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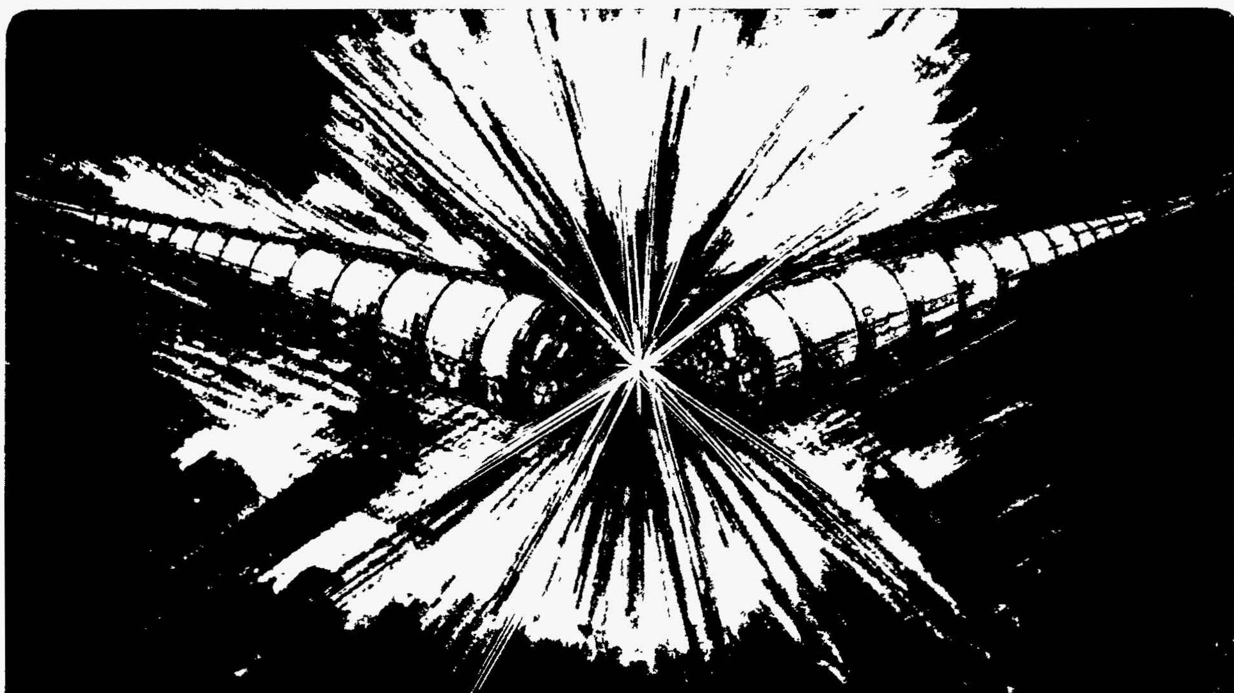

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The Vector Potential and Stored Energy of Thin Cosine(nθ) Helical Wiggler Magnet.*

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Abstract

Expressions for pure multipole field components that are present in helical devices have been derived from a current distribution on the surface of an infinitely thin cylinder[b] of radius R. The strength of such magnetic fields varies purely as a Fourier sinusoidal series of the longitudinal coordinate z in proportion to $\cos(n\theta - \omega_m z)$, where $\omega_m = \frac{(2m-1)\pi}{L}$, L denotes the *half-period* and m=1,2,3 etc. As an alternative to describing such field components as given by the negative gradient of a scalar potential function (Appendix A), one of course can derive these same fields as the curle of a vector potential function $\vec{A}$ — specifically one for which $\nabla \times \nabla \times \vec{A} = 0$ and $\nabla \cdot \vec{A} = 0$. It is noted that we seek a divergence-free vector that exhibits continuity in any of its components across the interface r=R, a feature that is free of possible concern when applying Stokes' theorem in connection with this form of vector potential. Alternative simpler forms of vector potential, that individually are divergence-free in their respective regions (r<R and r>R), do not exhibit full continuity on r=R and whose curl evaluations provide in these respective regions the correct components of magnetic field are not considered here. Such alternative forms must differ merely by the gradient of scalar functions that with the divergence-free property are required to be "harmonic" ($\nabla^2 \Psi = 0$).

A summary of the vector-potential derived in part one is given below. In part two we derive the magnetic field components and check the validity of $\nabla \cdot \vec{A} = 0$. In part three the stored energy is derived from the vector-potential (shown below) and finally in part four we reduce the problem dimensionality to the more familiar 2D results by extending the period to infinity.

r≤R :

$$\vec{A}_r = \frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\sin(n\theta - \omega_m z)$$

$$\vec{A}_\theta = -\frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\cos(n\theta - \omega_m z)$$

$$\vec{A}_z = -\sum_{n=1}\sum_{m=1}\frac{nG_{n,m}K_n(\omega_m R)}{\omega_m R K_n'(\omega_m R)}I_n(\omega_m r)\cos(n\theta - \omega_m z)$$

r≥R

$$\vec{A}_r = \frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}[I_{n+1}(\omega_m R)K_{n+1}(\omega_m r) - I_{n-1}(\omega_m R)K_{n-1}(\omega_m r)]\sin(n\theta - \omega_m z)$$

$$\vec{A}_\theta = -\frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}[I_{n+1}(\omega_m R)K_{n+1}(\omega_m r) + I_{n-1}(\omega_m R)K_{n-1}(\omega_m r)]\cos(n\theta - \omega_m z)$$

$$\vec{A}_z = -\sum_{n=1}\sum_{m=1}\frac{nG_{n,m}I_n(\omega_m R)}{\omega_m R K_n'(\omega_m R)}K_n(\omega_m r)\cos(n\theta - \omega_m z)$$

and the stored energy density

$$e = -\frac{1}{2\mu_0}\sum_n\sum_m\frac{G_{n,m}^2 I_n'}{R^2 K_n'}$$

Where I_n and K_n are the "modified" Bessel function of the first and second kind of order n, and the prime denotes differentiation of the Bessel function with respect to its argument.

