ∞ to ∞ in $\mathbb{C}$ parametrized by quantum area such that $\eta(0)=0$. Let $\mathcal{C}_a=(\eta([0,a],h,\eta|_{[0,a]})$ so the law of $\mathcal{C}_a$ is P_a, and let $(X_t,Y_t)_{[0,a]}$ be its boundary length process. Let $\widetilde{\eta}$ be the time-reversal of η, then by the reversibility of SLE from ∞ to ∞ in $\mathbb{C}$ we have $(\mathbb{C},h,\eta,0,\infty)/{\sim_\gamma}\overset{d}{=}(\mathbb{C},h,\widetilde{\eta},0,\infty)/{\sim_\gamma}$. Let

[2] This conditioning can be made sense via Bessel processes; see e.g. [14, Section 4.2].

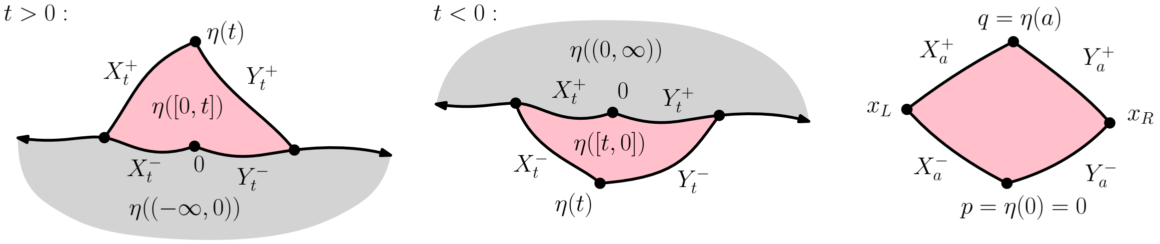

Figure 1: **Left:** Let $t>0$. Let X_t^- be the quantum length along $\partial(\eta((-\infty,0)))$ from 0 to the leftmost point of $\eta([0,t])\cap\partial(\eta((-\infty,0)))$, and X_t^+ the quantum length of the counterclockwise boundary arc of $\eta([0,t])$ from this point to $\eta(t)$. Likewise define Y_t^-, Y_t^+. **Middle:** When $t<0$ we let X_t^- be the quantum length along $\partial(\eta((-\infty,t)))$ from $\eta(t)$ to the leftmost point of $\eta([t,0])\cap\partial(\eta((-\infty,t)))$, and X_t^+ the quantum length of the counterclockwise boundary arc of $\eta([t,0])$ from this point to 0. Likewise define Y_t^-, Y_t^+. **Right:** An illustration of a quantum cell of quantum area a and its boundary lengths $X_a^+, X_a^-, Y_a^+, Y_a^-$.

$\widetilde{\eta}'(\cdot)=\widetilde{\eta}(\cdot-a)$ (so $\widetilde{\eta}'|_{[0,a]}$ is the time-reversal of $\eta|_{[0,a]}$), then [14, Lemma 8.3] implies $(\mathbb{C},h,\widetilde{\eta},0,\infty)/{\sim_\gamma}\overset{d}{=}(\mathbb{C},h,\widetilde{\eta}',\eta(a),\infty)/{\sim_\gamma}$, that is, $(\mathbb{C},h,\widetilde{\eta}',\eta(a),\infty)/{\sim_\gamma}$ is a quantum cone decorated by an independent SLE from ∞ to ∞ in $\mathbb{C}$. We conclude that the law of $\widetilde{\mathcal{C}}_a$ is also P_a, and directly from the definition of boundary length process, the boundary length process of $\widetilde{\mathcal{C}}_a$ is $(\widetilde{X}_t,\widetilde{Y}_t)_{[0,a]}$, so $F((\widetilde{X}_t,\widetilde{Y}_t)_{[0,a]})=\widetilde{\mathcal{C}}_a$ a.s.. □

Lemma 2.15 (Concatenation compatibility of F)**.** *Let $a_1,a_2>0$, and let $(X_t,Y_t)_{t\in\mathbb{R}}$ be as in (2.9). Let $\mathcal{C}_1=F((X_t,Y_t)_{[0,a_1]})$, let $\mathcal{C}_2=F((X_{t+a_1}-X_{a_1},Y_{t+a_1}-Y_{a_1})_{[0,a_2]})$, and let $\mathcal{C}=F((X_t,Y_t)_{[0,a_1+a_2]})$. Almost surely, $\mathcal{C}_1$ and $\mathcal{C}_2$ are the curve-decorated quantum surfaces obtained from $\mathcal{C}$ by restricting to the domains parametrized by its curve on the time intervals $[0,a_1]$ and $[a_1,a_1+a_2]$.*

Proof. This is immediate from the definition of F and the fact that if $(\mathbb{C},\phi,0,\infty)$ is an embedding of a γ-quantum cone and η is an independent space-filling SLE_κ from ∞ to ∞ parametrized by quantum area, then $(\mathbb{C},\phi,\eta,0,\infty)/{\sim_\gamma}\overset{d}{=}(\mathbb{C},\phi,\eta(\cdot+a_1),\eta(a_1),\infty)/{\sim_\gamma}$ [14, Lemma 8.3]. □

Finally, we recall the definition of the quantum sphere of [14]. This is a two-pointed quantum surface with finite quantum area.

Definition 2.16. *Let $\alpha<Q$. Let $(B_s)_{s\ge0}$ be a standard Brownian motion conditioned on $B_s-(Q-\alpha)s<0$ for all $s>0$, and let $(\widetilde{B}_s)_{s\ge0}$ be an independent copy of $(B_s)_{s\ge0}$. Let*

$$Y_t=\begin{cases}B_t-(Q-\alpha)t & \text{if } t\ge0\\ \widetilde{B}_{-t}+(Q-\alpha)t & \text{if } t<0\end{cases}$$

Let $h^1(z)=Y_{\mathrm{Re}\,z}$ for $z\in\mathcal{C}$, and let h^2 be independent of h^1 and have the law of the lateral component of the GFF on $\mathcal{C}$. Let $\hat{h}=h^1+h^2$. Let $\mathbf{c}\in\mathbb{R}$ be independently sampled from $\frac{\gamma}{2}e^{2(\alpha-Q)c}\,dc$. Let $\mathcal{M}_2^{\mathrm{sph}}(\alpha)$ be the infinite measure describing the law of the decorated quantum surface $(\mathcal{C},\hat{h}+\mathbf{c},-\infty,+\infty)/{\sim_\gamma}$.

2.5 LCFT and the quantum zipper

In this section we state a special case of the chordal quantum zipper for LCFT obtained in [1]. It will be used in Section 3.1 to derive a radial quantum zipper for LCFT.

Let $\kappa>8$ and $\gamma=\frac{4}{\sqrt{\kappa}}$. Let BM_κ denote the law of (one-sided) correlated two-dimensional Brownian motion $(X_t,Y_t)_{t\ge0}$ with $X_0=Y_0=0$ and covariance given by (2.9). Let $\alpha\in\mathbb{R}$, and sample $(\tilde{\psi}_0,(X_t,Y_t)_{t\ge0})\sim \mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(-\frac{\gamma}{2},0)}\times\mathrm{BM}_\kappa$. For each $s>0$ let $\mathcal{C}_s=F((X_\cdot,Y_\cdot)|_{[0,s]})$, and on the event that $-\inf_{u<s}X_u-\inf_{u<s}Y_u<\mathcal{L}_{\psi_0}(\mathbb{R})$, conformally weld $(\mathbb{H},\psi_0,i,0,\infty)/{\sim_\gamma}$ to $\mathcal{C}_s$ by identifying the first marked points of each quantum surface and identifying the two boundary arcs of $\mathcal{C}_s$ adjacent to its first marked point to a boundary interval of $(\mathbb{H},\psi_0,i,0)/{\sim_\gamma}$; this identification is via the quantum length measures of the two quantum surfaces. The resulting curve-decorated quantum surface has many possible embeddings in $\mathbb{H}$. Let $(\mathbb{H},\tilde{\psi}_s,\tilde{\eta}_s)$ be the unique embedding such that if $f:\mathbb{H}\to\mathbb{H}\backslash\tilde{\eta}_s$ is the conformal map which fixes ∞, sends the tip of $\tilde{\eta}_s$ to 0, and satisfies $f(z) = z + O(1)$ as $z \to \infty$, then $\tilde{\psi}_0 = f^{-1} \bullet_\gamma \tilde{\psi}_s$. In this way, we obtain a process $(\widetilde{\psi}_s, \widetilde{\eta}_s)$. Let (ψ_t, η_t) be the monotone reparametrization of the process such the half-plane capacity of the trace of η_t is $2t$.

Lemma 2.17. *For any stoppping time σ for the filtration $\mathcal{F}_t = \sigma(\eta_t)$, the law of $(\psi_\sigma, \eta_\sigma)$ is*

$$\frac{1}{\mathcal{Z}_\alpha(\widetilde{f}_{\mathbb{H},\sigma}(i))} \mathrm{LF}_{\mathbb{H}}^{(\alpha, \widetilde{f}_{\mathbb{H},\sigma}(i)),(-\frac{\gamma}{2},0)} \mathrm{rSLE}^\sigma_{\kappa, 2\sqrt{\kappa}\alpha}, \qquad \mathcal{Z}_\alpha(z) := (2\mathrm{Im}\, z)^{-\frac{\alpha^2}{2}} |z|^{\frac{2}{\sqrt{\kappa}}\alpha},$$

where $\mathrm{rSLE}^\sigma_{\kappa,\rho}$ denotes the law of centered reverse chordal $\mathrm{SLE}_\kappa(\rho)$ with the force point located at i run until the stopping time σ and $\widetilde{f}_{\mathbb{H},\sigma}$ is its associated reverse Loewner map.

Proof. This is the special case of [1, Theorem 1.8] where there is a single bulk insertion and a single boundary insertion, phrased in terms of centered reverse chordal $\mathrm{SLE}_\kappa(\rho)$ rather than reverse chordal $\mathrm{SLE}_\kappa(\rho)$. □

3 A radial mating-of-trees theorem

In this section, we prove our radial mating-of-trees result Theorem 3.1. Throughout this section, let $\gamma \in (0, \sqrt{2})$ and $\kappa = \frac{16}{\gamma^2} > 8$.

Sample $\phi \sim \mathrm{LF}_{\mathbb{D}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}$ conditioned on having quantum boundary length 1, let $A = \mathcal{A}_\phi(\mathbb{D})$, and let $\eta : [0, A] \to \overline{\mathbb{D}}$ be an independent radial SLE_κ in $\mathbb{D}$ from 1 to 0 parametrized by its $\mathcal{A}_\phi$-quantum area. There is a unique continuous process $(X_t, Y_t)_{[0,A]}$ starting at $(X_0, Y_0) = (0, 0)$ which keeps track of the local changes in the left and right LQG boundary lengths of $\mathbb{D}\backslash\eta([0, t])$ in the following sense. For any time $s \in (0, A)$ and any point $p \in \partial(\mathbb{D}\backslash\eta([0, s]))$ different from $\eta(s)$, let $\sigma > s$ be the next time η hits p. For each $t \in [s, \sigma)$, let X^s_t (resp. Y^s_t) be the quantum length of the clockwise (resp. counterclockwise) boundary arc of $\mathbb{D}\backslash\eta([0, t])$ from $\eta(t)$ to p. Then $(X_t - X_s, Y_t - Y_s)_{[s,\sigma)} = (X^s_t - X^s_s, Y^s_t - Y^s_s)_{[s,\sigma)}$. See Figure 2 (left, middle). This process can be constructed on the time interval $[0, A)$ by shifting the point p countably many times, and its value at A is defined by taking a limit. Note that these LQG lengths exist and are finite by local absolute continuity with respect to the setting of Theorem 2.12.

Theorem 3.1 ($\kappa > 8$ radial mating-of-trees)**.** *The process $(X_t, Y_t)_{0\le t\le A}$ has the law of 2-dimensional Brownian motion with covariance (2.9) stopped at the first time that $1 + X_\cdot + Y_\cdot = 0$. Moreover, for $0 \le s < t$, on the event that $t < A$ and $\eta([s, t])$ is simply connected, we have*

$$F((X_{\cdot+s} - X_s, Y_{\cdot+s} - Y_s)|_{[0,t-s]}) = (\eta([s, t]), \phi, \eta(\cdot + s)|_{[0,t-s]})/{\sim_\gamma} \quad \textit{almost surely.} \tag{3.1}$$

Here, F is as in Lemma 2.14.

We note that when $\eta([s, t])$ is not simply connected, then instead the right hand side of (3.1) is obtained from the left hand side by conformally welding its boundary to itself. In Theorem 3.1 the curve-decorated quantum surface $(\mathbb{D}, \phi, \eta, 0, 1)/{\sim_\gamma}$ can a.s. be recovered from $(X_t, Y_t)_{[0,A]}$ by conformally welding countably many simply connected quantum surfaces of the form $(\eta([s, t]), \phi, \eta(\cdot + s)|_{[0,t-s]})/{\sim_\gamma}$, each of which is measurable with respect to $(X_t, Y_t)_{0\le t\le A}$ by (3.1).

Corollary 3.2. *For $\phi \sim \mathrm{LF}_{\mathbb{D}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}$ conditioned on having boundary length 1, the quantum area $\mathcal{A}_\phi(\mathbb{D})$ has the law of the inverse gamma distribution with shape parameter $\frac{1}{2}$ and scale parameter $b = \frac{1}{8}\tan(\frac{\pi\gamma^2}{8})$, i.e., the law of $\mathcal{A}_\phi(\mathbb{D})$ is*

$$\mathbb{1}_{a>0}\sqrt{\frac{b}{\pi a^3}}e^{-\frac{b}{a}}\, da.$$

Proof. The law of $X_t + Y_t$ is Brownian motion with quadratic variation $(2\mathbbm{a}\sin(\frac{\pi\gamma^2}{8}))^2\, dt = 4\cot(\frac{\pi\gamma^2}{8})\, dt$, and $\mathcal{A}_\phi(\mathbb{D})$ equals the hitting time of -1. The claim then follows from the well-known law of Brownian motion first passage times. □

Remark 3.3. *Corollary 3.2, together with the result [37, Theorem 1.7] and the computation of [5, Section 4.4], can be used to compute the correlation function of LCFT on the disk with a bulk insertion $\alpha = Q - \frac{\gamma}{4}$ and a boundary insertion $\beta = \frac{3\gamma}{2}$. This gives an alternative derivation of a special case of [6, Theorem 1.2], i.e., proves a special case of the physical proposal by [21].*

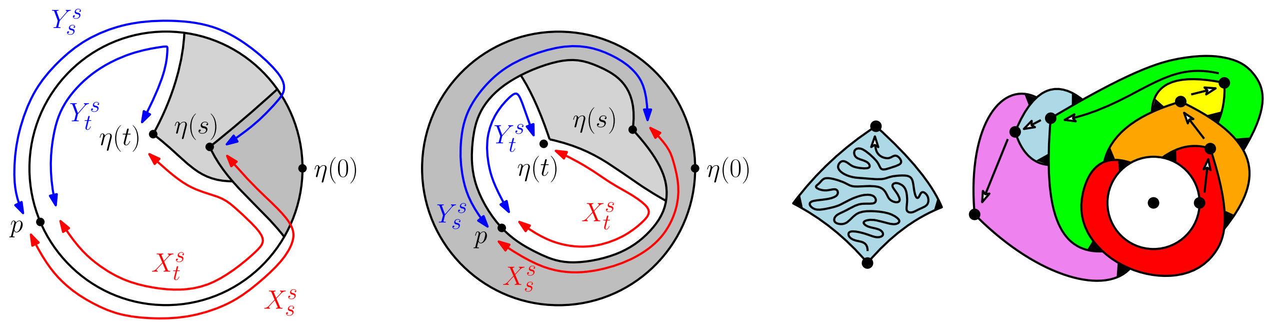

Figure 2: **Left:** The boundary length process $(X_t, Y_t)_{[0,A]}$ of Theorem 3.1 is characterized by $X_0 = Y_0 = 0$ and the property that for each time s and choice of boundary point $p \in \partial(\mathbb{D}\backslash\eta([0,s]))$ different from $\eta(s)$, for any time $t > s$ before the time η next hits p, we have $(X_t^s - X_s^s, Y_t^s - Y_s^s) = (X_t - X_s, Y_t - Y_s)$. Here $\eta([0,s])$ is shown in dark gray, and $\eta([s,t])$ is colored light grey. **Middle:** Another possible configuration. **Right:** Diagram for the definition of $(\widetilde{\psi}_s, \widetilde{\eta}_s)$ in Section 3.1. Each of the quantum surfaces $\mathcal{C}_j$ comes with four marked boundary points and a space-filling curve, as in Definition 2.13. We conformally weld $\mathcal{C}_1$ (red) to $\mathcal{D}_1 = (\mathbb{D}, \widetilde{\psi}_0, 0, 1)/{\sim_\gamma}$ along the two boundary arcs of $\mathcal{C}_1$ adjacent to the starting point of its space-filling curve, to obtain $\mathcal{D}_2$. Iterating this procedure (colors from red to purple, in order) gives $\mathcal{D}_k$; we concatenate its curves and forget all marked points except the bulk point from $\mathcal{D}_1$ (white) and the boundary endpoint of the curve of $\mathcal{C}_k$ (purple), to get the quantum surface $(\mathbb{D}, \widetilde{\psi}_s, \widetilde{\eta}_s, 0, 1)/{\sim_\gamma}$. Note that when each $\mathcal{C}_j$ is conformally welded, by construction it will not "wrap around" the whole boundary of the other quantum surface.

