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**Combinatorial Models for Invariant Subspaces, Irreducible Decompositions, and
Immersion Relations in Representation Theory**

By

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DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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Abstract

In this thesis, we apply combinatorial models to various problems concerning representations of classical Lie groups. Crystal graphs, developed by Kashiwara [26, 27], encode many important features of their corresponding Lie group representations, including invariant subspaces, weight spaces, irreducible subrepresentations, and characters. Our first application of crystal graphs analyzes the invariant subspace of a tensor product of the spin and vector representations of the special orthogonal group, which Westbury [73] showed is captured by the highest weight elements of weight zero in the corresponding crystal graph. We construct an injection from the highest weight elements of weight zero to chord diagrams which intertwines the promotion action on the invariant tensors with the rotation action on chord diagrams. Our second application of crystal graphs gives a novel approach to an outstanding open problem by Thrall to find a combinatorial description of the multiplicities appearing in the irreducible decomposition of the higher Lie modules of the general linear group. We show that the multiplicities are counted by the number of highest weight elements in crystals that we construct for the higher Lie modules corresponding to partitions of part size at most two, leveraging the quasicrystals developed by Cain-Malheiro [11] and Maas-Gariépy [37]. Furthermore, we prove that the leading term in the Schur expansion of the higher Lie characters is a two-part partition, and that the higher Lie characters have saturated Newton polytopes, as defined by Monical-Tokcan-Yong [40]. Finally, motivated by the notion of immersion given by Prasad-Raghunathan [50], we study immersion relations among irreducible polynomial representations of the general linear group. We model these relationships in a new partially ordered set called the immersion poset. Along with classifying certain maximal elements and cover relations in the immersion poset, we also explore lower intervals in the immersion poset in an application to a Schur-positivity question posed by Sundaram [64].

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CHAPTER 1

Introduction

1.1. Overview

Combinatorial models have proven to be powerful tools for understanding the structure of representations of groups. The rigidity and symmetry present in group actions motivate the use of combinatorics to encode enumerative information, such as the dimensions of weight spaces and multiplicities of irreducibles, as well as relationships between representations. In this thesis, we apply combinatorial structures including crystal graphs, symmetric functions, and tableaux to study certain representations of the general linear group, special orthogonal group, and symplectic group.

In the first part of this thesis, we apply crystal graphs to problems in representation theory. Crystal bases were developed by Kashiwara [26, 27] to mirror representations of quantum groups associated to Lie groups at the limit $q = 0$. Crystal graphs are the combinatorial structures of crystal bases that encode important features of the corresponding representations, including invariant subspaces, weight spaces, irreducible subrepresentations, and characters. The existence of crystal graphs allows many problems in representation theory to be reframed as a problem in combinatorics.

1.1.1. Invariant subspaces. We first use crystal graphs to analyze the invariant subspace of the tensor product $V^{\otimes n}$ of an irreducible representation V of a semisimple Lie group G , where G acts diagonally on $V^{\otimes n}$. Interest in invariant subspaces was initiated by Rumer, Teller, and Weyl [56] in their study of quantum mechanical descriptions of molecules. The symmetric group S_n also acts on the invariant subspace of $V^{\otimes n}$ by permuting tensor positions. By Schur–Weyl duality, this action commutes with the action of the Lie group.

Invariant subspaces of tensor products of irreducible representations are well modeled in the corresponding crystal graphs as highest weight elements of weight zero, $\mathcal{B}_0^{\otimes n}$ [49, 73]. Westbury [73] showed that the dimension of the invariant subspace of $V^{\otimes n}$ is equal to the cardinality of $\mathcal{B}_0^{\otimes n}$, and

furthermore that the action of the long cycle in S_n corresponds to the action of promotion on elements in $\mathcal{B}_0^{\otimes n}$. It is desirable to have a correspondence between $\mathcal{B}_0^{\otimes n}$ and diagram bases, such as chord diagrams, which intertwines promotion and rotation. This has been accomplished for type A vector and adjoint representations and type C vector representations [30, 47, 48, 49].

In Chapter 2, we construct such an injection from r -fans of Dyck paths (resp. vacillating tableaux) of length n into the set of chord diagrams on $[n]$ that intertwines the promotion action on the invariant tensors and the rotation action on chord diagrams. There is a natural correspondence between r -fans of Dyck paths (resp. vacillating tableaux) and highest weight elements in the tensor product of the spin (resp. vector) crystal of type B_r described in Section 2.1.3. We present this injection in two different ways: first, as fillings of promotion matrices [34] in Construction 2.2.3, and second, in terms of Fomin growth diagrams [15, 33, 54] in Constructions 2.2.16, 2.2.17, and 2.2.18. While the first description shows that the map intertwines promotion and rotation, the second description shows injectivity. Our proof strategy uses virtualization of crystals (see for example [10]) and results of [49] for oscillating tableaux of weight zero (or equivalently highest weight words of weight zero for the vector representation type C_r).

1.1.2. Thrall’s problem. The second application of crystal graphs is in a novel approach to an outstanding open problem by Thrall [67]. Thrall’s problem asks for a combinatorial description of the multiplicities in the irreducible decomposition of the higher Lie modules. Higher Lie modules are representations of the general linear group $\mathrm{GL}_m(\mathbb{C})$ which are graded pieces of the universal enveloping algebra of the free Lie algebra and indexed by partitions λ (specifics given in 3.1.1). Equivalently, Thrall’s problem studies the coefficients in the Schur expansion of the characters of the higher Lie modules. Variants of the higher Lie characters appear in multiple places, including the type A Orlik-Solomon algebra and the Whitney homology of Π_n [51].

It follows from the tensor product construction of the higher Lie modules that by applying the Littlewood–Richardson rule, Thrall’s problem can be reduced to the case of rectangular partitions $\lambda = \langle a^b \rangle$, written in multiplicity notation. The characters of the higher Lie modules for $\langle 1^n \rangle$ and $\langle 2^b \rangle$ come from the following plethystic substitutions, where h_k is the k -th complete homogeneous

symmetric function and $s_\mu(x_1, \dots, x_m)$ is the Schur polynomial indexed by the partition μ :

$$h_n[s_{(1)}(x_1, \dots, x_m)] = s_{(n)}(x_1, \dots, x_m),$$

$$h_b[s_{(1,1)}(x_1, \dots, x_m)] = \sum_{\substack{\mu \vdash 2b \\ \text{even columns}}} s_\mu(x_1, \dots, x_m).$$

(The plethysm $h_b[s_{(1,1)}]$ was given by Littlewood [35].) Kraśkiewicz and Weyman [32] solve the case of Thrall's problem when $\lambda = \langle a \rangle$ by showing that the coefficient of the Schur polynomial indexed by μ in the higher Lie character is counted by the number of standard Young tableaux of shape μ with major index statistic congruent to 1 mod a . This result was extended by Schocker [57], who gave a formula to compute the coefficients in the Schur expansion of all higher Lie characters. However, Schocker's formula involves subtractions coming from the number theoretic Möbius function and thus is not a purely combinatorial solution. Various coarsenings of Thrall's problem have been studied, including work by Désarémien and Wachs [13] and Hersh and Reiner [24], as well as generalizations to the free Lie superalgebra by Armon and Swanson [3].

In Theorem 3.2.3, we identify the largest partition in lexicographic ordering appearing in the Schur expansion of the higher Lie character associated to λ as a two part partition whose parts are (scaled) sums of the parts of λ . Monical, Tokcan, and Yong [40] introduced the study of polynomials which have saturated Newton polytopes, meaning every lattice point of the convex hull of the exponent vectors corresponds to a monomial in the polynomial. Schur functions, symmetric Macdonald polynomials, and Stanley symmetric polynomials have been shown to have saturated Newton polytope, and others have studied this property for Schubert polynomials, Grothendieck polynomials, chromatic symmetric functions, and the Kronecker product of two Schur functions [45]. We apply our result to prove that the higher Lie characters have saturated Newton polytope in Proposition 3.2.8.

In pursuit of Thrall's original problem, we develop a crystal base approach in Section 3.3. The monomial expansion of the higher Lie characters given by Gessel and Reutenauer [20] provides a combinatorial basis for the higher Lie modules that can be used as vertices in a type A crystal graph. By defining crystal operators, the multiplicities in the irreducible decomposition can be described as the multiplicities of the highest weight elements in the crystal. Classifying these highest weight elements thus gives the combinatorial description asked for by Thrall. Our strategy involves the

quasicrystals defined by Cain and Malheiro [11] and Maas–Gariépy [37], which refine connected components of type A crystals based on the fundamental quasisymmetric function expansion of their respective characters. In Construction 3.3.9, we stitch together appropriate quasicrystals in the crystal on words of length n to create type A crystals corresponding to the higher Lie modules with parts of size at most 2, recovering a known case of Thrall’s problem in Proposition 3.3.11.

1.1.3. Immersion poset. We next study irreducible representations of $\mathrm{GL}_N(\mathbb{C})$ via another combinatorial model that we introduce called the immersion poset. Specifically, we examine a partial ordering on homogeneous polynomial representations of $\mathrm{GL}_N(\mathbb{C})$ called immersion, defined by Prasad and Raghunathan [50]. Given two such finite-dimensional representations $\pi_1 : \mathrm{GL}_N(\mathbb{C}) \rightarrow \mathrm{GL}(W_1)$ and $\pi_2 : \mathrm{GL}_N(\mathbb{C}) \rightarrow \mathrm{GL}(W_2)$, the representation π_1 is immersed in the representation π_2 if the eigenvalues of $\pi_1(g)$, counting multiplicities, are contained in the eigenvalues of $\pi_2(g)$ for all $g \in \mathrm{GL}_N(\mathbb{C})$. Since the eigenvalues of $\pi_i(g)$ record the dimensions of the weight spaces in W_i , immersion of representations is a statement about the containment of weight spaces. Note that if π_1 is a subrepresentation of π_2 , then π_1 is immersed by π_2 , but the converse is not true.

In Chapter 4, we define the immersion poset to capture the immersion of finite-dimensional polynomial representations of $\mathrm{GL}_N(\mathbb{C})$. The immersion poset is a partial ordering on partitions with $\lambda \leq_I \mu$ if and only if the irreducible polynomial representations of $\mathrm{GL}_N(\mathbb{C})$ of highest weight λ and μ are appropriately immersed. The eigenvalues corresponding to these representations are stored in their symmetric function characters, which are the Schur polynomials in N variables indexed by their respective highest weights. Thus, we are able to study the combinatorics of semistandard Young tableaux in order to uncover immersion relations among these representations. We classify certain maximal elements using the hooklength formula (see Propositions 4.2.7, 4.2.8, 4.2.9, and 4.2.11) and identify cover relations in the immersion poset by creating explicit injections between sets of semistandard Young tableaux (for example, see Corollaries 4.2.19, 4.2.20, and 4.2.21).

We also explore an application of the immersion poset to a question posed by Sundaram [64] about the Schur-positivity of certain sums of power sum symmetric functions, relating the immersion order to representations of the symmetric group. In 1961, Solomon [59] proved that the sum of all power sum symmetric functions indexed by partitions of n is the character of the representation of the symmetric group acting on itself by conjugation. Sundaram’s question asks for which subsets of partitions of n is the sum of power sums Schur-positive, and hence the Frobenius

character of a representation of the symmetric group (see Section 4.3.1 for details). We investigate lower intervals in the immersion poset in Section 4.3.2, and prove that our conjectured intervals $\{(1^n) \leq_I \mu \leq_I (n-2, 1, 1)\}$ and $\{(1^n) \leq_I \mu \leq_I (n-2, 2)\}$ give rise to Schur-positivity.

1.2. Preliminaries

1.2.1. Partitions and tableaux. A *partition* $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ of an integer n is a sequence of weakly decreasing nonnegative integers which sum to n . The number of nonzero parts is called the length of λ , denoted $\ell(\lambda)$, and the sum of the parts is called the size of λ , denoted $|\lambda|$. Let $\mathcal{P}(n)$ denote the set of all integer partitions of n . The partition $\lambda = (\lambda_1, \lambda_2, \dots)$ can be represented by a *Ferrers diagram*, a left-justified array of boxes containing λ_i boxes in row i . A *semistandard Young tableau of shape λ* is a filling of the Ferrers diagram of λ with positive integers so that entries weakly increase across rows and strictly increase down columns. For a semistandard Young tableau T , the *weight of T* is the vector $\text{wt}(T) = (\alpha_1, \alpha_2, \dots)$ where α_i is the number of entries i appearing in T . If $\text{wt}(T) = (1, \dots, 1, 0, 0, \dots)$, then T is a *standard Young tableau*.

EXAMPLE 1.2.1. The partition $\lambda = (3, 3, 2) \vdash 8$ has Ferrers diagram

. Below, T_1 is a semistandard Young tableau of shape $(3, 3, 2)$ and weight $(2, 2, 1, 2, 1)$, and T_2 is a standard Young tableau of shape $(3, 3, 2)$.

$$T_1 = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 4 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & 8 \\ \hline 5 & 7 & \\ \hline \end{array}$$

Let $\lambda, \mu \in \mathcal{P}(n)$. We say $\lambda \leq_{lex} \mu$ in *lexicographic order* if there exists an integer k such that

$$\lambda_k < \mu_k \quad \text{and} \quad \lambda_i = \mu_i \quad \text{for all } 1 \leq i < k.$$

We say $\lambda \leq_D \mu$ in *dominance order* if

$$\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i \quad \text{for all } k \geq 1.$$

Both $(\leq_{lex}, \mathcal{P}(n))$ and $(\leq_D, \mathcal{P}(n))$ are examples of *partially ordered sets*, or *posets*, because for all $\lambda, \mu, \nu \in \mathcal{P}(n)$ the following are true.

$$\begin{array}{ll} \lambda \leq_{lex} \lambda & \lambda \leq_D \lambda \\ \lambda \leq_{lex} \mu \wedge \mu \leq_{lex} \lambda \implies \lambda = \mu & \lambda \leq_D \mu \wedge \mu \leq_D \lambda \implies \lambda = \mu \\ \lambda \leq_{lex} \mu \wedge \mu \leq_{lex} \nu \implies \lambda \leq_{lex} \nu & \lambda \leq_D \mu \wedge \mu \leq_D \nu \implies \lambda \leq_D \nu \end{array}$$

Since any two partitions are comparable under lexicographic order, $(\leq_{lex}, \mathcal{P}(n))$ is a total ordering on partitions. Dominance order $(\leq_D, \mathcal{P}(n))$ is not a total order when $n \geq 6$, since for example $(n-2, 1, 1)$ and $(n-3, 3)$ are incomparable.

1.2.2. Symmetric functions. The *symmetric group* S_n is the group of all permutations of the numbers $[n] := \{1, \dots, n\}$. Following notation from [60, Chapter 7], a *homogeneous symmetric function of degree n* is a formal power series in the variables $x = (x_1, x_2, \dots)$,

$$f(x) = \sum_{\alpha} c_{\alpha} x^{\alpha},$$

where the sum is over all weak compositions $\alpha = (\alpha_1, \alpha_2, \dots)$ of n , $c_{\alpha} \in \mathbb{Q}$, x^{α} is the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots$, and

$$f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$$

for any permutation σ of the positive integers. The set of all homogeneous symmetric functions of degree n is denoted Λ^n , and its dimension as a $\mathbb{Q}$ -vector space is given by the number of partitions of n . We next define five bases of Λ^n , each indexed by partitions.

Let $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$. Define the *monomial symmetric function* $m_{\lambda}(x) \in \Lambda^n$ by

$$m_{\lambda}(x) = \sum_{\alpha} x^{\alpha},$$

where the sum ranges over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \dots)$ of the parts of λ . The *elementary symmetric functions*, *complete homogeneous symmetric functions*, and *power sum symmetric*

functions are respectively defined as follows.

$$\begin{aligned} e_k(x) &= \sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k} \quad \text{for } k \geq 1, & e_\lambda(x) &= e_{\lambda_1}(x) e_{\lambda_2}(x) \cdots \\ h_k(x) &= \sum_{i_1 \leq \dots \leq i_k} x_{i_1} \cdots x_{i_k} \quad \text{for } k \geq 1, & h_\lambda(x) &= h_{\lambda_1}(x) h_{\lambda_2}(x) \cdots \\ p_k(x) &= \sum_{i \geq 1} x_i^k \quad \text{for } k \geq 1, & p_\lambda(x) &= p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots \end{aligned}$$

The *Schur function* $s_\lambda(x) \in \Lambda^n$ is defined combinatorially by

$$s_\lambda(x) = \sum_T x^{\text{wt}(T)},$$

where the sum ranges over all semistandard Young tableaux of shape λ . The monomial expansion of $s_\lambda(x)$ is given by

$$s_\lambda(x) = \sum_{\mu} K_{\lambda, \mu} m_{\mu}(x),$$

where the sum is over all permutations μ of size $|\lambda|$ and the coefficient $K_{\lambda, \mu}$ is the *Kostka number*, counting the number of semistandard Young tableaux of shape λ and weight μ .

THEOREM 1.2.2. *The sets*

$$\{m_\lambda \mid \lambda \vdash n\}, \quad \{e_\lambda \mid \lambda \vdash n\}, \quad \{h_\lambda \mid \lambda \vdash n\}, \quad \{p_\lambda \mid \lambda \vdash n\}, \quad \{s_\lambda \mid \lambda \vdash n\}$$

are bases for Λ^n .

For any basis $\{b_\lambda \mid \lambda \vdash n\}$ of Λ^n , we say that $f \in \Lambda^n$ is *b-positive* if the expansion of f in $\{b_\lambda \mid \lambda \vdash n\}$ contains only nonnegative integer coefficients. Specifically for the monomial and Schur bases, we say a symmetric function is *monomial-positive* or *Schur-positive*, respectively.

1.2.3. Representation theory of $\text{GL}_N(\mathbb{C})$. The *general linear group* $\text{GL}_N(\mathbb{C})$ is the group of all $N \times N$ invertible matrices with complex entries. The polynomial representation theory of $\text{GL}_N(\mathbb{C})$ was developed by Schur [58] and later popularized by Weyl [75] in his expository book on the representation theory of the classical groups. A *homogeneous polynomial representation of degree n* (φ, V) of $\text{GL}_N(\mathbb{C})$ consists of an M -dimensional vector space V and a group homomorphism $\varphi : \text{GL}_N(\mathbb{C}) \rightarrow \text{GL}_M(\mathbb{C})$ describing how $\text{GL}_N(\mathbb{C})$ acts on V . Further, for all $A \in \text{GL}_N(\mathbb{C})$, the entries of $\varphi(A)$ are polynomials in the entries of A and $\varphi(cA) = c^n \varphi(A)$ for all $c \in \mathbb{C} \setminus \{0\}$.

The *character* of (φ, V) is defined as the polynomial

$$\text{ch}(V)(x_1, \dots, x_N) = \sum_{x^a \in \mathcal{M}_\varphi} x^a,$$

where $\mathcal{M}_\varphi$ is a multiset of M monomials $x^a = x_1^{a_1} \cdots x_N^{a_N}$ of degree n (i.e. $\sum_{i=1}^N a_i = n$) such that if $A \in \text{GL}_N(\mathbb{C})$ has eigenvalues $\theta = (\theta_1, \dots, \theta_N)$, then $\varphi(A)$ has eigenvalues θ^a for all monomials in $\mathcal{M}_\varphi$ [60, Appendix 2]. Specifically, if $A \in \text{GL}_N(\mathbb{C})$ has eigenvalues $\theta = (\theta_1, \dots, \theta_N)$, then $\text{ch}(V)(\theta) = \text{tr}(\varphi(A))$, the trace of the action of A under φ . The homogeneous irreducible polynomial representations of degree n of $\text{GL}_N(\mathbb{C})$, also known as *Weyl modules* $(\rho_\lambda, W_\lambda(\mathbb{C}^N))$, are indexed by their highest weights, which are integer partitions $\lambda \vdash n$ with at most N nonzero parts. The corresponding irreducible characters are the *Schur polynomials* $s_\lambda(x_1, \dots, x_N)$, homogeneous symmetric polynomials of degree n in N variables $x_1, \dots, x_N$.

If (φ, V) is a finite-dimensional polynomial representation of $\text{GL}_N(\mathbb{C})$, then V can be decomposed into irreducible subrepresentations which are isomorphic to Weyl modules. Two polynomial representations of $\text{GL}_N(\mathbb{C})$ are isomorphic if and only if their characters are equal. Thus, finding the multiplicity of the irreducible $W_\lambda(\mathbb{C}^N)$ in V is equivalent to finding the coefficient of $s_\lambda(x_1, \dots, x_N)$ in the character of V ,

$$V = \bigoplus_{\lambda} W_\lambda(\mathbb{C}^N)^{\oplus c_\lambda} \iff \text{ch}(V)(x_1, \dots, x_N) = \sum_{\lambda} c_\lambda s_\lambda(x_1, \dots, x_N).$$

1.2.4. Crystal graphs. Kashiwara crystals are combinatorial models of representations of Lie groups. Informally, crystal graphs are directed, labeled graphs with data dependent on the particular representation and the root system and weight lattice of the given Cartan type. Each vertex is assigned a weight, and the edges in the graph mirror how the action in the representation moves between weight spaces. This is given precisely in the following definition.

DEFINITION 1.2.3 ([10, Bump-Schilling]). *Let Φ be a root system of rank r with index set I , simple roots $\{\alpha_1, \dots, \alpha_r\}$, and weight lattice Λ . A *crystal* of type Φ is a nonempty set $\mathcal{B}$ together with maps*

$$e_i, f_i : \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$$

$$\varepsilon_i, \varphi_i : \mathcal{B} \rightarrow \mathbb{Z} \sqcup \{-\infty\}$$

$$\text{wt} : \mathcal{B} \rightarrow \Lambda$$

for $i \in I$ and $0 \notin \mathcal{B}$, satisfying the following conditions:

A1. If $x, y \in \mathcal{B}$, then $e_i(x) = y$ if and only if $f_i(y) = x$. In this case, it is assumed that

$$\text{wt}(y) = \text{wt}(x) + \alpha_i, \quad \varepsilon_i(y) = \varepsilon_i(x) - 1, \quad \varphi_i(y) = \varphi_i(x) + 1$$

A2. We require that $\varphi_i(x) = \langle \text{wt}(x), \alpha_i^\vee \rangle + \varepsilon_i(x)$ for all $x \in \mathcal{B}$ and $i \in I$. In particular,

if $\varphi_i(x) = -\infty$, then $\varepsilon_i(x) = -\infty$ also, and we require $e_i(x) = f_i(x) = 0$.

We associate $\mathcal{B}$ with a directed graph with vertices in $\mathcal{B}$ and edges labeled by $i \in I$. If $f_i(x) = y$ for $x, y \in \mathcal{B}$, then we draw an edge $x \xrightarrow{i} y$. This is called the **crystal graph** of $\mathcal{B}$.

