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Waves in strongly nonlinear discrete systems

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The paper presents the main steps in the development of the strongly nonlinear wave dynamics of discrete systems. The initial motivation was prompted by the challenges in the design of barriers to mitigate high-amplitude compression pulses caused by impact or explosion. But this area poses a fundamental mathematical and physical problem and should be considered as a natural step in developing strongly nonlinear wave dynamics. Strong nonlinearity results in a highly tunable behaviour and allows design of systems with properties ranging from a weakly nonlinear regime, similar to the classical case of the Fermi–Pasta–Ulam lattice, or to a non-classical case of sonic vacuum. Strongly nonlinear systems support periodic waves and one of the fascinating results was a discovery of a strongly nonlinear solitary wave in sonic vacuum (a limiting case of a periodic wave) with properties very different from the Korteweg de Vries solitary wave. Shock-like oscillating and monotonous stationary stress waves can also be supported if the system is dissipative. The paper discusses the main theoretical and experimental results, focusing on travelling waves and possible future developments in the area of strongly nonlinear metamaterials.

This article is part of the theme issue 'Nonlinear energy transfer in dynamical and acoustical systems'.

1. Introduction

A motivation to study strongly nonlinear dynamics of discrete systems came from the practical problem related to development of mitigating media (e.g. bed of iron shots) to reduce effects of potentially dangerous high-amplitude compression pulses on structures (e.g. wall of blast chambers) generated by explosion [1–3]. This technical problem has the following major parameters

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(scales): (i) pressure of detonation products being in order of 100 kbar generates compression pulse with high amplitude, thus nonlinear interaction between iron shots (spheres) was expected; (ii) duration of the incoming pulse was very short, 10–100 µs, thus the problem must be considered as a wave propagation problem, and (iii) the granular bed should have the capability to mitigate multiple impacts, thus the behaviour of spheres should be close to linearly elastic.

The natural attempt in 1980 was to consider weakly nonlinear one-dimensional chain of elastic spheres arriving to the equations for an anharmonic chain and in continuum approximation to the corresponding Korteweg–de Vries (KdV) equation [1–3]. However, this approach has a problem when the amplitude of the disturbance is much larger than the initial precompression, resulting in losing one small parameter; the latter is absolutely necessary for the reduction of the equations for a discrete system to the KdV equation. It should be mentioned that even if a compression pulse propagating in a granular bed (diameter of iron shots is about 3 mm) originates from motion of a heavy plate at the top (initial velocity 1–10 m s^{-1}, mass 1000 kg, area 1 m^2), it should be considered as strongly nonlinear. Simple estimates of the static precompression on an imaginary single force chain in the granular bed is of the order 0.1 N and characteristic dynamic force between grains corresponding to these conditions is of the order of 100 N. Thus instead of a weakly nonlinear KdV type of equation, a more complex strongly nonlinear wave equation was introduced [1–3].

At that time I had an intuitive (not proved in any way) belief that solitary waves are a result of balance of dispersion and nonlinearity and the latter does not require the word 'weakly'. No mathematical proof existed to support such a belief. Moreover, the KdV solitary wave (excellent review of the discovery of this solitary wave in experiments by Russel [4,5] and following theoretical developments can be found in [6]) solution 'exploded' in the limiting case of zero precompression of a chain composed of spherical particles interacting by the nonlinearizable Hertz Law [7]. This case corresponds to the state of 'sonic vacuum', the term being coined in my 1992 short paper [8] is commonly used now. It took some time to find a simple analytical solution corresponding to the strongly nonlinear solitary wave propagating in a sonic vacuum, which was qualitatively different from the solitary wave supported by the KdV equation [1–3]. Now this solitary wave is commonly called the Nesterenko solitary wave (soliton and compacton), the term was coined by Coste, Falcon and Fauve in 1997 [9]. The analytical solution corresponding to the Nesterenko solitary wave was supported by numerical analysis of the discrete chain. Later it was discovered in experiments by different teams of independent researchers [9–15]. It is interesting that first presentation of theoretical and experimental results on wave propagation in a strongly nonlinear granular chain to a group of theoreticians in 1982 [16] did not attract significant attention. Only later this area became an active domain of theoretical and experimental research. During the last few decades multiple papers focused on strongly nonlinear wave dynamics (total number is close to 500), reviews [15,17–26], books [2,3,27,28] and even popular articles [29,30] were published.

Research in wave dynamics of strongly nonlinear discrete systems is a relatively new area and it is fascinating that Albert Einstein clearly anticipated a specific property of strongly nonlinear systems. In a well-known letter to Max Born (4 December 1926) [31], where he discussed doubts in quantum mechanics, Albert Einstein wrote 'Waves in three-dimensional space whose velocity is regulated by potential energy (for example, rubber bands) . . . '. It is clear that he had in mind a strongly nonlinear wave equation, because, in the linear case, velocity of sound c_0 (or similar pulse in another media) is not regulated by potential energy; it is simply constant. Rubber bands are mentioned probably as an example of a strongly nonlinear system experiencing large strains and strongly nonlinear behaviour. We know now that the wave speed of the strongly nonlinear Nesterenko solitary wave $V_{s,N}$ is indeed strongly regulated by potential energy U. For example, c_0 is equal to zero in the state of 'sonic vacuum' and for the specific case of a strongly nonlinear Hertzian chain $V_{s,N} \sim U^{1/10}$. It is interesting that Einstein did not connect strong nonlinearity with possible failure of the continuum approach (or I failed to find it), which he mentioned in his 1954 letter to Michelangelo Besso [32]. We know that, in the case of 'sonic vacuum', a continuum approximation predicts a solitary wave width equal to five particle sizes for the power interaction

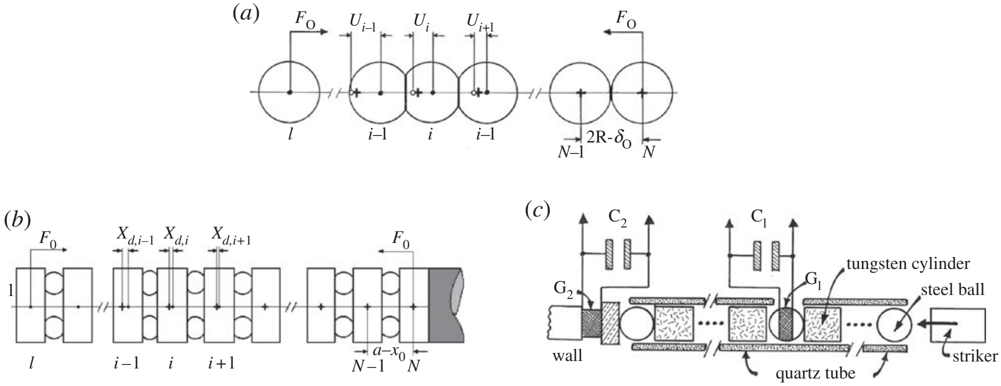

Figure 1. Contact deformation in pulse propagating in weakly precompressed chains of spherical beads (*a*), and in a chain of cylinders with O-rings responsible for strongly nonlinear double power-law interaction between cylinders (*b*) and non-compressed chain of spheres and tungsten cylinders with embedded gauges inside cylinders impacted by a striker (*c*).

law with exponent 3/2 (Hertz Law) or even close to the particle size at larger exponents [3,8,17,33–36]. This certainly raises a question of validity of the 'continuous structures' [32], which will be discussed later.