An alternative form for expressing the vector-potential as Bessel functions and their derivatives of order n only, is given in the text.

[b] Magnetic Field Components in a Sinusoidally Varying Helical Wiggler, LBL-35928, SC-MAG-464, July 1994.

Analysis

One may consider a direct evaluation of the vector-potential function through use of the integral formula[c]

$$\vec{A}(r,\theta,z) = \frac{\mu_0}{4\pi}\int\int \frac{J d\sigma_0}{|\vec{r}-\vec{r}_0|}$$

with subscripts 0 on the coordinates to indicate source-point locations. The integration in z_0 is taken to extend from $-\infty$ to $+\infty$ and the integration in θ_0 to extend over an interval of 2π. We have undertaken such an evaluation, using for the source an expression consistent with that cited previously in Ref.[b] :

$$\omega_m = \frac{(2m-1)\pi}{L} \quad \text{and} \quad G_{n,m} = n!R^n\left(\frac{2}{\omega_m R}\right)^n B_{n,m}$$

$$nB_{n,m} = \begin{Bmatrix} 1B_1 = B_m \text{ dipole field} \\ 2B_2 = G_m \text{ quad gradient} \\ 3B_3 = S_m \text{ Sextupole} \\ \dots \end{Bmatrix}$$

$$\vec{J}(\theta_0,z_0)|_{r=R} = -\frac{1}{\mu_0}\begin{Bmatrix} 0\hat{e}_{r_0} \\ \sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{R}\frac{1}{K_n'(\omega_m R)}\cos(n\theta_0-\omega_m z_0)\hat{e}_{\theta_0} \\ \sum_{n=1}\sum_{m=1}\frac{nG_{n,m}}{\omega_m R^2}\frac{1}{K_n'(\omega_m R)}\cos(n\theta_0-\omega_m z_0)\hat{e}_{z_0} \end{Bmatrix}$$

The pair of current density components satisfy the conservation condition $\nabla\cdot\vec{J}_s = \frac{\partial J_z}{\partial z_0} + \frac{1}{R}\frac{\partial J_\theta}{\partial\theta_0} = 0$ as required.

We may put

$$|\vec{r}-\vec{r}_0| = \sqrt{R^2+r^2+(z-z_0)^2-2Rr\cos(\theta-\theta_0)}$$

$$\begin{aligned} \hat{e}_{z0} &= \hat{e}_z \\ \hat{e}_{\theta_0} &= -\sin\theta_0\hat{e}_x + \cos\theta_0\hat{e}_y \\ &= -\sin\theta_0(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta) + \cos\theta_0(\sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta) \\ &= \sin(\theta-\theta_0)\hat{e}_r + \cos(\theta-\theta_0)\hat{e}_\theta \end{aligned}$$

and introduce working variables $t = \theta_0 - \theta$ and $s = z_0 - z$ for the purpose of performing the integration.

The z component of $\vec{A}$

The z component of the vector-potential may be written as :

$$\vec{A}_z = \frac{\mu_0}{4\pi}\int\int\left\{-\frac{1}{\mu_0}\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{\cos(n\theta_0-\omega_m z_0)}{\omega_m R^2 K_n'(\omega_m R)}\frac{R d\theta_0 dz_0}{\sqrt{R^2+r^2+(z-z_0)^2-2Rr\cos(\theta-\theta_0)}}\right\}$$

Employing the new working variables and the relation,

[c] Panofski and Phillips, Ed.2, Eq. (7–42), p.128

$$\cos(n\theta_0 - \omega_m z_0) = \cos(n\theta - \omega_m z)\cos(nt - \omega_m s) - \sin(n\theta - \omega_m z)\sin(nt - \omega_m s)$$

we may alternatively write,

$$\vec{A}_z = -\frac{R}{4\pi}\sum_{n=1}\sum_{m=1}\frac{nG_{n,m}}{\omega_m R^2 K_n'(\omega_m R)}\left[\begin{array}{l}\cos(n\theta - \omega_m z)\displaystyle\int\int\frac{\cos(nt-\omega_m s)dsdt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} \\ -\sin(n\theta - \omega_m z)\displaystyle\int\int\frac{\sin(nt-\omega_m s)dsdt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}}\end{array}\right]$$

and proceed with the evaluation of the two double integrals.

- ***First double integral***
  For the first double integral, by reference[d] and with the understanding that odd functions integrate to 0 over $\pm\infty$

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\cos(nt-\omega_m s)dsdt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} = \int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\cos nt\cos(\omega_m s)dsdt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} + \int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\sin nt\sin(\omega_m s)dsdt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}}$$
$$= 2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\cos(nt)dt + 0$$

In recognition of the "summation theorem"[e] and of the orthogonality properties of circular functions and since $I_{-n} = I_n$ and $K_{-n} = K_n$ [f] the above expression for the double integral can be reduced to (for r≤R)

$$2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\cos(nt)dt = 2\int_0^{2\pi}\sum_{k=-\infty}^{\infty} K_k(\omega_m R)I_k(\omega_m r)\cos(kt)\cos(nt)dt$$
$$= 4\int_0^{2\pi} K_n(\omega_m R)I_n(\omega_m r)\cos^2(nt)dt = 4\pi K_n(\omega_m R)I_n(\omega_m r)$$

- ***Second double integral***
  Similarly, we show that the second double integral vanishes.

