

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Combinatorial and Geometric Problems on Point Processes

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Mathematics

by

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Chair

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2007

DEDICATION

This dissertation is dedicated to Meghan, with love.

EPIGRAPH

Le secret d'ennuyer est celui de tout dire.

—Voltaire, Sept Discours en vers sur
l'Homme.

Ἄγεωμέτρητος μηδεὶς εἰσίτω.

—Motto over the entrance to Plato's
Academy.

TABLE OF CONTENTS

	Signature Page	iii
	Dedication	iv
	Epigraph	v
	Table of Contents	vi
	List of Figures	viii
	Acknowledgements	ix
	Vita and Publications	xi
	Abstract	xi
1	Mathematical Preliminaries	1
	1.1 Asymptotics	1
	1.2 Geometry	1
	1.3 Probability	3
	1.3.1 Random Measure Theory	3
	1.3.2 Concentration	5
	1.3.3 A Martingale Inequality	6
2	On Stochastic Geometry	8
	2.1 Random Polytopes	8
3	Random Circumscribing Polytopes	12
	3.1 Boundary Measure	14
	3.2 Caps and Visible Regions	14
	3.3 A Tail Bound	16
	3.3.1 Cap Covering	16
	3.3.2 Proof of Theorem 3.3.1	18
	3.3.3 Tail Bound Applications	20
	3.4 Variance Lower Bound	23
	3.4.1 A Construction	24
	3.4.2 Proof of Lemma 3.4.3	29
	3.5 A Poisson Central Limit Theorem	31
	3.5.1 Proof of Lemma 3.5.4	37
	3.6 A Stronger Tail Inequality	43
	3.6.1 Additional Geometry	44
	3.6.2 Proof	47

3.7	Further Directions	55
4	Two Geometric Random Graph Models	57
4.1	Local Structure in Complex Networks	58
4.2	Continuum Percolation and Random Geometric Graphs	60
5	A Random Geometric Tree Model	63
5.1	Definitions and Model	63
5.2	Results	64
5.3	Tools	68
5.4	Proofs	69
5.4.1	Proof of Theorem 5.2.1	69
5.4.2	Proof of Theorem 5.2.3	72
5.4.3	Proof of Theorem 5.2.4	73
5.4.4	Proof of Theorem 5.2.6, $\beta = 1/2$	74
5.5	Further Directions	82
6	A Random-Connection Model for Geometric Graphs	84
6.1	Model	85
6.1.1	Connection Functions	88
6.2	Combinatorial Properties	89
6.2.1	Edge Probability	89
6.2.2	Degree	94
6.2.3	Connection	96
6.3	Geometric Properties	99
6.3.1	Scaling	99
6.4	Further Directions	100
	Index	102
	Bibliography	103

LIST OF FIGURES

Figure 3.1: The visible region $G(x, s)$ for $x \in \partial K$	15
Figure 5.1: Determination of a local edge.	65
Figure 5.2: $\alpha = 3$	67
Figure 5.3: $\alpha = 0.5$	67
Figure 5.4: $\alpha = 1$	67
Figure 5.5: $\mathcal{T}_{\alpha, 10,000}$ instances.	67
Figure 5.6: γ -regions for γ small.	67
Figure 5.7: The intersection of two ellipses with a common focus and focal length forming an angle θ	78

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Chapter 3 consists of joint work with Van Vu being prepared for publication. The dissertation author was the primary author of this work. That work is based in part on Ross M. Richardson, Lei Wu, and Van H. Vu, *An inscribing model for random polytopes*, *Discrete and Computational Geometry*, to appear.

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Chapter 6 consists of joint work with Bill Aiello and Fan Chung being prepared for publication. The dissertation author was the primary author of this work.

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ABSTRACT OF THE DISSERTATION

Combinatorial and Geometric Problems on Point Processes

by

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In this dissertation we study three random structures derived from point processes.

We study the *circumscribed random polytope* P_n of a convex set $K \subset \mathbb{R}^d$ with smooth boundary, defined to be the intersection of n randomly chosen supporting half-spaces to K with boundary tangent to ∂K . Using cap covering techniques developed by Bárány and Larman [13] and sharp concentration arguments developed by R. et al. [87] we demonstrate that the *excess volume* $\text{Vol}_d(P_n \setminus K)$ is sharply concentrated about its mean, give an asymptotic characterization for the variance, and demonstrate that a Poissonized variant obeys a central limit theorem.

We introduce a random tree model with vertices in a convex set $K \subset \mathbb{R}^d$ and designated point $\mathfrak{o} \in K$, based on a related model of Fabrikant et al. [48]. Vertices v_i are chosen uniformly at random and added sequentially via a single edge to the prior vertex v_j which minimizes the function:

$$\alpha \|v_i - v_j\| + \|v_j - \mathfrak{o}\|.$$

We classify edges as *local* or *global* by comparison with the cord $\overline{\mathfrak{o}v_i}$, and analyze the number of local (global, respectively) edges based on the parameter α .

Finally, we introduce a new random graph model based on the *random-connection* model of continuum percolation [73]. The vertices are given by a point process and the probability that any two vertices forms an edge is given by some non-increasing function ρ of the distance between them. We focus primarily on the Riemannian

manifold case and functions ρ with polynomial tails. Based on the manifold and ρ , we give explicit formulæ for the probability that two points are connected by an edge, and determine the connectivity threshold for the graph. We additionally examine the geometric scaling of the graph on the cube $[0, 1]^d$ by looking at induced subgraphs in sub-cubes of length ε .

I Mathematical Preliminaries

In this chapter we establish notation, conventions, and collect some basic results in geometry and probability which are common to the remainder of the dissertation. The reader may be comfortable skimming the remainder of the chapter and returning for reference when necessary.

I.1 Asymptotics

The variable n will be preserved for a parameter which tends to infinity. Under this assumption, we shall primarily use the Bachmann [7] asymptotic notation. To wit, for functions $f(n), g(n)$ we write $f = O(g)$ to denote the existence of constants C, n_0 such that $f(n) \leq C g(n)$ for $n \geq n_0$. Similarly, we write $f = \Omega(g)$ if $g = O(f)$. We further write $f = o(g)$ if $\lim_{n \rightarrow \infty} f/g = 0$ and thus $f = \omega(g)$ if $g = o(f)$. The concept $f = o(g)$ may also be expressed succinctly using the Hardy notation wherein $f \ll g$ if and only if $f = o(g)$. Finally, we shall write $f \approx g$ to denote the relation $\lim_{n \rightarrow \infty} f/g = 1$.

When the asymptotic parameter is not n we shall attempt to make explicit our usage. The reader is referred to the excellent treatments on asymptotic notation found in [36, 54].

I.2 Geometry

We shall let d denote the dimension of the Euclidean space \mathbb{R}^d . When we speak of the *volume* of some $A \subset \mathbb{R}^d$, $\text{Vol}_d(A)$, we intend the d -dimensional Lebesgue measure. More generally, we may assume that $\text{Vol}_t, t \geq 0$ denotes the t -dimensional Hausdorff

measure (which coincides with Lebesgue measure for integral t) [51]. We shall implicitly invoke the fact that the measure induced by the Riemannian volume form of a $d - 1$ -dimensional manifold embedded in \mathbb{R}^d agrees with Hausdorff measure.

We next define a number of standard geometric structures. We denote by $B(x, r)$ the set of points of distance at most r from x in a given metric space. Let $\|\cdot\|$ denote the Euclidean (l_2) norm on \mathbb{R}^d . For $x, u \in \mathbb{R}^d$, we write

$$H^-(x, u) = \{y \in \mathbb{R}^d \mid (y - x) \cdot u \geq 0\},$$

and

$$H^+(x, u) = \{y \in \mathbb{R}^d \mid (y - x) \cdot u \leq 0\},$$

where $x \cdot y = \sum_{i=1}^d x_i y_i$ is the standard inner product on \mathbb{R}^d . If M is an oriented manifold of codimension one in \mathbb{R}^d , we denote by u_x the outward facing normal for the point $x \in M$. If M further bounds a compact set, we canonically choose the outward direction to point away from this set. For the case of fixed M , we will further abuse notation and write $H^-(x, b), x \in M, b \in \mathbb{R}$ for the set

$$\{y \in \mathbb{R}^d \mid (y - x) \cdot u_x + b \geq 0\}.$$

Hence, $H^-(x, 0)$ agrees with $H^-(x, u_x)$, and $H^-(x, b)$ is obtained by translating the half-space $H^-(x, u_x)$ a distance b in the direction $-u_x$.

We finish with some standard notions from discrete geometry. For two compact sets X, Y in a metric space, we construct the *Hausdorff distance*

$$\delta^H(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}.$$

For a finite set of points x_1, \dots, x_n in some metric space, we define the Voronoi cell (region) of x_i to be

$$\text{Vor}(x_i) = \{x \mid d(x, x_i) < d(x, x_j), j \neq i\}.$$

The collection of such cells constitutes a Voronoi diagram, and in \mathbb{R}^d is a partition of almost all points. Finally, for a set X in some affine space we let $\text{conv}(X)$ denote the

convex hull, i.e.

$$\text{conv}(X) := \bigcap_{\substack{H \text{ a (closed) halfspace} \\ X \subset H}} H.$$

We will further abuse this notation such that $\text{conv}(X_1, \dots, X_k) = \text{conv}\left(\bigcup_{i=1}^k X_i\right)$. Any remaining notation can be found in a standard reference on discrete or convex geometry, such as [58].

1.3 Probability

Our probabilistic notation is largely standard; we recommend the text [43]. The notation $\mathbb{P}, \mathbb{E}, \text{Var}$ etc. will be used unadorned whenever the underlying probability space is clear, and otherwise will be expanded and decorated as necessary. We shall write $\mathbb{1}_{\mathcal{A}}$ for the *indicator function* of the event \mathcal{A} . The addition of the letters p or d to our notation denotes “in probability” or “in distribution”, hence \xrightarrow{p} and $\stackrel{d}{=}$ denote convergence in probability and equal in distribution, respectively. Under the assumption $n \rightarrow \infty$ we say that a sequence of events $\{\mathcal{A}_n\}_{n=1}^{\infty}$ occurs asymptotically almost surely (a.a.s.) if $\mathbb{P}(\mathcal{A}_n) \rightarrow 0$.

One elementary inequality occurs frequently enough to merit mention. We have

$$e^{-nx}(1 - nx^2) \leq (1 - x)^n \leq e^{-nx}, \quad 0 \leq x < 1, n = 1, 2, \dots,$$

a proof of which may be found in [9], Lemma 5.

The next lemma is a convenient way to invoke a second moment argument, following [2], and is a consequence of Chebyshev’s inequality.

Lemma 1.3.1. *Let $X = X_1 + \dots + X_n$ where X_i is an indicator for the event A_i . If $\text{Var}[X] = o(\mathbb{E}[X]^2)$ then we have*

$$X = \mathbb{E}[X](1 + o(1)), \quad \text{a.a.s.}$$

1.3.1 Random Measure Theory

Although this dissertation is primarily concerned with structures constructed on random point sets, we shall—strictly speaking—require little more than the Kolmogorov Extension Theorem [43] to make rigorous our work. Indeed, to construct n iid

points chosen uniformly in $[0, 1]^d$ fits well within this framework. Despite this, we shall present the rudiments of *random measure theory* as it provides a general framework for constructing random point sets which is both elegant and easily includes every random set construction necessary for this dissertation. Our presentation follows that of Kallenberg [66].

Let \mathcal{S} be a locally compact second countable Hausdorff topological space (i.e. a *Polish* space). Let M consist of all measures on \mathcal{S} which are locally finite, i.e. $\mu \in M$ if $\mu B < \infty$ for every bounded[†] Borel set $B \in \mathcal{S}$. We further let N be the subset of M consisting of \mathbb{N} -valued measures. We may construct σ -algebras \mathcal{M} and \mathcal{N} to be the minimal σ -algebras which make the mappings $\mu \rightarrow \mu B$ measurable for every bounded Borel set B (where $\mu \in M$ or N , respectively).

A *random measure* is thus a measurable mapping from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into (M, \mathcal{M}) . If the codomain is instead (N, \mathcal{N}) , we obtain a *point process*. If δ_x is Dirac measure with support at x , on bounded Borel sets B one may represent a measure $\nu \in N$ as

$$\nu = \sum_{i=1}^{\nu(B)} \delta_{x_i},$$

and hence identify the measure ν with the multiset $\{x_1, \dots, x_{\nu(B)}\}$. For a given point process $\xi : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (N, \mathcal{N})$, we define the *intensity* $\mathbb{E}\xi$ of ξ to be the measure such that

$$(\mathbb{E}\xi)B = \mathbb{E}[\xi B]$$

for every bounded Borel set B . Viewing a point process as a random set, the intensity measure then gives the expected number of points landing in that set.

If we let $x \in \mathcal{S}$ be a fixed point, then the Dirac measure given by $\delta_x B = \mathbf{1}_B(x)$ provides the most basic example of a point process. If x is itself distributed according to the probability measure μ , then the point process $\xi = \delta_x$ has intensity $\mathbb{E}\xi = \mu$. One may add two point processes ξ_1, ξ_2 by $(\xi_1 + \xi_2)B = \xi_1 B + \xi_2 B$. Thus, we may obtain n iid points each distributed according to μ by the n -fold sum of independent copies of δ_x . The most studied point process is the *Poisson point process*. One may

[†] As a Polish space is *metrizable* it has at least one metric d which generates the topology. We may thus speak of a set being bounded if it is bounded in one, and hence all, such metrics.

specify this process up to distribution by the following properties: if μ is a locally finite measure on \mathcal{S} then

1. for mutually disjoint Borel sets A_1, \dots, A_k the random variables $\xi(A_1), \dots, \xi(A_k)$ are independent,² and
2. for a bounded Borel set A the random variable $\xi(A)$ is distributed as a Poisson random variable of mean $\mu(A)$.

As a matter of notation, if μ is Lebesgue measure we say that the Poisson point process is homogeneous, else it is non-homogeneous.

1.3.2 Concentration

The subject of *sharp concentration* of random variables is broad, and we make no attempt at a survey. The use of sharp concentration in this dissertation is limited to two inequalities. The first, referred to as a Chernoff bound or inequality, concerns the concentration of a sum of *independent* indicator random variables. The second is a recent martingale concentration inequality for product spaces.

The Chernoff Inequality

The Chernoff inequality has become a standard tool in the combinatorics and computer science literature [2, 74]. It addresses the question: “If X is a sum of n independent Bernoulli random variables, how close is X to its mean?” We shall list two versions: one for additive error and one for multiplicative error. Let us begin with the simpler version.

Lemma 1.3.2 (Multiplicative Chernoff). *Let $X = \sum_{i=1}^n X_i$, where X_i are independent Bernoulli random variables with $\mathbb{P}(X_i = 1) = p$. Then*

$$\mathbb{P}(X \leq (1 - \varepsilon)np) \leq e^{-\varepsilon^2 np/2}$$

² The independence requirement is in fact redundant in specifying a Poisson point process [35]. However, in most applications the independence property is the *raison d'être*, and so we include it.

and

$$\mathbb{P}(X \geq (1 + \varepsilon)np) \leq e^{-\varepsilon^2 np/3}.$$

One also has the following additive version, with the additional generalization that the Bernoulli summands may have distinct means. This formulation is found in [31].

Lemma 1.3.3 (Additive Chernoff). *Let X_1, \dots, X_n be independent random variables such that $\mathbb{P}(X_i = 1) = p_i$ and $\mathbb{P}(X_i = 0) = 1 - p_i$. Then if $X = \sum_{i=1}^n X_i$, we have the following two bounds:*

$$\mathbb{P}(X \leq \mathbb{E}[X] - \lambda) \leq \exp\left(\frac{-\lambda^2}{2\mathbb{E}[X]}\right), \quad (1.1)$$

$$\mathbb{P}(X \geq \mathbb{E}[X] + \lambda) \leq \exp\left(\frac{-\lambda^2}{2(\mathbb{E}[X] + \lambda/3)}\right). \quad (1.2)$$

1.3.3 A Martingale Inequality

The most technical concentration result we employ is a martingale inequality due to Vu in [104]. This inequality, in turn, is a continuous version of the *Divide-and-Conquer* martingale introduced by Kim and Vu in [68]. The reader familiar with techniques based on Azuma's inequality will find this tool familiar; we shall comment briefly on differences between these two results.

As a matter of notation, we let Ω be the product space of some probability spaces $\Omega_1, \dots, \Omega_n$. We denote by $t = (t_1, \dots, t_n)$ a point in Ω , and hence $t_i \in \Omega_i, i = 1, \dots, n$. If Z is a random variable which depends on t , then we define the (*absolute*) *martingale difference sequence*

$$G_i(t) = |\mathbb{E}[Z \mid t_1, \dots, t_i] - \mathbb{E}[Z \mid t_1, \dots, t_{i-1}]|.$$

For $i = 1$ we understand the second term on the right hand side to be $\mathbb{E}[Z]$. In particular, we note that $G_i(t)$ is a function of the first i coordinates of t . We then

compute the following:

$$\begin{aligned} V_i(t) &= \int_{\Omega_i} G_i(t)^2 dt_i \\ V(t) &= V_1(t) + \dots + V_n(t), \\ G_i^*(t) &= \sup_{t_i \in \Omega_i} G_i(t), \end{aligned}$$

and

$$G(t) = \max_{i=1}^n G_i^*(t).$$

As a check, the reader should verify that $V_i(t)$ and $G_i^*(t)$ depend only on the first $i - 1$ coordinates of t . We then have the following concentration lemma.

Lemma 1.3.4. *For positive λ, G_o , and V_o which satisfy $\lambda \leq V_o/4G_o^2$,*

$$\mathbb{P}(|Z - \mathbb{E}Z| \geq \sqrt{\lambda V_o}) \leq 2 \exp(-\lambda/4) + \mathbb{P}(V(t) \geq V_o \text{ or } G(t) \geq G_o).$$

If we compare the above result to Azuma's inequality [6], a few key differences emerge. One such difference is the appearance of an error term (the right most term in the above lemma), the function of which is to neglect large values of $G(t)$ and $V(t)$ which occur rarely. Such "relaxed" concentration inequalities have been used to great effect recently in the discrete setting, most notably in the work of Kim and Vu [68, 102, 101, 100] and Chung and Lu [32, 31]. The above lemma also differs in that the quantities $V_i(t)$ are based not on the supremum norm, as is the case with Azuma's inequality, but instead on the L^2 norm. For geometric problems it is often the case that one can show $V_o \ll nG_o^2$.

2 On Stochastic Geometry

The subject of *stochastic geometry* (see [105], [91] for surveys) dates back to the *needle problem* of Buffon. Buffon asks for the probability that a needle of length L , when dropped onto a plane with parallel lines separated by a distance D ($D > L$) lands on one of these lines. Successful solutions to this problem led to the development of *integral geometry*, wherein geometric objects are studied via the expected number of intersections with some random object (lines, flats, etc.) [105]. Originally the notions of integral geometry and stochastic geometry were closely related; the study of random lines, for example, was really viewed as a notion of averaging over certain compact Lie groups. The focus on the random geometric structure as an object of study, the subject of stochastic geometry, required the modern notion of *random points or sets* out of which random objects of varying shapes could be constructed.

2.1 Random Polytopes

The first, and most studied, object in stochastic geometry is the random polytope. Let us write \mathcal{K} (\mathcal{K}^d) for the set of compact, convex sets in \mathbb{R}^d (we shall generally suppress the dimension) with nonempty interior in \mathbb{R}^d . Henceforth these will be known as *compact bodies*. If $K \in \mathcal{K}$ and x_1, \dots, x_n are random independent points chosen according to the same distribution in K then the convex hull $K_n := \text{conv}(x_1, \dots, x_n)$ is referred to as a *random polytope* of K . The notion of a random polytope was first introduced in a way by J. J. Sylvester in the 1864 edition of *Educational Times* [96], question 1491 (a full account of this problem is given in [78]). In it, he wrote:

Show that the chance of four points forming the apices of a reentrant quadrilateral is $1/4$ if they be taken at random in an indefinite plane, but

$1/4 + e^2 + x^2$, where e is a finite constant and x a variable quantity, if they be limited by an area of any magnitude and of any form.

A number of prominent mathematicians of the day wrote in with “solutions”, giving the answers $1/4, 1/2, 1/3, 3/8, 35/12\pi^2$ and many others. Determining which of these was in fact *the* solution proved sufficiently challenging that J. J. Sylvester was led to conclude that “This problem does not admit of a determinate solution” [97]. A modern reader will quickly note that Sylvester’s problem is ill-posed. Indeed, the phrase “. . . if they be taken at random in an indefinite plane. . .” is problematic in that the plane has no canonical probability measure, and in particular no translation invariant probability measure. Indeed, the submitted solutions to Sylvester’s problem all implicitly assumed some model for selecting points in the plane, and naturally the answers obtained depended on such assumptions.

The modern theory of random polytopes was instigated by Rényi and Sulanke [84] and Stein [45]. Their work focused primarily on the expectation of *functionals* of random polytopes. The most studied of these are the number of vertices of K_n or the quantity $\text{Vol}_d(K \setminus K_n)$ denoting the *missed volume* of K_n . In their paper [84] Rényi and Sulanke discovered something startling: the rate of growth of the functionals depend on the smoothness of the boundary ∂K . The number of vertices of a random polytope in $K \subseteq \mathbb{R}^2$ is illustrative of this finding. For K with smooth boundary, the number of vertices is of the order $n^{1/3}$, whereas when K is a polytope this quantity is of the order $\log n$. The methods of Rényi and Sulanke were low-dimensional and relied on integral-geometric calculations and the linearity of expectation (see [12] for a survey). Their work was extended to higher dimensions with a host of intricate integral-geometric formulæ, as found in the work of Santaló [90].

In 1988, Bárány and Larman [13] freed the study of random polytopes from (increasingly complex) integral-geometric techniques with the introduction of more qualitative methods. They introduced the notion of an ε -cap of $K \in \mathcal{K}$, which is the intersection of K with a halfspace H such that $\text{Vol}_d(K \cap H) = \varepsilon$. The intersection of all ε -caps of K is then the ε -wet part of K , $K(\varepsilon)$, and the complement in K the ε -floating body. The fundamental result introduced in [13] is that the $1/n$ -wet part of K has volume of the same order as the missed volume. They also give a quantitative notion

of packing and covering the wet-part of K using caps¹, which is used to analyze the wet-part using the local geometry of the boundary ∂K .

With expectation questions largely settled, the major open question in random polytope theory is thus to understand the higher moments and concentration of key functionals about their expected values. We quote from the survey of Weil and Wieacker in the Handbook of Geometry [105]:

We finally emphasize that the results described so far give mean values hence first-order information on random sets and point processes. This is due to the geometric nature of the underlying integral geometric results. There are also some less geometric methods to obtain higher-order informations or distributions, but generally the determination of variance, e.g., is a major open problem.

The first real progress in studying higher moments of functionals of a random polytope was obtained by Reitzner in 2003 [80]. By the clever application of an inequality from the statistics literature, the Efron-Stein Jackknife inequality [46], Reitzner was able to give an upper bound on the variance of the missed volume using integral-geometric techniques (via a construction he also gives a lower bound of the same magnitude for the smooth case in [81]). This was a major breakthrough, but due to the special nature of the inequality it does not generalize to any moment beyond the second. In a completely different direction, Vu [104] in 2005 demonstrated that the missed volume (among other functionals) is *sharply concentrated* about its expectation. His method combined the “soft” geometric methods of Bárány and Larman [13] in conjunction with a continuous Azuma-Hoeffding-like concentration inequality [6] modeled after the Kim-Vu polynomial concentration inequality [68]. A central limit theorem was shown for the missed volume and other functionals of random polytope constructed from a Poisson point process by Reitzner [81] and later was de-Poissonized by Vu [103].

In addition to the random polytope K_n , other models have been examined. For K with sufficiently smooth boundary, the notion of an *inscribing random polytope* in which the vertices are chosen on the boundary ∂K was investigated in Schütt and Werner [93]. With over 100 pages of calculation, they provide an asymptotic formula

¹This is the notion of an *economic cap* covering.

for the expected value of the missed volume based on the geometry of ∂K , and in particular demonstrate that the missed volume in this model is asymptotically smaller than that of the standard random polytope. Richardson, Wu, and Vu [87] show how to apply the “soft” technique of [104] in this model for the missed volume, giving concentration and higher moment bounds as well as a central limit theorem in the Poisson case. Other models, such as the Gaussian random polytope, have seen recent progress [14, 61, 62].

3 Random Circumscribing Polytopes

We shall take $K \in \mathcal{K}$ and further assume that K has \mathcal{C}^2 boundary and nowhere vanishing Gaussian curvature in \mathbb{R}^d , $d \geq 2$. The set of such bodies we denote by \mathcal{K}_+^2 . Choose a set $X_n = \{x_1, \dots, x_n\}$ independently according to a probability measure on the boundary. Each point determines a unique supporting half-space of K which is tangent to this point; the union of all such half-spaces we denote by $P(X_n) = P_n$, calling it the *random circumscribed polytope*¹. This model is naturally dual to the previously studied [87] *random inscribed polytope*² and hence forms one of the key models in the random polytope theory.

The circumscribed model we discuss here has seen scant attention compared to most random polytope models. Böröczky [23] explored best approximating (non-random) circumscribed polytopes. With Reitzner [24], he also gave the first analysis of the random model we address here. They investigated the expectation of the volume, surface area, and mean width, and in particular they find

$$\mathbb{E} \text{Vol}_d(P_n \setminus K) = (1 + o(1))c_K n^{-\frac{2}{d-1}}.$$

Here, c_K is a constant depending only on K . This matches the result of Schütt and Werner in [93] for the missed volume of the inscribed model, and is suggestive of a general duality between the inscribed and circumscribed model.

The duality between the two models can be made explicit in one special case. Consider how the circumscribed model arises naturally from the inscribed model on the

¹Strictly speaking, the object P_n constructed in this manner may be an unbounded polyhedron. As we shall be interested only in the case of polytopes, we choose to intersect this object with a large fixed box B containing K (with volume C) whose boundary has positive Hausdorff distance to K . Clearly we can always find such a box. In the remainder it will become clear that the choice of box is unimportant. We shall make no further comment, assuming from here that P_n is always bounded in this manner.

²The random inscribed polytope is formed by the *convex hull* of X_n .

Euclidean ball $B(o, 1)$ centered at the origin. To wit, if $X_n = \{x_1, \dots, x_n\} \subset \partial K$ and we let K_n be the convex hull of these points, then examine the polar dual to K_n with respect to o ,

$$K_n^* = \{x \in \mathbb{R}_d \mid x \cdot y \leq 1 \text{ for all } y \in K_n\}.$$

In other words, K_n^* is the intersection of all half-spaces of the form $\{x \mid x \cdot y \leq 1\}$ for all $y \in K_n$. The inclusion $K_n^* \subset P_n$ is clear. Consider the reverse inclusion $P_n \subset K_n^*$. If $x \in P_n$, then we have $x \cdot x_i \leq 1, i = 1, \dots, n$. For any $y \in K_n = \text{conv}(x_1, \dots, x_n)$, we have that $y = c_1 x_1 + \dots + c_n x_n$ for $c_1 + \dots + c_n = 1$. Thus,

$$x \cdot y \leq c_1 x \cdot x_1 + \dots + c_n x \cdot x_n \leq 1,$$

hence $x \in K_n^*$ as y was arbitrary. Further, if $o \in K_n$ then $(K_n^*)^* = K_n$.

Let us couple K_n to P_n in the manner above, i.e. $P_n = K_n^*$. Now, consider the distance h from some point $x \in \partial K_n$ to $\partial B(o, 1) = S^{d-1}$. The above duality transform shows that the point $\frac{x}{\|x\|^2}$ is in ∂P_n . This point thus lies a distance $\frac{1}{1-h} = 1 + h + o(h)$ from the origin, and hence $h + o(h)$ from S^{d-1} .

Now consider $\text{Vol}_d(K \setminus K_n)$. We can express, using polar coordinates,

$$\text{Vol}_d(K \setminus K_n) = \int_{S^{d-1}} \int_0^1 \mathbb{1}_{\{x \in K \setminus K_n\}} r^{d-1} dr d\sigma(\zeta) = \int_{S^{d-1}} \int_{1-h(\zeta)}^1 r^{d-1} dr d\sigma(\zeta),$$

where σ is the surface measure on S^{d-1} and $h(\zeta)$ gives the distance from K to K_n along the ray through o with direction ζ . We thus have

$$\begin{aligned} (1-h)^{d-1} \int_{S^{d-1}} h(\zeta) d\sigma(\zeta) &\leq \int_{S^{d-1}} \int_{1-h(\zeta)}^1 r^{d-1} dr d\sigma(\zeta) = \text{Vol}_d(K \setminus K_n) \\ &\leq \int_{S^{d-1}} h(\zeta) d\sigma(\zeta) \end{aligned}$$

for $h = \max_{\zeta} h(\zeta)$. Similarly, if $h'(\zeta)$ is the distance from K to the boundary of P_n in the direction of ζ , then

$$\begin{aligned} \int_{S^{d-1}} \int_1^{1+h'(\zeta)} r^{d-1} dr d\sigma(\zeta) &= \text{Vol}_d(P_n \setminus K) \\ &\leq (1+h')^{d-1} \int_{S^{d-1}} h'(\zeta) d\sigma(\zeta) \end{aligned}$$

where $b' = \max_{\zeta} b'(\zeta)$.

Now, as $b'(\zeta) = b(\zeta) + o(b)$, the above implies that

$$\text{Vol}_d(P_n \setminus K) = (1 + o(1)) \text{Vol}_d(K \setminus K_n)$$

under the assumption $b \rightarrow 0$. As $b \rightarrow 0$ with high probability the above does as well.

In the remainder of this chapter we examine the higher moments and distribution of the *excess volume* $\text{Vol}_d(P_n \setminus K)$ of P_n . Though we make no more explicit connection between inscribed and circumscribed models, the asymptotics derived in this chapter will match those of [87]. Finally, let us emphasize that *all results hold under the assumption that K has \mathcal{C}^2 boundary and everywhere positive, hence bounded away from zero (due to compactness), Gaussian curvature.*³

3.1 Boundary Measure

As we select vertices on the boundary of ∂K , we require a probability measure on the boundary. One natural choice is the $(d - 1)$ -dimensional Hausdorff measure Vol_{d-1} , suitably scaled so as to be a probability measure. *We shall restrict our attention to probability measures ν such that*

$$d\nu = \rho d\text{Vol}_{d-1} \tag{3.1}$$

where the density $\rho : \partial K \rightarrow \mathbb{R}$ is a positive continuous function (which is to say that ν is equivalent to Hausdorff measure on ∂K). This class is large, and contains many measures of interest. A discussion of boundary measures in the context of random polytopes is contained in Schütt and Werner's tour-de-force [93] in which they investigate the expectation of the volume of a random inscribed polytope.