This article is not a comprehensive review of the rapidly developing area. It is focused mostly on travelling waves excited by impact, outlining suggestions for a future research and applications.

2. Discrete systems, continuum approximation

Examples of strongly nonlinear discrete systems are presented in figure 1. The one-dimensional chain of particles (spheres/cylinders) with initial distances between them being $2R$ or a, correspondingly, can be precompressed by a static force F_0, resulting in static displacements between particles centres δ_0 (figure 1*a*) or x_0 (figure 1*b*). The crosses in figure 1*a* correspond to the initial positions of the particle centres in the precompressed chain, the black circles are their positions in the pulse moving from left to right, and the open circles represent positions of centres in the uncompressed chain. Waves are commonly excited by strikers of different masses (as shown in figure 1*c*).

The chain schematically presented in figure 1*a* was the first strongly nonlinear system with Hertz's interaction law where Nesterenko solitary waves were discovered [1,10–12].

A periodic arrangement of cylinders and O-rings (figure 1*b*) has a stronger than Hertzian nonlinearity (double power law [37]), resulting in a higher level of tunability by static force [21,37–40]. The Buna-N rubber O-ring recovered its shape after a high-energy impact with a duration close to 10 ms (striker mass 14.4 kg, velocity $1\,\mathrm{m\,s^{-1}}$), absorbing energy of about 7 J [41]. O-rings can be used to design systems with a broad range of nonlinear interaction forces [21,37].

The chain presented in figure 1*c* allows changes in the mass of cylinders (e.g. assembling system with two masses in the cell) without changing the contact interaction or variations of elastic moduli of the elements without changes of their masses [3,12,42,43]. The systems presented in figure 1(*b*,*c*) can be beneficial for embedding resonators inside cylinders or piezo elements for pulse detection or harvesting mechanical energy of waves. Two- and three-dimensional ordered packing of spherical particles [44–47] may behave in a similar way to one-dimensional systems [1–3].

Novel strongly nonlinear systems were proposed: origami-based metamaterials [48], woodpile periodic structures [49], tensegrity metamaterials [50], locally resonant granular crystals [51,52], dense colloidal monolayers [53] and nanoscale (buckyballs) systems [54].

The Newton equations (equation (2.1)) for particles (spheres/cylinders) with a mass m interacting by general force f depending only on their relative displacements is presented as follows:

$$m\ddot{u}_i = f_{i,i-1}(u_{i-1} - u_i) - f_{i+1,i}(u_i - u_{i+1}), \quad N \geq i \geq 2, \tag{2.1}$$

where m is the mass of the particles, $f_{i,i-1}(u_{i-1} - u_i)$ is the force acting on the ith particle from $(i-1)$th particle, u_i is the displacement of the ith particle from the initial equilibrium position, which may include some initial displacements causing static deformation of the system, and N is the number of particles in the system. Equations for the first and the last particles should be written separately. Equation (2.1) is based on a few physically important assumptions—particles are considered as point masses and the interaction law is elastic. Validity of application of static laws between particles (e.g. Hertz Law in the case of elastic spheres) for dynamic conditions requires appropriate scaling of characteristic times related to wave propagation in the chain and wave propagation inside particles [1–3]. Its application for granular material acoustics was experimentally validated in [55].

Continuum approximations (equations (2.2) and (2.3)) for the general interaction law were presented in [3,19,56]:

$$\rho\xi_{tt} = \left\{ f + \frac{a^2}{24}[2f'\xi_{xx} + f''\xi_x^2] \right\}_{xx}. \tag{2.2}$$

A regularized wave equation, equation (2.3), [3] is presented below:

$$\rho\xi_{tt} = \left\{ f + \frac{a^2}{24}[2\rho\xi_{tt} - f''\xi_x^2] \right\}_{xx}, \tag{2.3}$$

where a is a lattice period, ρ is a linear density (m/a) and the prime denotes differentiation of the function $f(a\xi)$ with respect to strain $\xi = (-u_x) > 0$. They were helpful to proving the existence of a qualitatively new wave motion, introducing unique scaling and deriving exact solutions connecting main properties of waves with their amplitudes and material parameters, facilitating subsequent numerical analysis and experiments.

The introduction of these complex long wave approximations for a strongly nonlinear system was attempted with the hope that new wave solutions might be supported by a granular chain, which are qualitatively different from solutions of a weakly nonlinear equation. Technical details of the derivation of equations (2.2) and (2.3) can be found in [3,56].

The convective derivative in equations (2.2) and (2.3) was neglected, which is a correct assumption if the phase speed of a propagating disturbance is much larger than the particle velocities in the wave. For example, the wave speed of the Nesterenko solitary wave in the chain of steel spheres with a diameter of a few millimeters at a particle velocity of about $1\ \mathrm{m\,s^{-1}}$ is of the order of magnitude $10^3\ \mathrm{m\,s^{-1}}$. The corresponding estimates for the validity of this assumption in the general case of sonic vacuum can be found in [3].

Continuum approximations of discrete systems apparently have natural space and time-scale limitations. It should not be expected to be accurate in the range where assumptions used in the derivation of this equation are not valid. For example, equations (2.2) and (2.3), if applied for a chain of beads, cannot be expected to have a physical sense on the space scale smaller than the distance between particle centres or at a time scale much smaller than the characteristic time of wave propagation inside particles or at wave amplitudes resulting in plastic flow at particle contacts. Unfortunately, it is rather common, especially in some pure mathematical publications, to assume that long wave approximation for a discrete system must be applicable at any space and time scales, which is an unrealistic expectation from the physical point of view.

Equation (2.2) supports strongly nonlinear periodic waves in discrete systems with general strongly nonlinear interaction force. Their speed depends on the interaction law and at minimum and maximum values of strain [3,19]. Modulation stability of periodic solution in sonic vacuum with Hertzian interaction between elements was proved in [57].

