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Pattern Formation via Eigenstructure Assignment General Theory and Multi-agent Application

Andy Wu and Tetsuya Iwasaki

Abstract—Complex dynamical systems often exhibit formation of a pattern in observed variables in the steady state. An important special case is when the system consists of multiple subsystems (or “agents”) subjected to local interactions to reach consensus or an arbitrary pattern specified by their relative positioning in the state space. This paper formulates a general pattern formation problem as the design of a feedback controller such that selected outputs of a linear plant exponentially converge to $Re^{At}\rho_o$ for some vector ρ_o , with prescribed matrices R and Λ . We show that the problem reduces equivalently to an eigenstructure assignment problem, and provide a necessary and sufficient condition for existence of a feasible controller as well as a parameterization of all such controllers. This general theory is further specialized to give a complete solution to a heterogeneous multi-agent synchronization problem. Two numerical examples are provided to demonstrate the efficacy of the proposed design method: one illustrates the importance of adaptive pattern formation through sensory feedback and another suggests an extension for achieving stable limit cycles by additional nonlinearities.

I. INTRODUCTION

Eigenstructure assignment is a classical design methodology used to specify (a subset of) closed-loop eigenvalues and associated eigenvectors to achieve a desired behavior for a linear system. It was originally motivated by the exploitation of the extra freedom offered by multiple control inputs beyond assigning the full set of closed-loop eigenvalues by static state feedback [1]. The extra eigenvector specification allows for multi-mode decoupling/shaping, and was used extensively for flight control [2]–[4] and later for fault detection [5], [6].

As a theoretical basis, exact characterizations of assignable eigenvectors using the freedom beyond the pole placement were given in [7], [8] for the state feedback case. Later works, [9], [10], saw the use of static output feedback controllers and provided insights into assignable eigenstructures. However, the use of static gains in the absence of full state information considerably restricts the freedom in assigning eigenstructures. This motivated approximate eigenstructure assignment through projection [11]. More recently, [12] suggested the use of *dynamic* feedback with *partial* pole placement to leave more freedom for eigenvector assignment. However, the focus of the paper was not solely on the eigenstructure assignment and the issue was not addressed in depth. To date, a complete theory for eigenstructure assignment has not been developed.

Pattern formation problems for multi-agent systems have recently gained much attention in multiple contexts [13]–[17],

and their connections to eigenstructure assignment have been gradually recognized. Various dynamical properties for pattern formation, including consensus, synchronization, and coordinated oscillations, turned out to be related to eigenstructure assignment as we will briefly review in the following two paragraphs. Individual problems have been formulated and approached independently, but complete solutions are not yet available. A general theory of eigenstructure assignment for multi-agent systems, if developed, would shed new light on pattern formation problems.

The design of nonlinear oscillators for coordinated oscillations was considered in [16], using the biological structure of central pattern generators (CPGs). The problem was formulated as the search for a matrix that specifies the interconnections between agents (i.e. neurons) so that a stable limit cycle is achieved with a prescribed oscillation profile. It was shown that the problem reduces approximately to an eigenstructure assignment through multivariable harmonic balance, with eigenvalue and eigenvector specifying the frequency and relative amplitudes/phases, respectively. The idea was extended for feedback control design to generate coordinated oscillations for a linear plant [18], [19].

Another pattern formation problem of recent popularity in the controls community is the consensus of multiple agents. Earlier researches [14], [20] have noted the importance of the communication or graph structure in reaching consensus between agents. A fundamental observation is that the inter-agent coupling is designed so that the target state of consensus is achieved in the eigenspace of the corresponding graph Laplacian associated with the zero eigenvalue. Building on those developments, a linear homogeneous consensus problem was solved in [21], and later extended to the case of heterogeneous agents [17]. This reference showed an internal model principle in terms of a regulator equation, clearly indicating a strong connection to eigenstructure assignment [22]. However, the structure of the controller given by [17] is potentially restrictive as the control input to each agent is independent of sensory feedback from any other agent. This drawback has been overcome for some classes of heterogeneous agents, including integrators [23], minimum-phase agents [24], static state feedback homogenizable agents [25], and right-invertible agents [26], but a general theory without restrictions on agent dynamics has yet to be established.

In this paper, we formulate a general pattern formation problem of designing a linear static or dynamic output feedback controller for a linear time-invariant plant to achieve the following specifications: Given a matrix R and matrix with non-negative eigenvalues Λ , selected plant outputs exponentially

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converge to a trajectory $Re^{\Lambda t}\rho_o$ for some vector ρ_o dependent on the initial state. The spatial shape (relative positioning of output variables) is specified by R , and its time evolution (constancy, growth, or oscillation) is specified by Λ . Thus $Re^{\Lambda t}\rho_o$ can be viewed as a spatio-temporal pattern exhibited by the dynamical system. In the literature, pattern formations have often been considered for networks of multiple agents (e.g. vehicle formations [14], [27], [28], lattice and interconnected systems [13], [15]), but here we broaden the scope to include general unstructured and structured systems. Due to the linearity and autonomy of the system, the size of the pattern varies depending on the initial conditions through ρ_o . The dependence is inevitable within the linear framework, but we discuss possible extensions through a numerical example.

We first consider the general setting where the pattern $Re^{\Lambda t}\rho_o$ is to be formed through feedback interactions of dynamical elements, which may be completely unstructured. Our result establishes equivalence between the pattern formation problem and an eigenstructure assignment problem in which a subset of eigenvectors and eigenvalues are assigned to the closed-loop system, while unassigned eigenvalues are placed in the open left-half plane. We provide a necessary and sufficient condition for solvability of the eigenstructure assignment problem in terms of a regulator equation, and propose a parameterization of all feasible controllers which embeds an internal model of the desired dynamics explicitly or implicitly. Moreover, we prove separation principles, reducing the output feedback natural eigenstructure assignment to a double observer design plus full control, state feedback eigenstructure assignment.

The multi-agent pattern formation problem falls under a special case of our eigenstructure assignment framework for which a structured controller is designed for a structured plant. We consider the design of feedback controllers for linear heterogeneous agents with the objective that selected outputs of the k^{th} agent exponentially converge to $R_k e^{\Lambda t}\rho_o$ for some ρ_o , where information is exchanged between selected agents. Synchronization occurs in a special case where the same R_k is used for all agents. Our approach utilizes our general eigenstructure theory and separation principles to design local controllers that assign the desired dynamics to “homogenize” the modified agents, sharing the same spirit as in [26]. We then apply a standard result for synchronization of homogeneous agents [21] to achieve the desired formation between agents. Unlike the previous results, our controller solves the pattern formation problem with no restrictions on the agent dynamics, exploiting sensory feedback from neighboring agents. Various special cases for controller order reduction are discussed.

We illustrate the proposed design method through two numerical examples for a heterogeneous multi-agent system. First, we design a controller to achieve consensus to a common displacement trajectory with a constant speed. We then compare our controller to the one proposed by [17], demonstrating the importance of real-time generation of a consensus trajectory based on the sensory information when subjected to a disturbance. In the second example, we consider coordinated oscillations and touch on the possibility of expanding the linear eigenstructure framework to nonlinear pattern formations with

an additional nonlinearity to lock the amplitudes of oscillation and create a stable limit cycle.

An earlier version of our result, assigning a single pair of eigenvalue/eigenvector, appeared in [29] without proof. This paper generalizes the result in [29] to include the assignment of an arbitrary number of eigenvalues/eigenvectors and extends the multi-agent result to reflect the added complexity.

Notation: Let $\mathbb{R}$, $\mathbb{C}_-$, and $\mathbb{I}_n$ denote the sets of real numbers, complex numbers with negative real parts, and integers $\{1, \dots, n\}$, respectively. For a matrix M , the notations M^T , $M^\dagger$, and $\text{eig}(M)$ denote the transpose, the Moore-Penrose inverse, and the set of eigenvalues respectively. The symbol $\text{diag}(A_1, \dots, A_n)$ denotes the block-diagonal matrix with $A_1, \dots, A_n$ on the diagonal, and $\text{col}(B_1, \dots, B_n)$ denotes the matrix obtained by stacking $B_1, \dots, B_n$ in a column. For a state-space system $\dot{x} = Ax + Bu$, $y = Cx + Du$, the mapping from u to y is denoted by $y = \dot{P}u$ with $P := \begin{bmatrix} D & C \\ B & A \end{bmatrix}$. The same notation $y = \dot{P}u$ is used for the static case where $y = Du$ and $P := D$. For signals x and y , the notation $x(t) \rightarrow y(t)$ means that $\|x(t) - y(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

II. PROBLEM FORMULATION

A. Pattern Formation as Eigenstructure Assignment

Let a linear time-invariant plant be given by a state-space realization

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad z = Hx, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the measured output, and $z(t) \in \mathbb{R}^h$ is the performance output. Consider a static controller or a dynamic controller with state x_c

$$u = Ky \quad \text{or} \quad \begin{aligned} \dot{x}_c &= A_c x_c + B_c y, \\ u &= C_c x_c + D_c y, \end{aligned} \quad (2)$$

and let the closed-loop system be denoted by

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad z = H\mathbf{x}, \quad (3)$$

where $\mathbf{x} = x$ or $\text{col}(x, x_c)$ is the state vector.¹ Let a system with desired dynamics

$$\dot{\rho} = \Lambda\rho, \quad \zeta = R\rho, \quad (4)$$

be given, where $\rho(t) \in \mathbb{R}^r$ and $\zeta(t) \in \mathbb{R}^h$. We would like to design a controller so that the outputs of (3) and (4) match each other after transients die out. To make a precise problem statement, let us introduce the following.

Definition 1: Autonomous systems (3) and (4) are said to be asymptotically equivalent if the following conditions hold:

- (a) For each $\rho(0)$, there exists $\mathbf{x}(0)$ such that $z(t) \rightarrow \zeta(t)$.
- (b) For each $\mathbf{x}(0)$, there exists $\rho(0)$ such that $\zeta(t) \rightarrow z(t)$.

Properties (a) and (b) are closely related to the notions of system inclusion [30] and system equivalence [31]: system (3) is said to include (4) if (a) holds with $z(t) \rightarrow \zeta(t)$ replaced by $z(t) \equiv \zeta(t)$, and systems (3) and (4) are said to

¹We will keep this notation for the remainder of this paper. That is, variables with subscript c will refer to those associated with the controller unless otherwise noted.

be equivalent if the former includes the latter, and vice versa. Clearly, equivalent systems are asymptotically equivalent, but the converse is not true in general. However, the converse is also true if neither system has eigenvalues in the open left half plane because then there is no transient that dies out. Our interest is in the mixed case where (4) has no stable dynamics since it is meant to describe a nontrivial asymptotic behavior, but (3) may have both stable and unstable dynamics.