3.2 Caps and Visible Regions

We define the intersection of a half-space and K to be a *cap* of K , and in particular if the volume of this cap is ϵ we refer to such an object as an ϵ -cap. The intersection of a

³ In fact, via Alexandrov's theorem [1] almost every point on ∂K has a well-defined tangent plane (and generalized Gauß-Kronecker curvature) when K is convex. Indeed, we believe the assumption of \mathcal{C}^2 boundary can be weakened to non-smooth bodies with generalized curvature bounds (see [93] for a discussion).

half-space with ∂K forms a *boundary cap*, which we label an ϵ -boundary cap whenever the boundary measure of this set is ϵ . As the boundary structure of ∂K is smooth, we may also speak of a cap of height h at $x \in \partial K$:

$$C(x, h) = \{y \in K \mid u_x \cdot y \leq \delta(x) - h\},$$

where $\{y \cdot u_x = \delta(x)\}$ defines the tangent hyperplane $T_x \partial K$. Geometrically, this is constructed by translating $H^-(x)$ a distance h in the $-u_x$ direction and intersecting K with this half-space.

To work with circumscribed polytopes, we shall need an analogous notion to that of caps. For $x \in \partial K$ and $s > 0$, we let $G(x, s)$ denote the *visible region* of ∂K ,⁴ defined by

$$G(x, s) := \text{conv}(\{x + s u_x\} \cup K) \setminus K.$$

See Figure 3.1. The point $x + s u_x$ is the *peak* of the visible region. We shall also have

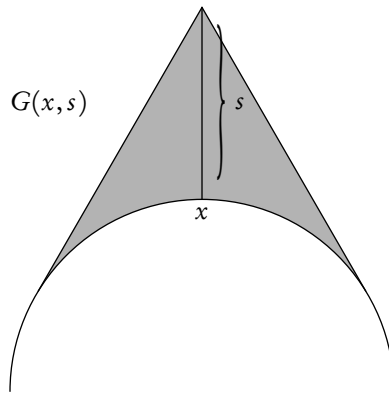


Figure 3.1: The visible region $G(x, s)$ for $x \in \partial K$. The southern edge of the shaded region constitutes a crown; the northernmost point of the shaded region a peak.

cause to look at the intersection of a visible region with the boundary ∂K , which we refer to as a *crown*. By an ϵ -visible region and ϵ -crown we indicate that the d ($d - 1$) dimensional volume of the visible region (crown) is ϵ . The notion of visible regions

⁴ This notion is closely related to that of an *illumination body*, due to Schütt [92]. One instead looks at the convex set

$$\{x \in \mathbb{R}^d \mid \text{Vol}_d(\text{conv}(\{x\} \cup K) \setminus K) \leq t\}.$$

In both cases, the key objects are the cones $\text{conv}(\{x\} \cup K) \setminus K$; we index here by height, whereas for Schütt the indexing is by volume.

and boundary caps are related as follows: for $s < s_0$ there are positive constants c and c' such that $G(x, s) \cap \partial K \subset C(x, cs) \cap \partial K$ and $G(x, s) \cap \partial K \supset C(x, c's) \cap \partial K$ (see Lemma 3.6.14). Further, for s sufficiently small $\text{Vol}_d(G(x, s)) = \Theta(s^{\frac{d+1}{2}})$, as can be shown by standard volume comparison techniques.

3.3 A Tail Bound

The main goal of this section is to demonstrate that the volume, and hence the missed volume, concentrates strongly about its expectation.

Theorem 3.3.1 (Tail Bound). *Fix $K \in \mathcal{K}_+^2$ with boundary measure as in Section 3.1. For $\alpha > 0$, there is a positive constant c such that we may take $V = cn^{-\frac{d+3}{d-1}} \ln^{-\frac{2(d+1)}{d-1}} n$, and for $0 \leq \lambda \leq \frac{2}{4}n$ we have*

$$\mathbb{P}(|\text{Vol}_d(P_n) - \mathbb{E} \text{Vol}_d(P_n)| \geq \sqrt{\lambda V}) \leq 2 \exp(-\lambda/4) + O(n^{-\alpha}).$$

In Section 3.6 we will prove the following stronger theorem:

Theorem 3.3.2. *For $d \geq 2$ there are constants c and c' such that if $V = cn^{-\frac{d+3}{d-1}}$ then for $0 \leq \lambda \leq c'n^{\frac{d^2-5d+1}{(3d+1)(d-1)}}$ we have*

$$\mathbb{P}(|\text{Vol}_d(P_n) - \mathbb{E} \text{Vol}_d(P_n)| \geq \sqrt{\lambda V}) \leq 2 \exp(-\lambda/4) + \exp(-\Omega(n^{\frac{d-1}{3d+1}})).$$

The proof of Theorem 3.3.1 and 3.3.2 are similar in spirit, however the bookkeeping involved in removing the logarithmic factor in V makes the proof of Theorem 3.3.2 rather taxing. The corollaries found in Section 3.3.3 can be proved with the above theorem at the cost of a small logarithmic factor.

3.3.1 Cap Covering

We next give a formulation of the notion of *cap covering*, which is our primary tool for analyzing the boundary structure of K . The proof is a standard application of packing and covering arguments for K with \mathcal{C}^2 boundary (see for instance [81] in the smooth case and [13] for the general case).

Lemma 3.3.3 (Cap Covering). *Given $m \geq m_0$ and $K \in \mathcal{K}_+^2$, there are points $y_1, \dots, y_m \in \partial K$, a height h_m , positive constants d_1, d_2 , and caps $C_i = C(y_i, h_m)$ and $\overline{C}_i = C(y_i, (2d_2/d_1)^2 h_m)$ with*

$$C_i \subset B(y_i, d_2 h_m^{1/2}) \subset \text{Vor}(y_i)$$

and

$$\text{Vor}(y_i) \cap \partial K \subset B(y_i, 2d_2 h_m^{1/2}) \cap \partial K \subset \overline{C}_i,$$

and

$$h_m = \Theta(m^{-\frac{2}{d-1}}).$$

Here $\text{Vor}(y_i)$ is the Voronoi cell of y_i . We further have constants d_4, d_5, d_6 , and d_7 such that

$$d_4 m^{-\frac{d+1}{d-1}} \leq \text{Vol}_d(C_i) \leq d_5 m^{-\frac{d+1}{d-1}}$$

and

$$d_6 m^{-1} \leq \text{Vol}_{d-1}(C_i \cap \partial K) \leq d_7 m^{-1}$$

for all $i = 1, \dots, m$.

The key feature is that one can place m points on the boundary ∂K and establish two systems of caps centered at these points: one mutually disjoint and the other covering the boundary. The height of these two systems is related by a constant, and so hence will the volumes be as well.

With the cap covering in hand we can establish some properties of a “typical” distribution of points on ∂K . Let us set for the moment

$$m = \left\lfloor \frac{d_6 n}{k \ln(n-1)} \right\rfloor$$

for k some constant we shall determine later. We shall create a system of caps according to Lemma 3.3.3. This argument will be employed again in Section 3.5. To summarize this construction, we select points $y_1, \dots, y_m \in \partial K$ and caps $C(y_i, h_m)$ such that the caps are disjoint and h_m is some function of m such that $h_m = \Theta(m^{-\frac{2}{d-1}})$. In particular, Lemma 3.3.3 guarantees that

$$\text{Vol}_{d-1}(C(y_i, h_m) \cap \partial K) \geq c_6 m^{-1} \geq \frac{k \ln n}{n-1}.$$

If we choose n points $X_n = \{x_1, \dots, x_n\}$ according to the distribution on the boundary, then the probability they all miss $C(y_i, h_m)$ is $O(n^{-k \min \rho})$. We denote by \mathcal{B} the event that some $C(y_i, h_m)$ contains no member of X_n . By the union bound we thus have

$$\mathbb{P}(\mathcal{B}) \leq \sum_{i=1}^m \mathbb{P}(X_n \cap C(y_i, h_m) = \emptyset) = O(n^{-k+1}). \quad (3.2)$$

Critically, we can raise the exponent while only modifying m by a linear factor.

Now, if \mathcal{B} does not hold, then in particular Lemma 3.6.15 gives that the Hausdorff distance $\delta = \delta^H(P_n, K) = O(m^{-\frac{2}{d-1}})$. Thus we know that $P_n \subset K + \delta B(o, 1) = K_\delta$. Consider the volume change $\text{Vol}_d(P_{n-1}) - \text{Vol}_d(P_n)$. We know that if $P_{n-1} \subseteq K_\delta$, then this volume difference is bounded by the volume of any cap $C_{K_\delta}(y, \delta)$ of K_δ . Letting $g(\delta) = \sup_{y \in \partial K_\delta} \text{Vol}_d(C_{K_\delta}(y, \delta))$, Lemma 3.6.11 shows that

$$g(\delta) = O(\delta^{\frac{d+1}{2}}) = O(m^{-\frac{d+1}{d-1}}). \quad (3.3)$$

For any finite set $L \subset \partial K$ and $x \in \partial K$ we let

$$\Delta_{x,L} = \text{Vol}_d(P(L) - P(L \cup \{x\})),$$

recalling that $P(L)$ denotes the circumscribed polytope determined by L . Thus if \mathcal{B}^c holds for X_n we have

$$\Delta_{x,L} \leq g(\delta). \quad (3.4)$$

3.3.2 Proof of Theorem 3.3.1

Recall that $P_n = P(X_n)$. In what follows, set $Y = \text{Vol}_d(P_n)$. We shall apply Lemma 1.3.4, setting $G_o = 3g(\delta)$ and $V_o = nG_o^2$. Our main task is to estimate the failure probability $\mathbb{P}(V(X_n) \geq V_o \text{ or } G(X_n) \geq G_o)$. We shall give a bound for $\mathbb{P}(V_i(X_n) \geq V_o \text{ or } G_i(X_n) \geq G_o)$, from which we can estimate

$$\mathbb{P}(V(X_n) \geq V_o \text{ or } G(X_n) \geq G_o) \leq n \mathbb{P}(V_i(X_n) \geq V_o/n \text{ or } G_i(X_n) \geq G_o)$$

by the union bound. Thus, we show that

$$\mathbb{P}(V_i(X_n) \geq V_o \text{ or } G_i(X_n) \geq G_o) \leq n^{-\alpha-1}, \quad i = 1, \dots, n.$$

Denote the product spaces spanned by $\{x_1, \dots, x_i\}$ and $\{x_i, \dots, x_n\}$ by $\Omega_{<i>}$ and $\Omega^{<i>}$, respectively. We shall use a coupling argument. If we let x be a point distributed according to the boundary measure on ∂K and let $Y' = \text{Vol}_d(P(\{x\} \cup X_n \setminus \{x_i\}))$, then

$$\begin{aligned} G_i(t) &= |\mathbb{E}[Y | x_1, \dots, x_i] - \mathbb{E}[Y | x_1, \dots, x_{i-1}]| \\ &\leq \mathbb{E}_x |\mathbb{E}[Y | x_1, \dots, x_i] - \mathbb{E}[Y' | x_1, \dots, x_{i-1}, x]|. \end{aligned}$$

We shall let $L = \{x_1, \dots, x_{i-1}\} \cup \{x_{i+1}, \dots, x_n\}$ which consists of both random and fixed coordinates of X_n . In particular, $\Delta_{x_i, L}$ is the region “cut off” from $P(L)$ by adding the point x_i . Thus, we can decompose

$$Y = \text{Vol}_d(P(L)) - \text{Vol}_d(\Delta_{x_i, L}),$$

and hence

$$\mathbb{E}[Y | x_1, \dots, x_{i-1}, x_i] = \mathbb{E}[\text{Vol}_d(P(L)) | x_1, \dots, x_{i-1}] - \mathbb{E}[\Delta_{x_i, L} | x_1, \dots, x_{i-1}].$$

We remark that the term involving $\text{Vol}_d(P(L))$ only depends on x_1, \dots, x_{i-1} .

Now we come to an inequality where we utilize our cap covering arguments:

$$\mathbb{E}[\Delta_{x_i, L} | x_1, \dots, x_{i-1}] \leq g(\delta) + \mathbb{P}[\mathcal{B} | x_1, \dots] C.$$

Let us explain the right hand side. If \mathcal{B}^c holds for L , then by (3.4), $\Delta_{x_i, L} \leq g(\delta)$, and we upper bound the quantity $\mathbb{P}[\mathcal{B}^c | x_1, \dots, x_i]$ by one. If \mathcal{B} holds for L , then we let C denote the volume of the large box we assumed to contain K .

Now, let us say that $\{x_1, \dots, x_{i-1}\}$ is *typical* if

$$\mathbb{P}_{\Omega^{<i>}}(\mathcal{B} | x_1, \dots, x_{i-1}) \leq n^{-4}.$$

Note the choice of n^{-4} obeys $n^{-4} = o(g(\delta))$ for $d \geq 2$ since $g(\delta) = O((n \ln^{-1} n)^{-\frac{d+1}{d-1}})$.

Typical Sets. We bound G_i and V_i under the assumption that x_1, \dots, x_{i-1} is typical. From the above we have

$$\begin{aligned} G_i(X_n) &= \mathbb{E}_x |\mathbb{E}[Y | x_1, \dots, x_i] - \mathbb{E}[Y' | x_1, \dots, x_{i-1}, x]| \\ &= \mathbb{E}_x |\mathbb{E}[\Delta_{x_i, L} | x_1, \dots, x_{i-1}] - \mathbb{E}[\Delta_{x_i, L} | x_1, \dots, x_{i-1}]| \\ &\leq \mathbb{E}[\Delta_{x_i, L} | x_1, \dots, x_{i-1}] + \mathbb{E}_x \mathbb{E}[\Delta_{x_i, L} | x_1, \dots, x_{i-1}] \\ &\leq 2g(\delta) + 2C n^{-7} \leq 3g(\delta) = G_o. \end{aligned}$$

The last line holds under the assumption that n is sufficiently large. Similarly, we estimate V_i as

$$\begin{aligned} V_i(X_n) &= \int_{\Omega_i} G_i(X_n)^2 dx_i \\ &\leq \int_{\Omega_i} 9g(\delta)^2 dx_i = 9g(\delta) = V_o/n. \end{aligned}$$

Nontypical Sets. We bound the probability that x_1, \dots, x_{i-1} is non-typical. To do so, we need a small probability result which says that a high probability event should remain high probability upon conditioning.

Lemma 3.3.4. *Let Ω' and Ω'' be probability spaces and Ω''' their product. Assume that an event A occurs in Ω''' with probability $1 - \epsilon'$ for $0 < \epsilon' < 1$. If $1 > \epsilon > \epsilon'$ we have*

$$\mathbb{P}_{\Omega'}(\mathbb{P}_{\Omega''}(A | x) \leq 1 - \epsilon) \leq \epsilon'/\epsilon.$$

Here, $x \in \Omega'$ and $\mathbb{P}_{\Omega'}$ and $\mathbb{P}_{\Omega''}$ are probabilities on Ω' and Ω'' , respectively.

Proof. We assume $\mathbb{P}_{\Omega''}(A) \geq 1 - \epsilon'$. Computing, we find

$$\mathbb{P}_{\Omega''}(A) = \int_{\Omega'} \mathbb{P}_{\Omega''}(A | x) dx \leq 1 - \epsilon \mathbb{P}_{\Omega'}(\mathbb{P}_{\Omega''}(A | x) \leq 1 - \epsilon).$$

The proof then follows by comparison. □

We thus have that

$$\begin{aligned} &\mathbb{P}_{\Omega_{<i-1>}}(\{x_1, \dots, x_{i-1}\} \text{ are non-typical}) \\ &= \mathbb{P}_{\Omega_{<i-1>}}(\mathbb{P}_{\Omega_{<i+1>}}(\mathcal{B} \text{ holds for } L | x_1, \dots, x_{i-1}) \geq n^{-7}) \leq n^{-k+5}. \end{aligned}$$

As we let k be a free parameter we choose it so that $k - 5 \geq \alpha + 1$. This finishes our theorem.

3.3.3 Tail Bound Applications

The tail bound has many possible applications. We first have a bound on the higher centered moments $M_k = \mathbb{E}[(Y - \mathbb{E} Y)^k]$, $k = 2, 3, \dots$

Corollary 3.3.5. *For any convex body K , the k th moment satisfies*

$$M_k = O\left(\left(n^{-\frac{d+1}{d-1}}\right)^{k/2}\right).$$

Proof of Corollary 3.3.5. We use Theorem 3.3.2, setting $Y = \text{Vol}_d(P_n)$.

Let $\lambda_o = cn^{\frac{d^2-5d+1}{(3d+1)(d-1)}}$. For any $\lambda > \lambda_o$ we have

$$\begin{aligned} \mathbb{P}(|Y - \mathbb{E}Y| \geq \sqrt{\lambda V}) &\leq \mathbb{P}(|Y - \mathbb{E}Y| \geq \sqrt{\lambda_o V}) \\ &\leq 2 \exp(\lambda_o/4) + \exp(-\Omega(n^{\frac{d-1}{3d+1}})). \end{aligned}$$

Hence for any $\lambda > o$ we obtain the (rather generous) bound

$$\mathbb{P}(|Y - \mathbb{E}Y| \geq \sqrt{\lambda V}) \leq 2 \exp(-\lambda/4) + 2 \exp(-\lambda_o/4) + 2 \exp(-\Omega(n^{\frac{d-1}{3d+1}})). \quad (3.5)$$

Our departure point for estimating M_k is the equality

$$M_k = \int_o^\infty x^k d\mathbb{P}(|Y - \mathbb{E}Y| < x).$$

We set $\mathbb{P}(|Y - \mathbb{E}Y| \geq x) = f(x)$. We further make the simplifying assumption that $\text{Vol}_d(K) = 1$. Thus, $|Y - \mathbb{E}Y| \leq 1$ which allows us to integrate only up to one.

$$\begin{aligned} M_k &= \int_o^1 x^k d\mathbb{P}(|Y - \mathbb{E}Y| < x) \\ &= - \int_o^1 x^k df(x) \\ &= (-x^k f(x))|_o^1 + \int_o^1 kx^{k-1} f(x) dx \\ &= \int_o^1 kx^{k-1} f(x) dx. \end{aligned}$$

We then reparameterize by $x = \sqrt{\lambda V}$ to obtain

$$\begin{aligned} \int_o^1 kx^{k-1} f(x) dx &\leq \int_o^{1/V} k(\lambda V)^{\frac{k-1}{2}} \mathbb{P}(|Y - \mathbb{E}Y| \geq \sqrt{\lambda V}) 2^{-1} V^{1/2} \lambda^{-1/2} d\lambda \\ &\leq k V^{k/2} \int_o^{1/V} \lambda^{k/2-1} \end{aligned} \quad (3.6)$$

$$\times \left(\exp(-\lambda/4) + \exp(-\lambda_o/4) + \exp(-\Omega(n^{\frac{d-1}{3d+1}})) \right) d\lambda. \quad (3.7)$$

The second line follows from our upper bound given in (3.5).

We thus integrate the three terms. The first yields

$$\int_0^{1/V} \lambda^{k/2-1} \exp(-\lambda/4) d\lambda \leq \int_0^\infty \lambda^{k/2-1} \exp(-\lambda/4) d\lambda = c_k.$$

Note that the constant c_k depends only on k .

Next, we show the other integrals are of lower order.

$$\int_0^{1/V} \lambda^{k/2-1} \exp(-\lambda_o/4) d\lambda \leq \exp(-\lambda_o/4) \frac{2}{k} (1/V)^{k/2}.$$

Now as $V = cn^{-\frac{d+3}{d-1}} \geq n^{-2}$ for n sufficiently large the above becomes

$$\exp(-\lambda_o/4) \frac{2}{k} (1/V)^{k/2} \leq \exp(-c' n^{\frac{d^2-5d+1}{(3d+1)(d-1)}}) \frac{2}{k} n^k = o(1)$$

for $d \geq 5$.

Similarly,

$$\int_0^{1/V} \lambda^{k/2-1} \exp(-\Omega(n^{\frac{d-1}{3d+1}})) d\lambda = o(1).$$

Collecting terms in our bound (3.7) we see that

$$M_k \leq (c_k + o(1))kV^{k/2}.$$

As $V = cn^{-\frac{d+3}{d-1}}$ this completes our proof. \square

Let us note that the above result holds only for $d \geq 5$. For smaller d one can use the weaker Theorem 3.3.1 to obtain an analogous result with a logarithmic factor. Specializing to $k = 2$, the above corollary gives an upper bound on the variance of Y . This provides half of Theorem 3.4.1.

We also obtain a result about the speed of convergence of the random variable $Y_n = \text{Vol}_d(P_n)$ (here stressing the dependence on n) to its expectation.

Corollary 3.3.6.

$$\lim_{n \rightarrow \infty} \left| \frac{Y_n}{\mathbb{E} Y_n} - 1 \right| f(n) = 0$$

almost surely for

$$f(n) = \delta(n)(n^{-\frac{d+3}{d-1}} \ln n)^{-1/2},$$

where here $\delta(n)$ is a function tending to zero arbitrarily slowly.

Proof of Corollary 3.3.6. We compute using Theorem 3.3.2. Set

$$f(n) = \frac{\delta(n) \mathbb{E} Y_n}{\lambda_n V_n}$$

where $\lambda_n = 8 \ln n$ and $V_n = c n^{-\frac{d+3}{d-1}}$.

$$\begin{aligned} \mathbb{P} \left(\left| \frac{Y_n}{\mathbb{E} Y_n} - 1 \right| f(n) \geq \delta(n) \right) &\leq \mathbb{P} \left(|Y_n - \mathbb{E} Y_n| \geq \mathbb{E} Y_n \sqrt{8 \ln n c n^{-\frac{d+3}{d-1}}} \right) \\ &\leq 2 \exp(-2 \ln n) + \exp(-\Omega(n^{\frac{d-1}{3d+1}})) \\ &\leq 3n^{-2} \end{aligned}$$

where the last line holds for n sufficiently large. As $\sum_n n^{-2}$ is finite, by the Borel-Cantelli lemma we see that $\left| \frac{Y_n}{\mathbb{E} Y_n} - 1 \right| f(n)$ must converge to zero almost surely. \square

3.4 Variance Lower Bound

In this section our goal is to construct a lower bound on the variance of $\text{Vol}_d(P_n)$ (and hence the missed volume as well). As noted, such a bound combined with Corollary 3.3.3 will prove the following:

Theorem 3.4.1 (Variance). *For $d \geq 5$. Then*

$$\text{Var}(P_n) = \Theta(n^{-\frac{d+3}{d-1}}).$$

Reitzner was the first to give a variance lower bound for a random polytope model [81], and his technique has been successfully extended to other models. The idea rests on the intuitive notion that for a given random polytope we can vary a set of points along the boundary which are mutually independent (i.e. their hyperplanes do not form a face of the polytope). If we do not vary them too much, this independence will be maintained and the contribution to the variance will be the sum of independent calculations. The details are somewhat more involved, and in particular we need good control over the boundary of K . For this, our central tool will be Lemma 3.6.9.

The proof consists of two parts. In the first, we check a number of geometric constructions necessary for the proof. In the second, we perform the relevant probability calculation.

3.4.1 A Construction

For a point $y \in \mathbb{R}^d$ we write $y = (y^1, \dots, y^d)$ for the relevant coordinates with respect to some fixed orthonormal basis e_1, \dots, e_d . We shall label the subspace spanned by the first $d - 1$ basis elements \mathbb{R}^{d-1} .

The Standard Paraboloid: We first construct a reference object. To do so, we let $b(y) = \frac{1}{2} \sum_{i=1}^{d-1} (y^i)^2$ denote the *standard quadratic form* on \mathbb{R}^{d-1} . We thus have in \mathbb{R}^d the *standard paraboloid* E

$$E = \{x \in \mathbb{R}^d \mid x^d \geq b(x)\},$$

and in general we define λE , $\lambda > 0$, to be

$$\lambda E = \{x \in \mathbb{R}^d \mid x^d \geq \lambda^{-1} b(x)\},$$

so that $\lambda_1 E \supseteq \lambda_2 E$ whenever $\lambda_1 \geq \lambda_2$.

Next, we consider the cap $C_E(o, 1)$, which is the intersection of E with the half-space $y^d \leq 1$. Let us form a simplex with base given by a $d - 1$ regular simplex located in $\partial E \cap \{y \mid y^d = h_d\}$ and apex at the origin, h_d to be chosen later. Let us further label the vertices of this simplex v_o, v_1, \dots, v_d , designating v_o to be the vertex located at the origin.

Now for $\delta > 0$ we consider the set of lines \mathcal{L} which are:

- incident to the ray $\{t e_d \mid t < 0\}$,
- incident to the set $\partial E \cap \{y \mid y^d = 1\}$,
- and tangent to the paraboloid $(1 + \delta)^{-1} E$.

We let $\text{conv}(\mathcal{L}) = PC_\delta$ be the δ -*protective cone*. Observe that PC_δ is a pointed cone with point on the ray $\{t e_d \mid t < 0\}$. We recall that $P_E(X)$ is the circumscribed polytope about E determined by a point set X . Note that $P_E(\{v_1, \dots, v_d\})$ thus forms a cone; we set h_d small enough such that $P_E(\{v_1, \dots, v_d\}) \cap \{y \mid y^d = h_d\} \subset \text{int } PC_\delta$. Finally, let $PC_\delta \setminus \text{int } E = S_\delta$. Geometrically, this represents everything the point of PC_δ can “see” up to the boundary of E . We then have the following lemma.

Lemma 3.4.2. *Let z be any point in $H^-(o, \mathfrak{I})$. If for some point $s \in S_\delta$ the line through s and z avoids the interior of $(\mathfrak{I} + \delta)^{-1}E$, then z avoids $\text{int } PC_\delta$.*

Proof. We may assume that the line through s, z is tangent to $(\mathfrak{I} + \delta)^{-1}E$, for otherwise we see that $[s + te_d, z]$ avoids $\text{int } (\mathfrak{I} + \delta)^{-1}E$ for small enough t , and we can just take s to be $s + te_d$ for some maximal t . Hence, the line is contained in some tangent hyperplane to $(\mathfrak{I} + \delta)^{-1}E$. By our construction of PC_δ and the convexity of $(\mathfrak{I} + \delta)^{-1}E$, the only tangent hyperplanes to $(\mathfrak{I} + \delta)^{-1}E$ which pass through S necessarily avoid $\text{int } PC_\delta$ in the region $H^-(o, \mathfrak{I})$. \square

We now proceed to our construction. We denote by $\text{proj} : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ the orthogonal projection operator which forgets the last coordinate. Let r, r' be positive constants, to be chosen later. We thus construct sets in \mathbb{R}^{d-1}

$$B_o = B(\text{proj}(v_o), r) \cap \mathbb{R}^{d-1}, \quad B_i = B(\text{proj}(v_i), r') \cap \mathbb{R}^{d-1}, \quad i = 1, \dots, d.$$

Next, a map $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ induces a map $\tilde{f} : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^d$ by

$$\tilde{f} : y \mapsto (y^1, \dots, y^{d-1}, f(y)).$$

We then use \tilde{b} to construct set $B'_i \subset \partial E$ by

$$B'_i = \tilde{b}(B_i), \quad i = 0, \dots, d.$$

For points $x_o, \dots, x_d \in E$ we set

$$\text{cut}_E(x_1, \dots, x_d; x_o) := P_E(\{x_1, \dots, x_d\}) - P_E(\{x_1, \dots, x_d, x_o\}).$$

For $x_i \in B'_i, i = 0, \dots, d$ we can choose r, r' sufficiently small such that

$$\text{cut}_E(x_1, \dots, x_d; x_o) \subset \text{int } PC_\delta.$$

This construction will serve as a model and computational tool.

The General Paraboloid: Let us now examine a general paraboloid

$$Q \equiv \left\{ x \in \mathbb{R}^d \mid x^d \geq \frac{1}{2} \sum_{i=1}^{d-1} k_i (x^i)^2 \right\}$$

which we assume to have positive principle curvatures k_i , and hence positive Gaussian curvature $\kappa = \prod k_i$ at the origin.

For a given $h > 0$, there is a unique linear map A which takes $C_E(0, 1)$ to $C_Q(0, h)$ which fixes the coordinate axes of \mathbb{R}^{d-1} .

We let D_i be the image of the $B_i, i = 0, \dots, d$ under this map. The Lebesgue measure of D_i on \mathbb{R}^{d-1} is given by $c_x h^{\frac{d-1}{2}}$ for some constant depending only on the curvature κ .

Now, for each point $x \in \partial K$ we let Q_x denote the osculating paraboloid of K at x . We shall identify the tangent hyperplane $T_x \partial K$ with \mathbb{R}^{d-1} in what follows.

Let us now denote D_i by $D_i(x)$ to keep track of the dependence on x . Now, as ∂K is locally represented by a convex function $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, we can form sets $D'_i(x)$ by

$$\tilde{f}(D'_i(x)) = D'_i(x), \quad i = 0, \dots, d.$$

It is important to observe that $D'_i(x)$ are *not in general the image of B'_i under A* , as ∂K and ∂Q_x do not in general coincide.

Because the curvature is bounded away from zero as well as from above, we have the estimate

$$c_1 h^{\frac{d-1}{2}} \leq \text{Vol}_{d-1}(D'_i(x)) \leq c_2 h^{\frac{d-1}{2}}. \quad (3.8)$$

The constants c_1 and c_2 depend only on K .

The following technical lemma, whose proof we delay until 3.4.2, shows that the variance scales appropriately.

Lemma 3.4.3. *There exists an r_0 such that for all $r_0 > r, r' > 0$ there is some $h_0 = h_0(r, r')$ where for any choice of points $x_i \in D'_i(x), i = 1, \dots, d$ and $h_0 > h > 0$ we have*

$$c_3 h^{d+1} \leq \text{Var}_{x_0}(\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; x_0))) \leq c_4 h^{d+1},$$

for some positive constants c_3, c_4 depending on K, r , and r' . Here the variance is taken choosing $x_0 \in B'_0(x)$ according to the distribution on the boundary, μ .

Recall here that h controls the scale of $D'_i(x)$.

Finally, we wish to ensure that $\text{cut}_K(x_1, \dots, x_d; x_0) \subset \text{int}A(PC_\delta)$ for $x_i \in B'_i(x)$. To do so, construct balls $B(v_i, \eta)$ for $\eta > 0$ about the vertices v_0, \dots, v_d in our model

paraboloid E . For points $x_i \in B(v_i, \eta)$ consider the intersection of half-spaces, (which is $\text{cut}_E(x_1, \dots, x_d; x_o)$ whenever the x_i are all on the boundary of E)

$$H^+(x_o, u_o) \cap H^-(x_1, u_1) \cap \dots \cap H^-(x_d, u_d)$$

where $u_i \cdot u_{v_i}^E \geq 1 - \eta$ (recalling that $u_{v_i}^E$ denotes the outward unit normal on E at v_i). By continuity we can assume that this object is contained in PC_δ for all $\eta > 0$ sufficiently small.

As A preserves inclusions, we thus find that for $x_i \in A(B(v_i, \eta)), i = 0, \dots, d$ and $u_i \cdot u_{A(v_i)}^{Q_\epsilon} \geq 1 - c_5 \eta$ we have the inclusion

$$H^+(x_o, u_o) \cap H^-(x_1, u_1) \cap \dots \cap H^-(x_d, u_d) \subset PC_\delta.$$

Here, c_5 is a positive constant depending only on the curvature of K . This follows since A can distort inner products by an amount depending only on the curvature of K .