In Section 3.1 we define a *radial quantum zipper* where, starting with a sample from $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1)}$, we grow the quantum surface by conformal welding with independent quantum cells, giving rise to a coupling of LCFT with *reverse* radial SLE. In Section 3.2 we prove Proposition 3.8 in which we decorate $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}$ by *forward* radial SLE and look at the quantum surfaces parametrized by the curve and its complement. Here, to switch between reverse and forward SLE, we use the fact that for any fixed time, the curve generated by centered reverse radial SLE has the law of forward radial SLE. In Section 3.3, since $\Delta_{Q+\frac{\gamma}{4}} = \Delta_{Q-\frac{\gamma}{4}}$ (with $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$), we can use Girsanov's theorem to obtain a variant of Proposition 3.8 about $\mathrm{LF}_{\mathbb{D}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}$ (Proposition 3.12), and hence Theorem 3.1.

3.1 A radial quantum zipper

Let $\kappa > 8$ and $\gamma = \frac{4}{\sqrt{\kappa}}$. The goal of this section is to prove Lemma 3.4, in which we define and study a quantum zipper process $(\psi_t, \eta_t)_{t\geq 0}$ where the marginal law of ψ_0 is $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1)}$ and the time-evolution corresponds to conformally welding quantum cells to the boundary of the quantum surface. The proof of Lemma 3.4 will depend on a result of [1] stated as Lemma 2.17.

Let BM_κ denote the law of (one-sided) correlated two-dimensional Brownian motion $(X_t, Y_t)_{t\geq 0}$ with $X_0 = Y_0 = 0$ and covariance given by (2.9). Sample $(\widetilde{\psi}_0, (X_t, Y_t)) \sim \mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1)} \times \mathrm{BM}_\kappa$, let $L_t = X_t + Y_t + \mathcal{L}_{\widetilde{\psi}_0}(\partial\mathbb{D})$, and let $\widetilde{\tau}$ be the first time t that $L_t = 0$. For $s \in (0, \widetilde{\tau})$ we define a random field and curve $(\widetilde{\psi}_s, \widetilde{\eta}_s)$ which correspond to "zipping up for quantum time s" as follows. See Figure 2 (right). Choose finitely many times $0 = s_1 < \cdots < s_k = s$ such that for $j < k$ we have $(X_{s_j} - \inf_{u\in[s_j,s_{j+1}]} X_u) + (Y_{s_j} - \inf_{u\in[s_j,s_{j+1}]} Y_u) < L_{s_j}$. For $j < k$ let $\mathcal{C}_j = F((X_{\cdot+s_j} - X_{s_j}, Y_{\cdot+s_j} - Y_{s_j})_{[0,s_{j+1}-s_j]})$. We iteratively define quantum surfaces with the disk topology decorated by a bulk point, a boundary point, and a curve as follows. Let $\mathcal{D}_1 = (\mathbb{D}, \widetilde{\psi}_0, 0, 1)/{\sim_\gamma}$, and iteratively for $j = 1, \ldots, k-1$, we conformally weld $\mathcal{C}_j$ to $\mathcal{D}_j$ to obtain $\mathcal{D}_{j+1}$. This is done by identifying the starting point of the curve of $\mathcal{C}_j$ with the boundary point of $\mathcal{D}_j$ and conformally welding the two boundary arcs of $\mathcal{C}_j$ adjacent to this point to $\mathcal{D}_j$ by quantum length (this is possible since by assumption the two boundary arcs have total quantum length smaller than the quantum boundary length of $\partial\mathcal{D}_j$). Doing this $k-1$ times produces $\mathcal{D}_k$, which we view as a quantum surface decorated by a bulk point (from $\mathcal{D}_1$), a curve (obtained by concatenating the $k-1$ curves from $\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}$), and a boundary point (the endpoint of the curve on the boundary). We orient the curve so that it starts on the boundary and ends in the bulk of $\partial\mathcal{D}_k$. Finally, we conformally embed $\mathcal{D}_k$ in $\mathbb{D}$, sending the bulk and boundary marked points to 0 and 1, to get $(\mathbb{D}, \widetilde{\psi}_s, \widetilde{\eta}_s, 0, 1)$. This gives our definition of $\widetilde{\psi}_s, \widetilde{\eta}_s$ for all $s < \widetilde{\tau}$; note that Lemma 2.15 implies this definition does not depend on the choice of $s_1, \dots, s_k$.

For each s, let $t(s)$ be the log conformal radius of $\mathbb{D}\backslash\widetilde{\eta}_s$ viewed from 0, i.e., $t(s) = -\log|g'(0)|$ where $g : \mathbb{D} \to \mathbb{D}\backslash\eta_s$ is any conformal map fixing 0. This gives a monotone reparametrization of the process which we denote by $(\psi_t, \eta_t)_{t\geq 0}$. We parameterize each curve $\eta_t : [0, t] \to \overline{\mathbb{D}}$ by log conformal radius, so $\eta_t(0) = 1$ and the conformal radius of $\mathbb{D}\backslash\eta_t([0, t'])$ viewed from 0 is $e^{-t'}$. We first give a description of the process $(\psi_t, \eta_t)_{t\geq 0}$ in terms of the Liouville field and reverse SLE. Recall $\bullet_\gamma$ from (2.1).

Lemma 3.4. *For $\kappa > 8$ and $\gamma = \frac{4}{\sqrt{\kappa}}$, let M be the law of the process $(\psi_t, \eta_t)_{t\geq 0}$ defined immediately above. Then*

i) For any a.s. finite stopping time τ for the filtration $\mathcal{F}_t$ generated by $(\eta_t)_{t\geq 0}$, the law of (ψ_τ, η_τ) is $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1)}\mathrm{rrSLE}_\kappa^\tau$, where $\mathrm{rrSLE}_\kappa^\tau$ denotes the law of centered reverse radial SLE_κ in $\mathbb{D}$ from 1 to 0 run until the stopping time τ.

ii) For $0 < t_1 < t_2$, let $\widetilde{f}_{t_1,t_2} : \mathbb{D} \to \mathbb{D}\backslash\eta_{t_2}([0, t_2 - t_1])$ be the conformal map fixing 0 with $\widetilde{f}_{t_1,t_2}(1) = \eta_{t_2}(t_2 - t_1)$, then $\psi_{t_1} = \widetilde{f}^{-1}_{t_1,t_2} \bullet_\gamma \psi_{t_2}$.

Remark 3.5. *[34, Theorem 5.1] constructed a process (ψ_t, η_t) satisfying the conclusions of Lemma 3.4 (but with the LQG field viewed modulo additive constant) by using a martingale argument to couple GFF and SLE. For our purposes, however, we crucially require the mating-of-trees description of M not present in [34].*

The insertions $(\alpha, \beta) = (Q + \frac{\gamma}{4}, -\frac{\gamma}{2})$ in the definition of M satisfy $\alpha + \frac{1}{2}\beta - Q = 0$, so the constant mode of the Liouville field has law $e^{(\alpha+\frac{\beta}{2}-Q)c}dc = dc$ (up to multiplicative constant). The translation invariance of this law makes the Liouville field closely related to the GFF modulo additive constant, and hence the GFF/SLE coupling of [34]. Moreover, the conformal invariance of the GFF modulo additive constant is the underlying reason why prefactors cancel in our subsequent argument (below (3.2)).

Proof of Lemma 3.4. From the definition of M, $(\mathbb{D}, \psi_{t_2}, 0, 1)/{\sim_\gamma}$ is obtained from conformally welding $(\mathbb{D}, \psi_{t_1}, 0, 1)/{\sim_\gamma}$ with another quantum surface, so $(\mathbb{D}\backslash\eta_{t_2}([0, t_2 - t_1]), \psi_{t_2}, 0, \eta_{t_2}(t_2 - t_1))/{\sim_\gamma} = (\mathbb{D}, \psi_{t_1}, 0, 1)/{\sim_\gamma}$. This gives $\psi_{t_1} = \widetilde{f}^{-1}_{t_1,t_2} \bullet_\gamma \psi_{t_2}$ so ii) holds.

For i), we first apply a change of coordinates from $(\mathbb{D}, 1, -1)$ to $(\mathbb{H}, 0, \infty)$ to change the radial process $(\psi_t, \eta_t)_{t\geq 0}$ into a chordal process $(\hat{\psi}_t, \hat{\eta}_t)_{t\geq 0}$ in $(\mathbb{H}, 0, \infty)$, apply Lemma 2.17 for the chordal process in $\mathbb{H}$, and finally convert back to the radial process in $\mathbb{D}$.

For a sample $(\psi_t, \eta_t)_{t\geq 0} \sim M$, let τ_0 be the time t that $\widetilde{f}_{0,t}(-1) = 1$, or in other words the time the boundary point $p_0 = -1$ of $(\mathbb{D}, \phi_0)$ intersects the zipped-in region (colored region in Figure 2 (right)). Let $g_0 : \mathbb{D} \to \mathbb{H}$ be the conformal map such that $g_0(0) = i$ and $g_0(1) = 0$. For $t < \tau_0$ let $p_t = \widetilde{f}_{0,t}(p_0) \in \partial\mathbb{D}\backslash\{1\}$, and let $g_t : \mathbb{D} \to \mathbb{H}$ be the conformal map such that $g_t(1) = 0, g_t(p_t) = \infty$, and $(g_t \circ \widetilde{f}_{0,t} \circ g_0^{-1})(z) = z + O(1)$ as $z \to \infty$. This gives us a process $(g_t \bullet_\gamma \psi_t, g_t \circ \eta_t)_{[0,\tau_0)}$ of (field, curve) pairs in $\mathbb{H}$; we reparametrize time to obtain a process $(\hat{\psi}_t, \hat{\eta}_t)_{[0,\infty)}$ such that the half-plane capacity of the trace of $\hat{\eta}_t$ is $2t$, and $\hat{\eta}_t : [0, t] \to \overline{\mathbb{H}}$ is parametrized by half-plane capacity. By Lemma 2.9, the law of $(\hat{\psi}_0, (X_t, Y_t)_{t\geq 0})$ is $\mathrm{LF}_{\mathbb{H}}^{(Q+\frac{\gamma}{4},i),(-\frac{\gamma}{2},0)} \times \mathrm{BM}_\kappa$, and by our choice of g_t the conformal maps $\widetilde{f}_{\mathbb{H},t} : \mathbb{H} \to \mathbb{H}\backslash\hat{\eta}_t([0, t])$ satisfying $\widetilde{f}_{\mathbb{H},t}(0) = \hat{\eta}_t(t)$ and $\widetilde{f}_{\mathbb{H},t}(z) = z + O(1)$ as $z \to \infty$ also satisfy $\hat{\psi}_0 = \widetilde{f}^{-1}_{\mathbb{H},t} \bullet_\gamma \hat{\psi}_t$.

By Lemma 2.15 the process $(\hat{\psi}_t, \hat{\eta}_t)$ is as described in Section 2.5, so by Lemma 2.17, for any stopping time σ for the filtration $\mathcal{F}_t = \sigma(\hat{\eta}_t)$, the law of $(\hat{\psi}_\sigma, \hat{\eta}_\sigma)$ is

$$\frac{1}{\mathcal{Z}_{Q+\frac{\gamma}{4}}(\widetilde{f}_{\mathbb{H},\sigma}(i))}\mathrm{LF}_{\mathbb{H}}^{(Q+\frac{\gamma}{4},\widetilde{f}_{\mathbb{H},\sigma}(i)),(-\frac{\gamma}{2},0)}\,\mathrm{rSLE}^\sigma_{\kappa,\kappa+6}, \qquad \mathcal{Z}_{Q+\frac{\gamma}{4}}(z) = (2\mathrm{Im}\, z)^{-\frac{(\kappa+6)^2}{8\kappa}}|z|^{\frac{\kappa+6}{\kappa}}. \tag{3.2}$$

Applying the conformal map $\mathbb{H} \to \mathbb{D}$ sending $\widetilde{f}_\sigma(i)$ to 0 and 0 to 1, and using the LCFT change of coordinates Lemma 2.9 and reverse SLE change of coordinates Lemma 2.10, we obtain i) for any stopping time τ with $\tau \leq \tau_0$ a.s.. (Note that the prefactor incurred from Lemma 2.9 cancels with the factor $1/\mathcal{Z}_{Q+\frac{\gamma}{4}}(\widetilde{f}_{\mathbb{H},\sigma}(i))$ from (3.2)).

As we will see, the above result can be iterated to get i) for all τ. By the previous paragraph, the law of $(\psi_{\tau_0}, \eta_{\tau_0})$ is $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1)} \times \mathrm{rrSLE}_\kappa^{\tau_0}$, so by the Markov property of Brownian motion, the law of $((\psi_{\tau_0+t}, \eta_{\tau_0+t}|_{[0,t]})_{t\geq 0}, \eta_{\tau_0})$ is $M \times \mathrm{rrSLE}_\kappa^{\tau_0}$. Define τ_1 for $(\psi_{\tau_0+t}, \eta_{\tau_0+t}|_{[0,t]})_{t\geq 0}$ in the same way that τ_0 was defined for $(\psi_t, \eta_t)_{t\geq 0}$, so τ_0, τ_1 are i.i.d.. Conditioning on η_{τ_0} and applying the result of the previous paragraph, we see that i) holds for any stopping time $\tau \leq \tau_0 + \tau_1$. Proceeding iteratively, we may define τ_k for all k, and i) holds for all $\tau \leq \sum_{i\leq k} \tau_i$. Since the τ_k are i.i.d. positive random variables we have $\sum_k \tau_k \to \infty$ a.s., completing the proof of i). □

The following lemma essentially tells us that if we run the process $(\psi_t, \eta_t)_{t\geq 0}$ until a random amount of quantum area has been added, if the added region is simply connected then it parametrizes a quantum cell independent of ψ_0.

The following lemma is the radial analog of [1, Proposition 5.7].