A limiting case of periodic solution of equation (2.2) is a stationary compression solitary wave in the case of 'normal' behaviour (elastic hardening) of the law of interaction ($f''(a\xi) > 0$). The speed of this supersonic compression solitary wave $V_{s,c}$ is related to initial (ξ_0) and maximum (ξ_m) strains:

$$V_{s,c}^2 = \frac{f(a\xi_m) - f(a\xi_0)}{\rho(\xi_m - \xi_0)} + \frac{a^2 f'(a\xi_m)\xi_{xx}(\xi_m)}{12\rho(\xi_m - \xi_0)}. \tag{2.4}$$

The convex behaviour of function f (elastic softening, $f''(a\xi) < 0$) is a condition for the existence of the rarefaction-type supersonic solitary solution with a speed $V_{s,r}$ [3,19]:

$$V_{s,r}^2 = \frac{f(a\xi_{min}) - f(a\xi_0)}{\rho(\xi_{min} - \xi_0)} + \frac{a^2 f'(a\xi_{min})\xi_{xx}(\xi_{min})}{12\rho(\xi_{min} - \xi_0)}. \tag{2.5}$$

Equations (2.4) and (2.5) demonstrate that solitary waves are expected in periodic chains with general strongly nonlinear interaction laws between discrete elements.

Power-law interaction ($f_{i-1,i} = B(u_{i-1} - u_i)^n$) with values of exponents $n > 1$ [3,8,18] allows propagation of stationary compression waves (special case $n = 3$ related to transverse vibrations of linear elastic unstressed fibre was first considered in [33]) and for $n < 1$ rarefaction waves are possible [3,18,58,59].

In the case $n > 1$, the speed of sound in precompressed chain c_0 (equation (2.6)) and of Nesterenko solitary wave $V_{s,N}$ in sonic vacuum related to maximum strain (ξ_m) and particle velocity (v_m), correspondingly ($c_0 = 0$, equation (2.7)) are given by the following formulas, where $c_n^2 = (B/m)a^{n+1}$:

$$c_0 = c_n\sqrt{n}\xi_0^{(n-1)/2}, \tag{2.6}$$

$$V_{s,N} = c_n\sqrt{\frac{2}{n+1}}\xi_m^{(n-1)/2} = \left(\frac{2c_n^2}{n+1}\right)^{1/(n+1)} v_m^{(n-1)/(n+1)}. \tag{2.7}$$

The length of the solitary wave L_n with a speed $V_{s,N}$ is represented by the following equation:

$$L_n = \frac{\pi a}{n-1}\sqrt{\frac{n(n+1)}{6}}. \tag{2.8}$$

The characteristic length of the Nesterenko solitary wave (L_n) is independent of its amplitude (even in case of general strongly nonlinear interaction law [3], unlike the KdV solitary wave) and this wave is probably the shortest travelling wave, which can be supported by a discrete system.

In the vicinity of $n = 1$, stationary rarefaction of compression solitary waves cannot propagate depending on values of n being larger or less than 1. At $n \to 1$ the speed of the solitary wave (equation (2.7)) in weakly precompressed sonic vacuum does not approach a constant value c_0 (equation (2.6)) and it has a characteristic space scale L_n (equation (2.8)) increasing with $n \to 1$. This behaviour was called a 'sonic catastrophe' [58].

The chain of elastic particles (spheres or spheres and cylinders) interacting by the Hertz Law ($n = 3/2$) is a special case investigated in great details in analytical, numerical approaches and in experiments. The solitary wave speed without restrictions on its amplitude in the precompressed chain can be expressed by the following equation, which follows directly from equation (2.2) [3,19]:

$$V_s = \frac{c}{(\xi_m - \xi_0)}\left\{\frac{2}{5}[3\xi_0^{5/2} + 2\xi_m^{5/2} - 5\xi_0^{3/2}\xi_m]\right\}^{1/2}. \tag{2.9}$$

The constant c (equation (2.10)) depends on elastic properties of the beads (Young modulus E, Poisson ratio ν and density ρ_0 of particles' material):

$$c^2 = \frac{2E}{\pi\rho_0(1-\nu^2)}. \tag{2.10}$$

In the case $\xi_m \to \xi_0$, equation (2.9) results in the sound of speed in precompressed chain c_0 (equation (2.11)):

$$c_0 = c\left(\frac{3}{2}\right)^{1/2}\xi_0^{1/4}. \tag{2.11}$$

In the next approximation, equation (2.9) is reduced to a speed of a weakly nonlinear solitary wave (with strain amplitude $\xi_m = \xi_0 + \Delta\xi_m$) corresponding to the KdV approximation, equation (2.12):

$$V_{\mathrm{s,KdV}} = c_0 + \frac{c_0 \Delta\xi_{\mathrm{m}}}{12\xi_0}. \tag{2.12}$$

In the opposite case of very small initial precompression approaching zero ($\xi_{\mathrm{m}} \gg \xi_0$ and maximum particle velocity υ_{m}) the speed of the Nesterenko solitary wave $V_{\mathrm{s,N}}$ in sonic vacuum with Hertzian interaction between particles is represented by equation (2.13) (it follows from equation (2.7) at $n = 3/2$):

$$V_{\mathrm{s,N}} = \left(\frac{4}{5}\right)^{1/2} c\xi_{\mathrm{m}}^{1/4} = \left(\frac{16}{25}\right)^{1/5} c^{4/5}\upsilon_{\mathrm{m}}^{1/5}. \tag{2.13}$$

The shape of this solitary wave in sonic vacuum is described by one hump of the following periodic function with characteristic length equal to $5a$:

$$\xi = \left(\frac{5V_{\mathrm{s,N}}^2}{4c^2}\right)^2 \cos^4\left(\frac{\sqrt{10}}{5a}x\right). \tag{2.14}$$

The Nesterenko solitary wave in sonic vacuum was introduced as one hump of the periodic function (equation (2.14)) using effective potential energy for precompressed chain with infinitesimally small initial strain $\xi_0 > 0$ [1–3]. It was assumed that a solitary wave propagating in sonic vacuum ($\xi_0 = 0$) will be almost identical to the solitary wave shape at infinitesimally small, but still finite, precompression $\xi_0 \ll \xi_{\mathrm{m}}$. Precompression ξ_0 was necessary in the derivation of long wave continuum equation. This approach was verified in numerical calculations by comparison of solitary wave propagating in a discrete chain with very small precompression with one hump of periodic solution (equation (2.14)) [3].

Chatterjee argued [60] that taking $\xi(x)$ to be given by the function described by equation (2.14) for x inside interval $-5\pi a/2\sqrt{10} < x < 5\pi a/2\sqrt{10}$, and setting $\xi(x) \equiv 0$ outside that interval, provides a function $\xi(x)$ that is three times differentiable as required, satisfies equation (2.2) everywhere (provided $\xi(x) \equiv 0$ is accepted as a valid solution), and satisfies the basic conditions on the travelling wave solution except for the strict inequality $\xi(t) > 0$.

It should be emphasized that strongly nonlinear Nesterenko solitary waves (compression or rarefaction) as a stationary solution of strongly nonlinear wave equation (equation (2.2)), qualitatively different than KdV solitary wave, exist for a broad range of strongly nonlinear interaction (their speeds are expressed by equations (2.4) and (2.5)). A strongly nonlinear Nesterenko solitary wave in the particular chain of particles interacting by the Hertz Law was just a first example of this amazing wave.