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\sin(nt-\omega_m s)dsdt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} = \int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\sin nt\cos(\omega_m s)dsdt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} - \int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\cos nt\sin(\omega_m s)dsdt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}}$$
$$= 2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\sin(nt)dt + 0$$

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d I.S. Gradshtyne and I.M. Ryzhik, "Table of Integrals ...", Eq. 3.754(2). p.419

e G.N. Watson, "Bessel Functions", Sec. 11.3, Eq. (8), p.361, with n=0.

f Abramowitz and Stegun, Chapter 9, Eqs. 9.9.6, p.375.

In recognition of the "summation theorem", the orthogonality properties of circular functions and $I_{-n} = I_n$ and $K_{-n} = K_n$, we may write for r≤R

$$2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\sin(nt)dt = 2\int_0^{2\pi}\sum_{k=-\infty}^{\infty} K_k(\omega_m R)I_k(\omega_m r)\cos(kt)\sin(nt)dt$$

$$= 4\int_0^{2\pi} K_n(\omega_m R)I_n(\omega_m r)\cos(nt)\sin(nt)dt = 0$$

The expression for the vector-potential may now be written for r≤R as,

$$\vec{A}_z = -\sum_{n=1}\sum_{m=1}\frac{nG_{n,m}K_n(\omega_m R)}{\omega_m RK_n'(\omega_m R)}I_n(\omega_m r)\cos(n\theta-\omega_m z)$$

and for r≥R, we may interchange the arguments of the Bessel functions and write,

$$\vec{A}_z = -\sum_{n=1}\sum_{m=1}\frac{nG_{n,m}I_n(\omega_m R)}{\omega_m RK_n'(\omega_m R)}K_n(\omega_m r)\cos(n\theta-\omega_m z)$$

and with evident continuity at the interface r=R.

The r component of $\vec{A}$

In developing the radial component of the vector-potential we shall employ similar technics to those previously applied for evaluating A_z. The r component of the vector-potential may be written as :

$$\vec{A}_r = \frac{\mu_0}{4\pi}\int\int\left\{-\frac{1}{\mu_0}\sum_{n=1}\sum_{m=1}G_{n,m}\frac{\cos(n\theta_0-\omega_m z_0)}{K_n'(\omega_m R)}\frac{\sin(\theta-\theta_0)d\theta_0 dz_0}{\sqrt{R^2+r^2+(z-z_0)^2-2Rr\cos(\theta-\theta_0)}}\right\}$$

with

$$\cos(n\theta_0-\omega_m z_0) = \cos(n\theta-\omega_m z)\cos(nt-\omega_m s) - \sin(n\theta-\omega_m z)\sin(nt-\omega_m s)$$

and using the working variables as before, we may alternatively write :

$$\vec{A}_r = -\frac{1}{4\pi}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}\left[\begin{array}{l} -\cos(n\theta-\omega_m z)\displaystyle\int\int\frac{\cos(nt-\omega_m s)\sin t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} \\ +\sin(n\theta-\omega_m z)\displaystyle\int\int\frac{\sin(nt-\omega_m s)\sin t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}}\end{array}\right]$$

- ***First double integral***
  We show that the first double integral in the above expression, vanishes. With the understanding that odd

functions integrate to 0 over $\pm\infty$,

$$\int_0^{2\pi}\int_{-\infty}^{\infty} \frac{\cos(nt-\omega_m s)\sin t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} = \int_0^{2\pi}\int_{-\infty}^{\infty} \frac{\cos nt\cos(\omega_m s)\sin t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} + \int_0^{2\pi}\int_{-\infty}^{\infty} \frac{\sin nt\sin(\omega_m s)\sin t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}}$$

$$= 2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\cos(nt)\sin t\,dt + 0$$

and in recognition of the "summation theorem", the orthogonality properties of circular functions, the identities $I_{-n} = I_n$ and $K_{-n} = K_n$ and the fact that

$$\cos nt\sin t = \frac{1}{2}[\sin(n+1)t - \sin(n-1)t]$$

the above expression for the double integral is identically 0, since

$$2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\cos(nt)\sin t\,dt = 2\int_0^{2\pi}\sum_{k=-\infty}^{\infty} K_k(\omega_m R)I_k(\omega_m r)\cos(kt)\cos(nt)\sin t\,dt$$

$$= 4\int_0^{2\pi} K_n(\omega_m R)I_n(\omega_m r)\cos(nt)\cos(nt)\sin t\,dt = 0$$

- ***Second double integral***
  Similarly for the second double integral :

$$\int_0^{2\pi}\int_{-\infty}^{\infty} \frac{\sin(nt-\omega_m s)\sin t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} = \int_0^{2\pi}\int_{-\infty}^{\infty} \frac{\sin nt\cos(\omega_m s)\sin t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} + \int_0^{2\pi}\int_{-\infty}^{\infty} \frac{\cos nt\sin(\omega_m s)\sin t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}}$$

$$= 2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\sin(nt)\sin t\,dt + 0$$