Thus, we will have the inclusion

$$\text{cut}_K(x_1, \dots, x_d; x_o) \subset A(PC_\delta), \quad x_i \in B'_i(x), \quad i = 0, \dots, d \quad (3.9)$$

if we can show that $B'_i(x) \subset A(B(v_i, \eta))$ and $u_{x_i}^K \cdot u_{A(v_i)}^{Q_\epsilon} \geq 1 - c_5 \eta$.

Considering our standard ellipsoid again, choose $\epsilon > 0$ such that

$$U_i = \left\{ (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid \text{proj}((x, y)) \subset B(\text{proj}(v_i), \eta/2) \right. \\ \left. \text{and } (1 + \epsilon)^{-1} b(x) \leq y \leq (1 + \epsilon) b(x) \right\} \subset B(v_i, \eta), i = 0, \dots, d.$$

Choosing h sufficiently small we can apply Lemma 3.6.9 to see that

$$(1 + \epsilon)^{-1} b(x) \leq \tilde{f}(x) \leq (1 + \epsilon) b(x).$$

Hence, in a small enough cap we have $B'_i(x) \subset A(B(v_i, \eta)), i = 0, \dots, n$ if $r, r' < \eta/2$. A similar application of Lemma 3.6.9 shows that for h sufficiently small we have $u_{x_i}^K \cdot u_{A(v_i)}^{Q_\epsilon} \geq 1 - c_5 \eta$. So, choosing r, r', h to satisfy the constraints discussed here and in Lemma 3.4.3 we have the inclusion (3.9) for $i = 0, \dots, d$.

Probability Calculation: Choose n random points $X_n = \{x_1, \dots, x_n\}$ independently along ∂K according to the boundary measure. Further, choose n points

$y_1, \dots, y_n \in \partial K$ and corresponding disjoint caps of height h_n according to Lemma 3.3.3. We further assume n is large enough so that h_n is smaller than h as chosen previously. Further, in each cap $C(y_j, h_n)$ of K establish sets $\{D_i(y_j)\}$ and $\{D'_i(y_j)\}$ as in the prior construction.

Next, we let $\mathcal{A}_j, j = 1, \dots, n$ be the event that exactly one point of $\{x_1, \dots, x_n\}$ is contained in $D_i(y_j), i = 0, \dots, d$. This probability we compute as

$$\begin{aligned} \mathbb{P}(\mathcal{A}_j) &= (n)_{d+1} \mathbb{P}(x_i \in D'_i(y_j), i = 0, \dots, d) \mathbb{P}(x_i \notin C(y_j, h_n) \cap \partial K, i \geq d+1) \\ &\leq (n)_{d+1} \prod_{i=0}^d \min_{\partial K} \rho \text{Vol}_{d-1}(D'_i(y_j)) \prod_{k=d+1}^n (1 - \max_{\partial K} \rho \text{Vol}_{d-1}(C(y_j, h_n) \cap \partial K)) \\ &\geq c_6 n^{d+1} n^{-d-1} (1 - c_7 n^{-1})^{n-d-1} \geq c_8 > 0, \end{aligned}$$

where here c_6, c_7, c_8 are positive constants and $(n)_k = \prod_{i=0}^{k-1} (n-i)$. This last line follows by noting that $h_n = \Theta(n^{-2/(d-1)})$ (see Lemma 3.3.3) and (3.8). This implies

$$\mathbb{E}\left[\sum_{j=1}^n \mathbb{1}_{\mathcal{A}_j}\right] = \sum_{j=1}^n \mathbb{P}(\mathcal{A}_j) \geq c_8 n.$$

We then have

$$\text{Var } Y = \mathbb{E} \text{Var}(Y | \mathcal{F}) + \text{Var} \mathbb{E}(Y | \mathcal{F}) \geq \mathbb{E} \text{Var}(Y | \mathcal{F}),$$

for some \mathcal{F} , where the equality is just the conditional variance formula.

We choose \mathcal{F} to denote the position of all random points $\{x_1, \dots, x_n\}$ except those $x_i \in D'_0(y_j)$ with $\mathbb{1}_{\mathcal{A}_j} = 1$. Under \mathcal{F} , examine the event $\mathbb{1}_{\mathcal{A}_i} = 1 = \mathbb{1}_{\mathcal{A}_j}$. Let x_i and x_j be the unique points in $D'_0(y_i)$ and $D'_0(y_j)$, respectively, and let $x_{i_k} \in D'_k(y_i), k = 1, \dots, d$. By assumption, (3.9) holds for both y_i and y_j (for an appropriate choice of points in $\{x_1, \dots, x_n\}$). Further, any hyperplane tangent to K through x_i must avoid the interior of $(1+\delta)^{-1}Q_{x_j}$, and hence avoids $\text{cut}_K(x_{i_1}, \dots, x_{i_d}; x_i)$ by Lemma 3.4.2. As this argument is symmetric in i and j , it is thus clear that $\text{Var}_{x_i}(Y)$ and $\text{Var}_{x_j}(Y)$ are independent, where here $\text{Var}_{x_i}(Y)$ represents the variance taken over $x_i \in D'_0(y_i)$.

We thus may write

$$\text{Var}(Y | \mathcal{F}) = \sum_{i=1}^n \text{Var}_{x_i}(Y) \mathbb{1}_{\mathcal{A}_i},$$

again assuming the variance is taken over $x_i \in D'_o(y_i)$.

Finally, we use Lemma 3.4.3 and the fact that $h_n = \Theta(n^{-2/(d-1)})$ to compute

$$\begin{aligned} \mathbb{E} \text{Var}(Y | \mathcal{F}) &= \mathbb{E} \left(\sum_{i=1}^n \text{Var}_{x_i}(Y | \mathcal{F}) \mathbb{1}_{\mathcal{A}_i} \right) \\ &\geq c_3 h_n^{d+1} \mathbb{E} \left(\sum_{i=1}^n \mathbb{1}_{\mathcal{A}_i} \right) \\ &\geq c_3 h_n^{d+1} c_8 n \\ &= \Omega(n^{-\frac{d+3}{d-1}}). \end{aligned}$$

Let us pause here to remark that this finishes the proof of theorem 3.4.1.

3.4.2 Proof of Lemma 3.4.3

We preserve the notation used earlier in this section. We identify $T_x \partial K$ with \mathbb{R}^{d-1} , and for h sufficiently small we represent the cap $C(x, h)$ locally using Lemma 3.6.9 by a convex function $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$. Recall that A is an affine map taking $C_E(o, 1)$ to $C_{Q_c}(x, h)$, and denote by $A' = A|_{\mathbb{R}^{d-1}}$.

We assume that $x_i \in D'_i(x), i = 1, \dots, d$. Then we have the expectation taken over $x_o \in D'_o(x)$ given by

$$\begin{aligned} &\mathbb{E}_{x_o}(\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; x_o))) \\ &= \frac{\int_{D_o(x)} \text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; \tilde{f}(Y))) \rho(\tilde{f}(Y)) \sqrt{1 + |\nabla f|^2} dY}{\int_{D_o(x)} \rho(\tilde{f}(Y)) \sqrt{1 + |\nabla f|^2} dY} \\ &= \frac{|\det A'| \int_{C_o} \text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; \tilde{f}(AX))) \rho(\tilde{f}(AX)) \sqrt{1 + |\nabla f|^2}(AX) dX}{|\det A'| \int_{D_o(x)} \rho(\tilde{f}(AX)) \sqrt{1 + |\nabla f|^2}(AX) dX} \end{aligned}$$

where here we use the change of variables formula. Next, we use the identity

$$\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; \tilde{f}(AX))) = |\det A| \text{Vol}_d \left(A^{-1}(\text{cut}_K(x_1, \dots, x_d; \tilde{f}(AX))) \right).$$

The object $A^{-1}(\text{cut}_K(x_1, \dots, x_d; \tilde{f}(AX)))$ depends on the positions of x_1, \dots, x_d as well as their normal vectors u_{x_1}, \dots, u_{x_d} to K . Using Lemma 3.6.9 we know that for $\delta > 0$ we

have $C_{(1+\delta)^{-1}Q_x}(x, h) \subset C_K(x, h) \subset C_{(1+\delta)Q_x}(x, h)$ by taking h sufficiently small (indeed for any x). Thus we have

$$C_{(1+\delta)^{-1}E}(o, 1) \subset C_{A^{-1}(K)}(o, 1) \subset C_{(1+\delta)E}(o, 1).$$

By construction, the points $A^{-1}(x_i), i = 0, \dots, d$ lie above the sets $B_i, i = 0, \dots, d$. By adjusting the parameters r and r' we thus can guarantee that the $A^{-1}(x_i)$ lie as close as desired to the v_i since δ can be made arbitrarily small (at the cost of h). Further, the normals to $A^{-1}(K)$ at the points $A^{-1}(x_i)$ can similarly be made close to the normals of E at the v_i .

Hence, we see that for any $\epsilon > 0$ we can make r and r' sufficiently small such that

$$\begin{aligned} \text{Vol}_d(\text{cut}_E(v_1, \dots, v_d; \tilde{b}(X)))(1 - \epsilon) &\leq \text{Vol}_d(A^{-1}(\text{cut}_K(x_1, \dots, x_d; \tilde{f}(AX)))) \\ &\leq \text{Vol}_d(\text{cut}_E(v_1, \dots, v_d; \tilde{b}(X)))(1 + \epsilon) \end{aligned}$$

for all h sufficiently small (but independent of x).

Thus

$$\begin{aligned} &\frac{\int_{C_o} \text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; \tilde{f}(AX))) \rho(\tilde{f}(AX)) \sqrt{1 + |\nabla f|^2}(AX) dX}{\int_{D_o(x)} \rho(\tilde{f}(AX)) \sqrt{1 + |\nabla f|^2}(AX) dX} \\ &= \frac{|\det A| \int_{C_o} \text{Vol}_d(A^{-1}(\text{cut}_K(x_1, \dots, x_d; \tilde{f}(AX)))) \rho(\tilde{f}(AX)) \sqrt{1 + |\nabla f|^2}(AX) dX}{\int_{D_o(x)} \rho(\tilde{f}(AX)) \sqrt{1 + |\nabla f|^2}(AX) dX} \\ &\leq \frac{(1 - \delta) |\det A| \int_{C_o} \text{Vol}_d(\text{cut}_E(v_1, \dots, v_d; \tilde{b}(X))) \rho(\tilde{f}(AX)) \sqrt{1 + |\nabla f|^2}(AX) dX}{\int_{D_o(x)} \rho(\tilde{f}(AX)) \sqrt{1 + |\nabla f|^2}(AX) dX}. \end{aligned}$$

Next, observe $1 \leq \sqrt{1 + |\nabla f|^2} \leq (1 + \delta)$ for h sufficiently small using Lemma 3.6.9. Further, as ρ is uniformly continuous on K we have

$$\frac{\min_{X \in C_o} \rho(AX)}{\max_{X \in C_o} \rho(AX)} \leq (1 - \delta)$$

for r sufficiently small. Thus, we have the lower bound

$$\begin{aligned} \mathbb{E}_{x_o}(\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; x_o))) &\geq \\ &\frac{(1 - \delta)^2 (1 + \delta)^{-1} |\det A| \int_{C_o} \text{Vol}_d(\text{cut}_E(v_1, \dots, v_d; \tilde{b}(X))) dX}{\int_{D_o(x)} dX}, \end{aligned}$$

and by the same type of argument

$$\mathbb{E}_{x_o}(\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; x_o))) \leq \frac{(1 + \delta)^2 |\det A| \int_{C_o} \text{Vol}_d(\text{cut}_E(v_1, \dots, v_d; \tilde{b}(X))) dX}{\int_{D_o(x)} dX}.$$

We let

$$\phi_1 = \frac{\int_{C_o} \text{Vol}_d(\text{cut}_E(v_1, \dots, v_d; \tilde{b}(X))) dX}{\int_{D_o(x)} dX}$$

and

$$\phi_2 = \frac{\int_{C_o} \text{Vol}_d(\text{cut}_E(v_1, \dots, v_d; \tilde{b}(X)))^2 dX}{\int_{D_o(x)} dX}.$$

Using this notation, we claim that we can also show

$$|\det A| \phi_2 (1 - \delta) \leq \mathbb{E}_{x_o}(\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; x_o)))^2 \leq |\det A| \phi_2 (1 + \delta).$$

Thus, we may bound

$$\begin{aligned} \text{Var}_{x_o}(\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; x_o))) &= \mathbb{E}_{x_o}(\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; x_o)))^2 - \mathbb{E}_{x_o}(\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; x_o)))^2 \\ &\geq |\det A| (\phi_2 (1 - \delta) - \phi_1 (1 + \delta)) = |\det A| ((\phi_2 - \phi_1) - \delta(\phi_2 - \phi_1)). \end{aligned}$$

By taking δ sufficiently small (making b perhaps smaller) the right hand side becomes positive, supposing that $\phi_2 - \phi_1$ is positive. But $\phi_2 - \phi_1$ is the variance of a random variable which is not almost everywhere constant, so in particular it is positive.

The upper bound on $\text{Var}_{x_o}(\text{Vol}_d(\text{cut}_K(x_1, \dots, x_d; x_o)))$ is similar, and this establishes the lemma.

3.5 A Poisson Central Limit Theorem

Given that the distribution of the mixed volume has sub-exponential tails, one may ask whether it obeys a central limit theorem. We show that, in a Poissonized variant of our model, this is the case.

If the boundary measure on K is denoted μ , then we shall construct a Poisson point process of intensity $n\mu$ on ∂K , denoted $\text{Pois}(n)$. Viewing $\text{Pois}(n)$ as a set of N points—here N is a Poisson random variable of mean n —we may form the *Poisson circumscribing polytope* $\Pi_n = P(\text{Pois}(n))$.

We may now state the main theorem of this section.

Theorem 3.5.1.

$$\left| \mathbb{P} \left(\frac{\text{Vol}_d(\Pi_n) - \mathbb{E} \text{Vol}_d(\Pi_n)}{\sqrt{\text{Var} \text{Vol}_d(\Pi_n)}} \leq x \right) - \Phi(x) \right| = O(n^{-\frac{1}{4}} \ln^{\frac{d+2}{d-1}} n). \quad (3.10)$$

We recall that $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ is the CDF of a standard Gaussian random variable.

Our method of proof relies on a central limit theorem, due to Baldi and Rinott [10], concerning dependency graphs. Consider a collection of random variables ξ_1, \dots, ξ_m . We form a graph $G = ([m], E)$. We say that G is a *dependency graph* if, given any two disjoint subsets $A_1, A_2 \subseteq [m]$, there is no edge in E from A_1 to A_2 , then the collection of random variables $\{\xi_i \mid i \in A_1\}$ and $\{\xi_i \mid i \in A_2\}$ are independent. We state here a form due to Baldi and Rinott, noting that Rinott gives an improved but more complicated version in [89].

Theorem 3.5.2 (Baldi-Rinott). *Let G be the dependency graph on random variables ξ_1, \dots, ξ_m . Set $\xi = \xi_1 + \dots + \xi_m$. If the maximal degree of G is at most D and we have the bound $|\xi_i| \leq B$ a.s., then*

$$\left| \mathbb{P} \left(\frac{\xi - \mathbb{E} \xi}{\sqrt{\text{Var}(\xi)}} \leq x \right) - \Phi(x) \right| \leq O(\sqrt{S}),$$

where $S = \frac{mD^2B^3}{\sqrt{\text{Var}(\xi)}}$.

Our strategy, in light of this theorem, is to divide the Poisson polytope Π_n into disjoint sets. This allows us to study the volume of Π_n as the sum of contributions from each disjoint set. The key to this scheme is the independence of the Poisson point process in disjoint sets. We shall see that this translates into very few dependences among the volume contributions.

We set, with foresight, the quantity

$$m = \left\lfloor \frac{d_6 \rho_o n}{k \ln n} \right\rfloor,$$

where d_6 is the constant given in Lemma 3.3.3, k is to be chosen, and $\rho_o = \min_{x \in \partial K} \rho(x)$. We then choose, according to Lemma 3.3.3, m points y_1, \dots, y_m along the boundary of K . These y_i dissect \mathbb{R}^d into Voronoi cells $\text{Vor}(y_i)$. We can thus form the quantities

$$Y_i = \text{Vol}_d(\text{Vor}(y_i) \cap \Pi_n) - \text{Vol}_d(\text{Vor}(y_i) \cap K), \quad i = 1, \dots, m,$$

and hence

$$Y = \sum_i Y_i = \text{Vol}_d(\Pi_n) - \text{Vol}_d(K). \quad (3.11)$$

Now, by Lemma 3.3.3 we see that each Voronoi cell $\text{Vor}(y_i)$ contains a cap C_i of K with volume given by

$$\text{Vol}_d(C_i) = \Theta(m^{-\frac{d+1}{d-1}}).$$

Hence, by Lemma 3.3.3

$$\mu(C_i \cap \partial K) \geq \rho_o d_6 m^{-1} \geq \frac{k \ln n}{n}. \quad (3.12)$$

We let $N_i, i = 1, \dots, m$, count the number of points given by the Poisson point process in $\Pi_n \cap \text{Vor}(y_i)$. This is given by a Poisson random variable with intensity $\lambda_i = n \mu(C_i \cap \partial K) \geq k \ln n$ according to (3.12). Hence,

$$\mathbb{P}(N_i = 0) = e^{-\lambda_i} = O(n^{-k}).$$

Observe also that

$$\mathbb{P}(N_i \geq 3\lambda_i) = \sum_{k=3\lambda_i}^{\infty} \frac{e^{-\lambda_i} \lambda_i^k}{k!} \leq \sum_{k=3\lambda_i}^{\infty} e^{-\lambda_i} \left(\frac{e\lambda_i}{k}\right)^k \leq \sum_{k=0}^{\infty} e^{-\lambda_i} \left(\frac{e}{3}\right)^k = O(n^{-\lambda_i}).$$

Thus, we obtain

$$\mathbb{P}(N_i \geq 3k \ln n) = O(n^{-k}).$$

Therefore, we can define \mathcal{A}^m to be the event that $1 \leq N_i \leq 3k \ln n, i = 1, \dots, m$, i.e. each cap contains a number of points from the Poisson process in this range. Hence, by the union bound we see that

$$\mathbb{P}(\mathcal{A}^m) = \mathbb{P}(1 \leq N_i \leq 3k \ln n) \geq 1 - \Omega(n^{-k+1}).$$

Next, we work conditioned upon \mathcal{A}^m . We show a central limit theorem for this restricted probability space, and following this we show that removing the condition does not have much effect on the overall estimate. But first, we need some notation. We shall use a tilde to denote the restricted probability space, i.e.

$$\tilde{\mathbb{P}}(\text{Vol}_d(\Pi_n) \leq x) = \mathbb{P}(\text{Vol}_d(\Pi_n) \leq x \mid \mathcal{A}^m),$$

and similarly for $\tilde{\mathbb{E}}$ and $\tilde{\text{Var}}$.

We have the following connection between \mathbb{P} and $\tilde{\mathbb{P}}$.

Lemma 3.5.3.

$$|\tilde{\mathbb{P}}(\text{Vol}_d(\Pi_n) \leq x) - \mathbb{P}(\text{Vol}_d(\Pi_n) \leq x)| = O(n^{-k+1}) \quad (3.13)$$

$$|\tilde{\mathbb{E}}(\text{Vol}_d(\Pi_n)) - \mathbb{E}(\text{Vol}_d(\Pi_n))| = O(n^{-k+1}) \quad (3.14)$$

$$|\tilde{\text{Var}}(\text{Vol}_d(\Pi_n)) - \text{Var}(\text{Vol}_d(\Pi_n))| = O(n^{-k+1}). \quad (3.15)$$

The proof of all three rely on the observation that for any events A and B we have $|\mathbb{P}(B \mid A) - \mathbb{P}(B)| \leq \mathbb{P}(\bar{A})$. Next, we require asymptotic bounds for the expectation and variance of Π_n .

Lemma 3.5.4. *We have*

$$\begin{aligned} \mathbb{E}[\text{Vol}_d(\Pi_n)] &= (1 + O(n^{-1} \ln^{5/2} n)) \mathbb{E}[\text{Vol}_d(P_n)] \\ &= (1 + O(n^{-1} \ln^{5/2} n))(1 - c(K, d)n^{-\frac{2}{d-1}}) \end{aligned} \quad (3.16)$$

where $c(K, d)$ is some constant depending only on K and d . Further,

$$\text{Var}(\text{Vol}_d(\Pi_n)) = \text{Var}(\text{Vol}_d(P_n))(1 + O(n^{-1/2} \ln^{\frac{3d+1}{2(d-1)}} n) + O(n^{-\frac{d+1}{4(d-1)}} \ln^{O(1)} n)). \quad (3.17)$$

Let us remark that the error terms are not strictly necessary to our Poisson CLT. However, these errors will be of critical importance in proving a CLT for P_n , and follow nicely from sharp concentration results we have already developed. The proof of this lemma is the primary content of the next section.

We are now set to prove a restricted central limit theorem.

Lemma 3.5.5.

$$\left| \tilde{\mathbb{P}} \left(\frac{\text{Vol}_d(\Pi_n) - \tilde{\mathbb{E}} \text{Vol}_d(\Pi_n)}{\sqrt{\tilde{\text{Var}} \text{Vol}_d(\Pi_n)}} \leq x \right) - \Phi(x) \right| = O \left(n^{-\frac{1}{4}} \ln^{\frac{d+2}{d-1}} n \right). \quad (3.18)$$

Proof. First, by (3.11) we see $\text{Vol}_d(\Pi_n) - \tilde{\mathbb{E}} \text{Vol}_d(\Pi_n) = \tilde{\mathbb{E}} Y - Y$. Further, $\tilde{\text{Var}} Y = \tilde{\text{Var}} \text{Vol}_d(\Pi_n)$. We can combine (3.15) and (3.17) with Theorem 3.4.1 to obtain $\tilde{\text{Var}} \text{Vol}_d(\Pi_n) = \Theta(n^{-\frac{d+2}{d-1}})$ for k sufficiently large. Hence, it will be enough to show that the central limit theorem holds for Y under $\tilde{\mathbb{P}}$.

Let us begin by defining the dependency graph on our random variables $(Y_i)_{i=1}^m$. Let $Y_i \sim Y_j$ if $\text{Vor}(y_i) \cap C(y_j, d_{14} m^{-\frac{2}{d-1}}) \neq \emptyset$, where here d_{14} is the constant given in Lemma 3.6.15. To see that this is indeed a dependency graph, we consider those $Y_i \not\sim Y_j$, so that $\text{Vor}(y_i) \cap C(y_j, d_{14} m^{-\frac{2}{d-1}}) = \emptyset$. We consider any two points p_1 and p_2 in the respective sets $\text{Vor}(y_i) \cap \partial K$ and $\text{Vor}(y_j) \cap \partial K$. Additionally, because we work under \mathcal{A}^m , there are points $(x_i)_{i=1}^m$ with $x_i \in C_i \cap \partial K$ from the Poisson point process. By Lemma 3.6.15, the supporting hyperplanes at p_1 and p_2 thus meet only at points greater than $\delta^H(K, P(\{x_1, \dots, x_m\}))$, hence any possible faces tangent to points p_1 and p_2 do not intersect. Hence, our variables Y_i and Y_j must be independent.

Finally, we estimate D and B in Theorem 3.5.2. For D , we note that Lemma 3.6.12 shows that any cap $C(y_i, d_{14} m^{-\frac{2}{d-1}})$, $i = 1, \dots, m$ intersects at most $O(1)$ Voronoi regions of the form $\text{Vor}(y_j)$, $j = 1, \dots, m$. Hence $D = O(1)$.

Next, by Lemma 3.3.3 we see that a cap of height at most $c_1 m^{-\frac{2}{d-1}}$ contains the intersection of $\text{Vor}(y_i) \cap \partial K$. Further, under \mathcal{A}^m the Hausdorff distance

$$\delta^H(K, P(\{x_1, \dots, x_m\})) = \delta(m) = c_2 m^{-\frac{2}{d-1}}$$

(here both c_1 and c_2 are constants depending only on K). Thus, we can bound Y_i by the volume cut off from the body $K + \delta(m)B(o, 1)$ by a cap of height $(c_1 + c_2)m^{-\frac{2}{d-1}}$.

Now by the second statement in Lemma 3.6.11 we see that this volume is $O(m^{-\frac{d+1}{d-1}})$, hence

$$Y_i = O(m^{-\frac{d+1}{d-1}}) = O\left(\left(\frac{\ln n}{n}\right)^{-\frac{d+1}{d-1}}\right) =: B.$$

We thus can conclude our lemma applying the theorem of Baldi and Rinott, noting that the error term is given by

$$S = \frac{mD^2B^3}{\sqrt{\text{Var}(\xi)^3}} = O(n^{-\frac{1}{2}}(\ln n)^{\frac{d+2}{d-1}}).$$

□

We are left with the task of removing the condition \mathcal{A}^m .

Proof of Theorem 3.5.1. First, to each real y we associate a \tilde{y} such that the following holds:

$$\mathbb{E}Y + y\sqrt{\text{Var}Y} = \tilde{\mathbb{E}}Y + \tilde{y}\sqrt{\tilde{\text{Var}}Y}.$$

We estimate

$$|y - \tilde{y}| \leq \left| y \frac{\sqrt{\text{Var}Y}}{\sqrt{\tilde{\text{Var}}Y}} - \tilde{y} \right| + \left| y - y \frac{\sqrt{\text{Var}Y}}{\sqrt{\tilde{\text{Var}}Y}} \right|$$

and use the (3.14), (3.15), and (3.17) to obtain.

$$|y - \tilde{y}| = O(n^{-k+1}) + |y|O(n^{-k+1+\frac{d+1}{2(d-1)}}). \quad (3.19)$$

Next, by (3.13) we have

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq \mathbb{E}Y + y\sqrt{\text{Var}Y}) = \tilde{\mathbb{P}}(Y \leq \tilde{\mathbb{E}}Y + \tilde{y}\sqrt{\tilde{\text{Var}}Y}) + O(n^{-4d+1}) \\ &= \Phi(\tilde{y}) + O(n^{-\frac{1}{4}} \ln^{\frac{d+2}{d-1}} n) + O(n^{-k+1}). \end{aligned}$$

Finally, examine the quantity $|\Phi(y) - \Phi(\tilde{y})|$. If $|y| \leq n$ then we have $|\Phi(y) - \Phi(\tilde{y})| \leq |y - \tilde{y}| = O(n^{-1})$ by (3.19). Additionally, there exists a positive constant $c < 1$ such that if $y > n$ then $\tilde{y} > cn$ and if $y < -n$ then $\tilde{y} < -cn$. Thus if $y > n$,

$$|\Phi(y) - \Phi(\tilde{y})| = \left| \int_{\tilde{y}}^y \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right| \leq \left| \int_{cy}^y \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right| = O(e^{-n^2c'}),$$

for some $c' > 0$ and similarly if $y < -n$.

Thus, $|F_Y(y) - \Phi(y)| = O(n^{-\frac{1}{4}} \ln^{\frac{d+2}{d-1}} n)$ by choosing k sufficiently large, which finishes the proof of Theorem 3.5.1. □

3.5.1 Proof of Lemma 3.5.4

We shall require the following technical result. A proof is provided at the end of this section.

Lemma 3.5.6. *For a fixed constant c ,*

$$\sum_{0 \leq m \leq \lfloor c\sqrt{n \ln n} \rfloor} \frac{e^{-n} n^{n+m}}{(n+m)!} - \frac{e^{-n} n^{n-m}}{(n-m)!} = O(n^{-1/2} \ln^2 n),$$

where the implicit constant depends only on the choice of c .

Proof of Lemma 3.5.4. We begin with a note on Poisson concentration. First, suppose n' is Poisson distributed with mean n . Then standard concentration inequalities give

$$\mathbb{P}\left(|n' - n| \geq A\sqrt{n \log n}\right) \leq n^{-A/4}, \quad A \geq 10. \quad (3.20)$$

The constants here are inconsequential; the important point is that by taking A sufficiently large we can guarantee that $n \in [n - A\sqrt{n \log n}, n + A\sqrt{n \log n}]$ with probability at least $1 - n^{-B}$ for B as large as desired.

Expectation: To prove the expectation result, examine the difference

$$\begin{aligned} & |\mathbb{E}[\text{Vol}_d(P_n \setminus K)] - \mathbb{E}[\text{Vol}_d(\Pi_n \setminus K)]| \\ &= \left| \mathbb{E}[\text{Vol}_d(P_n \setminus K)] - \sum_{m=-A\sqrt{n \ln n}}^{m=A\sqrt{n \ln n}} \mathbb{E}[\text{Vol}_d(P_{n+m} \setminus K)] \frac{e^{-n} n^{n+m}}{(n+m)!} - O(n^{-B}) \right|, \end{aligned} \quad (3.21)$$

where B depends only on the choice of A , using Poisson concentration.

Further, if ∂K has \mathcal{C}^k boundary then by [24] we can express

$$\begin{aligned} f(n) &:= \mathbb{E}[\text{Vol}_d(P_n \setminus K)] \\ &= c_2^K n^{-2/(d-1)} + c_3^K n^{-3/(d-1)} + \dots + c_{k-1}^K n^{-(k-1)/(d-1)} + O(n^{-k/(d-1)}) \end{aligned}$$

where the constants c_i^K depend only on K . Using a Taylor expansion for the non-error terms we find

$$\begin{aligned} f(n+m) &= \sum_{i=2}^{k-1} c_i^K n^{-i/(d-1)} + m \sum_{i=2}^{k-1} \frac{-i c_i^K}{d-1} n^{-i/(d-1)-1} \\ &\quad + O(m^2 n^{-2d/(d-1)}) + O((n+m)^{-k/(d-1)}) \end{aligned}$$

and

$$f(n-m) = \sum_{i=2}^{k-1} c_i^K n^{-i/(d-1)} - m \sum_{i=2}^{k-1} \frac{-i c_i^K}{d-1} n^{-i/(d-1)-1} \\ + O(m^2 n^{-2d/(d-1)}) + O((n-m)^{-k/(d-1)}).$$

We shall choose $k = d + 1$, which makes the error term $O(n^{-(d+1)/(d-1)})$.

Thus (3.21) becomes

$$\left| f(n) - \sum_{m=-A\sqrt{n \ln n}}^{m=A\sqrt{n \ln n}} f(n+m) \frac{e^{-n} n^{n+m}}{(n+m)!} - O(n^{-B}) - O(n^{-(d+1)/(d-1)}) \right| \\ = \left| f(n) O(n^{-B}) - \left(\sum_{i=2}^d \frac{-i c_i^K}{d-1} n^{-i/(d-1)-1} \right) \sum_{0 \leq m \leq A\sqrt{n \ln n}} m \left(\frac{e^{-n} n^{n+m}}{(n+m)!} - \frac{e^{-n} n^{n-m}}{(n-m)!} \right) \right. \\ \left. - O(n^{-B}) - O(n^{-(d+1)/(d-1)}) \right| \quad (3.22)$$

Now, we can make the n^{-B} terms negligible by choosing A sufficiently large. Bounding m by $A\sqrt{n \ln n}$ the term

$$\sum_{0 \leq m \leq A\sqrt{n \ln n}} m \left(\frac{e^{-n} n^{n+m}}{(n+m)!} - \frac{e^{-n} n^{n-m}}{(n-m)!} \right) = O(\ln^{5/2} n)$$

by Lemma 3.5.6. Thus, (3.22) is of order $O(n^{-(d+1)/(d-1)} \ln^{5/2} n)$, which gives our claimed result.