Lemma 3.6. *Let $\kappa > 8$ and $\gamma = \frac{4}{\sqrt{\kappa}}$. Sample $((\psi_t, \eta_t)_{t\geq 0}, A)$ from $M \times \mathbb{1}_{a>0} da$. Restrict to the event that there is a time $\tau > 0$ such that $\mathcal{A}_{\psi_\tau}(\eta_\tau([0,\tau])) = A$. Then the law of $(\psi_\tau, \eta_\tau, \tau)$ is $C \cdot \mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)} \times \mathrm{raSLE}_\kappa^t \mathbb{1}_{t>0} dt$ for some constant $C > 0$. Here raSLE_κ^t denotes the law of radial SLE_κ in $\mathbb{D}$ from 1 to 0 parametrized by log-conformal radius stopped at time t.*

Proof. Here is a proof sketch; just for this proof, we use the shorthand $s_+ := \max\{s, 0\}$. First, if we fix $\delta > 0$ and sample $((\psi_t, \eta_t)_{t\geq 0}, A, T) \sim \delta^{-1}\mathbb{1}_{T\in[\tau,\tau+\delta]} M \times \mathbb{1}_{A>0}\, dA \times dT$ then the marginal law of $((\psi_t, \eta_t)_{t\geq 0}, A)$ is $M \times \mathbb{1}_{a>0} da$, so the marginal law of $(\psi_\tau, \eta_\tau, \tau)$ is the same as in Lemma 3.6. In this new setup, the constraint $\{T \in [\tau, \tau+\delta]\} = \{\tau \in [(T-\delta)_+, T]\}$ is the same as

$$\mathcal{A}_{\psi_T}(\eta_T([0,T])) \geq A \geq \mathcal{A}_{\psi_{(T-\delta)_+}}(\eta_{(T-\delta)_+}([0,(T-\delta)_+])).$$

Note that the lower bound equals $\mathcal{A}_{\psi_T}(\eta_T([\delta \wedge T, T]))$ by ii) of Lemma 3.4, so using i) of Lemma 3.4, the law of (A, ψ_T, η_T, T) is then

$$\delta^{-1}\mathbb{1}_{a\in[\mathcal{A}_\psi(\eta([\delta\wedge t,t])),\mathcal{A}_\psi(\eta([0,t]))]} da \times \mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1)}(d\psi) \times \mathrm{rrSLE}_\kappa^t(d\eta)\mathbb{1}_{t>0}\, dt.$$

Let $z = \eta_T(T-\tau)$. Since $\mathcal{A}_{\psi_T}(\eta_T([T-\tau,T])) = A$, we have $\{T \in [\tau, \tau+\delta]\} = \{z \in \eta_T([0, \delta\wedge T])\}$. On the other hand, from the definition of τ, z can be viewed as a point sampled on $\eta_T([0,\delta\wedge T])$ according to the measure $\mathcal{A}_{\psi_T}$. Therefore the law of (z, ψ_T, η_T, T) is $\delta^{-1}\mathbb{1}_{z\in\eta([0,\delta\wedge t])}\mathcal{A}_\psi(dz)\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1)}(d\psi) \times \mathrm{raSLE}_\kappa^t(d\eta)\mathbb{1}_{t>0}\, dt$. Note we have obtained the term $\mathrm{raSLE}_\kappa^t \mathbb{1}_{t>0}\, dt$ using the symmetry between forward and reverse radial SLE_κ at fixed time t. Using Lemma 2.8, this law is

$$\delta^{-1}\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1),(\gamma,z)}(d\psi)\mathbb{1}_{z\in\eta([0,\delta\wedge t])}dz\, \mathrm{raSLE}_\kappa^t(d\eta)\mathbb{1}_{t>0}\, dt.$$

As $\delta \to 0$ we have $T - \tau \to 0$ so $z \to 1$, so in the limit the field has the singularity $\gamma G(\cdot,1) - \frac{\gamma}{4}G(\cdot,1) = \frac{1}{2}(\frac{3\gamma}{2})G(\cdot,1)$ at 1. This explains the term $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}$. The main difficulty in this argument is in taking limits of infinite measures; this is done by truncating on finite events and taking limits of finite measures.

The argument outlined above is implemented in the proof of [1, Proposition 5.7], a chordal analog of our desired result; we refer the reader there for details. The only part of that proof that does not immediately carry over to our setting is a certain finiteness claim [1, Lemma 5.8], whose analog in our setting can be stated as follows. For ρ the uniform probability measure on $\{z : |z| = \frac{1}{2}\}$ (the precise choice of ρ is unimportant), we have

$$(M \times \mathbb{1}_{a>0}da)[E_N] < \infty \text{ where } E_N := \{\tau, |(\psi_0,\rho)|, |(\psi_\tau,\rho)| < N\}. \tag{3.3}$$

Given this, the proof of our Lemma 3.6 is identical to that of [1, Proposition 5.7]. Thus it suffices to prove (3.3).

First, we observe $(M \times \mathbb{1}_{a>0}da)[E_N] \leq (M \times \mathbb{1}_{a>0}da)[\widetilde{E}_N] = M[\mathcal{A}_{\psi_N}(\eta_N([0,N]))\mathbb{1}_{|(\psi_0,\rho)|<N}]$ where $\widetilde{E}_N = \{\tau, |(\psi_0,\rho)| < N\}$. Now, our choice of parametrization implies the conformal radius of $\eta_N([0,N])$ viewed from 0 is e^{-N}, so the Koebe quarter theorem implies that the ball $B_{e^{-N}/4}(0)$ is contained in $\mathbb{D}\backslash\eta_N([0,N])$. By Lemma 3.4 the M-law of (ψ_N, η_N) is $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1)} \times \mathrm{raSLE}_\kappa^N$, so it suffices to show the finiteness of

$$(\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(-\frac{\gamma}{2},1)} \times \mathrm{raSLE}_\kappa^N)[\mathcal{A}_\psi(\mathbb{D}\backslash B_{e^{-N}/4}(0))\mathbb{1}_{|(f_N\bullet_\gamma\psi,\rho)|<N}], \tag{3.4}$$

where for the raSLE$_\kappa^N$ curve η the conformal map $f_N : \mathbb{D}\backslash\eta([0,N]) \to \mathbb{D}$ satsifies $f_N(0) = 0$ and $f_N(\eta(N)) = 1$. Writing $\mathbb{E}$ to denote expectation with respect to $(h,\eta) \sim P_\mathbb{D} \times \text{raSLE}_\kappa^N$ and $\widetilde{h} = h + (Q+\frac{\gamma}{4})G_\mathbb{D}(\cdot,0) - \frac{\gamma}{4}G_\mathbb{D}(\cdot,1)$, this equals

$$\mathbb{E}\big[\int_\mathbb{R} \mathcal{A}_{\widetilde{h}+c}(\mathbb{D}\backslash B_{e^{-N}/4}(0))\mathbb{1}_{|(f_N\bullet_\gamma\widetilde{h},\rho)+c|<N}\,dc\big] = \mathbb{E}\big[\int_{-(f_N\bullet_\gamma\widetilde{h},\rho)-N}^{-(f_N\bullet_\gamma\widetilde{h},\rho)+N} e^{\gamma c}\mathcal{A}_{\widetilde{h}}(\mathbb{D}\backslash B_{e^{-N}/4}(0))\,dc\big]$$
$$= \frac{1}{\gamma}(e^{\gamma N} - e^{-\gamma N})\mathbb{E}\big[e^{-\gamma(f_N\bullet_\gamma\widetilde{h},\rho)}\mathcal{A}_{\widetilde{h}}(\mathbb{D}\backslash B_{e^{-N}/4}(0))\big].$$

To see this is finite, first note that $Z := \mathbb{E}[e^{-\gamma(f_N\bullet_\gamma\widetilde{h},\rho)}] < \infty$ by standard conformal distortion estimates. Next, by Girsanov's theorem, the expression equals $\frac{1}{\gamma}(e^{\gamma N} - e^{-\gamma N})Z\mathbb{E}[\mathcal{A}_{\hat{h}}(\mathbb{D}\backslash B_{e^{-N}/4}(0)]$ where $\hat{h} = h + (Q+\frac{\gamma}{4})G_\mathbb{D}(\cdot,0) - \frac{\gamma}{4}G_\mathbb{D}(\cdot,1) - \gamma\int G_\mathbb{D}(\cdot,w)((f_N^{-1})_*\rho)(dw)$. To finish, we note that $\hat{h} - h$ is bounded above by a constant on $\mathbb{D}\backslash B_{e^{-N}/4}(0)$, and that $\mathbb{E}[\mathcal{A}_h(\mathbb{D}\backslash B_{e^{-N}/4}(0))] < \infty$ by standard GMC moment results, see for instance [38, Proposition 3.5]. We conclude that (3.4), and hence (3.3), is finite. □

Finally, between two "quantum typical" times for $(\psi_t, \eta_t) \sim M$, given the field and curve at the earlier time, on the event the zipped-in quantum surface is simply connected, it is a quantum cell with a boundary length restriction.

Lemma 3.7. *Let $\kappa > 8$ and $\gamma = \frac{4}{\sqrt{\kappa}}$, and fix $a_1, a_2 > 0$. Sample $(\psi_t,\eta_t)_{t\geq 0}$ from M and restrict to the event that there is a time $\tau_2 > 0$ such that $\mathcal{A}_{\psi_{\tau_2}}(\eta_{\tau_2}([0,\tau_2])) = a_1 + a_2$. Let τ_1 be the time that $\mathcal{A}_{\psi_{\tau_1}}(\eta_{\tau_1}([0,\tau_1])) = a_1$. Conditioned on $(\psi_{\tau_1},\eta_{\tau_1})$, the law of $(\eta_{\tau_2}([0,\tau_2-\tau_1]),\psi_{\tau_2},\eta_{\tau_2}|_{[0,\tau_2-\tau_1]})/{\sim_\gamma}$ restricted to the event $\{\eta_{\tau_2}([0,\tau_2-\tau_1])$ is simply connected$\}$ is*

$$\mathbb{1}_{X^+_{a_2}(\mathcal{C})+Y^+_{a_2}(\mathcal{C})<\mathcal{L}_{\psi_{\tau_1}}(\partial\mathbb{D})}P_{a_2}(d\mathcal{C})$$

where $X^+_{a_2}$ and $Y^+_{a_2}$ are as in Definition 2.13.

Proof. Let $(X_t,Y_t)_{t\geq 0}$ be the process in the definition of M, then the law of $\widetilde{\mathcal{C}} := F((X_{\cdot+a_1}, Y_{\cdot+a_1}))_{[0,a_2]}$ is P_{a_2}, and reversing the orientation of the curve of $\widetilde{\mathcal{C}}$ gives $\mathcal{C} := (\eta_{\tau_2}([0,\tau_2-\tau_1]),\psi_{\tau_2},\eta_{\tau_2}|_{[0,\tau_2-\tau_1]})/{\sim_\gamma}$. By construction $\{\eta_{\tau_2}([0,\tau_2-\tau_1])$ is simply connected$\} = \{X^-_{a_2}(\widetilde{\mathcal{C}}) + Y^-_{a_2}(\widetilde{\mathcal{C}}) < \mathcal{L}_{\psi_{\tau_1}}(\partial\mathbb{D})\}$, and since $X^-_{a_2}(\widetilde{\mathcal{C}}) = Y^+_{a_2}(\mathcal{C})$ and $Y^-_{a_2}(\widetilde{\mathcal{C}}) = X^+_{a_2}(\mathcal{C})$, this event equals $\{X^+_{a_2}(\mathcal{C}) + Y^+_{a_2}(\mathcal{C}) < \mathcal{L}_{\psi_{\tau_1}}(\partial\mathbb{D})\}$ as needed. □

3.2 Cutting an infinite volume LCFT disk until a quantum typical time

The aim of this section is to prove Proposition 3.8 below. We write raSLE$_\kappa^t$ for the law of radial SLE$_\kappa$ in $\mathbb{D}$ from 1 to 0 stopped at time t, and raSLE$_\kappa^z$ for the law of radial SLE$_\kappa$ in $\mathbb{D}$ from 1 to 0 stopped when it hits $z \in \overline{\mathbb{D}}\backslash\{0\}$.

Proposition 3.8. *Suppose $\kappa > 8$ and $\gamma = \frac{4}{\sqrt{\kappa}}$. Sample (ϕ,η,A) from the measure*

$$\text{LF}_\mathbb{D}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)} \times \text{raSLE}_\kappa \times \mathbb{1}_{a>0}da \tag{3.5}$$

and parametrize η by its $\mathcal{A}_\phi$ quantum area. For $a \geq 0$, let $f_a : \mathbb{D}\backslash\eta([0,a]) \to \mathbb{D}$ be the conformal map such that $f_a(0) = 0$ and $f_a(\eta(a)) = 1$. Let $\phi_a = f_a \bullet_\gamma \phi$, $\widetilde{\eta}_a = f_a \circ \eta|_{[a,\infty)}$, and $\mathcal{C}_a = (\eta([0,a]),\phi,\eta|_{[0,a]})/{\sim_\gamma}$. Then the law of $(\phi_A,\widetilde{\eta}_A,A)$ is given by[3]

$$\text{LF}_\mathbb{D}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)} \times \text{raSLE}_\kappa \times \mathbb{1}_{a>0}da. \tag{3.6}$$

Moreover, the law of $(\phi_A,\widetilde{\eta}_A,\mathcal{C}_A,A)$ restricted to the event that $\eta([0,A])$ is simply connected is given by

$$\mathbb{1}_{X^+_a(\mathcal{C}_a)+Y^+_a(\mathcal{C}_a)<\mathcal{L}_{\phi_a}(\partial\mathbb{D})}\text{LF}_\mathbb{D}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi_a) \times \text{raSLE}_\kappa \times P_a(d\mathcal{C}_a)\,\mathbb{1}_{a>0}da. \tag{3.7}$$

where X^+_a, Y^+_a are as in Definition 2.13.

[3]There is a slight abuse of notation here: the curve $\widetilde{\eta}_A$ should be viewed as parametrized by log-conformal radius rather than by quantum area for (3.6) to hold. We do this because this section is already notationally dense.

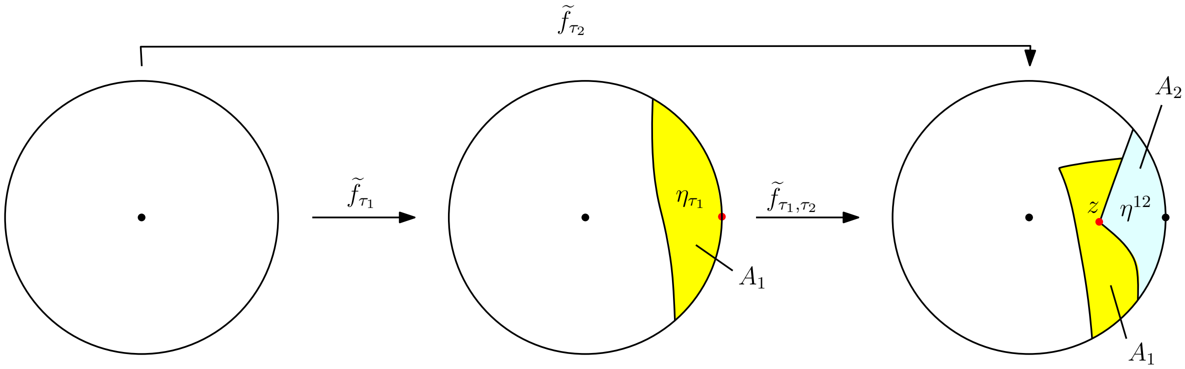

Figure 3: Setup for the proof of Proposition 3.8. We sample A_1, A_2 from $\mathbb{1}_{A_1,A_2>0} dA_1 dA_2$. The middle panel corresponds to the time τ_1 where the yellow quantum cell (filled with the curve η_{τ_1}) has quantum area A_1 has been "zipped in", while in the right panel we continue to time τ_2 where we have conformally welded the blue quantum cell (filled with the curve η^{12}) with quantum area A_2. In the right cell, the curve η_{τ_2} is the concatenation of the curves in the blue and yellow cells, and z corresponds to the point $\eta_{\tau_2}(\tau_2 - \tau_1)$.

Proof of Proposition 3.8. Here is a proof outline. Consider the setup in Figure 3 where we conformally weld quantum cells of areas A_1 and A_2 in the definition of M. Since A_1, A_2 are sampled from Lebesgue measure, the point z can be viewed as being sampled from $\mathcal{A}_{\psi_{\tau_2}}$-measure on $\eta_{\tau_2}([0,\tau_2])$, and the radial SLE_κ η_{τ_2} can be decomposed into $\widetilde{f}_{\tau_1,\tau_2} \circ \eta_{\tau_1}$ and η^{12}. Then by Lemma 3.6 $(\psi_{\tau_2}, \eta^{12}, A_2) \overset{d}{=} (\phi, \eta|_{[0,A]}, A)$. Passing to the middle panel via $\widetilde{f}^{-1}_{\tau_1,\tau_2}$ gives a description of the law of (ϕ_A, η_A, A) in terms of that of $(\psi_{\tau_1}, \eta_{\tau_1}, \tau_1)$, and the conclusion follows from another application of Lemma 3.6.