REMARK 1.2.4. In this thesis, only crystals of the classical root systems of type A_r, B_r, C_r and D_r are discussed. Furthermore, we only consider Stembridge crystals of types A and D , and virtualizable crystals of types B and C , both of which satisfy additional axioms to A1 and A2 in Definition 1.2.3. Together, these so called **normal crystals** form the class of crystals which are in correspondence with representations of the Lie groups $GL_{r+1}(\mathbb{C})$, $SO_{2r+1}(\mathbb{C})$, $Sp_{2r}(\mathbb{C})$, and $SO_{2r}(\mathbb{C})$. For precise details, see Bump-Schilling Chapters 4 and 5.

EXAMPLE 1.2.5. For type A_{m-1} representations of $GL_m(\mathbb{C})$, the weight lattice is $\mathbb{Z}^m$ and the simple roots are $\alpha_1, \dots, \alpha_{m-1}$ where $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ and $\mathbf{e}_i$ is the standard basis vector with 1 in the i -th entry and 0 else. Type A_{m-1} **standard crystal graph** looks like

$$\mathbb{B} = \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \dots \xrightarrow{m-2} \boxed{m-1} \xrightarrow{m-1} \boxed{m}.$$

The other crystal maps are defined as

$$\varphi_i \left(\boxed{j} \right) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}, \quad \varepsilon_i \left(\boxed{j} \right) = \begin{cases} 1 & \text{if } i = j + 1 \\ 0 & \text{else} \end{cases}, \quad \text{wt} \left(\boxed{j} \right) = \mathbf{e}_j,$$

where $1 \leq i \leq m-1$ and $1 \leq j \leq m$. This crystal corresponds to the **standard representation** of $GL_m(\mathbb{C})$, where $A \in GL_m(\mathbb{C})$ acts on $v \in \mathbb{C}^m$ by usual matrix multiplication.

DEFINITION 1.2.6 ([10, Bump-Schilling]). Let $\mathcal{B}$ and $\mathcal{C}$ be two crystals associated to the same root system Φ and index set I . A **crystal morphism** is a map $\psi : \mathcal{B} \rightarrow \mathcal{C} \sqcup \{0\}$ such that

- if $b \in \mathcal{B}$ and $\psi(b) \in \mathcal{C}$, then (a) $\text{wt}(\psi(b)) = \text{wt}(b)$, (b) $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$ for all $i \in I$, and (c) $\varphi_i(\psi(b)) = \varphi_i(b)$ for all $i \in I$;
- if $b, e_i(b) \in \mathcal{B}$ such that $\psi(b), \psi(e_i(b)) \in \mathcal{C}$, then we have $\psi(e_i(b)) = e_i(\psi(b))$;
- if $b, f_i(b) \in \mathcal{B}$ such that $\psi(b), \psi(f_i(b)) \in \mathcal{C}$, then we have $\psi(f_i(b)) = f_i(\psi(b))$.

DEFINITION 1.2.7 ([10, Bump-Schilling]). Let $\mathcal{B}$ and $\mathcal{C}$ be two crystals associated to the same root system Φ . The **tensor product** $\mathcal{B} \otimes \mathcal{C}$ is the set $\{x \otimes y \mid x \in \mathcal{B}, y \in \mathcal{C}\}$ together with the following maps:

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{if } \varphi_i(y) \leq \varepsilon_i(x) \\ x \otimes f_i(y) & \text{if } \varphi_i(y) > \varepsilon_i(x) \end{cases} \quad \begin{aligned} \varphi_i(x \otimes y) &= \max(\varphi_i(x), \varphi_i(y) + \langle \text{wt}(x), \alpha_i^\vee \rangle) \\ \varepsilon_i(x \otimes y) &= \max(\varepsilon_i(y), \varepsilon_i(x) + \langle \text{wt}(y), \alpha_i^\vee \rangle) \\ \text{wt}(x \otimes y) &= \text{wt}(x) + \text{wt}(y) \end{aligned}$$

$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \text{if } \varphi_i(y) < \varepsilon_i(x) \\ x \otimes e_i(y) & \text{if } \varphi_i(y) \geq \varepsilon_i(x) \end{cases}$$

Kashiwara crystals respect tensor products, meaning that if the crystals $\mathcal{B}$ and $\mathcal{C}$ correspond to representations V and W , respectively, then the tensor product crystal $\mathcal{B} \otimes \mathcal{C}$ corresponds to the tensor product representation $V \otimes W$.

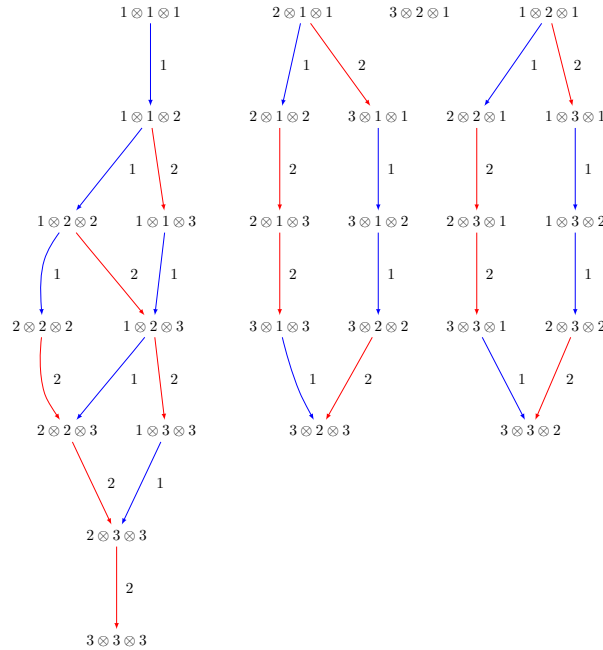


FIGURE 1.1. The type A_2 crystal on words of length 3, $\mathbb{B}^{\otimes 3}$.

EXAMPLE 1.2.8. The type A_{m-1} crystal $\mathbb{B}^{\otimes n}$ created by taking the n -fold tensor product of the standard crystal $\mathbb{B}$ corresponds to $T^n(\mathbb{C}^m)$, the n -th degree piece of the tensor algebra of $\mathbb{C}^m$, where $A \in GL_m(\mathbb{C})$ acts diagonally on $v_1 \otimes \cdots \otimes v_n \in T^n(\mathbb{C}^m)$. Figure 1.1 shows the type A_2 crystal $\mathbb{B}^{\otimes 3}$, called the [crystal on words](#) of length 3.

Connected crystal graphs correspond to an irreducible representation of a Lie group. An element $v \in \mathcal{B}$ of a crystal is called a [highest weight element](#) if $e_i(v) = 0$ for all $i \in I$. Every connected crystal has a unique highest weight element [10, Theorem 5.20], and the connected components of highest weight λ are denoted $\mathcal{B}(\lambda)$. The [crystal character](#) of $\mathcal{B}$ is defined as

$$\text{ch}(\mathcal{B}) = \sum_{v \in \mathcal{B}} x^{\text{wt}(v)}.$$

The characters of connected crystals with highest weight element of weight λ are equal to the character of the corresponding irreducible representation of highest weight λ [10, Corollary 13.9]. For example, the type A_{m-1} highest weights are partitions of n with at most m parts. For such a partition λ , we have

$$\text{ch}(\mathcal{B}(\lambda))(x_1, \dots, x_m) = s_\lambda(x_1, \dots, x_m) = \text{ch}(W_\lambda(\mathbb{C}^m))(x_1, \dots, x_m).$$

Therefore, if $\mathcal{B}$ is the crystal corresponding to the representation V of $GL_m(\mathbb{C})$, then the character of V is given by

$$\text{ch}(V)(x_1, \dots, x_m) = \sum_{\lambda} c_{\lambda} s_{\lambda}(x_1, \dots, x_m),$$

where $c_{\lambda} := \#\{v \in \mathcal{B} \mid e_i(v) = 0 \text{ for } 1 \leq i \leq m-1 \text{ and } \text{wt}(v) = \lambda\}$. To summarize, classifying the highest weight elements of weight λ in $\mathcal{B}$ gives a combinatorial description of the multiplicity of the irreducible representation of highest weight λ in V .

EXAMPLE 1.2.9. Figure 1.1 shows that the type A_2 regular representation $T^3(\mathbb{C}^3)$ decomposes into irreducibles $W_{(1,1,1)}(\mathbb{C}^3) \oplus W_{(2,1)}(\mathbb{C}^3)^{\oplus 2} \oplus W_{(3)}(\mathbb{C}^3)$ because the crystal on words of length 3 $\mathbb{B}^{\otimes 3}$ decomposes into connected components $\mathcal{B}((1, 1, 1)) \oplus \mathcal{B}((2, 1))^{\oplus 2} \oplus \mathcal{B}((3))$.

CHAPTER 2

Diagrammatic bases for invariant tensors

This chapter is based on work in collaboration with Joseph Pappé, Stephan Pfannerer, and Anne Schilling published in [46]. My specific contributions include the crystal virtualization in Section 2.1.2, highest weight elements of weight zero in Section 2.1.3, Fomin growth diagrams in Section 2.2.2, and the proof of Proposition 2.3.10.

2.1. Crystal Bases

In this section we provide further background on crystal bases. In Section 2.1.1 we define crystals corresponding to the type B_r spin and vector representations and to the type C_r vector representation. In Section 2.1.2 we define virtual crystals and provide a virtualization of type B_r spin and vector crystals into certain type C_r vector crystals. In Section 2.1.3 we identify the highest weight elements of weight zero in these crystals with sequences of tableaux. In Section 2.1.4 we define promotion on crystals in two ways, using the crystal commutor and local rules.

2.1.1. Crystals of type B_r and type C_r . Here we define certain crystals for the root systems B_r and C_r explicitly. Let $\mathbf{e}_i \in \mathbb{Z}^r$ be the i -th unit vector with 1 in position i and 0 everywhere else.

DEFINITION 2.1.1. *The **spin crystal** of type B_r , denoted by $\mathcal{B}_{\text{spin}}$, consists of all r -tuples $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_r)$, where $\epsilon_i \in \{\pm\}$. The weight of ϵ is*

$$\text{wt}(\epsilon) = \frac{1}{2} \sum_{i=1}^r \epsilon_i \mathbf{e}_i.$$

The crystal operator f_r annihilates ϵ unless $\epsilon_r = +$. If $\epsilon_r = +$, f_r acts on ϵ by changing ϵ_r from $+$ to $-$ and leaving all other entries unchanged. The crystal operator f_i for $1 \leq i < r$ annihilates ϵ unless $\epsilon_i = +$ and $\epsilon_{i+1} = -$. In the latter case, f_i acts on ϵ by changing ϵ_i to $-$ and ϵ_{i+1} to $+$. Similarly, the crystal operator e_r annihilates ϵ unless $\epsilon_r = -$. If $\epsilon_r = -$, e_r acts on ϵ by changing ϵ_r from $-$ to $+$. The crystal operator e_i for $1 \leq i < r$ annihilates ϵ unless $\epsilon_i = -$ and $\epsilon_{i+1} = +$. In the latter case, e_i acts on ϵ by changing ϵ_i to $+$ and ϵ_{i+1} to $-$.

The crystal $\mathcal{B}_{\text{spin}}$ of type B_3 is depicted in Figure 2.1.

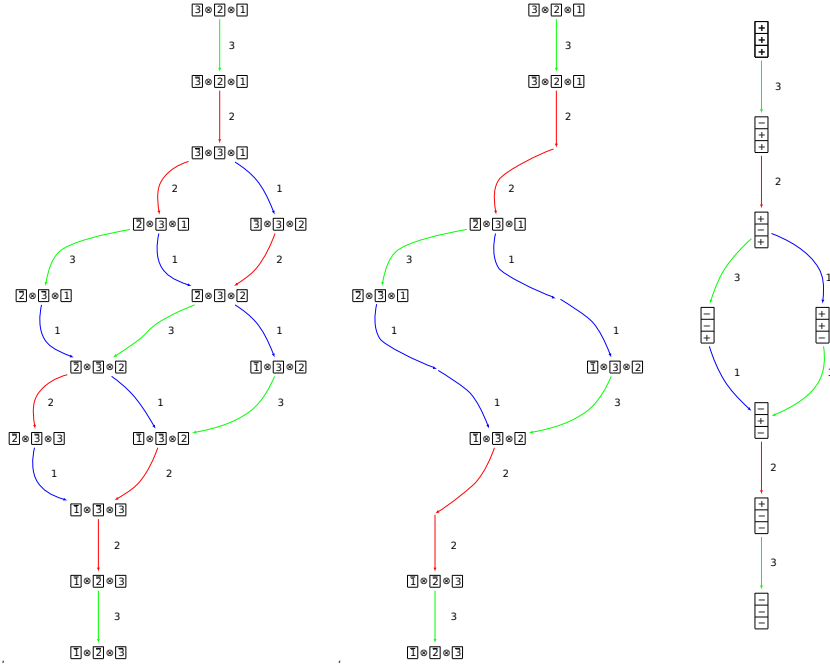


FIGURE 2.1. Left: One component of the crystal $\widehat{\mathcal{V}} = \mathcal{C}_{\square}^{\otimes 3}$ of type C_3 . Middle: The virtual crystal $\mathcal{V}$ inside $\widehat{\mathcal{V}}$ of type B_3 . Right: The spin crystal $\mathcal{B}_{\text{spin}}$ of type B_3 .

DEFINITION 2.1.2. Here we define the **crystals for the vector representation** of type B_r and C_r .

- (1) The crystal $\mathcal{C}_{\square}$ of type C_r consists of the elements $\{1, 2, \dots, r, \bar{r}, \dots, \bar{2}, \bar{1}\}$. The crystal operator f_i for $1 \leq i < r$ maps i to $i+1$, maps $\overline{i+1}$ to $\bar{i}$ and annihilates all other elements. The crystal operator f_r maps r to $\bar{r}$ and annihilates all other elements. Similarly, the crystal operator e_i for $1 \leq i < r$ maps $i+1$ to i , maps $\bar{i}$ to $\overline{i+1}$ and annihilates all other elements. The crystal operator e_r maps $\bar{r}$ to r and annihilates all other elements. Furthermore, $\text{wt}(i) = \mathbf{e}_i$ and $\text{wt}(\bar{i}) = -\mathbf{e}_i$.
- (2) The crystal $\mathcal{B}_{\square}$ of type B_r consists of the elements $\{1, 2, \dots, r, 0, \bar{r}, \dots, \bar{2}, \bar{1}\}$. The crystal operator f_i for $1 \leq i < r$ maps i to $i+1$, maps $\overline{i+1}$ to $\bar{i}$ and annihilates all other elements. The crystal operator f_r maps r to 0 , 0 to $\bar{r}$ and annihilates all other elements. Similarly, the crystal operator e_i for $1 \leq i < r$ maps $i+1$ to i , maps $\bar{i}$ to $\overline{i+1}$ and annihilates all other elements. The crystal operator e_r maps $\bar{r}$ to 0 , 0 to r and annihilates all other elements. Furthermore, $\text{wt}(i) = \mathbf{e}_i$ and $\text{wt}(\bar{i}) = -\mathbf{e}_i$ for $i \neq 0$ and $\text{wt}(0) = 0$.

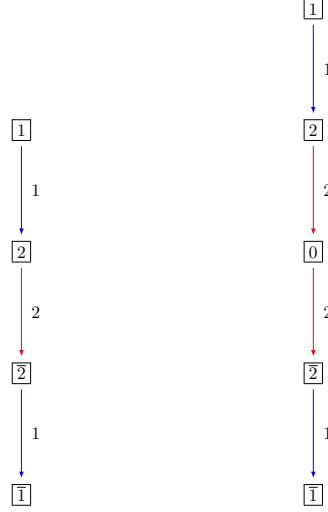


FIGURE 2.2. Left: The crystal $\mathcal{C}_{\square}$ of type C_2 . Right: The crystal $\mathcal{B}_{\square}$ of type B_2 .

The crystals $\mathcal{C}_{\square}$ for type C_2 and $\mathcal{B}_{\square}$ for type B_2 are depicted in Figure 2.2.

2.1.2. Virtual crystals. Stembridge [62] characterized crystals which are associated with quantum group representations for simply-laced root systems in terms of local rules on the crystal graph. Crystals for non-simply-laced root systems can be constructed using virtual crystals, see [10, Chapter 5].

In this paper, we utilize virtual crystals to construct Fomin growth diagrams and the promotion operators for type B_r using results for type C_r . Hence let us briefly review the set-up for virtual crystals. Let $X \hookrightarrow Y$ be an embedding of Lie algebras such that the fundamental weights ω_i and simple roots α_i map as follows

$$\begin{aligned}\omega_i^X &\mapsto \gamma_i \sum_{j \in \sigma(i)} \omega_j^Y, \\ \alpha_i^X &\mapsto \gamma_i \sum_{j \in \sigma(i)} \alpha_j^Y.\end{aligned}$$

Here γ_i is a multiplication factor, $\sigma: I^X \rightarrow I^Y / \text{aut}$ is a bijection and aut is an automorphism on the Dynkin diagram for Y .

Let $\widehat{\mathcal{V}}$ be an ambient crystal associated to the Lie algebra Y . In [10, Chapter 5] it is assumed that $\widehat{\mathcal{V}}$ is a crystal for a simply-laced root system. However, in general it may be assumed that $\widehat{\mathcal{V}}$ is a crystal corresponding to a quantum group representation (which is the case in our setting).

DEFINITION 2.1.3. *If there is an embedding of Lie algebras $X \hookrightarrow Y$, then $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a **virtual crystal** for the root system Φ^X if*

- V1.** *The ambient crystal $\widehat{\mathcal{V}}$ is a Stembridge crystal or a crystal associated to a representation for the root system Φ^Y with crystal operators $\widehat{e}_i, \widehat{f}_i, \widehat{\varepsilon}_i, \widehat{\varphi}_i$ for $i \in I^Y$ and weight function $\widehat{\text{wt}}$.*
- V2.** *If $b \in \mathcal{V}$ and $i \in I^X$, then $\widehat{\varepsilon}_j(b)$ has the same value for all $j \in \sigma(i)$ and that value is a multiple of γ_i . The same is true for $\widehat{\varphi}_j(b)$.*
- V3.** *The subset $\mathcal{V} \sqcup \{\emptyset\} \subseteq \widehat{\mathcal{V}} \sqcup \{\emptyset\}$ is closed under the virtual crystal operators*

$$e_i := \prod_{j \in \sigma(i)} \widehat{e}_j^{\gamma_i} \quad \text{and} \quad f_i := \prod_{j \in \sigma(i)} \widehat{f}_j^{\gamma_i}.$$

Furthermore, for all $b \in \mathcal{V}$

$$\varepsilon_i(b) = \max\{k \geq 0 \mid e_i^k(b) \neq \emptyset\} \quad \text{and} \quad \varphi_i(b) = \max\{k \geq 0 \mid f_i^k(b) \neq \emptyset\}.$$

The tensor product of two virtual crystals for the same embedding $X \hookrightarrow Y$ is again a virtual crystal (see for example [10, Theorem 5.8]).

2.1.2.1. *Virtual crystal $B_r \hookrightarrow C_r$ spin to vector.* We will now apply the theory of virtual crystals to the embedding $B_r \hookrightarrow C_r$. In this setting $I^{C_r} = I^{B_r} = \{1, 2, \dots, r\}$, $\sigma(i) = \{i\}$, $\gamma_i = 2$ for $1 \leq i < r$ and $\gamma_r = 1$. We consider as the ambient crystal

$$\widehat{\mathcal{V}} = \mathcal{C}_{\square}^{\otimes r}.$$

Define an ordering $<$ on the set $[r] \cup [\bar{r}]$ as follows:

$$1 < 2 < \dots < r < \bar{r} < \dots < \bar{1}.$$

Denote by $|\cdot|$ the map from $[r] \cup [\bar{r}]$ to $[r]$ that sends letters to their corresponding unbarred values.

DEFINITION 2.1.4. *Let $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ be given by*

$$\mathcal{V} := \{v_r \otimes v_{r-1} \otimes \dots \otimes v_1 \in \widehat{\mathcal{V}} \mid v_i > v_j \text{ and } |v_i| \neq |v_j| \text{ for all } i > j\}.$$

Let $f_i = \widehat{f}_i^2$, $e_i = \widehat{e}_i^2$ for $1 \leq i < r$ and $f_r = \widehat{f}_r$, $e_r = \widehat{e}_r$.

LEMMA 2.1.5. $\mathcal{V} \sqcup \{\emptyset\}$ is closed under the operators f_i and e_i for $1 \leq i \leq r$.

PROOF. Let $v = v_r \otimes v_{r-1} \otimes \cdots \otimes v_1 \in \mathcal{V}$. We break into cases depending on the value of i .

Assume that $i = r$. By the definition of $\mathcal{V}$, v must either contain an r or $\bar{r}$, but not both. If v contains an r , then this r must be to the left of all other unbarred letters and to the right of all barred letters. As f_r changes the r to a $\bar{r}$, $f_r(v)$ is still in $\mathcal{V}$. If v contains an $\bar{r}$, then $f_r(v) = \emptyset \in \mathcal{V} \sqcup \{\emptyset\}$.

Assume that $i \neq r$. Note that the conditions imposed on v imply that there exists exactly two indices j and k such that $|v_j| = i$ and $|v_k| = i + 1$. By the ordering imposed on v , v can only be in the following forms:

- $\cdots \otimes i + 1 \otimes i \otimes \cdots$
- $\cdots \otimes \bar{i} \otimes \overline{i + 1} \otimes \cdots$
- $\cdots \otimes \bar{i} \otimes \cdots \otimes i + 1 \otimes \cdots$
- $\cdots \otimes \overline{i + 1} \otimes \cdots \otimes i \otimes \cdots$

For the first three cases, $f_i(v) = \emptyset$. When v is of the form $\cdots \otimes \overline{i + 1} \otimes \cdots \otimes i \otimes \cdots$, f_i replaces the $\overline{i + 1}$ with $\bar{i}$ and the i with $i + 1$. Since v does not contain an $\bar{i}$ nor an $i + 1$, $f_i(v)$ is an element of $\mathcal{V}$.

The fact that $e_i(v) \in \mathcal{V}$ for all $i \in 1 \leq i \leq r$ follows similarly. Thus, $\mathcal{V}$ is closed under the operators f_i and e_i . $\square$

LEMMA 2.1.6. All elements of $\mathcal{V}$ are in the connected component of $\widehat{\mathcal{V}}$ with highest weight element $r \otimes r - 1 \otimes \cdots \otimes 1$.

PROOF. Clearly $r \otimes r - 1 \otimes \cdots \otimes 1$ is a highest weight element of $\widehat{\mathcal{V}}$ and the only element in $\mathcal{V}$ without any barred letters.