Variance: We shall borrow a construction from the proof of Theorem 3.5.5, namely the event \mathcal{A}^m . To summarize the earlier argument, under \mathcal{A}^m the volume cut off from P_n by a single tangent hyperplane to ∂K is bounded by $O(n^{-\frac{d+1}{d-1}} \ln^{\frac{d+1}{d-1}})$. Further, $\mathbb{P}(\mathcal{A}^m) \geq 1 - O(n^{-k})$ for any fixed k .

Next, let $n' > n$. We adopt a coupling argument. Let $S = \{x_1, \dots, x_n\}$ denote a set of n points in $\Omega = \partial K^n$, and $S' = S \cup \{x_{n+1}, \dots, x_{n'}\} = S \cup T$ a set of n' points in $\Omega' = \partial K^{n'}$. The key is that the first n points of S' agree with S . In this way, $Y_n = \text{Vol}_d(P(S))$ and $Y_{n'} = \text{Vol}_d(P(S'))$ (and similarly for the case $n' \leq n$). We denote by $\mathbf{x} = (x_1, \dots, x_{n'})$ a point in Ω' .

We shall assume that $|n' - n| \leq A\sqrt{n \ln n}$ for some constant to be chosen. Let us decompose

$$Y_{n'} = Y_n - Z$$

where evidently $Z = \text{Vol}_d(P(S)) \setminus \text{Vol}_d(P(S'))$. We show first a concentration result for Z , using a martingale argument following that of Theorem 3.3.1.

We begin with the following claim.

Claim 3.5.7. *For some $1 \leq i \leq n$, the probability that the hyperplane tangent to x_i intersects any hyperplane tangent to one of $x_{n+1}, \dots, x_{n'}$ at a distance less than h from K is at most $(n' - n)O(h^{\frac{d-1}{2}})$, assuming h is sufficiently small.*

Proof. Assume the hyperplanes of x_i and x_j intersect for some $n+1 \leq j \leq n'$. If the point of intersection, say y , is at a height h from K , then the visible region of y on the boundary is contained in some $O(h^{\frac{d-1}{2}})$ boundary cap. The distance $|x_i - x_j|$ is thus $O(\sqrt{h})$, and thus x_i must lie in some $O(h^{\frac{d-1}{2}})$ boundary cap about x_j .

Thus, the probability that x_i avoids such caps about each $x_{n+1}, \dots, x_{n'}$ is $(n' - n)O(h^{\frac{d-1}{2}})$. \square

Claim 3.5.8. *If $|n' - n| \leq A\sqrt{n \ln n}$ we have*

$$\text{Var}(Z) = O\left(n^{-\frac{3d+5}{2(d-1)}} \ln^{\frac{2(d+1)}{d-1}} n\right).$$

Proof of claim 3.5.8. We apply Lemma 1.3.4 with the space $\partial K^{n'}$.

There are two regimes we need to bound. For $1 \leq i \leq n$ we have

$$\begin{aligned} G_i(\mathbf{x}) &= |\mathbb{E}[Z \mid x_1, \dots, x_i] - \mathbb{E}[Z \mid x_1, \dots, x_{i-1}]| \\ &\leq \mathbb{E}_x |\mathbb{E}[Z \mid x_1, \dots, x_i] - \mathbb{E}[Z \mid x_1, \dots, x_{i-1}, x]| \end{aligned}$$

where here we let x be a random point replacing x_i in the second expression.

Now, denote by $L = \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ and recall $T = \{x_{n+1}, \dots, x_{n'}\}$. We then define

$$W_{x,L} = \text{Vol}_d(P(L)) - \text{Vol}_d(P(L \cup T)) - (\text{Vol}_d(P(L \cup \{x\})) - \text{Vol}_d(P(L \cup \{x\} \cup T)))$$

which allows us to write

$$\begin{aligned} \mathbb{E}[Z \mid x_1, \dots, x_i] &= \mathbb{E}[\text{Vol}_d(P(L)) - \text{Vol}_d(P(L \cup T)) \mid x_1, \dots, x_{i-1}] \\ &\quad - \mathbb{E}[W_{x,L} \mid x_1, \dots, x_{i-1}], \quad i = 1, \dots, n. \end{aligned} \quad (3.23)$$

Now, if \mathcal{A}^m holds, then by Lemma 3.6.15 then the Hausdorff distance between P_n and K is at most $O((n \ln^{-1} n)^{-\frac{2}{d-1}})$. Hence, by Claim 3.5.8 the probability that the tangent hyperplane at x_i intersects any tangent hyperplane at one of $x_{n+1}, \dots, x_{n'}$ is $O(|n' - n|(n \ln^{-1} n)^{-1})$, and we label such an event with \mathcal{E}' . Without such an intersection, we have $W_{x,L} = 0$. We can thus upper bound $W_{x,L}$ with the following

$$W_{x,L} \leq O((n \ln^{-1} n)^{-\frac{d+1}{d-1}}) \mathbb{1}_{\mathcal{A}^m \wedge E} + C \mathbb{1}_{(\mathcal{A}^m \wedge \mathcal{E}')^c}.$$

As the position of $x_{n+1}, \dots, x_{n'}$ is independent of that of x_i , the events \mathcal{A}^m and \mathcal{E}' are independent. Thus we have

$$\mathbb{E}[W_{x,L} \mid x_1, \dots, x_{i-1}] \leq O((n \ln^{-1} n)^{-\frac{d+1}{d-1}}) \mathbb{P}(\mathcal{A}^m \mid x_1, \dots, x_{i-1}) \mathbb{P}(\mathcal{E}' \mid x_1, \dots, x_{i-1}) \quad (3.24)$$

$$\begin{aligned} &+ C \mathbb{P}((\mathcal{A}^m \wedge \mathcal{E}')^c \mid x_1, \dots, x_{i-1}) \\ &\leq O((n \ln^{-1} n)^{-\frac{d+1}{d-1}}) \mathbb{P}(E) + C \mathbb{P}((\mathcal{A}^m \wedge \mathcal{E}')^c \mid x_1, \dots, x_{i-1}) \\ &\leq O((n \ln^{-1} n)^{-\frac{d+1}{d-1}}) O(|n' - n|(n \ln^{-1} n)^{-1}) \quad (3.25) \end{aligned}$$

$$\begin{aligned} &+ C \mathbb{P}((\mathcal{A}^m \wedge \mathcal{E}')^c \mid x_1, \dots, x_{i-1}) \\ &= O(n^{-\frac{d+1}{d-1} - \frac{1}{2}} \ln^{\frac{d+1}{d-1} + \frac{3}{2}}) + C \mathbb{P}((\mathcal{A}^m \wedge \mathcal{E}')^c \mid x_1, \dots, x_{i-1}). \quad (3.26) \end{aligned}$$

In the last line we use the assumption that $|n' - n| = O(\sqrt{n \ln n})$.

Thus we have

$$\begin{aligned} G_i(\mathbf{x}) &\leq \mathbb{E}_x |\mathbb{E}[Z \mid x_1, \dots, x_i] - \mathbb{E}[Z \mid x_1, \dots, x_{i-1}, x]| \\ \text{by (3.23)} &= \mathbb{E}_x |\mathbb{E}[W_{x_i,L} \mid x_1, \dots, x_i] - \mathbb{E}[W_{x,L} \mid x_1, \dots, x_{i-1}, x]| \\ &= \mathbb{E}[W_{x_i,L} \mid x_1, \dots, x_i] + \mathbb{E}_x \mathbb{E}[W_{x,L} \mid x_1, \dots, x_{i-1}, x] \\ \text{by (3.26)} &\leq 2(O(n^{-\frac{d+1}{d-1} - \frac{1}{2}} \ln^{\frac{d+1}{d-1} + \frac{3}{2}}) + C \mathbb{P}((\mathcal{A}^m \wedge \mathcal{E}')^c \mid x_1, \dots, x_{i-1})). \end{aligned}$$

Now, our argument follows that of Theorem 3.3.1. We say $\{x_1, \dots, x_{i-1}\}$ is typical if $\mathbb{P}((\mathcal{A}^m \wedge \mathcal{E}')^c \mid x_1, \dots, x_{i-1}) \leq n^{-4}$. For a typical set

$$G_i(\mathbf{x}) \leq c n^{-\frac{d+1}{d-1} - \frac{1}{2}} \ln^{\frac{d+1}{d-1} + \frac{3}{2}} n$$

for some absolute constant c .

Similarly,

$$V_i(\mathbf{x}) = \int_{\partial K} G_i(\mathbf{x})^2 dx_i \leq c^2 n^{-1 - \frac{2(d+1)}{d-1}} \ln^{\frac{2(d+1)}{d-1} + 3} n$$

for typical sets.

The probability that x_1, \dots, x_{i-1} is not typical is n^{-k} for any fixed k one may choose, by an application of Lemma 3.3.4.

For the regime $i > n$, our argument is exactly as in Theorem 3.3.1. That is, we show that for typical sets (with n^{-7} instead of n^{-4}) we obtain

$$G_i(\mathbf{x}) \leq O(n^{-\frac{d+1}{d-1}} \ln^{\frac{d+1}{d-1}} n)$$

and

$$V_i(\mathbf{x}) \leq O(n^{-\frac{2(d+1)}{d-1}} \ln^{\frac{2(d+1)}{d-1}} n).$$

We thus find that for typical sets we have

$$V(\mathbf{x}) = \sum_{i=1}^n V_i(\mathbf{x}) + \sum_{i=n+1}^{n'} V_i(\mathbf{x}) \leq n O(n^{-1 - \frac{2(d+1)}{d-1}} \ln^{\frac{2(d+1)}{d-1} + 3} n) \quad (3.27)$$

$$\begin{aligned} &+ A \sqrt{n \ln n} O(n^{-\frac{2(d+1)}{d-1}} \ln^{\frac{2(d+1)}{d-1}} n) \\ &= O(n^{-\frac{3d+5}{2(d-1)}} \ln^{\frac{2(d+1)}{d-1}} n). \end{aligned} \quad (3.28)$$

Thus, we find a concentration result of the form

$$\mathbb{P}(|Z - \mathbb{E}Z| \sqrt{\lambda V_0}) \leq 2 \exp(-\lambda/4) + O(n^{-k}), \quad \lambda \leq c' \sqrt{n} \ln^{O(1)} n$$

for any fixed k and for V_0 given by (3.28). This implies, by an elementary argument (see the proof of Corollary 3.3.5) that $\text{Var}(Z) = O(n^{-\frac{3d+5}{2(d-1)}} \ln^{\frac{2(d+1)}{d-1}} n)$. \square

Now, note that

$$\begin{aligned} \text{Var}(P_{n'}) &= \text{Var}(P_n - Z) = \text{Var}(P_n) + \text{Var}(Z) + \sqrt{\text{Var}(P_n) \text{Var}(Z)} \text{Cor}(P_n, Z) \\ &\leq \text{Var}(P_n) + \text{Var}(Z) + \sqrt{\text{Var}(P_n) \text{Var}(Z)}, \end{aligned}$$

as $|\text{Cor}(P_n, Z)| \leq 1$.

By the above claim we thus have

$$\text{Var}(P_{n'}) = \text{Var}(P_n) (1 + O(n^{-\frac{d+1}{4(d-1)}} \ln^{O(1)} n)) \quad (3.29)$$

if $|n' - n| \leq A\sqrt{n \ln n}$.

Finally, we denote by the event $\mathcal{E}_{n'}$ the event that the number of points in Π_n takes the value n' , and \mathcal{E}_o the event that such a random variable avoids the interval $[n - A\sqrt{n \ln n}, n + A\sqrt{n \ln n}]$. By choosing A sufficiently large we have $\mathbb{P}(\mathcal{E}_o) = O(n^{-k})$ for any fixed k . Thus, the conditional variance formula gives

$$\text{Var}(\text{Vol}_d(\Pi_n)) = \mathbb{E}_{n'} \text{Var}(\text{Vol}_d(\Pi_n) | \mathcal{E}_{n'}) + \text{Var}_{n'} \mathbb{E}[\text{Vol}_d(\Pi_n) | \mathcal{E}_{n'}],$$

where we let $n' \in [n - A\sqrt{n \ln n}, n + A\sqrt{n \ln n}]$ or $n' = o$.

Now, as $\text{Vol}_d(\Pi_n) = \text{Vol}_d(K'_n)$ on $\mathcal{E}_{n'}$, we can apply (3.16) and (3.29) to conclude

$$\text{Var}(\text{Vol}_d(\Pi_n)) = \text{Var}(\text{Vol}_d(P_n))(1 + O(n^{-1/2} \ln^{\frac{3d+1}{2(d-1)}} n) + O(n^{-\frac{d+1}{4(d-1)}} \ln^{O(1)} n)). \quad \square$$

Proof of Lemma 3.5.6. We can factor the summand above to give

$$\frac{e^{-n} n^{n-m}}{(n-m)!} \left(\frac{n^{2m}}{(n+m)_{2m}} - 1 \right),$$

where $(n)_k = n \cdot (n-1) \cdots (n-k+1)$, and thus we concentrate on the quantity $\frac{n^m}{(n-m)_{2m+1}}$.

If $I \subseteq [-m, m]$, we let $\prod I = \prod_{i \in I} i$, then observe

$$\left(\frac{n^m}{(n-m)_{2m+1}} \right)^{-1} = \prod_{k=-m}^m \left(1 + \frac{k}{n} \right) = 1 + n^{-1} \sum_{k=-m}^m k + \sum_{j=2}^{2m+1} n^{-j} \sum_{\substack{I \subseteq [-m, m] \\ |I|=j}} I.$$

By the symmetry of the interval, $\sum_{k=-m}^m k = 0$. Indeed, for j odd, it is easy to see that the sum $\sum_{\substack{I \subseteq [-m, m] \\ |I|=j}} I = 0$ (indeed, for a term of the form $k_1 k_2 \cdots k_j$ there is a term corresponding to $(-k_1)(-k_2) \cdots (-k_j) = -k_1 k_2 \cdots k_j$, so the terms cancel).

For $j = 2l > 0$ and even, similar cancellation arguments show that

$$\sum_{\substack{I \subseteq [-m, m] \\ |I|=j}} I = \sum_{i_1 < i_2 < \dots < i_l \in [-m, m]} (-2)^l i_1^2 i_2^2 \cdots i_l^2. \quad (3.30)$$

We can thus bound the magnitude of the above sum (3.30) by $2^l m^{2l} (2m+1)^l \leq 5^l (m^{3l})$ for n sufficiently large (we can replace 5 here with any constant greater than 4).

Thus, we have

$$\begin{aligned}
\left| 1 - \prod_{k=-m}^m \left(1 + \frac{k}{n} \right) \right| &= \sum_{l=1}^{\lfloor (2m+1)/2 \rfloor} n^{-2l} \sum_{\substack{I \subset [-m, m] \\ |I|=2l}} I \\
&\leq \sum_{l=1}^{\infty} \frac{5^l m^{3l}}{n^{2l}} \\
&= \frac{1}{1 - \frac{5m^3}{n^2}} - 1 = O(m^3 n^{-2}).
\end{aligned}$$

Thus, we have that

$$\left| \frac{n^m}{(n+m)_{2m+1}} - 1 \right| = \left| \frac{1}{1 + O(m^3 n^{-2})} - 1 \right| = O(m^3 n^{-2}) \quad (3.31)$$

and so

$$\begin{aligned}
\sum_{0 \leq m \leq \lfloor c\sqrt{n \ln n} \rfloor} \frac{e^{-n} n^{n+m}}{(n+m)!} - \frac{e^{-n} n^{n-m}}{(n-m)!} &\leq (2m+1) \times \\
&\quad \max_{0 \leq m \leq \lfloor c\sqrt{n \ln n} \rfloor} \frac{e^{-n} n^{n-m}}{(n-m)!} \left| \frac{n^m}{(n+m)_{2m+1}} - 1 \right| \\
&\quad (\text{by (3.31)}) \leq O(m^4 n^{-2}) \max_{0 \leq m \leq \lfloor c\sqrt{n \ln n} \rfloor} \frac{e^{-n} n^{n-m}}{(n-m)!} \\
&\quad (\text{by Stirling's approximation}) = O(m^4 n^{-2}) O(n^{-1/2}) = O(n^{-1/2} \ln^2 n). \quad \square
\end{aligned}$$

3.6 A Stronger Tail Inequality

The proof of Theorem 3.3.2 differs in concept from the proof of Theorem 3.3.1 by taking more careful account of the configurations x and L where $\Delta_{x,L}$ is large. The technique we use was first introduced in [104] for the traditional random polytope model. The key geometric objects were caps, but used in a different manner.

Our proof requires some preliminaries, and we break it into three pieces. We first discuss some additional geometry necessary for the circumscribed case. We then prove a key estimate, and finally, we apply our results to Lemma 1.3.4.

First, we state a more technical version of our theorem, which is what we prove here.

Theorem 3.6.1. *For a given K there exists constants c, c', ϵ_o such that for $c \ln n/n \leq \epsilon \leq \epsilon_o$, $V_o \geq cn^{-\frac{d+3}{d-1}}$, $G_o \geq c' \epsilon^{\frac{d+1}{d-1}}$ then for $0 \leq \lambda \leq V_o/4G_o^2$ we have*

$$\mathbb{P}(|\text{Vol}_d(P_n) - \mathbb{E} \text{Vol}_d(P_n)| \geq \sqrt{\lambda V_o}) \leq 2 \exp(-\lambda/4) + p_{\text{err}}.$$

Here,

$$p_{\text{err}} = \exp(-\Omega(\epsilon n)) + \exp(-\Omega(n^{\frac{d-1}{3d+1}})).$$

Note that we can set $\epsilon = n^{\frac{2d+2}{3d+1}}$ in the above so that the exponents of both exponentials in p_{err} are the same. Thus, $p_{\text{err}} = \exp(-\Omega(n^{\frac{d-1}{3d+1}}))$ for this choice, and we find $\lambda \leq V_o/4G_o^2 \leq c'' n^{\frac{d^2-5d+1}{(3d+1)(d-1)}}$. Note that the exponent is positive for $d \geq 5$.

3.6.1 Additional Geometry

We shall require a few general facts regarding the geometry of the boundary. In what follows we shall implicitly invoke the fact that a δ -visible region $G(x, s)$ induces a crown 1) of diameter $\Theta(\delta^{\frac{1}{d+1}})$ and 2) projection along the line normal to ∂K at x of length $\Theta(s) = \Theta(\delta^{\frac{2}{d+1}})$. These estimates can be proved, for example, by an appeal to 3.6.9. Here and in what follows we assume that δ is sufficiently small.

Next, we need a ‘‘economical cap covering’’ for crowns. We have

Lemma 3.6.2. *There are constants c_1, c_2, c_3 such that for any $\delta > 0$ there exists disjoint crowns C_1, \dots, C_m , $m \leq c_1 \delta^{-1}$ with the property that any δ -crown contains at least one C_i . Further, $c_2 \delta \leq \text{Vol}_{d-1}(C_i) \leq c_3 \delta$, $i = 1, \dots, m$.*

Lemma 3.6.3. *There are constants c_4, c_5, c_6 such that for any $\delta > 0$ there exists crowns C_1, \dots, C_m , $m \leq c_4 \delta^{-1}$ with the property that any δ -crown is contained inside some C_i . Further, $c_5 \delta \leq \text{Vol}_{d-1}(C_i) \leq c_6 \delta$, $i = 1, \dots, m$.*

Remark 3.6.4. The proof of these two lemmas is via a simple packing argument and our note on the diameter of crowns. The term ‘‘economical cap covering’’ is due to Bárány and Larman [13] referring to a similar result regarding cap systems. In the case of caps, the smoothness assumption is not necessary and the proof requires a more intricate analysis using so-called Macbeath regions. The interested reader is referred to the recent survey [12].

If L is a finite subset of ∂K , we shall let $E_{\delta,L}$ be the union of all δ -crowns.

Lemma 3.6.5. *There is a positive constant c_7 such that for any finite set L and sufficiently small $\delta > 0$, $E_{\delta,L}$ contains at least $\lfloor c_7 \delta^{-1} \text{Vol}_{d-1}(E_{\delta,L}) \rfloor$ disjoint δ -crowns which avoid L .*

Proof. Let C_1, \dots, C_m be a maximal system of disjoint δ -crowns in ∂K . Further, let \mathcal{C}_i be the union of all δ -crowns intersecting C_i . By our note on the diameter of δ -crowns, $\text{Vol}_{d-1}(\mathcal{C}_i) = O(\delta)$.

Now, by maximality we have $E_{\delta,L} \subset \bigcup_{i=1}^m \mathcal{C}_i$, and hence

$$\text{Vol}_{d-1}(E_{\delta,L}) \leq \sum_{i=1}^m \text{Vol}_d(\mathcal{C}_i) = O(m\delta),$$

hence $m = \Omega(\delta^{-1} \text{Vol}_{d-1}(E_{\delta,L}))$. □

Lemma 3.6.6. *There exists a constant c_8 such that for a finite set L and any x such that $\Delta_{x,L} \geq \delta$, we have $x \in E_{c_8 \delta^{\frac{d-1}{d+1}}, L}$.*

Proof. Assume that any visible region $G(y, s)$ containing x and avoiding L has volume at most ε , and hence the diameter of $G(y, s) \cap \partial K$ is $O(\varepsilon^{\frac{1}{d+1}})$.

Next, note that any point in the interior of $P(L) - P(L \cup \{x\})$ is the peak of some visible region which both contains x (by convexity) and avoids L (because it is an interior point). Thus, the union

$$\Upsilon(x, L) = \bigcup_{\substack{G(y,s) \ni x \\ G(y,s) \cap L = \emptyset}} G(y, s)$$

of all visible regions with these two properties contains all but a measure zero set of $P(L) - P(L \cup \{x\})$. Further, the diameter of $\Upsilon(x, L) \cap \partial K$ is $O(\varepsilon^{\frac{1}{d+1}})$ since this is true for each visible region in the union and every such region shares x . As $\Upsilon(x, L)$ does not extend more than $O(\varepsilon^{\frac{2}{d+1}})$ from x in the direction of u_x (as none of the visible regions may), this shows that $\text{Vol}_d(\Upsilon(x, L)) = O(\varepsilon)$, and hence $\text{Vol}_d(\Delta_{x,L}) \leq \text{Vol}_d(\Upsilon(x, L)) = O(\varepsilon)$.

We have thus shown that there is some positive constant c such that if $\Delta_{x,L} \geq \varepsilon$ then some visible region $G(y, s)$ containing x and avoiding L has volume at least $c\varepsilon$ for a fixed positive c . Hence, by Lemma 3.6.14 there is a crown of volume at least $c_8 \varepsilon^{\frac{d-1}{d+1}}$ which contains x and avoids L for some c_8 . \square

For a finite set $L \subset \partial K$ and point x , we say that x is δ -large with respect to L if $\Delta_{x,L} \geq \delta$. Put

$$U_{\delta,L} = \{x \in \partial K \mid x \text{ is } \delta \text{ large with respect to } L\}.$$

The next lemma shows that for large δ , $\text{Vol}_{d-1}(U_{\delta,L})$ is rarely large.

Lemma 3.6.7. *Let L be a set of n iid points selected on ∂K according to the boundary measure. There exist constants c, c', c'' , and c''' such that if*

$$T \geq \max \left\{ c''' \exp(-c' \delta^{\frac{d-1}{d+1}} n), c'' \delta^{\frac{d-1}{d+1}} \right\}$$

then

$$\mathbb{P}(\text{Vol}_{d-1}(U_{L,\delta}) \geq T) \leq \exp(-cnT).$$

Proof. Our goal is to show that the set $U_{L,\delta}$ is small. As $U_{\delta,L} \subset E_{c_8 \delta^{\frac{d-1}{d+1}}, L}$ by Lemma 3.6.6, we shall instead bound the probability

$$\mathbb{P} \left(\text{Vol}_{d-1} \left(E_{c_8 \delta^{\frac{d-1}{d+1}}, L} \right) \geq T \right).$$

Using Lemma 3.6.2, let us fix a system of crowns C_1, \dots, C_m with the property that any $c_8 \delta^{\frac{d-1}{d+1}}$ crown contains at least one C_i . Note that this choice is independent of the set L .

Now we shall assume that $\text{Vol}_{d-1} \left(E_{c_8 \delta^{\frac{d-1}{d+1}}, L} \right) \geq T$. In this case we may find a maximum disjoint set of $c_8 \delta^{\frac{d-1}{d+1}}$ -crowns D_1, \dots, D_l contained in $E_{c_8 \delta^{\frac{d-1}{d+1}}, L}$ by Lemma 3.6.5, hence $l = \Omega(\delta^{-\frac{d-1}{d+1}} T)$. To guarantee that l is at least one we shall assume $T \geq c'' \delta^{\frac{d-1}{d+1}}$. By construction every D_i contains some crown C_{j_i} , thus implying that there is a size l subset of C_1, \dots, C_m which avoids L .

The probability that the n points of L miss some fixed C_i is at most

$$\left(1 - c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}}\right)^n \leq \exp\left(-c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}} n\right),$$

where $\rho_o = \min_{x \in \partial K} \rho(x)$. Using the union bound, the probability that n random points miss some size l subsystem of C_1, \dots, C_m is at most

$$\begin{aligned} \binom{m}{l} \exp\left(-c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}} n\right)^l &= \binom{m}{l} \exp\left(-c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}} n l\right) \\ &\leq \left(\frac{em}{l}\right)^l \exp\left(-c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}} n l\right) \\ &= \exp\left(l \left(-c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}} n + \ln \frac{em}{l}\right)\right). \end{aligned}$$

Now, as $l = \Omega\left(\delta^{-\frac{d-1}{d+1}} T\right)$ and $m = O\left(\delta^{-\frac{d-1}{d+1}}\right)$, we have

$$\ln \frac{em}{l} \leq \ln c_9 T^{-1}$$

for some $c_9 > 0$. Further, we have

$$\ln c_9 T^{-1} \leq \frac{1}{2} c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}} n$$

if we assume $T \geq c_9 \exp\left(-\frac{1}{2} c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}} n\right)$. Thus, we see that

$$\exp\left(l \left(-c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}} n + \ln \frac{em}{l}\right)\right) \leq \exp\left(-\frac{1}{2} c_2 c_8 \rho_o \delta^{\frac{d-1}{d+1}} n l\right) \leq \exp(-cnT)$$

for some positive constant c . □

3.6.2 Proof

We begin by mimicking the proof of Theorem 3.3.1. We thus need to bound

$$\mathbb{P}(V(t) \geq V_o \text{ or } G(t) \geq G_o).$$

We do this by bounding separately the quantities

$$\mathbb{P}(V(t) \geq V_o) \text{ and } \mathbb{P}(G(t) \geq G_o).$$

The quantity $\mathbb{P}(G(t) \geq G_o)$ is handled as in the proof of Theorem 3.3.1. In particular, if $\epsilon_o \geq \epsilon = \Omega(\frac{\ln n}{n})$ then $\mathbb{P}(G(t) \geq G_o) \leq \exp(-\Omega(\epsilon n))$ for a choice of $G_o = \Omega(\epsilon^{\frac{d+1}{d-1}})$. This follows by using the same argument as in Theorem 3.3.1 with an appropriate choice of m , the number of caps involved.

Now let us focus on $\mathbb{P}(V(t) \geq V_o)$. Thus, we shall set $m = n^\beta$, $0 < \beta < 1$. This gives a cap system as in Lemma 3.3.3. We label the event that given n randomly selected points $\{x_1, \dots, x_n\}$ from the boundary one such cap has one or less points as \mathcal{B} . We see that $\mathbb{P}(\mathcal{B}) = \exp(-\Omega(n^{1-\beta}))$. The key property we need is that under \mathcal{B}^c , if $L = \{x_1, \dots, x_{i-1}\} \cup \{x_{i+1}, \dots, x_n\}$ then $\Delta_{L, x_i} \leq O(n^{-\frac{\beta(d+1)}{d-1}})$ for any i .

Setting $V_o = n^{-\frac{d+1}{d-1}}$, we thus focus on the following claim.

Claim 3.6.8. Fix $\frac{2d+2}{3d+1} \leq \beta < 1$. For each $1 \leq i \leq n$,

$$\mathbb{P}(V_i(t) \geq n^{-1} V_o) \leq \exp(-\Omega(n^{\frac{d-1}{3d+1}}))$$

The bound $\mathbb{P}(V(t) \geq V_o)$ then follows from the union bound.

Proof of claim 3.6.8. Now, there is some constant c such that $\Delta_{x,L} \leq cn^{-\frac{\beta(d+1)}{d-1}}$ whenever \mathcal{B}^c holds.

We begin by setting $T_o = c'$, $\delta_o = c'n^{-\frac{d+1}{d-1}}$, $\delta_j = 2^j \delta_o$, and $T_j = (1+j)^{-2} 4^{-j} T_o$, c' to be determined. We further fix j_o to be the minimum integer such that $\delta_{j_o} \geq cn^{-\frac{\beta(d+1)}{d-1}}$. One can check that for $j \leq j_o$ the pairs (T_j, δ_j) satisfy the hypotheses of Lemma 3.6.7, setting c' sufficiently large.

Now, recall from the proof of Theorem 3.3.1 that

$$G_i(t) \leq \mathbb{E}[\Delta_{x_i, L} | x_1, \dots, x_{i-1}] + \mathbb{E}_x \mathbb{E}[\Delta_{x, L} | x_1, \dots, x_{i-1}].$$

We thus compute

$$\begin{aligned} V_i(t) &= \int_{\partial K} G_i(t)^2 dx_i \\ &\leq \int_{\partial K} (\mathbb{E}[\Delta_{x_i, L} | x_1, \dots, x_{i-1}] + \mathbb{E}_x \mathbb{E}[\Delta_{x, L} | x_1, \dots, x_{i-1}])^2 dx_i \\ &\leq 4 \int_{\partial K} \mathbb{E}[\Delta_{x_i, L} | x_1, \dots, x_{i-1}]^2 dx_i. \end{aligned}$$

In the last line we simply expand the square and apply Hölder's inequality. \square

Next, we say a set $L = \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$ is *good* if $\text{Vol}_{d-1}(U_{L,\delta}) \leq T_j$ for $1 \leq j \leq j_0$ and if \mathcal{B}^c holds for L . We say a set $\{x_1, \dots, x_{i-1}\}$ is *typical* if

$$\mathbb{P}_{\Omega^{<i+1>}}(L \text{ is not good} \mid x_1, \dots, x_{i-1}) \leq n^{-7}.$$

We now conclude by first bounding $V_i(t)$ on typical sets and then bounding the probability associated to non-typical sets.