To streamline notation in this proof, we will often use the same notation for a random object as in the description of its law (in the indented equations), or similar notation (e.g. use $d\psi_{t_2}$ in a description of the law of ψ_{τ_2}). To begin with, sample $(\{(\psi_t, \eta_t)_{t\ge 0}\}, A_1, A_2)$ from $M \times \mathbb{1}_{A_1,A_2>0} dA_1 dA_2$, and let τ_1 (resp. τ_2) be the time t when $\mathcal{A}_{\psi_t}(\eta_t([0,t]))$ equals A_1 (resp. $A_1 + A_2$). We restrict to the event E that these times exist ($\tau_1 < \tau_2 < \infty$). Let $z = \eta_{\tau_2}(\tau_2 - \tau_1)$, $S = A_1 + A_2$, and $\eta^{12} = \eta_{\tau_2}|_{[0,\tau_2-\tau_1]}$. Then the law of $((\psi_t, \eta_t)_{t\ge0}, A_1, S)$ is $\mathbb{1}_E M \times \mathbb{1}_{A_1\in[0,S]} dA_1 \mathbb{1}_{S>0} dS$, so by Lemma 3.6 applied to $((\psi_t,\eta_t)_{t\ge0}, S)$, the law of $(A_1, \psi_{\tau_2}, \eta_{\tau_2}, \tau_2)$ is

$$C \cdot \mathbb{1}_{A_1 \in [0, \mathcal{A}_{\psi_{t_2}}(\eta_{t_2}([0,t_2]))]} dA_1 \, \mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\psi_{t_2}) \, \mathrm{raSLE}_\kappa^{t_2}(d\eta_{t_2}) \, \mathbb{1}_{t_2>0} dt_2.$$

Since z is the point where η_{τ_2} covers $S - A_1$ units of quantum area when hitting z, it follows that the law of $(z, \psi_{\tau_2}, \eta_{\tau_2}, \tau_2)$ is

$$C \cdot \mathbb{1}_{z\in\eta_{t_2}([0,t_2])} \mathcal{A}_{\psi_{t_2}}(dz) \, \mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\psi_{t_2}) \, \mathrm{raSLE}_\kappa^{t_2}(d\eta_{t_2}) \, \mathbb{1}_{t_2>0} dt_2.$$

Then by Lemma 3.9 below, the law of $(z, \psi_{\tau_2}, \eta^{12}, (\eta_{\tau_1}, \tau_1))$ is

$$\mathcal{A}_{\psi_{t_2}}(dz) \mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\psi_{t_2}) \, \mathrm{raSLE}_\kappa^{z}(d\eta^{12}) \times [C \cdot \mathrm{raSLE}_\kappa^{t_1}(d\eta_{t_1}) \, \mathbb{1}_{t_1>0} dt_1] \tag{3.8}$$

where raSLE_κ^z is as defined before Proposition 3.8.

Since (ϕ, η, A) is sampled from (3.5) and η is parametrized by quantum area, the law of $(\eta(A), \phi, \eta|_{[0,A]})$ is $\mathcal{A}_\phi(du) \mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi) \, \mathrm{raSLE}_\kappa^u(d\eta)$. Then, by the domain Markov property of radial SLE_κ, if we instead sample (ϕ, η, A, t') from

$$\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi) \times \mathrm{raSLE}_\kappa(d\eta) \times \mathbb{1}_{a>0} \, da \times [C \mathbb{1}_{t>0} \, dt] \tag{3.9}$$

(or "independently sample t' from $[C\mathbb{1}_{t>0} dt]$") then the law of $(\eta(A), \phi, \eta|_{[0,A]}, (\widetilde{\eta}_A|_{[0,t']}, t'))$ is

$$\mathcal{A}_\phi(du) \mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi) \, \mathrm{raSLE}_\kappa^u(d\eta) \times [C \cdot \mathrm{raSLE}_\kappa^t \mathbb{1}_{t>0} \, dt].$$

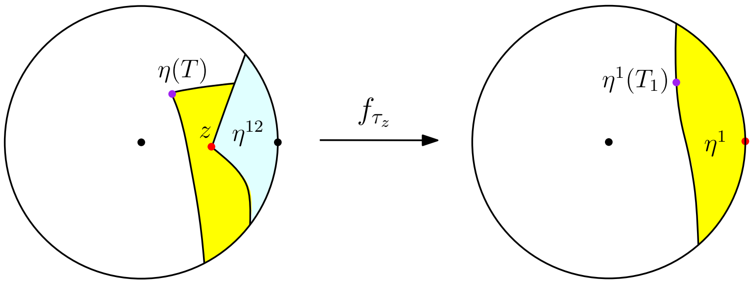

Figure 4: An illustration of Lemma 3.9, where z is fixed, T is sampled from Lebesgue measure restricted to the event that the radial SLE_κ curve η covers z at time $\tau_z < T$. Let $T_1 = T - \tau_z$ and f_{τ_z} be the centered Loewner map at time τ_1. We show that the law of $(\eta^{12}, (\eta^1, T_1))$ is $\mathrm{raSLE}_\kappa^z \times [\mathrm{raSLE}_\kappa^{t_1} \mathbb{1}_{t_1>0} dt_1]$.

This law agrees with (3.8) up to renaming random variables, so $(\eta(A), \phi, \eta|_{[0,A]}, \widetilde{\eta}_A|_{[0,t']}, t') \stackrel{d}{=} (z, \psi_{\tau_2}, \eta^{12}, \eta_{\tau_1}, \tau_1)$. Since $A_2 = \mathcal{A}_{\psi_{\tau_2}}(\eta^{12})$, $\psi_{\tau_1} = \widetilde{f}_{\tau_1,\tau_2}^{-1} \bullet_\gamma \psi_{\tau_2}$ where $\widetilde{f}_{\tau_1,\tau_2} : \mathbb{D} \to \mathbb{D} \backslash \eta^{12}$ is the conformal map fixing 0 and sending 1 to the tip of η^{12}, it follows that $(\phi_A, A, \widetilde{\eta}_A|_{[0,t']}, t') \stackrel{d}{=} (\psi_{\tau_1}, A_2, \eta_{\tau_1}, \tau_1)$.

On the other hand, by Lemma 3.6, the law of $(\psi_{\tau_1}, A_2, \eta_{\tau_1}, \tau_1)$ is

$$\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\psi_{t_1})\, \mathbb{1}_{A_2>0} dA_2 \times \mathrm{raSLE}_\kappa^{t_1}(d\eta_{t_1})\, [C\mathbb{1}_{t_1>0} dt_1]. \tag{3.10}$$

Note the term $[C\mathbb{1}_{t_1>0}dt_1]$ above corresponds to $[C\mathbb{1}_{t>0}dt]$ in (3.9), so by varying t', for (ϕ, η, A) sampled from (3.5) the law of $(\phi_A, \widetilde{\eta}_A, A)$ is given by (3.7). This concludes the proof of the first claim.

For the second claim, we repeat the above except we restrict to the event $F := \{\eta^{12}$ is simply connected$\}$ throughout. Then the law of $(z, \psi_{\tau_2}, \eta^{12}, \eta_{\tau_1}, \tau_1)$ is $\mathbb{1}_F$ times (3.8), and by Lemmas 3.6 and 3.7, the law of $(\psi_{\tau_1}, \eta_{\tau_1}, (\eta^{12}([0, \tau_2 - \tau_1]), \psi_{\tau_2}, \eta^{12})/{\sim_\gamma}, A_2, \tau_1)$ is

$$\mathbb{1}_{X^+_{A_2}(\mathcal{C}) + Y^+_{A_2}(\mathcal{C}) < \mathcal{L}_{\psi_{t_1}}(\partial\mathbb{D})} \mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\psi_{t_1})\, \mathrm{raSLE}_\kappa^{t_1}(d\eta_{t_1})\, P_{A_2}(d\mathcal{C})\, \mathbb{1}_{A_2>0} dA_2\, C\mathbb{1}_{t_1>0} dt_1,$$

c.f. (3.10). The same argument as that of the first claim then gives the second claim. □

In the above proof we needed the following lemma, see Figure 4.

Lemma 3.9. *Fix $z \in \mathbb{D}$, and sample (η, T) from $\mathbb{1}_{z\in\eta([0,t])}\mathrm{raSLE}_\kappa(d\eta)\mathbb{1}_{t>0}\,dt$ where η is parametrized by log-conformal radius seen from 0. Let τ_z be the time when η hits z, $T_1 = T - \tau_z$ and $\eta^{12} = \eta|_{[0,\tau_z]}$. Let $f_{\tau_z} : \mathbb{D}\backslash\eta([0,\tau_z]) \to \mathbb{D}$ be the centered Loewner map of η at time τ_z, and $\eta^1 = f_{\tau_z} \circ \eta(\cdot + \tau_z)|_{[0,T_1]}$. Then the law of $(\eta^{12}, (\eta^1, T_1))$ is $\mathrm{raSLE}_\kappa^z \times [\mathrm{raSLE}_\kappa^{t_1} \mathbb{1}_{t_1>0}dt_1]$, where raSLE_κ^z is the law of radial SLE_κ run until it hits z, and $\mathrm{raSLE}_\kappa^{t_1}$ is the law of the radial SLE_κ curve stopped at the time when the log-conformal radius seen from 0 equals t_1 as in Lemma 3.6.*

Proof. By a change of variables, the law of (η^{12}, T_1) is $\mathrm{raSLE}_\kappa^z(d\eta^{12}) \times \mathbb{1}_{t_1>0}dt$. By the domain Markov property of radial SLE, conditioned on η^{12} and T_1, the law of η^1 is $\mathrm{raSLE}_\kappa^{T_1}$. This finishes the proof. □

3.3 Proof of Theorem 3.1

In this section we prove Theorem 3.1. We first use Proposition 3.8 about $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}$ to obtain an analogous result for $\mathrm{LF}_{\mathbb{D}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}$ (Proposition 3.12). The idea is to weight the field to change $Q + \frac{\gamma}{4}$ into $Q - \frac{\gamma}{4}$ via the Girsanov theorem; this is done in Lemmas 3.10 and 3.11.

Let $B_\varepsilon(0) := \{z \in \mathbb{C} : |z| < \varepsilon\}$, and let θ_ε denote the uniform probability measure on the circle $\partial B_\varepsilon(0)$.

Lemma 3.10. *Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ and $\varepsilon \in (0,1)$. Let $\widetilde{\mathrm{LF}}_{\mathbb{D},\varepsilon}^{(\alpha_2,0),(\beta,1)}$ be the law of $\psi|_{\mathbb{D}\backslash B_\varepsilon(0)}$, where $\psi \sim \mathrm{LF}_{\mathbb{D}}^{(\alpha_2,0),(\beta,1)}$. Sample ϕ from the measure $\mathrm{LF}_{\mathbb{D}}^{(\alpha_1,0),(\beta,1)}$ and weight its law by $\varepsilon^{\frac{1}{2}(\alpha_2^2-\alpha_1^2)}e^{(\alpha_2-\alpha_1)(\phi,\theta_\varepsilon)}$. Then the law of $\phi|_{\mathbb{D}\backslash B_\varepsilon(0)}$ is $\widetilde{\mathrm{LF}}_{\mathbb{D},\varepsilon}^{(\alpha_2,0),(\beta,1)}$.*

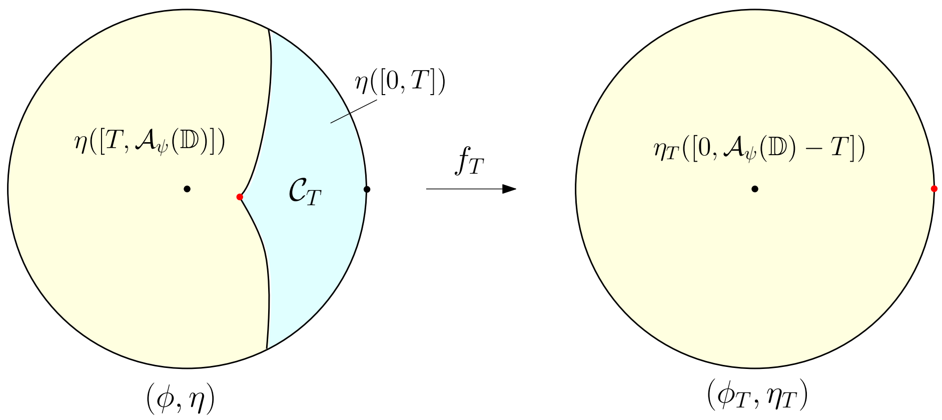

Figure 5: An illustration for Proposition 3.12. We prove that by cutting the quantum disk $(\mathbb{D},\phi,0,1)$ with the radial SLE_κ curve η up to quantum time T restricted to the event $\eta([0,T])$ is simply connected, one gets an independent pair of a quantum cell $\mathcal{C}_T$ and a quantum disk $(\mathbb{D},\phi_T,0,1)$ after restricting to the event $\{X_T^+(\mathcal{C})+Y_T^+(\mathcal{C})<\mathcal{L}_{\phi_T}(\partial\mathbb{D})\}$.

Proof. Recall $P_{\mathbb{D}}$ is the law of the free boundary GFF on $\mathbb{D}$ normalized to have average 0 on $\partial\mathbb{D}$. By Girsanov's theorem, for h sampled from $P_{\mathbb{D}}$ weighted by $\varepsilon^{\frac{1}{2}\alpha^2}e^{\alpha(h,\theta_\varepsilon)}$, we have $h|_{\mathbb{D}\backslash B_\varepsilon(0)}\stackrel{d}{=}(h'-\alpha\log|\cdot|)|_{\mathbb{D}\backslash B_\varepsilon(0)}$ where $h'\sim P_{\mathbb{D}}$. In other words, this weighting introduces an α-log singularity at 0. Using the above and keeping track of the terms that arise in the definition of the Liouville field, the lemma follows from a direct computation. See [5, Lemma 4.7] for details in the case where $\alpha_1=\beta=\gamma$; the argument is identical in our setting. □

Lemma 3.11. *Let $\alpha_1,\alpha_2,\beta\in\mathbb{R}$ and $\varepsilon\in(0,1)$. Let $z\in\mathbb{D}\backslash\{0\}$ and let $K\subset\overline{\mathbb{D}}$ be a compact set such that $\mathbb{D}\backslash K$ is simply connected, contains 0, and has z on its boundary. Let $f:\mathbb{D}\backslash K\to\mathbb{D}$ be the conformal map such that $f(0)=0$ and $f(z)=1$. Let $\widetilde{\mathrm{LF}}_{\mathbb{D},K,\varepsilon}^{(\alpha_2,0),(\gamma,z),(\beta,1)}$ be the law of $\psi|_{\mathbb{D}\backslash f^{-1}(B_\varepsilon(0))}$ where $\psi\sim\mathrm{LF}_{\mathbb{D}}^{(\alpha_2,0),(\gamma,z),(\beta,1)}$.*

- *Define the pushforward measure $\hat\theta_\varepsilon=f_*^{-1}\theta_\varepsilon$. For $\phi\sim\mathrm{LF}_{\mathbb{D}}^{(\alpha_1,0),(\gamma,z),(\beta,1)}$ with its law weighted by $(|(f^{-1})'(0)|\varepsilon)^{\frac{1}{2}(\alpha_2^2-\alpha_1^2)}e^{(\alpha_2-\alpha_1)(\phi,\hat\theta_\varepsilon)}$, the law of $\phi|_{\mathbb{D}\backslash f^{-1}(B_\varepsilon(0))}$ is $\widetilde{\mathrm{LF}}_{\mathbb{D},K,\varepsilon}^{(\alpha_2,0),(\gamma,z),(\beta,1)}$.*
- *Suppose $\alpha_1+\alpha_2=2Q$. For $\phi\sim\mathrm{LF}_{\mathbb{D}}^{(\alpha_1,0),(\gamma,z),(\beta,1)}$ with its law weighted by $\varepsilon^{\frac{1}{2}(\alpha_2^2-\alpha_1^2)}e^{(\alpha_2-\alpha_1)(f\bullet_\gamma\phi,\theta_\varepsilon)}$, the law of $\phi|_{\mathbb{D}\backslash f^{-1}(B_\varepsilon(0))}$ is $\widetilde{\mathrm{LF}}_{\mathbb{D},K,\varepsilon}^{(\alpha_2,0),(\gamma,z),(\beta,1)}$.*

Proof. The first claim follows from the same argument as that of Lemma 3.10. Indeed, $|(f^{-1})'(0)|\varepsilon$ is the conformal radius of $f^{-1}(\partial B_\varepsilon(0))$ viewed from 0 and $\hat\theta_\varepsilon$ is a probability measure on $f(\partial B_\varepsilon(0))$, and these play the role of ε and θ_ε in Lemma 3.10. See [5, Lemma 4.8] for details. For the second claim, note that

$$(f\bullet_\gamma\phi,\theta_\varepsilon)=(\phi\circ f^{-1}+Q\log|(f^{-1})'|,\theta_\varepsilon)=(\phi,f_*^{-1}\theta_\varepsilon)+Q(\log|(f^{-1})'|,\theta_\varepsilon)=(\phi,\hat\theta_\varepsilon)+Q\log|(f^{-1})'(0)|.$$

Since $\alpha_1+\alpha_2=2Q$ implies $(\alpha_2-\alpha_1)Q=\frac{1}{2}(\alpha_2^2-\alpha_1^2)$, we conclude $(|(f^{-1})'(0)|\varepsilon)^{\frac{1}{2}(\alpha_2^2-\alpha_1^2)}e^{(\alpha_2-\alpha_1)(\phi,\hat\theta_\varepsilon)}=\varepsilon^{\frac{1}{2}(\alpha_2^2-\alpha_1^2)}e^{(\alpha_2-\alpha_1)(f\bullet_\gamma\phi,\theta_\varepsilon)}$. This with the first claim gives the second claim. □

Proposition 3.12. *Let (ϕ,η,T) be a sample from $\mathbb{1}_{0<t<\mathcal{A}_\phi(\mathbb{D})}\mathrm{LF}_{\mathbb{D}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi)\times\mathrm{raSLE}_\kappa(d\eta)\times dt$ and parametrize η by its $\mathcal{A}_\phi$ quantum area. For $t>0$, let $f_t:\mathbb{D}\backslash\eta([0,t])\to\mathbb{D}$ be the conformal map fixing 0 such that $f_t(\eta(t))=1$. Let $\phi_t=f_t\bullet_\gamma\phi$, $\eta_t(s)=f_t(\eta(s+t))$ for $0\le s\le\mathcal{A}_\phi(\mathbb{D})-t$, and $\mathcal{C}_t=(\eta([0,t]),\phi,\eta|_{[0,t]})/{\sim_\gamma}$. Restricted to the event that $\eta([0,T])$ is simply connected, the law of $(\phi_T,\eta_T,\mathcal{C}_T,T)$ is*

$$\mathbb{1}_{X_t^+(\mathcal{C})+Y_t^+(\mathcal{C})<\mathcal{L}_{\phi_t}(\partial\mathbb{D})}\mathrm{LF}_{\mathbb{D}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi_t)\times\mathrm{raSLE}_\kappa(d\eta)\times P_t(d\mathcal{C})\,\mathbb{1}_{t>0}dt,\tag{3.11}$$

where $X_t^+(\mathcal{C}),Y_t^+(\mathcal{C})$ are as in Definition 2.13.