Consider $v = v_r \otimes \cdots \otimes v_1 \in \mathcal{V}$ containing a barred letter. Observe that the number of barred letters in $e_i(v)$ is at most the number of barred letters in v whenever $e_i(v) \neq \emptyset$. Since $\widehat{\mathcal{V}}$ is finite and $\mathcal{V}$ is closed under e_i , it suffices to show that $e_i(v) \neq \emptyset$ for some i . Let v_j denote the rightmost tensor factor in v that is a barred letter, and let $i = |v_j|$. We break into cases depending on the value of i .

If $i = r$, then $v_j = \bar{r}$ and v cannot contain an r . This implies that $e_r(v) \neq \emptyset$ as it acts on v by replacing v_j by r . The number of barred letters has decreased by one.

If $i \neq r$, then $v_j = \bar{i}$. As v_j is the rightmost barred letter in v , v must be of the form $\cdots \otimes \bar{i} \otimes \cdots \otimes i+1 \otimes \cdots$. Thus, e_i acts by changing $\bar{i}$ to $\overline{i+1}$ and $i+1$ to i . Note that the rightmost barred letter is closer to $\bar{r}$. $\square$

DEFINITION 2.1.7. Let $\Psi: \mathcal{B}_{\text{spin}} \rightarrow \mathcal{V}$ be the map

$$\Psi(\epsilon_1 \epsilon_2 \cdots \epsilon_r) = v_r \otimes v_{r-1} \otimes \cdots \otimes v_1,$$

where $v_r > v_{r-1} > \cdots > v_1$ such that if $\epsilon_i = +$ then v contains an i and if $\epsilon_i = -$ then v contains an $\bar{i}$ for all $1 \leq i \leq r$.

LEMMA 2.1.8. The map Ψ is a bijective map that intertwines the crystal operators on $\mathcal{B}_{\text{spin}}$ and $\mathcal{V}$.

PROOF. From the definition of Ψ , it is clearly bijective. Let $\epsilon = \epsilon_1 \epsilon_2 \cdots \epsilon_r \in \mathcal{B}_{\text{spin}}$. Since the raising and lowering operators of a crystal are partial inverses, it suffices to prove that $f_i(\epsilon) \neq \emptyset$ if and only if $f_i(\Psi(\epsilon)) \neq \emptyset$ and $\Psi(f_i(\epsilon)) = f_i(\Psi(\epsilon))$ whenever $f_i(\epsilon) \neq \emptyset$.

Assume that $f_i(\Psi(\epsilon)) \neq \emptyset$. If $i = r$, then $\Psi(\epsilon)$ contains an r implying $\epsilon_r = +$. Therefore $f_r(\epsilon) \neq \emptyset$. If $i \neq r$, then ϵ contains both an i and an $\overline{i+1}$. Thus, $\epsilon_i = +$ and $\epsilon_{i+1} = -$ implying $f_i(\epsilon) \neq \emptyset$.

Assume that $f_i(\epsilon) \neq \emptyset$. If $i = r$, then $\epsilon_r = +$ and f_r acts on ϵ by replacing ϵ_r with a $-$. This implies that $\Psi(f_r(\epsilon))$ can be obtained from $\Psi(\epsilon)$ by changing the r to $\bar{r}$, which agrees with the action of f_r . Therefore $\Psi(f_r(\epsilon)) = f_r(\Psi(\epsilon))$. If $i \neq r$, then ϵ_i must be a $+$ and ϵ_{i+1} must be a $-$. Thus, f_i swaps the signs of ϵ_i and ϵ_{i+1} . Since $\epsilon_i = +$ and $\epsilon_{i+1} = -$, $\Psi(\epsilon)$ must contain both an $\overline{i+1}$ and an i . This implies $\Psi(f_i(\epsilon))$ can be obtained from $\Psi(\epsilon)$ by replacing the $\overline{i+1}$ with $\bar{i}$ and the i with $i+1$. Observe that f_i acts on $\Psi(\epsilon)$ in exactly the same manner. Hence, $\Psi(f_i(\epsilon)) = f_i(\Psi(\epsilon))$. $\square$

PROPOSITION 2.1.9. $\mathcal{V}$ is a virtual crystal for the embedding of Lie algebras $B_r \hookrightarrow C_r$.

PROOF. The ambient crystal $\widehat{\mathcal{V}}$ is a crystal coming from a representation (see for example [10]), ensuring **V1**. Using Lemmas 2.1.5 and 2.1.8, we have $\Psi(\mathcal{B}_{\text{spin}}) = \mathcal{V}$ is closed under the crystal operators f_i and e_i . Since $\mathcal{B}_{\text{spin}}$ and $\widehat{\mathcal{V}}$ are both seminormal, the string lengths of $\mathcal{B}_{\text{spin}}$ are the

same as the string lengths in $\mathcal{V}$, showing **V3**. It is also not hard to see from Definition 2.1.4, that $\widehat{\varphi}_i(v), \widehat{\varepsilon}_i(v) \in 2\mathbb{Z}$ for $v \in \mathcal{V}$ and $1 \leq i < r$, proving **V2**. $\square$

An example of the virtual crystal construction for $\mathcal{B}_{\text{spin}}$ is given in Figure 2.1. The virtual crystal of this section also follows from [28]. An affine version of this virtual crystal construction (which implies the one in this section) has appeared in [17, Lemma 4.2].

2.1.2.2. *Virtual crystal $B_r \hookrightarrow C_r$ vector to vector.* The crystal $\mathcal{B}_{\square}$ of Definition 2.1.2 can be realized as a virtual crystal inside the ambient crystal $\widehat{\mathcal{V}} = \mathcal{C}_{\square}^{\otimes 2}$.

DEFINITION 2.1.10. Define $\mathcal{V} \subseteq \widehat{\mathcal{V}} = \mathcal{C}_{\square}^{\otimes 2}$ of type C_r as

$$\mathcal{V} = \{a \otimes a \mid 1 \leq a \leq r\} \cup \{\bar{a} \otimes \bar{a} \mid 1 \leq a \leq r\} \cup \{r \otimes \bar{r}\}$$

with $f_i = \widehat{f}_i^2$, $e_i = \widehat{e}_i^2$ for $1 \leq i < r$ and $f_r = \widehat{f}_r$, $e_r = \widehat{e}_r$.

LEMMA 2.1.11. $\mathcal{V} \sqcup \{\emptyset\}$ of Definition 2.1.10 is closed under the operators f_i and e_i for $1 \leq i \leq r$ and all elements in $\mathcal{V}$ are in the connected component of $\widehat{\mathcal{V}}$ with highest weight $1 \otimes 1$.

PROOF. We leave this to the reader to check. $\square$

DEFINITION 2.1.12. Let $\Psi: \mathcal{B}_{\square} \rightarrow \mathcal{V}$ be the map $\Psi(a) = a \otimes a$ and $\Psi(\bar{a}) = \bar{a} \otimes \bar{a}$ for $1 \leq a \leq r$ and $\Psi(0) = r \otimes \bar{r}$.

LEMMA 2.1.13. The map Ψ of Definition 2.1.12 is a bijective map that intertwines the crystal operators on $\mathcal{B}_{\square}$ and $\mathcal{V}$.

PROOF. We leave this to the reader to check. $\square$

PROPOSITION 2.1.14. $\mathcal{V}$ of Definition 2.1.10 is a virtual crystal for the embedding of Lie algebras $B_r \hookrightarrow C_r$.

PROOF. We leave this to the reader to check. $\square$

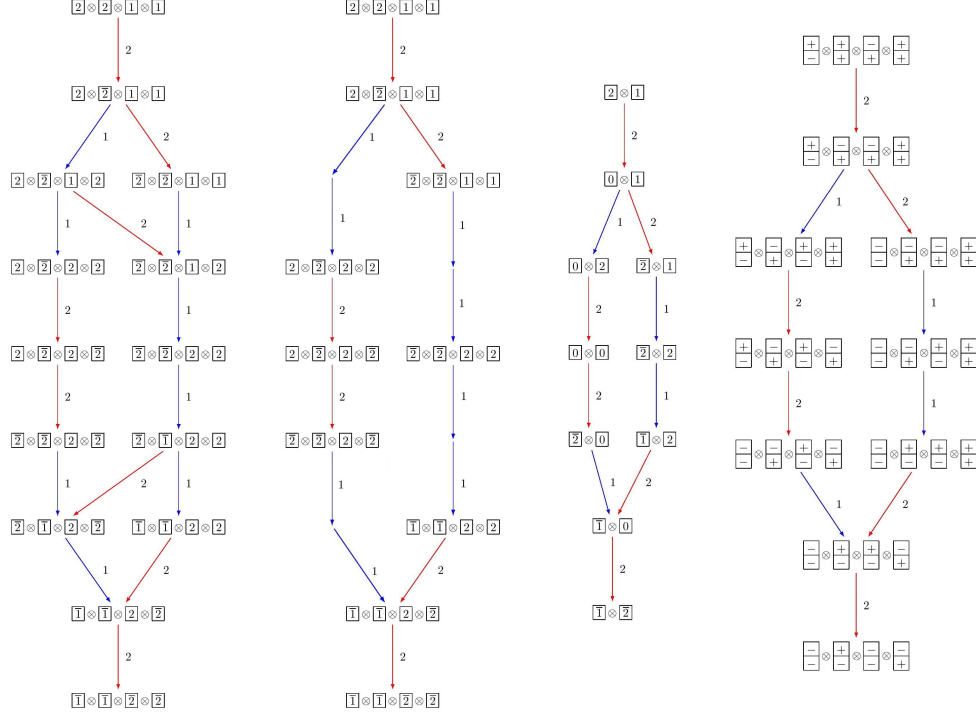


FIGURE 2.3. Far Left: One connected component $\hat{\mathcal{S}}$ of the crystal $\hat{\mathcal{V}}^{\otimes 2} = (\mathcal{C}_{\square}^{\otimes 2})^{\otimes 2}$ of type C_2 . Middle Left: The connected component $\mathcal{S}$ of the virtual crystal $\mathcal{V}^{\otimes 2}$ inside $\mathcal{S}$ induced by Definition 2.1.10. Middle Right: The corresponding connected component $\mathcal{T}$ of the crystal $\mathcal{B}_{\square}^{\otimes 2}$ of type B_2 that corresponds to $\mathcal{S}$ under the embedding given in Definition 2.1.12. Far Right: The connected component $\mathcal{U}$ of $(\mathcal{B}_{\text{spin}} \otimes \mathcal{B}_{\text{spin}})^{\otimes 2}$ of type B_2 corresponding to $\mathcal{T}$ under the isomorphism given in Figure 2.4.

An example of the virtual crystal construction for $\mathcal{B}_{\square}$ is given in Figure 2.3. The virtual crystal of this section also follows from [28]. An affine version of this virtual crystal construction (which implies the one in this section) has appeared in [17, Theorem 4.8].

2.1.3. Highest weights of weight zero. A weight $\lambda \in \Lambda$ is called *minuscule* if $\langle \lambda, \alpha^{\vee} \rangle \in \{0, \pm 1\}$ for all coroots $\alpha^{\vee}$. A crystal $\mathcal{B}$ is called *minuscule* if $\text{wt}(b)$ is minuscule for all $b \in \mathcal{B}$. Note that $\mathcal{B}_{\text{spin}}$ is a minuscule crystal (see for example [10, Chapter 5.4]).

A weight λ is called *dominant* if $\langle \lambda, \alpha_i^{\vee} \rangle \geq 0$ for all $i \in I$. Let $\Lambda^+ \subseteq \Lambda$ denote the set of all dominant weights. Except for spin weights, dominant weights can be identified with partitions, where the fundamental weight ω_h corresponds to a column of height h in the partition. A *partition* λ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell})$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell} \geq 0$. We identify partitions that differ by trailing zeroes. That is, $(3, 2, 0, 0)$ is identified with the partition $(3, 2)$.

Let $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n$ be minuscule crystals. For any highest weight element

$$u = u_n \otimes \cdots \otimes u_1 \in \mathcal{B}_n \otimes \cdots \otimes \mathcal{B}_1$$

we may bijectively associate a sequence of dominant weights $\emptyset = \mu^0, \mu^1, \dots, \mu^n$, where $\mu^q := \sum_{i=1}^q \text{wt}(u_i)$. The final weight $\mu := \mu^n$ of such a sequence is also the weight of the crystal element u . If μ is zero, u is a *highest weight element of weight zero*.

Note that the number of highest weight elements of weight zero in a tensor product of crystals is equal to the dimension of the invariant subspace, see for example [49, 73].

2.1.3.1. *Oscillating tableaux*. Oscillating tableaux were introduced by Sundaram [63].

DEFINITION 2.1.15 (Sundaram [63]). An *r -symplectic oscillating tableau* $\mathbf{O}$ of length n and shape μ is a sequence of partitions

$$\mathbf{O} = (\emptyset = \mu^0, \mu^1, \dots, \mu^n = \mu)$$

such that the Ferrers diagrams of two consecutive partitions differ by exactly one cell, and each partition μ^i has at most r nonzero parts.

The r -symplectic oscillating tableaux of length n and shape μ are in bijection with highest weight elements in $\mathcal{C}_{\square}^{\otimes n}$ of type C_r and weight μ . This can be seen by induction on n . For $n = 1$, the only highest weight element is 1 and the only oscillating tableau is $(\emptyset, \square)$. Suppose the claim is true for $n - 1$. If $u = b \otimes u_0 \in \mathcal{C}_{\square}^{\otimes n}$ is highest weight, then $u_0 \in \mathcal{C}_{\square}^{\otimes(n-1)}$ must be highest weight and hence by induction corresponds to an oscillating tableau $(\emptyset = \mu^0, \mu^1, \dots, \mu^{n-1})$. The element b is either an unbarred or barred letter. If b is the unbarred letter a , μ^n differs from μ^{n-1} by a box in row a . If b is the barred letter $\bar{a}$, μ^n has one less box in row a than μ^{n-1} . More precisely, for a highest weight element $b_n \otimes \cdots \otimes b_1 \in \mathcal{C}_{\square}^{\otimes n}$, the corresponding oscillating tableau satisfies $\mu^q = \sum_{i=1}^q \text{wt}(b_i)$. This map can be reversed and it is not hard to see that the result is a highest weight element using the tensor product rule.

2.1.3.2. *r -fans of Dyck paths*. Next we relate highest weight elements of weight zero in $\mathcal{B}_{\text{spin}}^{\otimes n}$ of type B_r and r -fans of Dyck paths. A *Dyck path* of length n is a path from $(0, 0)$ to $(n, 0)$ consisting of up-steps $(1, 1)$ and down-steps $(1, -1)$ which never crosses the line $y = 0$.

DEFINITION 2.1.16. An r -fan of Dyck paths F of length n is a sequence

$$F = (\emptyset = \mu^0, \mu^1, \dots, \mu^n = \emptyset)$$

of partitions μ^i with at most r parts such that the Ferrers diagram of two consecutive partitions differs by exactly one cell in each part. In other words, μ^i differs from μ^{i+1} by $(\pm 1, \pm 1, \dots, \pm 1)$ for $0 \leq i < n$.

EXAMPLE 2.1.17. For $r = 3$ and $n = 4$, the following is a 3-fan of Dyck paths

$$F = ((000), (111), (220), (111), (000)).$$

Since $\mathcal{B}_{\text{spin}}$ of type B_r is minuscule, by the above discussion $\epsilon = \epsilon_n \otimes \dots \otimes \epsilon_1 \in \mathcal{B}_{\text{spin}}^{\otimes n}$ is highest weight if and only if $\sum_{i=1}^q \text{wt}(\epsilon_i)$ is dominant for all $1 \leq q \leq n$. Hence highest weight elements of weight zero can be identified with an r -fan of Dyck paths of length n : the j -th entry of ϵ_i is $+$ if and only if the j -th Dyck path has an up-step at position i . In particular, for a highest weight element ϵ of weight zero, the sequence of dominant weights $\mu^q := \sum_{i=1}^q 2\text{wt}(\epsilon_i)$ for $0 \leq q \leq n$ defines an r -fan of Dyck paths consistent with Definition 2.1.16.

A similar bijection was given in [41].

EXAMPLE 2.1.18. The 3-fan of Dyck paths of Example 2.1.17 corresponds to the element

$$\epsilon = (-, -, -) \otimes (-, -, +) \otimes (+, +, -) \otimes (+, +, +) \in \mathcal{B}_{\text{spin}}^{\otimes 4}.$$

Following Definition 2.1.7, we obtain an embedding from the set of r -fans of Dyck paths into the set of oscillating tableaux.

DEFINITION 2.1.19. For an r -fan of Dyck paths $F = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^n = \emptyset)$ we define the oscillating tableau $\iota_{F \rightarrow O}(F) = (\emptyset = \mu^0, \dots, \mu^n = \emptyset)$ as follows. Let $v^t = \Psi(\lambda^t - \lambda^{t-1})$ for $1 \leq t \leq n$ with Ψ as in Definition 2.1.7. Then

$$\mu^{tr+s} = \lambda^t + \sum_{i=1}^s \text{wt}(v_i^{t+1}) \quad \text{for } 0 \leq t < n, 0 \leq s < r.$$

2.1.3.3. *Vacillating tableaux.* Next we define *vacillating tableaux* which correspond to highest weight elements in $\mathcal{B}_{\square}^{\otimes n}$ of type B_r .

DEFINITION 2.1.20. A $(2r + 1)$ -orthogonal vacillating tableau of length n is a sequence of partitions $\mathbf{V} = (\emptyset = \lambda^0, \dots, \lambda^n)$ such that:

- (1) λ^i has at most r parts.
- (2) Two consecutive partitions either differ by a box or are equal.
- (3) If two consecutive partitions are equal, then all their r parts are greater than 0.

We call λ^n the **weight** of $\mathbf{V}$.

A highest weight element $u = u_n \otimes \dots \otimes u_1 \in \mathcal{B}_{\square}^{\otimes n}$ of type B_r corresponds to the $(2r + 1)$ -vacillating tableau $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^n)$, where $\lambda^q = \sum_{i=1}^q \text{wt}(u_i)$.

Note that $\mathcal{B}_{\square}$ is not minuscule. The crystal $\mathcal{B}_{\square}$ is isomorphic to the component with highest weight element $(+, -, \dots, -) \otimes (+, \dots, +)$ in $\mathcal{B}_{\text{spin}} \otimes \mathcal{B}_{\text{spin}}$, see Figure 2.4. From this we obtain a map from the set of vacillating tableaux of weight zero and length n into the set of fans of Dyck paths of length $2n$ that we now explain. Denote by $\mathbf{1}$ the vector $\mathbf{e}_1 + \mathbf{e}_2 + \dots + \mathbf{e}_r$ and write $\rho < \nu$ if $\nu = \rho + \mathbf{e}_i$ for some i .

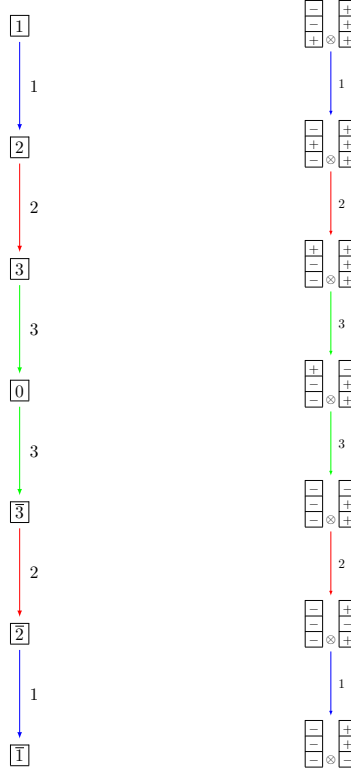


FIGURE 2.4. Left: $\mathcal{B}_{\square}$ of type B_3 , Right: The component in $\mathcal{B}_{\text{spin}} \otimes \mathcal{B}_{\text{spin}}$ of type B_3 isomorphic to $\mathcal{B}_{\square}$.

DEFINITION 2.1.21. For a vacillating tableau of weight zero $V = (\emptyset = \lambda^0, \dots, \lambda^n = \emptyset)$ we define the fan of Dyck paths $\iota_{V \rightarrow F}(V) = (\emptyset = \mu^0, \dots, \mu^{2n} = \emptyset)$ as follows:

$$\mu^{2i} = 2 \cdot \lambda^i$$

$$\mu^{2i-1} = \begin{cases} 2 \cdot \lambda^{i-1} + \mathbf{1} & \text{if } \lambda^{i-1} < \lambda^i, \\ 2 \cdot \lambda^i + \mathbf{1} & \text{if } \lambda^{i-1} > \lambda^i, \\ 2 \cdot \lambda^{i-1} + \mathbf{1} - 2\mathbf{e}_r & \text{if } \lambda^{i-1} = \lambda^i. \end{cases}$$

Similarly, following Definition 2.1.12, we obtain an embedding from the set of vacillating tableaux of weight zero into the set of oscillating tableaux.

DEFINITION 2.1.22. For a vacillating tableau of weight zero $V = (\emptyset = \lambda^0, \dots, \lambda^n = \emptyset)$ we define the oscillating tableau $\iota_{V \rightarrow O}(V) = (\emptyset = \mu^0, \dots, \mu^{2n} = \emptyset)$ as follows:

$$\mu^{2i} = 2 \cdot \lambda^i$$

$$\mu^{2i-1} = \lambda^{i-1} + \lambda^i + \begin{cases} 0 & \text{if } \lambda^{i-1} \neq \lambda^i, \\ -\mathbf{e}_r & \text{if } \lambda^{i-1} = \lambda^i. \end{cases}$$

2.1.4. Promotion.

2.1.4.1. *Promotion via crystal commutator.* For finite crystals B_λ of classical type of highest weight λ , Henriques and Kamnitzer [23] introduced the crystal commutator as follows. Let $\eta_{B_\lambda}: B_\lambda \rightarrow B_\lambda$ be the Lusztig involution, which maps the highest weight vector to the lowest weight vector and interchanges the crystal operators f_i with $e_{i'}$, where $w_0(\alpha_i) = -\alpha_{i'}$ under the longest element w_0 . This can be extended to tensor products of such crystals by mapping each connected component to itself using the above. Then the *crystal commutator* is defined as

$$\sigma: B_\lambda \otimes B_\mu \rightarrow B_\mu \otimes B_\lambda$$

$$b \otimes c \mapsto \eta_{B_\mu \otimes B_\lambda}(\eta_{B_\mu}(c) \otimes \eta_{B_\lambda}(b)).$$

If we want to emphasize the crystals involved, we write $\sigma_{A,B}: A \otimes B \rightarrow B \otimes A$.

Following [16, 73, 74], we define the promotion operator using the crystal commutator.