Different approaches to the continuum description of discrete systems with various power-law interaction exponents n were presented in [35] based on expansion of differences. This approach results in a travelling wave equation similar to the $K(n,n)$ equation formally introduced in [61]. For a Hertzian interaction both approaches demonstrated a similar shape of solitary waves with increased deviations at larger exponent n. The limits of validity of continuum approximation for strongly nonlinear discrete systems is still an open mathematical problem.

Introduction of dissipation in equation (2.2) results in stationary compression (normal elastic behaviour) or rarefaction shock-like stress waves for discrete systems with abnormal elastic interaction [3,19,62,63]. For general interaction laws the speed of the corresponding shock-like stress wave is larger than the speed of the solitary wave, if the steady strains behind the compression (rarefaction) shock-like wave and in the compression (rarefaction) soliton crest are equal [3,19].

In the chain with Hertzian interaction ($n = 3/2$) the relation between wave speed V_{sh} and particle velocity υ_f (or strain ξ_f) behind a shock-like stress wave in sonic vacuum is represented by the following equation:

$$V_{\mathrm{sh}} = c^{4/5}\upsilon_f^{1/5} = c\xi_f^{1/4}. \tag{2.15}$$

This speed is larger than the speed of the Nesterenko solitary wave $V_{\mathrm{s,N}}$ if $\xi_{\mathrm{m}} = \xi_f$, reflecting a relation for a general interaction law emphasized above. This shock-like stationary stress wave in the case of very weak dissipation is oscillatory [62,63] with the ratio of maximum strain (particle velocity) in the leading peak to the corresponding values behind a shock being equal $(5/4)^2$ [3]. This ratio can be exceeded on the initial non-stationary stage of wave propagation.

As mentioned above in the case of the small wave amplitude in comparison with static precompression, equation (2.2) can be reduced to the KdV equation [3], which corresponds to the Fermi–Pasta–Ulam (FPU) problem [64]. The celebrated weakly nonlinear FPU problem explored a long-standing paradigm expressed by D. Bernoulli's remark in 1741: 'for the elongation will not be proportional to the extending force ... and everything must be irregular' [65]. This paradigm was dominant in physics until the middle of the twentieth century. Its validity was first explored in numerical calculations by FPU using the first computers developed for the purposes of the Manhatten project. It is clearly expressed by citations from the FPU paper [64]: 'This report is intended to be the first one of a series dealing with the behaviour of certain nonlinear physical systems where the nonlinearity is introduced as *a perturbation to a primary linear problem*'. Thus FPU considered that due to weak nonlinearity, added to a primary linear problem, the periodic linear solution for the string might assume more and more complicated shapes; in other words 'everything must be irregular', as D. Bernoulli suggested. It is clear that FPU explored if weak nonlinearity can be a probable source of phonon thermalization.

The FPU paper did not pose a strongly nonlinear or fully nonlinear physical problem, being focused on a weakly nonlinear case. On the other hand, the Hertzian granular chain is characterized by the 'nonlinearizable' interaction law in the case of zero precompression and basic excitations are not phonons, which are replaced by Nesterenko solitary waves.

Nonlinearity in the sonic vacuum case cannot be introduced as a perturbation to a primary linear problem. However, if a granular chain is precompressed and amplitudes of dynamic displacements are small in comparison with static displacements caused by precompression, the sonic vacuum problem is transformed into a weakly nonlinear (FPU) problem. Thus the weakly nonlinear FPU problem is a partial case of the granular chain when amplitude of the dynamic strain is much smaller than the initial precompression.

3. Periodic waves in discrete chain and in continuum

A periodic wave can be supported by strongly nonlinear discrete systems and corresponding solutions of equation (2.2) were compared with numerical calculations of quasi-periodic waves for granular chains with Hertzian contact. It was observed that the periodic waves in the numerical calculations are very close to results from the long-wave approximation [66]. Envelope soliton solutions in precompressed granular chains are considered in [67].

Non-dissipative strongly nonlinear two-mass chains (dimer systems consisting of alternatively arranged particles, e.g. spheres and cylinders with two different masses presented in figure 1*c*) demonstrate a qualitatively new behaviour (frequency band gaps) in comparison with uniform chains. It is important that the band gaps in these systems can be tuned by external static force. A corresponding quasi-harmonic nonlinear and strongly nonlinear excitations can be effectively mitigated after propagation only through four to eight cells as was demonstrated in numerical calculations and in experiments. Systems which are able to transform nonlinear and strongly nonlinear waves at short distances are important for practical applications such as attenuation of high-amplitude pulses. The frequencies of a band gap in strongly nonlinear two-mass systems are close to predicted for the linear elastic interaction and they can be tuned into the audible range [68,69].

4. Solitons in discrete one-dimensional chain versus waves in continuum

Wave equation (2.2) predicted stationary strongly nonlinear periodic waves, solitary waves and shock-like stress waves (if dissipation was included), and helped to identify their properties, equations (2.4, 2.5, 2.6–2.15) [3]. Dimensionless analysis of equations for the discrete chain of identical particles at zero precompression [1,3] demonstrated that spatial size of the assumed

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stationary solitary wave does not depend on its amplitude, and similar dependence of solitary speed on its amplitude found in strongly nonlinear long-wave approximation.

The mathematical proof of the existence of a solitary wave in a discrete chain [70] was published after discovery of a solitary wave using long-wave approximation for power-law chains in numerical calculations and experiments with the Hertzian chain [1,2,8,10,11–13,15,17]. The condition for the existence of strongly nonlinear solitary waves in continuum ($f''(a\xi) > 0$) for compression solitary wave) [3,19] is identical to the conditions of solitary wave existence in discrete chains [70]. But the latter proof did not provide specific properties of the solitary waves and it did not allow one to conclude if a solitary wave in a strongly nonlinear system is qualitatively different from the KdV solitary wave. A precompressed discrete chain does support the KdV solitary wave at a relatively small amplitude of the wave and the formal proof in effect could be interpreted as the proof of the existence of the wave close to a well-known KdV solitary wave.

Non-stationary behaviour of strongly nonlinear discrete systems was investigated in numerical simulations. Nesterenko solitary waves being a stationary solution of the wave equation (equation (2.2)) and solitary waves emerging from the striker impact on the system of discrete particles (equation (2.1)) in the Hertzian chain are close to each other [1–3,34,35,71]. They have the same characteristic space and temporal width, and similar dependence of phase speed on maximum velocity and strain.

In addition to confirmation of the existence of a qualitatively new solitary wave found initially as a stationary solution of long-wave approximation (equation (2.2)), numerical simulations provided insight which was not possible in the analysis of the stationary solutions. For example it was found that

(a) Nesterenko solitary waves can be formed very close to the impacted end. This observation was crucial for the design of the experimental set-up and experimental confirmation of theoretical and numerical results soon after the publication of the first paper [1,10–12];
(b) solitary waves re-emerge after their interaction with phase shift. Very small amplitude additional solitary waves were observed after collision, leading to specific equilibrium in the system without phonons explored by Sen *et al*. [22];
(c) the number of solitary waves was dependent on the mass of the striker [1–3,10–12];
(d) introduction of weak randomness of particle radii did not completely destroy a localized wave obtained in a perfectly periodic system, but caused its decay due to generation of the oscillation left behind a propagating wave [1–3];
(e) particle velocity behind a wave in a random, non-dissipative chain did not demonstrate equilibration to the velocity of the piston [1–3].