Proof. See Figure 5. Sample $(\widetilde{\phi}, z, \widetilde{\eta})$ from

$$\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\gamma,z),(\frac{3\gamma}{2},1)}(d\widetilde{\phi})\,\mathrm{raSLE}_\kappa(d\widetilde{\eta})\,\mathbb{1}_{z\in\mathbb{D}}dz$$

and parametrize $\widetilde{\eta}$ by $\mathcal{A}_\phi$-quantum area. Let A be the time such that $\widetilde{\eta}(A) = z$, let $\widetilde{\eta}^z = \widetilde{\eta}|_{[0,A]}$, let $f : \mathbb{D}\backslash\widetilde{\eta}^z \to \mathbb{D}$ be the conformal map such that $f(0) = 0$ and $f(z) = 1$, let $\widetilde{\phi}_A = f \bullet_\gamma \widetilde{\phi}$, let $\widetilde{\eta}_A = f \circ \widetilde{\eta}(\cdot + A)$ and let $\widetilde{\mathcal{C}}_A = (\widetilde{\eta}^z([0, A]), \widetilde{\phi}, \widetilde{\eta}^z)/{\sim_\gamma}$. By Lemma 2.8 the law of $(\widetilde{\phi}, \widetilde{\eta}, A)$ is $\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\widetilde{\phi}) \times \mathrm{raSLE}_\kappa(d\widetilde{\eta}) \times \mathbb{1}_{a>0}da$ so Proposition 3.8 implies the law of $(\widetilde{\phi}_A, \widetilde{\eta}_A, \widetilde{\mathcal{C}}_A, A)$ restricted to the event $\{\widetilde{\eta}^z$ is simply connected$\}$ is

$$\mathbb{1}_{\widetilde{X}_a^+(\widetilde{\mathcal{C}})+\widetilde{Y}_a^+(\widetilde{\mathcal{C}})<\mathcal{L}_{\widetilde{\phi}_a}(\partial\mathbb{D})}\mathrm{LF}_{\mathbb{D}}^{(Q+\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\widetilde{\phi}_a) \times \mathrm{raSLE}_\kappa \times P_a(d\widetilde{\mathcal{C}})\mathbb{1}_{a>0}da.$$

Now for $\varepsilon > 0$, let θ_ε be the uniform probability measure on $\partial B_\varepsilon(0)$. Let $\alpha_1 = Q + \frac{\gamma}{4}$ and $\alpha_2 = Q - \frac{\gamma}{4}$. Weight the law of $(\widetilde{\phi}, z, \widetilde{\eta})$ by $\varepsilon^{\frac{\alpha_2^2-\alpha_1^2}{2}}e^{(\alpha_2-\alpha_1)(\widetilde{\phi},\theta_\varepsilon)}$. By Lemma 3.11 the law of $(\widetilde{\phi}|_{\mathbb{D}\backslash\widetilde{f}_z^{-1}(B_\varepsilon(0))}, z, \widetilde{\eta})$ under this weighting is

$$\widetilde{\mathrm{LF}}_{\mathbb{D},\widetilde{\eta}^z,\varepsilon}^{(Q-\frac{\gamma}{4},0),(\gamma,z),(\frac{3\gamma}{2},1)}(d\widetilde{\phi})\,\mathrm{raSLE}_\kappa(d\widetilde{\eta})\,\mathbb{1}_{z\in\mathbb{D}}dz.$$

On the other hand, by Lemma 3.10, the weighted law of $(\widetilde{\phi}_A|_{\mathbb{D}\backslash B_\varepsilon(0)}, \widetilde{\eta}_A, \widetilde{\mathcal{C}}_A, A)$ restricted to the event $\{\widetilde{\eta}^z$ is simply connected$\}$ is

$$\mathbb{1}_{\widetilde{X}_a^+(\widetilde{\mathcal{C}})+\widetilde{Y}_a^+(\widetilde{\mathcal{C}})<\mathcal{L}_{\widetilde{\phi}_a}(\partial\mathbb{D})}\widetilde{\mathrm{LF}}_{\mathbb{D},\varepsilon}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\widetilde{\phi}_a) \times \mathrm{raSLE}_\kappa \times P_a(d\widetilde{\mathcal{C}})\mathbb{1}_{a>0}da. \tag{3.12}$$

To rephrase, if $(\widetilde{\phi}, z, \widetilde{\eta})$ is sampled from

$$\mathrm{LF}_{\mathbb{D}}^{(Q-\frac{\gamma}{4},0),(\gamma,z),(\frac{3\gamma}{2},1)}(d\widetilde{\phi})\,\mathrm{raSLE}_\kappa(d\widetilde{\eta})\,\mathbb{1}_{z\in\mathbb{D}}dz \tag{3.13}$$

with $\widetilde{\eta}$ parametrized by quantum area, and A is the time when $\widetilde{\eta}$ hits z, then on the event where $\widetilde{\eta}^z$ is simply connected, the law of $(\widetilde{\phi}_A|_{\mathbb{D}\backslash B_\varepsilon(0)}, \widetilde{\eta}_A, \widetilde{\mathcal{C}}_A, A)$ is given by (3.12). Sending $\varepsilon \to 0$, the same statement holds for $\varepsilon = 0$ when (3.12) is replaced by (3.11). On the other hand, by Lemma 2.8, the law of $(\phi, \eta, \eta(T))$ is given by (3.13) (up to renaming of variables). We conclude the proof by observing that the pair $(\phi, \eta, \eta(T))$ and the pair (ϕ, η, T) uniquely determine each other. □

Recall the disintegration by quantum boundary length $\{\mathrm{LF}_{\mathbb{D},\ell}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}\}_{\ell>0}$ from Lemma 2.7.

Corollary 3.13. *Fix $t, \ell_0 > 0$. Let (ϕ, η) be a sample from $\mathbb{1}_{\mathcal{A}_\phi(\mathbb{D})>t}\mathrm{LF}_{\mathbb{D},\ell_0}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi) \times \mathrm{raSLE}_\kappa(d\eta)$ and parametrize η by its $\mathcal{A}_\phi$ quantum area. Let f_t, ϕ_t, η_t and $\mathcal{C}_t$ be determined by (ϕ, η) in the same way as Proposition 3.12. Then on the event that $\eta([0, t])$ is simply connected, the law of $(\phi_t, \mathcal{C}_t, \eta_t)$ is*

$$\mathbb{1}_{X_t^-(\mathcal{C})+Y_t^-(\mathcal{C})<\ell_0}\mathrm{LF}_{\mathbb{D},\ell_0+X_t(\mathcal{C})+Y_t(\mathcal{C})}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi_t)\,P_t(d\mathcal{C}) \times \mathrm{raSLE}_\kappa(d\eta), \tag{3.14}$$

where $X_t(\mathcal{C}), Y_t(\mathcal{C}), X_t^-(\mathcal{C}), Y_t^-(\mathcal{C})$ are as in Definition 2.13.

Proof. If we do not fix the boundary length of ϕ, i.e., we instead assume that (ϕ, η) is sampled from $\mathbb{1}_{\mathcal{A}_\phi(\mathbb{D})>t}\mathrm{LF}_{\mathbb{D}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi) \times \mathrm{raSLE}_\kappa(d\eta)$, then it follows from Proposition 3.12 by disintegrating on the value of T that the law of $(\phi_t, \mathcal{C}_t, \eta_t)$ is

$$\mathbb{1}_{X_t^+(\mathcal{C})+Y_t^+(\mathcal{C})<\mathcal{L}_{\phi_t}(\partial\mathbb{D})}\mathrm{LF}_{\mathbb{D}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi_t)\,P_t(d\mathcal{C}) \times \mathrm{raSLE}_\kappa(d\eta). \tag{3.15}$$

Now we disintegrate over $\mathcal{L}_\phi(\partial\mathbb{D})$, and the claim follows from $\mathcal{L}_{\phi_t}(\partial\mathbb{D}) = \mathcal{L}_\phi(\partial\mathbb{D}) + X_t(\mathcal{C}) + Y_t(\mathcal{C})$ and $\{X_t^+(\mathcal{C}) + Y_t^+(\mathcal{C}) < \mathcal{L}_{\phi_t}(\partial\mathbb{D})\} = \{X_t^-(\mathcal{C}) + Y_t^-(\mathcal{C}) < \mathcal{L}_\phi(\partial\mathbb{D})\}$. □

Proof of Theorem 3.1. Recall that BM_κ is the law of correlated two-dimensional Brownian motion $(\widetilde{X}_t, \widetilde{Y}_t)_{t\geq0}$ with $\widetilde{X}_0 = \widetilde{Y}_0 = 0$ and covariance given by (2.9). Sample $(\widetilde{X}_t, \widetilde{Y}_t)_{t\geq0}$ from BM_κ and let $\widetilde{\tau}$ be the first time t that $1 + \widetilde{X}_t + \widetilde{Y}_t = 0$. Our first goal is to show that $(X_t, Y_t)_{[0,A]} \stackrel{d}{=} (\widetilde{X}_t, \widetilde{Y}_t)_{[0,\widetilde{\tau}]}$. To that end, we will show that $(X_s,Y_s)_{[0,\tau_1]} \overset{d}{=} (\widetilde{X}_s,\widetilde{Y}_s)_{[0,\widetilde{\tau}_1]}$ for suitable stopping times $\tau_1,\widetilde{\tau}_1$ corresponding to "wrapping around", then iterate to conclude. Afterwards, we establish (3.1) to complete the proof.

Recall that $Z := |\mathrm{LF}_{\mathbb{D},\ell}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}|$ does not depend on ℓ (Lemma 2.7). Suppose ϕ is a sample from $Z^{-1}\mathrm{LF}_{\mathbb{D},1}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}$, and η is an independent radial SLE_κ process from 1 to 0 parametrized by its $\mathcal{A}_\phi$ quantum area. Fix $t>0$ and let f_t,ϕ_t,η_t and $\mathcal{C}_t$ be determined by (ϕ,η) in the same way as Proposition 3.12. By Corollary 3.13, when restricted to the event F_t that $\mathcal{A}_\phi(\mathbb{D})>t$ and $\eta([0,t])$ is simply connected, the joint law of $(\phi_t,\mathcal{C}_t,\eta_t)$ is

$$\mathbb{1}_{X_t^-(\mathcal{C})+Y_t^-(\mathcal{C})<1} Z^{-1}\mathrm{LF}_{\mathbb{D},1+X_t(\mathcal{C})+Y_t(\mathcal{C})}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi_t)\, P_t(d\mathcal{C}_t)\times \mathrm{raSLE}_\kappa(d\eta_t),$$

so the joint law of $(\phi_t,(X_\cdot,Y_\cdot)|_{[0,t]},\eta_t)$ is

$$Z^{-1}\mathrm{LF}_{\mathbb{D},1+X_t+Y_t}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi_t)\ \times \mathbb{1}_{\widetilde{F}_t}\mathrm{BM}_\kappa^t(d(X_\cdot,Y_\cdot))\times \mathrm{raSLE}_\kappa(d\eta_t)$$

where BM_κ^t is the law of a sample from BM_κ restricted to the time interval $[0,t]$, and $\widetilde{F}_t=\{-\inf_{[0,t]}X_\cdot-\inf_{[0,t]}Y_\cdot>-1\}$. Since $|Z^{-1}\mathrm{LF}_{\mathbb{D},1+X_t+Y_t}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}|=|\mathrm{raSLE}_\kappa|=1$ regardless of the value of $1+X_t+Y_t$, the marginal law of $(X_\cdot,Y_\cdot)|_{[0,t]}$ restricted to F_t is $\mathbb{1}_{\widetilde{F}_t}\mathrm{BM}_\kappa^t$. Since t is arbitrary, we conclude that $(X_\cdot,Y_\cdot)_{[0,\tau_1]}\overset{d}{=}(\widetilde{X}_\cdot,\widetilde{Y}_\cdot)_{[0,\widetilde{\tau}_1]}$ where $\tau_1=\inf\{s:-\inf_{[0,s]}X_\cdot-\inf_{[0,s]}Y_\cdot\le -1\}$ and $\widetilde{\tau}_1=\inf\{s:-\inf_{[0,s]}\widetilde{X}_\cdot-\inf_{[0,s]}\widetilde{Y}_\cdot\le -1\}$.

Next, let τ_2 (resp. $\widetilde{\tau}_2$) be the first time $t>\tau_1$ (resp. $t>\widetilde{\tau}_1$) that $\inf_{\tau_1<s<t}X_s+\inf_{\tau_1<s<t}Y_s=-1$ (resp. $\inf_{\widetilde{\tau}_1<s<t}\widetilde{X}_s+\inf_{\widetilde{\tau}_1<s<t}\widetilde{Y}_s=-1$). We will show that $(X_\cdot,Y_\cdot)_{[0,\tau_2]}\overset{d}{=}(\widetilde{X}_\cdot,\widetilde{Y}_\cdot)_{[0,\widetilde{\tau}_2]}$. Fix $t_1>0$ and condition on $\{t_1<\tau_1\}$. Then the conditional law of (ϕ_{t_1},η_{t_1}) given $\mathcal{C}_{t_1}$ is $Z^{-1}\mathrm{LF}_{\mathbb{D},1+X_{t_1}+Y_{t_1}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}(d\phi)\times\mathrm{raSLE}_\kappa$, and the boundary length process of (ϕ_{t_1},η_{t_1}) is specified by $(X_t-X_{t_1},Y_t-Y_{t_1})_{t_1\le t\le A}$. Therefore following the same reasoning, if we let σ_2 (resp. $\widetilde{\sigma}_2$) be the first time t such that $\inf_{t_1<s<t}(X_s-X_{t_1})+\inf_{t_1<s<t}(Y_s-Y_{t_1})=-1-X_{t_1}-Y_{t_1}$ (resp. $\inf_{t_1<s<t}(\widetilde{X}_s-\widetilde{X}_{t_1})+\inf_{t_1<s<t}(\widetilde{Y}_s-\widetilde{Y}_{t_1})=-1-\widetilde{X}_{t_1}-\widetilde{Y}_{t_1}$), then $(X_t-X_{t_1},Y_t-Y_{t_1})_{t_1\le t\le\sigma_2}$ is independent of $(X_s,Y_s)_{0\le s\le t_1}$ and agrees in law with $(\widetilde{X}_t-\widetilde{X}_{t_1},\widetilde{Y}_t-\widetilde{Y}_{t_1})_{t_1\le t\le\widetilde{\sigma}_2}$ conditioned on $\{t_1<\widetilde{\tau}_1\}$. This implies that conditioned on $\{t_1<\tau_1\}$, the law of $(X_s,Y_s)_{0\le s\le\sigma_2}$ agrees with that of $(\widetilde{X}_s,\widetilde{Y}_s)_{0\le s\le\widetilde{\sigma}_2}$ conditioned on $\{t_1<\widetilde{\tau}_1\}$. Since t_1 is arbitrary, we conclude $(X_\cdot,Y_\cdot)_{[0,\tau_2]}\overset{d}{=}(\widetilde{X}_\cdot,\widetilde{Y}_\cdot)_{[0,\widetilde{\tau}_2]}$.