DEFINITION 2.1.23. Let C be a crystal and $u \in C^{\otimes n}$ a highest weight element. Then [promotion](#) pr on u is defined as $\sigma_{C^{\otimes n-1}, C}(u)$.

REMARK 2.1.24. Note that inverse promotion is given by $\sigma_{C, C^{\otimes n-1}}(u)$. The conventions in the literature about what is called promotion and what is called inverse promotion are not always consistent. Our convention here agrees with the definition of promotion on posets that removes the letters 1 and slides letters (see for example [\[5, 61\]](#)). The convention here is the opposite of the convention on tableaux which removes the largest letter and uses jeu de taquin slides (see for example [\[6, 53\]](#)).

EXAMPLE 2.1.25. Consider the crystal $C = B_{\square}$ of type A_2 (see [\[10\]](#)). Then

$$u = 1 \otimes 3 \otimes 2 \otimes 2 \otimes 1 \otimes 1 \in C^{\otimes 6}$$

is highest weight and

$$\sigma_{C^{\otimes 5}, C}(u) = 2 \otimes 1 \otimes 3 \otimes 1 \otimes 2 \otimes 1.$$

The recording tableaux for the RSK insertion of the words 132211 and 213121 (from right to left) are

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & \\ \hline 5 & & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & \\ \hline 4 & & \\ \hline \end{array}$$

which are related by the usual (inverse) promotion operator (removing the letter 1, doing jeu-de-taquin slides, filling the empty cell with the largest letter plus one and subtracting 1 from all entries) on standard tableaux.

EXAMPLE 2.1.26. Promotion on the element ϵ in Example [2.1.18](#) is

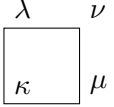
$$\sigma_{\mathcal{B}_{\text{spin}}^{\otimes 3}, \mathcal{B}_{\text{spin}}}(\epsilon) = (-, -, -) \otimes (-, +, +) \otimes (+, -, -) \otimes (+, +, +).$$

Note that if $\Psi: C \rightarrow \mathcal{V} \subseteq \widehat{\mathcal{V}}$ is a virtual embedding, then virtualization intertwines with promotion

$$(2.1) \quad \Psi \circ \sigma_{C^{\otimes n-1}, C} = \sigma_{\widehat{\mathcal{V}}^{\otimes n-1}, \widehat{\mathcal{V}}} \circ \Psi$$

by Axioms V2 and V3 in Definition 2.1.3 as long as the folding σ and the multiplication factors γ_i respect the map $w_0(\alpha_i) = -\alpha_{i'}$. This is the case for the virtualizations in this paper.

2.1.4.2. *Promotion via local rules.* Adapting local rules of van Leeuwen [68], Lenart [34] gave a combinatorial realization of the crystal commutor $\sigma_{A,B}$ by constructing an equivalent bijection between the highest weight elements of $A \otimes B$ and $B \otimes A$ respectively. The *local rules* of Lenart [34]

can be stated as follows: four weight vectors $\lambda, \mu, \kappa, \nu \in \Lambda$ depicted in a square diagram  satisfy the local rule, if $\mu = \text{dom}_W(\kappa + \nu - \lambda)$, where W is the Weyl group of the root system Φ underlying A and B . Furthermore, $\text{dom}_W(\rho)$ is the dominant weight in the Weyl orbit of ρ .

THEOREM 2.1.27 ([34, Theorem 4.4]). *Let A and B be crystals embedded into tensor products $A_\ell \otimes \cdots \otimes A_1$ and $B_k \otimes \cdots \otimes B_1$ of crystals of minuscule representations, respectively. Let $w = w_{k+\ell} \otimes \cdots \otimes w_1$ be a highest weight element in $A \otimes B$ with corresponding tableau ($\emptyset = \mu^0, \mu^1, \dots, \mu^{k+\ell} = \mu$) Then $\sigma_{A,B}(w)$ can be computed as follows. Create a $k \times \ell$ grid of squares as in (2.2), labelling the edges along the left border with $w_1, \dots, w_k$ and the edges along the top border with $w_{k+1}, \dots, w_{k+\ell}$:*

$$(2.2) \quad \begin{array}{ccccccc} & & \mu^k & \xrightarrow{w_{k+1}} & \cdots & \xrightarrow{w_{k+\ell}} & \mu^{k+\ell} \\ & \uparrow w_k & & \uparrow & & \uparrow & \uparrow \hat{w}_{k+\ell} \\ & \mu^{k-1} & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & & \\ & \vdots & & & & & \\ & \mu^1 & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & & \\ w_1 & \uparrow & & \uparrow & & \uparrow & \uparrow \hat{w}_{1+\ell} \\ \mu^0 & \xrightarrow{\hat{w}_1} & \hat{\mu}^1 & \cdots & \xrightarrow{\hat{w}_\ell} & \hat{\mu}^\ell \end{array}$$

For each square use the local rule to compute the weight vectors on the square's corners. Given a horizontal edge from κ to μ in the j th column, label the edge by the element in A_j with weight $\mu - \kappa$. Similarly, given a vertical edge from μ to ν in the i th row, label the edge by the element in B_i with weight $\nu - \mu$. The labels $\hat{w}_{k+\ell} \dots \hat{w}_1$ of the edges along the right and the bottom border of the grid then form $\sigma_{A,B}(w)$ with corresponding tableau ($\emptyset = \mu^0, \hat{\mu}^1, \dots, \hat{\mu}^{k+\ell-1}, \mu^{k+\ell} = \mu$).

EXAMPLE 2.1.28. *Performing Lenart's local rules on the elements in Example 2.1.25 gives*

$$\begin{array}{cccccccc}
(1, 0, 0) & \xrightarrow{1} & (2, 0, 0) & \xrightarrow{2} & (2, 1, 0) & \xrightarrow{2} & (2, 2, 0) & \xrightarrow{3} & (2, 2, 1) & \xrightarrow{1} & (3, 2, 1) \\
1 \uparrow & & 1 \uparrow & & 1 \uparrow & & 2 \uparrow & & 2 \uparrow & & 2 \uparrow \\
(0, 0, 0) & \xrightarrow{1} & (1, 0, 0) & \xrightarrow{2} & (1, 1, 0) & \xrightarrow{1} & (2, 1, 0) & \xrightarrow{3} & (2, 1, 1) & \xrightarrow{1} & (3, 1, 1)
\end{array}$$

which recovers $\sigma_{C^{\otimes 5}, C}(1 \otimes 3 \otimes 2 \otimes 2 \otimes 1 \otimes 1) = 2 \otimes 1 \otimes 3 \otimes 1 \otimes 2 \otimes 1$.

2.2. Chord Diagrams

In this section, we describe an injection from the set of r -fans of Dyck paths (resp. vacillating tableaux) of length n into the set of chord diagrams on $[n]$ that intertwines promotion and rotation. We present this injection in two ways. In Section 2.2.1, we construct an adjacency matrix from the fillings of a promotion scheme applied to a highest weight word of weight zero, and show in Proposition 2.2.13 that this map intertwines promotion and rotation. In Section 2.2.2, we describe the invertible algorithms which construct Fomin growth diagrams corresponding to highest weight words of weight zero. In the next section, we prove that these two constructions coincide.

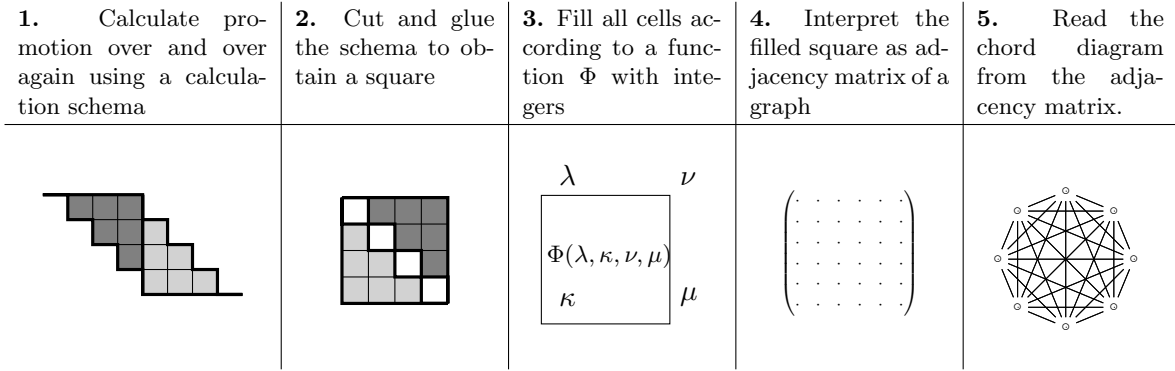


FIGURE 2.5. Overview of the steps in our map

2.2.1. Promotion matrices. In this section we describe a map from highest weight words of weight zero to chord diagrams that intertwines promotion and rotation.

We start with the definition of chord diagrams and their rotation.

DEFINITION 2.2.1. A **chord diagram** of size n is a graph with n vertices depicted on a circle which are labelled $1, \dots, n$ in counter-clockwise orientation.

The **rotation** of a chord diagram is obtained by rotating all edges clockwise by $\frac{2\pi}{n}$ around the center of the diagram.

In our setting all chord diagrams are undirected graphs with possibly multiple edges between the same two vertices. We can therefore identify chord diagrams with their **adjacency matrix**. The adjacency matrix is a symmetric $n \times n$ matrix $M = (m_{ij})_{1 \leq i, j \leq n}$ with nonnegative integer entries and m_{ij} denotes the number of edges between vertex i and vertex j .

PROPOSITION 2.2.2. *Let M be the adjacency matrix of a chord diagram G . Denote by $\text{rot } M$ the toroidal shift of M , that is, the matrix obtained from M by first cutting the top row and pasting it to the bottom and then cutting the leftmost column and pasting it to the right.*

Then $\text{rot } M$ is the adjacency matrix corresponding to the rotation of G .

Let us now outline the idea to construct such a rotation and promotion intertwining map and then provide the details on the individual steps on the examples of oscillating tableaux, r -fans of Dyck paths and vacillating tableaux. A visual guideline can be seen in Figure 2.5.

CONSTRUCTION 2.2.3. *The construction is given as follows:*

- Step 1:** *Iteratively calculate promotion of a highest weight word of weight zero and length n using Lenart's schema (2.2) a total of n times.*
- Step 2:** *Group the results into a square grid, called the **promotion matrix**.*
- Step 3:** *Fill the cells of the square grid with certain nonnegative integers according to a filling rule Φ that only depends on the four corners of the cells in the schema (2.2).*
- Step 4:** *Regard the filling as the adjacency matrix of a graph, which is the chord diagram.*

We now discuss the filling rules in the various cases. Note that the filling rules are new even in the case of oscillating tableaux as the proofs in [49] did not follow this construction.

2.2.1.1. Chord diagrams for oscillating tableaux. Recall that the Weyl group of type C_r is the hyperoctahedral group r of signed permutations of $\{\pm 1, \pm 2, \dots, \pm r\}$. Weights are elements in $\mathbb{Z}^r$ and dominant weights are weakly decreasing integer vectors with nonnegative entries (or equivalently partitions). Thus, the dominant representative $\text{dom}_r(\lambda)$ of a weight λ is obtained by sorting the absolute values of its entries into weakly decreasing order.

We slightly modify Lenart's schema for the crystal commutor (2.2) by omitting edge labels as only the weights on the corners are needed. Additionally, given an oscillating tableau $\mathbf{O} = (\emptyset = \mu^0, \mu^1, \dots, \mu^n = \mu)$, we start each row with the zero weight $\emptyset$ and end each row with the weight μ , which makes it easier to iteratively use this schema to calculate promotion. This way the promotion of the oscillating tableau $\mathbf{O} = (\emptyset = \mu^0, \mu^1, \dots, \mu^n = \mu)$ is the unique sequence $(\emptyset = \hat{\mu}^0, \hat{\mu}^1, \dots, \hat{\mu}^n = \mu)$, such that all squares in the diagram



satisfy the local rule of Section 2.1.4.

Using this schema we iteratively calculate promotion a total of n times and depict the results in a diagram as seen in Figure 2.6 on the left. This diagram consists of n promotion schemas glued together. As $\mathbf{pr}^n = \text{id}$, the labels on the top and the bottom row must be equal to $\mu^0, \dots, \mu^n$.

We now transform this diagram by copying everything to the right of the n -th column into the triangular empty space on the left, see Figure 2.6. In this way the labels on the right corners of the n -th column are duplicated. We obtain an $n \times n$ grid, where each corner of a cell is labelled with a dominant weight and the labels on the top and bottom border are equal and the labels on the left and right border are equal. This grid is called the *promotion matrix* of $\mathbf{O}$.

To obtain an adjacency matrix, we fill the cells of this diagram with nonnegative integers according to the following rule.

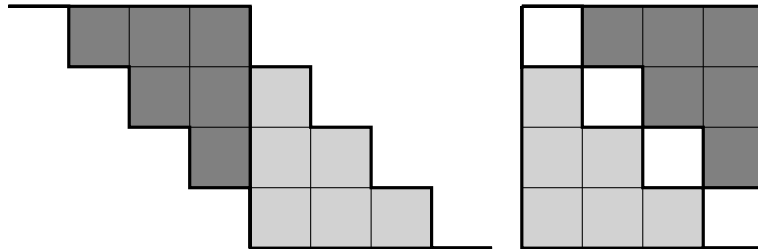


FIGURE 2.6. The transformation into a promotion matrix. The highlighted part is cut away and glued on the left.

DEFINITION 2.2.4. The [filling rule](#) for oscillating tableaux is

$$(2.3) \quad \Phi(\lambda, \kappa, \nu, \mu) = \begin{cases} 1 & \text{if } \kappa + \nu - \lambda \text{ contains a negative entry,} \\ 0 & \text{else,} \end{cases}$$

where the cells are labelled as depicted below:

$$(2.4) \quad \begin{array}{ccc} & \lambda & \nu \\ & \boxed{\Phi(\lambda, \kappa, \nu, \mu)} & \\ \kappa & & \mu \end{array} .$$

DEFINITION 2.2.5. Denote by M_O the function that maps an r -symplectic oscillating tableau of length n to an $n \times n$ adjacency matrix using Construction [2.2.3](#) and the filling rule [\(2.3\)](#).

Next, we generalize the above construction for r -fans of Dyck paths and vacillating tableaux.

2.2.1.2. *Chord diagrams for r -fans of Dyck paths.* Given an r -fan of Dyck paths $F = (\emptyset = \mu^0, \mu^1, \dots, \mu^n = \emptyset)$, we construct an adjacency matrix via Construction [2.2.3](#) using the following filling rule:

DEFINITION 2.2.6. The [filling rule](#) for fans of Dyck paths is

$$(2.5) \quad \Phi(\lambda, \kappa, \nu, \mu) = \text{number of negative entries in } \kappa + \nu - \lambda,$$

where the cells are labelled as in [\(2.4\)](#).

REMARK 2.2.7. Note that for oscillating tableaux at most one negative entry can occur. Thus the filling rule [\(2.5\)](#) for fans of Dyck paths is a natural generalization of the rule [\(2.3\)](#).

DEFINITION 2.2.8. Denote by M_F the function that maps an r -fan of Dyck paths of length n to an $n \times n$ adjacency matrix using Construction [2.2.3](#) and the filling rule [\(2.5\)](#).

EXAMPLE 2.2.9. Consider the following fan corresponding to the sequence of vectors

$$F = (000, 111, 222, 311, 422, 331, 222, 111, 000).$$

(1) We apply promotion a total of $n = 8$ times, to obtain the full orbit.

```

000 111 222 311 422 331 222 111 000
  000 111 200 311 220 111 000 111 000
    000 111 222 311 220 111 222 111 000
      000 111 200 111 200 311 200 111 000
        000 111 220 311 422 311 222 111 000
          000 111 220 331 220 311 200 111 000
            000 111 222 111 220 111 220 111 000
              000 111 000 111 200 311 220 111 000
                000 111 222 311 422 331 222 111 000 .

```

(2) We group the results into the promotion matrix and fill the cells of the square grid according to Φ . For better readability we omitted zeros.

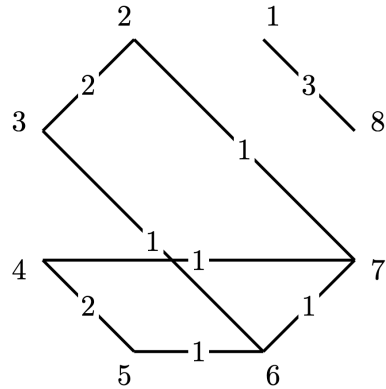
```

000 111 222 311 422 331 222 111 000
111 000 111 200 311 220 111 000 111
222 111 000 111 222 311 220 111 222
311 200 111 000 111 200 111 200 311
422 311 222 111 000 111 220 311 422
331 220 311 200 111 000 111 220 331
222 111 220 111 220 111 000 111 222
111 000 111 200 311 220 111 000 111
000 111 222 311 422 331 222 111 000

```

(3) Regard the filling as the adjacency matrix of a graph, the chord diagram.

$$M_F(F) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



2.2.1.3. *Chord diagrams for vacillating tableaux.* Note that $\mathcal{B}_\square$ is not minuscule and thus Theorem 2.1.27 is not directly applicable. Using Definition 2.1.12 we can embed $\mathcal{B}_\square$ in $\mathcal{C}_\square^{\otimes 2}$ which gives a map $\iota_{V \rightarrow O}$ from vacillating tableaux to oscillating tableaux of twice the length which commutes with the crystal commutor. That is

$$(2.6) \quad \iota_{V \rightarrow O} \circ \text{pr}_{\mathcal{B}_\square} = \iota_{V \rightarrow O} \circ \sigma_{\mathcal{B}_\square^{\otimes n-1}, \mathcal{B}_\square} = \sigma_{(\mathcal{C}_\square^{\otimes 2})^{\otimes n-1}, \mathcal{C}_\square^{\otimes 2}} \circ \iota_{V \rightarrow O}.$$

This follows directly from the properties of virtualization.

Let V be a vacillating tableau of length n and weight zero. Let $O = (\emptyset = \mu^0, \mu^1, \dots, \mu^{2n} = \emptyset)$ be the corresponding oscillating tableau using $\iota_{V \rightarrow O}$. Then we obtain the promotion of V using the following schema

$$(2.7) \quad \begin{array}{ccccccc} \mu^0 & \mu^1 & \mu^2 & \mu^3 & \dots & \mu^{2n-1} & \mu^{2n} \\ & & \boxed{\mu^1} & & & \boxed{\mu^{2n-1}} & \\ & & \boxed{\mu^0} & & & \boxed{\hat{\mu}^{2n-3}} & \hat{\mu}^{2n-2} \\ & & \hat{\mu}^1 & & & \hat{\mu}^{2n-1} & \hat{\mu}^{2n} \end{array}$$

Following Construction 2.2.3, we apply promotion a total of n times and use the cut-and-glue procedure to obtain a $2n \times 2n$ square. We fill the squares using the filling rule for oscillating tableaux as given by (2.3).

To obtain an $n \times n$ adjacency matrix, we subdivide the $2n \times 2n$ matrix into 2×2 blocks and take the sum of each block.

DEFINITION 2.2.10. Denote by $M_{V \rightarrow O}$ the function that maps a vacillating tableau V of weight zero of length n to an $n \times n$ adjacency matrix using $\iota_{V \rightarrow O}$, Schema (2.7), Construction 2.2.3, filling rule (2.3), and block sums.

EXAMPLE 2.2.11. Consider the vacillating tableau of length 9

$$V = (000, 100, 200, 210, 211, 111, 111, 110, 100, 000).$$

We first embed V into an oscillating tableau using the bijection Ψ from $\mathcal{B}_\square$ to $\mathcal{V}$ given in Definition 2.1.12. Specifically, we use Ψ to establish a correspondence between the highest weight element

$$\iota_{V \rightarrow O}(V) = (000, 100, 200, 300, 400, 410, 420, 421, 422, 322, 222, 221, 222, 221, 220, 210, 200, 100, 000).$$

(1) We apply promotion a total of $n = 9$ times on the above schema ($2n = 18$ times on the oscillating tableau $\iota_{V \rightarrow O}(V)$), to obtain the full orbit. Below the first iteration of promotion, we show all 9 applications of promotion.