In recognition of the "summation theorem", the orthogonality properties of circular functions, $I_{-n} = I_n$ and $K_{-n} = K_n$ and the fact that

$$\sin nt\sin t = -\frac{1}{2}[\cos(n+1)t - \cos(n-1)t]$$

we may write for $r \le R$,

$$2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\sin(nt)\sin t\,dt = 2\int_0^{2\pi}\sum_{k=-\infty}^{\infty} K_k(\omega_m R)I_k(\omega_m r)\cos(kt)\sin(nt)\sin t\,dt$$

$$= -\int_0^{2\pi}\sum_{k=-\infty}^{\infty} K_k(\omega_m R)I_k(\omega_m r)\cos(kt)[\cos(n+1)t - \cos(n-1)t]dt$$

$$= -2\pi[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]$$

(care has been taken to verify that the above is true for the case n=1 as well)

The final expression for $\vec{A}_r$ in the region r≤R can now be written as

$$\vec{A}_r = \frac{1}{2}\sum_{n=1}\sum_{m=1} \frac{G_{n,m}}{K_n'(\omega_m R)}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\sin(n\theta - \omega_m z)$$

and for the region r≥R, we interchange the arguments of the Bessel functions and write,

$$\vec{A}_r = \frac{1}{2}\sum_{n=1}\sum_{m=1} \frac{G_{n,m}}{K_n'(\omega_m R)}[I_{n+1}(\omega_m R)K_{n+1}(\omega_m r) - I_{n-1}(\omega_m R)K_{n-1}(\omega_m r)]\sin(n\theta - \omega_m z)$$

with evident continuity at the interface r=R.
Alternatively with the relation :

$$K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r) = 2n\left[\frac{K_n'(\omega_m R)I_n(\omega_m r)}{(\omega_m r)} + \frac{K_n(\omega_m R)I_n'(\omega_m r)}{(\omega_m R)}\right]$$

we may express the vector-potential in terms of Bessel functions and their derivatives of order n only.

The θ component of $\vec{A}$

The vector-potential in the θ direction is :

$$\vec{A}_\theta = \frac{\mu_0}{4\pi}\int\int\left\{-\frac{1}{\mu_0}\sum_{n=1}\sum_{m=1} G_{n,m}\frac{\cos(n\theta_0 - \omega_m z_0)}{K_n'(\omega_m R)}\frac{\cos(\theta-\theta_0)d\theta_0 dz_0}{\sqrt{R^2+r^2+(z-z_0)^2-2Rr\cos(\theta-\theta_0)}}\right\}$$

With

$$\cos(n\theta_0 - \omega_m z_0) = \cos(n\theta - \omega_m z)\cos(nt - \omega_m s) - \sin(n\theta - \omega_m z)\sin(nt - \omega_m s)$$

and use of the working variables, we alternatively write :

$$\vec{A}_\theta = -\frac{1}{4\pi}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}\left[\begin{array}{l}\cos(n\theta - \omega_m z)\int\int\frac{\cos(nt-\omega_m s)\cos t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} \\ -\sin(n\theta - \omega_m z)\int\int\frac{\sin(nt-\omega_m s)\cos t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}}\end{array}\right]$$

- ***First double integral***
  For the first double integral and in recognition of the "summation theorem" and that odd functions integrate to 0 over $\pm\infty$

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\cos(nt-\omega_m s)\cos t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} = \int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\cos nt\cos(\omega_m s)\cos t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} + \int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\sin nt\sin(\omega_m s)\cos t\,ds\,dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}}$$

$$= 2\int_0^{2\pi}\cos(nt)\cos t K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)dt + 0$$

Again, using the orthogonality properties of circular functions, the identities $I_{-n} = I_n$, $K_{-n} = K_n$ and the fact that

$$\cos nt \cos t = \frac{1}{2}[\cos (n+1)t + \cos (n-1)t]$$

the above expression for the double integral reduces, for r≤R, to

$$2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\cos(nt)\cos t dt = 2\int_0^{2\pi}\sum_{k=-\infty}^{\infty} K_k(\omega_m R) I_k(\omega_k r)\cos kt \cos(nt)\cos t dt$$

$$= 2\int_0^{2\pi}\left[K_{n+1}(\omega_m R) I_{n+1}(\omega_k r)\cos^2(n+1)t + K_{n-1}(\omega_m R) I_{n-1}(\omega_k r)\cos^2(n-1)t\right] dt$$

$$= 2\pi[K_{n+1}(\omega_m R) I_{n+1}(\omega_k r) + K_{n-1}(\omega_m R) I_{n-1}(\omega_k r)]$$

- ***Second double integral***
  Similarly, we demonstrate that the second double integral vanishes,

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\sin(nt-\omega_m s)\cos t ds dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} = \int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\sin nt\cos(\omega_m s)\cos t ds dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}} - \int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\cos nt\sin(\omega_m s)\cos t ds dt}{\sqrt{R^2+r^2+s^2-2Rr\cos t}}$$

$$= 2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\sin(nt)\cos t dt + 0$$

and in recognition of the orthogonality properties of circular functions, $I_{-n} = I_n$ and $K_{-n} = K_n$ and the fact that

$$\sin nt \cos t = \frac{1}{2}[\sin (n+1)t + \sin (n-1)t]$$

we may write for r≤R,

$$2\int_0^{2\pi} K_0\left(\omega_m\sqrt{R^2+r^2-2Rr\cos t}\right)\sin(nt)\cos t dt = 2\int_0^{2\pi}\sum_{k=-\infty}^{\infty} K_k(\omega_m R) I_k(\omega_m r)\cos(kt)\sin(nt)\cos t dt$$

$$= -\int_0^{2\pi}\sum_{k=-\infty}^{\infty} K_k(\omega_m R) I_k(\omega_m r)\cos(kt)[\sin(n+1)t + \sin$$

$$= 0$$

The final expression for $\vec{A}_\theta$ in the region r≤R can now be written as

$$\vec{A}_\theta = -\frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}[K_{n+1}(\omega_m R) I_{n+1}(\omega_m r) + K_{n-1}(\omega_m R) I_{n-1}(\omega_m r)]\cos(n\theta - \omega_m z)$$

and for the region r≥R, we may interchange the arguments of the Bessel functions and write

$$\vec{A}_\theta = -\frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}[I_{n+1}(\omega_m R)K_{n+1}(\omega_m r) + I_{n-1}(\omega_m R)K_{n-1}(\omega_m r)]\cos(n\theta - \omega_m z)$$

with evident continuity at the interface r=R.