Arguing similarly, if we iteratively define τ_n (resp. $\widetilde{\tau}_n$) to be the first time $t>\tau_{n-1}$ (resp. $t>\widetilde{\tau}_{n-1}$) such that $\inf_{\tau_{n-1}<s<t}X_s+\inf_{\tau_{n-1}<s<t}Y_s=-1$ (resp. $\inf_{\widetilde{\tau}_{n-1}<s<t}\widetilde{X}_s+\inf_{\widetilde{\tau}_{n-1}<s<t}\widetilde{Y}_s=-1$), then $(X_\cdot,Y_\cdot)_{[0,\tau_n]}\overset{d}{=}(\widetilde{X}_\cdot,\widetilde{Y}_\cdot)_{[0,\widetilde{\tau}_n]}$ for all n. Since $\lim_{n\to\infty}\tau_n=A$ and $\lim_{n\to\infty}\widetilde{\tau}_n=\widetilde{\tau}$ where $\widetilde{\tau}=\inf\{t>0:1+\widetilde{X}_t+\widetilde{Y}_t=0\}$, it follows that $(X_t,Y_t)_{0\le t\le A}\overset{d}{=}(\widetilde{X}_t,\widetilde{Y}_t)_{0\le t\le\widetilde{\tau}}$. This proves the first claim.

Finally, we prove (3.1), which is immediate from Corollary 3.13 when $s=0$. We first claim that for each fixed $s>0$, conditioned on the event $s<\mathcal{A}_\phi(\mathbb{D})$ and $(X_\cdot,Y_\cdot)|_{[0,s]}$, the law of (ϕ_s,η_s) is $Z^{-1}\mathrm{LF}_{\mathbb{D},1+X_s+Y_s}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}\times\mathrm{raSLE}_\kappa$. To see this, fix $n>0$. For $1\le k\le 2^n$, let $E_{n,k,s}$ be the event where $\frac{ks}{2^n}<\mathcal{A}_\phi(\mathbb{D})$ and for each $1\le j\le k$, $\eta([\frac{(j-1)s}{2^n},\frac{js}{2^n}])$ is simply connected. Then conditioned on $E_{n,1,s}$ and $(X_\cdot,Y_\cdot)|_{[0,\frac{s}{2^n}]}$, by Corollary 3.13 the law of $(\phi_{\frac{s}{2^n}},\eta_{\frac{s}{2^n}})$ is $Z^{-1}\mathrm{LF}_{\mathbb{D},1+X_{\frac{s}{2^n}}+Y_{\frac{s}{2^n}}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}\times\mathrm{raSLE}_\kappa$. Applying Corollary 3.13 once more to $(\phi_{\frac{s}{2^n}},\eta_{\frac{s}{2^n}})$, we see that conditioned on $E_{n,2,s}$ and $(X_\cdot,Y_\cdot)|_{[0,\frac{2s}{2^n}]}$, the law of $(\phi_{\frac{2s}{2^n}},\eta_{\frac{2s}{2^n}})$ is $Z^{-1}\mathrm{LF}_{\mathbb{D},1+X_{\frac{2s}{2^n}}+Y_{\frac{2s}{2^n}}}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}\times\mathrm{raSLE}_\kappa$. By iterating this argument 2^n times, conditioned on $E_{n,2^n,s}$, the law of (ϕ_s,η_s) is $Z^{-1}\mathrm{LF}_{\mathbb{D},1+X_s+Y_s}^{(Q-\frac{\gamma}{4},0),(\frac{3\gamma}{2},1)}\times\mathrm{raSLE}_\kappa$. On the other hand, using the continuity of the curve η, conditioned on $\mathcal{A}_\phi(\mathbb{D})>s$ the event $E_{n,2^n,s}$ holds with probability $1-o_1(n)$ as $n\to\infty$. Now we can apply Corollary 3.13 to (ϕ_s,η_s) and conclude that conditioned on the event that $t<\mathcal{A}_\phi(\mathbb{D})$ and $\eta_s([0,t-s])$ is simply connected, the law of $(\eta_s([0,t-s]),\phi_s,\eta_s|_{[0,t-s]})/{\sim_\gamma}$ is absolutely continuous with respect to P_{t-s}. Therefore $F((X_{\cdot+s}-X_s,Y_{\cdot+s}-Y_s)|_{[0,t-s]})=(\eta_s([0,t-s]),\phi_s,\eta_s|_{[0,t-s]})/{\sim_\gamma}$ a.s.. By definition $(\eta([s,t]),\phi,\eta(\cdot+s)|_{[0,t-s]})/{\sim_\gamma}=(\eta_s([0,t-s]),\phi_s,\eta_s|_{[0,t-s]})/{\sim_\gamma}$, so (3.1) holds.

□

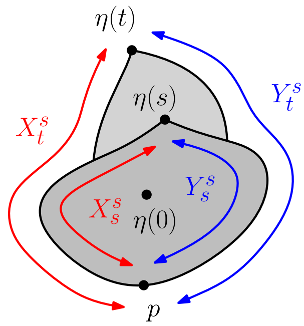

Figure 6: The boundary length process $(X_t, Y_t)_{[0,A]}$ of Theorem 4.1 is characterized by $X_0 = Y_0 = 0$ and the property that for each time s and choice of boundary point $p \in \partial\eta([0,s])$ not equal to $\eta(s)$, for any time $t > s$ before the time η next hits p, we have $(X_t^s - X_s^s, Y_t^s - Y_s^s) = (X_t - X_s, Y_t - Y_s)$, where $(X_\cdot^s, Y_\cdot^s)$ is shown in red and blue. Here $\eta([0,s])$ is shown in dark gray, and $\eta([s,t])$ is colored light grey.

4 A spherical mating-of-trees

In this section we prove Theorem 1.1. The key ingredient is the following spherical mating-of-trees result of independent interest. Recall $\mathcal{M}_2^{\mathrm{sph}}(\alpha)$ is the law of the quantum sphere from Definition 2.16.

Suppose $\kappa \geq 8$ and $\gamma = \frac{4}{\sqrt{\kappa}}$. Let $(\mathbb{C}, \phi, \infty, 0)$ be an embedding of a sample from $\mathcal{M}_2^{\mathrm{sph}}(Q - \frac{\gamma}{4})$ conditioned to have quantum area at least 1; write $A = \mathcal{A}_\phi(\mathbb{C})$. Let $\eta : [0, A] \to \hat{\mathbb{C}}$ be an independent whole-plane SLE_κ from 0 to ∞ parametrized by quantum area. There is a unique continuous process $(X_t, Y_t)_{[0,A]}$ starting at $(X_0, Y_0) = (0, 0)$ which keeps track of the changes in the left and right boundary lengths of $\eta([0,t])$, in the following sense. For any $s \in (0, A)$ and point $p \in \partial(\eta([0,s]))$ different from $\eta(s)$, let $\sigma > s$ be the next time η hits p. For each $t \in (s, \sigma)$ let X_t^s (resp. Y_t^s) be the quantum length of the counterclockwise (resp. clockwise) boundary arc of $\eta([0,s])$ from $\eta(s)$ to p. Then $(X_t^s - X_s^s, Y_t^s - Y_s^s)_{(s,\sigma)} = (X_t - X_s, Y_t - Y_s)_{(s,\sigma)}$. See Figure 6 for an illustration. To justify the existence and uniqueness of $(X_t, Y_t)_{[0,A]}$, similarly to the radial case in Theorem 3.1 we can define a process $(\widetilde{X}_t, \widetilde{Y}_t)_{(0,A)}$ with this property and which is unique up to additive constant. We extend it to $(\widetilde{X}_t, \widetilde{Y}_t)_{[0,A]}$ by continuity, and thus uniquely fix $(X_t, Y_t)_{[0,A]} = (\widetilde{X}_t - \widetilde{X}_0, \widetilde{Y}_t - \widetilde{Y}_0)_{[0,A]}$.

Theorem 4.1 (Spherical mating-of-trees)**.** *Let $(L_t, Z_t) = (X_t + Y_t, X_t - Y_t)$. Then L_t has the law of a Brownian excursion with quadratic variation $(2\mathbb{a}\sin(\frac{\pi\gamma^2}{8}))^2\,dt$ conditioned to have duration at least 1, and given the process (L_t) with random duration τ, the process $(Z_t)_{[0,\tau]}$ is conditionally independent Brownian motion with quadratic variation $(2\mathbb{a}\cos(\frac{\pi\gamma^2}{8}))^2\,dt$ run for time τ. Here $\mathbb{a}$ is as in (2.9). Moreover, for any $0 < s < t$, on the event that $t < \tau$ and $\eta([s,t])$ is simply connected, we have*

$$F((X_{\cdot+s} - X_s, Y_{\cdot+s} - Y_s)_{[0,t-s]}) = (\eta([s,t]), \phi, \eta(\cdot + s)|_{[0,t-s]})/{\sim_\gamma} \tag{4.1}$$

where F is the map from Lemma 2.14.

We note that L_t is the quantum length of $\partial(\eta([0,t]))$ for all t.

To prove Theorem 4.1, we start with the radial mating of trees Theorem 3.1, and condition on having quantum area at least 1 but having small boundary length $\ell \ll 1$ (event F_ℓ from (4.3)). Lemma 4.2 below implies that when $\ell \to 0$ the limiting boundary length process is that of Theorem 4.1. On the other hand, when $\ell \to 0$ the curve-decorated quantum surface converges to the conditioned quantum sphere decorated by independent whole-plane SLE (Proposition 4.4). Combining these two facts gives Theorem 4.1.

Lemma 4.2. *Let $\ell > 0$ and let L_t^ℓ be Brownian motion starting at ℓ and having quadratic variation $(2\mathbb{a}\sin(\frac{\pi\gamma^2}{8}))^2\,dt$, run until the time τ that it first hits 0. Given $(L_t^\ell)_{[0,\tau]}$ let Z_t^ℓ be an independent Brownian motion with quadratic variation $(2\mathbb{a}\cos(\frac{\pi\gamma^2}{8}))^2\,dt$ run for time τ. As $\ell \to 0$, the process (L_t^ℓ, Z_t^ℓ) conditioned on $\tau \geq 1$ converges in distribution to the Brownian process described in Theorem 4.1.*

Proof. This is immediate from the limiting construction of the Brownian excursion. □

Given Theorem 4.1, the proof of Theorem 1.1 goes as follows. Let $\mathfrak{S}$ be the SLE-decorated quantum surface in Theorem 4.1. Since its boundary length process agrees in law with its time-reversal, we have $\mathfrak{S} \stackrel{d}{=} \widetilde{\mathfrak{S}}$ where $\widetilde{\mathfrak{S}}$ is obtained from $\mathfrak{S}$ by switching its two points and reversing its curve. This implies that the law of the curve is reversible, as desired.

In Section 4.1 we show that a certain quantum sphere can be obtained from a disk by taking a limit (Proposition 4.4). In Section 4.2 we use this to obtain Theorem 4.1 and then Theorem 1.1.

4.1 Pinching an LCFT disk to get an LCFT sphere

The goal of this section is to prove Proposition 4.4 which states that a Liouville field on the disk conditioned to have area at least 1 and boundary length ℓ converges as $\ell \to 0$ to a sample from $\mathcal{M}_2^{\mathrm{sph}}(Q - \frac{\gamma}{4})$ conditioned to have area at least 1. Although the statement of Proposition 4.4 does not involve SLE or mating-of-trees, our arguments will use these to establish that the field remains "well behaved" near the boundary despite the conditioning on low probability events.

Instead of working in the domains $\mathbb{C}$ and $\mathbb{D}$, we will parametrize by the horizontal cylinder $\mathcal{C} := (\mathbb{R} \times [0, 2\pi])/\sim$ and half-cylinder $\mathcal{C}_+ := ([0, \infty) \times [0, 2\pi])/\sim$ where the upper and lower boundaries are identified by $x \sim x + 2\pi i$. This simplifies our exposition later.

We first define the Liouville field on $\mathcal{C}_+$. Let $f : \mathcal{C}_+ \to \mathbb{D}$ be the map such that $f(z) = e^{-z}$.

Definition 4.3. *For $\alpha, \beta \in \mathbb{R}$ and $\ell > 0$, define* $\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)} := f^{-1} \bullet_\gamma \mathrm{LF}_{\mathbb{D},\ell}^{(\alpha,0),(\beta,1)}$.

$\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$ inherits the following Markov property from $\mathrm{LF}_{\mathbb{D},\ell}^{(\alpha,0),(\beta,1)}$. For $\phi \sim \mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$, conditioned on $\phi|_{\partial \mathcal{C}_+}$ we have

$$\phi \stackrel{d}{=} \mathfrak{h} + h_0 - (Q - \alpha)\Re\cdot, \tag{4.2}$$

where $\mathfrak{h}$ is the harmonic function on $\mathcal{C}_+$ with boundary conditions $\phi|_{\mathcal{C}_+}$, and h_0 is a Dirichlet GFF on $\mathcal{C}_+$.

We define a probability measure $\mathcal{L}$ on fields on $\mathcal{C}$ as follows. Consider $(\hat{h}, \mathbf{c})$ sampled as in Definition 2.16 with $\alpha = Q - \frac{\gamma}{4}$ and conditioned on the event that $\mathcal{A}_{\hat{h}+\mathbf{c}}(\mathcal{C}) > 1$, let $\sigma \in \mathbb{R}$ satisfy $\mathcal{A}_{\hat{h}+\mathbf{c}}([\sigma, +\infty) \times [0, 2\pi]) = \frac{1}{2}$, let $\phi' = \hat{h}(\cdot + \sigma) + \mathbf{c}$, and let $\mathcal{L}$ be the law of ϕ'. Thus, $\phi' \sim \mathcal{L}$ corresponds to a sample from $\mathcal{M}_2^{\mathrm{sph}}(\alpha)$ conditioned to have quantum area greater than 1, embedded such that $\mathcal{A}_\phi(\mathcal{C}_+) = \frac{1}{2}$.

The main result of this section is that for small ℓ, a field sampled from $\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$ conditioned on F_ℓ resembles a quantum sphere conditioned to have quantum area at least 1.

Proposition 4.4. *Let $(\alpha, \beta) = (Q - \frac{\gamma}{4}, \frac{3\gamma}{2})$ and $\ell > 0$. Sample ϕ from $\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$ conditioned on*

$$F_\ell := \{\mathcal{A}_\phi(\mathcal{C}_+) > 1\}. \tag{4.3}$$

Let $\sigma > 0$ satisfy $\mathcal{A}_\phi(\mathcal{C}_+ + \sigma) = \frac{1}{2}$ and let $\widetilde{\phi} = \phi(\cdot + \sigma)$. For any $U \subset \mathcal{C}$ bounded away from $-\infty$, as $\ell \to 0$ the field $\widetilde{\phi}|_U$ converges in distribution to $\phi'|_U$ where $\phi' \sim \mathcal{L}$.

We first state a version of Proposition 4.4 where we additionally condition on the field near $\partial \mathcal{C}_+$ not behaving too wildly, in the sense that it has "scale ℓ" observables near $\partial \mathcal{C}_+$.

Lemma 4.5. *Let $(\alpha, \beta) = (Q - \frac{\gamma}{4}, \frac{3\gamma}{2})$. Fix a nonnegative smooth function ρ in $\mathcal{C}$ supported on $[1, 2] \times [0, \pi]$, such that ρ is constant on each vertical segment[4] $\{t\} \times [0, \pi]$ and $\int \rho = 1$. Let $K, \ell > 0$. Sample a field ϕ from $\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$ conditioned on*

$$E_{\ell,K} := F_\ell \cap \{\mathcal{A}_{\phi - \frac{2}{\gamma}\log \ell}([0, 1] \times [0, 2\pi]) < K \text{ and } |(\phi, \rho) - \frac{2}{\gamma}\log \ell| < K\}. \tag{4.4}$$

Let $\sigma > 0$ satisfy $\mathcal{A}_\phi(\mathcal{C}_+ + \sigma) = \frac{1}{2}$ and let $\widetilde{\phi} = \phi(\cdot + \sigma)$. For any $U \subset \mathcal{C}$ bounded away from $-\infty$, as $\ell \to 0$ the field $\widetilde{\phi}|_U$ converges in distribution to $\phi'|_U$ where $\phi' \sim \mathcal{L}$.

[4] This is convenient for the proof of Lemma 4.5 since (ϕ, ρ) only depends on the projection of ϕ to $H_{\mathrm{av}}(\mathcal{C})$.