Typical Sets: Assuming that $\{x_1, \dots, x_{i-1}\}$ is typical, we then have

$$\int_{\partial K} \mathbb{E}(\Delta_{x_i,L}^2 \mid x_1, \dots, x_{i-1})^2 dx_i = \int_{\Omega^{<i+1>}} \int_{\partial K} \Delta_{x_i,L}^2 dx_i dt^{<i+1>}$$

We further split $\Omega^{<i+1>}$ into two disjoint subsets Ω_G and Ω_B consisting of those $\{x_{i+1}, \dots, x_n\}$ such that L is good and not good (bad), respectively. Hence,

$$\int_{\Omega^{<i+1>}} \int_{\partial K} \Delta_{x_i,L}^2 dx_i dt^{<i+1>} = \int_{\Omega_G} \int_{\partial K} \Delta_{x_i,L}^2 dx_i dt^{<i+1>} + \int_{\Omega_B} \int_{\partial K} \Delta_{x_i,L}^2 dx_i dt^{<i+1>}.$$

If C denotes the large box we assume to enclose K , then we may bound the second term above by

$$\int_{\Omega_B} \int_{\partial K} \Delta_{x_i,L}^2 dx_i dt^{<i+1>} \leq C \mathbb{P}(L \text{ is not good} \mid x_1, \dots, x_{i-1}) \leq C n^{-7}.$$

Finally, we estimate

$$\begin{aligned} \int_{\partial K} \Delta_{x_i,L}^2 dx_i &\leq \delta_o^2 + \sum_{j=0}^{\infty} \delta_{j+1}^2 \text{Vol}_{d-1}(U_{L,\delta_j}) \\ &\leq \delta_o^2 T_o + \sum_{j=1}^{j_0} \delta_{j+1}^2 T_j \\ &= (c')^2 n^{-\frac{2(d+1)}{d-1}} + \sum_{j=0}^{j_0} (c')^2 4^{j+1} n^{-\frac{2(d+1)}{d-1}} (j+1)^{-2} 4^{-j} \\ &\leq (c')^2 n^{-\frac{d+1}{d-1}} + (c')^2 n^{-\frac{2(d+1)}{d-1}} \sum_{j=0}^{\infty} 4(j+1)^{-2} \\ &= (c')^2 (1 + 4\pi^2/6) n^{-\frac{2(d+1)}{d-1}}. \end{aligned}$$

Here we use the fact that L is good. In particular, note that by construction we can ignore terms involving δ_j for $j > j_0$ since $\Delta_{x,L} \leq \delta_{j_0}$ whenever L is nice.

Thus, we have

$$\int_{\Omega_G} \int_{\partial K} \Delta_{x_i, L}^2 dx_i dt^{<i+1>} \leq \int_{\partial K} \Delta_{x_i, L}^2 dx_i \leq O(n^{-\frac{2(d+1)}{d-1}})$$

and hence

$$\begin{aligned} \int_{\Omega^{<i+1>}} \int_{\partial K} \Delta_{x_i, L}^2 dx_i dt^{<i+1>} &= \int_{\Omega_G} \int_{\partial K} \Delta_{x_i, L}^2 dx_i dt^{<i+1>} + \int_{\Omega_B} \int_{\partial K} \Delta_{x_i, L}^2 dx_i dt^{<i+1>} \\ &\leq O(n^{-\frac{2(d+1)}{d-1}}) + Cn^{-7} = O(n^{-\frac{2(d+1)}{d-1}}). \end{aligned}$$

Thus we have $V_i(t) \leq n^{-1}V_0$ for a sufficiently large constant c' .

Non-typical Sets: The probability that a set L is not good is bounded by

$$\mathbb{P}(\Omega(-n^{1-\beta})) + \sum_{j=0}^{j_0} \mathbb{P}(-\Omega(nT_j)).$$

The first term comes from the bound on $\mathbb{P}(\mathcal{B})$ and the latter comes from Lemma 3.6.7.

It is clear that $nT_j \geq nT_{j_0} = \Omega(n^{1-2(1-\beta)\frac{d+1}{d-1}})$. Setting $\beta = \frac{2d+2}{3d+1}$ forces the exponent in both error terms to be the same, so we have

$$\mathbb{P}(\Omega(-n^{1-\beta})) + \sum_{j=0}^{j_0} \mathbb{P}(-\Omega(nT_j)) = \exp(-\Omega(n^{\frac{d-1}{3d+1}})).$$

Finally, an application of Lemma 3.3.4 combined with the above shows

$$\mathbb{P}(L \text{ is not good} \mid t_1, \dots, t_{i-1}) \leq n^7 \exp(-\Omega(n^{\frac{d-1}{3d+1}})) = \exp(-\Omega(n^{\frac{d-1}{3d+1}}))$$

for an appropriate constant.

Appendix: Geometry

For $K \in \mathcal{K}_+^2$, at each point $x \in \partial K$ there is a unique paraboloid Q_x , given by a quadratic form b_x , osculating ∂K at x . We may describe Q_x and b_x by identifying the tangent hyperplane of ∂K at x with \mathbb{R}^{d-1} , which gives each point $y \in \mathbb{R}^{d-1}$ the form (y^1, \dots, y^{d-1}) . For some neighborhood about x , we can represent ∂K as the graph of

a \mathcal{C}^2 , convex function $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, i.e. near x each point in ∂K can be written in the form $(y, f_x(y))$. Thus, we may write

$$b_x(y) = \frac{1}{2} \sum_{1 \leq i, j \leq d-1} \frac{\partial^2 f_x}{\partial y^i \partial y^j}(\mathfrak{o}) y^i y^j,$$

and

$$Q_x = \{(y, z) \mid z \geq b_x(y), y \in \mathbb{R}^{d-1}, z \in \mathbb{R}\}.$$

The main thrust of the above is that these paraboloids approximate the boundary structure. The formulation given here is due to Reitzner, who provides a proof in [79]. The spirit here is that not only are paraboloids good approximations for the boundary, but these approximations are uniform.

Lemma 3.6.9. *Let $K \in \mathcal{K}_+^2$ and choose $\delta > 0$ sufficiently small. Then there exists a $\lambda > 0$, depending only on δ and K , such that for each point $x \in \partial K$ the following holds: If we identify the tangent hyperplane to ∂K at x with \mathbb{R}^{d-1} , then we may define the λ -neighborhood U^λ of $x \in \partial K$ by $\text{proj } U^\lambda = B(\mathfrak{o}, \lambda)$. U^λ can be represented by a convex function $f_x(y) \in \mathcal{C}^2$, for $y \in B(\mathfrak{o}, \lambda)$.*

$$(1 + \delta)^{-1} b_x(y) \leq f_x(y) \leq (1 + \delta) b_x(y), \quad (3.32)$$

$$\sqrt{1 + |\nabla f_x(y)|^2} \leq (1 + \delta) \quad (3.33)$$

and

$$(1 + \delta)^{-1} 2b_x(y) \leq (y, \mathfrak{o}) \cdot n_K(y) \leq (1 + \delta) 2b_x(y), \quad (3.34)$$

for $y \in B(\mathfrak{o}, \lambda)$, where here b_x is as above and $n_K(y)$ is the outer normal of ∂K at the point $(y, f_x(y))$.

The remaining lemmata are mostly standard in the literature (see [24, 81, 79, 13]). Proofs are given only for those which are new to this work.

Lemma 3.6.10. *Given $K \in \mathcal{K}_+^2$, there exist constants d_1, d_2 such that for each cap $C(x, h)$ with $h \leq h_\circ$, we have*

$$\partial K \cap B(x, d_1 h^{\frac{1}{2}}) \subset C(x, h) \subset B(x, d_2 h^{\frac{1}{2}}).$$

This is a direct consequence of 3.6.9.

Lemma 3.6.11. *Given $K \in \mathcal{K}_+^2$, there exists a constant d_3 such that for each cap $C(x, h)$ with $h \leq h_\circ$, we have*

$$\text{Vol}_d(C(x, h)) \leq d_3 h^{\frac{d+1}{2}}.$$

Fix λ_\circ . For $\lambda < \lambda_\circ$ we have

$$\text{Vol}_d(C_{K_\lambda}(y, h)) \leq d_3 h^{\frac{d+1}{2}}$$

where here $K_\lambda = K + \lambda B(o, 1)$ and $y \in \partial K_\lambda$.

Proof sketch. The first statement is by volume comparison, using Lemma 3.6.9, say. For the second statement, we note that there exists some R depending only on K_{λ_\circ} such that at every point $x \in \partial K_{\lambda_\circ}$ there is a ball of radius R tangent to K_{λ_\circ} at x which contains K_{λ_\circ} .

Now, for $\lambda < \lambda_\circ$, translate the ball in the direction of the inner normal at x such that the distance from x to the boundary of the ball is λ . We observe that this ball contains K_λ , and hence the volume of a cap of height h of K_λ is bounded by the volume of a cap of height h of the ball of radius R . As this ball is a fixed convex body, the first statement shows that such a cap has volume $O(h^{\frac{d+1}{2}})$. \square

Lemma 3.6.12. *Let K, m be given, and $y_i, i = 1, \dots, m$ be chosen as in Lemma 3.3.3. The number of Voronoi cells $\text{Vor}(y_i)$ intersecting a cap $C(y_i, h)$ is $O((h^{\frac{1}{2}} m^{\frac{1}{d-1}} + 1)^{d+1})$, $i = 1, \dots, m$.*

This is essentially a calculation based on Lemma 3.3.3.

Lemma 3.6.13. *For a given $K \in \mathcal{K}_+^2$, there exists constants d_8, d_9, d_{10}, d_{11} such that for all $0 < \epsilon < \epsilon_\circ$ we have that for any ϵ -cap C of K ,*

$$d_8^{-1} \epsilon^{(d-1)/(d+1)} \leq \mu(C \cap \partial K) \leq d_9 \epsilon^{(d-1)/(d+1)}$$

and for any ϵ -boundary cap C' of K ,

$$d_{10} \epsilon^{(d+1)/(d-1)} \leq \text{Vol}_d(C') \leq d_{11} \epsilon^{(d+1)/(d-1)}.$$

This estimate again follows from a computation using Lemma 3.6.9

Lemma 3.6.14. *For any $x \in \partial K$, we have constants $d_{12} < d_{13}$ and h_0 such that if $h < h_0$ then*

$$C(x, d_{12}, h) \cap \partial K \subset G(x, h) \cap \partial K \subset C(x, d_{13}, h) \cap \partial K. \quad (3.35)$$

Proof. By Lemma 3.6.9, we can represent the boundary of ∂K by some convex function $f_x : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, where we identify \mathbb{R}^{d-1} with the tangent hyperplane at x . Further, for $\|y\| \leq y_0$ we have the bounds

$$c_1 \|y\|^2 \leq f_x(y) \leq c_2 \|y\|^2, \quad y \in \mathbb{R}^{d-1} \quad (3.36)$$

for some positive constants c_1 and c_2 , which follows by the bound on principle curvatures. We shall call the paraboloids given by $y \mapsto c_1 \|y\|^2$ and $y \mapsto c_2 \|y\|^2$ P_1 and P_2 , respectively. Note that the bound y_0 holds for all $x \in \partial K$.

Now, if u_x is the outward unit normal of ∂K at x , let $z = x + hu_x$. In the coordinate system given by setting $T_x \partial K = \mathbb{R}^{d-1}$, we thus identify $z = (0, -h)$. Simple geometry shows that the line l through z which is tangent to the paraboloid given by P_2 intersects P_1 at a height $\frac{c_1}{c_2}h$. Thus $G(x, h)$ must contain a boundary cap of height $\frac{c_1}{c_2}h$, since the line l must intersect ∂K at a height at least as large as where it intersects P_1 .

Next, the line l must intersect ∂K twice. The height of the point farther from \mathbb{R}^{d-1} is thus an upper bound on the height of some cap which contains all of $G(x, h) \cap \partial K$ so long as ∂K remains bounded as in (3.36). We know that this is the case as long as $G(x, h) \cap \partial K$ is contained in the graph of $B(0, y_0)$. So we first show that for h sufficiently small this must be the case. Observe that the intersection of l and P_2 occurs at $\|y\| = \sqrt{\frac{h}{2c_2 - c_1}}$, which establishes this claim. Second, the height of this intersection is thus $\frac{c_2 h}{2c_2 - c_1}$, which is an upper bound on the intersections of l and ∂K . Thus for all h such that $\sqrt{\frac{h}{2c_2 - c_1}} < y_0$ we see that $G(x, h) \cap \partial K$ is contained in a cap of height $\frac{c_2 h}{2c_2 - c_1}$. \square

Lemma 3.6.15. *Let m, K and $y_i, C_i, i = 1, \dots, m$ be chosen as in Lemma 3.3.3. Choose on the boundary within each cap C_i an arbitrary point x_i (i.e. $x_i \in C_i \cap \partial K$), then*

$$\delta^H(K, P(\{x_1, \dots, x_m\})) = O(m^{-\frac{2}{d-1}}),$$

and there is a constant d_8 such that for any $y \in \partial K$ with $y \notin C(y_i, d_{14} m^{-\frac{2}{d-1}})$, the supporting hyperplanes of y and x_i intersect at a distance greater than $\delta^H(K, P(\{x_1, \dots, x_m\}))$.

Proof. Say $x \in \partial K$ is such that $x' = x + \frac{16d_2^2}{d_1^2 d_{12}} h_m u_x \in P(\{x_1, \dots, x_m\})$. By Lemma 3.6.14 $G(x, h) \cap \partial K$ contains $C(x, d_{12} h) \cap \partial K$. Further, by Lemma 3.6.10, this cap contains $B(x, d_1 d_{12}^{1/2} h^{1/2})$. Now by Lemma 3.3.3 the circumradius of a Voronoi region intersected with the boundary is at most $2d_2 h_m^{1/2}$, and by our choice of h we have

$$d_1 d_{12}^{1/2} h^{1/2} > 4d_2 h_m^{1/2}.$$

Thus, the ball contains the intersection of some Voronoi region with the boundary, and hence some $C_i, i = 1, \dots, m$. However, we assumed that there was some point x_i in C_i , so critically the segment from x' to x_i intersects K only at x_i . This must also be true then for $x'' = x' + \epsilon u_x$ for any ϵ . However, the line through x'' and x_i can not be tangent to ∂K as the curvature of ∂K is bounded, and hence x'' and K must lie on opposite sides of the tangent plane $T_x \partial K$. Thus, x'' is not a point of $P(\{x_1, \dots, x_m\})$, and hence no point of $P(\{x_1, \dots, x_m\})$ can be located a distance greater than $\frac{16d_2^2}{d_1^2 d_{12}} h_m$ from K . Noting that $h_m = O(m^{-\frac{2}{d-1}})$, this establishes the first claim.

For the remainder, set $\delta(m) = \frac{16d_2^2}{d_1^2 d_{12}} h_m$. Let x' be any point in the set $T_x \partial K \cap K_{\delta(m)}$. The same argument in Lemma 3.6.11 can be used to show that there is some constant c depending only on K such that $\|x - x'\| < c \delta(m)^{1/2}$. Hence, if any two hyperplanes tangent to $z_1, z_2 \in \partial K$ intersect in $K_{\delta(m)}$ then the distance

$$\|z_1 - z_2\| < 2c \delta(m)^{1/2} \tag{3.37}$$

by the triangle inequality.

Now let y be as in the second claim, i.e. $y \notin C(y_i, d_{14} m^{-\frac{2}{d-1}})$ where d_{14} will be fixed later. First, the distance $\|y_i - y\| \geq d_1 d_{14}^{1/2} m^{-\frac{1}{d-1}}$ by Lemma 3.6.10. Next, $\|x_i - y_i\| \leq 2d_2 h_m^{1/2}$ by Lemma 3.3.3. Hence by the triangle inequality we have

$$\|y_i - y\| \leq \|x_i - y\| + \|x_i - y_i\|$$

and thus

$$\|x_i - y\| \geq d_1 d_{14}^{1/2} m^{-\frac{1}{d-1}} - 2d_2 h_m^{1/2}.$$

Lemma 3.3.3 gives that $h_m^{1/2} \geq c' m^{-\frac{1}{d-1}}$ for some constant c' , and hence by choosing d_{14} sufficiently large shows that $\|x_i - y\| > 2c \delta(m)^{1/2} = 2c \frac{4d_2}{d_1 d_{12}^{1/2}} h_m^{1/2}$ since $h_m = \Omega(m^{-\frac{1}{d-1}})$. But any two points whose tangent hyperplanes intersect in $K_{\delta(m)}$ must obey (3.37), and thus x_i and y fail to do so.

□

3.7 Further Directions

- The largest outstanding question for the circumscribed model, and indeed the inscribed model, is the question of the existence of a central limit theorem. While we are able in this work and in [87] to demonstrate a CLT for the Poissonized version of our polytope model, it remains unclear if this result can be pushed onto the primary model.

For the basic random polytope model (in which vertices are chosen uniformly in K , not ∂K), Vu demonstrates how to “de-Poissonize” the CLT proved in [81]. His method is typical of such an argument (see also [8] for a related use). First he shows that the number of vertices of the random polytope (and the Poisson version) is distributed as a binomial (Poisson, respectively) random variable of identical mean. He then couples the events $\{\#\text{vertices} = k\}$ and appeals to a classical result on the approximation of a Binomial by a Poisson [71].

In the circumscribed and inscribed case, no such coupling is possible. Indeed, in the circumscribed case the Poisson polytope may have a number of vertices differing from the non-Poisson version of order \sqrt{n} , and the volume contributed by \sqrt{n} points is unfortunately of the same order as the variance.

- Given the discussion of duality between inscribed and circumscribed models, we may ask if a more formal link between the two models may be constructed. Thus far, each of the major polytope models (basic, Gaussian, inscribed, and

circumscribed) have been analyzed with similar but distinct geometric tools. A unified approach to all four problems remains elusive.

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4 Two Geometric Random Graph Models

Random graph theory emerged in 1960 from the papers of Erdős and Rényi [47], in which they introduced the fundamental model $G_{n,p}$ and $G_{n,m}$.¹ The model $G_{n,p}$ is a distribution on all graphs on the vertex set $[n]$ in which each edge in $\binom{[n]}{2}$ occurs independently with probability p . This simple model has become a fundamental tool in the analysis of algorithms, combinatorics, and more recently complex network modeling (see [33, 44, 19, 74, 2] for surveys). While largely unknown relative to the success of purely combinatorial random graphs, the theory of random *geometric* graphs² also traces its birth to this time period. Combining the then recent theory of percolation—due to Broadbent and Hammersley in 1957 [25, 60]—with the advent of random graphs, E. N. Gilbert [53] established the study of boolean percolation in \mathbb{R}^2 using a Poisson point process, wherein any two points in the process that are a distance less than some threshold r are joined by an edge. The study of *finite* random geometric graphs has emerged more recently; a detailed history is given in [75].

In the following two chapters we introduced two random graph models which demonstrate the utility of geometry in random graphs. The first model is used to

¹ The model $G_{n,p}$ was actually introduced by Gilbert in [52], which predates [19] by a year. However, the study of random graphs using *probabilistic tools* was the innovation of Erdős and Rényi from which most of the modern theory of random graphs has developed. The methods of Gilbert, in contrast, were purely enumerative.

² It is unfortunate that the term “random geometric graph” is increasingly taken in the literature to imply a specific model, namely that which is analyzed in [75]. For the purposes of this dissertation we shall break with this convention and intend “random geometric graph” to be used in the more inclusive sense of “random graphs constructed in reference to some underlying geometry.” When we have cause to refer to the specific model found in [75] we shall use the term *boolean model* in analogy to the notion of boolean percolation.

understand the notion of *locality* in large networks. The second provides a generalization of both the boolean and Erdős-Rényi models in which the dimension of the underlying space plays a more prominent role.

4.1 Local Structure in Complex Networks

The drive to understand the structure of large, complex networks such as the World Wide Web is perhaps the most powerful contemporary application of random graph theory. In addition to combinatorial results about connectivity, expansion, and diameter for example, the study of complex networks has led researchers to explore and quantify phenomena such as clustering and power-law degree sequences. This latter phenomenon of power-law degree distributions, the empirical fact that large networks arising in fields as disparate as biology [106, 107], linguistics [108], and Internet networking [49] all display degree distributions with

$$\text{\#vertices of degree } i \propto i^{-\alpha}$$

for some α , has driven the development of new models and tools as well as served as a touchstone for this rapid development.³ The reader is directed to the two recent texts [33, 44] for a more comprehensive treatment of the subject of complex networks and power laws (in particular, chapter 1 of [33] gives a concise history of power laws and generative models).

In Chapter 5, we shall investigate a network model in order to understand the notion of *locality*. Consider the network formed by all friends in the United States. The network appears substantially different in terms of connectivity, diameter, etc., when viewed (geographically) in a small town as compared with a state or the entire nation. The geography, and hence geometry, of this network determine a natural notion of *local* and *global* structure.

Network models that allow for a distinction of local and global structure have been investigated for some time. One of the earliest such papers is [20], in which the

³ The centrality of the power-law in the study of complex networks is itself a controversial topic. See for instance [42, 98].

authors analyze the union of an n -cycle and a random matching, showing that it has both a linear number of edges and small diameter.

From an algorithmic perspective, Kleinberg in [69] proposed a simple local/global network model consisting of an $n \times n$ planar grid, to which one adds a single random edge at each vertex. The edge is chosen with probability proportional to some power of the inverse l_1 distance. He demonstrates that only for power 2 can an algorithm find short paths given only local information; for other exponents the random component is either too short or too uniformly distributed. In a similar vein, the authors of [72] show that for an arbitrarily populated grid model in which link probabilities are determined via local density (sparse regions have higher link probability than dense regions), computable short paths exist. The authors of [3] replace the local grid with a graph that satisfies prescribed local flow constraints, and the global graph with a power-law $G(\mathbf{w})$ random graph (see [33]). In this way they obtain both local clustering and small diameter, and they further provide an algorithm for separating the local and global components.

The model introduced in Chapter 5 is a variant of that originally proposed in [48] (the FKP model). In their model, a random tree is formed in the unit square, where each new vertex v_t is attached to the prior vertex v_i that minimizes $\alpha \|v_i - v_t\| + d(v_i, o)$, where here $\|\cdot\|$ denotes the Euclidean distance and $d(\cdot, o)$ the graph theoretic distance to the root. This model is an example of the *highly optimized tolerance* (HOT) paradigm, due to Carlson and Doyle [26, 27] and popularized in [28], characterized by optimization under a random hazard. They show that for $\alpha = o(\sqrt{n})$ the graph is essentially star-like while for $\alpha = \omega(\sqrt{n})$ the graph behaves as a random recursive tree [94]. In [42] the authors show that the HOT paradigm does a substantially better job of modeling real, designed networks in comparison to models which are constructed without any geometric information (most notable preferential attachment).

Finally, we mention the work of Flaxman et al. in [50]. They explore a model which combines aspects of the preferential attachment model [11] with the boolean model [75] to give a graph with a definite local structure as well as power-law degree distribution.

4.2 Continuum Percolation and Random Geometric Graphs

It is now common when one speaks of random geometric graphs to refer to the objects investigated by Penrose in [75]. In particular, if $r > 0$ is given, one may form a random graph by placing n iid points onto the unit cube $[0, 1]^d$ uniformly. Every pair of points whose Euclidean distance is at most r then form an edge. This model, sometimes referred to as the $G_{n,r}$ model (with the understanding that r may be a function of n), will be referenced here exclusively as the *boolean* random graph model. Many variations on this basic model are possible. Instead of measuring distances using the Euclidean metric, one may use a *toroidal* metric to avoid boundary effects [76]. One may also study the graph built on a Poisson point process of mean n with uniform intensity measure in order to introduce more independence into the model. Indeed, the choice of space $[0, 1]^d$ may be replaced with a general metric space. We shall refer to all such variants as “the” boolean model, adding qualifiers as necessary.

Percolation, as introduced by Broadbent and Hammersley in [25], studies networks generated on infinite vertex sets and hence provides a natural counterpoint to the theory of random graphs. Discrete percolation studies the random network generated on the set \mathbb{Z}^d in which every pair of vertices of distance 1 has an edge independently with probability p . The main question addressed is the calculation of $\mathbb{P}(C(o) = \infty)$ where $C(x)$ is the size of the connected component containing x . The literature on percolation is now vast; for an introduction see [56].

The topic of *continuum percolation* asks for the emergence of an infinite component when the vertex set is given by a point process. Specifically, we can define the (following Meester and Roy [73]) *boolean percolation* model on \mathbb{R}^d as the graph with vertices given by a homogeneous Poisson process and edges determined by any pair of distance at most r (this is the model introduced by Gilbert in [73]). If we add the point o to the point process, the question of percolation then becomes the determination of when $C(o)$ is infinite, as a function of r . Meester and Roy also introduce a generalized model which they label the *random-connection percolation* model. Given some non-increasing function $f : [0, \infty) \rightarrow [0, 1]$ they place an edge between each

pair (x_i, x_j) of vertices in the underlying point process independently with probability $f(\|x_i - x_j\|)$.

The study of the boolean geometric random graph model *per se* was preceded by work on linkages between random points. Kester [67] and Jammalamadaka and Janson [64] studied the interpoint distances of n random points. The nearest neighbor linkage, wherein n random points are joined to their closest neighbors, was investigated by Dette and Henze [37] and Steele and Tierney [95]. Similarly, the Euclidean minimum spanning tree⁴ was investigated for random points in the cube by Jaillet [63] and Penrose [76].

The work of Appel and Russo in 1997 [4] provides the first appearance of a graph on n uniform vertices in which two vertices of distance less than some threshold are connected (though they utilize the l_∞ distance). They provide rates of convergence for the maximum degree of the resulting graph and examine related quantities including the clique number, chromatic number, and independence number. In a later work they examine the *connectivity threshold*, the smallest threshold distance at which the graph becomes connected a.a.s. In a different direction, Penrose et al. [40] introduce the boolean model $G_{n,r}$ in order to study various geometric layout problems, borrowing from the earlier work of Appel and Russo as well as the tools developed in [76, 77]. At the same time, Gupta and Kumar [59] brought the study of $G_{n,r}$ into the wireless networking community, analyzing the connectivity threshold for $G_{n,r}$ using tools from the percolation literature. The recent book by Penrose summarizes much of this development.

Recent work has suggested new directions in geometric random graph theory and deficiencies with the $G_{n,r}$ model. The authors of [18] investigate the adjacency spectrum of $G_{n,r}$ both analytically and numerically. They show that the spectrum is skew, and in particular deviates strongly from the spectrum of both $G_{n,p}$ and “real-world” graphs. Scheinerman and his students have investigated *dot-product graphs*, in which two random vectors are joined by an edge if their dot product exceeds a certain threshold [99]. The authors of [39] introduce the notion of *dynamic* random geometric graphs in which the vertex set moves in each time step as a method to understand

⁴The connected graph on n fixed points in \mathbb{R}^d with minimum total edge length.

disconnection in wireless networks.

5 A Random Geometric Tree Model

Consider the problem of providing telephone service to some central hub. Each customer has a given position in the plane, and thus a distance from the hub. If we are allowed to extend a single connection from each new customer to an existing customer, then choosing the nearest neighbor clearly optimizes the amount of new wire we have to string. On the other hand, connecting to an existing customer who is farther away from the hub than our new customer may lead to attenuated service, and hence we may wish to choose a customer who is closer to the hub. If we weigh these two costs and choose a customer which optimizes our total cost function, the behavior will clearly depend on the relative weighting given to each cost.

We shall construct a simple geometric tree model motivated by the above example. We shall say that a customer is linked by a *local edge* (resp. *global edge*) if the edge length is short (long, respectively) *relative to the distance between the customer and the hub*. In this way we obtain a meaningful description of the local behavior which respects the length scale of each vertex. The above example shows that, depending on how we choose to weigh edge costs, both completely local and completely global behavior are possible. In this paper, we shall quantify these ideas, and in particular we shall look at the transition from global to local behavior based on relative costs.

5.1 Definitions and Model

We define a random graph model \mathcal{T}_α ($\mathcal{T}_{\alpha,n}$ when we wish to stress the dependence on n) with positive parameter α . We denote by K some compact, convex set in \mathbb{R}^d and \mathfrak{o} a fixed point interior to K . Without loss of generality, we assume the volume of K to be one.

For a given natural number n , we construct $\mathcal{T}_{\alpha,n}$ as follows: The vertices of $\mathcal{T}_{\alpha,n}$, denoted by $V_n = \{v_1, \dots, v_n\}$, are chosen independently and uniformly in K .¹ For each vertex $v_i, i = 1, \dots, n$, associate to it a function

$$\phi_i(x) = \alpha \|v_i - x\| + \|x - \mathfrak{o}\|. \quad (5.1)$$

For the model $\mathcal{T}_{\alpha,n}$, we associate to each vertex $v_i, i = 1, \dots, n$, a unique edge e_i with source v_i and target v_{t_i} given by

$$v_{t_i} := \operatorname{argmin}_{j < i} \phi_i(v_j).$$

We shall assume in the sequel that such a minimizer is unique, as this happens with probability one.

Finally, for an edge e_i with target v_j , we call e_i β -local (or $(1 - \beta)$ -global) for $0 \leq \beta \leq 1$ if the Euclidean edge length of the orthogonal projection of $\overline{v_i v_j}$ onto the segment $\overline{\mathfrak{o} v_i}$ is at most $\beta \|v_i - \mathfrak{o}\|$. Equivalently, the target of the edge is contained in the half-space

$$H(v_i, \beta) := \{x \mid (x - \mathfrak{o}) \cdot (v_i - \mathfrak{o}) \geq (1 - \beta) \|v_i - \mathfrak{o}\|^2\}.$$

If not otherwise stated we take $\beta = \frac{1}{2}$. See Figure 5.1. We shall concern ourselves with the fraction $\rho(\beta)$ of β -local edges in \mathcal{T} .

Let us remark here that the definition of a local edge is not entirely natural, but instead is given as a technical convenience. Indeed, it could be the case under our definition that the target of v_i lies a distance much longer than $\|\mathfrak{o} - v_i\|$ from v_i . It shall become clear in the course of the proofs that this is not generally the case; for the moment, however, we shall simply find it convenient.

5.2 Results

Our aim is to analyze the model \mathcal{T}_α . In order to understand the basic combinatorial properties of our model as well as to compare it to the FKP model, we analyze

¹As is usual in the theory of random graphs, we shall adopt the point of view that $\mathcal{T}_{\alpha,n}$ and $\mathcal{T}_{\alpha,m}$ are constructed on the same probability space such that $V_n \cap V_m = V_n$ if $n \leq m$. As the vertices completely determine the graph, the subgraph of $\mathcal{T}_{\alpha,m}$ induced by V_n is thus $\mathcal{T}_{\alpha,n}$, and hence we view \mathcal{T} as being built one vertex and edge at a time.