Alternatively with the relation :

$$K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}(\omega_m r) = -2\left[K_n'(\omega_m R)I_n'(\omega_m r) + \frac{n^2}{(\omega_m R)(\omega_m r)}K_n(\omega_m R)I_n(\omega_m r)\right]$$

we may express the vector-potential in terms of Bessel functions and their derivatives of order n only.

The Magnetic Field Components

We shall proceed and derive the magnetic field directly from the vector-potential, a process that may as well serve as a check for such a field when compared with similar results obtained from the scalar potential as shown in reference[b].

Accordingly,

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r}\begin{pmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & A_\theta & A_z \end{pmatrix}$$

The z component of $\vec{B}$

With the vector-potential derived earlier, we proceed in deriving the field expression in the region r≤R

$$B_z = \left(\nabla \times \vec{A}\right)_z = \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]$$

$$\frac{\partial(rA_\theta)}{\partial r} = -\frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\cos(n\theta - \omega_m z) +$$

$$-\frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}(\omega_m r)}{K_n'(\omega_m R)}\left[K_{n+1}(\omega_m R)I_{n+1}'(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}'(\omega_m r)\right]\cos(n\theta - \omega_m z)$$

$$\frac{\partial A_r}{\partial \theta} = \frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{nG_{n,m}}{K_n'(\omega_m R)}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\cos(n\theta - \omega_m z)$$

Therefore :

$$B_z = -\frac{1}{2}\sum_{n=1}\sum_{m=1}\frac{\omega_m G_{n,m}}{K_n'(\omega_m R)(\omega_m r)}\{K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}(\omega_m r) +$$

$$+ (\omega_m r)\left[K_{n+1}(\omega_m R)I_{n+1}'(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}'(\omega_m r)\right] +$$

$$+ n[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\}\cos(n\theta - \omega_m z)$$

Applying the relations :

$$xI'_{n+1} + (n+1)I_{n+1} = xI_n$$
$$xI'_{n-1} - (n-1)I_{n-1} = xI_n$$
$$K_{n+1} + K_{n-1} = -2K'_n$$

(where x corresponds to the Bessel function argument), the squiggly brackets reduces to : $-2(\omega_m r)K'_n I_n$ and the field in the region r≤R is :

$$B_z = \sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n(\omega_m r)\cos(n\theta - \omega_m z)$$

Similarly applying the above procedure to the region r≥R ,

$$B_z = \sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{I'_n(\omega_m R)}{K'_n(\omega_m R)} K_n(\omega_m r)\cos(n\theta - \omega_m z)$$

as it showed be.

The r component of $\vec{B}$

We continue and derive the radial field expression in the region r≤R,

$$B_r = \left(\nabla \times \vec{A}\right)_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}$$

$$\frac{1}{r}\frac{\partial A_z}{\partial \theta} = \sum_{n=1}\sum_{m=1} \frac{n^2 G_{n,m}\omega_m K_n(\omega_m R)}{K'_n(\omega_m R)(\omega_m R)(\omega_m r)} I_n(\omega_m r)\sin(n\theta - \omega_m z)$$
$$\frac{\partial A_\theta}{\partial z} = -\frac{1}{2}\sum_{n=1}\sum_{m=1} \frac{G_{n,m}\omega_m}{K'_n(\omega_m R)}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\sin(n\theta - \omega_m z)$$

Therefore,

$$B_r = \frac{1}{2}\sum_{n=1}\sum_{m=1} \frac{G_{n,m}\omega_m}{K'_n(\omega_m R)}[\frac{2n^2}{(\omega_m R)(\omega_m r)} K_n(\omega_m R)I_n(\omega_m r) +$$
$$+ K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\sin(n\theta - \omega_m z)$$
$$= \frac{1}{2}\sum_{n=1}\sum_{m=1} \frac{G_{n,m}\omega_m}{K'_n(\omega_m R)}\frac{1}{2}[(K_{n+1}(\omega_m R) + K_{n-1}(\omega_m R))(I_{n+1}(\omega_m r) + I_{n-1}(\omega_m r))]\sin(n\theta - \omega_m z)$$

and introducing the relations

$$2I'_n = I_{n-1} + I_{n+1}$$
$$-2K'_n = K_{n-1} + K_{n+1}$$
$$2nI_n = a(I_{n-1} - I_{n+1}) \quad ; \quad a = \omega_m r$$
$$-2nK_n = b(K_{n-1} - K_{n+1}) \quad ; \quad b = \omega_m R$$

(with two different arguments) the radial field component in the region r≤R is :

$$B_r = -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n'(\omega_m r)\sin(n\theta - \omega_m z)$$

and similarly applying the above procedure to the outer region r≥R we get,

$$B_r = -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{I_n'(\omega_m R)}{K_n'(\omega_m R)} K_n'(\omega_m r)\sin(n\theta - \omega_m z)$$

as it showed be.