The statement of Lemma 4.5 is parallel to that of [36, Proposition 4.1], except that we condition on an event measurable with respect to $\phi|_{[0,2]\times[0,2\pi]}$ (the second set in RHS of (4.4)), while they more strongly assert the asymptotic independence of $\phi|_{[0,2]\times[0,2\pi]}$ and $\widetilde{\phi}|_U$ (or rather, the corresponding fields in their setting). Using the Markov property (4.2) of $\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$, the proof of Lemma 4.5 is identical to the proof of [36, Proposition 4.1], so we omit it.

Next, we show that conditioned on F_ℓ, with high probability $E_{\ell,K}$ occurs. To that end, we will control the field near $\partial\mathcal{C}_+$ when we condition on F_ℓ by using the following lemma. Any planar domain A with the annulus topology is conformally equivalent to $\{z : 1 < |z| < e^{2\pi M}\}$ for some unique $M > 0$; this M is called the *modulus* of A, and we denote it by $\mathrm{Mod}(A)$.

Lemma 4.6. *Let $(\alpha, \beta) = (Q - \frac{\gamma}{4}, \frac{3\gamma}{2})$ and $n \geq 1$. Consider the setting of Theorem 3.1, except we embed in $(\mathcal{C}_+, +\infty, 0)$ rather than $(\mathbb{D}, 0, 1)$, so ϕ is sampled from $\mathrm{LF}_{\mathcal{C}_+,1}^{(\alpha,+\infty),(\beta,0)}$ and η is an independent radial* SLE *in $(\mathcal{C}_+, +\infty, 0)$. Let $(L_t, Z_t) = (1 + X_t + Y_t, X_t - Y_t)$ and let $\tau_x = \inf\{t : L_t = x\}$. Conditioned on $\{\tau_{2^n} < \tau_0\}$, the explored region $A = \eta([0, \tau_{2^n}])$ is annular with probability $1 - o_n(1)$, and its modulus tends to ∞ in probability as $n \to \infty$.*

Proof. First consider $n = 1$. Condition on $\tau_2 < \tau_0$ and let $A_1 = \eta([0, \tau_2])$. Since Brownian motion stays arbitrarily close to any deterministic path with positive probability, A_1 is annular with positive probability. Thus there exists $m_0 > 0$ such that the event $E_1 = \{A_1 \text{ annular and } \mathrm{Mod}(A_1) > m_0\}$ has conditional probability $p > 0$ given $\tau_2 < \tau_0$.

Now consider general $n \geq 1$. Condition on $\tau_{2^n} < \tau_0$, and for $1 \leq i \leq n$ define $A_i = \eta([\tau_{2^{i-1}}, \tau_{2^i}])$ and $E_i = \{A_i \text{ annular and } \mathrm{Mod}(A_i) > m_0\}$. By the scale invariance and strong Markov property of Brownian motion, the events $E_1, \dots, E_n$ are conditionally independent and each occur with probability p. Let I be the random set of i such that E_i holds, then $|I| \to \infty$ in probability as $n \to \infty$, so in particular $\mathbb{P}[A \text{ annular}] \geq \mathbb{P}[|I| \geq 1] \to 1$ as $n \to \infty$. Finally, by the subadditivity of moduli, on the event $\{A \text{ annular}\}$ we have

$$\mathrm{Mod}(A) \geq \sum_{i\in I} \mathrm{Mod}(A_i) \geq m_0|I|.$$

Since $|I| \to \infty$ in probability, $\mathrm{Mod}(A) \to \infty$ in probability as desired. □

Lemma 4.6 states that on the rare event that the boundary length hits 2^n, with high probability the explored region A at this hitting time is an annulus with large modulus. Next, we give a uniform bound on the field for all embeddings of $(A, \phi, 0)/\sim$ in $\mathcal{C}_+$ having $\partial\mathcal{C}_+$ as a boundary component.

Lemma 4.7. *There is an absolute constant $m > 0$ such that the following holds. Fix $n \geq 1$ and let ρ be as defined as in Lemma 4.5. In the setting of Lemma 4.6, condition on $\{\tau_{2^n} < \tau_0\}$ and on $\mathrm{Mod}(A) > m$. Let $\widetilde{A} \subset \mathcal{C}_+$ be any bounded annulus having $\partial\mathcal{C}_+$ as a boundary component such that $\mathrm{Mod}(\widetilde{A}) = \mathrm{Mod}(A)$. Let $\widetilde{\phi}$ be the field on $\widetilde{A}$ such that $(\widetilde{A}, \widetilde{\phi}, 0)/{\sim_\gamma} = (A, \phi, 0)/{\sim_\gamma}$, then*

$$\sup |(\widetilde{\phi}, \rho)| < \infty \qquad \textit{almost surely,} \tag{4.5}$$

where the supremum is taken over all choices of $\widetilde{A}$.

Proof. We first fix a canonical embedding $(\widetilde{A}_0, \widetilde{\phi}_0, 0)$ by specifying that A_0 is concentric, i.e., $A_0 = [0, t] \times [0, 2\pi]$ where $t = 2\pi\mathrm{Mod}(A) > 2\pi m$. For $b > 0$, let H_b be the set of nonnegative smooth functions f in $\mathcal{C}$ supported in $[\frac{1}{2}, 3] \times [0, 2\pi]$ with $f \geq 0$, $\int f(x)\,dx = 1$ and $\|f'\|_\infty \leq b$. Since $\widetilde{\phi}_0$ is locally absolutely continuous with respect to a GFF, as explained in the paragraph just after [14, Proposition 9.19] we have $\sup_{f\in H_b} |(\widetilde{\phi}_0, f)| < \infty$ almost surely.

For any other embedding $(\widetilde{A}, \widetilde{\phi}, 0)$ let $g : \widetilde{A}_0 \to \widetilde{A}$ be the conformal map such that $\widetilde{\phi} = g \bullet_\gamma \widetilde{\phi}_0$, then

$$(\widetilde{\phi}, \rho) = (\widetilde{\phi}_0 \circ g^{-1} + Q \log |(g^{-1})'|, \rho) = (\widetilde{\phi}_0, |g'|^2 \rho \circ g) + Q(\log |(g^{-1})'|, \rho).$$

Assuming the absolute constant m is chosen sufficiently large, conformal distortion estimates (e.g. [16, Theorem 5]) give $\sup_{[\frac{1}{2},3]\times[0,2\pi]} |g' - 1| < \frac{1}{10}$. Thus, $|(\widetilde{\phi}_0, |g'|^2\rho \circ g)| \leq \sup_{f\in F_b} |(\widetilde{\phi}_0, f)|$ for some b depending only on ρ and $|Q(\log |(g^{-1})'|, \rho)| \leq 10Q$, giving the desired uniform bound for $|(\widetilde{\phi}, \rho)|$. □

Now, we will prove that conditioned on F_ℓ, the event $E_{\ell,K}$ is likely. Briefly, conditioning on F_ℓ, Theorem 3.1 gives a description of the quantum surface near $\partial\mathcal{C}_+$ which we use to bound the field average near $\partial\mathcal{C}_+$ via Lemma 4.7.

Proposition 4.8. *Let $(\alpha,\beta) = (Q-\frac{\gamma}{4}, \frac{3\gamma}{2})$. For each $\delta>0$ there exists $K_0>0$ such that for all $K>K_0$*

$$\liminf_{\ell\to 0} \mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}[E_{\ell,K} \mid F_\ell] > 1-\delta.$$

Proof. Fix $n=n(\delta)\geq 1$ sufficiently large such that in the setting of Lemma 4.6 we have $\mathbb{P}[\mathrm{Mod}(A)\geq m] \geq 1-\frac{\delta}{4}$, where m is the absolute constant in Lemma 4.7.

Sample $\phi \sim \mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$, and independently sample a radial SLE_κ curve η in $(\mathcal{C}_+,0)$ targeting $+\infty$ and parametrized by $\mathcal{A}_\phi$. The law of $\phi^0 := \phi - \frac{2}{\gamma}\log\ell$ is $\mathrm{LF}_{\mathcal{C}_+,1}^{(\alpha,+\infty),(\beta,0)}$. Let (X_t,Y_t) be the boundary length process for (ϕ^0,η) as in Lemma 4.6, let $(L_t,Z_t) = (1+X_t+Y_t, X_t-Y_t)$, and let τ_x be the time L_t first hits x. By Lemma 4.2 we have $\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}[\tau_{2^n}<\tau_0 \mid F_\ell] = 1-o_\ell(1)$, and furthermore conditioning on $\{\tau_{2^n}<\tau_0\}\cap F_\ell$ the conditional law of $(L_t,Z_t)_{[0,\tau_{2^n}]}$ is within $1-o_\ell(1)$ in total variation distance of the corresponding process of Lemma 4.6. We conclude that conditioned on F_ℓ, the conditional law of the quantum surface $\mathcal{A} := (\eta([0,\ell^2\tau_{2^n}]), \phi - \frac{2}{\gamma}\log\ell, 0)$ is within $1-o_\ell(1)$ in total variation distance of the quantum surface of Lemma 4.6, and hence within $1-\frac{\delta}{2}-o_\ell(1)$ in total variation distance of the quantum surface of Lemma 4.7. Choose K_0 sufficiently large that in Lemma 4.7 the finite constant in (4.5) is bounded by $K_0-\log 2$ with probability at least $1-\frac{\delta}{4}$, and the quantum area of the annular quantum surface is bounded by K_0 with probability $1-\frac{\delta}{4}$. Then for $\phi\sim\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$ conditioned on F_ℓ, with probability at least $1-\delta-o_\ell(1)$ we have $|(\phi-\frac{2}{\gamma}\log\ell,\rho)|<K_0-\log 2$ and $\mathcal{A}_{\phi-\frac{2}{\gamma}\log\ell}([0,1]\times[0,2\pi])<K_0$. We are done.

□

Proof of Proposition 4.4. The result is immediate from Lemma 4.5 and Proposition 4.8. □

4.2 Proofs of Theorems 4.1 and 1.1

Proof of Theorem 4.1. Let $(L_t^\infty, Z_t^\infty)_{[0,\tau^\infty]}$ have the law of the Brownian process described in Theorem 4.1.

Step 1: Constructing a pair $(\widetilde{\phi}^\infty,\widetilde{\eta}^\infty)$ with boundary length process (L_t^∞,Z_t^∞). For $x>0$ let τ_x^∞ be the first time L_t^∞ hits x (or, if no such time exists, $\tau_x^\infty=\infty$). For each ℓ of the form 2^{-n} such that $\tau_\ell^\infty\neq\infty$, by Theorem 3.1 a.s. there is a corresponding SLE-decorated quantum surface $\mathcal{D}_\ell^\infty$ associated to the process $(L_{t-\tau_\ell^\infty}, Z_{t-\tau_\ell^\infty})_{[0,\tau^\infty-\tau_\ell^\infty]}$, and the $\mathcal{D}_\ell^\infty$ are consistent in the sense that for $\ell'<\ell$ the decorated quantum surface $\mathcal{D}_\ell^\infty$ arises as a sub-surface of $\mathcal{D}_{\ell'}^\infty$. Thus by the Kolmogorov extension theorem there is a curve-decorated quantum surface $(\mathcal{C},\widetilde{\phi}^\infty,\widetilde{\eta}^\infty,-\infty,+\infty)$ such that for all $\ell=2^{-n}$ such that $\tau_\ell^\infty\neq\infty$, we have $\mathcal{D}_\ell^\infty = (\widetilde{\eta}^\infty([\tau_\ell^\infty,\tau^\infty]),\widetilde{\phi}^\infty,\widetilde{\eta}^\infty(\cdot+\tau_\ell^\infty)|_{[0,\tau^\infty-\tau_\ell^\infty]},\widetilde{\eta}^\infty(\tau_\ell^\infty),+\infty)$.

Let $\phi'\sim\mathcal{L}$ as in Proposition 4.4, so $(\mathcal{C},\phi',-\infty,+\infty)/{\sim_\gamma}$ has the law of $\mathcal{M}_2^{\mathrm{sph}}(\alpha)$ conditioned to have quantum area greater than 1. Independently let η' be whole-plane SLE_κ from $-\infty$ to $+\infty$ in $\mathcal{C}$.

Step 2: (ϕ',η') is the $\ell\to 0$ limit of $\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$ decorated by independent radial SLE. By Proposition 4.4, for $\phi^\ell\sim\mathrm{LF}_{\mathcal{C}_+,\ell}^{(\alpha,+\infty),(\beta,0)}$ conditioned on F_ℓ, with $\sigma_\ell>0$ satisfying $\mathcal{A}_{\phi^\ell}(\mathcal{C}_++\sigma_\ell)=\frac{1}{2}$ and $\widetilde{\phi}^\ell=\phi^\ell(\cdot+\sigma_\ell)$, for any $U\subset\mathcal{C}$ bounded away from $-\infty$, as $\ell\to 0$ the field $\widetilde{\phi}^\ell|_U$ converges in distribution to $\phi'|_U$. Note that $\sigma_\ell\to\infty$ in probability as $\ell\to 0$ (e.g. by taking $U=[-N,\infty)\times[0,2\pi]$ in the above statement).

Next, sample a radial SLE_κ curve η^ℓ in $(\mathcal{C}_+,0,+\infty)$ independently of ϕ^ℓ and parametrize it by quantum area. Let $\widetilde{\eta}^\ell=\eta^\ell+\sigma_\ell$, and for each neighborhood U of $+\infty$ bounded away from $-\infty$ define the curve $\widetilde{\eta}_U:[0,\infty)\to\mathcal{C}$ by $\widetilde{\eta}_U:=\widetilde{\eta}(\cdot+\sigma_U)$ where σ_U is the first time $\widetilde{\eta}$ hits $\overline{U}$. Since whole-plane SLE_κ is the local limit of radial SLE_κ as the domain tends to the whole plane, the curve $\widetilde{\eta}_U$ converges in the topology of uniform convergence on compact sets to $\eta'_U:=\eta'(\cdot+\sigma'_U)$, where σ'_U is the time η' first hits $\overline{U}$.

Thus, in the setup of Theorem 3.1 with boundary length ℓ rather than 1, conditioned on having quantum area at least 1, as $\ell\to 0$ the field and curve $\widetilde{\phi}^\ell,\widetilde{\eta}^\ell$ converge in law to ϕ',η' above.

Step 3: Showing $(\mathcal{C}, \widetilde{\phi}^\infty, \widetilde{\eta}^\infty, -\infty, +\infty)/{\sim_\gamma} \stackrel{d}{=} (\mathcal{C}, \phi', \eta', -\infty, +\infty)/{\sim_\gamma}$. For ℓ of the form 2^{-n}, let (X^ℓ_t, Y^ℓ_t) be the boundary length process associated to $\widetilde{\phi}^\ell, \widetilde{\eta}^\ell$ defined above, and let $(L^\ell_t, Z^\ell_t) = (X^\ell_t - Y^\ell_t, X^\ell_t + Y^\ell_t)$. Since $\tau^\infty_\ell \to 0$ in probability as $\ell \to 0$, we can couple $(L^\infty_{t-\tau^\infty_\ell}, Z^\infty_{t-\tau^\infty_\ell})$ to agree with (L^ℓ_t, Z^ℓ_t) with probability $1 - o_\ell(1)$. On this event $(\widetilde{\eta}^\infty([\tau^\infty_\ell, T^\infty]), \widetilde{\phi}^\infty, \widetilde{\eta}^\infty(\cdot + \tau^\infty_\ell)|_{[0,\tau^\infty - \tau^\infty_\ell]}, \widetilde{\eta}^\infty(\tau^\infty_\ell), +\infty)/{\sim_\gamma} = (\mathcal{C}_+ - \sigma_\ell, \widetilde{\phi}^\ell, \widetilde{\eta}^\ell, 0, +\infty)/{\sim_\gamma}$; let f_ℓ be the conformal map sending $\widetilde{\eta}^\infty([\tau^\infty_\ell, \tau^\infty])$ to $\mathcal{C}_+ - \sigma_\ell$ such that $f_\ell(\widetilde{\eta}^\infty(\tau^\infty_\ell)) = -\sigma_\ell$ and $f_\ell(+\infty) = +\infty$. Since for any N the regions $\mathcal{C} \backslash \widetilde{\eta}^\infty([\tau^\infty_\ell, \tau^\infty])$ and $\mathcal{C} \backslash (\mathcal{C}_+ - \sigma_\ell)$ are subsets of $(-\infty, N) \times [0, 2\pi]$ in probability as $\ell \to \infty$, standard conformal distortion estimates give that for every neighborhood U of $+\infty$ bounded away from $-\infty$ we have $\sup_U |f'_\ell - 1| \to 0$ in probability. This implies that there is a coupling of (ϕ', η') with $(\widetilde{\phi}^\infty, \widetilde{\eta}^\infty)$ and a random rotation $f_\infty : \mathcal{C} \to \mathcal{C}$ of the cylinder (i.e. conformal map fixing $\pm\infty$ with $\operatorname{Re} f_\infty(z) = \operatorname{Re} z$ for all z) such that $\widetilde{\phi}^\infty = f_\infty \bullet_\gamma (\phi')$ and $\widetilde{\eta}^\infty = f_\infty(\eta')$ a.s., completing the step.