Problem 1. Consider the plant (1) and the desired dynamics (4). Assume that (A, B) is stabilizable, (H, A) and (C, A) are detectable, (R, Λ) is observable, and all the eigenvalues of Λ have nonnegative real-parts. Find a necessary and sufficient condition for the existence of a controller of the form (2) such that the closed-loop system (3) satisfies properties (a) and (b) in Definition 1 and (c) $(\mathcal{H}, \mathcal{A})$ is detectable. Parametrize the set of all such controllers.

Properties (a) and (b) ensure that the closed-loop behavior of $z(t)$ matches the desired dynamics $\zeta(t)$ in the steady state, regardless of the initial conditions. It is reasonable to require Property (c) for the control design since otherwise there is an unstable mode that makes an internal variable diverge while the output specifications (b) and (a) are satisfied.

Problem 1 can be viewed as a pattern formation problem. In particular, the signal $z(t)$ is required to converge to the pattern specified by $\zeta(t) = Re^{\Lambda t} \rho(0)$ for some $\rho(0)$. A pattern is defined as relative behaviors of the entries of $\zeta(t)$. There are two primary types of patterns captured by the linear framework depending on the eigenvalues of Λ ; oscillation (complex) and constant (real). The relative amplitudes (and phases) of the entries of $\zeta(t)$ are specified by R , and the absolute amplitudes can be constant or growing over time, as specified by the real part of the eigenvalues and their multiplicity. The actual values at particular time instants depend on the initial state $\rho(0)$.

An oscillation pattern can be specified by (R, Λ) with

$$\Lambda := \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad R_k := [a_k \cos(b_k) \quad a_k \sin(b_k)] \quad (5)$$

where $\omega, a_k, b_k \in \mathbb{R}$ are constants, and R_k is the k^{th} row of R . In particular, the pattern $\zeta = Re^{\Lambda t} \rho(0)$ takes the form

$$\zeta_k(t) = \alpha_k \sin(\omega t + \beta_k), \quad k \in \mathbb{I}_h, \quad (6)$$

where $\alpha_k = \gamma_a a_k$ and $\beta_k = b_k + \gamma_b$ with (γ_a, γ_b) dependent on the initial condition $\rho(0)$. Note that a_k and b_k , respectively, determine the relative amplitude and phase between ζ_k . The absolute amplitudes and phases depend on the initial state.

A non-oscillatory pattern can be specified by

$$\Lambda := \lambda \geq 0, \quad R_k := a_k, \quad k \in \mathbb{I}_h,$$

where $\lambda, a_k \in \mathbb{R}$ are constant parameters. The resulting pattern is given by $\zeta_k = \gamma_a a_k e^{\lambda t}$ with γ_a being the initial condition $\rho(0)$. This is a trajectory that either exponentially diverges ($\lambda > 0$) or converges to a constant value ($\lambda = 0$). As with the previous oscillation case, a_k can only specify the steady-state values of an output relative to other outputs. For instance, a star-shape formation of multiple vehicles can be specified by a_k if $z(t)$ represents their physical coordinates, but the size of the star, γ_a , will depend on the initial configuration.

It is also possible to combine the basic specifications to obtain different steady-state behaviors. For example, assigning two zero eigenvalues as a double integrator with specifications

$$\Lambda := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad R := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (7)$$

achieves $\zeta_1 = \gamma_v t + \gamma_p$ and $\zeta_2 = \gamma_v$ for some scalars $\text{col}(\gamma_p, \gamma_v) := \rho(0)$, specifying the (initial) position and velocity. Other variations include cases where some outputs converge to constants with prescribed relative magnitudes, while other outputs converge to oscillations with prescribed frequency and relative phases and amplitudes.

For the control design in Problem 1, we require $z(t) \rightarrow \zeta(t)$ for some solution $\zeta(t)$ of (4) without specifying the value of $\rho(0)$. This means that the spatio-temporal pattern of $z(t)$ is specified, but the size of the pattern depends on the initial state $\mathbf{x}(0)$. The dependence is unavoidable since the closed-loop system is linear and unforced. We will later show in an example (of coordinated oscillations) that the size (amplitude) can be locked with the addition of a nonlinearity.

The specifications of Problem 1 can be characterized by the closed-loop eigenvalues and eigenvectors as follows.

Lemma 1: Consider Problem 1. For a given controller, the closed-loop system (3) satisfies the design specifications (a)–(c) if and only if the following statements hold.

(i) There exists a full column-rank matrix $\mathcal{V}$ such that

$$\mathcal{A}\mathcal{V} = \mathcal{V}\Lambda, \quad \mathcal{H}\mathcal{V} = R. \quad (8)$$

(ii) $\text{eig}(\mathcal{A}) \setminus \text{eig}(\Lambda) \subset \mathbb{C}_-$.

Proof. See Appendix. ■

Based on the absence of stable dynamics in (4), the proof shows that property (a) implies inclusion of (4) by (3), which in turn is equivalent to condition (i) as stated in [30]. Condition (i) requires that the spectrum of the closed-loop matrix $\mathcal{A}$ contains the eigenvalues of Λ , and the associated eigenvectors $\mathcal{V}$ give the prescribed output eigenstructure R . Thus, the desired trajectory $\zeta(t)$ generated by (4) can be reproduced by the closed-loop system. Condition (ii) constrains the closed-loop eigenvalues not associated with Λ to the open left-half plane. This ensures that the output $z(t)$ coincides with $\zeta(t)$ after transients die out. Problem 1 is now formulated as an eigenstructure assignment where the controller parameters are sought to satisfy conditions (i) and (ii).

Definition 2: A controller is said to solve Problem 1 with eigenmatrix $\mathcal{V}$ if the closed-loop system satisfies conditions (i) and (ii) in Lemma 1.

When the controller is dynamic with n_c dimensional state space, the eigenmatrix can be partitioned as $\mathcal{V} = \text{col}(X, \Xi)$, where $X \in \mathbb{R}^{n \times r}$ and $\Xi \in \mathbb{R}^{n_c \times r}$ specify the eigenstructures for the plant and controller states, respectively. For brevity, we may use the same notation for the case of static controller ($n_c = 0$), with the understanding that Ξ is an empty $(0 \times r)$ matrix and any terms containing Ξ are considered absent.

B. Controller Equivalence

We aim to parametrize the set of all controllers solving Problem 1. However, some controllers may be considered

equivalent for the purpose of achieving the design specifications. It turns out that removal of such redundancy simplifies the controller parametrization. The purpose of this section is to formally define the notion of equivalence and identify a subset of feasible controllers such that all feasible controllers can be covered by those equivalent to members of this subset.

In many linear control designs, specifications are given in terms of the closed-loop transfer function with internal stability. In this case, controllers are equivalent if they are stabilizable and detectable realizations of the same transfer function. However, this notion of equivalence does not apply to feasible controllers for Problem 1. For example, a solution to Problem 1 may be given as an open-loop controller with uncontrollable and unstable modes that comprise an internal model for the desired dynamics (this case will be discussed later). It turns out that the following definition of equivalence for state space systems is appropriate for our purposes.

Definition 3: Input-output systems $y_i = \dot{P}_i u_i$ with $i = 1, 2$ are said to be asymptotically equivalent if their autonomous dynamics under $u_i(t) \equiv 0$ are asymptotically equivalent in the sense of Definition 1 and their transfer functions are identical.

It can be shown (similarly to the proof of Lemma 1) that two systems are asymptotically equivalent if and only if their minimal forms are related by a state coordinate transformation, where a minimal form is defined as follows. Consider a Kalman canonical form of a system as shown on the left of

$$\left[\begin{array}{cccc|c} A_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 \\ A_{31} & A_{32} & A_3 & 0 & B_3 \\ A_{41} & A_{42} & A_{43} & A_4 & B_4 \\ \hline C_1 & C_2 & C_3 & 0 & D \end{array} \right] \rightarrow \left[\begin{array}{cc|c} A_2 & 0 & 0 \\ A_{32} & A_3 & B_3 \\ \hline C_2 & C_3 & D \end{array} \right]$$

where A_4 contains all the unobservable modes, (A_3, B_3, C_3) is controllable and observable, A_1 is Hurwitz, and A_2 has all the eigenvalues in the closed right half plane. Then a minimal form is obtained by removing the stable uncontrollable state x_1 and unobservable state x_4 as shown on the right of the above equation.

When two controllers are asymptotically equivalent, they produce the same control input $u(t)$ for the same sensory signal $y(t)$ after transients die out, regardless of the initial state. This is because the difference between the two, i.e. unobservable modes and stable uncontrollable modes, do not affect $u(t)$ in the steady state. With this property, a controller asymptotically equivalent to a feasible controller also solves Problem 1. Moreover, they share the same plant eigenmatrix.

Lemma 2: Let $\dot{\mathcal{K}}_a$ be a controller that solves Problem 1 with eigenmatrix $\mathcal{V}_a := \text{col}(X, \Xi_a)$. Let $\dot{\mathcal{K}}_b$ be a controller asymptotically equivalent to $\dot{\mathcal{K}}_a$. Then $\dot{\mathcal{K}}_b$ solves Problem 1 with eigenmatrix $\mathcal{V}_b := \text{col}(X, \Xi_b)$ for some Ξ_b .

Proof. It can be seen from the specifications that $\dot{\mathcal{K}}_b$ solves Problem 1 since stable uncontrollable modes of the controller will not affect the steady state behavior, unobservable modes are invisible in $z(t)$, and addition/removal of stable uncontrollable or unobservable modes preserves detectability of $(\mathcal{H}, \mathcal{A})$. Writing equation (8) in Kalman canonical form verifies that $\dot{\mathcal{K}}_a$ and $\dot{\mathcal{K}}_b$ share the same plant eigenmatrix X . ■

The following result shows that the controller eigenmatrix Ξ can be normalized for the purpose of solving Problem 1.

Lemma 3: Suppose $\dot{\mathcal{K}}_a$ solves Problem 1 with $\mathcal{V}_a := \text{col}(X, \Xi_a)$. Then there is a controller $\dot{\mathcal{K}}_b$ asymptotically equivalent to $\dot{\mathcal{K}}_a$, that solves Problem 1 with eigenmatrix $\mathcal{V}_b := \text{col}(X, \Xi_b)$ where $\Xi_b := \text{col}(I, 0)$.

Proof. Consider the case where $\dot{\mathcal{K}}_a$ in the statement is dynamic; the static case can be shown similarly. Let x_a be the state vector of $\dot{\mathcal{K}}_a$, and define $\dot{\mathcal{K}}_c$ with state vector $x_c := \text{col}(x_a, x_o)$ by adding stable unobservable modes

$$\dot{x}_o = Ax_o + Bu + G(Cx_o - y)$$

to $\dot{\mathcal{K}}_a$, where G is a matrix such that $A + GC$ is Hurwitz. It can then be verified that the closed-loop system (3) satisfies (8) with eigenmatrix $\mathcal{V}_c := \text{col}(X, \Xi_c)$ where $\Xi_c := \text{col}(\Xi_a, X)$.