The θ component of $\vec{B}$

In deriving the azimuthal field component in the region r≤R,

$$B_\theta = \left(\nabla \times \vec{A}\right)_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$\frac{\partial A_r}{\partial z} = -\frac{1}{2}\sum_{n=1}\sum_{m=1} \frac{G_{n,m}\omega_m}{K_n'(\omega_m R)}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\cos(n\theta - \omega_m z)$$

$$\frac{\partial A_z}{\partial r} = -\sum_{n=1}\sum_{m=1} \frac{nG_{n,m}K_n(\omega_m R)}{RK_n'(\omega_m R)} I_n'(\omega_m r)\cos(n\theta - \omega_m z)$$

Therefore,

$$B_\theta = -\sum_{n=1}\sum_{m=1} \frac{nG_{n,m}}{RK_n'(\omega_m R)}\left\{\frac{\omega_m R}{2n}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)] - K_n(\omega_m R)I_n'(\omega_m r)\right\}$$
$$\times \cos(n\theta - \omega_m z)$$

and introducing the relation

$$K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r) =$$
$$= 2n\left[\frac{K_n'(\omega_m R)I_n(\omega_m r)}{\omega_m r} + \frac{K_n(\omega_m R)I_n'(\omega_m r)}{\omega_m R}\right]$$

the θ field component in the region r≤R is :

$$B_\theta = -\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{1}{r}I_n(\omega_m r)\cos(n\theta - \omega_m z)$$

and similarly by applying the above procedure to the outer region r≥R we get, as it should be.

$$B_\theta = -\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{I_n'(\omega_m R)}{K_n'(\omega_m R)}\frac{1}{r}K_n(\omega_m r)\cos(n\theta - \omega_m z)$$

as it showed be.

The divergence of $\vec{A}$ — a check

As a final check, we shall show that the divergence of $\vec{A}$ vanish throughout space including r=R,

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} = 0$$

We first check the divergence in the region r≤R

$$\frac{1}{r}\frac{\partial (rA_r)}{\partial r} = \frac{1}{2r}\sum_{n=1}\sum_{m=1}\frac{G_{n,m}}{K_n'(\omega_m R)}\{[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]+$$

$$+\omega_m r\left[K_{n+1}(\omega_m R)I_{n+1}'(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}'(\omega_m r)\right]\}\sin(n\theta - \omega_m z)$$

$$\frac{1}{r}\frac{\partial A_\theta}{\partial \theta} = \frac{1}{2r}\sum_{n=1}\sum_{m=1}\frac{nG_{n,m}}{K_n'(\omega_m R)}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]\sin(n\theta - \omega_m z)$$

$$\frac{\partial A_z}{\partial z} = -\sum_{n=1}\sum_{m=1}\frac{nG_{n,m}K_n(\omega_m R)}{RK_n'(\omega_m R)}I_n(\omega_m r)\sin(n\theta - \omega_m z)$$

Therefor :

$$\nabla \cdot \vec{A} = \sum_{n=1}\sum_{m=1}\frac{nG_{n,m}}{2RK_n'(\omega_m R)}\{\frac{R}{nr}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)]+$$

$$+\frac{\omega_m R}{n}\left[K_{n+1}(\omega_m R)I_{n+1}'(\omega_m r) - K_{n-1}(\omega_m R)I_{n-1}'(\omega_m r)\right]+$$

$$+\frac{R}{r}[K_{n+1}(\omega_m R)I_{n+1}(\omega_m r) + K_{n-1}(\omega_m R)I_{n-1}(\omega_m r)] - 2K_n(\omega_m R)I_n(\omega_m r)\}\sin(n\theta - \omega_m z)$$

$$= \sum_{n=1}\sum_{m=1}\frac{nG_{n,m}}{2RK_n'(\omega_m R)}\{\frac{R}{nr}K_{n+1}(\omega_m R)\left[(n+1)I_{n+1}(\omega_m r) + (\omega_m r)I_{n+1}'(\omega_m r)\right]-$$

$$-\frac{R}{nr}K_{n-1}(\omega_m R)\left[-(n-1)I_{n-1}(\omega_m r) + (\omega_m r)I_{n-1}'(\omega_m r)\right] - 2K_n(\omega_m R)I_n(\omega_m r)\}\sin(n\theta - \omega_m z)$$

Applying the identities with different arguments a,b :

$$(n+1)I_{n+1} + aI_{n+1}' = aI_n \ ; \ a = \omega_m r$$
$$-(n-1)I_{n-1} + aI_{n-1}' = aI_n$$
$$b(K_{n+1} - K_{n-1}) = 2nK_n \ ; \ b = \omega_m R$$

the divergence vanish,

$$\nabla \cdot \vec{A} = \sum_{n=1}\sum_{m=1}\frac{nG_{n,m}}{2RK_n'(\omega_m R)}[2K_nI_n - 2K_nI_n]\sin(n\theta - \omega_m z) = 0$$

In the region r≥R we interchange the I's with the K's and make use of,

$$(n+1)K_{n+1} + aK_{n+1}' = -aK_n \ ; \ a = \omega_m r$$
$$-(n-1)K_{n-1} + aK_{n-1}' = -aK_n$$
$$-b(I_{n+1} - I_{n-1}) = 2nI_n \ ; \ b = \omega_m R$$

the divergence vanish as well,

$$\nabla \cdot \vec{A} = \sum_{n=1}\sum_{m=1}\frac{nG_{n,m}}{2RK_n'(\omega_m R)}[2K_nI_n - 2K_nI_n]\sin(n\theta - \omega_m z) = 0$$

The stored energy in multipole helical windings

Now, it is only a hop skip and a jump to apply the vector-potential in calculating the stored energy in helical devices. From the definition of the energy,

$$E = \frac{1}{2}\int\int\int \vec{J}\cdot\vec{A}dv$$

We recognize that we need integrate the vector product over the current surface only and divide the stored energy by the volume of integration taken here as extending over the period 2L.

$$e = \frac{E}{V} = \frac{1}{2(\pi R^2 2L)}\int_0^{2\pi}\int_{-L}^{L} \vec{J}\cdot\vec{A}d\sigma$$

(the current density is generally per unit area but when applied to thin windings is per unit length, the energy density is $\frac{J}{m^3} = \frac{T\cdot A}{m}$).