Conclusion. $(\mathcal{C}, \widetilde{\phi}^\infty, \widetilde{\eta}^\infty, -\infty, +\infty)/{\sim_\gamma}$ has the law of the curve-decorated quantum surface of Theorem 4.1 (Step 3), and its boundary length process is as desired (Step 1). The measurability claim (4.1) is immediate from that of Theorem 3.1 and the construction of Step 1. □

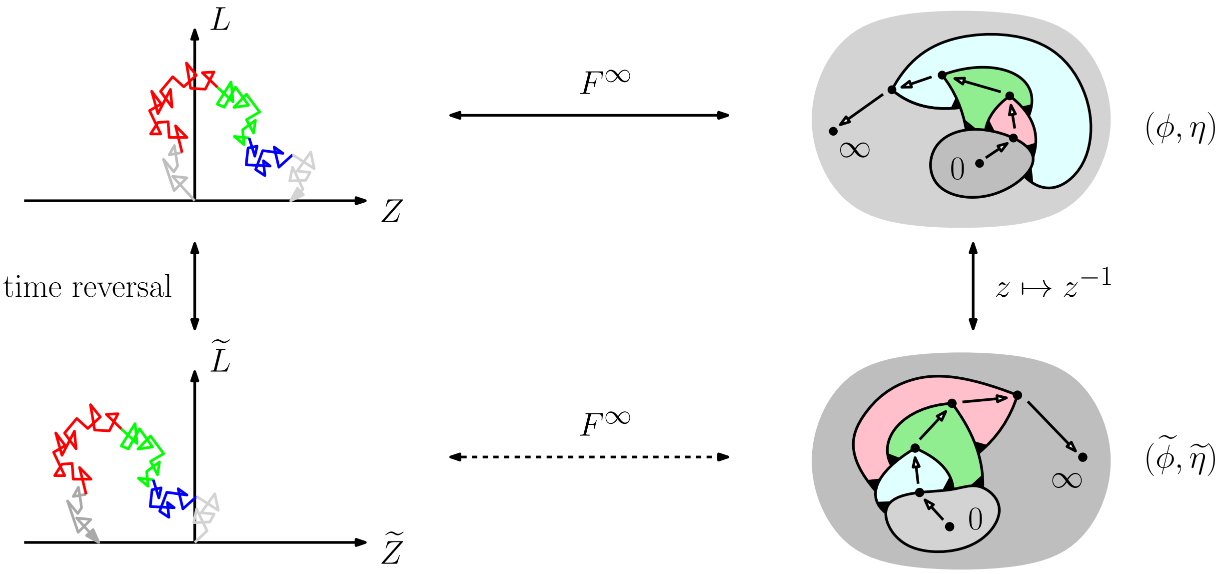

Figure 7: Proof of Theorem 1.1. **Top:** $(\mathbb{C}, \phi, 0, \infty)$ is an embedding of a sample from $\mathcal{M}^{\mathrm{sph}}_2(Q - \frac{\gamma}{4})$ and η is an independent whole-plane SLE from 0 to ∞. Let (L, Z) be its boundary length process. The grey Brownian motion segments are $(L, Z)_{[0,\varepsilon]}$ and $(L, Z)_{[\tau-\varepsilon,\tau]}$. The map F identifies the red, green and blue Brownian motion segments with the corresponding quantum cells. **Right:** $(\widetilde{\phi}, \widetilde{\eta})$ is obtained from (ϕ, η) by inverting the plane and orienting $\widetilde{\eta}$ so it is a curve from 0 to ∞. **Left:** $(\widetilde{L}, \widetilde{Z})$ is the time-reversal of (L, Z) (translated to start at 0). By reversibility of Brownian motion we have $(L, Z) \stackrel{d}{=} (\widetilde{L}, \widetilde{Z})$. **Bottom:** By the reversibility of F, the map F identifies the red, green and blue Brownian motion segments with the corresponding quantum cells. Sending $\varepsilon \to 0$, we see $F^\infty((\widetilde{L}, \widetilde{Z})) = (\mathbb{C}, \widetilde{\phi}, \widetilde{\eta}, \infty, 0)/{\sim_\gamma}$, so $(\mathbb{C}, \phi, \eta, \infty, 0)/{\sim_\gamma} = F^\infty((L, Z)) \stackrel{d}{=} F^\infty((\widetilde{L}, \widetilde{Z})) = (\mathbb{C}, \widetilde{\phi}, \widetilde{\eta}, \infty, 0)/{\sim_\gamma}$. This implies reversibility of whole-plane SLE.

Proof of Theorem 1.1. In the setting of Theorem 4.1, the decorated quantum surface $(\mathbb{C}, \phi, \eta, \infty, 0)/{\sim_\gamma}$ is measurable with respect to (L, Z). Indeed, let $(X_t, Y_t) = (\frac{1}{2}(L_t + Z_t), \frac{1}{2}(L_t - Z_t))$ and let $(t_n)_{n\in\mathbb{Z}}$ be an increasing collection of rational times in $(0, \tau)$ such that $\lim_{n\to-\infty} t_n = 0$, $\lim_{n\to\infty} t_n = \tau$, and for each n we have $(X_{t_n} - \inf_{[t_n,t_{n+1}]} X_\cdot) + (Y_{t_n} - \inf_{[t_n,t_{n+1}]} Y_\cdot) < L_{t_n}$ (i.e., $\eta([t_n, t_{n+1}])$ is simply connected). Then $(\mathbb{C}, \phi, \eta, \infty, 0)/{\sim_\gamma}$ is the conformal welding of $\mathcal{C}_n := F((X_{\cdot+t_n} - X_{t_n}, Y_{\cdot+t_n} - Y_{t_n})|_{[0,t_{n+1}-t_n]})$ for $n \in \mathbb{Z}$, where, similarly as in Figure 2 (right), the first marked point of $\mathcal{C}_{n+1}$ is identified with the second marked point of $\mathcal{C}_n$, and the two boundary arcs of $\mathcal{C}_{n+1}$ adjacent to its first marked point are conformally welded according to quantum length to the boundary of the conformal welding of $\bigcup_{i\le n}\mathcal{C}_i$. Let F^∞ be the map sending the process $(L_t,Z_t)_{[0,\tau]}$ to the decorated quantum surface $(\mathbb{C},\phi,\eta,\infty,0)/{\sim_\gamma}$.

See Figure 7. Let $(\mathbb{C},\phi,\infty,0)$ be an embedding of a sample from $\mathcal{M}_2^{\mathrm{sph}}(Q-\frac{\gamma}{4})$ and let η be an independent whole-plane SLE. Let (L_t,Z_t) be the boundary length process associated to $(\mathbb{C},\phi,\eta,\infty,0)$ as in Theorem 4.1, and let τ be its random duration. Let $\mathrm{Inv}(z)=z^{-1}$, let $\widetilde{\phi}=\mathrm{Inv}\bullet_\gamma\phi$, let $\widetilde{\eta}$ be the time-reversal of $\mathrm{Inv}\circ\eta$ (so $\widetilde{\eta}$ is also a curve from 0 to ∞), and let $(\widetilde{L}_t,\widetilde{Z}_t):=(L_{\tau-t},Z_{\tau-t}-Z_\tau)$ be the time-reversal of (L_t,Z_t). Let $\mathfrak{S}=(\mathbb{C},\phi,\eta,\infty,0)$ and $\widetilde{\mathfrak{S}}=(\mathbb{C},\widetilde{\phi},\widetilde{\eta},\infty,0)$. In the next paragraph we will show that $\mathfrak{S}\overset{d}{=}\widetilde{\mathfrak{S}}$; this is the crux of the argument.

By definition $F^\infty((L,Z))=\mathfrak{S}$. We pick $\varepsilon,n>0$ and equally divide $(\varepsilon,\tau-\varepsilon)$ into n intervals $I_1,...,I_n$. We restrict to the event E_n that $\eta(I_k)$ is simply connected for $k=1,...,n$. Let $\mathcal{C}_k=(\eta(I_k),\phi,\eta|_{I_k})/{\sim_\gamma}$ and $\widetilde{\mathcal{C}}_k=(\widetilde{\eta}(I_k),\widetilde{\phi},\widetilde{\eta}|_{I_k})/{\sim_\gamma}$, so by definition of $(\widetilde{\phi},\widetilde{\eta})$ the decorated quantum surface $\widetilde{\mathcal{C}}_{n+1-k}$ agrees with $\mathcal{C}_k$ with its curve reversed. For an interval $I=[a,b]$, define $F'((L,Z)|_I):=F((X_{\cdot+a}-X_a,Y_{\cdot+a}-Y_a)|_{[0,b-a]})$ where $(X_\cdot,Y_\cdot)=(\frac{1}{2}(L_\cdot+Z_\cdot),\frac{1}{2}(L_\cdot-Z_\cdot))$. By Theorem 4.1 $F'((L,Z)|_{I_k})=\mathcal{C}_k$, so by reversibility of F (Lemma 2.14) and the fact that $(L,Z)|_{I_k}$ and $(\widetilde{L},\widetilde{Z}-Z_\tau)|_{I_{n+1-k}}$ differ by time-reversal, we have $F((\widetilde{L},\widetilde{Z})|_{I_{n+1-k}})=\widetilde{\mathcal{C}}_{n+1-k}$. Consequently, on the event E_n, the boundary length process of $(\widetilde{\phi},\widetilde{\eta})$ restricted to $(\varepsilon,\tau-\varepsilon)$ agrees with $(\frac{1}{2}(\widetilde{L}+\widetilde{Z}),\frac{1}{2}(\widetilde{L}-\widetilde{Z}))|_{[\varepsilon,\tau-\varepsilon]}$ up to additive constant. Therefore, by first sending $n\to\infty$ and then $\varepsilon\to0$, we see that $F^\infty((\widetilde{L},\widetilde{Z}))=\widetilde{\mathfrak{S}}$ a.s.. The reversibility of Brownian motion yields $(L_t,Z_t)\overset{d}{=}(\widetilde{L}_t,\widetilde{Z}_t)$, and combining with $(F^\infty((L,Z)),F^\infty((\widetilde{L},\widetilde{Z}))=(\mathfrak{S},\widetilde{\mathfrak{S}})$, we conclude $\mathfrak{S}\overset{d}{=}\widetilde{\mathfrak{S}}$.

Let $r>0$ be such that $\mathcal{A}_\phi(r\mathbb{D})=\mathcal{A}_\phi(\mathbb{C}\backslash r\mathbb{D})$, let θ be uniformly sampled from $[0,2\pi)$ independently of (ϕ,η), define $f(z)=r^{-1}e^{i\theta}z$, and set $\phi_0=f\bullet_\gamma\phi$ and $\eta_0=f\circ\eta$. Likewise define $\widetilde{\phi}_0,\widetilde{\eta}_0$ by applying the same embedding procedure for $\widetilde{\phi},\widetilde{\eta}$. Since $\mathfrak{S}\overset{d}{=}\widetilde{\mathfrak{S}}$ we have $(\phi_0,\eta_0)\overset{d}{=}(\widetilde{\phi}_0,\widetilde{\eta}_0)$. Since ϕ and η are independent, and whole-plane SLE is invariant in law under dilations and rotations of the plane, the law of η_0 is whole-plane SLE. Likewise $\widetilde{\eta}_0$ has the law of the time-reversal of whole-plane SLE after applying Inv. The statement $\eta_0\overset{d}{=}\widetilde{\eta}_0$ is thus the desired reversibility of whole-plane SLE for $\kappa>8$. □

5 Open problems

The convergence of lattice statistical physics models to SLE was a primary reason to expect the reversibility of chordal SLE_κ for $\kappa\le8$. Conversely, Theorem 1.1 suggests the following question.

Problem 5.1. *Find a lattice statistical physics model whose scaling limit is whole-plane* SLE_κ *for some* $\kappa>8$.

Questions of this sort are sometimes easier when the underlying lattice is random, i.e., is a random planar map. Some random planar maps decorated by statistical physics models can be encoded by a pair of trees, which in turn may be described by a random walk on the 2D lattice. If this random walk converges in the scaling limit to Brownian motion with covariance given by (2.9), then we say the corresponding decorated random planar map converges in the *peanosphere topology* to $(\gamma=\frac{4}{\sqrt{\kappa}})$-LQG decorated by space-filling SLE_κ. In the case when the SLE_κ is a space-filling loop in $\hat{\mathbb{C}}$ from ∞ to ∞, such convergences are known for random planar maps decorated by bipolar orientations ($\kappa=12$) [24], Schnyder woods ($\kappa=16$) [30], or a variant of spanning trees ($\kappa>8$) [19]; see the survey [18] for examples where $\kappa\le8$. The next problem asks for such a result for whole-plane SLE_κ where $\kappa>8$.

Problem 5.2. *Exhibit a random planar map decorated by a statistical physics model which can be encoded by a random walk converging in the limit to* (X_t,Y_t) *defined in Theorem 3.1 or in Theorem 4.1. In other words, find a random planar map model which converges in the peanosphere sense to LQG decorated by radial or whole-plane SLE.*

One of the most natural variants of SLE_κ is the $\mathrm{SLE}_\kappa(\underline{\rho})$ process [26, 12, 31]; other important variants include multiple SLE [9, 25, 13] and the conformal loop ensemble [43, 46]. For $\kappa\in(0,8]$, the time reversal of chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ has been solved when the sum of the weights is larger than $(-2)\vee(\frac{\kappa}{2}-4)$ [32, 54, 50], while [35, Theorem 1.18] gives a criterion for the reversibility of $\mathrm{SLE}_\kappa(\rho^-;\rho^+)$ curves when $\kappa>8$. On the whole plane side, the most natural variant of whole-plane SLE_κ is whole-plane $\mathrm{SLE}_\kappa(\rho)$ for $\rho>-2$, which agrees with whole-plane SLE_κ when $\rho=0$ (see e.g. [35, Section 2.1.3]). Miller and Sheffield showed that when $\kappa \in (0,4]$ and $\rho > -2$, or $\kappa \in (4,8]$ and $\rho \geq \frac{\kappa}{2} - 4$, whole-plane $\mathrm{SLE}_\kappa(\rho)$ is reversible [35, Theorem 1.20]. They also show that when $\kappa > 8$ and $\rho \geq \frac{\kappa}{2} - 4$ whole-plane $\mathrm{SLE}_\kappa(\rho)$ is not reversible [35, Remark 1.21]. They do not treat the regime where $\kappa > 4$ and $\rho \in (-2, \frac{\kappa}{2} - 4)$ because it is not as natural in the imaginary geometry framework, see [35, Remark 1.22]. On the other hand, Theorem 1.1 gives reversibility when $\kappa > 8$ and $\rho = 0$ even though it falls into this regime, so there is still hope for reversibility in this range.

Problem 5.3. *When $\kappa > 4$, for which $\rho \in (-2, \frac{\kappa}{2} - 4)$ is $\mathrm{SLE}_\kappa(\rho)$ reversible?*

A further generalization of whole-plane $\mathrm{SLE}_\kappa(\rho)$ can be obtained by adding a constant drift term to the driving function. Zhan [53] showed that when $\kappa \in (0,4]$, $\rho = 0$ and any constant drift is chosen, the curve is reversible. Miller and Sheffield [35, Theorem 1.20] showed that when $\kappa \in (0,4]$ and $\rho > -2$, or $\kappa \in (4,8]$ and $\rho \geq \frac{\kappa}{2} - 4$, for any chosen drift the curve is reversible.

Problem 5.4. *When $\kappa > 4$ and $\rho \in (-2, \frac{\kappa}{2} - 4)$, what choices of drift coefficient give a reversible curve?*

The statement of Theorem 1.1 involves only SLE, but our arguments depend on couplings with LQG.

Problem 5.5. *Find a proof of Theorem 1.1 not using mating-of-trees.*

It seems likely that a solution to Problem 5.5 would represent a significant step towards solving Problems 5.3 and 5.4.

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