Now, since $\mathcal{V}_a$ is a full column-rank matrix, Ξ_c also has full column rank. Hence, there is a nonsingular matrix T that transforms Ξ_c into the form $T\Xi_c = \Xi_b := \text{col}(I, 0)$. Then, defining $\dot{\mathcal{K}}_b$ with state vector $x_b := Tx_c$ by the coordinate transformation T on the controller $\dot{\mathcal{K}}_c$, one can verify that $\dot{\mathcal{K}}_b$ solves Problem 1 with $\mathcal{V}_b := \text{col}(X, \Xi_b)$. ■

By Lemma 3, $\mathcal{V}$ in (8) can be assumed to have the structure $\mathcal{V} = \text{col}(X, \Xi_o)$ with $X \in \mathbb{R}^{n \times r}$ and $\Xi_o := \text{col}(I_r, 0)$ without loss of generality when searching for the controllers that solve Problem 1. We will thus restrict our attention to the subset of feasible controllers having this structure for the eigenmatrix.

III. GENERAL EIGENSTRUCTURE ASSIGNMENT

A. Separation Principle and State Feedback

Separation principles hold for various control problems including stabilization and H_2 optimal control, where the output feedback design reduces to independent designs of an observer and a state feedback gain. This section shows that a separation principle also holds for the output feedback eigenstructure assignment, where the problem is broken down into a standard observer design and a state feedback eigenstructure assignment. We will also show how the state feedback problem can be solved, and discuss a subtlety unique to our problem. To this end, let us first state the following.

Theorem 1 (Separation Principle): There exists an output feedback controller $u = \dot{\mathcal{K}}_s y$ that solves Problem 1 if and only if there exists a state feedback controller $u = \dot{\mathcal{K}}_s x$ that solves Problem 1 with $C = I$. In particular, an output feedback controller that solves Problem 1 is given by

$$u = \dot{\mathcal{K}}_s \hat{x}, \quad \dot{\hat{x}} = A\hat{x} + Bu + G(C\hat{x} - y), \quad (9)$$

where G is a matrix such that $A + GC$ is Hurwitz, and $\dot{\mathcal{K}}_s$ is a state feedback controller that solves Problem 1 with $C = I$.

Proof. Suppose there exists an output feedback controller $u = \dot{K}_o y$ that solves Problem 1. The closed-loop system is given by (3) with $\mathcal{A} = A + BK_o C$, where $A = \text{diag}(A, 0)$, $B = \text{diag}(B, I)$, and $C = \text{diag}(C, I)$, with identity and zero matrices of dimension equal to the controller order. We then see that the state feedback controller $u = \dot{K}_s x$ with $\dot{K}_s := K_o C$ solves Problem 1. Thus, if there exists an output feedback controller that solves Problem 1, then there exists a state feedback controller that solves Problem 1 with $C = I$.

To show the converse, suppose there exists a state feedback controller $u = \mathcal{K}_s x$ that solves Problem 1 with $C = I$, i.e., conditions (i) and (ii) of Lemma 1 are satisfied for $\mathcal{A} = \mathcal{A}_s := A + B\mathcal{K}_s$ and some matrix $\mathcal{V} = \mathcal{V}_s := \text{col}(X, \Xi_s)$. With this $\mathcal{K}_s x$, consider the controller in (9) and the closed-loop system

$$\begin{bmatrix} \dot{\mathbf{x}}_s \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_s & * \\ 0 & A + GC \end{bmatrix} \begin{bmatrix} \mathbf{x}_s \\ e \end{bmatrix}. \quad (10)$$

where $e := \hat{x} - x$, $\mathbf{x}_s := \text{col}(x, x_s)$, and x_s is the state vector associated with the dynamics of $\mathcal{K}_s$. Because $\mathcal{A} := \mathcal{A}_s$ satisfies conditions (i) and (ii) of Lemma 1 and $A + GC$ is Hurwitz, the closed-loop eigenvalues are the eigenvalues of Λ with the rest having negative real-parts. Thus, condition (ii) is satisfied. Furthermore, the eigenvectors of (10) associated with Λ are given by $\text{col}(\mathcal{V}_s, 0)$, implying that condition (i) is satisfied with eigenvector $\mathcal{V} := \text{col}(\mathcal{V}_s, X)$ for the closed-loop system with the original state coordinates $\mathbf{x} := \text{col}(\mathbf{x}_s, \hat{x})$. ■

Since the observer design is trivial, Problem 1 is essentially reduced to a state feedback eigenstructure assignment. The following is the solution to the *static* state feedback case.

Theorem 2 (Static State Feedback): Consider Problem 1 with $C = I$. The following statements are equivalent.

- (i) There exists a static state feedback controller $u = Kx$ that solves Problem 1.
- (ii) There exist matrices F and full column-rank X satisfying

$$AX + BF = X\Lambda, \quad HX = R. \quad (11)$$

In this case, all such controllers are parametrized by

$$u = Kx, \quad K := FX^\dagger + ZY^\dagger, \quad (12)$$

where Z is an arbitrary matrix such that

$$\text{eig}(Y^\dagger AY + Y^\dagger BZ) \subset \mathbb{C}_-, \quad (13)$$

and Y is the orthogonal complement of X .

Proof. Suppose (i) holds. Then the conditions in Lemma 1 are satisfied for $\mathcal{A} := A + BK$, $\mathcal{H} := H$, and $\mathcal{V} = X$ with some full column-rank matrix X . Defining $F = KX$, the conditions in (11) are satisfied. Thus, we have (i) $\Rightarrow$ (ii). To see the converse, define a state feedback controller by (12) and note that Z satisfying (13) always exists because stabilizability of (A, B) implies stabilizability of $(Y^\dagger AY, Y^\dagger B)$. Applying a similarity transformation to $A + BK$, we have

$$\begin{bmatrix} X^\dagger \\ Y^\dagger \end{bmatrix} (A + BK) \begin{bmatrix} X & Y \end{bmatrix} = \begin{bmatrix} \Lambda & * \\ 0 & Y^\dagger (AY + BZ) \end{bmatrix}, \quad (14)$$

where $*$ denotes irrelevant entries. Because $Y^\dagger (AY + BZ)$ is Hurwitz, we see that (12) is a static controller that solves Problem 1 and therefore, (ii) $\Rightarrow$ (i). It remains to show that all feasible state feedback gains are given by (12) for some F and Z satisfying (11) and (13). Let $u = Kx$ be a controller solving Problem 1 with eigenmatrix X . Define $F := KX$ and $Z := KY$ so that (12) holds. Then, F satisfies (11), which further implies that (14) holds, and thus Z satisfies (13). ■

Condition (11) is similar to the regulator equation that gives solvability of the output regulation problem [32], and has arisen in the context of eigenstructure assignment [22] and output synchronization [17], topics associated with Problem 1.

Solvability of Problem 1 under static feedback requires existence of F and full column-rank X satisfying (11), but the full rank requirement is restrictive in general. Consider, for example the case in which

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

The target trajectory specified by (4) is given by

$$\zeta(t) = Re^{\Lambda t} \rho(0) = (a + bt) \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{matrix} a := \rho_1(0) + 2\rho_2(0), \\ b := \rho_2(0). \end{matrix}$$

In this case, there is a unique solution (X, F) satisfying (11):

$$X = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 3 \end{bmatrix}.$$

The matrix X does not have full column rank, implying that there is no *static* state feedback controller to achieve the output pattern formation. However, the existence of solution (X, F) to (11) turns out to imply that the problem can be solved by a *dynamic* state feedback controller. This is shown as a special case of the general result presented in the next section.

B. General Solution

This section provides a general and complete solution to Problem 1 with no assumptions on the controller structure (e.g. observer-based). The following result gives a necessary and sufficient condition for solvability of Problem 1, and a parametrization of all feasible controllers.

Theorem 3 (General Dynamic Output Feedback): The following statements are equivalent.

- (i) There exists a controller that solves Problem 1.
- (ii) There exist matrices F and X such that (11) holds.

In this case, all such controllers are captured as those that are asymptotically equivalent to the class of controllers parametrized by

$$\begin{bmatrix} u \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} F \\ \Lambda \end{bmatrix} \xi + \dot{Q}(y - CX\xi). \quad (15)$$

where matrices F and X satisfy (11), and $\dot{Q}$ stabilizes the augmented plant $(A, \mathfrak{B}, C)$ with $\mathfrak{B} := \begin{bmatrix} B & -X \end{bmatrix}$.

Proof. See Appendix. ■

The controller in (15) is illustrated in Fig. 1. We see that the internal model of the targeted eigenstructure is embedded in the controller. The mechanisms underlying the pattern formation is most evident when the plant is stable. In this situation, choosing $\dot{Q}$ to be zero makes the transfer function from y to u zero and reduces (15) to the open-loop control

$$u = F\xi, \quad \dot{\xi} = \Lambda\xi,$$

and the control input u to the plant is nonzero when $\xi(0) \neq 0$ because of the internal model. With this controller, the closed-loop system satisfies the eigenstructure condition

$$\begin{bmatrix} A & BF \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} X \\ I \end{bmatrix} = \begin{bmatrix} X \\ I \end{bmatrix} \Lambda, \quad HX = R,$$

and the reference signal $F\xi$ drives the stable plant as

$$\begin{aligned} \dot{x} &= Ax + BF\xi \\ \Rightarrow \dot{x} &= Ax + (X\Lambda - AX)\xi \\ \Rightarrow \dot{x} - X\dot{\xi} &= A(x - X\xi) \\ \Rightarrow \|x(t) - X\xi(t)\| &\rightarrow 0 \\ \Rightarrow z(t) = Hx(t) &\rightarrow HXe^{\Lambda t}\xi(0) = Re^{\Lambda t}\xi(0). \end{aligned}$$

Thus the rate of convergence $z(t) \rightarrow \zeta(t)$ is determined by A . In the general case with nonzero $\dot{Q}$, we have similar equations:

$$\begin{aligned} \dot{x} - X\dot{\xi} &= (A + \mathfrak{B}\dot{Q}C)(x - X\xi), \\ \dot{\xi} &= \Lambda\xi + \dot{Q}_2C(x - X\xi), \end{aligned} \quad (16)$$

where $\dot{Q} = \text{col}(\dot{Q}_1, \dot{Q}_2)$, and the rate of convergence is dictated by the dynamics of $A + \mathfrak{B}\dot{Q}C$ instead of A .