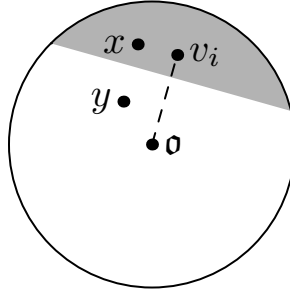


Figure 5.1: The shaded region represents $H(v_i, 1/4)$ where K is a sphere. Note that x and y (as possible targets of v_i) give $1/4$ local and global edges, respectively.

the diameter and degree distribution. The FKP model was shown to exhibit both star-like and exponential-tailed degree distributions based on the parameter α . However, the authors of [48] also asserted a third range of behavior ranging from the regime where α was constant to $O(\sqrt{n})$ in which the degree distribution had a power-law tail. Though consistent with the work of [48], it was subsequently shown in [17] that up to $O(\sqrt{n}/(\ln n)^4)$ this power law held only over a vanishingly small portion of the tail, and indeed the actual behavior was similar to that of part 1 of Theorem 5.2.1. We show below that for any fixed value of $\alpha \neq 1$ the degree distribution is consistent with either the star-like or exponential-tail behavior, leaving unexamined for the moment the value $\alpha = 1$.

We begin by establishing a few results about the structure of $\mathcal{T}_{\alpha,n}$ when α is bounded away from one. When $\alpha < 1$ almost all vertices are leaves connected to a vertex near o , whereas when $\alpha > 1$ the degree distribution of each vertex has an exponential tail.

Theorem 5.2.1 (Degree Distribution). *We have the following for $\mathcal{T}_{\alpha,n}$:*

1. *If $\alpha < \delta < 1$ for δ fixed, then a.a.s. the number of vertices of degree greater than one is $O(\lg n)$.*
2. *If $\alpha > \delta > 1$ for δ fixed, then there exists a constant $c > 0$ such that for any vertex $v_i, i = 1, \dots, n$, we have*

$$\mathbb{P}[\deg(v_i) \geq D] = O(n \exp(-cD)). \quad (5.2)$$

In particular, the maximum degree is $O(\ln n)$ a.a.s.

Remark 5.2.2. Under the assumption that $\alpha \rightarrow \infty$ sufficiently fast ($\omega(\ln n)$ say), one can show a matching exponential lower bound in (5.2), which follows by a modification of an argument found in [17].

Despite the change in degree distribution, the diameter is well-behaved for all parameter ranges. We have

Theorem 5.2.3 (Diameter). *For any $\alpha > 0$, the diameter of \mathcal{T} is $\Theta(\ln n)$ a.a.s.*

We now turn to the primary goal of this work, analyzing the relative number of local edges in our graph as a function of α . For the case of α bounded away from 1 we will see that \mathcal{T}_α consists almost entirely of local (global, respectively) edges.

Theorem 5.2.4 (Edge Length). *Fix $0 < \beta < 1$.*

1. *If $\alpha > \delta > 1$, then $\rho(\beta) \rightarrow 1$, and*
2. *if $\alpha < \delta < 1$, then $\rho(\beta) \rightarrow 0$.*

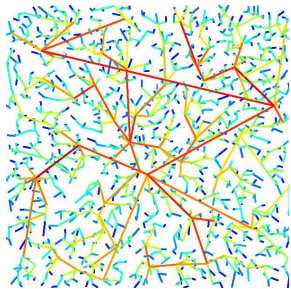
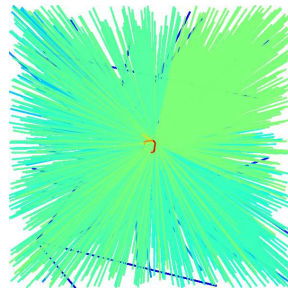
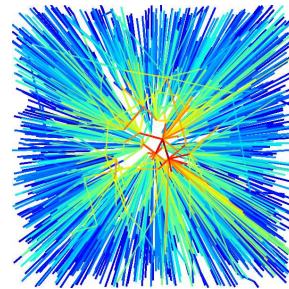
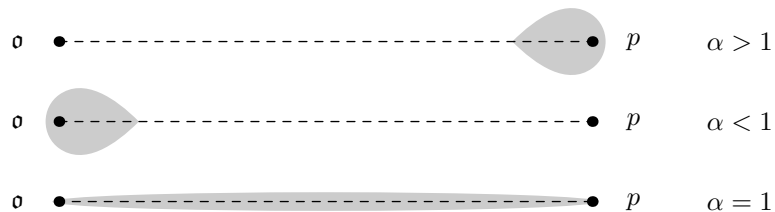
Remark 5.2.5. In particular, we find that when α is bounded away from one the distribution of edge lengths is governed primarily by the geometry of the extreme cases, $\alpha = 0$ and $\alpha \rightarrow \infty$. Hence, as $\rho(\beta)$ is a rough measure of the local tendency of the graph, we see that the graph is entirely local or global in this case.

When $\alpha > 1$ ($\alpha < 1$, respectively) but not bounded away from one, what can be said about $\rho(\beta)$? If β is a function of α we can have both $\rho(\beta) \rightarrow 0$ and $\rho(\beta) \rightarrow 1$, and indeed $\rho(\beta)$ can be bounded away from both 0 and 1. While the relationship between α and β that forces these behaviors can be determined to some extent, the attendant details outweigh the utility of such a result. Thus, we focus on our main theorem, which already suggests some of this behavior.

Our main theorem asserts that around $\alpha = 1$, our model \mathcal{T}_α consists of both local and global edges of roughly the same number, as measured by our parameter $\rho(\beta)$. We work in the unit volume ball in \mathbb{R}^2 , for simplicity, though the same result should hold for general K .

Theorem 5.2.6 ($\alpha = 1$). *Set $K = B(0, \pi^{-1/2}) \subseteq \mathbb{R}^2$ with $0 = 0$. If $\alpha = 1$, we have*

$$\rho(1/2) \rightarrow 1/2, \quad \text{a.a.s.}$$

Figure 5.2: $\alpha = 3$ Figure 5.3: $\alpha = 0.5$ Figure 5.4: $\alpha = 1$ Figure 5.5: $\mathcal{T}_{\alpha, 10,000}$ instances.Figure 5.6: γ -regions for γ small.

Additionally, if β is fixed and $|\alpha - 1| = o(n^{-2})$ then there exists constants $0 < c < c' < 1$ depending on β such

$$c < \mathbb{E}[\rho(\beta)] < c'.$$

Remark 5.2.7. The proof of Theorem 5.2.6 gives the following heuristic explanation for this behavior: for a point v_i in K with $\|v_i - \mathbf{o}\| = l$, the edge e_i has length uniformly chosen in $[0, l]$ and is contained in the thin tube about $\overline{\mathbf{o}v_i}$. It is thus tempting to conjecture that $\rho(x) \approx x$, but the proof of Theorem 5.2.6 does not seem to generalize to this case.

5.3 Tools

All of the subsequent analysis in this paper relies on the notion of influence *regions*. Set $\gamma > 0$. Then the set of points

$$U(p, \gamma) = \{q \mid \|\sigma - q\| + \alpha\|q - p\| \leq (\min(1, \alpha) + \gamma)\|p - \sigma\|\}$$

forms the γ -influence region (or γ -region) about p . We then have:

Lemma 5.3.1 (Convexity). *The region $U(p, \gamma)$ is convex for any choice of p and $\gamma > 0$.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2 \in K$ be such that

$$\|\mathbf{x}_i - \sigma\| + \alpha\|\mathbf{x}_i - p\| = (\min(1, \alpha) + \gamma)\|p - \sigma\|, \quad i = 1, 2, \quad (5.3)$$

which is to say that they lie on the boundary of $U(p, \gamma)$. Let $\mathbf{z} = \lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$. Then

$$\|\mathbf{z} - p\| = \|\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 - p\| \leq \lambda\|\mathbf{x}_1 - p\| + (1 - \lambda)\|\mathbf{x}_2 - p\|,$$

and similarly for $\|\mathbf{z} - \sigma\|$. Thus,

$$\begin{aligned} \|\mathbf{z} - \sigma\| + \alpha\|\mathbf{z} - p\| &\leq (\lambda\|\mathbf{x}_1 - \sigma\| + (1 - \lambda)\|\mathbf{x}_2 - \sigma\|) \\ &\quad + \alpha(\lambda\|\mathbf{x}_1 - p\| + (1 - \lambda)\|\mathbf{x}_2 - p\|) \\ &\text{(by (5.3))} = (\min(1, \alpha) + \gamma)\|p - \sigma\|. \end{aligned} \quad \square$$

For the special case $\alpha = 1$, the γ -region is simply an ellipse with foci σ and p and with major axis length $\|\sigma - p\|(1 + \gamma)/2$. For $\alpha < 1$, as $\gamma \rightarrow \infty$ the γ region approaches that of an ellipse with foci σ and p . However, for γ sufficiently small, the γ -region localizes about the point σ . Specifically, the γ -region forms a convex region about σ , the boundary of which is at maximum and minimum distance from σ along the line through σ and p . The case $\alpha > 1$ is similar, but in this case the γ -region concentrates about p . See Figure 5.6.

We can further elucidate the structure of this region by computing the radii of the smallest enclosing circle and the largest inscribing circle of $U(p, \gamma)$. We summarize this as follows:

Lemma 5.3.2. *Let p be a point of distance r to \mathfrak{o} .*

Assume $\alpha < 1$. Then we have

$$B\left(\mathfrak{o}, \frac{r(1+\gamma-\alpha)}{1+\alpha}\right) \subset U(p, \gamma) \subset B\left(\mathfrak{o}, \frac{r(1+\gamma-\alpha)}{1-\alpha}\right),$$

and in particular

$$\frac{\pi_d}{(1+\alpha)^d} \leq \frac{\text{Vol}_d(U(p, \gamma))}{(r(1+\gamma-\alpha))^d} \leq \frac{\pi_d}{(1-\alpha)^d}. \quad (5.4)$$

Let $\alpha > 1$. Then we have

$$B\left(\mathfrak{o}, \frac{r\gamma}{1+\alpha}\right) \subset U(p, \gamma) \subset B\left(\mathfrak{o}, \frac{r\gamma}{1-\alpha}\right),$$

and

$$\frac{\pi_d}{(1+\alpha)^d} \leq \frac{\text{Vol}_d(U(p, \gamma))}{(r\gamma)^d} \leq \frac{\pi_d}{(1-\alpha)^d}.$$

Here, $\pi_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ denotes the volume of a unit ball in \mathbb{R}^d .

Proof. For $\alpha > 1$, consider the ball centered at p of minimum radius $xr, x \leq 1$ that includes $U(p, \gamma)$. As $U(p, \gamma)$ and this ball intersect along the line $p\mathfrak{o}$, we obtain the equation $(1-x)r + \alpha xr = (1+\gamma)r$, hence $xr = \gamma r / (1-\alpha)$. The other cases are similar. \square

Remark 5.3.3. It is worth observing that whenever $\alpha \neq 1$, the ratio of the radius of a ball which inscribes $U(p, \gamma)$ to the radius of a ball which circumscribes $U(p, \gamma)$ can be made at most $\frac{\alpha+1}{|\alpha-1|}$ for any γ . As a consequence, for α bounded away from one this will allow us to conclude that the length of e_i is of the same order as the nearest neighbor distance of v_i . See Lemmas 5.4.1 and 5.4.5.

5.4 Proofs

5.4.1 Proof of Theorem 5.2.1

Proof of case $\alpha < \delta < 1$:

For $i = 1, \dots, \lfloor \lg n \rfloor$ we let $r_i = \left(\frac{2i}{\pi_d (\lg e)^{(2^i-1)}} \right)^{1/d}$. We claim that

$$\mathbb{P}(V_{2^i-1} \cap B(\mathfrak{o}, r_i) = \emptyset \text{ for some } i = \lfloor \lg \lg n \rfloor, \dots, \lfloor \lg n \rfloor) = o(1).$$

To see this, we use the union bound to compute

$$\mathbb{P} \left(\bigcup_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} \{V_{2^{i-1}} \cap B(\mathfrak{o}, r_i) = \emptyset\} \right) \leq \sum_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} (1 - \pi_d r_i^d)^{2^{i-1}} \quad (5.5)$$

$$\leq \sum_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} \exp(-\pi_d r_i^d (2^i - 1)) \quad (5.6)$$

$$\leq \lg n (\lg n)^{-3/2} = o(1), \quad (5.7)$$

since $\exp(-2 \lfloor \lg \lg n \rfloor / \lg e) \leq (\lg n)^{-3/2}$ for n sufficiently large.

Next, observe that the point closest to \mathfrak{o} gives a bound on the target for any vertex.

Lemma 5.4.1. *Fix $2 \leq i \leq n$. If $r = \min_{j < i} \|\mathfrak{o} - v_j\|$, then $\|\mathfrak{o} - v_{t_i}\| < r \left(\frac{1+\delta}{1-\delta} \right)$.*

Proof. Let v_j be the point which obtains the minimum distance r . By Lemma 5.3.2, there is a γ such that $B(\mathfrak{o}, r) \subset U(v_j, \gamma) \subset B(\mathfrak{o}, r \frac{1+\alpha}{1-\alpha})$. Thus, any point v_k falling outside $B(\mathfrak{o}, r \frac{\alpha+1}{\alpha-1})$ must have $\phi_i(v_k) > \phi_i(v_j)$, and hence the target of v_i must obey $\|\mathfrak{o} - v_{t_i}\| \leq r \frac{1+\alpha}{1-\alpha} < r \frac{1+\delta}{1-\delta}$. \square

Now, for each $i = \lfloor \lg \lg n \rfloor, \dots, \lfloor \lg n \rfloor$ we can compute the number of points in $\{v_{2^i}, \dots, v_{2^{i+1}-1}\}$ which fall inside $B(\mathfrak{o}, r_i \left(\frac{1+\delta}{1-\delta} \right))$ (which for n sufficiently large lies entirely inside K). The expected such number is $2^i \pi_d \left(r_i \frac{1+\delta}{1-\delta} \right)^d = \frac{2^i}{\lg e} \binom{2^i}{2^i-1} \left(\frac{1+\delta}{1-\delta} \right)^d$. By Lemma 1.3.3, the probability that the actual number exceeds the expectation by more than ci for some constant $c > 0$ is bounded by

$$\exp \left(\frac{-c^2 i^2}{2 \left(\frac{2^i}{\lg e} \binom{2^i}{2^i-1} \left(\frac{1+\delta}{1-\delta} \right)^d + ci/3 \right)} \right),$$

which is $o((\lg n)^{-1})$ for c sufficiently large and $i \geq \lfloor \lg \lg n \rfloor$. Hence, we may assume that for all such i the number of points in $\{v_{2^i}, \dots, v_{2^{i+1}-1}\}$ falling inside $B(\mathfrak{o}, r_i \frac{1+\delta}{1-\delta})$ is $\frac{2^i}{\lg e} \binom{2^i}{2^i-1} \left(\frac{1+\delta}{1-\delta} \right)^d + ci$ with probability tending to one. Summing gives

$$\sum_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} i \left(\left(\frac{2^i}{2^i-1} \right) \left(\frac{1+\delta}{1-\delta} \right)^d + c \right) = O(\lg n)$$

such vertices. From Equation 5.7 and Lemma 5.4.1 the remaining vertices among $\{v_{2^{\lfloor \lg n \rfloor}}, \dots, v_n\}$ have degree one. Hence, the total number of vertices of degree greater than one is $O(\lg n)$.

Proof of case $\alpha > \delta > 1$:

The method of proof here is an extension of that found in [48].

Fix the point v_i . We consider each edge in which v_i participates to be either *short* or *long* based on whether it is less than $r = (\pi_d(n-1))^{-1/d}$. The number of short edges of v_i is bounded by the number of vertices which fall inside the ball of radius r . Lemma 1.3.3 shows that

$$\mathbb{P}(\#\text{short edges} \geq D/2) \leq \exp\left(\frac{(D/2-1)^2}{4/3+D/3}\right).$$

Now, pick ϵ and θ sufficiently small such that

$$\frac{(1+\epsilon)}{\delta} + \sqrt{2((1+\epsilon)^2 - \cos\theta)} < 1, \quad (5.8)$$

observing that the left hand side decreases in both ϵ and θ . We then have the following proposition.

Proposition 5.4.2. *For $l > 0$, let v_j, v_k lie in the annulus about v_i of radii l and $(1+\epsilon)l$ for any $l > 0$. If $\angle v_j v_i v_k < \theta$ then $\phi_j(v_k) < \phi_j(v_i)$, i.e. v_j prefers v_k to v_i .*

Proof. We abbreviate $d_{xy} := \|x - y\|$. By the law of cosines,

$$d_{v_j v_k}^2 \leq d_{v_j v_i}^2 + d_{v_k v_i}^2 - 2d_{v_j v_i} d_{v_k v_i} \cos \angle v_j v_i v_k \leq 2(1+\epsilon)^2 l^2 - 2l^2 \cos \theta$$

hence $d_{v_j v_k} \leq l \sqrt{2((1+\epsilon)^2 - \cos \theta)}$. Further, $d_{v_i v_j} \geq l$. Hence, by (5.8)

$$(1+\epsilon)l < \delta(l - l \sqrt{2((1+\epsilon)^2 - \cos \theta)}) \leq \delta(d_{v_i v_j} - d_{v_j v_k}) < \alpha(d_{v_i v_j} - d_{v_j v_k}).$$

On the other hand, $d_{v_i v_0} - d_{v_j v_0} \leq (1+\epsilon)l$, so $d_{v_k v_0} - d_{v_i v_0} < \alpha(d_{v_i v_j} - d_{v_j v_k})$, i.e.

$$d_{v_k v_0} + \alpha d_{v_j v_k} < d_{v_i v_0} + \alpha d_{v_i v_j},$$

our conclusion. □

In \mathbb{R}^d , no more than $N_o = N_o(d, \theta)$ points can be placed such that the angle $\angle x v_i y > \theta$ for every pair x, y . This follows by a simple packing argument. As a result, in any annulus as above there can be at most N_o vertices linked to v_i . Now, if L_x denotes the number of vertices linked to v_i in the annulus of radii x and $(1 + \epsilon)x$ centered at v_i , we thus can count the total number of long edges as

$$\sum_{i=\lfloor \log_{1+\epsilon} \zeta_i \rfloor}^{\lfloor \log_{1+\epsilon} \text{diam}K \rfloor} L_{(1+\epsilon)^i} \leq N_o(\log_{1+\epsilon} \text{diam}K - \log_{1+\epsilon} \zeta_i + 1),$$

where here $\zeta_i = \min_j \{ \|v_i - v_j\|, r \}$. Now, $\mathbb{P}(N_o(\log_{1+\epsilon} \text{diam}K - \log_{1+\epsilon} \zeta_i + 1) \geq D/2)$ is just $\mathbb{P}(\zeta_i \leq \text{diam}K(1 + \epsilon)^{1-D/2N_o})$. As our points are chosen independently, the union bound gives $\mathbb{P}(\zeta_i \leq \text{diam}K(1 + \epsilon)^{1-D/2N_o}) \leq (n - 1)\pi_d(\text{diam}K(1 + \epsilon)^{1-D/2N_o})^d$. Thus, $\mathbb{P}(\#\text{long edges} \geq D/2) = O(n \exp(-cD))$ for some constant c .

As $\mathbb{P}(d(v_i) \geq D) \leq \mathbb{P}(\#\text{short edges} \geq D/2) + \mathbb{P}(\#\text{long edges} \geq D/2)$, we thus have $\mathbb{P}(d(v_i) \geq D) = O(n \exp(-c'D))$ for some $c' > 0$.

5.4.2 Proof of Theorem 5.2.3

The upper bound follows from a simple adaptation of 2 in [17], so we focus on the lower bound. We shall use the following, which is adapted from Lemma 1 in [17].

Lemma 5.4.3. *The probability that for $1 < i_1 < \dots < i_k$ the path $v_{i_k} \rightarrow v_{i_{k-1}} \rightarrow \dots \rightarrow v_{i_1} \rightarrow v_1$ exists in \mathcal{T} is $\frac{1}{(i_1-1)(i_2-1)\dots(i_{k-1}-1)}$.*

Proof. We shall fix the positions—but not the labels—of vertices in $V_{i_{k-1}}$. As edges are determined only from the positions, the target of v_{i_k} is any member of $V_{i_{k-1}}$, chosen uniformly with probability $1/(i_k - 1)$. If we condition on the target being $v_{i_{k-1}}$, we may view the vertex labels prior to $v_{i_{k-1}}$ as still unchosen, and thus we may repeat our argument to obtain the lemma. \square

To prove our lower bound, we bound the length of the $v_n \rightarrow v_1$ path in \mathcal{T}_α . Note that the probability that the length of this path is at most L is given by

$$\frac{1}{n-1} + \sum_{r=2}^L \sum_{1 < i_1 < \dots < i_{r-1} < n} \frac{1}{(i_1-1)\dots(i_{r-1}-1)(n-1)} \leq \frac{1}{n-1} \sum_{r=1}^L \frac{(1 + \ln n)^{r-1}}{r!}. \quad (5.9)$$

Now, if $a_r = \frac{(1+\ln n)^{r-1}}{r!}$, then $a_{r+1} = a_r \frac{1+\ln n}{r+1}$, and hence for a given n the sequence $\{a_r\}_{r=1}^\infty$ is unimodal with maximum $a_{\lfloor \ln n \rfloor}$. If we let $L = \lfloor \ln n / 2 \rfloor$, then Stirling's approximation shows a_L (and hence all lower terms) to be $o(n/\ln n)$, and hence (5.9) tends to zero. Thus, with probability tending to one the path from $v_n \rightarrow v_1$ is greater than $L = \lfloor \ln n / 2 \rfloor$.

5.4.3 Proof of Theorem 5.2.4

We shall handle only the case $\alpha > \delta > 1$, as the other case is similar.

We will require a small technical fact, which is a (more or less) direct consequence of the compactness of K .

Claim 5.4.4. *Let $x \in K$. Then there exists an $r_0 > 0$ and $\eta > 0$ such that $\|(\|K \cap B(x, r)\|) \geq \eta \pi_d r^d$ for all $r \leq r_0$.*

Now, for each $i = 2, \dots, n$ we shall set $r_i = \left(\frac{2 \ln(i-1)}{\eta \pi_d (i-1)} \right)^{1/d}$. The probability that all vertices in V_{i-1} are a distance at least r_i from v_i is $(1 - \|(\|B(v_i, r_i) \cap K)\|)^{i-1} \leq (1 - \eta \pi_d r^d)^{i-1} \leq \exp(-2 \ln(i-1)) = (i-1)^{-2}$. Hence, we may assume that for each $i \geq i_0$, $V_{i-1} \cap B(v_i, r_i) \neq \emptyset$ with probability $\sum_{i=i_0}^n (i-1)^{-2}$, which tends to zero as $i_0 = i_0(n) \rightarrow \infty$.

The following fact is a companion to Lemma 5.4.1 and is proved in the same way.

Claim 5.4.5. *Fix $2 \leq i \leq n$. If $r = \min_{j < i} \|v_i - v_j\|$ then $\|v_i - v_{t_i}\| < r \left(\frac{\delta+1}{\delta-1} \right)$.*

In particular, note that if $B(v_i, r_i) \cap V_{i-1} \neq \emptyset$ then $\|v_i - v_{t_i}\| < r_i \left(\frac{\delta+1}{\delta-1} \right)$. If, in addition, $\|v_i - \mathfrak{o}\| > \beta^{-1} r_i \left(\frac{\delta+1}{\delta-1} \right)$ then e_i is β -local. The probability that v_i falls in $B(\mathfrak{o}, \beta^{-1} r_i \left(\frac{\delta+1}{\delta-1} \right))$ is bounded by $\frac{\beta^{-d} \ln(i-1)}{\eta (i-1)}$. Thus, the expected number of such "close" points is at most $\frac{\beta^{-d}}{\eta} \sum_{i=1}^n \frac{\ln(i-1)}{i-1} = O((\ln n)^2)$. By Lemma 1.3.3, the actual number of such points is $O((\ln n)^2)$ with probability tending to one (taking $\lambda = c(\ln n)^{2/3}$, say, for c sufficiently large). Thus, the only edges which can fail to be β -local are these "close" points as well as the first i_0 . Setting $i_0 = O((\ln n)^2)$ thus allows us to conclude that the number of local edges is $n(1 - o(1))$ with probability tending to one.

5.4.4 Proof of Theorem 5.2.6, $\beta = 1/2$

Our argument will be via the second moment method, using Lemma 1.3.1. The key idea is that for $\alpha = 1$ the γ -region for each point $v_i \in V_n$ is (with high probability) an *ellipse*. In particular, reflecting all points except v_i in such an ellipse about the minor axis does not alter their values with respect to ϕ_i . Thus, the probability that any given edge is local is about half. Unfortunately, as two γ -regions always overlap (they always contain the point \mathfrak{o} for example), we must overcome issues of dependency in applying Lemma 1.3.1.

We shall let X_i be the indicator that vertex v_i is local, hence $X = \sum_{i=1}^n X_i$ counts the number of local vertices. For a given point $x \in K$, we shall let $E(x, t)$ denote the (closed) ellipse with foci x and \mathfrak{o} and area t .

We shall partition the vertices V_n into *epochs*, indexed by $0, 1, \dots, \lfloor \lg n \rfloor$, where vertex v_i belongs to epoch $l(i) := \lfloor \lg i \rfloor$. To each epoch we associate a set of parameters which are given (with foresight) by

- $t_i = 2^{-i/2}$,
- $r_i = (1 - 2^{-1/7})^{i/2}$,
- $\theta_i = (1 - 2^{-1/7})^i$,
- and $h_i = \left(\frac{2-\varepsilon}{4}\right)^{i/2}$ for some $0 < \varepsilon < 2 - \frac{1}{1-2^{-1/7}}$.

Given these parameters, we shall construct the following events:

- $\mathcal{A}(v_i, l): E(v_i, t_l) \subseteq K$,
- $\mathcal{B}(v_i, l): |(E(v_i, t_l) - B(\mathfrak{o}, h_l) - B(v_i, h_l)) \cap V_i| > 1, \|v_i - \mathfrak{o}\| > r_l, E(v_i, t_l) \cap B(\mathfrak{o}, h_l) \cap V_{i-1} = \emptyset, \text{ and } E(v_i, t_l) \cap B(v_i, h_l) \cap V_{i-1} = \emptyset$,
- $\mathcal{C}(v_i, v_j, l): \not\prec v_i \mathfrak{o} v_j > \theta_l$.

The following geometric lemma will be of critical importance.

Lemma 5.4.6. *If $t = o(r^2)$ and $r \rightarrow 0$ then there exists a constant $c > 0$ such that the distance from \mathfrak{o} to any point in $E(v_i, t) \cap E(v_j, t)$ is bounded by $c \frac{t^2}{r^3(1 - \cos \frac{\theta}{2})}$.*

Let us further define the event

$$\mathcal{D}(v_i, v_j, l) = \mathcal{A}(v_i, l) \cap \mathcal{A}(v_j, l) \cap \mathcal{B}(v_i, l) \cap \mathcal{B}(v_j, l) \cap \mathcal{C}(v_i, v_j, l).$$

We then have the following technical lemma:

Lemma 5.4.7. *The following estimates hold:*

$$\sum_{i=1}^n \mathbb{P}(\overline{\mathcal{A}(v_i, l(i)) \cap \mathcal{B}(v_i, l(i))}) = o(n), \quad (5.10)$$

$$\sum_{i=1}^n \sum_{j>i}^n \mathbb{P}(\overline{\mathcal{D}(v_i, v_j, l(j))}) = o(n^2), \quad (5.11)$$

and

$$\sum_{i=1}^n \sum_{j>i}^n \mathbb{P}(\overline{\mathcal{A}(v_i, l(j)) \cap \mathcal{B}(v_i, l(j))}) + \mathbb{P}(\overline{\mathcal{A}(v_j, l(j)) \cap \mathcal{B}(v_j, l(j))}) = o(n^2). \quad (5.12)$$

Here, the $\overline{\mathcal{E}}$ denotes the complement of an event \mathcal{E} . We can now prove Theorem 5.2.6.

Proof. Expectation: Fix $1 \leq i \leq n$. We shall construct a map T^i of the underlying probability space K^n . On the complement of the set $\mathcal{A}(v_i, l(i)) \cap \mathcal{B}(v_i, l(i))$, T^i acts as the identity. Otherwise, for each vertex $v_j \neq v_i$ the map T^i is the identity if $v_j \notin E(v_i, t_{l(i)})$ and otherwise reflects v_j about the minor axis of $E(v_i, t_{l(i)})$. The assumption $\mathcal{A}(v_i, l(i))$ makes this operation well defined. Further, T^i is probability preserving as rigid reflection preserves Lebesgue measure.

Under $\mathcal{B}(v_i, l(i))$ the target of v_i lies inside $E(v_i, t_{l(i)}) - B(o, h_{l(i)}) - B(v_i, h_{l(i)})$. As $E(v_i, t_{l(i)}) - B(o, h_{l(i)}) - B(v_i, h_{l(i)})$ is symmetric about the minor axis of $E(v_i, t_{l(i)})$, T^i exchanges the events $X_i = 1$ and $X_i = 0$ under $\mathcal{A}(v_i, l(i)) \cap \mathcal{B}(v_i, l(i))$. Hence, we have

$$\mathbb{P}(X_i = 1) = 1/2 + O(1 - \mathbb{P}(\mathcal{A}(v_i, l(i)) \cap \mathcal{B}(v_i, l(i)))).$$

Thus, we can compute

$$\mathbb{E}[X] = \sum_{i=1}^n 1/2 + O(1 - \mathbb{P}(\mathcal{A}(v_i, l(i)) \cap \mathcal{B}(v_i, l(i)))) = n/2 + o(n), \quad (5.13)$$

where the last equality follows from (5.10).

Variance: Computation of the variance is similar. We shall fix $1 \leq i < j \leq n$. We now construct two maps S_l^i on K^n for each l . The map S_l^i fixes all those v_j which fall outside $E(v_i, t_l)$ as well as v_i . Those vertices which fall inside $E(v_i, t_l)$ are reflected about the minor axis of this ellipse.