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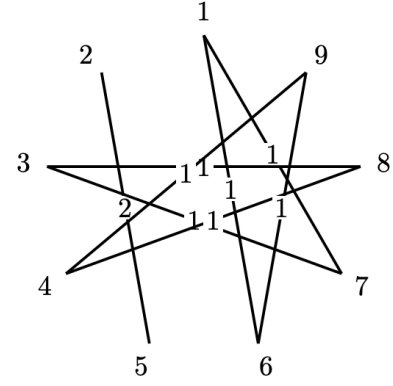
000 100 200 300 400 410 420 421 422 322 222 221 222 221 220 210 200 100 000
      100 200 300 310 320 321 322 222 221 220 221 220 221 220 221 210 200 100 000
      000 100 200 210 220 221 222 221 220 221 222 221 222 221 220 210 200 100 000
        100 110 210 211 221 220 221 222 322 321 322 321 320 310 300 200 100
          000 100 200 210 220 221 222 322 422 421 422 421 420 410 400 300 200 100 000
            100 110 210 211 221 321 421 420 421 420 421 411 410 310 210 110 100
              000 100 200 210 220 320 420 421 422 421 422 421 420 320 220 210 200 100 000
                100 110 210 310 410 411 421 420 421 420 421 321 221 211 210 110 100
                  000 100 200 300 400 410 420 421 422 421 422 322 222 221 220 210 200 100 000
                    100 200 300 310 320 321 322 321 322 222 221 220 221 211 210 110 100
                      000 100 200 210 220 221 222 221 220 221 222 322 321 320 310 300 200 100
                        100 110 210 211 221 220 221 220 221 222 322 422 421 420 410 400 300 200 100 000
                          000 100 200 210 220 221 222 221 222 322 422 421 420 421 411 410 310 210 110 100
                            000 100 200 210 220 221 222 322 422 421 422 421 420 421 420 320 220 210 200 100 000
                              100 110 210 211 221 321 421 420 421 420 421 321 221 211 210 110 100
                                000 100 200 210 220 320 420 421 422 421 422 322 222 221 220 210 200 100 000
                                  100 110 210 310 410 411 421 420 421 321 221 220 221 211 210 110 100
                                    000 100 200 300 400 410 420 421 422 322 222 221 222 221 220 210 200 100 000

```

(2) We group the results into the promotion matrix and fill the cells of the square grid according to Φ in (2.3). For better readability, we subdivided the diagram into 2×2 blocks and took the sum of the entries in each block, as well as omitted the zeros.

000	200	400	420	422	222	222	220	200	000
						1	1		
200	000	200	220	222	220	222	222	220	200
				2					
400	200	000	200	220	222	422	422	420	400
							1	1	
420	220	200	000	200	220	420	422	422	420
								1	1
422	222	220	200	000	200	400	420	422	422
		2							
222	220	222	220	200	000	200	220	222	222
	1								1
222	222	422	420	400	200	000	200	220	222
	1		1						
220	222	422	422	420	220	200	000	200	220
			1	1					
200	220	420	422	422	222	220	200	000	200
				1		1			
000	200	400	420	422	222	222	220	200	000

(3) Regard the filling as the adjacency matrix of a graph, the chord diagram.

$$M_{V \rightarrow O}(V) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$


Alternatively, we may obtain an adjacency matrix by embedding $\mathcal{B}_{\square}$ as a connected component of $\mathcal{B}_{\text{spin}}^{\otimes 2}$ (see Section 2.1.3.3). As discussed in Definition 2.1.21, this embedding gives rise to the map $\iota_{V \rightarrow F}$ from vacillating tableaux to r -fans of Dyck paths of twice the length. From the r -fans of Dyck paths, we apply M_F to obtain a $2n \times 2n$ matrix. Subdividing this matrix into 2×2 blocks and taking block sums produces an $n \times n$ adjacency matrix for vacillating tableaux.

DEFINITION 2.2.12. Denote by $M_{V \rightarrow F}$ the function that maps a vacillating tableau V of weight zero and length n to an $n \times n$ adjacency matrix using $\iota_{V \rightarrow F}$, Construction 2.2.3, filling rule (2.5), and block sums.

2.2.1.4. *Promotion and rotation.* For the various maps M_X with $X \in \{O, F, V \rightarrow O, V \rightarrow F\}$ constructed in this section, we obtain the following main result.

PROPOSITION 2.2.13. The map M_X for $X \in \{O, F, V \rightarrow O, V \rightarrow F\}$ intertwines promotion and rotation, that is

$$M_X \circ \text{pr} = \text{rot} \circ M_X.$$

PROOF. Let T be either a fan of Dyck paths, an oscillating tableau of weight zero or a vacillating tableau of weight zero of length n and denote by $\widehat{T}$ its promotion.

For $0 \leq i, j < n$ let $\mu^{i,j}$ be the $(j-i)$ -th entry of $\text{pr}^i(T)$, where indexing starts with zero and is understood modulo n . For $1 \leq i, j \leq n$ denote by $m_{i,j}$ the entry in the i -th row and j -th column of $M_X(T)$. Similarly, denote by $\widehat{\mu}^{i,j}$ the $(j-i)$ -th entry of $\text{pr}^i(\widehat{T})$ and by $\widehat{m}_{i,j}$ the i -th row and j -th column of $M_X(\widehat{T})$.

In all of our constructions $m_{i,j}$ depends on the four partitions $\mu^{i-1,j-1}$, $\mu^{i,j-1}$, $\mu^{i-1,j}$ and $\mu^{i,j}$ via some function

$$m_{i,j} = \tilde{\Phi}(\mu^{i-1,j-1}, \mu^{i,j-1}, \mu^{i-1,j}, \mu^{i,j}).$$

Analogously we have

$$\hat{m}_{i,j} = \tilde{\Phi}(\hat{\mu}^{i-1,j-1}, \hat{\mu}^{i,j-1}, \hat{\mu}^{i-1,j}, \hat{\mu}^{i,j}).$$

A simple calculation gives

$$\begin{aligned} \hat{m}_{i,j} &= \tilde{\Phi}(\hat{\mu}^{i-1,j-1}, \hat{\mu}^{i,j-1}, \hat{\mu}^{i-1,j}, \hat{\mu}^{i,j}) \\ &= \tilde{\Phi}(\mu^{i,j}, \mu^{i+1,j}, \mu^{i,j+1}, \mu^{i+1,j+1}) = m_{i+1,j+1}, \end{aligned}$$

where indices are understood modulo n . Thus, $M_X(\hat{T}) = \text{rot}(M_X(T))$. $\square$

Note that the promotion matrix $M_X(T)$ is sometimes referred to as the *promotion-evacuation diagram* of T as it also encodes information about the evacuation of T . Following [49], a generalization of Schützenberger’s evacuation operator can be defined on crystals as follows.

DEFINITION 2.2.14. *Let C be a crystal and $u \in C^{\otimes n}$ a highest weight element. Then **evacuation** evac on u is defined as*

$$(1_{C^{\otimes n-2}} \otimes \text{pr}) \circ \cdots \circ (1_C \otimes \text{pr}) \circ \text{pr}(u),$$

where $(1_{C^{\otimes n-m}} \otimes \text{pr})(w_n \otimes \cdots \otimes w_2 \otimes w_1) = w_n \otimes \cdots \otimes w_{m+1} \otimes \text{pr}(w_m \otimes \cdots \otimes w_1)$.

Given a tableau T corresponding to a highest weight element u , we denote by $\text{evac}(T)$ the tableau associated to the highest weight element $\text{evac}(u)$.

PROPOSITION 2.2.15. *The map M_X for $X \in \{O, F, V \rightarrow O, V \rightarrow F\}$ intertwines evacuation and the anti-transpose, that is*

$$M_X \circ \text{evac} = \text{antr} \circ M_X,$$

where the anti-transpose antr of a matrix is its transpose over its anti-diagonal.

PROOF. Let T be either a fan of Dyck paths, an oscillating tableau of weight zero, or a vacillating tableau of weight zero of length n . From the definition of evac and the construction of M_X , we have that $\text{evac}(T)$ is precisely the tableau obtained by reading the right border of M_X from bottom to top. Note that in order to prove the statement for $M_{V \rightarrow O}$ it suffices to show it for M_O as Ψ

intertwines $\sigma_{\mathcal{B}_{\square}^{\otimes m}, \mathcal{B}_{\square}}$ and $\sigma_{(\mathcal{C}_{\square}^{\otimes 2})^{\otimes m}, \mathcal{C}_{\square}^{\otimes 2}}$ for all $m \geq 1$ by Equation (2.1), where Ψ is the virtualization map given in Definition 2.1.12. Similarly, in order to prove the statement for $M_{V \rightarrow F}$ it suffices to prove it for M_F .

Consider partitions $\lambda, \kappa, \nu, \mu$ labelling the corner of a cell in M_X as in (2.4), where $X \in \{O, F\}$. By [68, Lemma 4.1.2], we have $\mu = \text{dom}_W(\kappa + \nu - \lambda)$ if and only if $\lambda = \text{dom}_W(\kappa + \nu - \mu)$ as $\mathcal{B}_{\text{spin}}$ and $\mathcal{C}_{\square}$ are minuscule. This implies that partitions labelling the corners of every cell in $M_X \circ \text{evac}$ and $\text{antr} \circ M_X$ are equal.

To complete the proof we show that filling rules $\Phi(\lambda, \kappa, \nu, \mu)$ given in (2.3) and (2.5) satisfy $\Phi(\lambda, \kappa, \nu, \mu) = \Phi(\mu, \kappa, \nu, \lambda)$. As partitions connected by a vertical or horizontal edge in M_O differ by exactly one box, we have that $\Phi(\lambda, \kappa, \nu, \mu) = 1$ if and only if $\lambda = \mu = (\lambda_1, \dots, \lambda_i, 0, \dots, 0)$, $\lambda_i = 1$ for some i , and $\kappa = \nu = (\lambda_1, \dots, \lambda_{i-1}, 0, 0, \dots, 0)$. Thus, the filling rule for oscillating tableaux satisfies $\Phi(\lambda, \kappa, \nu, \mu) = \Phi(\mu, \kappa, \nu, \lambda)$. By a similar argument the filling rule for fans of Dyck paths also satisfies the desired symmetry. $\square$

2.2.2. Fomin growth diagrams. Generally speaking, a *Fomin growth diagram* is a means to bijectively map sequences of partitions satisfying certain constraints to fillings of a Ferrers shape with nonnegative integers [15, 33, 54, 69]. In this setting, we draw the Ferrers shape in French notation (to fix how the growth diagrams are arranged).

To map a filling of a Ferrers shape to a sequence of partitions we iteratively label all corners of cells of the shape with partitions by certain local rules. Given a cell, where already all three partitions on the left and bottom corners are known, the forward rules determine the fourth partition on the top right corner based on the filling of the cell. Conversely, given the three partitions on the top and right corners of a cell, the backwards rules determine the last partition and the filling of the cell. When defining the local rules we label the cells as seen in Figure 2.7.

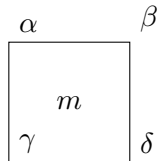


FIGURE 2.7. A cell of a growth diagram filled with a nonnegative integer m

For partitions δ and α , we define their union $\delta \cup \alpha$ to be the partition containing $\delta_i + \alpha_i$ cells in row i , where δ_i and α_i denote the number of cells in row i of δ and α respectively. Recall that we pad partitions with 0's if necessary. We denote $\delta \cup \delta$ by 2δ . We define the intersection of two partitions $\delta \cap \alpha$ to be the partition containing $\min\{\delta_i, \alpha_i\}$ cells in row i .

We begin by describing the local rules for a filling of a Ferrers shape with at most one 1 in each row and in each column and 0's everywhere else (omitted for readability). Moreover, we require that any two adjacent partitions in the labelling of our growth diagram (for example, $\gamma \rightarrow \alpha$ and $\gamma \rightarrow \delta$ in Figure 2.7) must either coincide or the one at the head of the arrow is obtained from the other by adding a unit vector. We record the local forward rules and local backward rules for this case of 0/1 filling, which are stated explicitly in [33, p. 4-5].

Given a 0/1 filling of a Ferrers shape and partitions labelling the bottom and left side of the Ferrers shape, we apply the following *local forward rules* to complete the labelling.

- (F1) If $\gamma = \delta = \alpha$, and there is no 1 in the cell, then $\beta = \gamma$.
- (F2) If $\gamma = \delta \neq \alpha$, then $\beta = \alpha$.
- (F3) If $\gamma = \alpha \neq \delta$, then $\beta = \delta$.
- (F4) If γ, δ, α are pairwise different, then $\beta = \delta \cup \alpha$.
- (F5) If $\gamma \neq \delta = \alpha$, then β is formed by adding a square to the $(k+1)$ -st row of $\delta = \alpha$, given that $\delta = \alpha$ and γ differ in the k -th row.
- (F6) If $\gamma = \delta = \alpha$, and if there is a 1 in the cell, then β is formed by adding a square to the first row of $\gamma = \delta = \alpha$.

Given a Ferrers shape and partitions labelling the top and right side, we apply the following *local backward rules* to complete the labelling and recover the filling.

- (B1) If $\beta = \delta = \alpha$, then $\gamma = \beta$.
- (B2) If $\beta = \delta \neq \alpha$, then $\gamma = \alpha$.
- (B3) If $\beta = \alpha \neq \delta$, then $\gamma = \delta$.
- (B4) If β, δ, α are pairwise different, then $\gamma = \delta \cap \alpha$.
- (B5) If $\beta \neq \delta = \alpha$, then γ is formed by deleting a square from the $(k-1)$ -st row of $\delta = \alpha$, given that $\delta = \alpha$ and β differ in the k -th row with $k \geq 2$.
- (B6) If $\beta \neq \delta = \alpha$, and if β and $\delta = \alpha$ differ in the first row, then $\gamma = \delta = \alpha$ and the cell is filled with a 1.

CONSTRUCTION 2.2.16 ([49]). Let $\mathbf{O} = (\emptyset = \mu^0, \mu^1, \dots, \mu^n = \emptyset)$ be an oscillating tableau. The associated triangular growth diagram is the Ferrers shape $(n-1, n-2, \dots, 2, 1, 0)$. Label the cells according to the following specification:

- (1) Label the north-east corners of the cells on the main diagonal from the top-left to the bottom-right with the partitions in $\mathbf{O}$.
- (2) For each $i \in \{0, \dots, n-1\}$ label the corner on the first subdiagonal adjacent to the labels μ^i and μ^{i+1} with the partition $\mu^i \cap \mu^{i+1}$.
- (3) Use the backwards rules B1-B6 to obtain all other labels and the fillings of the cells.

We denote by $G_{\mathbf{O}}(\mathbf{O})$ the symmetric $n \times n$ matrix one obtains from the filling of the growth diagram by putting zeros in the unfilled cells and along the diagonal and completing this to a symmetric matrix.

Starting from a filling of a growth diagram one obtains the oscillating tableau by setting all vectors on corners on the bottom and left border of the diagram to be the empty partition and applying the forwards growth rules F1-F6.

Next, we will extend these local rules to any filling of a Ferrers shape with nonnegative integers.

2.2.3. Fomin growth diagrams: Rule Burge. Given a filling of a Ferrers shape $(\lambda_1, \dots, \lambda_\ell)$ with nonnegative integers, we produce a “blow up” construction of the original shape for the Burge variant which contains south-east chains of 1’s, as done by [33]. We begin by separating entries. If a cell is filled with a positive entry m , we replace the cell with an $m \times m$ grid of cells with 1’s along the diagonal (from top-left to bottom-right). If there exist several nonzero entries in one column, we arrange the grids of cells also from top-left to bottom-right, so that the 1’s form a south-east chain in each column. We make the same arrangements for the rows, also establishing a south-east chain in each row. The resulting blow up Ferrers diagram then contains c_j columns in the original j -th column, where c_j is equal to the sum of the entries in column j or 1 if the j -th column contains only 0’s, and r_i rows in the original i -th row, where r_i is equal to the sum of the entries in row i or 1 if the i -th row contains only 0’s. See Figure 2.8.

Since the filling of the blow up growth diagram consists of 1’s and 0’s, we now apply the forward local rules. To start, we label all of the corners of the cells on the left side and the bottom side of the blow up growth diagram by $\emptyset$. Then we apply the forward local rules to determine

the partition labels of the other corners, using the 0/1 filling and partitions defined in previous iterations of the forward local rule. Finally, we “shrink back” the labelled blow up growth diagram to obtain a labelling of the original Ferrers diagram by only considering the partitions labelling positions $\{(c_1 + \dots + c_j, r_i + \dots + r_\ell) \mid 1 \leq i \leq \ell, 1 \leq j \leq \lambda_{\ell-i+1}\}$. These positions are precisely the intersections of the bolded black lines in Figure 2.8. To shrink back, we ignore the labels on intersections involving any blue lines in the blow up growth diagram and assign the partition labelling $(c_1 + \dots + c_j, r_i + \dots + r_\ell)$ to the position $(j, \ell - i + 1)$ in the original Ferrers diagram. The resulting labelling has the property that partitions on adjacent corners differ by a vertical strip [33, Theorem 11].

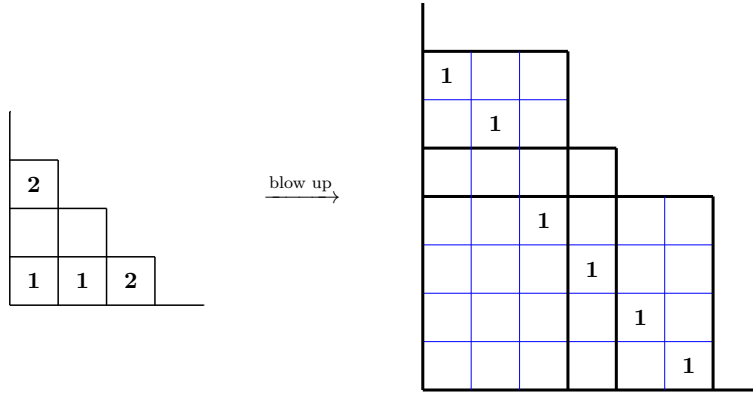


FIGURE 2.8. An example of the blow up construction for Burge rules replacing positive integer entries with south-east chains of 1’s in each column and row.

We now describe the direct Burge forward and backwards rules [33, Section 4.4]. Consider a cell filled by a nonnegative integer m , and labelled by the partitions γ, δ, α , where $\gamma \subset \delta$ and $\gamma \subset \alpha$, α/γ and δ/γ are vertical strips. Moreover, denote by 1_A the truth function

$$1_A = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

Then β is determined by the following procedure:

Burge F0: Set $\text{CARRY} := m$ and $i := 1$.

Burge F1: Set $\beta_i := \max\{\delta_i, \alpha_i\} + \min\{1_{\gamma_i=\delta_i=\alpha_i}, \text{CARRY}\}$

Burge F2: If $\beta_i = 0$, then stop and return $\beta = (\beta_1, \beta_2, \dots, \beta_{i-1})$. If not, then set $\text{CARRY} := \text{CARRY} - \min\{1_{\gamma_i=\delta_i=\alpha_i}, \text{CARRY}\} + \min\{\delta_i, \alpha_i\} - \gamma_i$ and $i := i + 1$ and go to F1.

Note that this algorithm is reversible. Given β, δ, α such that β/δ and β/α are vertical strips, the backwards algorithm is defined by the following rules:

Burge B0: Set $i := \max\{j \mid \beta_j \text{ is positive}\}$ and $\text{CARRY} := 0$.

Burge B1: Set $\gamma_i := \min\{\delta_i, \alpha_i\} - \min\{1_{\gamma_i=\alpha_i=\beta_i}, \text{CARRY}\}$.

Burge B2: Set $\text{CARRY} := \text{CARRY} - \min\{1_{\beta_i=\delta_i=\alpha_i}, \text{CARRY}\} + \beta_i - \max\{\delta_i, \alpha_i\}$ and $i := i - 1$.

If $i = 0$, then stop and return $\gamma = (\gamma_1, \gamma_2, \dots)$ and $m = \text{CARRY}$. If not, got to B1.

CONSTRUCTION 2.2.17. Let $\mathbf{F} = (\emptyset = \mu^0, \mu^1, \dots, \mu^n = \emptyset)$ be an r -fan of Dyck paths. The associated triangular growth diagram is the Ferrers shape $(n-1, n-2, \dots, 2, 1, 0)$. Label the cells according to the following specification:

- (1) Label the north-east corners of the cells on the main diagonal from the top-left to the bottom-right with the partitions in $\mathbf{F}$.
- (2) For each $i \in \{0, \dots, n-1\}$ label the corner on the first subdiagonal adjacent to the labels μ^i and μ^{i+1} with the partition $\mu^i \cap \mu^{i+1}$.
- (3) Use the backwards rules Burge B0, B1 and B2 to obtain all other labels and the fillings of the cells.

We denote by $\mathbf{G}_F(\mathbf{F})$ the symmetric $n \times n$ matrix one obtains from the filling of the growth diagram by putting zeros in the unfilled cells and along the diagonal and completing this to a symmetric matrix.

Starting from a filling of a growth diagram one obtains the r -fan by filling the cells of a growth diagram, setting all vectors on corners on the bottom and left border of the diagram to be the empty partition and applying the forwards growth rules Burge F0-F2.

An example is given in Figure 2.9.

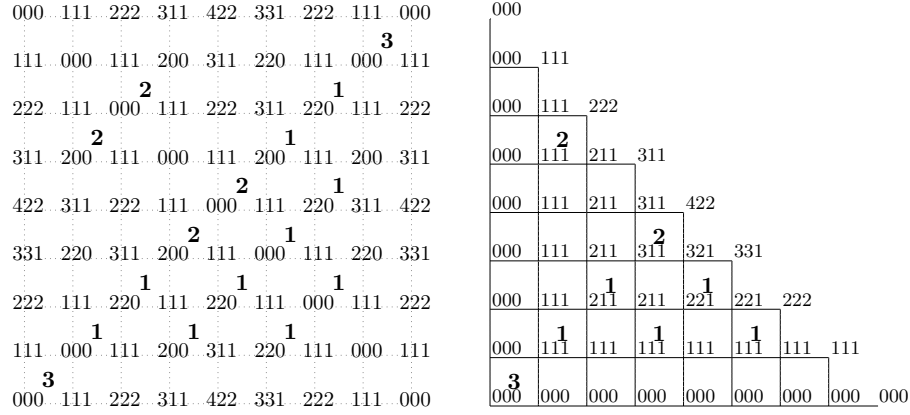


FIGURE 2.9. On the left the filled promotion matrix of $F = (000, 111, 222, 311, 422, 331, 222, 111, 000)$. On the right the triangular growth diagram for the same fan.