The series solution for a stationary Nesterenko solitary wave in a discrete Hertzian chain, more accurate than equation (2.14), is presented in [22]. An asymptotic description of the tail of the soliton in a discrete chain and a new asymptotic solution for the full solitary wave was first developed in [60]. Very detailed and accurate description of properties of Nesterenko solitary waves for Hertzian interaction can be found in [71–73] and comparative discussions of most of the theoretical approaches in the literature can be found in [26].

The comparison of the speed of a Nesterenko solitary wave propagating in sonic vacuum with power-law interaction between particles found in continuum approximation (equation (2.7)) with its speed determined in numerical calculations and using the binary collision model demonstrated that equation (2.7) gives quantitatively accurate results for the pulse velocity for relatively soft potentials, $1 < n \leq 2.5$ [34]. The binary collision model is quantitatively correct for relatively hard potentials, $n \geq 2.5$. For very large values of the power exponent ($n \sim 100$) equation (2.7) gives values of speed which are approaching 0.883 of the value obtained in numerical calculations. It is truly amazing and unexpected that even at these values of exponent n, where the characteristic length of the solitary wave is close to the lattice period a (equation (2.8)), the long-wave approximation gives a correct order of magnitude estimate of the pulse speed and even correctly predicts the qualitative change of speed behaviour with increase of exponent n [34].

The solitary waves propagating in weakly precompressed granular chains with power-law interaction with the exponents n being in the interval $1 < k < 3.5$ can be approximated by Nesterenko's compacton in sonic vacuum, if the ratio of the initial strain to the peak strain is less than 4%. The shapes of solitary waves in a weakly precompressed chain are noticeably different from that of the Nesterenko's compacton in sonic vacuum if the speed ratio between them exceeds 10% [36].

It should be mentioned that theoretical and numerical approaches focused on the research of a new type of solitary wave in non-dissipative systems are physically relevant only because the distance of their formation from the impacted end is rather short (obviously, this condition was not in the formal mathematical proofs of existence and it was not predicted based on stationary solution of long wave equation 2.2). If formation of strongly nonlinear solitary waves would require very long distances of propagation (e.g. a few thousands of particles), then even at weak dissipation, typical for Hertzian contacts of steel spheres, these waves would not be observable and become irrelevant for experimental research and applications.

Limits of continuum descriptions of a discrete system are very fascinating and important subject especially due to rapid development of metamaterials, which in many cases are macro/mesoscopically discrete systems. It is interesting that Albert Einstein was rather pessimistic about the ability of the continuum approach. He wrote in a 1954 (1 year before his death on 18 April 1955) in a letter to his friend Michelangelo Besso: 'I consider it quite possible that physics cannot be based on the field concept, i.e. on continuous structures. In that case nothing remains of my entire castle in the air, gravitation theory included, and of the rest of modern physics' [32].

This pessimism is quite appropriate especially for strongly nonlinear systems, where a continuum wave equation results in the spatial size of stationary solutions comparable to the characteristic size of the system, e.g. five-particle sizes for a discrete system with Hertzian interaction and even in less spatial sizes for power-law interaction with larger exponents [3,34,35]. But results for a strongly nonlinear discrete system based on the continuum approach are relevant even at characteristic sizes of waves comparable to the distance between particles! It is reasonable to conclude that Einstein was probably too pessimistic about the validity of 'continuous structures'.

It was shown in [59] that a strongly nonlinear stationary solution corresponding to a solitary rarefaction wave in continuum approximation for materials with elastic softening ($f''(a\xi) < 0$) has a shape and similar characteristic spatial scale to a stationary wave in a discrete system. The width of this wave is scaled with the diameter of the particle and weakly depends on the force exponent (unlike the case with a strongly nonlinear compression wave).

A system of particles (e.g. cylinders) where O-rings are strongly nonlinear elements allows one to design discrete systems with almost any interaction law between masses, including combination of normal and abnormal elastic behaviour. The latter case results in a kinked solitary wave [21]. This is unusual because 'abnormal' systems alone do not support stationary compression solitary waves.

5. Train of solitary waves

Striker impact with significantly larger mass than the mass of particles in a granular chain excites a train of solitary waves [1–3,10–12]. Figure 2 presents a train of Nesterenko solitary waves reflected from the steel wall excited in the chain of 40 steel spheres with diameter 4.75 mm by striker impact with a mass equal 5 m and velocity $0.5\,\mathrm{m\,s^{-1}}$ (m is a mass of the steel sphere).

There are a few attempts to predict amplitudes of these waves based on conservation laws without solving numerically equations for dynamics of granular chains. One approach to predict amplitudes of solitary waves at large distances from the impacted end is based on imaginary steps in the transformation of linear momentum and energy of the striker to the solitary waves

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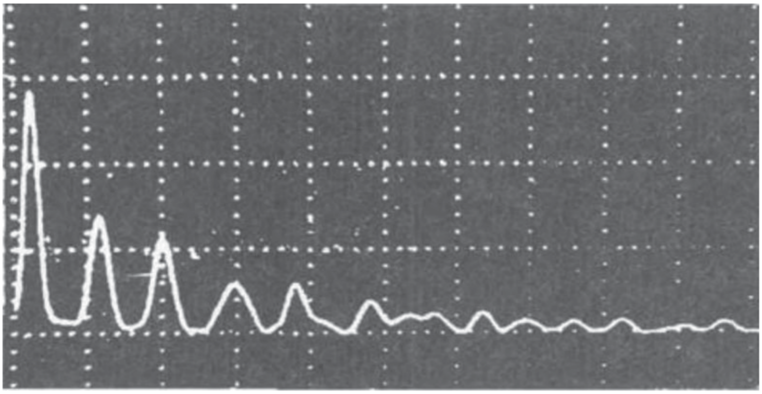

Figure 2. Train of Nesterenko solitary waves reflected from a steel wall. The time scale is 50 µs per large horizontal division, vertical scale 18.3 N between large divisions.

considered as imaginary quasi-particles being initially at rest with an effective mass equal to 1.4 m [24,43]. It is similar to that used in [74] to predict relative amplitudes in the train of solitary waves generated at the interface of two sonic vacua. This imaginary path conserves linear momentum and energy of the system, predicting approximate amplitudes of the solitary wave but it is not unique. Based on this approach, the corresponding equation for the momenta of the *n*th solitary wave is

$$P_n = \frac{2P_0(B-1)^{n-1}}{(B+1)^n}, \tag{5.1}$$

where P_0 is the initial linear momentum of the striker, and $n = 1,2,3\ldots$, are numbers of solitary waves, number 1 corresponds to the leading solitary wave, and $B = M_{\mathrm{imp}}/m_{\mathrm{eff}}$. Because the effective mass of the solitary wave, considered a quasi-particle, depends on the interaction law, amplitudes of solitary waves in the train depend on the interaction law between particles in the system.