The most general expression for $\vec{J}\cdot\vec{A}$ on the surface r=R is :

$$e = \frac{1}{4\pi L\mu_0}\int_0^{2\pi}\int_{-L}^{L}\sum_n\sum_m\sum_i\sum_j \frac{niG_{n,m}G_{i,j}}{(\omega_m R^2)(\omega_j R^2)K_n' K_i'}\cos(n\theta-\omega_m z)\cos(i\theta-\omega_j z)\times$$
$$\times\left\{\frac{(\omega_m R)(\omega_j R)}{2ni}[K_{i+1}I_{i+1}+K_{i-1}I_{i-1}]+K_iI_i\right\}d\theta dz$$

We shall omit writing the argument $\omega_m R$ in both Bessel functions I and K.

Making use of the orthogonality properties of circular functions,

$$\int_0^{2\pi}\int_{-L}^{L}\cos(n\theta-\omega_m z)\cos(i\theta-\omega_j z)Rd\theta dz = \begin{cases} 2\pi RL & ;\ n=i\ ,\ m=j \\ 0 & ;\ n\neq i\ ,\ m\neq j\end{cases}$$

the only terms that due not vanish, are for n=i and m=j, therefor :

$$e = \frac{1}{2\mu_0}\sum_n\sum_m \frac{n^2G_{n,m}^2}{(\omega_m R^2)^2(K_n')^2}\left[\frac{(\omega_m R)^2}{2n^2}(K_{i+1}I_{i+1}+K_{i-1}I_{i-1})+K_iI_i\right]$$

The term in the above bracket may be reduced by applying the relation :

$$\left[\frac{(\omega_m R)^2}{2n^2}(K_{i+1}I_{i+1}+K_{i-1}I_{i-1})+K_iI_i\right] = -\frac{(\omega_m R)^2}{n^2}K_n'I_n'$$

resulting in :

$$\boxed{e = -\frac{1}{2\mu_0}\sum_n\sum_m \frac{G_{n,m}^2 I_n'}{R^2K_n'}}$$

We express the energy density in terms of current density using the relations from reference[g]

$$J_\xi = J_{0\xi}\cos(n\theta - \omega_m z)$$
$$J_{0\xi} = -\frac{G_{n,m}\sqrt{n^2 + (\omega_m R)^2}}{\mu_0\omega_m R^2 K_n'}$$

where ξ denotes the direction of current flow in the helix. In terms of the total current per pole I_{pole}, we have :

$$I_{pole} = J_{0\xi}\frac{2R}{\sqrt{n^2 + (\omega_m R)^2}} = -\frac{2G_{n,m}}{\mu_0\omega_m R K_n'}$$

and inversely we may write,

$$G_{n,m} = -\frac{J_{0\xi}\mu_0\omega_m R^2 K_n'}{\sqrt{n^2 + (\omega_m R)^2}}$$
$$G_{n,m} = -\frac{I_{pole}\mu_0\omega_m R K_n'}{2}$$

so that the energy density can be written as :

$$e = -\frac{\mu_0}{2}\sum_n\sum_m \frac{J_{0\xi}^2(\omega_m R)^2}{n^2 + (\omega_m R)^2} K_n' I_n'$$
$$e = -\frac{\mu_0}{8}\sum_n\sum_m I_{pole}^2\omega_m^2 K_n' I_n'$$

The limiting 2 dimensional case

As a farther simplification and a check, we reduce the results obtained for helical devices by extending the periodicity to infinity, $lim_{L\to\infty}\omega_m = 0$, and compare those with more familiar 2D cases of multipole magnets.

With

$$s \to 0$$
$$I_n(s) \to \frac{1}{n!}\left(\frac{s}{2}\right)^n$$
$$K_n(s) \to \frac{(n-1)!}{2}\left(\frac{s}{2}\right)^{-n}$$
$$I_n'(s) \to \frac{1}{2(n-1)!}\left(\frac{s}{2}\right)^{n-1}$$
$$K_n'(s) \to -\frac{n!}{4}\left(\frac{s}{2}\right)^{-(n+1)}$$

[g] Forces in a Thin Cosine(nθ) Helical Wiggler, LBL-36988, SC-MAG-495, March 1995.

The 2D vector-potential reduces to :

$$\vec{A}_z = B_n r^n \cos n\theta \qquad \text{for } r \leq R$$

$$\vec{A}_z = B_n R^n \left(\frac{R}{r}\right)^n \cos n\theta \qquad for\ r \geq R$$

$$\vec{A}_r = \vec{A}_\theta = 0$$

and the stored energy is :

$$J_{02d} = \lim_{\omega_m \to \infty} J_{0\xi}$$

$$e_{2d} = \frac{\mu_0}{4} \frac{J_{02d}^2}{n}$$

$$e_{2d} = \frac{\mu_0}{16} \frac{n}{R^2} I_{pole}^2$$

$$e_{2d} = \frac{n}{\mu_0} R^{2(n-1)} B_n^2$$

Example — dipole, n=1

We have calculated and plotted (using mathcad) the magnitude of the vector-potential components for a dipole n=1, with a single period m=1, $\omega_1 = \frac{\pi}{L}$, and a half period length of L=2.0 cm. As a parameter we varied the winding radius, R=1.0, 1.5, 2.0, and 2.5 cm.