2.2.4. Fomin growth diagrams: Rule RSK. Given a filling of a Ferrers shape $(\lambda_1, \dots, \lambda_\ell)$ with nonnegative integers, we produce a “blow up” construction of the original shape for the RSK variant which contains north-east chains of 1’s, as done by [33]. We begin by separating entries. If a cell is filled with positive entry m , we replace the cell with an $m \times m$ grid of cells with 1’s along the off-diagonal (from bottom-left to top-right). If there exist several nonzero entries in one column, we arrange the grids of cells also from bottom-left to top-right, so that the 1’s form a north-east chain in each column. We make the same arrangements for the rows, also establishing a north-east chain in each row. The resulting blow up Ferrers diagram then contains c_j columns in the original j -th column, where c_j is equal to the sum of the entries in column j or 1 if the j -th column contains only 0’s, and r_i rows in the original i -th row, where r_i is equal to the sum of the entries in row i or 1 if the i -th row contains only 0’s.

Since the filling of the blow up growth diagram consists of 1’s and 0’s, we now apply the forward local rules. To start, we label all of the corners of the cells on the left side and the bottom side of the blow up growth diagram by $\emptyset$. Then, we apply the forward local rules to determine the partition labels of the other corners, using the 0/1 filling and partitions defined in previous iterations of the forward local rule. Finally, we “shrink back” the labelled blow up growth diagram to obtain a labelling of the original Ferrers diagram by only partitions labelling positions $\{(c_1 + \dots + c_j, r_i + \dots + r_\ell) \mid 1 \leq i \leq \ell, 1 \leq j \leq \lambda_{\ell-i+1}\}$. To shrink back, we assign the partition labelling $(c_1 + \dots + c_j, r_i + \dots + r_\ell)$ in the blow up growth diagram to the position $(j, \ell - i + 1)$ in

the original Ferrers diagram. The resulting labelling has the property that partitions on adjacent corners differ by a horizontal strip [33, Theorem 7].

The direct RSK forward rules are as follows [33, Section 4.1]: Consider a cell as in Figure 2.7 filled by a nonnegative integer m , and labelled by the partitions γ, δ, α , where $\gamma \subset \delta$ and $\gamma \subset \alpha$, α/γ and δ/γ are horizontal strips. Then β is determined by the following procedure:

RSK F0: Set $\text{CARRY} := m$ and $i := 1$.

RSK F1: Set $\beta_i := \max\{\delta_i, \alpha_i\} + \text{CARRY}$

RSK F2: If $\beta_i = 0$, then stop and return $\beta = (\beta_1, \beta_2, \dots, \beta_{i-1})$. If not, then set $\text{CARRY} := \min\{\delta_i, \alpha_i\} - \gamma_i$ and $i := i + 1$ and go to F1.

Note that this algorithm is reversible. Given β, δ, α such that β/δ and β/α are horizontal strips, the backwards algorithm is defined by the following rules:

RSK B0: Set $i := \max\{j \mid \beta_j \text{ is positive}\}$ and $\text{CARRY} := 0$.

RSK B1: Set $\gamma_i := \min\{\delta_i, \alpha_i\} - \text{CARRY}$.

RSK B2: Set $\text{CARRY} := \beta_i - \max\{\delta_i, \alpha_i\}$ and $i := i - 1$. If $i = 0$, then stop and return $\gamma = (\gamma_1, \gamma_2, \dots)$ and $m = \text{CARRY}$. If not, got to B1.

CONSTRUCTION 2.2.18. Let $V = (\emptyset = \mu^0, \mu^1, \dots, \mu^n = \emptyset)$ be a vacillating tableau of weight zero. The associated triangular growth diagram is the Ferrers shape $(n-1, n-2, \dots, 2, 1, 0)$. Label the cells according to the following specification:

- (1) Label the north-east corners of the cells on the main diagonal from the top-left to the bottom-right with the partitions $2\mu^i$.
- (2) For each $i \in \{0, \dots, n-1\}$ label the corner on the first subdiagonal adjacent to the labels $2\mu^i$ and $2\mu^{i+1}$ with the partition $2(\mu^i \cap \mu^{i+1})$ when $\mu^i \neq \mu^{i+1}$ and the partition obtained by removing a cell from the final row of $2\mu^i$ when $\mu^i = \mu^{i+1}$.
- (3) Use the backwards rules RSK B0, B1 and B2 to obtain all other labels and the fillings of the cells.

We denote by $G_V(V)$ the symmetric $n \times n$ matrix one obtains from the filling of the growth diagram by putting zeros in the unfilled cells and along the diagonal and completing this to a symmetric matrix.

Starting from a filling of a growth diagram one obtains the vacillating tableau by setting all vectors on corners on the bottom and left border of the diagram to be the empty partition and applying the forwards growth rules *RSK F0-F2*.

The triangular growth diagram of the vacillating tableau from Example 2.2.11 is depicted in Figure 2.10.

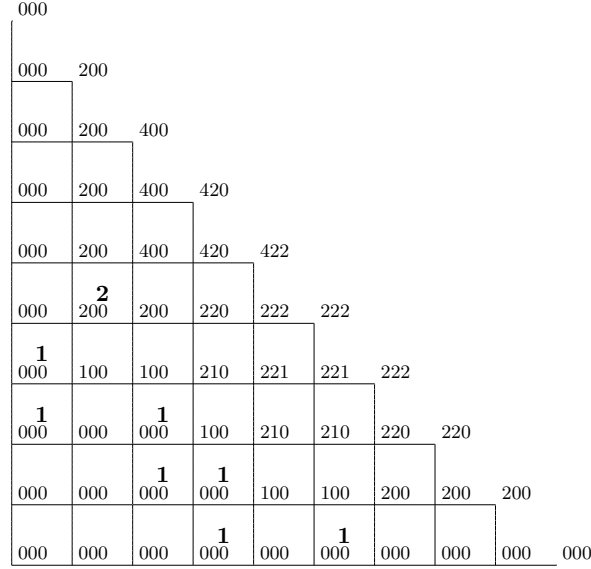


FIGURE 2.10. The triangular growth diagram for the vacillating tableau $V = (000, 100, 200, 210, 211, 111, 111, 110, 100, 000)$.

2.3. Main Results

In this section, we state and prove our main results for oscillating tableaux, fans of Dyck paths, and vacillating tableaux. In particular, we show in Theorems 2.3.4, 2.3.5 and 2.3.11 that the fillings of the growth diagrams coincide with the fillings of the promotion–evacuation diagrams. This in turn shows that the maps M_F , $M_{V \rightarrow O}$ and $M_{V \rightarrow F}$ are injective. Having these injective maps to chord diagrams gives a first step towards a diagrammatic basis for the invariant subspaces.

We begin by defining the following notation used later in this section. Let $M = (a_{i,j})_{i,j=1}^{kn}$ be a $kn \times kn$ matrix. It will often be convenient to consider M as the block matrix $(B_{i,j}^{(k)})_{i,j=1}^n$, where $B_{i,j}^{(k)}$ is the $k \times k$ matrix given by $(a_{p,q})_{p=k(i-1)+1, q=k(j-1)+1}^{ki, kj}$. We also follow the convention that for all $p, q > n$ we have $B_{p,q}^{(k)} := B_{i,j}^{(k)}$, where $p \equiv i \pmod n$ and $q \equiv j \pmod n$.

DEFINITION 2.3.1. For a $kn \times kn$ matrix M with block matrix decomposition given by $(B_{i,j}^{(k)})_{i,j=1}^n$, denote by $\text{blocksum}_k(M)$ the $n \times n$ matrix $(b_{i,j})_{i,j=1}^n$, where $b_{i,j}$ is equal to the sum of all entries in $B_{i,j}^{(k)}$.

Given an $n \times n$ matrix $M = (a_{i,j})_{i,j=1}^n$, we recursively define its skewed partial row sums $r_{i,j}$ by setting $r_{i,i} = 0$ for all $1 \leq i \leq n$ and letting $r_{i,j+1} = r_{i,j} + a_{i,j}$ for $1 \leq j \leq n-1$. Note that as before, we use the convention that $a_{p,q} = a_{i,j}$ whenever $p \equiv i \pmod n$ and $q \equiv j \pmod n$. Similarly, the skewed partial column sums $c_{i,j}$ can be defined. Partial inverses to blocksum_k are given by $\text{blowup}_k^{\text{SE}}$ and $\text{blowup}_k^{\text{NE}}$ which we presently define.

DEFINITION 2.3.2. Let $M = (a_{i,j})_{i,j=1}^n$ be a matrix with nonnegative integer entries such that for each row and for each column the sum of the entries is k . Let $r_{i,j}$ and $c_{i,j}$ be its skewed partial row and column sums respectively. Let $B_{i,j}^{\text{SE}}$ be the $k \times k$ matrix, where $B_{i,j}^{\text{SE}}$ is the zero-matrix if $a_{i,j} = 0$ and a zero-one-matrix if $a_{i,j} \neq 0$ consisting of 1's in positions $(r_{i,j} + 1, c_{i,j} + 1), \dots, (r_{i,j} + a_{i,j}, c_{i,j} + a_{i,j})$ and zeros elsewhere. We define $\text{blowup}^{\text{SE}}(M)$ to be the block matrix $(B_{i,j}^{\text{SE}})_{i,j=1}^n$.

Similarly, let $B_{i,j}^{\text{NE}}$ be the $k \times k$ matrix, where $B_{i,j}^{\text{NE}}$ is the zero-matrix if $a_{i,j} = 0$ and a zero-one-matrix if $a_{i,j} \neq 0$ consisting of 1's in positions $(k - r_{i,j}, k - c_{i,j} - a_{i,j} + 1), \dots, (k - r_{i,j} - (a_{i,j} - 1), k - c_{i,j})$ and zeros elsewhere. We define $\text{blowup}^{\text{NE}}(M)$ to be the block matrix $(B_{i,j}^{\text{NE}})_{i,j=1}^n$.

REMARK 2.3.3. Note that $\text{blowup}^{\text{SE}}(M)$ and $\text{blowup}^{\text{NE}}(M)$ are the unique $kn \times kn$ zero-one-matrices whose blocksum_k equals M and for all $1 \leq i \leq n$, the nonzero entries in the matrices

$$[B_{i,i}, B_{i,i+1}, B_{i,i+2}, \dots, B_{i,i+n-1}] \quad \text{and}$$

$$[B_{i,i}, B_{i+1,i}, B_{i+2,i}, \dots, B_{i+n-1,i}]$$

form a south-east chain or a north-east chain, respectively.

2.3.1. Results for oscillating tableaux. The next result was not stated explicitly in [49], but can be deduced from the proof in the paper.

THEOREM 2.3.4. For an oscillating tableau of weight zero O the fillings of the growth diagram (Construction 2.2.16) and the fillings of the promotion-evacuation (Construction 2.2.3) diagram coincide, that is

$$G_O(O) = M_O(O).$$

PROOF. This follows from the proof of [49, Corollary 6.17, Lemma 6.26]. $\square$

2.3.2. Results for r -fans of Dyck paths. We state our main results.

THEOREM 2.3.5. *For an r -fan of Dyck paths F*

$$G_F(F) = M_F(F).$$

In other words, the fillings of its growths diagram (Construction 2.2.17) and the fillings of the promotion-evacuation diagram coincide.

In particular we obtain the corollary:

COROLLARY 2.3.6. *The map M_F is injective.*

We now state and prove some results which are needed for the proof of Theorem 2.3.5.

LEMMA 2.3.7. *Let F be an r -fan of Dyck paths of length n . Then*

$$\iota_{F \rightarrow O} \circ \text{pr}_{\mathcal{B}_{\text{spin}}}^r(F) = \text{pr}_{\mathcal{C}_{\square}}^r \circ \iota_{F \rightarrow O}(F).$$

PROOF. Let $\iota_{F \rightarrow O}(F) = \mu = (\emptyset = \mu^{(0,0)}, \dots, \mu^{(0,rn)} = \emptyset)$. We first prove that $\text{pr}_{\mathcal{C}_{\square}}^r(\mu) = \text{pr}_{\mathcal{C}_{\square}^{\otimes r}}(\mu)$. Let $\text{pr}_{\mathcal{C}_{\square}}^i(\mu) = (\emptyset = \mu^{(i,0)}, \dots, \mu^{(i,rn)} = \emptyset)$. From the definition of $\iota_{F \rightarrow O}$, we have $\mu^{(0,k)} = (1^k)$ for all $0 \leq k \leq r$ where (1^0) denotes the empty partition $\emptyset$. Using the local rules for promotion and induction, we see that the sequence of partitions $(\mu^{(k,0)}, \dots, \mu^{(k,r-k)})$ is equal to $((1^0), \dots, (1^{r-k}))$ for all $0 \leq k \leq r$. This implies the following equality

$$\begin{aligned} \mu &= ((1^0), (1^1), \dots, (1^r), \mu^{(0,r+1)}, \dots, \mu^{(0,rn)}) \\ &= (\mu^{(r,0)}, \mu^{(r-1,1)}, \dots, \mu^{(0,r)}, \mu^{(0,r+1)}, \dots, \mu^{(0,rn)}). \end{aligned}$$

By a similar argument, the sequence of partitions $(\mu^{(k,rn-k)}, \dots, \mu^{(k,rn)})$ is equal to $((1^k), \dots, (1^0))$ for all $1 \leq k \leq r$ implying

$$\begin{aligned} \text{pr}_{\mathcal{C}_{\square}}^r(\mu) &= (\mu^{(r,0)}, \mu^{(r,1)}, \dots, \mu^{(r,r(n-1)-1)}, (1^r), (1^r - 1), \dots, (1^0)) \\ &= (\mu^{(r,0)}, \mu^{(r,1)}, \dots, \mu^{(r,rn-r-1)}, \mu^{(r,rn-r)}, \mu^{(r-1,rn-r-1)}, \dots, \mu^{(0,r)}). \end{aligned}$$

By Theorem 2.1.27, we obtain the desired equality

$$\begin{aligned} \text{pr}_{\mathcal{C}_{\square}^{\otimes r}}(\mu) &= \text{pr}_{\mathcal{C}_{\square}^{\otimes r}}(\mu^{(r,0)}, \mu^{(r-1,1)}, \dots, \mu^{(0,r)}, \mu^{(0,r+1)}, \dots, \mu^{(0,rn)}) \\ &= (\mu^{(r,0)}, \mu^{(r,1)}, \dots, \mu^{(r,r(n-1))}, \mu^{(r-1,r(n-1)+1)}, \dots, \mu^{(0,rn)}) = \text{pr}_{\mathcal{C}_{\square}^r}(\mu). \end{aligned}$$

Let $w = w_n \otimes w_{n-1} \otimes \dots \otimes w_1 \in \mathcal{B}_{\text{spin}}^{\otimes n}$ and $v = v_{rn} \otimes v_{rn-1} \otimes \dots \otimes v_1 \in (\mathcal{C}_{\square}^{\otimes r})^{\otimes n}$ be the highest weight crystal elements associated to $\mathbf{F}$ and μ , respectively. In order to show $\iota_{F \rightarrow O} \circ \text{pr}_{\mathcal{B}_{\text{spin}}}(\mathbf{F}) = \text{pr}_{\mathcal{C}_{\square}^{\otimes r}}(\mu)$, it suffices to show that $\Psi(\text{pr}_{\mathcal{B}_{\text{spin}}}(w)) = \text{pr}_{\mathcal{C}_{\square}^{\otimes r}}(v)$, where Ψ is the crystal isomorphism defined in Definition 2.1.7. Let $\mathcal{V} \subseteq \mathcal{C}_{\square}^{\otimes r}$ be the virtual crystal defined in Definition 2.1.4. As Ψ is a crystal isomorphism, we have $\Psi(\text{pr}_{\mathcal{B}_{\text{spin}}}(w)) = \text{pr}_{\mathcal{V}}(\Psi(w)) = \text{pr}_{\mathcal{V}}(v)$. As Lusztig's involution for crystals of type B_r and C_r interchanges the crystal operators f_i and e_i , the virtualization induced by the embedding $B_r \hookrightarrow C_r$ commutes with Lusztig's involution. In addition virtualization is preserved under tensor products (see for example [10, Theorem 5.8]). Thus, we have $\text{pr}_{\mathcal{V}}(v) = \text{pr}_{\mathcal{C}_{\square}^{\otimes r}}(v)$. $\square$

LEMMA 2.3.8. *Let $\mathbf{F}$ be an r -fan of Dyck paths with length n , and let $(B_{i,j}^{(r)})_{i,j=1}^n$ be the block matrix decomposition of the $rn \times rn$ adjacency matrix $\mathbf{M}_O(\iota_{F \rightarrow O}\mathbf{F})$. Then for all $1 \leq i \leq n$, the nonzero entries in the matrices*

$$\begin{aligned} &[B_{i,i+1}^{(r)}, B_{i,i+2}^{(r)}, \dots, B_{i,i+n-1}^{(r)}] \quad \text{and} \\ &[B_{i+1,i}^{(r)}, B_{i+2,i}^{(r)}, \dots, B_{i+n-1,i}^{(r)}] \end{aligned}$$

form a south-east chain of r 1's.

PROOF. By the definition of oscillating tableaux and the local rules for promotion, $\mathbf{M}_O$ is a zero-one matrix. From Lemma 2.3.7, Proposition 2.2.2, and Proposition 2.2.13, it suffices to prove that the nonzero entries in $[B_{n,n+1}^{(r)}, B_{n,n+2}^{(r)}, \dots, B_{n,2n-1}^{(r)}]$ and $[B_{2,1}^{(r)}, B_{3,1}^{(r)}, \dots, B_{n,1}^{(r)}]^T$ form a south-east chain. Recall that by construction, the Fomin growth diagram of $\iota_{F \rightarrow O}(\mathbf{F})$ is a triangle diagram with the entries of $\iota_{F \rightarrow O}(\mathbf{F})$ labelling its diagonal. As $\mathbf{F}$ is an r -fan of Dyck paths, the partition (1^r) sits at the corners $(r, r(n-1))$ and $(r(n-1), r)$ in the Fomin growth diagram of $\iota_{F \rightarrow O}(\mathbf{F})$. By Theorem 2.3.4, we have $\mathbf{M}_O(\iota_{F \rightarrow O}(\mathbf{F})) = \mathbf{G}_O(\iota_{F \rightarrow O}(\mathbf{F}))$. This implies that the filling of the leftmost r columns and bottommost r rows match $\mathbf{M}_O(\iota_{F \rightarrow O}(\mathbf{F}))$. As all the entries of $\mathbf{M}_O(\iota_{F \rightarrow O}(\mathbf{F}))$ are either 0 or 1, we have by [33, Theorem 2] that there are exactly r 1's forming a south-east chain in the leftmost r columns and in the bottommost r rows. $\square$

REMARK 2.3.9. The proof of Lemma 2.3.8 implies that the diagonal block matrices $B_{i,i}^{(r)}$ of $M_O(\iota_{F \rightarrow O} F)$ are all zero matrices.

PROPOSITION 2.3.10. Let F be an r -fan of Dyck paths of length n . Then

$$M_F(F) = \text{blocksum}_r(M_O(\iota_{F \rightarrow O}(F))).$$

Moreover,

$$\text{blowup}_r^{\text{SE}}(M_F(\mathcal{F})) = M_O(\iota_{F \rightarrow O}(\mathcal{F})).$$

PROOF. By Remark 2.3.9, the diagonal entries of $M_F(F)$ and $\text{blocksum}_r(M_O(\iota_{F \rightarrow O}(F)))$ are all zero. Let $a_{i,j}$ with $i \neq j$ be the entry in $M_F(F)$ that is the filling of the cell labelled by $\begin{array}{c} \lambda \quad \nu \\ \boxed{\kappa} \quad \mu \end{array}$ in the promotion matrix of F . To show that the number of 1's appearing in $B_{i,j}^{(r)}$ of $M_O(\iota_{F \rightarrow O}(F))$ is also equal to $a_{i,j}$, we first compute $a_{i,j}$ for $i \neq j$. By Definition 2.5, $a_{i,j}$ is the number of negative entries in $\kappa + \nu - \lambda$. Since λ, ν and κ, μ are consecutive partitions in an r -fan of Dyck paths, we know that they differ by a vector of the form $(\pm 1, \dots, \pm 1)$. We may write $\nu - \lambda$ and $\mu - \kappa$ as

$$\nu - \lambda = \mathbf{e}_{i_1} + \dots + \mathbf{e}_{i_k} - \mathbf{e}_{i_{k+1}} - \dots - \mathbf{e}_{i_r},$$

$$\mu - \kappa = \mathbf{e}_{j_1} + \dots + \mathbf{e}_{j_m} - \mathbf{e}_{j_{m+1}} - \dots - \mathbf{e}_{j_r},$$

where

$$\{i_1, \dots, i_r\} = [r] = \{j_1, \dots, j_r\},$$

$$i_1 < \dots < i_k \text{ and } i_{k+1} > \dots > i_r,$$

$$j_1 < \dots < j_m \text{ and } j_{m+1} > \dots > j_r.$$

By the definition of μ from the local rules of Lenart [34] (see Section 2.1.4), we have

$$\begin{aligned} \mu &= \text{dom}_r(\kappa + \nu - \lambda) \\ &= \text{dom}_r(\kappa + \mathbf{e}_{i_1} + \dots + \mathbf{e}_{i_k} - \mathbf{e}_{i_{k+1}} - \dots - \mathbf{e}_{i_r}). \end{aligned}$$

Recall that dom_r applied to a weight sorts the absolute values of the entries of the weight into weakly decreasing order. In particular, $\text{dom}_r(\kappa + \mathbf{e}_{i_1} + \dots + \mathbf{e}_{i_k} - \mathbf{e}_{i_{k+1}} - \dots - \mathbf{e}_{i_r})$ will change all of the -1 entries of $\kappa + \mathbf{e}_{i_1} + \dots + \mathbf{e}_{i_k} - \mathbf{e}_{i_{k+1}} - \dots - \mathbf{e}_{i_r}$ to $+1$ and then sort all entries into

weakly decreasing order (note that sorting will not change the number of cells). We thus have two equations for μ :

$$\begin{aligned}\mu &= \text{dom}_r(\kappa + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_k} - \mathbf{e}_{i_{k+1}} - \cdots - \mathbf{e}_{i_r}) \\ &= \kappa + \mathbf{e}_{j_1} + \cdots + \mathbf{e}_{j_m} - \mathbf{e}_{j_{m+1}} - \cdots - \mathbf{e}_{j_r}.\end{aligned}$$

Therefore, dom_r changed $m - k$ negative entries in $\kappa + \nu - \lambda$ to $+1$ in μ , showing that $a_{i,j} = m - k$.

From the virtualization given in Definition 2.1.7, the partitions labelling the top of the first row of cells in $B_{i,j}^{(r)}$ are $\lambda, \lambda^{(1)}, \dots, \lambda^{(r-1)}, \nu$, where $\lambda^{(\ell)} = \lambda + \mathbf{e}_{i_1} + \cdots \pm \mathbf{e}_{i_\ell}$. Similarly, the partitions labelling the bottom of the r -th row of cells in $B_{i,j}^{(r)}$ are $\kappa, \kappa^{(1)}, \dots, \kappa^{(r-1)}, \mu$, where $\kappa^{(\ell)} = \kappa + \mathbf{e}_{j_1} + \cdots \pm \mathbf{e}_{j_\ell}$. In particular, we have

$$\begin{aligned}\lambda &\subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k-1)} \subset \lambda^{(k)} \supset \lambda^{(k+1)} \supset \cdots \supset \lambda^{(r-1)} \supset \nu, \\ \kappa &\subset \kappa^{(1)} \subset \cdots \subset \kappa^{(m-1)} \subset \kappa^{(m)} \supset \kappa^{(m+1)} \supset \cdots \supset \kappa^{(r-1)} \supset \mu.\end{aligned}$$

Let $\begin{array}{cc} \lambda' & \nu' \\ \boxed{\kappa'} & \mu' \end{array}$ label a cell in the first row of $B_{i,j}^{(r)}$, and note that the pairs λ', ν' and κ', μ' differ by a unit vector since they are adjacent partitions in an oscillating tableau. It is impossible for

the inclusions $\begin{array}{cc} \lambda' \subset \nu' \\ \boxed{\kappa'} \supset \mu' \end{array}$ since $\lambda' \subset \nu'$ implies $\kappa' + \nu' - \lambda' = \kappa' + \mathbf{e}_i$ for some i , and by definition