Another approach to find properties of solitary waves generated by impulse forces acting on a granular chain was recently introduced in [73].

Discrete periodic materials with an abnormal normal power-law relationship between force and displacement ($n < 1$) support rarefaction solitary waves [3,18,58,59]. A fascinating property of such systems is that a striker initially generates a non-steady compression pulse which disintegrates into a leading rarefaction wave and oscillatory tail with decaying amplitude. This creates a possibility of development of the ultimate protection shield without even using energy dissipation mechanisms or an additional source of impact mitigation [48,50,59]. However, unlike sonic vacuum the discrete systems with abnormal behaviour require a large number of cells for effective transformation of incoming pulses. It is necessary to develop configurations of these systems which are effective with practically acceptable numbers of unit cells.

6. Interfaces

Strongly nonlinear materials in a sonic vacuum state have zero acoustic impedance (product of density and speed of sound). In classical acoustics this parameter determines the outcome of incident wave interactions with interfaces. Thus a new approach to wave interaction with interfaces of sonic vacua must be developed especially taking into account that, in this case, the breakdown of the continuum approach may result in non-classical behaviour [74,75].

The nature of transmitted and reflected pulses is dramatically different depending on the direction of its propagation with respect to the interface between 'light' and 'heavy' sonic vacua [76]. It allows the confinement of an impulse in a particular region of the shielding granular medium [77–79].

An energy- and linear-momentum-conserving approach can predict the amplitudes of the transmitted solitary waves generated when an incident solitary-wavefront, parallel to the interface, moves from a 'heavy' to a 'light' granular system [74]. When incident wave approaches interface at an oblique angle, the angles of refraction and reflection are captured by a granular analogue of Snell's Law in which the solitary-wave speed is replacing the speed of sound [74].

A strong sensitivity of the reflected and transmitted energy from the interface of two granular systems to the initial precompression was called the 'acoustic diode' effect [75]. It can be applied to manipulate the signals' delay and reflection at will, and decompositions/scrambling of security-related information. The interface between a solid conical rod and precompressed granular chain demonstrated strong non-reciprocal acoustic propagation without change of the frequency of the incident wave in a weakly nonlinear regime [80].

7. Shock-like stress waves in strongly nonlinear discrete systems

A shock-like, non-stationary wave with an expanding leading oscillatory part was observed in the numerical calculations of a non-dissipative piston problem [1–3]. The amplitude of the particle velocity at the leading stationary solitary wave, established at a distance of about 100 particles from the impacted end, was twice the piston velocity. The speed of this non-dissipative non-stationary wave V^{nst} with an established stationary leading solitary wave can be derived (equation (7.1)) using the equation for Nesterenko solitary waves (equation (2.13)) with maximum particle velocity υ_{m} equal to $2\upsilon_f$, where υ_f is a velocity of the piston:

$$V^{\mathrm{nst}} = \left(\frac{16}{25}\right)^{1/5} c^{4/5}(2\upsilon_f)^{1/5}. \tag{7.1}$$

It is clear that speed V^{nst} of the non-dissipative, non-stationary wave is not equal to the speed of stationary shock-like stress wave V_{sh} (equation (2.15)) despite the fact that, in both cases, the final particle velocity υ_f is the same. Also the shock-like stationary stress wave, predicted in continuum approximation at weak dissipation, has the ratio of particle velocity in the leading peak to υ_f equal to $(5/4)^2$, being less than 2. Detailed analysis of the structure of a stationary shock-like stress wave in sonic vacuum in a discrete system in comparison with the continuum approach in case of viscous dissipation and the criterion for the transition from oscillatory to the monotonous wave profiles are presented in [62,63].

Dissipation in some strongly nonlinear metamaterials may be neglected because the discrete nature of metamaterials and strong nonlinearity dominate when waves travel relatively short distances. At the same time decay of waves can be essential due to viscoplastic deformation between particles (spheres or cylinders) [81,82], friction between particles and the holder, expulsion of fluid between interacting particles [63] or energy leaking from the main pulse due to disorder [1–3]. Several dissipation models were compared in [26].

A significant dissipation may change the shape of shock from oscillatory to monotonous [62,63]. The oscillatory shock-like stress wave was observed in the chain of steel spheres with mainly elastic behaviour (figure 3*a*), but a monotonous stress wave was propagated in the chain of plastically deformed lead spheres with similar sizes (figure 3*b*) impacted by the same striker [3,11,18].

Plastic deformation of contacts for lead spheres resulted not only in the qualitatively different stress wave structure, but also caused the essential decrease of force amplitude acting on the wall (17 times), and an increase of shock rise time (40 times) (figure 3).

A dissipation may facilitate an unusual two-wave structure when a system is excited by the δ function force [42,43,83,84], and prevent generation of multiple solitary waves by the striker with a mass significantly larger than mass of particles in the chain [38–40].

Moreover, dissipation may bring new mechanisms contributing to strong nonlinearity. Recently metamaterials using O-rings as strongly nonlinear elements (e.g. Nitrile rubber O-ring) were investigated [38–40]. Unlike the Hertz contact law between spherical grains, toroidal O-rings

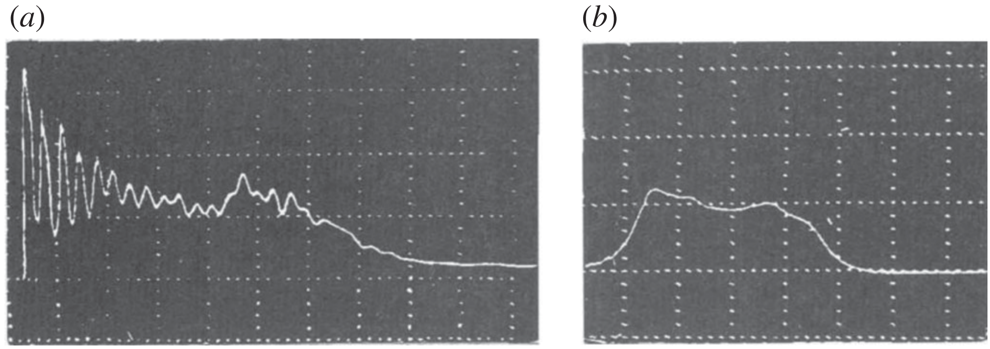

Figure 3. Oscillatory 'shock' wave in the chain of steel spheres (*a*) and monotonous 'shock' (*b*) in the chain of lead spheres reflected from a steel wall. Number of spheres in both chains 20, striker mass 30 m, where *m* is the mass of steel sphere, velocity 1 m s^{-1}. Diameter of spheres in both cases was 4.75 mm. Time scale: (*a*) 50 µs, (*b*) 200 µs per large horizontal division, vertical scale 92N (*a*) and 18.5N (*b*) between large vertical divisions. (Online version in colour.)

obey a more complicated force–displacement interaction law–double power law under static compression [85]. At relatively large deformation (when the second term with exponent $n = 6$ provides a significant contribution to the compression force) this relationship ensures significantly stronger nonlinearity than the Hertz Law. These double power-law metamaterials are a few times more tunable at higher preload than granular chains of linear elastic spherical particles and their acoustic impedance can be increased by a factor of three to four at very moderate static precompression [38].