If we set $l_0 = \lg \lg n$, we can see that for $l \geq l_0$ we have $r_l \rightarrow 0$ and $t_l = o(r_l^2)$. Hence, by our choice of the parameters t_l, r_l, h_l , and θ_l as well as Lemma 5.4.6 we see that on the event $\mathcal{D}(v_i, v_j, l(j))$ the regions $E(v_i, t_{l(j)}) - B(v_i, h_{l(j)}) - B(o, h_{l(j)})$ and $E(v_j, t_{l(j)}) - B(v_j, h_{l(j)}) - B(o, h_{l(j)})$ are disjoint so long as $l(j) \geq l_0$, and hence $S_{l(j)}^i$ and $S_{l(j)}^j$ commute. Further, both these maps are probability preserving on $\mathcal{D}(v_i, v_j, l(j))$, and so we see that the events $\{X_i = \varepsilon_1 \wedge X_j = \varepsilon_2\} \cap \mathcal{D}(v_i, v_j, l(j))$ consist of the orbit of any one of them under the group generated by $S_{l(j)}^i$ and $S_{l(j)}^j$, and hence are equiprobable. Thus, $\mathbb{E}[X_i X_j] \leq 1/4 + \mathbb{P}(\overline{\mathcal{D}(v_i, v_j, l(j))})$. As a result, we have that

$$\begin{aligned} \text{Cov}[X_i X_j] &= \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j] \leq 1/4 + \mathbb{P}(\overline{\mathcal{D}(v_i, v_j, l(j))}) \\ &- (1/2 - 1/2 \mathbb{P}(\overline{\mathcal{A}(v_i, l(j)) \cap \mathcal{B}(v_i, l(j))})) \cdot (1/2 - 1/2 \mathbb{P}(\overline{\mathcal{A}(v_j, l(j)) \cap \mathcal{B}(v_j, l(j))})) \\ &\leq \mathbb{P}(\overline{\mathcal{D}(v_i, v_j, l(j))}) \\ &+ 1/4 \left(\mathbb{P}(\overline{\mathcal{A}(v_i, l(j)) \cap \mathcal{B}(v_i, l(j))}) + \mathbb{P}(\overline{\mathcal{A}(v_j, l(j)) \cap \mathcal{B}(v_j, l(j))}) \right). \end{aligned} \quad (5.14)$$

Hence, we can compute

$$\begin{aligned} \text{Var}[X] &= \sum_{i=1}^n \text{Var}[X_i] + 2 \sum_{i=1}^n \sum_{j>i}^n \text{Cov}[X_i X_j] \quad (5.15) \\ &\leq n + 2 \sum_{i=1}^n \sum_{j>i}^n \text{Cov}[X_i X_j] \\ (\text{by (5.14)}) &\leq n + 2 \sum_{i=1}^n \sum_{j>i}^n \left(\mathbb{P}(\overline{\mathcal{D}(v_i, v_j, l(j))}) \right. \\ &\quad \left. + 1/4 \left(\mathbb{P}(\overline{\mathcal{A}(v_i, l(j)) \cap \mathcal{B}(v_i, l(j))}) \right. \right. \\ &\quad \left. \left. + \mathbb{P}(\overline{\mathcal{A}(v_j, l(j)) \cap \mathcal{B}(v_j, l(j))}) \right) \right) \quad (5.16) \end{aligned}$$

$$(\text{by (5.11) and (5.12)}) = o(n^2). \quad (5.17)$$

Thus, (5.13) and (5.17) and Lemma 1.3.1 allow us to conclude that $X = \mathbb{E}[X](1 + o(1)) = n/2(1 + o(1))$ with probability tending to one. \square

The rest of the proof is devoted to establishing Lemmas 5.4.6 and 5.4.7. We first require a few geometric estimates.

Lemma 5.4.8. *Let $t \rightarrow 0$. If v is chosen uniformly in K then $E(v, t) \subseteq K$ with probability $1 - O(t^2)$.*

Proof. We bound the failure probability. Note that $E(v, t) \not\subseteq K$ is equivalent to the event that the semi-major axis length of $E(v, t)$ exceeds half the distance of v to \mathfrak{o} (the focal distance) by an amount greater than the distance from v_i to the boundary of K . This is established on the ball by noting that the curvature of the boundary is one everywhere, whereas the curvature at the extremes of the major axis of (γ -region) ellipses tends to infinity.

We label the semi-major and semi-minor axis lengths a and b and let $r = \frac{\|v-\mathfrak{o}\|}{2}$, such that $a^2 = b^2 + r^2$.

Let us examine the excess $a - r$. Note that $(a^2 - r^2) = (a + r)(a - r) \geq 2r(a - r)$, hence $a - r \leq \frac{a^2 - r^2}{2r}$. Now the area of our ellipse is t , thus $\pi a \sqrt{a^2 - r^2} = \pi a b = t$. An appeal to the quadratic formula yields

$$a^2 = \frac{1}{2} \left(r^2 + r^2 \sqrt{1 + \frac{4t^2}{\pi^2 r^4}} \right) \leq \left(r^2 + \frac{t^2}{\pi^2 r^2} \right). \quad (5.18)$$

Here, we use the fact that for all $x > 0$ we have $\sqrt{1+x} \leq 1 + x/2$. Thus we have $\frac{a^2 - r^2}{2r} \leq \frac{t^2}{2\pi^2 r^3}$.

Fixing $r > r' > 0$ for some constant $r' < \frac{1}{2\sqrt{\pi}}$, we are thus assured that if v_i falls a distance at least $\frac{t^2}{2\pi^2(r')^3}(1 + o(1))$ from the boundary of K then $E(v, t) \subseteq K$. For $r \leq r'$, $E(v, t)$ intersects the boundary of k only if $2a \geq 1/\sqrt{\pi}$, since the sum of the distances from any point in $\partial E(v, t)$ is $2a$. Now, as $a^2 = t^2/(\pi^2 a^2) + r^2 \leq t^2/(\pi^2 r^2) + r^2 \leq t^2/(\pi^2 (r')^2) + (r')^2$, we see that for t sufficiently small we have $a < 1/(2\sqrt{\pi})$

Thus the only chance of failure comes from a point landing too close to the boundary of K , and as above this gives a failure probability of $O(t^2)$. \square

Proof of Lemma 5.4.6. For the moment, assume that v_i and v_j have the same length to the center \mathfrak{o} . Label the common focal length r (half the distance to center), and label the farthest point of intersection of the two ellipses h . See Figure 5.7.

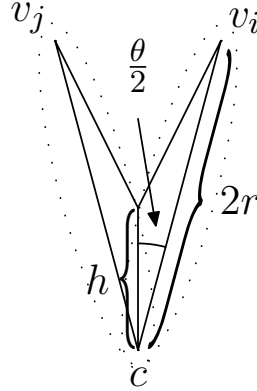


Figure 5.7: The intersection of two ellipses with a common focus and focal length forming an angle θ .

Consider the triangle on the right of figure 5.7. Note that the sum of the smaller two sides must equal the major axis length of the ellipse. Letting a be the semi-major axis length and x be the unlabeled edge in the figure, this implies that $2a = x + h$. Next, the law of cosines tells us that $h^2 + 4r^2 - 4rh \cos \frac{\theta}{2} = x^2$. Using our relation between x and h we obtain $h = \frac{a^2 - r^2}{a - r \cos \frac{\theta}{2}}$.

Recall that in Lemma 5.4.8 we already established the necessary asymptotics, so by (5.18) we have

$$a^2 = r^2 \left(1 + \frac{t^2}{\pi^2 r^4} (1 + o(1)) \right),$$

under the assumption that $t^2/r^4 = o(1)$, i.e. $t = o(r^2)$. Taking square roots we obtain the similar expansion $a = r \left(1 + \frac{t^2}{2\pi^2 r^4} (1 + o(1)) \right)$, again under the assumption $t = o(r^2)$. Thus we conclude that

$$h = \frac{a^2 - r^2}{a - r \cos \frac{\theta}{2}} \leq c \frac{t^2}{r^2 (1 - \cos \frac{\theta}{2})},$$

for some $c > 0$.

The case of v_i and v_j at different distances from \mathfrak{o} follows by noting that lengthening one ellipse along its major axis while keeping the other fixed (maintaining the area in both) causes h to decrease. \square

Now we need estimates on $\mathcal{A}(v_i)$, $\mathcal{B}(v_i)$, etc.

Lemma 5.4.9. *For $l = \lfloor \lg \lg n \rfloor, \dots, \lfloor \lg n \rfloor$ and $v_i, v_j \in V_n$, there exist constants $c_1, c_2, c_3, c_4 > 0$ such that*

$$\mathbb{P}(\overline{\mathcal{A}(v_i, l)}) \leq c_1 t_l^2, \quad (5.19)$$

$$\mathbb{P}(\overline{\mathcal{B}(v_i, l)}) \leq c_2 r_l^2 + (1 - t_l(1 + o(1)))^{2^{l(i)}-1} + c_3 2^{l(i)} h_l^2, \quad (5.20)$$

$$\mathbb{P}(\overline{\mathcal{C}(v_i, v_j, l)}) = \theta_l. \quad (5.21)$$

In particular,

$$\begin{aligned} \mathbb{P}(\overline{\mathcal{D}(v_i, v_j, l)}) &\leq 2c_1 t_l^2 + (1 - t_l(1 - o(1)))^{2^{l(i)}-1} + (1 - t_l(1 - o(1)))^{2^{l(j)}-1} \\ &\quad + c_3 2^{l(i)} h_l^2 + c_3 2^{l(j)} h_l^2 + 2c_2 r_l^2 + c_4 \theta_i. \end{aligned} \quad (5.22)$$

Proof. The estimate on $\mathbb{P}(\overline{\mathcal{A}(v_i, l)})$ follows from Lemma 5.4.8.

The probability $\|v_i - \mathbf{o}\| \leq r_l$ is πr_l^2 . The probability that some point $v_{i'}$ such that $i' < i$ falls in $E(v_i, t_l) \cap B(v_i, h_l)$ or $E(v_i, t_l) \cap B(\mathbf{o}, h_l)$ is at most $(i-1)2\pi h_l^2 \leq 2^{l(i)} 2\pi h_l^2$. Finally, the probability that every point $v_{i'}$ such that $i' < i$ misses $E(v_i, t_l)$ is at most $(1 - \text{Area}(E(v_i, t_l)))^{i-1} \leq (1 - t_l(1 - o(1)))^{2^{l(i)}-1}$. Here, we use the fact that $2^{l(i)} - 1 \leq i$ and the bound $\text{Area}(E(v_i, t_l)) = t_l(1 - o(1))$ (here the $o(1)$ term is due to those v_i which fall near the boundary of K). These facts combine to give the estimate on $\mathbb{P}(\overline{\mathcal{B}(v_i, l)})$.

The equation $\mathbb{P}(\overline{\mathcal{C}(v_i, v_j, l)})$ comes simply from the symmetry of K . \square

We are now in a position to verify Lemma 5.4.7.

Proof of Lemma 5.4.7. We shall focus on the Equation 5.11, the methods of which also give (5.10) and (5.12). Thus, we estimate $\sum_{i=1}^n \sum_{j>i}^n \mathbb{P}(\overline{\mathcal{D}(v_i, v_j, l(j))})$. As the right hand side of all estimates in Lemma 5.4.9 depend only on relevant epochs, we can

obtain the upper bound

$$\begin{aligned}
\sum_{i=1}^n \sum_{j>i}^n \mathbb{P}(\overline{\mathcal{D}(v_i, v_j, l(j))}) &\leq \sum_{i=\lfloor \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j=i+1}^{\lfloor \lg n \rfloor} 2^j \mathbb{P}(\overline{\mathcal{D}(v_{2^i}, v_{2^j})}) + O(n \lg n) \\
\text{(by Lemma 5.4.9)} &\leq \sum_{i=\lfloor \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j=i+1}^{\lfloor \lg n \rfloor} 2^j \left(2c_1 t_j^2 + (1 - t_j(1 - o(1)))^{2^{j-1}} \right. \\
&\quad \left. + (1 - t_j(1 - o(1)))^{2^{j-1}} + c_3 2^i h_j^2 + c_3 2^j h_j^2 + 2c_2 r_j^2 + c_4 \theta_j \right) \\
&\quad + O(n \lg n). \tag{5.23}
\end{aligned}$$

Here, the $O(n \lg n)$ term comes from neglecting those epochs for which Lemma 5.4.9 does not apply. As our goal is to bound the above by a $o(n^2)$ term we shall further neglect this error term and focus on the sum.

We evaluate the sum on each term. First, consider

$$\sum_{i=\lfloor \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j t_j^2 \leq \sum_{i=1}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j t_j^2 = \sum_{i=1}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j 2^{-j} = O(n \lg n).$$

Here, and in the remainder, it is understood that $i, j \leq \lfloor \lg n \rfloor$.

Similarly, we have

$$\sum_{i=\lfloor \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j r_j^2 \leq \sum_{i=1}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j (1 - 2^{-1/7})^j = O(n),$$

as the inner sum is bounded by a geometric sum with modulus less than one. The same estimate thus also shows $\sum_{i=\lfloor \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j \theta_j = O(n)$.

Next, we compute

$$\sum_{i=\lfloor \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j (2^j h_j^2) \leq \sum_{i=1}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} (2 - \varepsilon)^j = o(n^2),$$

given that the inner sum is bounded by $\sum_{j=1}^{\lfloor \lg n \rfloor} (2 - \varepsilon)^j = o(n)$.

Finally, we check the sum

$$\sum_{i=\lfloor \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j (1 - t_j(1 - o(1)))^{2^{j-1}}.$$

We shall use the bound $t_j(1 - o(1)) \geq t_j/2$ for $j \geq \lfloor \lg \lg n \rfloor$. Thus,

$$\begin{aligned}
\sum_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j (1 - t_j(1 - o(1)))^{2^i-1} &\leq \sum_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j \exp(-t_j(1 - o(1))(2^i - 1)) \\
&\leq \sum_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j \exp(-t_i(2^i - 1)/2) \\
&\leq O(n) \sum_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \exp(-2^{-i/2}(2^i - 1)/2) \\
&\leq O(n) \sum_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \exp(-2^{i/2-2}) = O(n),
\end{aligned}$$

by comparison to $\sum_{i=1}^{\infty} 2^i e^{2^{i/2-2}} < \infty$. Noting that $(1 - t_j(1 - o(1)))^{2^i-1} \geq (1 - t_j(1 - o(1)))^{2^j-1}$ as $j > i$, the estimate shows that $\sum_{i=\lfloor \lg \lg n \rfloor}^{\lfloor \lg n \rfloor} 2^i \sum_{j>i} 2^j (1 - t_j(1 - o(1)))^{2^j-1} = O(n)$ as well.

Thus, as (5.23) is a finite sum of sums each bounded by $o(n^2)$, (5.11) holds. The other estimates fall to the same methods. \square

Remainder of proof of Theorem 5.2.6. With β fixed, we shall assume in what follows that $|\alpha - 1| = o(\gamma)$ for a parameter $\gamma \rightarrow 0$. Further, we assume that p is a point such that $\|\sigma - p\| = r$ is bounded away from zero.

We consider the γ -region about p , and construct inscribed and circumscribed figures about the γ -region. Let l_1 be the intersection of the line through σ and p and the γ region. The length of l_1 is thus $(1 + o(1))r$. Let l_2 be the segment perpendicular to l_1 which intersects the midpoint of $\overline{p\sigma}$. The length of l_2 is $(1 + o(1))r\sqrt{\gamma}$. The convex hull of $l_1 \cup l_2$ forms a rhombus R_1 contained entirely in the γ -influence region with area $(1 + o(1))r^2\sqrt{\gamma}$. We can also form a circumscribed rectangle R_2 which is axis-parallel to the segments l_1 and l_2 , and which has asymptotically twice the area of the rhombus.

Now, let p be one of our randomly chosen points v_i , and further assume it has distance $\|v_i - \sigma\|$ bounded away from 0 and $1/\sqrt{\pi}$, which happens with positive probability. Next, set $\gamma = (nr)^{-2}$, which causes the rhombus and rectangle to have areas

$(1 + o(1))n^{-1}$ and $(1 + o(1))2n^{-1}$, respectively. Further, for n sufficiently large the rhombus, ellipse, and rectangle all lie in K . Thus, the event that one of $\{v_1, \dots, v_n\} - \{v_i\}$ falls in the rhombus and all others avoid the rectangle is at least

$$\binom{n}{1} \text{Area}(R_1)(1 - \text{Area}(R_2))^{n-2} = \binom{n}{1} n^{-1}(1 + o(1))(1 - 2n^{-1}(1 + o(1)))^{n-2} > c_1 > 0,$$

for some positive constant c_1 .

Now, for γ bounded above and hence the length of l_2 bounded above, we see that

$$\frac{\text{Area}(H(v_i, \beta) \cap R_1)}{\text{Area}(R_1)} > c_2 > 0.$$

Summarizing, the probability that there is exactly one point in the rhombus apart from v_i and that all other points fall outside the rectangle is bounded below by some positive constant. Further, the probability that a point chosen uniformly in the rhombus falls in $H(v_i, \beta)$ is bounded below by a positive constant. Thus, the total probability that the target of v_i is local is bounded below by a positive constant, which shows that $\mathbb{E}[X_i] > c > 0$ for some constant c , hence $\mathbb{E}[\rho(\beta)] > c > 0$. The upper bound is similar. □

5.5 Further Directions

- In both \mathcal{T} and the FKP model, the weight α is a fixed function of n . A. Flaxman (personal communication) suggests looking at a non-homogeneous variant of these models. To wit, for the \mathcal{T} model he suggests setting

$$\phi_i(x) = \alpha(i) \|v_i - x\| + \|x - \mathbf{o}\|,$$

where here $\alpha(i)$ varies with i .

- The results of Theorem 5.2.1 and Theorem 5.2.6 are similar to an infinite Pólya urn model studied in [30]. In this model, at each time step a ball is added to an existing urn with probability $1 - p$, else a new urn is created. If a ball is to be placed into an existing urn, then each urn is chosen with probability

proportional to m^γ , where m is the number of balls in the urn. Under the regimes $\gamma > 1$, $\gamma < 1$, and $\gamma = 1$ the bin distributions are exponential, dominated by a single bin, and power law a.a.s. The question of why both combinatorial and geometric selection rules produce similar phenomena remains open.

- While random graphs in metric spaces already provide the correct setting for notions of locality, the model \mathcal{T}_α shows that random graphs models with edges chosen in a biased, or spatially inhomogeneous manner, allow for more complex behavior (compare \mathcal{T}_α to the homogeneous geometric random graph model of Penrose). One outstanding question available in this setting is understanding the extent to which dimensionality can be defined intrinsically for complex network models, and how dimensionality influences the degree distribution, edge locality, etc.

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6 A Random-Connection Model for Geometric Graphs

We present here a new random geometric graph model, which we label the *random-connection* model in analogy to the related model in continuum percolation. Much as the $G_{n,r}$ random graph model can be obtained from boolean percolation by restricting to a compact subset of \mathbb{R}^d , our model here can be obtained from the more general random-connection model of continuum percolation by restriction to a compact subset. Loosely speaking, our model is constructed as follows: given some function ρ , and n random points in a metric space, we link points x_i and x_j with probability $\rho(d(x_i, x_j))$, where here d is the metric. We shall insist that ρ is non-increasing and hence the probability that two vertices share an edge should not increase if they are made to lie further apart.

The motivation for analyzing such a model comes from a number of directions. Our model will be general enough to include the $G_{n,r}$ and $G_{n,p}$ models, and in particular we are interested in models which incorporate features of each. Chen in [29] investigates a family of models which interpolate between these two, though the analysis is not entirely rigorous. Bollobás et al.[22] propose a very general sparse random graph model in which each pair of vertices (x_i, x_j) are connected by an edge independently with probability $\chi(x_i, x_j)$ (admissibility criteria on χ force the resulting graph to be sparse). They study the distribution of degrees and the emergence of a giant component as a function of χ . In [21] the same authors demonstrate that percolation in the boolean model can be analyzed in the context of the general graph model presented in [22]. Finally, there is the question of scale. Graphs are said to be “scale-free” in time or space if they generate “similar” graphs on a wide range of time or

length scales.¹ While graphs which are “scale-free” in time have been investigated to some extent, random graph models which are “scale-free” in space have received little attention.

We present here a preliminary report on some properties of our model. In particular, we focus on the relation between the geometry of the underlying metric space and the function ρ . Our goals are twofold: first, we investigate the connection probability for a given edge, which allows us to attack the question of connectivity in our model (note that our model is not limited to sparse graphs). Second, we investigate how our model scales at different lengths, and in particular how the function ρ affects this scaling.

6.1 Model

We shall require a few definitions. Our terminology follows that found in [22].

Definition 6.1.1 (Ground space.). *We say that (\mathcal{S}, d, μ) is a ground space if:*

1. (\mathcal{S}, d) is a separable, complete metric space with metric d ,²
2. μ is a Borel probability measure on \mathcal{S} .

We next need a notion of random points in a ground space.

Definition 6.1.2 (Vertex space.). *Let N_n be a sequence of positive integer-valued random variables such that $\mathbb{E}[N_n] = n$ for $n = 1, 2, \dots$. We say that $(\mathcal{S}, d, \mu, \{\mathbf{x}_{N_n}\}_{n=1}^{\infty})$ is a vertex space if:*

1. (\mathcal{S}, d, μ) is a ground space and,
2. For each n , $\{\mathbf{x}_1, \dots, \mathbf{x}_{N_n}\} \subseteq \mathcal{S}$ is a sequence of N_n iid points distributed according to μ .

¹ The proper definition of the term “scale-free” is a contentious issue in itself, and we shall make no more attempt at a rigorous definition. See Chapter 3.5 in [33] for a more complete discussion.

² It is common in random measure theory to understand \mathcal{S} to be metrizable but make no use of any particular metric. It is worth emphasizing that this will *not* be the case here. Rather, the choice of metric will be seen to play an explicit role in the structure of the graph model we present.

Further, for each Borel subset A of \mathcal{S} and $n = 1, 2, \dots$ we let

$$\nu_n(A) := \#\{i \mid \mathbf{x}_i \in A, i \leq N_n\}$$

be a (random) counting measure on \mathcal{S} .

Note that, given the existence of a ground space and positive integer-valued random variable N_n , the existence of a vertex space can be given constructively. Specifically, one can construct the sequence $\{\mathbf{x}_i\}_{i=1}^\infty$ of iid points by the Kolmogorov Extension Theorem [43] and consider the first N_n points to be the sequence required. Such a construction may also be used to *couple* the sequences $\{\mathbf{x}_1, \dots, \mathbf{x}_{N_n}\}$, as well as being used to canonically label the points. Also, note that the counting measures ν_n have the property that $\mathbb{E}[\nu_n(A)] = n\mu(A)$ for every Borel subset A .³

Example 1 (Uniform vertex space.). *Let (\mathcal{S}, d, μ) be a ground space. If we let $N_n = n$ in the above construction, then this corresponds to choosing n points according to the n -fold product of μ . We refer to this vertex space as the uniform vertex space on (\mathcal{S}, d, μ) , and write $\mathcal{X}_n = \mathcal{X}_n(\mathcal{S}, \mu)$.*

Example 2 (Poisson vertex space.). *Again, let (\mathcal{S}, d, μ) be a ground space. If we let $N_n \sim \text{Poi}(n)$, then we have the Poisson vertex space $\mathcal{P}_n = \mathcal{P}_n(\mathcal{S}, \mu)$. The Poisson vertex space has the important property that for each n and disjoint Borel subsets $A, B \subseteq \mathcal{S}$, $\nu_n(A)$ and $\nu_n(B)$ are independent Poisson random variables of mean $n\mu(A)$ and $n\mu(B)$, respectively.*

Remark 6.1.3. Observe that the uniform and Poisson vertex spaces correspond to uniform and Poisson *point processes* (see [66, 73]), which is how they are generally referred to in the percolation literature. In particular, the independent Poisson distributions of $\nu_n(A)$ and $\nu_n(B)$ are a standard part of Poisson point process theory (though this can also be verified via straightforward but tedious calculation). We shall use the notion of vertex spaces for maximum clarity of exposition as well as consistency with [22].

In what follows we shall construct a threshold function which depends only on the metric distance between two points in \mathcal{S} .

³ The measure $n\mu$ is called the *intensity measure* in random measure theory.

Definition 6.1.4 (Threshold function.). *We call a non-increasing function $\rho : [0, \infty) \rightarrow [0, 1]$ a threshold function. For a fixed, positive sequence r_n and threshold function ρ , we define the threshold sequence*

$$\rho_n(x) := \rho(x/r_n),^4 \quad n = 1, 2, \dots$$

Typically we will assume $r_n \rightarrow 0$.

We are now prepared to define our random graph model.

Definition 6.1.5 (Model.). *Let $(\mathcal{S}, d, \mu, \{\mathbf{x}_{N_n}\})$ denote a vertex space, and let ρ_n be a threshold sequence determined by threshold ρ and sequence r_n . Define the (sequence of) random graphs $G_n = G_n(\mathcal{S}, d, \mu, \{\mathbf{x}_{N_n}\}, \rho, \{r_n\})$ to have vertex set $X_n = \{x_1, \dots, x_{N_n}\}$. Further, for any $\mathbf{x}_i, \mathbf{x}_j \in X_n$ let $\mathbf{x}_i \mathbf{x}_j \in E(G_n)$ occur independently with probability $\rho_n(d(\mathbf{x}_i, \mathbf{x}_j))$.*

Remark 6.1.6. For the remainder we shall implicitly invoke the fact that $\rho_n \circ d : \mathcal{S} \times \mathcal{S} \rightarrow [0, 1]$ is $\mu \otimes \mu$ measurable. This follows as ρ_n is non-increasing and d is continuous. No further mention of measurability will thus be made.

The key feature of our model is the assumption that the connection probability of any edge is determined completely by the distance of the endpoints under the metric d . We list a few key examples.

Example 3 (Cube.). *We shall refer to the space $[0, 1]^d$ as the d -cube or d -torus depending on the use of the Euclidean or toroidal metric,⁵ respectively. In either case the space $([0, 1]^d, d, \lambda)$ is then a ground space if λ is Lebesgue measure and d one of the two metrics. If one uses the threshold function $\rho(x) = \mathbb{1}_{\{x \leq 1\}}$ we obtain the boolean model since $\rho_n(x) = \mathbb{1}_{\{x \leq r_n\}}$. On the other hand, equipping the cube with the discrete metric (i.e. $d(x, y) = 1$ if $x \neq y$) and setting $\rho(x) = 1/x$ gives the Erdős-Rényi random graph $G_{n,p}$ with probability one if $r_n = p$.*

⁴ In the statistics literature we would say that $\{\rho_n\}$ constitutes a *scale family*.

⁵ The toroidal metric can be written explicitly as [76]:

$$d_T(v_i, v_j) = \min_{z \in \mathbb{Z}^d} \|v_i - v_j - z\|.$$

The key property of the toroidal metric is that it removes boundary effects; the natural action of \mathbb{R}^d on $[0, 1]^d$ is transitive.

Example 4 (Sphere.). *We shall consider the (unit) d -sphere $S^d \subset \mathbb{R}^{d+1}$. The sphere has a natural metric induced by the length of the shortest geodesic between two points. To obtain a ground space we may let σ denote the normalized Riemannian surface measure on S^d . We shall find that the spherical symmetry of S^d makes it the most natural space for calculation.*

For the above models, indeed even in the boolean model on the cube, it is clear that not all edges occur independently. In particular, we record the following fact which follows directly from the definition of our model.

Proposition 6.1.7 (Independence). *Let $\{e_i\}_{i=1}^m \subset \binom{V(G_n)}{2}$ and let $\{X_i\}_{i=1}^m$ be the corresponding indicators. Then $\{X_i\}_{i=1}^m$ are jointly independent if $\{e_i\}_{i=1}^m$ form a forest.*

6.1.1 Connection Functions

Set for the moment $X = \mathcal{X}_n$ on the cube $[0, 1]^2$, and let ρ be a connection function. The two choices $\rho(x) = \mathbb{1}_{x \leq r}$ and $\rho(x) = p$, where r and p are functions of n , give the boolean model [75] and the $G_{n,p}$ Erdős-Rényi model, respectively. In terms of its connection function, the boolean model is characterized primarily by having finite—typically vanishing—support. The consequences of this choice include the fact that many possible induced subgraphs are prohibited (e.g. high-valency stars). The Erdős-Rényi graph, as we have constructed it here, completely ignores the underlying geometry and hence the connection function is constant.

By allowing a wide choice of connection functions, we may obtain graphs which retain some features of either extreme. We thus introduce two classes of interest.

Example 5 (Polynomial Tails.). *We shall say that the function ρ has a polynomial tail of degree α if $\lim_{x \rightarrow \infty} \rho(x)x^\alpha$ is a positive constant. The function $f^\alpha = \mathbb{1} \wedge x^{-\alpha}$ is the simplest such function. The function $g^\alpha = \left(\frac{1}{1+x}\right)^\alpha$ is a bounded, differentiable version of f^α . Consider the scale families $\{f_n^\alpha\}$ and $\{g_n^\alpha\}$ and the (sequence of) random graphs $\{G(X, f_n^\alpha)\}$ and $\{G(X, g_n^\alpha)\}$. Both share the property that for vertices of distance $x \ll r_n$ the probability of sharing an edge is bounded below by a positive constant, whereas for $x \gg r_n$ the probability decreases as $x^{-\alpha}$.*

Example 6 (Exponential Tails.). *In contrast to the polynomial case, we shall say that the function ρ has a exponential tail if there is some $c > 0$ such that $\rho(x) = O(e^{-cx})$ as $x \rightarrow \infty$.*

6.2 Combinatorial Properties

Many possible combinatorial questions exist about our geometric random graph model, e.g. the emergence of a giant component. In the abstract setting some of these questions are addressed in [22]. Our goal is orthogonal to such an approach, however, in that we seek to understand the quantitative interplay between the geometry of \mathcal{S} and the threshold ρ .

In this section we address the edge probabilities, degrees of a given point, and the connectivity threshold. Most of our focus is on the case of \mathcal{S} in the class of d -dimensional manifolds (or manifolds-with-boundary).

6.2.1 Edge Probability

We shall henceforth let $G_n = G_n(\mathcal{S}, d, \mu, \{\mathbf{x}_{N_n}\}, \rho, \{r_n\})$. We define the *expected edge probability* to be

$$p_n = \iint_{\mathcal{S} \times \mathcal{S}} \rho_n(d(\mathbf{x}, \mathbf{y})) d\mu(\mathbf{x}) d\mu(\mathbf{y}).$$

The nomenclature is obvious. We have

Proposition 6.2.1.

$$\mathbb{P}(\mathbf{x}_i \mathbf{x}_j \in E(G_n) \mid i, j \leq N_n) = p_n$$

and

$$\mathbb{E}[e(G_n)] = \frac{p_n}{2} \mathbb{E}[(N_n)_2],$$

where $\mathbb{E}[(X)_2]$ is the second factorial moment of a random variable X .

Proof. The first equation follows from the definition of the model. For the second equation, we may write $e(G_n) = \sum_{1 \leq i < j \leq N_n} \mathbb{1}_{\{\mathbf{x}_i \sim \mathbf{x}_j\}}$. By linearity of expectation we have $\mathbb{E}[e(G_n)] = p_n \mathbb{E} \left[\binom{N_n}{2} \right] = \frac{p_n}{2} \mathbb{E}[(N_n)_2]$. \square

Example 7. For a uniform vertex space the number of edges is $p_n \binom{n}{2}$. For a Poisson vertex space the number is $\frac{n^2 p_n}{2}$ as the k^{th} factorial moment of a $\text{Poi}(\lambda)$ is λ^k .