Fig. 1. Block diagram of controller in (15) where $\Upsilon(s) := (sI - \Lambda)^{-1}$

Given the parameterization in (15), the maximum controller order necessary to solve Problem 1 is determined as sum of the dimension of $\xi(t) \in \mathbb{R}^r$, and the order of the system $\dot{Q}$ that stabilizes the augmented plant. Under state feedback, $\dot{Q}$ can be a static gain, and the controller order is r . Under output feedback, there always exists a stabilizing controller of order less than or equal to the plant order, n , so Problem 1 can be solved with a controller of order less than or equal to $n + r$.

In these cases, the internal model explicitly appears in the controller (15), accounting for part of the controller state space of dimension r . However, when X is full column-rank, the internal model can be made implicit by embedding it as part of the plant or observer state space, thereby reducing the controller order. In particular, Problem 1 can be solved with a static gain under state feedback (Theorem 2) or an observer-based controller of the plant order under output feedback (Theorem 1). These results are special cases of Theorem 3.

The state feedback result in Theorem 2 is obtained by setting $\dot{Q}$ to be a static gain of the form $\text{col}(K, J)$ in Theorem 3. Let K be chosen as in (12), and J be any matrix that makes $\Lambda - JX$ Hurwitz. This choice of $\dot{Q}$ stabilizes the augmented plant with $C = I$, and reduces the controller in (15) to

$$u = Kx, \quad \dot{\xi} = \Lambda\xi + J(x - X\xi).$$

The dynamics of ξ are made stable by J , and are unobservable from u . Therefore, ξ can be removed to yield the static control $u = Kx$. Note that the internal model (Λ, X) is no longer explicit but is embedded in the plant state space as in (14).

The observer-based controller (combination of Theorems 1 and 2) can also be parameterized by (15) with a specific choice of $\dot{Q}$. In particular, $q = \dot{Q}w$ is a dynamic system of the form

$$\begin{aligned} \dot{x}_q &= Y^\dagger(A + BK + GC)Yx_q - Y^\dagger Gw \\ q &= \begin{bmatrix} KY \\ X^\dagger(A + BK + GC)Y \end{bmatrix} x_q - \begin{bmatrix} 0 \\ X^\dagger G \end{bmatrix} w \end{aligned}$$

where observer gain G and state feedback gain K are specified as in Theorems 1 and 2. It can be shown that this $\dot{Q}$ of order $n - r$ stabilizes the augmented plant. Under a change of coordinates defined by $\hat{x} = X\xi + Yx_q$ on the state $\text{col}(\xi, x_q)$ of (15), we can obtain the controller in (9) with $\hat{K}_s := K$.

C. Further Separation for Natural Eigenstructure Assignment

Consider the case where the target eigenstructure (R, Λ) is already embedded in the plant as part of its natural dynamics:

$$AX = X\Lambda, \quad HX = R, \quad (17)$$

with some full column-rank X . The goal is to stabilize the remaining dynamics without affecting the natural eigenstructure. Since solvability (11) is confirmed with $F = 0$, Theorem 3 can be used to solve Problem 1. However, property (17) turns out to permit an additional separation principle. We start with the simplest case of full control ($B = I$), state feedback ($C = I$).

Lemma 4 (Full Control, State Feedback): Consider Problem 1 with $B = C = I$. Suppose there exists a full column-rank matrix X satisfying (17). Then, the controller $u = -Lx$ solves Problem 1 if L is chosen such that $LX = 0$ and $A - L$ has eigenvalues $\text{eig}(\Lambda)$ with the rest having negative real parts.

Proof. Noting that the closed-loop system satisfies

$$(A - L)X = X\Lambda,$$

the result follows directly from Lemma 1. $\blacksquare$

The following result shows a separation principle for the state feedback case where Problem 1 is reduced to a standard state feedback stabilization and an eigenstructure assignment with full control, state feedback.

Lemma 5: Consider Problem 1 with state feedback ($C = I$) and suppose there exists a full column-rank matrix X satisfying (17). Let K be a state feedback gain to stabilize $A + BK$, and introduce the pre-compensator

$$u = Kx_s, \quad \dot{x}_s = Ax_s + Bu + v, \quad (18)$$

such that the augmented plant is described by

$$\dot{\varepsilon} = A\varepsilon + v, \quad \varepsilon := x - x_s,$$

where v is the new control input. Then, the controller (18) with $v = -L\varepsilon$ solves Problem 1 if L is a static gain satisfying the conditions in Lemma 4.

Proof. The closed-loop system can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_s \end{bmatrix} = \begin{bmatrix} A & BK \\ L & A + BK - L \end{bmatrix} \begin{bmatrix} x \\ x_s \end{bmatrix}.$$

By Lemma 4, we have $LX = 0$ and the above system satisfies condition (i) of Lemma 1 with eigenmatrix $\mathcal{V} := \text{col}(X, 0)$.

Defining $e_s = x - x_s$, the system can be rewritten as

$$\begin{bmatrix} \dot{e}_s \\ \dot{x}_s \end{bmatrix} = \begin{bmatrix} A - L & 0 \\ L & A + BK \end{bmatrix} \begin{bmatrix} e_s \\ x_s \end{bmatrix}.$$

By definition, $A + BK$ is Hurwitz and the eigenvalues of $A - L$ not shared by Λ are also stable. Thus, condition (ii) of Lemma 1 is satisfied and the controller solves Problem 1. ■

The controller in Lemma 5 is a dynamic state feedback controller with an observer-based structure:

$$u = Kx_s, \quad \dot{x}_s = Ax_s + Bu + L(x - x_s). \quad (19)$$

While a state observer within a state feedback controller is unusual, this control architecture is beneficial for the eigenstructure assignment. In fact, the x_s dynamics do not constitute an observer in a strict sense; x_s does not converge to x since the eigenvalues of Λ are embedded in the error dynamics $\dot{e}_s = (A - L)e_s$. Using the core idea from Lemma 5, let us state the general separation result for the output feedback case.

Theorem 4: [Separation with Double Observers] *Consider Problem 1 and suppose there exists a full column-rank matrix X satisfying (17). Then, a feasible controller is given by*

$$u = Kx_s, \quad \begin{aligned} \dot{x}_s &= Ax_s + Bu + L(x_o - x_s), \\ \dot{x}_o &= Ax_o + Bu + G(Cx_o - y), \end{aligned} \quad (20)$$

where K and G stabilize $A + BK$ and $A + GC$, and $u = -Lx$ solves Problem 1 with $B = C = I$ as described in Lemma 4.

Proof. The result follows directly from an application of Theorem 1 and Lemma 5. ■

Theorem 4 shows that the output feedback problem can be separated into three independent designs; state feedback stabilization with K , standard observer with G , and state feedback, full control eigenstructure assignment with L . The resulting controller has a double observer structure, and is not of minimal order due to the dynamics associated with x_s . Although the natural eigenstructure assignment can be solved without the x_s dynamics by choosing the static gain K from Theorem 2, the use of an additional observer shifts the search for a controller that solves Problem 1 for the general plant (A, B, C) to a search for a controller that solves Problem 1 for the plant with $B = C = I$. The controller gain, L is not restricted by the plant input/output structure, permitting flexibility in the controller structure. The benefit of this property becomes apparent when it is applied to multi-agent problems for which a communication topology is a common constraint. The basic idea is to use block-diagonal K and G to stabilize the agents by local minor feedback, and then add inter-agent coupling through L to achieve coordination between agents. We will explore this idea in the next section.

IV. STRUCTURED EIGENSTRUCTURE ASSIGNMENT FOR MULTI-AGENT SYSTEMS

We now consider a special class of the eigenstructure assignment problem; the design of a structured controller for pattern formation between heterogeneous linear agents. In particular, the control objective is to make selected outputs of every agent converge to desired trajectories described by (4). We assume that agents must respect communication constraints;

the control input to a specific agent may depend only on information from itself and neighboring agents.

Consider the set of N agents, where the dynamics of the k^{th} agent ($k \in \mathbb{I}_N$) are given by

$$\dot{x}_k = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \quad z_k = H_k x_k. \quad (21)$$

We assume that (A_k, B_k) is stabilizable, and (H_k, A_k) and (C_k, A_k) are detectable. For each agent $k \in \mathbb{I}_N$, we consider a local controller $\mathcal{K}_k$ that generates control input u_k to agent k and information signal ε_k to be shared by its neighboring agents. Each local controller $\mathcal{K}_k$ is assumed to have access to the measured output y_k of its own agent (“introspective” as termed in [26]) as well as to the signals ε_i transmitted from its neighbors $i \in \mathbb{N}_k$ specified by an index set $\mathbb{N}_k \subseteq \mathbb{I}_N$ (by definition, $k \notin \mathbb{N}_k$). Thus the controller takes the form

$$\begin{bmatrix} u_k \\ \varepsilon_k \end{bmatrix} = \mathcal{K}_k \begin{bmatrix} y_k \\ v_k \end{bmatrix}, \quad v_k = -L\varepsilon, \quad (22)$$

where v_k is a linear combination of ε_i with $i \in \mathbb{N}_k \cup \{k\}$, specified by a structured inter-agent coupling matrix L .

Let a matrix with non-negative eigenvalues $\Lambda \in \mathbb{R}^{r \times r}$, and matrices $R_k \in \mathbb{R}^{h_k \times r}$ for $k \in \mathbb{I}_N$ be given. The problem is to design local controllers $\mathcal{K}_k$ with $k \in \mathbb{I}_N$ and inter-agent coupling L such that the outputs of the agents satisfy $z_k(t) \rightarrow R_k e^{\Lambda t} \rho_o$ for some $\rho_o \in \mathbb{R}^r$ depending on the initial state of the agents. This is a special case of Problem 1; multi-agent system (21) can be described by (1) with x and A defined by

$$x := \text{col}(x_1, \dots, x_N), \quad A = \text{diag}(A_1, \dots, A_N), \quad (23)$$

and other signals and matrices defined similarly.² Thus, we will solve Problem 1 with the block-diagonal plant (21) to find the structured controller (22) that assigns the eigenstructure (Λ, R) with $R := \text{col}(R_1, \dots, R_N)$.