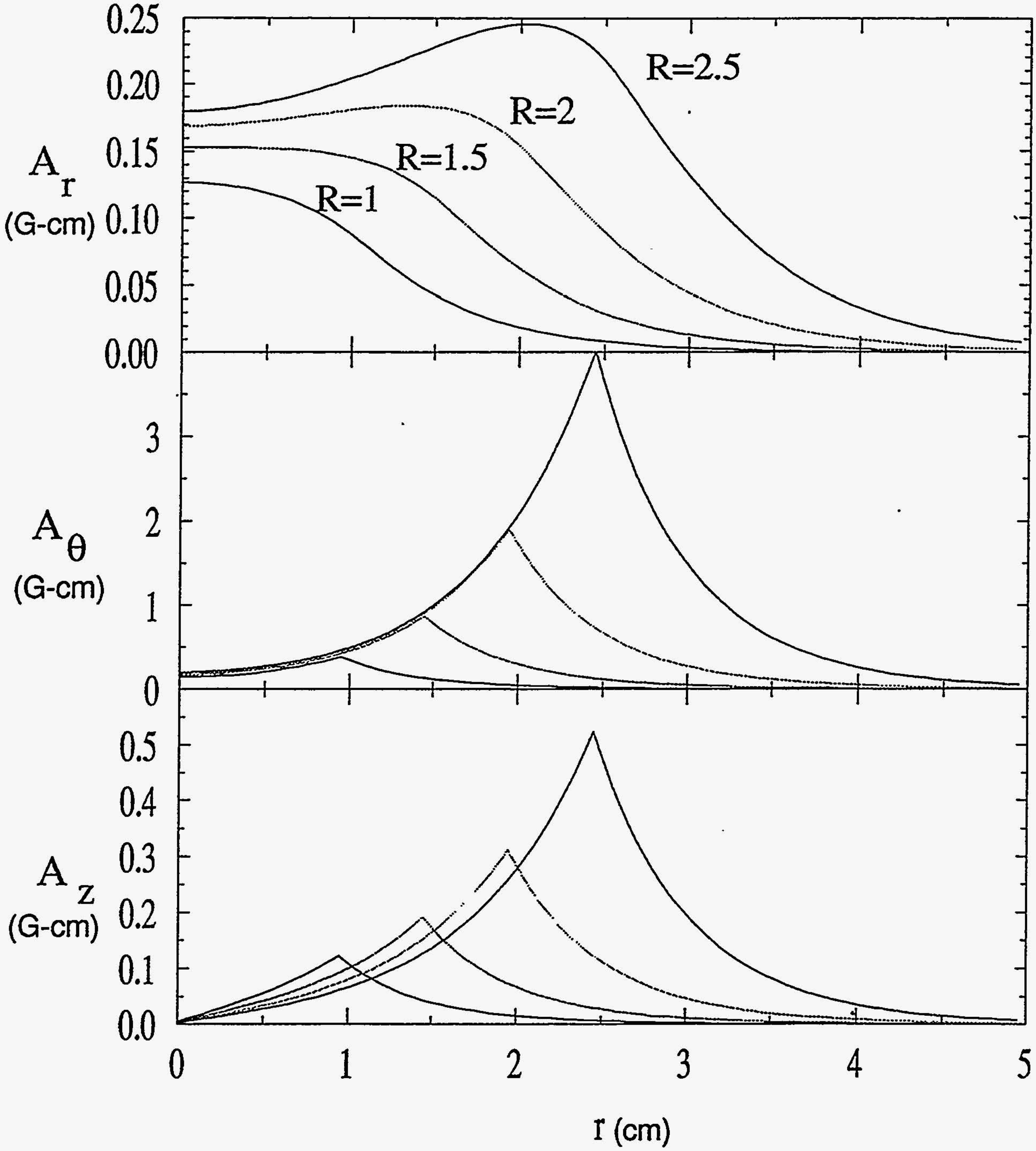

Figure 1 The magnitude of the three vector-potential components in a dipole helix with a period 2L=4.0 cm, z=0 (for A_r θ=90 and for A_θ and A_z at θ=0).

Appendix A Field Components

The field components, derived from a scalar potential, in the region interior to the windings $r<R$ (from reference[b]):

$$B_r = -\frac{\partial V}{\partial r} = -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n'(\omega_m r)\sin(n\theta - \omega_m z)$$

$$B_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{1}{r}I_n(\omega_m r)\cos(n\theta - \omega_m z)$$

$$B_z = -\frac{\partial V}{\partial z} = \sum_{n=1}\sum_{m=1} G_{n,m}\omega_m I_n(\omega_m r)\cos(n\theta - \omega_m z)$$

The field components in the region exterior to the windings $r>R$ are :

$$B_r = -\sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{I_n'(\omega_m R)}{K_n'(\omega_m R)} K_n'(\omega_m r)\sin(n\theta - \omega_m z)$$

$$B_\theta = -\sum_{n=1}\sum_{m=1} nG_{n,m}\frac{I_n'(\omega_m R)}{K_n'(\omega_m R)}\frac{1}{r}K_n(\omega_m r)\cos(n\theta - \omega_m z)$$

$$B_z = \sum_{n=1}\sum_{m=1} G_{n,m}\omega_m \frac{I_n'(\omega_m R)}{K_n'(\omega_m R)} K_n(\omega_m r)\cos(n\theta - \omega_m z)$$