The setting we are most interested in is that of \mathcal{S} which are smooth manifolds (or manifolds-with-boundary). In this case we have the following (for terminology see [41]):

Theorem 6.2.2. Let \mathcal{S} denote a d -dimensional, compact, connected, manifold-with-boundary with smooth interior and \mathcal{C}^2 boundary. Denote by μ the normalized Riemannian volume. Further, let ρ be a polynomial tail of degree $\alpha \geq 1$ such that $\rho(x)x^\alpha \rightarrow c_\rho > 0$ as $x \rightarrow \infty$. If $d : \mathcal{S} \times \mathcal{S} \rightarrow [0, \infty)$ is the distance function induced by the Riemannian metric, then for any $r_n \rightarrow 0$ we have

1. If $\alpha > d$:

$$\lim_{n \rightarrow \infty} r_n^{-d} p_n = \frac{1}{\text{Vol}_d(\mathcal{S})} \int_{\mathbb{R}^d} \rho(|x|) dx.$$

2. If $\alpha = d$:

$$\lim_{n \rightarrow \infty} \left(r_n^d \ln(r_n^{-1}) \right)^{-1} p_n = \frac{\pi_d}{\text{Vol}_d(\mathcal{S})} c_\rho.$$

3. If $\alpha < d$:

$$\lim_{n \rightarrow \infty} r_n^{-\alpha} p_n = c_\rho \int_{\mathcal{S}} \int_{\mathcal{S}} d(\mathbf{x}, \mathbf{y})^{-\alpha} d\mu(\mathbf{x}) d\mu(\mathbf{y}).$$

Here, $\pi_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of a unit ball in \mathbb{R}^d . In particular, the integral on the right hand side above exists.

Proof. In what follows we shall drop the subscript n on r_n . Fix $\varepsilon > 0$. Put

$$U_\delta = \{x \in \mathcal{S} \mid \text{The injectivity radius of } x > \delta\}.$$

As the injectivity radius of every non-boundary point is positive, it is clear that $U_\delta \uparrow \mathcal{S} \setminus \partial \mathcal{S}$ as $\delta \rightarrow 0$. We shall assume without loss of generality that the Riemannian volume $\text{Vol}_d(\mathcal{S}) = 1$. Thus, there is some $\delta > 0$ such that $\text{Vol}_d(U_\delta) \geq 1 - \varepsilon$.

For $x \in U_\delta$, consider the integral

$$\int_{\mathcal{S}} \rho(d(\mathbf{x}, \mathbf{y})/r) d\mu(\mathbf{y}). \quad (6.1)$$

We shall estimate the above integral as

$$\int_{B(x,\delta)} \rho(d(\mathbf{x},\mathbf{y})/r) d\mu(\mathbf{y}) + \int_{\mathcal{S} \setminus B(x,\delta)} \rho(d(\mathbf{x},\mathbf{y})/r) d\mu(\mathbf{y}). \quad (6.2)$$

The second integrand is bounded by $\rho(\delta/r)$. By our assumption on the tail of ρ we may assume $\rho(\delta/r) \leq c(\delta/r)^{-\alpha} = O(r^\alpha)$ for some positive constant c .

For the first integrand, we may write

$$\int_{B(x,\delta)} \rho(d(\mathbf{x},\mathbf{y})/r) d\mu(\mathbf{y}) = \int_0^\delta \rho(t/r) dV_x(t),$$

where $V_x(t)$ is the volume of a geodesic ball of radius t centered at x . By a theorem of A. Gray [55], we have

$$V_x(t) = \frac{\pi_d t^d}{d} \left(1 - \frac{\tau(x)}{6(d+2)} t^2 + O(t^4) \right), \quad (6.3)$$

where $\tau(x)$ is the trace of the Ricci curvature at x and the hidden constant depends on the scalar curvatures at x and $\pi_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of a unit ball in \mathbb{R}^d . Now, as \mathcal{S} is compact the Ricci curvature is bounded, hence $|\frac{d}{dt} V_x(t) - \pi_d t^{d-1}| < C t^{d+1}$ for some constant C depending on \mathcal{S} . Hence

$$\begin{aligned} \left| \int_0^\delta \rho(t/r) \frac{d}{dt} V_x(t) dt - \int_0^\delta \rho(t/r) \pi_d t^{d-1} dt \right| &\leq C \int_0^\delta \rho(t/r) t^{d+1} dt \\ &= C \int_0^{\delta/r} \rho(u) r^{d+2} du \leq r^{d+2} C \int_0^{\delta/r} u^{-\alpha} du = r^{d+2} c', \end{aligned}$$

where the last line follows from some $c' > 0$ by the tail bound on ρ and C .

Now, another application of the change of variables formula gives

$$\int_0^\delta \rho(t/r) \pi_d t^{d-1} dt = \int_0^{\delta/r} \rho(u) \pi_d r^d u^{d-1} du$$

By the tail bound on ρ , if $\alpha > d$ then

$$\left| \int_0^{\delta/r} \rho(u) \pi_d r^d u^{d-1} du - \int_0^\infty \rho(u) \pi_d r^d u^{d-1} du \right| \leq \varepsilon r^d$$

where in particular both terms of the difference exist. As a result, we can write

$$\begin{aligned} \lim_{r \rightarrow 0} r^{-d} \int_{\mathcal{S}} \rho(d(\mathbf{x}, \mathbf{y})/r) d\mu(\mathbf{y}) &= (1 + O(\varepsilon)) \int_0^\infty \rho(u) \pi_d u^{d-1} du \\ &= (1 + O(\varepsilon)) \int_{\mathbb{R}^d} \rho(|x|) dx, \end{aligned} \quad (6.4)$$

for any ε . Thus

$$\begin{aligned} \lim_{r \rightarrow 0} r^{-d} \int_{\mathcal{S}} \int_{\mathcal{S}} \rho(d(\mathbf{x}, \mathbf{y})/r) d\mu(\mathbf{y}) d\mu(\mathbf{x}) \\ \geq \lim_{r \rightarrow 0} \int_{U_\delta} r^{-d} \int_{\mathcal{S}} \rho(d(\mathbf{x}, \mathbf{y})/r) d\mu(\mathbf{y}) d\mu(\mathbf{x}) \\ = \text{Vol}_d(U_\delta) (1 + O(\varepsilon)) \int_{\mathbb{R}^d} \rho(|x|) dx. \end{aligned}$$

The equality follows from bounded convergence. As ε is arbitrary, the right-hand side is lower-bounded by $\int_{\mathbb{R}^d} \rho(|x|) dx$. For $x \notin U_\delta$, Equation 6.4 is an upper-bound,⁶ and hence the lower bound is the truth.

Similarly, if $\alpha = d$ we have

$$\int_0^{\delta/r} \rho(u) \pi_d r^d u^{d-1} du = (1 + O(\varepsilon)) c_\rho \pi_d r^d \ln(r^{-1}),$$

where $c_\rho = \lim_{x \rightarrow \infty} \rho(x) x^{-\alpha}$. Noting that the second term in 6.2 is $o(r \ln(1/r))$, we can repeat the above to give the formula claimed for $\alpha = d$.

For $\alpha < d$

$$\int_0^{\delta/r} \rho(u) \pi_d r^d u^{d-1} du = (1 + O(\varepsilon)) c_\rho \pi_d (d - \alpha)^{-1} \delta^{d-\alpha} r^\alpha.$$

Hence,

$$\lim_{r \rightarrow 0} r^{-\alpha} \int_{\mathcal{S}} \int_{B(x, \delta)} \rho(d(\mathbf{x}, \mathbf{y})/r) d\mu(\mathbf{y}) d\mu(\mathbf{x}) \leq (1 + O(\varepsilon)) c_\rho \pi_d (d - \alpha)^{-1} \delta^{d-\alpha}$$

⁶ In fact, the existence of the appropriate limit in this case is more delicate, and requires one to analyze points near the boundary. Critically, as \mathcal{S} is compact the scalar curvature is bounded above and below and hence the only obstruction to large injectivity radius are points near the boundary (the assumption of \mathcal{C}^2 boundary guarantees this, though less stringent conditions are probably sufficient). We will forego the additional bookkeeping.

by dominated convergence (again, the inequality follows from considering points $x \notin U_\delta$ and is an equality for the manifold case for δ sufficiently small). On the other hand, we have

$$\lim_{r \rightarrow 0} r^{-\alpha} \iint_{\{d(\mathbf{x}, \mathbf{y}) \geq \delta\}} \rho(d(\mathbf{x}, \mathbf{y})/r) d\mu(\mathbf{x}) d\mu(\mathbf{y}) = \iint_{\{d(\mathbf{x}, \mathbf{y}) \geq \delta\}} c_\rho d(\mathbf{x}, \mathbf{y})^{-\alpha} d\mu(\mathbf{x}) d\mu(\mathbf{y}).$$

As $\rho(\delta/r)r^{-\alpha}$ is bounded for r sufficiently small the above also follows via dominated convergence. Hence, as ε was arbitrary and $\delta \downarrow 0$ with $\varepsilon \downarrow 0$,

$$\begin{aligned} \lim_{r \rightarrow 0} r^{-\alpha} \int_{\mathcal{S}} \int_{\mathcal{S}} \rho(d(\mathbf{x}, \mathbf{y})/r) d\mu(\mathbf{x}) d\mu(\mathbf{y}) &= \lim_{\delta \rightarrow 0} c_\rho \iint_{d(\mathbf{x}, \mathbf{y}) \geq \delta} d(\mathbf{x}, \mathbf{y})^{-\alpha} d\mu(\mathbf{x}) d\mu(\mathbf{y}) \\ &= c_\rho \int_{\mathcal{S}} \int_{\mathcal{S}} d(\mathbf{x}, \mathbf{y})^{-\alpha} d\mu(\mathbf{x}) d\mu(\mathbf{y}). \end{aligned}$$

□

Remark 6.2.3. If one assumes only that $\rho(x)x^\alpha \rightarrow 0$ for some $\alpha > d$, the first formula of the theorem still holds. This holds in particular, if ρ has exponential tails.

Theorem 6.2.2 thus demonstrates how the threshold function ρ and the space \mathcal{S} interact. We see that for $\alpha > d$, the function ρ determines the edge probability entirely. If $\alpha < d$, the edge probability is determined only by the geometry of \mathcal{S} and the asymptotics of ρ . At the boundary, $\alpha = d$, only the volume of \mathcal{S} and the asymptotics of ρ determine the edge probability.

It is worth noting that in the case of random-connection percolation as studied in [73], functions with polynomial tails of exponent $\alpha \leq d$ are disallowed since the relevant integrals do not converge. In reference to Theorem 6.2.2 this shows that random-connection percolation is essentially a local theory, not unlike boolean percolation. For compact \mathcal{S} and finite point processes those thresholds with $\alpha \leq d$ the situation is different; the probability that a given edge $x_i x_j$ occurs if $d(x, y) > \delta$ for some fixed δ is bounded from below by a positive constant.

Example 8. For the sphere $\mathcal{S} = S^2$ with threshold function $\rho(x) = (1+x)^{-\alpha}$ we can

compute directly.⁷ We have⁸

$$p_\alpha = \iint_{\mathcal{S} \times \mathcal{S}} \rho_n(d(\mathbf{x}, \mathbf{y})) d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) = \int_0^\pi \left(\frac{r_n}{r_n + x} \right)^\alpha \frac{\sin x}{2} dx.$$

For $r_n \rightarrow 0$

$$\begin{aligned} (\alpha = 1) \quad p_n &= \frac{r_n \text{Si}(\pi)}{2} + \frac{r_n^2}{2} (-1 + \gamma + \ln(r_n) - \text{Ci}(\pi)) + \frac{r_n^3}{2} \left(-\frac{\text{Si}(\pi)}{2} + \frac{1}{2\pi} \right) + O(r_n^4), \\ (\alpha = 2) \quad p_n &= \frac{r_n^2}{2} ((-\gamma - \ln(r_n) + \text{Ci}(\pi)) + (\text{Si}(\pi) - \pi^{-1}) r_n) + O(r_n^4), \\ (\alpha = 3) \quad p_n &= \frac{1}{4} r_n^2 + 1/4 \frac{1 - \text{Si}(\pi)\pi}{\pi} r_n^3 + O(r_n^4). \end{aligned}$$

6.2.2 Degree

Let us turn to the question of degree. We have the following basic result.

Proposition 6.2.4. *Let G_n be as in Definition 6.1.5. Then*

$$\mathbb{E}[\deg(\mathbf{x}_i) \mid i \leq N_n] = p_n \mathbb{E}[N_n - 1 \mid i \leq N_n], \quad i = 1, 2, \dots$$

Hence, for the uniform vertex space the degree of a vertex \mathbf{x}_i has expected degree $(n - 1)p_n$ if $i \leq n$.

Recall that in the boolean model, the regimes given by $r_n^d n \rightarrow c_1$ and $r_n^d \frac{n}{\ln n} \rightarrow c_2$ for positive constants c_1 and c_2 are referred to as the *thermodynamic limit* and *connectivity threshold*, respectively. By Theorem 6.2.2 and the above, this corresponds to constant and logarithmic expected degrees. Expanding this terminology to the case $1 \leq \alpha < d$, we shall say that r_n is in the thermodynamic limit if $r_n^\alpha n \rightarrow c_3 > 0$ and that r_n is in the connectivity threshold if $r_n^\alpha \frac{n}{\ln n} \rightarrow c_4 > 0$. One may extend the case $\alpha = d$ in the obvious manner.

Having quantified the expected degree of a vertex and the scaling regimes of interest, the next question is the full distribution of $\deg(x_i)$. For the space \mathcal{X}_n , Proposition 6.1.7 gives the answer: $\deg(x_i) \sim \text{Bin}(n - 1, p_n)$ so long as $i \leq n$. For general vertex spaces this is simply $\text{Bin}(N_n - 1, p_n)_{\mathbb{1}_{\{i \leq N_n\}}}$.

⁷Via Maple[®].

⁸ Recall the following notation: $\text{Si}(z) = \int_0^z \frac{\sin(t)}{t} dt$, $\text{Ci}(z) = \gamma + \ln z + \int_0^z \frac{\cos(t) - 1}{t} dt$ where γ is the Euler-Mascheroni constant. Observe that $\text{Ci}(z)$ has a logarithmic singularity at $z = 0$.

Next, one may ask for the number of vertices of degree k in a graph. In the thermodynamic limit this question is answered by Bollobás et al (see chapter 13, [22]) for a special class of cutoff functions. Using their techniques one can show:

Theorem 6.2.5 (Bollobás et al. [22]). *Let \mathcal{S} as in Theorem 6.2.2, ρ a threshold with $\rho(x)x^\alpha \rightarrow c_\rho$, $1 \leq \alpha < d$, and $r_n^\alpha n \rightarrow c > 0$. Let G_n be a random graph on this ground state. If we denote by Z_k the vertices of degree k in G_n , then*

$$Z_k/n \xrightarrow{\mathbb{P}} \int_{\mathcal{S}} \frac{\lambda(\mathbf{x})^k}{k!} e^{-\lambda(\mathbf{x})} d\mu(\mathbf{x}),$$

where $\lambda(\mathbf{x}) = c_\rho \int_{\mathcal{S}} d(\mathbf{x}, \mathbf{y})^{-\alpha} d\mu(\mathbf{y})$.

Hence, in the thermodynamic limit the number of vertices of degree k tends in probability to a mixed Poisson distribution.

For our model, determining Z_k for general thresholds and scales is more challenging. We show here how to apply the Chen-Stein method to vertices of degree 0, in anticipation of the connection result in the next section. Recall that the *total variation distance* for \mathbb{N} -valued random variables is given by

$$d_{\text{TV}}(X, Y) = \sup_{A \subseteq \mathbb{N}} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| = \frac{1}{2} \sum_{n \in \mathbb{N}} |\mathbb{P}(X = n) - \mathbb{P}(Y = n)|.$$

In [15] Barbour et al. given a number of formulations under which a sum of indicators tends in distribution to a Poisson random variable. We shall make use of the following coupling result:

Lemma 6.2.6 (Poisson Approximation [15]). *Let $W = \sum_{a \in A} I_a$ be a sum of indicators I_a indexed to some finite set A . For each a , let U_a and V_a be coupled such that $U_a \stackrel{d}{=} W$ and $V_a + 1 \stackrel{d}{=} W$ conditioned on $I_a = 1$ if such a coupling exists. Setting $\lambda = \sum_{a \in A} \pi_a$ with $\pi_a = \mathbb{E}[I_a]$ we have*

$$d_{\text{TV}}(W, \text{Poi}(\lambda)) \leq \lambda^{-1} (1 - e^{-\lambda}) \sum_{a \in A} \pi_a \mathbb{E}[|U_a - V_a|].$$

Remark 6.2.7. The bound on the right hand side depends on the quality of the coupling.

We then have

Proposition 6.2.8. *Let G_n be of the form given in Definition 6.1.5 and fix $N_n = n$ (i.e. the uniform model). If Z_o denotes the number of vertices of degree zero then*

$$d_{\text{TV}}(Z_o, \text{Poi}(n)) \leq (1 - e^{n(1-p_n)^{n-1}}) ((1 - p_n)^{n-1} + (n-1)p_n(1 - p_n)^{n-2}).$$

Proof. Writing $Z_o = \sum_{i=1}^n \mathbb{1}_{\{\deg(v_i)=0\}}$. For each i , we then form the modified sum V_i by removing all edges adjacent to x_i and counting the number of degree zero indicators. We then observe that Z_o and U_i are coupled as in Lemma 6.2.6, and hence to apply the lemma we need only compute $\mathbb{E}|W - V_i|$.

Comparing where the relevant indicators differ, we have

$$\begin{aligned} \mathbb{E}|W - V_i| &\leq \mathbb{P}(\deg(x_i) = 0) + \sum_{j \neq i} \mathbb{P}(\deg(x_j) = 1 \wedge x_i x_j \in E(G_n)) \\ &= (1 - p_n)^{n-1} + (n-1)p_n(1 - p_n)^{n-2}. \end{aligned}$$

Here, the last line follows by Proposition 6.1.7. As $\mathbb{P}(\deg(x_i) = 1) = (1 - p_n)^{n-1}$, Lemma 6.2.6 yields our result. \square

Remark 6.2.9. The above proposition should generalize to the non-constant N_n case so long as N_n is sufficiently concentrated about n .

Example 9. *In the uniform model, \mathcal{X}_n , if $p_n = \frac{\ln n + c + o(1)}{n}$ then $\mathbb{P}(Z_o > 0) \rightarrow 1 - e^{-e^{-c}}$ as $n \rightarrow \infty$. Hence, if $c \rightarrow \infty$ a.a.s. no vertex is isolated, whereas if $c \rightarrow 0$ a.a.s. isolated vertices exist.*

6.2.3 Connection

A necessary condition for the connectivity of a graph G_n is the disappearance of isolated vertices. Hence, as shown in Example 9, G_n may not be connected if $p_n \ll \frac{\ln n}{n}$. If G_n obeys the hypotheses of Theorem 6.2.2 (that is, the d -manifold case), we see that lack of connectivity is determined by the scales r_n . If the connection function ρ has polynomial tail of degree α , then we observe that connectivity is prohibited

if $r_n \ll \left(\frac{\ln n}{n}\right)^{1/(\alpha \wedge d)}$ in the case $\alpha \neq d$. Curiously, if $\alpha = d$ then $p_n = \Theta\left(\frac{\ln n}{n}\right)$ for $r_n = n^{-1/d}$.

We can demonstrate a connectivity result for the manifold case. The idea is to use the classical connectivity result for $G_{n,p}$ due to Erdős and Rényi [47]. Our result is as follows:

Theorem 6.2.10. *Let \mathcal{S} be a compact, connected d -manifold, μ normalized Riemannian volume, and ρ a threshold with polynomial tail of degree α .*

Let $G_n = (\mathcal{S}, d, \mu, \mathcal{X}_n, \rho, r_n)$. There is a positive constant C such that if $r_n \geq C \left(\frac{\ln n}{n}\right)^{\frac{1}{\alpha \wedge d}}$ then G_n is connected a.a.s.

We need the following formulation of connectivity in $G_{n,p}$, taken from [19]:

Theorem 6.2.11 (Erdős and Rényi (1961)). *If $p = \frac{\ln n + c + o(1)}{n}$ for some $c \in \mathbb{R}$ then $G_{n,p}$ is connected with probability tending to $e^{-e^{-c}}$.*

We shall need some simple structural lemmas.

Lemma 6.2.12. *\mathcal{S} can be covered by a collection $\{B(p_i, \delta)\}_{i=1}^m$, where $m = O(\delta^{-d})$ as $\delta \rightarrow 0$.*

Proof. The proof is via packing. Construct a maximal packing of δ balls in \mathcal{S} . By (6.3) the volume of any such ball is $\Theta(\delta^d)$. By the disjointness of the balls, we have $m \text{Vol}_d(B(p_i, \delta)) \leq \text{Vol}_d(\mathcal{S})$ hence $m = O(\delta^{-d})$. The set of balls $\{B(p_i, \delta)\}$ then forms a covering of \mathcal{S} by maximality of the packing and the triangle equality. \square

Lemma 6.2.13. *For $c > 0$, there is a positive constant c' such that for $\delta \geq c' \left(\frac{\ln n}{n}\right)^{1/d}$ the number of points in a fixed ball of radius δ is at least $\ln n$.*

Proof. The probability that a single point falls in a ball of radius δ is at least $\pi_d \delta^d / 2$ by (6.3). Hence, if X is the number of points falling in one such ball then by the Chernoff bound (Lemma 1.3.2) we have

$$\mathbb{P}\left(X \leq \pi_d \delta^d / 4\right) \leq e^{-n \delta^d \pi_d / 8},$$

since the volume of any δ ball is at least $\pi_d \delta^d / 2$ for δ sufficiently small. Setting c' sufficiently large bounds the right hand side by n^{-c} . \square

Proof of Theorem 6.2.10. We begin with the case $\alpha \geq d$. Let us fix a collection of balls $\{B(p_i, \delta)\}_{i=1}^m$ according to Lemma 6.2.12 where $\delta = r_n$. By Lemma 6.2.13 and the union bound $n_i = v_n(B(p_i, \delta)) \geq \frac{2}{\rho(2)} \ln n$ for each $1 \leq i \leq m$ (setting $c = 2$, say) with failure probability $o(1)$. Consider the probability that two points in any such δ ball share an edge. As the diameter of any ball is 2δ this probability is bounded below by $\rho_n(2\delta) = \rho(2) > 0$.

Consider, for each $1 \leq i \leq m$, the subgraph of G_n restricted to the ball $B(p_i, \delta)$. Couple this subgraph with an Erdős-Rényi random graph $G_{n_i, \rho(2)}$ on the same vertex set, noting that connection in $G_{n_i, \rho(2)}$ will imply connection in the subgraph of G_n restricted to $G_\delta(p_i)$. If

$$\rho(2) = \frac{\ln n_i + c + o(1)}{n_i}$$

then

$$c = \rho(2)n_i - \ln n_i - o(1) \geq 2 \ln n - \ln \left(\frac{2}{\rho(2)} \ln n \right) - o(1)$$

as $n_i \geq \ln n$ and $\rho(2)x - \ln x$ is increasing for x sufficiently large. Thus, our coupled random graph $G_{n_i, \rho(2)}$ is connected by Theorem 6.2.10 with failure probability

$$1 - e^{-e^{-c}} = O(-e^{-c}) = O(n^{-2} \ln n).$$

By the union bound, G_n restricted to any $B(p_i, \delta)$ is thus connected with failure probability $o(1)$.

Finally, we show that any two points in \mathcal{X}_n are connected. Let \mathbf{x} and \mathbf{y} be vertices in \mathcal{X}_n connected by a path $\gamma : [0, 1] \rightarrow \mathcal{S}$ with $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{y}$ (since \mathcal{S} is path connected). Let $\{B(p_i, \delta)\}_{i \in I}$ be the collection of δ balls which intersects the trace of γ . If (1) $\bigcup_{i \in I} B(p_i, \delta) \supseteq \gamma([0, 1])$ and if (2) G_n restricted to $B(p_i, \delta) \cup B(p_j, \delta)$ is connected whenever $B(p_i, \delta) \cap B(p_j, \delta) \neq \emptyset$, then \mathbf{x} is connected to \mathbf{y} in G_n . The first condition holds since the $B(p_i, \delta)$ cover \mathcal{S} . For the second condition, let $B(p_i, \delta) \cap B(p_j, \delta) \neq \emptyset$. If they share a point, their union induces a connected component in G_n since each ball individually does. Otherwise, the probability that there is no edge from $B(p_i, \delta)$ to $B(p_j, \delta)$ is at most

$$(1 - \rho(4))^{v_n(B(p_i, \delta))v_n(B(p_j, \delta))} \leq (1 - \rho(4))^{\Omega((\ln n)^2)}.$$

As $m = o(n)$ the union bound shows that this happens for all such pairs with failure probability $o(1)$. Thus all pairs of points on \mathcal{X}_n are connected a.a.s.

Turning to the case $1 \leq \alpha \leq d$, we may make a direct comparison with the Erdős-Rényi result. In particular, the probability that any pair of points in \mathcal{X}_n is connected is bounded below by $\rho_n(\text{diam}(\mathcal{S})) \geq (1 - \varepsilon) c_\rho r_n^\alpha \text{diam}(\mathcal{S})^{-\alpha}$ for any $\varepsilon > 0$. Setting $r_n \geq C \left(\frac{\ln n}{n}\right)^{1/\alpha}$ for C sufficiently large shows that this probability exceeds the connection threshold in Theorem 6.2.11. \square

For $\alpha = d$ we see that connectivity coincides with the disappearance of degree zero vertices, analogous to situation for $G_{n,p}$ and the boolean model. Indeed, if $\alpha > d$ the proof above demonstrates that the graph G_n is highly locally connected as in the boolean model; global connection occurs when sufficiently large connected neighborhoods overlap. The case $\alpha < d$ is the other extreme, not sufficiently localized to differ substantially from the ER model. However, when $\alpha = d$ the above theorem gives connection well beyond the disappearance of isolated vertices. We offer the following conjecture:

Conjecture 6.2.14. *For G_n as in Theorem 6.2.10, if $\alpha = d$ there is some constant C such that $r_n \geq C n^{-1/d}$ implies G_n is a.a.s. connected.*

6.3 Geometric Properties

6.3.1 Scaling

Consider G_n constructed on some ground space (\mathcal{S}, d, μ) . For a fixed point $\mathbf{x} \in \mathcal{S}$, how does the graph G_n restricted to $B(p, \varepsilon) = \{\mathbf{y} \in \mathcal{S} \mid d(\mathbf{x}, \mathbf{y}) < \varepsilon\}$ behave?

The d -cube provides the simplest setting in which to address this question. We shall ask for the behavior of an $\varepsilon \times \cdots \times \varepsilon$ sub-cube of $[0, 1]^d$. We then have the following:

Proposition 6.3.1. *Fix the ground space to be the standard d -cube. For a sequence r_n and threshold ρ with polynomial tail of degree α , form the random graph G_n on the vertex*

space \mathcal{P}_n . For any sub-cube C of $[0, 1]^d$ of side-length ε , let \tilde{G}_n denote the subgraph of G_n induced by the vertices of \mathcal{P}_n falling in C .

The graph \tilde{G}_n is equal (up to homothety) in distribution to a graph on the d -cube with the same threshold ρ , vertex space $\mathcal{P}_{\varepsilon^d n}$, and modified scale sequence r_n/ε .

Proof. Clearly, the vertex space on C has intensity $\varepsilon^d n$ as the full vertex space is Poisson. If we rescale C via homothety to the standard unit cube $[0, 1]^d$, then two points $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$ are connected in $[0, 1]^d$ if and only if their inverse image under homothety, \mathbf{x} and \mathbf{y} , are connected in G_n . This occurs with probability $\rho_n(d(\mathbf{x}, \mathbf{y})) = \rho_n(\varepsilon d(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})) = \rho(\varepsilon d(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})/r_n)$. \square

How does \tilde{G}_n compare to G_n in the above? Consider the degree. By Theorem 6.2.2 and Proposition 6.2.8, for $1 \gg \varepsilon \gg r_n$,

$$\mathbb{E}[\deg_{\tilde{G}_n}(x_1) \mid \text{Poi}(\varepsilon^d n) \geq 1] \approx \varepsilon^{d-\alpha \wedge d} \mathbb{E}[\deg_{G_n}(x_1) \mid \text{Poi}(n) \geq 1], \quad \alpha \neq d.$$

Hence, if $\alpha < d$ the degree in \tilde{G}_n of any fixed vertex tends to a vanishing fraction of the total degree. For $\alpha > d$ the degree in \tilde{G}_n is asymptotically the degree in the entire graph, indicating that all but a vanishing fraction of edges fall inside a ball of radius $\varepsilon \gg r_n$. The case $\alpha = d$ is more subtle. Due to the logarithmic term, if $\varepsilon = \Theta(r_n^\gamma)$, $0 < \gamma < 1$, then

$$\mathbb{E}[\deg_{\tilde{G}_n}(x_1) \mid \text{Poi}(\varepsilon^d n) \geq 1] \approx (1 - \gamma) \mathbb{E}[\deg_{G_n}(x_1) \mid \text{Poi}(n) \geq 1].$$

6.4 Further Directions

- Little is known about the degree distribution in this model. In particular, the dependence in this model will require new tools to overcome. The results cited by Bollobás et al. [22] do not extend to dense graphs.
- While we have addressed rather easily the notion of subgraphs induced in small neighborhoods, there is an alternate notion of scaling which we have yet to address. For the graph on $[0, 1]^d$, one can divide the cube into smaller cubes of length ε . Each cube then becomes a vertex, and we add an edge between two

cubes if they share a certain number of edges in the original graph (that is, if the number of edges with endpoints in both cubes is above a given threshold). This notion of “coarsening” is a fairly natural way to simplify the original graph.

- The question of connectivity for graphs on d -dimensional manifolds and connection functions with polynomial tails of degree d remains open. While we do not yet know how to attack this problem, we take note of a related phenomenon. Kleinberg in [69] investigates a simple grid model on which random edges are placed with probability proportional to some inverse polynomial power of the graph-theoretic (l_1) distance. His results show that algorithmic short routing is possible only when this power agrees with the dimension of the grid.

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Index

- $C(x, h)$
 - see boundary cap, 15
- H^+ , 2, 27
- H^- , 2, 27
- P_n , 12
- Π_n , 32
- Pois
 - see Poisson point process, 32
- \mathcal{P}
 - see Poisson vertex space, 86
- \mathcal{X}
 - see uniform vertex space, 86
- α
 - see \mathcal{T}_α , 63
- π_d , 69
- Vor, 17, 33, 35, 52
- $\tilde{\mathbb{E}}$, 34
- $\tilde{\mathbb{P}}$, 34
- $\tilde{\text{Var}}$, 34
- u
 - see unit normal, 27
- boundary cap, 15
- cap, 14
- convergence, speed of, 22
- crown, 15
- dependency graph, 32
- epochs, 74, 79, 80
- excess volume, 14
- Graph Model
 - $G_{n,p}$, 57
 - $\mathcal{T}_{\alpha,n}$
 - see \mathcal{T}_α , 63
 - \mathcal{T}_α , 63
- ground space, 85
- influence regions, 68, 81
- intensity, 4, 32, 33, 86, 100
- long, 72
- long edge, 71, 72
- missed volume, 9
- Poisson point process, 4, 32
- short edge, 71, 72
- unit normal, 27
- vertex space, 85
 - Poisson, 86
 - uniform, 86
- visible region, 15, 39
- Voronoi cell
 - see Vor, 17

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