Before proceeding further, let us introduce some notation. We define $\mathbb{L} \subset \mathbb{R}^{N \times N}$ as the set of Laplacian matrices $\mathcal{L}$ for directed graphs, containing a spanning tree, with positive weights. That is, $\mathcal{L} \in \mathbb{L}$ if and only if $\mathcal{L}$ satisfies the following properties: (a) the row sum is equal to zero, (b) all the off-diagonal entries are nonpositive, and (c) at least one of the cofactors is nonzero. It is well known that such $\mathcal{L} \in \mathbb{L}$ has a simple eigenvalue at the origin, and the rest of the eigenvalues are in the open right-half plane [20]. For a given $\Lambda \in \mathbb{R}^{r \times r}$, we denote by $\mathbb{L}_\Lambda$ the set of $\mathcal{L} \in \mathbb{L}$ such that the smallest real-part of the nonzero eigenvalues of $\mathcal{L}$ is greater than the largest real part of the eigenvalues of Λ . For the control design, we assume that the directed graph specified by $\mathbb{N}_k$ contains a spanning tree, and consider the Laplacian $\mathcal{L} \in \mathbb{L}$ satisfying

$$\begin{aligned} \mathcal{L}_{k\ell} &\leq 0 \text{ for } \ell \in \mathbb{N}_k, \\ \mathcal{L}_{k\ell} &= 0 \text{ for } \ell \notin \mathbb{N}_k \cup \{k\}, \quad \mathcal{L}_{kk} = - \sum_{\ell \in \mathbb{N}_k} \mathcal{L}_{k\ell}. \end{aligned} \quad (24)$$

where the digraph topology is specified by $\mathbb{N}_k$ for $k \in \mathbb{I}_N$.

²For the remainder of this paper, we will use this notation. That is, when constants and signals are defined by capital and lower case letters with subscripts $k \in \mathbb{I}_N$ (e.g., A_k and x_k), the same symbols without the subscripts (e.g., A and x) denote the corresponding block-diagonal matrix and the column vector defined similarly to (23). This notation is used whenever a symbol appears with and without the subscript k , except for R , X and F that are defined by stacking R_k , X_k and F_k , respectively, in a column.

The following result is instrumental for later developments.
Lemma 6: Let $\Lambda \in \mathbb{R}^{r \times r}$ and $\mathcal{L} \in \mathbb{L}_\Lambda$ be given. Then

$$\begin{aligned} (\Lambda - L)\mathcal{J} &= \mathcal{J}\Lambda, \\ \Lambda &:= I_N \otimes \Lambda, \quad L := \mathcal{L} \otimes I_r, \quad \mathcal{J} := \text{col}(I_r, \dots, I_r) \end{aligned} \quad (25)$$

holds, and the eigenvalues of $\Lambda - L$ other than those of Λ have negative real parts. Consequently, we have

$$\dot{\varepsilon} = (\Lambda - L)\varepsilon \Rightarrow \varepsilon(t) \rightarrow \mathcal{J}e^{\Lambda t}c \quad (26)$$

for some constant vector c dependent on $\varepsilon(0)$.

Proof. See Appendix. ■

This result is directly useful for solving the pattern formation problem for multi-agents with homogeneous dynamics.

Lemma 7 (Homogeneous Multi-agents): Consider Problem 1 for the multi-agent plant given by (21). Suppose $A_k = \Lambda$ and $C_k = H_k = R_k = I$ for all $k \in \mathbb{I}_N$. Then the controller (19) solves Problem 1 if

$$K := \text{diag}(K_1, \dots, K_N), \quad L := \mathcal{L} \otimes I,$$

where $\mathcal{L} \in \mathbb{L}_\Lambda$ and $\Lambda + B_k K_k$ is Hurwitz.

Proof. Note that $X = \mathcal{J} := \text{col}(I, \dots, I)$ satisfies (17). By definition, $A + BK$ is Hurwitz and from Lemma 6, $A - L$ shares eigenvalues with Λ , and the rest in the open left-half plane. Moreover, $LX = 0$ holds since the row sum of $\mathcal{L}$ is zero. Thus, the result now follows from Lemma 5. ■

The essential part of the result has appeared in [21]. Our contribution is to explicitly show the underlying principles for the control law within the framework of natural eigenstructure assignment. More importantly, this result is essential for the heterogeneous multi-agent pattern formation problem because the general heterogeneous problem can be reduced to a homogeneous one through local feedback as we show below.

Theorem 5: Consider Problem 1 for the multi-agent plant given by (21) with signals and matrices defined as in (23). The following statements are equivalent.

- (i) Problem 1 is solvable by a controller (2).
- (ii) There exist F_k and X_k such that

$$A_k X_k + B_k F_k = X_k \Lambda, \quad H_k X_k = R_k, \quad k \in \mathbb{I}_N. \quad (27)$$

Suppose these conditions hold, let $\mathcal{L} \in \mathbb{L}_\Lambda$ be given, and for each $k \in \mathbb{I}_N$, let G_k , Φ_k , K_k , and J_k be chosen to make the following three matrices Hurwitz:

$$A_k + G_k C_k, \quad \Lambda + \Gamma_k \Phi_k, \quad \Omega_k := A_k + B_k K_k - X_k J_k, \quad (28)$$

where Γ_k is defined using the unique solution M_k to the Sylvester equation:

$$\Lambda M_k - M_k \Omega_k = J_k, \quad \Gamma_k = [M_k B_k \quad I - M_k X_k].$$

Then the distributed controller given by

$$\dot{\hat{x}}_k = A_k \hat{x}_k + B_k u_k + G_k (C_k \hat{x}_k - y_k) \quad (29)$$

$$\dot{\hat{\eta}}_k = \Lambda \hat{\eta}_k + \Gamma_k w_k + v_k, \quad w_k = \Phi_k \hat{\eta}_k, \quad \varepsilon_k = \hat{\eta}_k - \eta_k, \quad (30)$$

$$\begin{bmatrix} \dot{\eta}_k \\ \dot{u}_k \\ \dot{\xi}_k \end{bmatrix} = \begin{bmatrix} I \\ F_k \\ \Lambda \end{bmatrix} \xi_k + \begin{bmatrix} M_k \\ K_k \\ J_k \end{bmatrix} (\hat{x}_k - X_k \xi_k) + \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} w_k, \quad (31)$$

$$v = -L\varepsilon, \quad L = \mathcal{L} \otimes I_r, \quad (32)$$

solves Problem 1, where $k \in \mathbb{I}_N$.

Proof. See Appendix. ■

The pattern formation mechanism underlying the controller in Theorem 5 is best illustrated in Figs. 2, 3, and 4 and works in two distinct steps; the first is a local controller design (Figs. 2 and 3) to assign the eigenstructure of the desired dynamics to each agent and homogenize the group, and the second is the inter-agent coupling to reach the desired formation through information exchange between neighboring agents (Fig. 4). To better see this, consider the case in which full-state feedback is available ($C_k = I$) and the observer in (29) is replaced by $\hat{x}_k = x_k$. We first apply Theorem 3 to assign the eigenstructure (Λ, X_k) to each agent with $Q_k := \text{col}(K_k, J_k)$ and introduce an auxiliary input w_k . This local controller is given by (31) with each agent under minor feedback described by

$$\begin{bmatrix} \dot{x}_k \\ \dot{\xi}_k \end{bmatrix} = \begin{bmatrix} A_k + B_k K_k & B_k (F_k - K_k X_k) \\ J_k & \Lambda - J_k X_k \end{bmatrix} \begin{bmatrix} x_k \\ \xi_k \end{bmatrix} + B_k w_k.$$

where $B_k := \text{diag}(B_k, I)$. By applying a coordinate transformation (see Lemma 8 in the appendix) defined by

$$\begin{bmatrix} \eta_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} M_k & I - M_k X_k \\ I & -X_k \end{bmatrix} \begin{bmatrix} x_k \\ \xi_k \end{bmatrix},$$

dynamics for desired formation and the rest are decoupled as

$$\begin{bmatrix} \dot{\eta}_k \\ \dot{\theta}_k \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Omega_k \end{bmatrix} \begin{bmatrix} \eta_k \\ \theta_k \end{bmatrix} + \begin{bmatrix} \Gamma_k \\ \mathcal{B}_k \end{bmatrix} w_k,$$

where $\mathcal{B}_k := [B_k \quad -X_k]$. Leaving the stable dynamics θ_k unobservable, the agent with the local controller is given by $\dot{S}_k$ in Fig. 2. The modified agents now share homogeneous dynamics Λ . Using the idea in Lemma 5 to achieve separation, we introduce the pre-compensator in (30) and obtain the augmented agents $\dot{S}_o$ as in Fig. 3. At this point, the agents $\dot{S}_o$ are set up so that $z_k(t) \rightarrow R_k e^{\Lambda t} \varepsilon_o$ holds if $\varepsilon_k(t) \rightarrow -e^{\Lambda t} \varepsilon_o$ for some ε_o . We now see that the original problem with heterogeneous agents reduced to a problem with full control, state feedback, homogeneous agents $\dot{S}_o$. By Lemma 6, this problem can be solved by a constant Laplacian gain control L , giving inter-agent coupling (Fig. 4).

Using the idea of equivalence, some of dynamics of the controller in Theorem 5 can be removed if certain conditions are satisfied, thereby reducing the controller order.

Corollary 1: In Theorem 5, the local controller dynamics for each individual agent can be simplified under certain conditions as follows:

- (i) If $C_k = I$, then the observer dynamics in (29) can be replaced by $\hat{x}_k := x_k$.
- (ii) If Γ_k is full row-rank, then (30) can be replaced by

$$w_k = -\Gamma_k^\dagger v_k, \quad \varepsilon_k = -\eta_k. \quad (33)$$

- (iii) If X_k is full column-rank, then (31) can be replaced by

$$\eta_k = M_k \hat{x}_k, \quad u_k = K_k \hat{x}_k + \begin{bmatrix} I & 0 \end{bmatrix} w_k.$$

where K_k is defined by (12) with subscript k added.

Proof. We show (i) first. Let the controller in Theorem 5 be described by (29) and $u_k = \hat{K}_k \hat{x}_k$, where $\hat{K}_k$ contains (30) and (31). Suppose $C_k = I$ and define $e_k = \hat{x}_k - x_k$. Then

$$\dot{e}_k = (A_k + G_k)e_k, \quad u_k = \hat{K}_k(e_k + x_k).$$

Fig. 2. Local feedback for quasi-homogeneous state feedback agents

Fig. 3. Local feedback for homogeneous full control, state feedback agents

Fig. 4. Inter-agent coupling for coordination

By Lemma 2, since the dynamics for e_k are stable and uncontrollable, we can remove them from the controller. Thus the controller becomes $u_k = \hat{K}_k x_k$.