O-rings also demonstrated a complex dynamic behaviour under impact with dramatically increased elastic modulus in comparison with the static response [38]. In the paper [41], the dissipative model for dynamic deformation of O-rings was introduced with strongly nonlinear dependence on the precompression and linear dependence on the relative velocity. Strong dissipation in a discrete system with O-rings can facilitate propagation of simple waves instead of solitary waves. Simple waves in highly dissipative Hertzian chains were considered in [86].

Amazing behaviour of stress pulses in strongly nonlinear non-dissipative dimer chains excited by δ-force was observed numerically in [87,88] and confirmed experimentally in [81]. A true solitary wave similar to the Nesterenko solitary wave was observed only at certain discrete mass ratios of light to heavy spheres (e.g. 0.3428, 0.1548 and 0.0901). There is an optimal mass ratio which results in the most effective pulse mitigation in this non-dissipative system. However, a certain level of dissipation may dramatically change the behaviour of the two-mass chain [42,43]. Viscous damping (at the damping ratio 6 kg s^{-1} characteristic for investigated system) eliminates the process of gap openings and corresponding time scales (gap opening divided by particle velocity) characteristic for non-dissipative chains. As a result, mitigation properties of uniform chains in comparison with the two-mass chain may be better with damping coefficient in the interval 10–100 kg s^{-1}. At the higher level of viscous dissipation the pulses in both systems are of similar width, resulting in a similar decay of pulses travelling the same number of contacts [42,43].

Highly dissipative, strongly nonlinear metamaterials can be instrumental for fast transformation/attenuation of wave shape generated by impact or contact explosion. For example, layers of granular materials and foams with relatively large thickness (at small thickness they can amplify the amplitude of the reflected pulse) are successfully used to attenuate blast loading precisely due to their ability to transform an incoming high-amplitude short duration pulse into a longer ramped pulse with significantly smaller amplitude [2,3,89].

8. Breathers in strongly nonlinear granular chains

The previous discussion was mostly focused on travelling strongly nonlinear waves excited by impact. Recently, the different types of solutions—breathers, supported by the strongly nonlinear granular chains with Hertzian interaction, were reported [52,90,91]. Non-existence of breathers for an uncompressed chain of beads interacting via Hertz's contact forces was proved in [90]. An additional on-site potential allows the existence of breathers in such systems.

Two types of configurations were observed in an analytical approach and in numerical calculations in a locally resonant granular system with harmonic internal resonators: (a) small-amplitude periodic travelling waves and (b) dark breather solutions [91]. It is important to emphasize that the latter were observed in the case of complete *absence* of precompression in the system. Authors also observed that, at sufficiently large mass ratio and suitable initial data, the system supports long-lived bright breather solutions which eventually disintegrate.

The existence of discrete breathers in a one-dimensional, mass-in-mass chain with linear intersite coupling and nonlinear, precompressed Hertzian local resonators was theoretically proved in [90]. Numerical calculations were used to compute a family of breathers.

9. Mass with mass strongly nonlinear granular chains

Dynamics of significantly modified classical granular chains—mass-with-mass strongly nonlinear granular chains, were recently analysed [92–94]. The dynamic response of a granular chain of beads in a state of sonic vacuum in which a mass-with-mass single defect (one of the beads has an attached linear resonator) is present was studied numerically and analytically in [92]. This set-up allows one to control the transmitted and reflected energy of a mechanical pulse by changing the ratio between the harmonic resonator mass and the bead mass. If the defect-to-bead mass ratio is small, the incident Nesterenko solitary wave remains essentially unaltered with small parts of its energy being reflected and trapped in the form of localized oscillation. In the case of very large mass ratio, the reflection is more significant than the transmission and a considerable amount of trapping occurs. An interesting phenomenon was discovered in numerical simulations—the energy trapped in the mass-with-mass defect shows a non-monotonic dependence on the mass ratio.

Wave propagation in a chain of spherical elements containing an internal resonator in linear and nonlinear regimes was investigated theoretically and in experiments demonstrating a wide band gap in the audible regime [93]. This allows filtering of mechanical waves between 3 and 8.5 kHz and above 17 kHz, and this frequency range can be tuned by approximately 1 kHz using a static precompression.

A theoretical and numerical analysis of mass with mass granular chains was conducted in [94]. The authors demonstrated that under suitable 'anti-resonance' conditions bell-shaped travelling-wave solutions similar to Nesterenko solitary waves in the standard granular chain with elastic Hertzian contacts are supported by this more complex system. It is interesting that if these conditions are violated, then non-monotonic waves bearing non-vanishing tails may exist [94].

10. Strongly nonlinear waves in three-dimensional granular beds

Three-dimensional granular beds are very efficient mitigators of high-amplitude compression pulses caused by the impact of contact explosion. Granular beds made of cast iron shots were proposed as supporting structures for explosive welding of complex shapes, such as blades of hydroturbines (with a size of the order 15–20 m^2) helping to minimize residual strains in cladded products and attain high-quality welding [95]. They were successfully used as supporting structures for explosive working in blast chambers as schematically illustrated in figure 4. Granular beds not only effectively mitigate high-amplitude shock waves caused by contact explosions. It is also practically important that they did not create dust particles because metal spheres (iron shots) were ductile, unlike sand particles. Iron shots are relatively cheap, being a waste of metallurgical plants, and subjects placed on them practically stayed in place after an explosive event.

The main mechanism of the effectiveness of the granular beds made from iron shots, in addition to their shock mitigation capability due to 'friction between iron beads', was explained

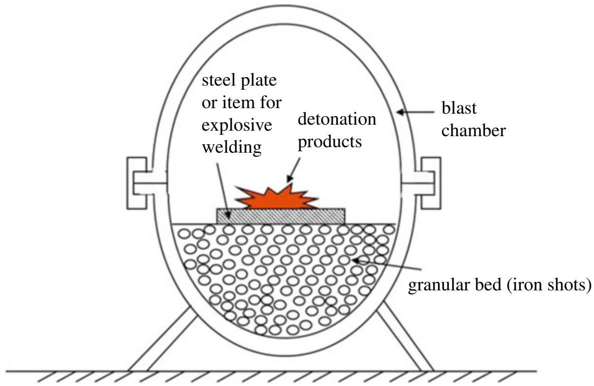

Figure 4. Blast chamber with a granular bed made from iron shots as a supporting barrier.

by assuming that 'the material of the shot have physico-mechanical properties substantially close to those of the workpiece being clad' and a granular bed was 'practically incompressible' [95]. It was also argued in [95] that the effectiveness of granular beds made from iron shots is facilitated by close values of acoustic impedances of the explosively cladded metal plates and acoustic impedance of iron shots. In fact an acoustic impedance of the granular bed, considered as a continuum, is much smaller than the acoustic impedance of the beads' material—solid iron. In case of zero precompression it is equal to zero (emphasized by the term sonic vacuum) or very small at weak precompression as was explained above. The exact mechanism of the effectiveness of granular beds and how to optimize their performance are still unclear, and are very interesting and practically important subjects of research.