$\mu' = \text{dom}_r(\kappa' + \mathbf{e}_i) = \kappa' + \mathbf{e}_i$ which contradicts $\mu' \subset \kappa'$. When $\begin{array}{cc} \lambda' \supset \nu' \\ \boxed{\kappa'} \subset \mu' \end{array}$ occurs, we know that $\kappa' + \nu' - \lambda' = \kappa' - \mathbf{e}_i$ for some i since $\nu' \subset \lambda'$. Since $\kappa' \subset \mu' = \text{dom}_r(\kappa' - \mathbf{e}_i)$, it must be that $\mu' = \kappa' + \mathbf{e}_i$ and therefore $\kappa' - \mathbf{e}_i$ contained a negative entry. Therefore, when $\lambda' \supset \nu'$ and $\kappa' \subset \mu'$

there is a 1 filling the cell. Conversely, when there is a 1 filling a cell labelled $\begin{array}{cc} \lambda' & \nu' \\ \boxed{\kappa'} & \mu' \end{array}$, then there is a negative in $\kappa' + \nu' - \lambda' = \kappa' \pm \mathbf{e}_i$ for some i , which is only possible when $\kappa' + \nu' - \lambda' = \kappa' - \mathbf{e}_i$. As a result, $\kappa' \subset \mu'$ and $\lambda' \supset \nu'$.

By Theorem 2.3.4, each row and each column in $\mathbf{M}_O(\iota_{F \rightarrow O}(\mathbf{F}))$ contains exactly one 1. Therefore there is at most one cell in the first row of $B_{i,j}^{(r)}$ where the containment between the top and bottom pairs of partitions is flipped. By the cases described above, containment between pairs of

partitions labelling the bottom of the first row of cells in $B_{i,j}^{(r)}$ either exactly matches the containment between pairs of partitions labelling the top of the first row or the switch in containment in the bottom occurs immediately to the right of the switch in containment in the top. The same outcome is observed recursively in the remaining rows of cells in $B_{i,j}^{(r)}$. Since we already knew the labels of the bottom of the r -th row to be increasing up to $\kappa^{(m)}$, we conclude that the number of 1's appearing in $B_{i,j}^{(r)}$ is equal to $m - k$, which we showed above is equal to $a_{i,j}$. Therefore, $M_F(F) = \text{blocksum}_r(M_O(\iota_{F \rightarrow O}(F)))$. Further, since the 1's in $M_O(\iota_{F \rightarrow O}(F))$ form a south-east chain, by Remark 2.3.3 we have $\text{blowup}_r^{\text{SE}}(M_F(F)) = M_O(\iota_{F \rightarrow O}(F))$. $\square$

We can now prove Theorem 2.3.5.

PROOF. Let $F = (\mu^0, \dots, \mu^n)$ be an r -fan of Dyck paths of length n . We have

$$\begin{aligned} M_F(F) &= \text{blocksum}_r(M_O(\iota_{F \rightarrow O}(F))) && \text{by Proposition 2.3.10} \\ &= \text{blocksum}_r(G_O(\iota_{F \rightarrow O}(F))) && \text{by Theorem 2.3.4.} \end{aligned}$$

It remains to show that $\text{blocksum}_r(G_O(\iota_{F \rightarrow O}(F))) = G_F(F)$.

The diagonal entries of $\text{blocksum}_r(G_O(\iota_{F \rightarrow O}(F)))$ and $G_F(F)$ are all zero by Remark 2.3.9 and by definition of G_F respectively. As G_O and G_F are symmetric matrices, it suffices to show that the lower triangular entries of $\text{blocksum}_r(G_O(\iota_{F \rightarrow O}(F)))$ and $G_F(F)$ agree. Let G denote the triangular growth diagram associated with $\iota_{F \rightarrow O}(F)$. By the definition of $\iota_{F \rightarrow O}$ and Construction 2.2.16, the coordinate $(kr, (n-k)r)$ is labelled with partition μ^k for $0 \leq k \leq n$. As G has a 0/1 filling, the local rules guarantee that the partition ν^k labelling the coordinate $(kr, (n-k-1)r)$ of G is contained within the partition $\mu^k \cap \mu^{k+1}$ for $0 \leq k \leq n-1$. Moreover, $|\mu^k/\nu^k| + |\mu^{k+1}/\nu^k|$ is equal to the total number of 1's lying in either a column from $kr+1$ to $(k+1)r$ or in a row from $(n-k-1)r+1$ to $(n-k)r$. From Lemma 2.3.8 and the fact that G_O is symmetric, there exist exactly r such 1's which implies $|\mu^k/\nu^k| + |\mu^{k+1}/\nu^k| = r$. Since μ^k and μ^{k+1} differ by exactly k boxes, $\nu^k = \mu^k \cap \mu^{k+1}$ for all $0 \leq k \leq n-1$.

Let H denote the triangular growth diagram with filling given by the lower triangular entries of $\text{blocksum}_r(G_O(\iota_{F \rightarrow O}(F)))$ and local rules given by the Burge rules. From Lemma 2.3.8,

$$\text{blowup}_r^{\text{SE}}(\text{blocksum}_r(G_O(\iota_{F \rightarrow O}(F)))) = G_O(\iota_{F \rightarrow O}(F)).$$

A result by Krattenthaler [33] implies that the labellings of the hypotenuse of H are given by $(\mu^0, \nu^0, \mu^1, \dots, \nu^{n-1}, \mu^n)$. As the Burge rules are injective and the growth diagram associated to $\mathbf{F}$ under Construction 2.2.17 has hypotenuse labelled by $(\mu^0, \mu^0 \cap \mu^1, \mu^1, \dots, \mu^{n-1} \cap \mu^n, \mu^n)$, the lower triangular entries of $\text{blocksum}_r(\mathbf{G}_O(\iota_{F \rightarrow O}(\mathbf{F})))$ and $\mathbf{G}_F(\mathbf{F})$ are equal. $\square$

2.3.3. Results for vacillating tableaux. We state our main results.

THEOREM 2.3.11. *For a vacillating tableau $\mathbf{V}$*

$$\mathbf{G}_V(\mathbf{V}) = \mathbf{M}_{V \rightarrow O}(\mathbf{V}) = \mathbf{M}_{V \rightarrow F}(\mathbf{V}).$$

In other words, the filling of the growth diagram (see Construction 2.2.18), the filling of the promotion matrix $\mathbf{M}_{V \rightarrow O}(\mathbf{V})$, and the filling of the promotion matrix $\mathbf{M}_{V \rightarrow F}(\mathbf{V})$ coincide.

In particular we obtain the corollary:

COROLLARY 2.3.12. *The maps $\mathbf{M}_{V \rightarrow O}$ and $\mathbf{M}_{V \rightarrow F}$ are injective.*

We will first prove the second equality in Theorem 2.3.11. To do so, we need the following lemma.

LEMMA 2.3.13. *We have the following:*

- (1) $\mathbf{M}_{V \rightarrow O} = \text{blocksum}_2 \circ \mathbf{M}_O \circ \iota_{V \rightarrow O}$.
- (2) Denote by $\mathbf{E}$ the $r \times r$ identity matrix, then

$$\mathbf{M}_{V \rightarrow F} + 2(r-1)\mathbf{E} = \text{blocksum}_2 \circ \mathbf{M}_F \circ \iota_{V \rightarrow F}.$$

PROOF. Let $\mathbf{V}$ be a vacillating tableau of length n and weight zero and let $X \in \{O, F\}$. Denote by $\mathbf{T} = (\emptyset = \mu^0, \mu^1, \dots, \mu^{2n} = \emptyset)$ the corresponding oscillating tableau (resp. r -fan of Dyck path) to $\mathbf{V}$ using $\iota_{V \rightarrow X}$.

Recall that $\mathbf{M}_{V \rightarrow X}$ is defined using the Schema (2.7) to calculate promotion. Let $\hat{\mu}^1, \dots, \hat{\mu}^{2n-1}$ be the partitions in the middle row in of this schema.

Note that we have $\mu^2 = \hat{\mu}^{2n-2} = 2\mathbf{e}_1$ and

$$\mu^1 = \hat{\mu}^1 = \hat{\mu}^{2n-1} = \hat{\mu}^{2n-1} = \begin{cases} \mathbf{e}_1 & \text{if } X = O, \\ \mathbf{1} & \text{if } X = F. \end{cases}$$

It is easy to see that the squares

$$\begin{array}{|c|c|} \hline \mu^1 & \mu^2 \\ \hline \emptyset & \hat{\mu}^1 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline \hat{\mu}^{2n-1} & \emptyset \\ \hline \hat{\mu}^{2n-2} & \hat{\mu}^{2n-1} \\ \hline \end{array}$$

satisfy the local rule and

$$\Phi(\mu^1, \emptyset, \mu^2, \hat{\mu}^1) = \Phi(\hat{\mu}^{2n-1}, \hat{\mu}^{2n-2}, \emptyset, \hat{\mu}^{2n-1}) = \begin{cases} 0 & \text{if } X = O, \\ r-1 & \text{if } X = F. \end{cases}$$

Thus we have

$$\text{pr}_X(\iota_{V \rightarrow X}(\mathbf{V})) = (\emptyset, \hat{\mu}_1, \dots, \hat{\mu}_{2n-1}, \emptyset)$$

and obtain $\mathbf{M}_{V \rightarrow X} + 1_{X=F} \cdot 2(r-1)E = \text{blocksum}_2 \circ \mathbf{M}_X \circ \iota_{V \rightarrow X}$. $\square$

The following relates the growth diagrams for $\iota_{V \rightarrow O}(\mathbf{V})$ and $\iota_{V \rightarrow F}(\mathbf{V})$.

LEMMA 2.3.14. *Denote by S the $2r \times 2r$ block diagonal matrix consisting of r copies of the block $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ along the diagonal and zeros everywhere else. Then*

$$\mathbf{G}_F \circ \iota_{V \rightarrow F} = \mathbf{G}_O \circ \iota_{V \rightarrow O} + (r-1)S.$$

PROOF. Let $\mathbf{V} = (\lambda^0, \dots, \lambda^n)$ be a vacillating tableau of weight zero. Denote with $\mathbf{O} = (\mu^0, \dots, \mu^{2n}) = \iota_{V \rightarrow O}(\mathbf{V})$ the corresponding oscillating tableaux and denote with $\mathbf{F} = (\nu^0, \dots, \nu^{2n}) = \iota_{V \rightarrow O}(F)$ the r -fan of Dyck paths.

Consider the portion of the growth diagram for the oscillating tableau involving only $(\mu^{2i-2}, \mu^{2i-1}, \mu^{2i})$ and the portion of the growth diagram for the fan of Dyck paths involving only $(\nu^{2i-2}, \nu^{2i-1}, \nu^{2i})$. We label the partitions as follows.

$$(2.8) \quad \begin{array}{ccc} \mu^{2i-2} & & \nu^{2i-2} \\ | & & | \\ \alpha & \mu^{2i-1} & \hat{\alpha} & \nu^{2i-1} \\ | & \text{---} & | & \text{---} \\ \begin{array}{|c|c|} \hline \gamma & m \\ \hline \end{array} & \delta & \begin{array}{|c|c|} \hline \hat{\gamma} & n \\ \hline \end{array} & \hat{\delta} & \nu^{2i} \end{array}$$

Claim: We have $\mu^{2i-2} = \nu^{2i-2}$, $\mu^{2i} = \nu^{2i}$, $\alpha = \hat{\alpha}$, $\gamma = \hat{\gamma}$, $\delta = \hat{\delta}$, $m = 0$ and $n = r - 1$. Moreover all partitions on consecutive corners on the lower left border of the diagrams in (2.8) differ by at most one cell.

We consider the three cases $\lambda^{i-1} = \lambda^i$, $\lambda^{i-1} \subset \lambda^i$ and $\lambda^{i-1} \supset \lambda^i$.

By Definition 2.1.21, Construction 2.2.16, Definition 2.1.22 and Construction 2.2.17 we have

$$\begin{aligned}\mu^{2i-2} &= \nu^{2i-2} = 2\lambda^{i-1}, & \mu^{2i} &= \nu^{2i} = 2\lambda^i, \\ \alpha &= \mu^{2i-2} \cap \mu^{2i-1}, & \delta &= \mu^{2i-1} \cap \mu^{2i}, \\ \hat{\alpha} &= \nu^{2i-2} \cap \nu^{2i-1}, & \hat{\delta} &= \nu^{2i-1} \cap \nu^{2i}.\end{aligned}$$

Case I. Assume $\lambda^{i-1} = \lambda^i$. In this case we have $\mu^{2i-1} = 2\lambda^i - \mathbf{e}_r$ and $\nu^{2i-1} = 2\lambda^i + \mathbf{1} - 2\mathbf{e}_r$ and get

$$\begin{aligned}\alpha &= \delta = (2\lambda^i) \cap (2\lambda^i - \mathbf{e}_r) = 2\lambda^i - \mathbf{e}_r, \\ \hat{\alpha} &= \hat{\delta} = (2\lambda^i) \cap (2\lambda^i + \mathbf{1} - 2\mathbf{e}_r) = 2\lambda^i - \mathbf{e}_r.\end{aligned}$$

Using the backwards rules for growth diagrams we obtain

$$\gamma = \hat{\gamma} = 2\lambda^i - \mathbf{e}_r, \quad m = 0 \quad \text{and} \quad n = r - 1.$$

Case II. Assume $\lambda^{i-1} \subset \lambda^i$. In this case we have $\mu^{2i-1} = \lambda^{i-1} + \lambda^i$ and $\nu^{2i-1} = 2\lambda^{i-1} + \mathbf{1}$. Furthermore we obtain

$$\begin{aligned}\alpha &= (2\lambda^{i-1}) \cap (\lambda^{i-1} + \lambda^i) = 2\lambda^{i-1}, \\ \hat{\alpha} &= (2\lambda^{i-1}) \cap (2\lambda^{i-1} + \mathbf{1}) = 2\lambda^{i-1}, \\ \delta &= (\lambda^{i-1} + \lambda^i) \cap (2\lambda^i) = \lambda^{i-1} + \lambda^i, \\ \hat{\delta} &= (2\lambda^{i-1} + \mathbf{1}) \cap (2\lambda^i) = \lambda^{i-1} + \lambda^i.\end{aligned}$$

Using the backwards rules for growth diagrams we obtain

$$\gamma = \hat{\gamma} = 2\lambda^{i-1}, \quad m = 0 \quad \text{and} \quad n = r - 1.$$

Case III. Assume $\lambda^{i-1} \supset \lambda^i$. This case is symmetric to Case II.

This proves the claim.

The rest of the growth diagrams must agree, as the Burge growth rules and Fomin growth rules agree in the case where labels on consecutive corners differ by at most one cell. $\square$

Note that Lemma 2.3.14 implies

$$(2.9) \quad \text{blocksum}_2 \circ G_F \circ \iota_{V \rightarrow F} = \text{blocksum}_2 \circ G_O \circ \iota_{V \rightarrow O} + 2(r-1)E.$$

Now we can prove the second identity of Theorem 2.3.11.

PROOF. We have

$$\begin{aligned} M_{V \rightarrow O} &= \text{blocksum}_2 \circ M_O \circ \iota_{V \rightarrow O} && \text{by Lemma 2.3.13 (i)} \\ &= \text{blocksum}_2 \circ G_O \circ \iota_{V \rightarrow O} && \text{by Theorem 2.3.4} \\ &= \text{blocksum}_2 \circ G_F \circ \iota_{V \rightarrow F} - 2(r-1)E && \text{by Equation (2.9)} \\ &= \text{blocksum}_2 \circ M_F \circ \iota_{V \rightarrow F} - 2(r-1)E && \text{by Theorem 2.3.5} \\ &= M_{V \rightarrow F} && \text{by Lemma 2.3.13 (ii).} \end{aligned}$$

$\square$

It is possible to invert Lemma 2.3.13 (i) as follows.

LEMMA 2.3.15. *Let V be a vacillating tableau of weight zero with length n , and let $(B_{i,j}^{(2)})_{i,j=1}^n$ be the block matrix decomposition of the $2n \times 2n$ adjacency matrix $M_O(\iota_{V \rightarrow O}V)$. Then for all $1 \leq i \leq n$, the nonzero entries in the matrices*

$$\begin{aligned} &[B_{i,i+1}^{(2)}, B_{i,i+2}^{(2)}, \dots, B_{i,i+n-1}^{(2)}] \quad \text{and} \\ &[B_{i+1,i}^{(2)}, B_{i+2,i}^{(2)}, \dots, B_{i+n-1,i}^{(2)}] \end{aligned}$$

form a north-east chain. In particular, we have

$$\text{blowup}_2^{\text{NE}} \circ M_{V \rightarrow O} = M_O \circ \iota_{V \rightarrow O}.$$

PROOF. From Propositions 2.2.2 and 2.2.13, it suffices to prove that the nonzero entries in $[B_{n,n+1}^{(2)}, B_{n,n+2}^{(2)}, \dots, B_{n,2n-1}^{(2)}]$ and $[B_{2,1}^{(2)}, B_{3,1}^{(2)}, \dots, B_{n,1}^{(2)}]^T$ form a south-east chain. Recall that by construction, the Fomin growth diagram of $\iota_{V \rightarrow O}(V)$ is a triangle diagram with the entries of

$\iota_{V \rightarrow O}(\mathbf{V})$ labelling its diagonal. As $\mathbf{V}$ is a vacillating tableau of weight zero, the partition (2) sits at the corners $(2, 2(n-1))$ and $(2(n-1), 2)$ in the Fomin growth diagram of $\iota_{V \rightarrow O}(\mathbf{V})$. By Theorem 2.3.4, we have $M_O(\iota_{V \rightarrow O}(\mathbf{V})) = G_O(\iota_{V \rightarrow O}(\mathbf{V}))$. This implies that the filling of the first 2 columns and first 2 rows match $M_O(\iota_{V \rightarrow O}(\mathbf{V}))$. As all the entries of $M_O(\iota_{V \rightarrow O}(\mathbf{V}))$ are either 0 or 1, we have that all the nonzero entries in the first 2 rows and the first 2 rows form a north-east chain by [33, Theorem 2]. $\square$

We can now prove the first part of Theorem 2.3.11.

PROOF. Putting together the current results we obtain:

$$\begin{aligned} \text{blowup}_2^{\text{NE}} \circ M_{V \rightarrow O} &= M_O \circ \iota_{V \rightarrow O} && \text{by Lemma 2.3.15} \\ &= G_O \circ \iota_{V \rightarrow O} && \text{by Theorem 2.3.4.} \end{aligned}$$

It thus remains to show: $G_V = \text{blocksum}_2 \circ G_O \circ \iota_{V \rightarrow O}$. Let $\mathbf{V}$ be a fixed vacillating tableau of weight zero and length n . Let $O = \iota_{V \rightarrow O}(\mathbf{V})$. Let $M = (m_{i,j})_{1 \leq i,j \leq 2n} = G_O(O)$ and let $B_{i,j}^{(2)}$ be its block matrix decomposition. Let $\alpha_{i,j}$ for $0 \leq j \leq i \leq 2n$ be the partition in the i -th row and j -th column in the growth diagram of O . Above calculation shows that the nonzero entries in the matrices

$$\begin{aligned} &[B_{i,i+1}^{(2)}, B_{i,i+2}^{(2)}, \dots, B_{i,i+n-1}^{(2)}] \quad \text{and} \\ &[B_{i+1,i}^{(2)}, B_{i+2,i}^{(2)}, \dots, B_{i+n-1,i}^{(2)}] \end{aligned}$$

form north-east chains.

Thus the squares

$$\begin{array}{c} \alpha_{2i,2j} \qquad \alpha_{2i,2(j+1)} \\ \boxed{\phantom{\alpha_{2(i+1),2j}}} \\ \alpha_{2(i+1),2j} \qquad \alpha_{2(i+1),2(j+1)} \end{array}$$

with entry $m_{2i,2j} + m_{2i+1,2j} + m_{2i,2j+1} + m_{2i+1,2j+1}$ satisfy the rules RSK F0-F2 and RSK B0-B2. As in proof of Lemma 2.3.14, the entries of the first subdiagonal of M are zero. Hence M is uniquely determined by the labels $\alpha_{2i,2i}$ and $\alpha_{2i,2i+1}$. Again by proof of Lemma 2.3.14 we have $\alpha_{2i,2i} = 2\lambda^i$ and $\alpha_{2i,2i+1} = (2\lambda^i) \cup (2\lambda^{i+1})$. As these partitions agree with the labels in Construction 2.2.18, we get $G_V(\mathbf{V}) = \text{blocksum}_2(G_O(O))$. $\square$

PROBLEM 2.3.16. Find a characterization of the image of the injective maps M_F , $M_{V \rightarrow O}$ and $M_{V \rightarrow F}$.

REMARK 2.3.17. For M_O the solution to the above problem is known (see [49]). The set of r -symplectic oscillating tableaux of weight zero are in bijection with the set of $(r + 1)$ -noncrossing perfect matchings of $\{1, 2, \dots, n\}$.

CHAPTER 3

A crystal base approach to Thrall's problem

This chapter is based on collaborations with Joseph Pappé and Anne Schilling.

3.1. Background and Definitions

3.1.1. Representation theory of higher Lie modules. We begin by summarizing the construction of the higher Lie modules as $\mathrm{GL}_m(\mathbb{C})$ representations as given by Thrall [67]. Let $V = \{\mathbf{1}, \dots, \mathbf{m}\}$ be an alphabet of size m , and let $T(V) := \bigoplus_{n \geq 0} V^{\otimes n}$ be the tensor algebra of V . The *free Lie algebra on V* , denoted $\mathcal{L}(V)$, is the Lie subalgebra of $T(V)$ generated by V and all bracketings $[a, b] := a \otimes b - b \otimes a$. The free Lie algebra inherits a natural grading, with n -th degree piece $\mathcal{L}_n(V) := V^{\otimes n} \cap \mathcal{L}(V)$, the span of all bracketings of length n .

Under the natural $\mathrm{GL}_m(\mathbb{C})$ action on $\mathbb{C}[V]$, the matrix $X = (x_{i,j}) \in \mathrm{GL}_m(\mathbb{C})$ acts on $\mathbf{k} \in V$ by $X\mathbf{k} = x_{1,k}\mathbf{1} + \dots + x_{m,k}\mathbf{m}$. This representation is known as the *standard representation* of $\mathrm{GL}_m(\mathbb{C})$. Any representation V of $\mathrm{GL}_m(\mathbb{C})$ can be extended to $T(V)$ by allowing X to act diagonally,

$$X(v_1 \otimes \dots \otimes v_k) = Xv_1 \otimes \dots \otimes Xv_k.$$

The free Lie algebra of V is a naturally graded $\mathrm{GL}_m(\mathbb{C})$ -subrepresentation of $T(V)$ with graded components $\mathcal{L}_n(V)$ called the *Lie modules*.

In order to reach the higher Lie modules studied by Thrall, we define universal enveloping algebras. The *universal enveloping algebra* of the free Lie algebra $\mathcal{L}(V)$, denoted $\mathcal{U}(\mathcal{L}(V))$, is the associative algebra such that

- (1) there exists an embedding ι from $\mathcal{L}(V)$ into $\mathcal{U}(\mathcal{L}(V))$ such that

$$\iota([v, w]) = \iota(v)\iota(w) - \iota(w)\iota(v),$$

- (2) $\mathcal{U}(\mathcal{L}(V))$ is generated by elements $\iota(w)$ for $w \in \mathcal{L}(V)$, and

- (3) $\mathcal{U}(\mathcal{L}(V))$ satisfies the universal property that for any associative algebra $\mathcal{A}$ with a Lie algebra homomorphism $j : \mathcal{L}(V) \rightarrow \mathcal{A}$, there exists a unique algebra homomorphism $\phi : \mathcal{U}(\mathcal{L}(V)) \rightarrow \mathcal{A}$ such that $\phi(\iota(w)) = j(w)$ for all $w \in \mathcal{L}(V)$.

Universal enveloping algebras have an explicit basis given by the Poincaré–Birkhoff–Witt Theorem. For each $n \geq 0$, fix $B_n = \{v_{n,1}, \dots, v_{n,r_n}\}$ to be an ordered basis for $\mathcal{L}_n(V)$. Then the set

$$\{v_{n,j} \mid n \geq 0, 1 \leq j \leq r_n\}$$

forms an ordered basis for $\mathcal{L}(V)$. The Poincaré–Birkhoff–Witt basis for $\mathcal{U}(\mathcal{L}(V))$ consists of elements of the form

$$\prod_{n \geq 0} \prod_{1 \leq j \leq r_n} \iota(v_{n,j})^{k_{n,j}}$$

such that each $k_{n,j}$ is a nonnegative integer and only finitely many are positive [22, Theorem 9.10].