We provide only an outline of the proof for (ii) for space considerations. First, it can be verified that Theorem 5 holds when Φ_k that makes $\Lambda + \Gamma_k \Phi_k$ Hurwitz is replaced by Υ and Ψ that make $\Lambda + \Gamma(\Upsilon + \Psi)$ Hurwitz and satisfy $\Psi\mathcal{I} = 0$, and the second equation in (30) is replaced by $w = \Upsilon\hat{\eta} + \Psi\eta$. Let $\mathbb{K} \subseteq \mathbb{I}_N$ be such that Γ_k is full rank for $k \in \mathbb{K}$. Then choose Φ as the block diagonal matrix defined in Theorem 5 and

$$\begin{aligned} \Upsilon &:= \Phi - \Psi, \\ \Psi &:= \text{col}(\Psi_1, \dots, \Psi_N), \quad \Psi_k := \begin{cases} -\Gamma_k^\dagger L_k & k \in \mathbb{K}, \\ 0 & k \notin \mathbb{K}, \end{cases} \end{aligned}$$

where L_k is the row block of L corresponding to agent k . Then it can be shown that the dynamics of $\hat{\eta}_k$ are stable, uncontrollable and, thus removable.

To show (iii), suppose X_k is full column rank, and define K_k as described. Then there exists J_k that makes Ω_k Hurwitz. In particular, it can be shown through a transformation like (14) that Ω_k is Hurwitz if and only if $\Lambda - J_k X_k$ is Hurwitz. Moreover, $A_k + B_k K_k$ contains the desired eigenstructure (X_k, Λ) and the rest of the eigenvalues are in the open left-half plane. Therefore $A_k + B_k K_k$ can be block-diagonalized into $\text{diag}(\Lambda, H_k)$ for some Hurwitz matrix H_k via a similarity transformation of the form $[X_k \quad *]$. Let M_k be the matrix formed by the first r rows of the inverse of this transformation matrix. Then it can be verified that M_k is the unique solution to the Sylvester equation in Theorem 5, and yields $\eta_k = M_k \hat{x}_k$ in (31) due to $M_k X_k = I$. Since u_k does not directly depend on ξ_k due to $F_k = K_k X_k$, we see that ξ_k has stable, unobservable dynamics which can be removed from the controller. ■

Corollary 2: In Theorem 5, suppose X_k is full column-rank and Γ_k has full row-rank for all $k \in \mathbb{I}_N$. Then Problem 1 can be solved by a static gain under state feedback, or by an observer-based controller under output feedback:

$$\dot{\hat{x}}_k = A_k \hat{x}_k + B_k u_k + G_k (C_k \hat{x}_k - y_k), \quad k \in \mathbb{I}_N \quad (34)$$

$$u = \mathcal{K} \hat{x}, \quad \mathcal{K} := K - (MB)^\dagger LM, \quad L = \mathcal{L} \otimes I_r \quad (35)$$

where $\mathcal{L}$, G_k , X_k , F_k , and Ω_k are specified in Theorem 5, Y_k is the orthogonal complement of X_k , Z_k is chosen such that $Y_k^\dagger \mathcal{U}_k$ is Hurwitz, and

$$M_k := X_k^\dagger + T_k Y_k^\dagger, \quad \mathcal{U}_k := A_k Y_k + B_k Z_k,$$

$$\Lambda T_k - T_k \Omega_k = X_k^\dagger \mathcal{U}_k, \quad K_k := F_k X_k^\dagger + Z_k Y_k^\dagger.$$

Proof. See Appendix. ■

Condition (27) for solvability of the multi-agent pattern formation problem has been obtained in a slightly different setting outside of the eigenstructure assignment framework [17]. The condition was interpreted as the internal model of a virtual exosystem embedded in the dynamics of each agent through local feedback. Our contributions beyond [17] include the general controller formula fully supported by analytical understanding of the controller architecture as described earlier. In fact, the main result (Theorem 5) of [17] can be seen as a special case of the controller from Theorem 5.

To reproduce their controller, we set $J_k = 0$, which results in $M_k = 0$ and $\eta_k = \xi_k$. Let Φ_k have the form $\text{col}(0, \mathcal{P}_k)$. Then the dynamics of $\hat{\eta}$ and ξ reduce to

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\hat{\eta}} \end{bmatrix} = \begin{bmatrix} \Lambda - L & 0 \\ -L & \Lambda + \mathcal{P} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \hat{\eta} \end{bmatrix}, \quad \Lambda := I \otimes \Lambda. \quad (36)$$

The dynamics of $\hat{\eta}$ can be removed from the controller because

$$\varepsilon_k(t) \rightarrow e^{\Lambda t} c, \quad L\varepsilon(t) \rightarrow 0, \quad \hat{\eta}(t) \rightarrow 0,$$

as seen from Lemma 6 and stability of $\Lambda + \mathcal{P}_k$, and the resulting controller is given by the observer (29) and

$$\dot{\varepsilon} = (\Lambda - L)\varepsilon, \quad u_k = F_k \varepsilon_k + K_k (\hat{x}_k - X_k \varepsilon_k), \quad (37)$$

where ε_k is redefined as $-\varepsilon_k$, giving the controller in [17].

In this controller, the reference trajectory ε is generated by copies of local internal models $\dot{\varepsilon}_k = \Lambda \varepsilon_k$, interacting with each other through the Laplacian coupling L to achieve coordination $\varepsilon_k(t) \rightarrow \varepsilon_\ell(t)$ for all $k, \ell \in \mathbb{I}_N$. The observer-based local feedback makes x_k track $X_k \varepsilon_k$ by a mechanism similar to (16) so that $z(t) \rightarrow Re^{\Lambda t} c$. Unlike the general controller from Theorem 5, the reference generator receives no feedback which prevents the reference command from being adjusted in real time, possibly leading to undesirable results.

V. DESIGN EXAMPLES

A. Multi-agent Mechanical System

We consider a system of four linear heterogeneous agents ($N = 4$), each of which consists of point masses connected in series by linear springs, constrained to move along a straight, frictionless line. For $k \in \mathbb{I}_N$, we define the k^{th} agent to be $k + 1$ masses connected by k springs, actuated by a horizontal

force, u_k , on the first mass (Fig. 5). With uniform mass m and stiffness σ , the equations of motion for agent k is given by

$$\begin{aligned} m\ddot{x}_k^1 &= -\sigma(x_k^1 - x_k^2) + u_k, \\ m\ddot{x}_k^i &= \sigma(x_k^{i-1} - 2x_k^i + x_k^{i+1}), \quad i \in \mathbb{I}_k \setminus \{1\}, \\ m\ddot{x}_k^{k+1} &= -\sigma(x_k^{k+1} - x_k^k) \end{aligned}$$

where x_k^i is the displacement of the i^{th} mass in the k^{th} agent, and we set $\sigma = 2$ and $m = 1$. Let the system be described by the first equation in (21) with state vector $x_k := \text{col}(x_k, \dot{x}_k) \in \mathbb{R}^{2(k+1)}$, and consider the state feedback case $C_k = I$.

Fig. 5. The k^{th} agent

To illustrate the utility of the proposed theory, we apply the result of Theorem 5 to the multi-agent system and consider two design objectives. The first is the consensus of all agents towards a constant velocity and linearly growing displacement. The response of the controller in Theorem 5 is compared with the controller (37) proposed by [17] when the plant is subjected to a disturbance. The second is on coordinated displacement oscillations with prescribed frequency, amplitudes, and relative phases. A stable limit cycle is obtained by adding a nonlinearity to the distributed controller at a root of the directed graph. In both examples, the graph is designed with nearest neighbor coupling and uniform connectivity weights, $\mathcal{L}_{i,i+1} = \mathcal{L}_{i+1,i} = -\mu$ for $i \in \mathbb{I}_{N-1}$, with $\mu = 5$ in the first example and $\mu = 10$ in the second.

B. Control for Consensus

We consider a consensus problem, where the design goal is to make the first masses of all the agents move together at a constant speed, i.e., $z_k(t) \rightarrow \text{col}(\gamma_v t + \gamma_p, \gamma_v)$ for some constants γ_p and γ_v , where $z_k := \text{col}(x_k^1, \dot{x}_k^1)$. The corresponding desired dynamics are described by (4) with Λ in (7) and

$$R_k = I_2, \quad H_k = \begin{bmatrix} 1 & 0_{1 \times k} & 0 & 0_{1 \times k} \\ 0 & 0_{1 \times k} & 1 & 0_{1 \times k} \end{bmatrix}, \quad k \in \mathbb{I}_4.$$

where $0_{1 \times k}$ is the $1 \times k$ zero vector.

We design two consensus controllers, one by Theorem 5 and another by (37) from [17]. To ensure parity between the controllers, we used the same X_k and F_k (minimum norm solution to (27)) in both controllers as well as the same Laplacian matrix $\mathcal{L}$. The only remaining freedom was in the choice of static gains Φ_k and $\text{col}(K_k, J_k)$ for our controller and K_k (with $J_k = 0$) for the controller (37) in [17], to stabilize the matrices in (28). For each gain, the problem is in the form of state feedback stabilization to choose control gain $\mathfrak{K}$ to make $\mathfrak{A} + \mathfrak{B}\mathfrak{K}$ Hurwitz. We used the optimal LQR gain, $\mathfrak{K}_o$, that minimizes the cost function $\int_0^\infty (\|x(t)\|^2 + \|u(t)\|^2/10^3) dt$ for the modified plant $\dot{x} = (\mathfrak{A} + I)x + \mathfrak{B}u$. The identity was added to $\mathfrak{A}$ to ensure that the eigenvalues of $\mathfrak{A} + \mathfrak{B}\mathfrak{K}$ have real-part less than -1 to increase the rate of convergence.

We tested the two controllers by simulations of the closed-loop systems, starting with randomly generated initial conditions. The numerical experiments were conducted for 10,000 random initial conditions with values uniformly distributed between -1 and 1. We compared the two with respect to settling time, t_s , and the maximum input, $u_{\max}$, which are defined by

$$\begin{aligned} t_s &:= \min t_o \quad \text{s.t.} \quad \max_{k \in \mathbb{I}_4} |\dot{x}_k^1(t) - \gamma_v| \leq 0.05 \quad \forall t \geq t_o, \\ u_{\max} &:= \max_{k \in \mathbb{I}_4, t \geq 0} |u_k(t)|, \end{aligned}$$

where γ_v is the steady state velocity. The results of the 10,000 simulations for each case are shown in Fig. 6 with additional information on averages given in Table I.

Fig. 6. Settling time versus max input for Theorem 5 (Blue) and [17] (Red)

TABLE I
COMPARISON OF AVERAGE SETTLING TIME, $\bar{t}_s$, AND MAX INPUT, $\bar{u}_{\max}$.

Controller	$\bar{t}_s$	$\bar{u}_{\max}$
Wieland <i>et al.</i> 2011 [17]	7.08	11116
Theorem 5	7.65	3656

We see that the settling times for both controllers are comparable, with a slight advantage (roughly 8% on average) to the controller by [17]. The primary difference is the amount of control effort required to reach consensus. The controller from Theorem 5 outperforms the controller from [17] by a factor of 3. We attribute this difference to the use of feedback in the controller from Theorem 5. Because the reference generator from [17] receives no feedback, the target consensus trajectory is determined only by the initial conditions of the controller. In contrast, the controller from Theorem 5 reaches the target consensus trajectory through distributed communications based on the sensory feedback from the plant.