Granular beds incorporate one-dimensional elements—'force' chains. Nonlinear force propagation into a granular bed caused by the striker's impact was experimentally studied in [96], and the results were explained using nonlinear grain-scale force relation similar to one-dimensional structures.

Attenuating high-amplitude pulses resulting in elasto-plastic deformation of beads in ordered two-dimensional square and hexagonal packing were investigated in experiments and in a molecular dynamics simulation [47,82]. In numerical simulations it was found that even small amounts of disorder in a two-dimensional packing significantly affect both wave transit time and peak force.

The highly heterogeneous, random mesostructure of 'force' chains determines dynamic force transmission even under loading by contact explosion, as in blast chambers (figure 4). In the experimental set-up, a cover steel plate with a thickness of 10 mm was placed between an explosive charge (density $1\,\mathrm{g\,cm^{-3}}$, detonation speed $4\,\mathrm{km\,s^{-1}}$) and a granular bed made from iron shots (diameter of spheres about 3 mm) with heights of 50 and 100 mm. The estimated initial velocity of the cover steel plate was $10\,\mathrm{m\,s^{-1}}$. A broad spectrum of indentation diameters on the lead plate placed at the bottom of the granular bed on the supporting massive plate is shown in figure 5.

It is estimated, using an approach similar to that applied in the Brinell hardness test, that the ratio of corresponding contact forces, resulted in indentation diameters of 0.5 and 3 mm, is about 20, suggesting a strong heterogeneity of force transmission inside the granular bed in these experiments.

Some indentations are about 6 mm apart corresponding to the distance between spheres' centres. It is clear that distances between other indentations are significantly larger. Thus figure 5 demonstrates that not all spheres being in contact with the lead plate are participating in force transmission and the number of active force chains is reduced with reduction of pulse amplitude.

One-dimensional chains of spheres are a reasonable approach to model dynamic behaviour of ordered three-dimensional packing. But three-dimensional disordered granular beds made

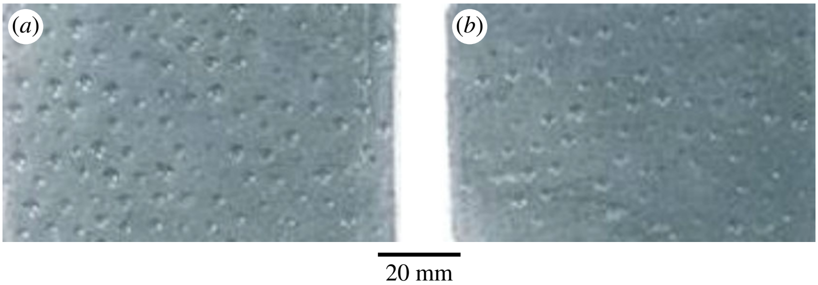

Figure 5. Indentations on the lead plate at the bottom of the granular bed made from iron shots at different distances from contact explosion: 50 mm (*a*) and 100 mm (*b*). (Online version in colour.)

from spherical particles of different sizes demonstrate significantly different behaviour [3]. For example, the reflected pressure profiles from the rigid wall in cast iron shots with diameters of spheres 3–4 mm and its one-dimensional model at analogous conditions of loading (the same velocity of the piston and its mass per unit surface) are qualitatively different. The time of force increase in the case of a one-dimensional chain in numerical calculations was 6 to 10 times less than that obtained in experiments with cast iron shots. This almost an order of magnitude difference in front width cannot be explained in the frame of the one-dimensional approach even taking into account the random size of iron shots. This indicates that three-dimensional granular beds have essentially different dynamic behaviour in comparison with the one-dimensional chains. It can be associated with the transition time for configuration change in the packing from one state to another, depending on the applied pressure and its time dependence. The more dispersed character of shock impulses in granular beds can be explained by additional dispersion due to friction processes between spherical particles. It is also possible that energy is dissipated as a result of excitation of the rotational degree of particle movement. The simplest way to describe such behaviour might be an introduction of artificial 'viscosity' [3].

11. Conclusion

Discrete strongly nonlinear systems (e.g. granular materials, systems with strongly nonlinear elements (O-rings, tensegrity structures) represent a new class of metamaterials being a natural extension from classical weakly nonlinear behaviour to the strongly nonlinear case. The corresponding theoretical approach should be considered as the natural step in developing strongly nonlinear wave dynamics. Low-dimensional or highly ordered three-dimensional weakly dissipative structures support Nesterenko solitary waves, which are compact with space scale independent of amplitude and dictated by mesostructured and interaction law. Unlike linear and weakly nonlinear systems they exhibit highly tunable behaviour, sensitive to low amplitude of an external mechanical force. They are the only known systems that can be tuned from a strongly nonlinear regime (nonlinearizable case of 'sonic vacuum' at zero precompression) to weakly nonlinear and linear regimes by application of moderate external static force with small changes of overall dimensions. Multiscale systems based on the tensegrity concept even allow tuning from strongly nonlinear elastically stiffening behaviour to the elastically softening regime. Discrete systems combining hardening (normal) and softening (abnormal) behaviours at different strains can be realized in experiments, but it is a mathematically challenging problem to describe them in long-wave approximation. A very important task is to identify the similar phenomena in different mechanical and electromagnetic metamaterials.

Wave interactions with interfaces of strongly nonlinear discrete systems or with interfaces between them and linear elastic materials result in a new phenomenon, e.g. breakdown of the continuum approximation, and in the 'acoustic diode' behaviour. Interaction of the wave with the

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interface of two sonic vacua dramatically depends on the incident direction and applied external precompression, which can be used for targeted energy transfer.

Recent theoretical and experimental development directed towards a strongly nonlinear mass in a mass system and to the analysis of breathers is very promising and opens new opportunities in vibration control and impact mitigation.

Disordered three-dimensional granular beds are highly nonlinear according to a few physically different reasons including strong nonlinearity of the Hertz Law and structural rearrangements under applied dynamic loads. Dissipation caused by contact viscoelastic deformation of polymer elements (e.g., O-rings) is a challenging and important problem for design of dissipative metamaterials for mitigation of high-amplitude pulses. The synthesis of all components of highly nonlinear behaviour is a very exciting area for future research, which has a strong potential for designing metamaterials with highly nonlinear properties desirable for various applications.

Competing interests. I declare I have no competing interests.

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