The advantage of feedback in our controller is best exemplified when there is a disturbance that pushes the states exactly onto a consensus trajectory. Because the agents are not subjected to friction, the control input is nonzero only when the agents are not in consensus. If the agents have reached consensus, then the masses remain in consensus when subjected to equal forcing. Without feedback, the reference generator of [17] is unaware of this and the controller attempts to return to the same reference trajectory determined by initial conditions, requiring nontrivial control inputs. On the other hand, when there is feedback as in Theorem 5, the controller recognizes that the agents are still in consensus after the disturbance, requiring little effort to remain in consensus.

To illustrate this case, we chose the initial conditions so that the states for both closed-loop systems initially converge

to similar consensus trajectories; the velocities of the agents converge to $\gamma_v = -0.0235$ and -0.0257 for the controllers in Theorem 5 and [17], respectively (Fig. 7). After reaching consensus, an impulse disturbance of 0.05 Newtons was applied at $t = 9$ s to every mass of each agent in the same direction; essentially an addition of equal velocity to every mass. Since this is a consensus trajectory satisfying (4), the controller in Theorem 5 does (almost) nothing as seen in Fig. 8. In contrast, the controller from [17] does not notice the disturbance due to the lack of feedback, and insists on the original trajectory at $\gamma_v = -0.0257$. This results in the large transient, especially in the control input which spikes to over 1800 in amplitude.

Fig. 7. Position comparison of controller in Theorem 5 (left) and controller proposed by [17] (right) subjected to a disturbance at $t = 9$ s.

Fig. 8. Control input comparison of controller in Theorem 5 (left) and controller proposed by [17] (right) subjected to a disturbance at $t = 9$ s.

This example illustrates the importance of having feedback in a consensus controller so that the controller allows the agents to adapt their reference trajectories in real-time.

C. Control for Coordinated Oscillations

Here we design a controller so that the first mass of every agent oscillates with a specified frequency, amplitude, and relative phase between agents. In particular, we aim to achieve $\mathbf{x}_k^1(t) \rightarrow \zeta_k(t + t_o)$ for some constant $t_o \in \mathbb{R}$ depending on the initial state, where $\zeta_k(t)$ is given by (6) with

$$\omega = 5, \quad \alpha_k = (5 - k)/2, \quad \beta_k = (\pi/3)(k - 1)$$

for $k \in \mathbb{I}_4$. Specifying (R, Λ) as in (5) with $a_k := \alpha_k$ and $b_k := \beta_k$, the linear controller in Theorem 5 will achieve $\mathbf{x}_k^1(t) \rightarrow \gamma_a \zeta_k(t + \gamma_b/\omega)$ for some constants (γ_a, γ_b) as discussed earlier. While we allow an arbitrary time shift t_o , it is desired to fix the amplitude to $\gamma_a = 1$ so that $\zeta_k(t)$ is achieved as part of an orbitally stable limit cycle.

To remedy this, we added a local nonlinear feedback to the control of the first agent's first mass:

$$u_1 = \tilde{u}_1 + c(\alpha_1^2 - \tilde{\alpha}_1^2)\dot{\mathbf{x}}_1^1, \quad \tilde{\alpha}_1^2 := (\mathbf{x}_1^1)^2 - (\dot{\mathbf{x}}_1^1/\omega)^2,$$

where c is a positive constant, $\alpha_1 = 2$ is the desired oscillation amplitude for $\mathbf{x}_1^1$, and $\tilde{u}_1$ is the force input generated by the original linear controller from Theorem 5. Note that $\tilde{\alpha}_1(t)$ represents the actual amplitude at a given time if $\mathbf{x}_1^1$ is sinusoidal. This nonlinearity does not affect the communication structure of the controller because it requires only information already available to the first agent. The additional feedback stabilizes the amplitude of $\mathbf{x}_1^1$ to α_1 with nonlinear damping that is negative when $\tilde{\alpha}_1 < \alpha_1$ and positive when $\tilde{\alpha}_1 > \alpha_1$; this amplitude regulation propagates to the other agents through proper inter-agent coupling so that $\alpha_k = (a_k/a_1)\alpha_1$. In general, this nonlinearity must be applied on an agent at a root of a spanning tree of the graph, otherwise, there is an agent that does not receive information about the desired amplitude.

The closed-loop simulation result with $c = 20$ is shown in Fig. 9. We start at zero initial state with the exception of a displacement of -1 on the first agent's first mass. Figure 9 shows that the agents converge to the limit cycle of specified amplitudes and relative phases. While we provide no rigorous proof to guarantee that the agents will converge to a limit cycle, this example shows the potential for eigenstructure theory to encompass pattern formations beyond the linear domain. It would be possible to expand these linear results to nonlinear theories for pattern formation.

Fig. 9. Limit cycle entrainment with the addition of a nonlinearity

VI. CONCLUSION

We considered the problem of designing a controller such that selected outputs of a linear plant exponentially converge to $Re^{\Lambda t} \rho_o$ for some vector ρ_o , with prescribed matrix, R , and matrix with non-negative eigenvalues, Λ . We demonstrated the equivalence of this problem to that of an eigenstructure assignment problem and provided necessary and sufficient conditions for solvability along with a parameterization of all feasible solutions. Using this parameterization, we showed that under a mild condition and linear independence of the eigenvectors associated with Λ , the problem could be solved with a static gain under state feedback and a dynamic controller of plant order, n , under output feedback. Moreover, when the condition for linear independence is not satisfied, the problem can be solved with a controller of order less than or equal to $n + r$, where r is the dimension of the desired dynamics Λ .

Recognizing that the multi-agent pattern formation problem is a special case of the eigenstructure assignment problem, we proceeded to apply the aforementioned result to design a

structured controller for pattern formation of heterogeneous agents. In our method, we first performed a local controller design such that the desired eigenstructure was assigned to each agent, and the heterogeneous agent dynamics were homogenized. The inter-agent coupling is then designed by a Laplacian matrix representing a directed graph containing a spanning tree, to achieve coordination through exchange of relative information. The general result is shown to include an important existing result [17] as a special case, and have an additional capability of adaptive pattern formation through reference generator placed within the feedback loop.

Two numerical examples were provided for pattern formation control of multiple heterogeneous agents to substantiate the utility of the proposed method. In the first, our controller was compared to the one proposed by [17] in order to assert the importance of feedback in reaching a consensus. In the second, we designed a linear distributed controller to achieve coordinated phase-locked oscillations for heterogeneous agents, and further added a local nonlinear feedback to lock the amplitudes of oscillation. The second example touched on the possibility of further research in expanding the linear eigenstructure framework to nonlinear pattern formation problems.

APPENDIX

BLOCK-DIAGONALIZING TRANSFORMATION

Lemma 8: Let matrices (A, B, C, D) and (X, Λ) be given. Suppose A , D , and Λ are square, and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ I \end{bmatrix} = \begin{bmatrix} X \\ I \end{bmatrix} \Lambda, \quad \text{eig}(\Lambda) \cap \text{eig}(\Omega) = \emptyset, \\ \Omega := A - XC.$$

Then there exists a unique matrix M satisfying

$$\Lambda M - M \Omega = C,$$

and the matrix can be block-diagonalized by the following similarity transformation

$$\begin{bmatrix} M & I - MX \\ I & -X \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & I - XM \\ I & -M \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ 0 & \Omega \end{bmatrix}.$$

Proof. The result can be verified by direct calculations. ■

PROOF OF LEMMA 1

Since properties (a)–(c) and (i)–(ii) are preserved under state coordinate transformations, we may assume that system (3) is given in a Kalman canonical form,

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 & 0 \\ 0 & \mathcal{A}_2 & 0 \\ * & * & \mathcal{A}_3 \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} \mathcal{H}_1 & \mathcal{H}_2 & 0 \end{bmatrix},$$

where $(\mathcal{H}_i, \mathcal{A}_i)$ for $i = 1, 2$ are observable, and all the eigenvalues of $\mathcal{A}_1$ and $\mathcal{A}_2$ are, respectively, in the closed right half plane and open left half plane. Let the state vector be partitioned accordingly; $\mathbf{x} = \text{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. Noting that

$$z(t) = \mathcal{H}_1 e^{\mathcal{A}_1 t} \mathbf{x}_1(0) + \mathcal{H}_2 e^{\mathcal{A}_2 t} \mathbf{x}_2(0), \quad \zeta(t) = R e^{\Lambda t} \rho(0),$$

the convergence $z(t) \rightarrow \zeta(t)$ occurs if and only if

$$\mathcal{H}_1 e^{\mathcal{A}_1 t} \mathbf{x}_1(0) = R e^{\Lambda t} \rho(0), \quad \forall t \in \mathbb{R}. \quad (38)$$

Suppose properties (a)–(c) hold. Let $\rho(0)$ be the i^{th} column of the identity matrix and m_i be the corresponding $\mathbf{x}_1(0)$ satisfying (38). Define M as the matrix having m_i as the i^{th} column. Similarly, define N by $n_i := \rho(0)$ that satisfies (38) when $\mathbf{x}_1(0)$ is the i^{th} column of the identity matrix. Then

$$\mathcal{H}_1 e^{\mathcal{A}_1 t} M = R e^{\Lambda t}, \quad \mathcal{H}_1 e^{\mathcal{A}_1 t} = R e^{\Lambda t} N, \quad \forall t \in \mathbb{R},$$

from which we obtain

$$\mathcal{H}_1 e^{\mathcal{A}_1 t} (I - MN) = 0, \quad R e^{\Lambda t} (I - NM) = 0, \quad \forall t \in \mathbb{R}.$$

By observability of $(\mathcal{H}_1, \mathcal{A}_1)$ and (R, Λ) , matrices M and N are square and $MN = NM = I$. We now see that $(\mathcal{A}_1, M, \mathcal{H}_1)$ and (Λ, I, R) are minimal realizations of the same transfer function, and hence are related by the state coordinate transformation:

$$\mathcal{A}_1 M = M \Lambda, \quad \mathcal{H}_1 M = R.$$

Condition (i) is then satisfied by $\mathcal{V} = \text{col}(M, 0, 0)$. Condition (ii) follows by noting that $\mathcal{A}_2$ is Hurwitz by definition and $\mathcal{A}_3$ is Hurwitz due to property (c).

To show the converse, suppose (i) and (ii) hold. Note that

$$e^{\mathcal{A} t} \rightarrow \mathcal{V} e^{\Lambda t} \mathcal{U}, \quad z(t) = \mathcal{H} e^{\mathcal{A} t} \mathbf{x}(0) \rightarrow R e^{\Lambda t} \mathcal{U} \mathbf{x}(0),$$

where the rows of $\mathcal{U}$ span the left eigenspace corresponding to Λ , i.e., $\mathcal{U} \Lambda = \Lambda \mathcal{U}$, and are normalized such that $\mathcal{U} \mathcal{V} = I$. Hence (b) holds for $\rho(0) := \mathcal{U} \mathbf{x}(0)$ and (a) holds for $\mathbf{x}(0) := \mathcal{V} \rho(0)$. To show (c), let (λ, v) be an arbitrary eigenvalue/eigenvector pair of Λ . Then (i) implies

$$(\mathcal{A} - \lambda I)(\mathcal{V} v) = 0, \quad \mathcal{H}(\mathcal{V} v) = R v.$$

Since $\mathcal{V}$ has full column-rank and $v \neq 0$, we have $\mathcal{V} v \neq 0$. Moreover, since (R, Λ) is observable, $R v \neq 0$. Hence λ is an observable mode of $(\mathcal{H}, \mathcal{A})$, i.e., all the eigenvalues of $\mathcal{A}$ shared with Λ are observable. Therefore, all the unobservable modes of $(\mathcal{H}, \mathcal{A})$ must be stable due to (ii), proving detectability of $(\mathcal{H}, \mathcal{A})$. Thus we can conclude that (i)–(ii) $\Rightarrow$ (a)–(c).

PROOF OF THEOREM 3

Based on Lemma 3, it suffices to characterize the set of all controllers that solve Problem 1 with eigenmatrices of the form $\mathcal{V} = \text{col}(X, \Xi_o)$ where $\Xi_o := \text{col}(I, 0)$. A controller $\hat{\mathcal{K}}$ belongs to the set if and only if $HX = R$ and

$$\begin{bmatrix} A + BD_1C & BC_1 & BC_2 \\ B_1C & A_{11} & A_{12} \\ B_2C & A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\ = \begin{bmatrix} X & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \Lambda & * & * \\ 0 & \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{21} & \Omega_{22} \end{bmatrix} \quad (39)$$

hold for some entries $*$, matrix X , and Hurwitz matrix Ω , where A_{ij} , B_i , C_i , and D_1 for $i, j = 1, 2$ are partitioned blocks of the state space matrices of $\hat{\mathcal{K}}$, and Ω_{ij} for $i, j = 1, 2$ are partitioned blocks of Ω . Define

$$Q := \begin{bmatrix} D_q & C_q \\ B_q & A_q \end{bmatrix} := \left[\begin{array}{c|c} D_1 & C_2 \\ \hline B_1 & A_{12} \\ \hline B_2 & A_{22} \end{array} \right], \quad F := C_1 + D_1 C X.$$

Then condition (39) holds for some entries $*$ if and only if

$$\begin{aligned} AX + BF &= X\Lambda, \\ B_1CX + A_{11} &= \Lambda, \\ B_2CX + A_{21} &= 0, \end{aligned} \quad \Omega = \begin{bmatrix} A + \mathfrak{B}D_qC & \mathfrak{B}C_q \\ B_qC & A_q \end{bmatrix}$$

hold. Note that the closed-loop system of the augmented plant $(A, \mathfrak{B}, C)$ and $\dot{Q}$ is given by $\dot{\mathbf{x}} = \Omega\mathbf{x}$ with $\mathbf{x} := \text{col}(x, q)$ where q is the state vector of $\dot{Q}$. Hence, Ω is Hurwitz if and only if $\dot{Q}$ stabilizes $(A, \mathfrak{B}, C)$. $\mathcal{K}$ can now be expressed as

$$\begin{bmatrix} u \\ \dot{\xi} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} D_1 & F - D_1CX & C_2 \\ B_1 & \Lambda - B_1CX & A_{12} \\ B_2 & -B_2CX & A_{22} \end{bmatrix} \begin{bmatrix} y \\ \xi \\ q \end{bmatrix}.$$

Finally, it is straightforward to verify that this controller can be expressed as (15). This proves the result.

PROOF OF LEMMA 6

The following lemmas are useful.

Lemma 9: [33] For square matrices A and B of dimensions n and m , the eigenvalues of $A \otimes B$ are given by $\lambda_i \mu_j$ with $i \in \mathbb{I}_n$ and $j \in \mathbb{I}_m$, where λ_i and μ_i are the eigenvalues of A and B , respectively.

Lemma 10: [20] The Laplacian, $\mathcal{L}$, of a directed graph with a directed spanning tree and adjacency matrix with non-negative weights has a simple eigenvalue at the origin with the rest in the open right-half plane.

Lemma 11: [34] If matrices $A, B \in \mathbb{C}^{n \times n}$ commute, then they can be simultaneously triangularized. That is, there exists a $U \in \mathbb{C}^{n \times n}$ such that both $U^{-1}AU$ and $U^{-1}BU$ are upper triangular matrices.

We now prove Lemma 6. Equation (25) is verified by noting that $\Lambda J = J\Lambda$ and $LJ = 0$. By Lemma 9, the eigenvalues of L are those of $\mathcal{L}$ with each having a multiplicity of r . Furthermore, Lemma 10 implies that L has r eigenvalues at the origin with the others having strictly positive real part. Additionally, the eigenvalues of Λ are those of Λ , each repeated N times. By Lemma 11 we can define a U such that $U^{-1}\Lambda U$ and $U^{-1}LU$ are both upper triangular since Λ and L commute. This implies that the eigenvalues of $\Lambda - L$ are the difference of the eigenvalues of Λ and L , i.e., $\lambda_i - \mu_j$ with λ_i for $i \in \mathbb{I}_r$ from Λ and μ_j for $j \in \mathbb{I}_N$ from $\mathcal{L}$. Thus, r eigenvalues of $\Lambda - L$ coincide with those of Λ , and the remaining have strictly negative real parts due to $\mathcal{L} \in \mathbb{L}_\Lambda$. Finally, (26) follows from a standard linear system theory.

PROOF OF THEOREM 5

The equivalence (i) $\Leftrightarrow$ (ii) follows directly from Theorem 3. It remains to show that if the conditions in (27) hold, the controller presented in (29)-(32) yields the closed-loop system satisfying the conditions in Lemma 1 and solves Problem 1. In view of Theorem 1, it suffices to show that the state feedback controller $u = \mathcal{K}x$ with $\mathcal{K}$ specified by (30)-(32) with $\hat{x}_k$ replaced by x_k solves Problem 1.

The closed-loop system in (3) is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \\ \dot{\hat{\eta}} \end{bmatrix} = \begin{bmatrix} A + BK & B(\mathcal{F} - K\mathcal{X}) & B\mathcal{O} \\ J & \Lambda - J\mathcal{X} & \mathcal{P} \\ LM & L(I - M\mathcal{X}) & \Lambda + \Gamma\Phi - L \end{bmatrix} \begin{bmatrix} x \\ \xi \\ \hat{\eta} \end{bmatrix}$$

where $\mathcal{F}$, $\mathcal{X}$, $\mathcal{O}$, and $\mathcal{P}$ are block diagonal matrices with F_k , X_k , $\mathcal{O}_k$, and $\mathcal{P}_k$ on the diagonal, respectively, and $\mathcal{O}_k$ and $\mathcal{P}_k$ are defined by $\Phi_k = \text{col}(\mathcal{O}_k, \mathcal{P}_k)$. It can readily be verified through direct calculations that the system satisfies condition (i) of Lemma 1 with $\mathcal{V} = \text{col}(X, J, 0)$, where we note that $LJ = 0$, $\Lambda J = J\Lambda$, $\mathcal{F}J = F$, and $\mathcal{X}J = X$.

Using a coordinate transformation $(x, \xi, \hat{\eta}) \leftrightarrow (\eta, e, \varepsilon)$, the closed-loop system can be rewritten as

$$\begin{bmatrix} \dot{\eta} \\ \dot{e} \\ \dot{\varepsilon} \end{bmatrix} = \begin{bmatrix} \Lambda + \Gamma\Phi & 0 & \Gamma\Phi \\ \mathfrak{B}\Phi & \Omega & \mathfrak{B}\Phi \\ 0 & 0 & \Lambda - L \end{bmatrix} \begin{bmatrix} \eta \\ e \\ \varepsilon \end{bmatrix}, \quad \begin{aligned} \eta &:= \xi + Me, \\ e &:= x - \mathcal{X}\xi, \\ \varepsilon &:= \hat{\eta} - \eta. \end{aligned}$$

where $\mathfrak{B}$ is the block diagonal matrix with $\mathfrak{B}_k := [B_k - X_k]$ on the diagonal. We then see that condition (ii) of Lemma 1 is satisfied due to the Hurwitz requirement for (28) and Lemma 6.

PROOF OF COROLLARY 2

It suffices to show that the state feedback controller $u = \mathcal{K}x$ with $\mathcal{K}$ specified by (35) solves Problem 1.

Suppose there exist F_k , and full column-rank X_k satisfying (27). Define a controller $u = \mathcal{K}x$ by (35) and the subsequent definitions in Corollary 2. The closed-loop system satisfies

$$(A + B\mathcal{K})X = AX + BKX = AX + BF = X\Lambda,$$

where we note that $LMX = LJ = 0$ for $J := \text{col}(I_r, \dots, I_r)$, $X := \text{col}(X_1, \dots, X_N)$ and $F := \text{col}(F_1, \dots, F_N)$. Thus, condition (i) of Lemma 1 holds for $\mathcal{V} = X$. To confirm that all other eigenvalues are in the open left-half plane, define X , Y , M , Π , and ϑ to be the block diagonal matrices with X_k , Y_k , M_k , Π_k , and ϑ_k on the diagonal, where $\Pi_k := Y_k - X_k T_k$. Then applying a similarity transformation, we obtain

$$\begin{bmatrix} M \\ Y^\dagger \end{bmatrix} (A + B\mathcal{K}) \begin{bmatrix} X & \Pi \end{bmatrix} = \begin{bmatrix} \Lambda - L & 0 \\ * & \vartheta \end{bmatrix}$$

where we note that MB has full row rank because $M_k B_k$ has full row rank, and

$$\begin{bmatrix} X^\dagger \\ Y^\dagger \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} = I \Rightarrow \begin{bmatrix} M \\ Y^\dagger \end{bmatrix} \begin{bmatrix} X & \Pi \end{bmatrix} = I.$$

By Lemma 6, if $\mathcal{L}$ has sufficiently strong weights, then the matrix $\Lambda - L$ has eigenvalues in Λ with the rest in the open left half plane. Since ϑ is Hurwitz, we now see that condition (ii) of Lemma 1 is satisfied. Therefore, we conclude that the controller provided by in Corollary 2 solves Problem 1.

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