A similarly structured basis exists for the k -th symmetric power of $\mathcal{L}_n(V)$, $\text{Sym}^k(\mathcal{L}_n(V))$, that lists basis vectors of $\mathcal{L}_n(V)$ in weakly increasing order in the k -fold simple tensors. Thus we have

$$\mathcal{U}(\mathcal{L}(V)) \cong \bigoplus_{\lambda = \langle 1^{m_1}, 2^{m_2}, \dots \rangle} \text{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \text{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \dots$$

as graded $\text{GL}_m(\mathbb{C})$ -representations, where the sum is over all partitions λ having m_i parts of size i . The piece

$$\mathcal{L}_\lambda(V) := \text{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \text{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \dots,$$

is called the *higher Lie module* associated to $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots \rangle$. Thrall’s original paper studied the decomposition of $\mathcal{L}_\lambda(V)$ into irreducible $\text{GL}_m(\mathbb{C})$ -representations [67]. Phrased equivalently in terms of symmetric functions, Thrall’s problem asks for a combinatorial interpretation of the coefficients d_λ^μ in the Schur expansion of the higher Lie character associated to λ ,

$$\text{ch}(\mathcal{L}_\lambda(V)) = \sum_{\mu} d_\lambda^\mu s_\mu(x_1, \dots, x_m).$$

Schocker [57] gives a formula to compute the multiplicities in $\mathcal{L}_{\langle a^b \rangle}(V)$ that involves division and subtractions coming from the Möbius function. Results by Klyachko [31] and Bergeron–Bergeron–Garsia [7] identify the Schur–Weyl duals of $\mathcal{L}_n(V)$ and $\mathcal{L}_{\langle a^b \rangle}(V)$, respectively. For background on the higher Lie modules as representations of the symmetric group, see [2]. For further details on

the free Lie algebra we refer the reader to [52] and for summaries of the current status of Thrall's problem, see [2, 51].

3.1.2. Known cases. The Littlewood–Richardson rule simplifies Thrall's problem to the case of a rectangular partitions $\lambda = \langle a^b \rangle$ containing b parts of size a . These are the known cases for Thrall's problem:

(1) $\lambda = \langle 1^n \rangle$: Since $\mathcal{L}_{\langle 1^n \rangle}(V) = \text{Sym}^n(V)$, we have

$$\text{ch}(\mathcal{L}_{\langle 1^n \rangle}(V)) = s_{(n)}(x_1, \dots, x_m).$$

(2) $\lambda = \langle 2^b \rangle$: Since $\mathcal{L}_{\langle 2^b \rangle}(V) = \text{Sym}^b(\mathcal{L}_2(V))$ and $\text{ch}(\mathcal{L}_2(V)) = s_{(1,1)}(x_1, \dots, x_m)$, by Littlewood [35] we have

$$\text{ch}(\mathcal{L}_{\langle 2^b \rangle}(V)) = h_b[\text{ch}(\mathcal{L}_2(V))] = \sum_{\substack{\mu \vdash 2b \\ \text{even columns}}} s_{\mu}(x_1, \dots, x_m).$$

(3) $\lambda = \langle a^1 \rangle$: Kraśkiewicz and Weyman [32] showed that the coefficients in the character of the Lie module $\mathcal{L}_a(V)$ count the number of standard Young tableaux with *major index statistic* congruent to 1 mod a . Specifically,

$$\text{ch}(\mathcal{L}_a(V)) = \sum_{\mu \vdash a} d_{\langle a^1 \rangle}^{\mu} s_{\mu}(x_1, \dots, x_m) \text{ where } d_{\langle a^1 \rangle}^{\mu} = \#\{Q \in \text{SYT}(\mu) \mid \text{maj}(Q) \equiv 1 \pmod{a}\}.$$

Combining the results above and employing the Littlewood–Richardson rule, Thrall's problem is solved for all higher Lie modules associated to partitions $\lambda = \langle 1^n, 2^b, k_1^1, \dots, k_{\ell}^1 \rangle$ when $n, b \geq 0$ and $k_i \geq 3$ are distinct. In particular this includes all partitions with distinct parts and all hook shapes.

3.1.3. Fundamental quasi-symmetric functions. Let $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots, n^{m_n} \rangle$ be a partition of n . For $\pi \in S_n$, we say that π has *cycle type* λ if π has m_i cycles of length i , written $\text{type}(\pi) = \lambda$. Let C_{λ} be the conjugacy class of S_n indexed by λ . Recall that $\pi \in C_{\lambda}$ if and only if $\text{type}(\pi) = \lambda$.

The *descent set* of $\pi \in S_n$ is

$$D(\pi) := \{j \in [n-1] \mid \pi(j) > \pi(j+1)\}.$$

Subsets of $[n - 1]$ are in one-to-one correspondence with compositions of n , $\alpha \models n$, given by the following map:

$$(3.1) \quad D = \{z_1 < z_2 < z_3 < \dots < z_k\} \mapsto C(D) := (z_1, z_2 - z_1, z_3 - z_2, \dots, n - z_k),$$

$$(3.2) \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \mapsto D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \dots + \alpha_{k-1}\}.$$

Gessel [18] introduced the *fundamental quasi-symmetric function* F_α indexed by $\alpha \models n$,

$$(3.3) \quad F_\alpha := \sum_{\substack{1 \leq i_1 \leq \dots \leq i_n : \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} x_{i_1} \cdots x_{i_n}.$$

By the correspondence above, we may also index fundamental quasi-symmetric functions by sets. We specifically consider those indexed by descent sets of permutations $\pi \in S_n$, $F_{D(\pi)}$, where $D(\alpha)$ is replaced by $D(\pi)$ in Equation (3.3).

In 1993, Gessel and Reutenauer [20] described the fundamental quasi-symmetric expansion of the higher Lie characters in a combinatorial way as a sum over a conjugacy class of S_n .

THEOREM 3.1.1 ([20, Theorem 3.6]). *For a partition $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots, n^{m_n} \rangle \vdash n$,*

$$\text{ch}(\mathcal{L}_\lambda(V)) = \sum_{\pi \in C_\lambda} F_{D(\pi)}(x_1, \dots, x_m).$$

Thrall's problem asks for a combinatorial description of the coefficients appearing in the Schur function expansion of $\text{ch}(\mathcal{L}_\lambda(V))$. If the character of the higher Lie module corresponding to λ expands

$$(3.4) \quad \text{ch}(\mathcal{L}_\lambda(V)) = \sum_{\mu \vdash n} d_\lambda^\mu s_\mu(x_1, \dots, x_m),$$

then by Theorem 3.1.1 we can instead study the known fundamental quasi-symmetric expansion

$$(3.5) \quad \sum_{\mu \vdash n} d_\lambda^\mu s_\mu(x_1, \dots, x_m) = \sum_{\alpha \models n} c_\lambda^\alpha F_\alpha(x_1, \dots, x_m).$$

We fix the notation from Equation (3.5) throughout for d_λ^μ and

$$c_\lambda^\alpha := \#\{\pi \in C_\lambda \mid C(D(\pi)) = \alpha\}.$$

REMARK 3.1.2. *Transforming the fundamental quasi-symmetric expansion of a symmetric function into the Schur expansion is a notoriously difficult problem. For example, quasi-symmetric expansions of LLT polynomials, modified Macdonald polynomials, and plethysm of two Schur functions exist while their Schur expansions remain unknown in general. Many tools and frameworks have been developed to address this problem, including among others [4, 8, 12, 14, 42].*

If μ is the largest partition in lexicographic order such that $d_\lambda^\mu > 0$ for a fixed partition λ , we call μ the *leading partition* and $d_\lambda^\mu s_\mu(x_1, \dots, x_m)$ the *leading term* of $\text{ch}(\mathcal{L}_\lambda(V))$. Let $\text{supp}(\text{ch}(\mathcal{L}_\lambda(V))) := \{\mu \mid d_\lambda^\mu > 0\}$ be the support of the higher Lie character in the Schur expansion.

The expansion of a Schur function s_μ in terms of fundamental quasi-symmetric functions is given by Gessel [18, 19] as

$$(3.6) \quad s_\mu = \sum_{T \in \text{SYT}(\mu)} F_{D(T)},$$

where $\text{SYT}(\mu)$ is the set of standard Young tableaux of shape μ and $D(T)$ is the descent set of the standard Young tableau T , the collection of letters i where the letter $i+1$ appears in a lower row in T (in English notation). From the expansion of s_μ above, we observe the following lemmas which appear in [42] as Corollary 3.5 and 3.6.

LEMMA 3.1.3 ([42]). *For any partition ν ,*

$$s_\nu = F_\nu + \text{terms that are strictly smaller than } \nu \text{ in lexicographic order.}$$

LEMMA 3.1.4 ([42]). *If*

$$f = \sum_{\mu \vdash n} b_\mu s_\mu = \sum_{\alpha \models n} c_\alpha F_\alpha$$

is a symmetric function, let ν be the largest element in lexicographic order on compositions such that $c_\nu > 0$. Then $\nu \vdash n$ and $b_\nu = c_\nu$.

Therefore, if ν is the largest partition in lexicographic order such that $c_\lambda^\nu > 0$ in (3.5), then $d_\lambda^\nu s_\nu(x_1, \dots, x_m)$ is the leading term in $\text{ch}(\mathcal{L}_\lambda(V))$, and further $d_\lambda^\nu = c_\lambda^\nu$.

3.2. The Leading Term

3.2.1. The lexicographic largest term. We start by considering the rectangular partition $\lambda = \langle a^b \rangle$ for some $a, b \geq 1$. In the case where $a = 1$, it is already known that $\text{ch}(\mathcal{L}_{\langle 1^b \rangle}(V)) = s_{(b)}(x_1, \dots, x_m)$, so the leading term is not a mystery. Recall from (3.5) that c_λ^α is the number of permutations of cycle type λ with descent set $D(\pi)$ corresponding to α . Since $\text{type}(\pi) = \text{type}(\pi^{-1})$ for all permutations π , we can compute c_λ^α by finding all permutations π of cycle type λ with inverse descent set $D(\pi^{-1})$ corresponding to α . We will use square (resp. round) brackets to denote the one-line (resp. cycle) notation of π .

LEMMA 3.2.1. *Let $\pi \in S_{ab}$ with $a \geq 2$ and descent set of π^{-1} given by $D(\pi^{-1}) = \{z_1 < z_2 < \dots < z_k\} \subseteq [ab - 1]$. If $\text{type}(\pi) = \langle a^b \rangle$, then $z_1 \leq ab - b$.*

PROOF. Since $D(\pi^{-1}) = \{z_1 < z_2 < \dots < z_k\}$, the one-line notation of π contains $1, 2, \dots, z_1$ as an increasing substring reading left to right, with $z_1 + 1$ appearing to the left of z_1 in π , creating the first descent of π^{-1} . Suppose by contradiction that $\text{type}(\pi) = \langle a^b \rangle$ and $z_1 > ab - b$. Since there are b cycles in π , by pigeon-hole principle there exists a cycle $(x_1, x_2, \dots, x_a)$ in π such that $x_i \leq ab - b + 1$ for all $1 \leq i \leq a$. Without loss of generality, $x_1 < x_i$ for $2 \leq i \leq a$. However, this implies that x_2 appears to the left of x_1 in the one-line notation of π , a contradiction since both $x_1, x_2 \leq ab - b + 1 \leq z_1$ and thus part of the increasing substring $1, 2, \dots, z_1$ in the one-line notation of π . $\square$

LEMMA 3.2.2. *Let $a \geq 2, b \geq 1$. Then*

$$c_{\langle a^b \rangle}^{(ab-b, b)} = 1.$$

PROOF. Suppose there exists $\pi \in C_{\langle a^b \rangle}$ with $D(\pi^{-1}) = \{ab - b\}$. Written in one-line notation, π contains two increasing substrings:

$$\begin{aligned} &1, 2, \dots, ab - b \\ &ab - b + 1, ab - b + 2, \dots, ab. \end{aligned}$$

Thus $\pi(1) = ab - b + 1$. In order for the first substring to be increasing, we must have $\pi^{-1}(1) \leq b + 1$ and

$$\begin{aligned}\pi(ab - b + 1) &\geq ab - 2b + 1 = (a - 2)b + 1, \\ \pi^2(ab - b + 1) &\geq ab - 3b + 1 = (a - 3)b + 1, \\ &\vdots \\ \pi^{-1}(1) = \pi^{a-2}(ab - b + 1) &\geq ab - (a - 1)b + 1 = b + 1.\end{aligned}$$

Therefore $\pi^{-1}(1) = b + 1$, which determines the remaining one-line notation of π :

$$\begin{aligned}\pi &= [ab - b + 1 \ ab - b + 2 \ \cdots \ ab \ 1 \ 2 \ \cdots \ ab - b] \\ &= (1, (a - 1)b + 1, (a - 2)b + 1, \dots, 2b + 1, b + 1) \\ &\quad (2, (a - 1)b + 2, (a - 2)b + 2, \dots, 2b + 2, b + 2) \\ &\quad \vdots \\ &\quad (b, ab, (a - 1)b, (a - 2)b, \dots, 2b).\end{aligned}$$

Thus by construction, there is exactly one permutation of cycle type $\langle a^b \rangle$ with inverse descent set $\{ab - b\}$. $\square$

THEOREM 3.2.3. *The leading term in the Schur expansion of $\text{ch}(\mathcal{L}_{\langle a^b \rangle}(V))$ is $s_{(ab-b, b)}(x_1, \dots, x_m)$.*

PROOF. Combining Lemmas 3.2.1 and 3.2.2 shows that $(ab - b, b)$ is the largest partition in lexicographic order in $\text{supp}(\text{ch}(\mathcal{L}_{\langle a^b \rangle}(V)))$. $\square$

REMARK 3.2.4. *When $b = 1$, the leading term in $\text{ch}(\mathcal{L}_a(V))$ is $s_{(a-1, 1)}(x_1, \dots, x_m)$ by Theorem 3.2.3. This is also quickly seen from the result by Krařkiewicz and Weyman [32], since there is one standard Young tableau of shape $(a - 1, 1)$ with major index congruent to 1 mod a (the lone entry in the second row must be 2) and any standard Young tableau of shape (a) has major index 0.*

REMARK 3.2.5. *The formula by Schocker [57] for the Schur coefficients $d_{\langle a^b \rangle}^\mu$ in (3.4) involves a sum over partitions of b , the number theoretic Möbius function, and major index statistic. Schocker's formula when $b = 1$ immediately returns the combinatorial description for coefficients*

of the Lie modules found by Krařkiewicz and Weyman, however it appears difficult to compute the leading coefficient in general for the higher Lie modules when $b \geq 2$ due to the subtractions present.

We extend the result of Theorem 3.2.3 to the general case of Thrall's problem using Littlewood–Richardson.

COROLLARY 3.2.6. *Let $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots \rangle$. Then the leading term in the Schur expansion of $\text{ch}(\mathcal{L}_\lambda(V))$ is $s_{\text{lp}(\lambda)}(x_1, \dots, x_m)$ where*

$$(3.7) \quad \begin{aligned} \text{lp}(\lambda) &= (m_1 + \sum_{k=2} (k-1)m_k, \sum_{k=2} m_k) \\ &= (m_1 + m_2 + 2m_3 + 3m_4 + \dots, m_2 + m_3 + m_4 + \dots) \end{aligned}$$

PROOF. By the Littlewood–Richardson rule, the lexicographically largest term in $s_\nu s_\mu$ is $s_{\nu+\mu}$ where $\nu + \mu = (\nu_1 + \mu_1, \dots)$. Recall that the higher Lie module indexed by $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots \rangle$ is given by the following tensor product:

$$\mathcal{L}_\lambda(V) = \mathcal{L}_{\langle 1^{m_1} \rangle} \otimes \mathcal{L}_{\langle 2^{m_2} \rangle} \otimes \dots$$

By Theorem 3.2.3,

$$\begin{aligned} \text{ch}(\mathcal{L}_\lambda(V)) &= \text{ch}(\mathcal{L}_{\langle 1^{m_1} \rangle}(V)) \cdot \text{ch}(\mathcal{L}_{\langle 2^{m_2} \rangle}(V)) \cdots \\ &= \prod_k \sum_{\mu \vdash km_k} d_{\langle k^{m_k} \rangle}^\mu s_\mu(x_1, \dots, x_m) \\ &= s_{(m_1)}(x_1, \dots, x_m) \prod_{k=2} d_{\langle k^{m_k} \rangle}^{(km_k - m_k, m_k)} s_{(km_k - m_k, m_k)}(x_1, \dots, x_m) + \text{lex lower terms} \\ &= \left(\prod_{k=2} d_{\langle k^{m_k} \rangle}^{(km_k - m_k, m_k)} \right) s_\mu(x_1, \dots, x_m) + \text{lex lower terms} \end{aligned}$$

where $\mu = \text{lp}(\lambda)$ is as in Equation (3.7). □

3.2.2. Newton Polytopes. Another consequence of Theorem 3.2.3 is that the higher Lie characters have saturated Newton polytopes. Monical, Tokcan, and Yong [40] introduced the study of Newton polytopes of prominent polynomials in algebraic combinatorics. A polynomial has saturated Newton polytope (SNP) if every lattice point of the convex hull of its exponent vectors corresponds to a monomial. In their original paper, they show that Schur functions, symmetric Macdonald polynomials, and Stanley symmetric polynomial have SNP [40]. Others have studied the

SNP property for Schubert polynomials, Grothendieck polynomials, chromatic symmetric functions, and the Kronecker product of two Schur functions (see [45] for an overview). In particular, SNP is a necessary condition for types of log-concavity [9, 38].

The Newton polytope of a polynomial $f(x_1, \dots, x_m) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^m} c_\alpha x^\alpha$ is the convex hull of its exponent vectors,

$$N_m(f) = \text{convex}(\{\alpha \mid c_\alpha \neq 0\}) \subseteq \mathbb{R}^m.$$

Newton polytopes were first introduced in [40]. We say that f has saturated Newton polytope if and only if $c_\beta \neq 0$ for all $\beta \in N_m(f) \cap \mathbb{Z}^m$, meaning that the set of exponent vectors with nonzero coefficient coincides with $N_m(f)$.

PROPOSITION 3.2.7 ([40] Proposition 2.5). *Suppose f is a homogeneous symmetric function of degree n in m variables such that*

$$f = \sum_{\mu \vdash n} c_\mu s_\mu.$$

Suppose there exists $\nu \vdash n$ with $c_\nu \neq 0$ and $c_\mu \neq 0$ only if $\mu \leq_D \nu$. If $m < \ell(\nu)$, $f = 0$. Otherwise:

- (1) $N_m(f)$ is the ν -permutahedron in $\mathbb{R}^m$ (i.e. the convex hull of the S_m orbit of ν in $\mathbb{R}^m$).
- (2) The vertices of $N_m(f)$ are rearrangements of ν .
- (3) If moreover $c_\mu \geq 0$ for all μ , then f has saturated Newton polytope.

PROPOSITION 3.2.8. *The higher Lie characters have saturated Newton polytope.*

PROOF. Corollary 3.2.6 identifies the lexicographically largest partition appearing in the higher Lie character corresponding to $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots \rangle$ as the two part partition $\text{lp}(\lambda) = (\tilde{\lambda}_1, \tilde{\lambda}_2)$ in Equation (3.7). For any $\mu \in \text{supp}(\text{ch}(\mathcal{L}_\lambda(V)))$, we have $\mu_1 \leq \tilde{\lambda}_1$ and $\mu_1 + \mu_2 \leq \tilde{\lambda}_1 + \tilde{\lambda}_2 = |\lambda|$. Therefore $\mu \leq_D \text{lp}(\lambda)$. Since $\text{ch}(\mathcal{L}_\lambda(V))$ is Schur-positive, by Proposition 3.2.7 (3) $\text{ch}(\mathcal{L}_\lambda(V))$ has saturated Newton polytope. $\square$

REMARK 3.2.9. *Proposition 3.2.8 is a necessary condition for other properties, such as $\text{ch}(\mathcal{L}_\lambda(V))$ being M -convex, being Lorentzian and satisfying types of log-concavity (see for example [38]). It would be interesting to investigate if certain sequences of coefficients of $\text{ch}(\mathcal{L}_\lambda(V))$ are log-concave. For example, Adin, Hegedüs, and Roichman [1] conjecture that for every $\lambda = \langle r^s \rangle$ with even $r \neq 6$, the multiplicities of the Schur functions indexed by hook-shaped partitions in $\text{ch}(\mathcal{L}_\lambda(V))$ form a log-concave sequence, and confirmed this for all $r \leq 40, s \leq 5$.*

3.3. The Crystal Base Approach

We outline an approach to Thrall’s problem that proposes constructing a type A_{m-1} crystal graph for the higher Lie module $\mathcal{L}_{\langle a^b \rangle}(V)$. In Section 3.3.1 we describe the monomial expansion of the higher Lie characters in terms of multisets of primitive necklaces given by Gessel and Reutenauer [20]. In Section 3.3.2 we define the type A_{m-1} quasicrystals introduced by Cain and Malheiro [11] and Maas–Gariépy [37]. In Section 3.3.3, we apply Gessel’s necklace bijection to define a crystal structure on multisets of type $\langle 1^a, 2^b \rangle$ and recover a known case of Thrall’s problem. In Section 3.3.4, we construct the analogous dual equivalence graph for the type $\langle 1^a, 2^b \rangle$ case.

3.3.1. A combinatorial basis for higher Lie modules. The objective of the crystal base approach to Thrall’s problem is to define a type A_{m-1} -crystal structure on a combinatorial basis for the higher Lie module $\mathcal{L}_{\langle a^b \rangle}(V)$ and classify the highest weight elements. The multiplicity of the irreducible $\mathrm{GL}_m(\mathbb{C})$ representation of highest weight μ is then counted by the number of highest weight elements of weight μ in the crystal. Throughout, let $\mathbb{B}$ denote the standard crystal of type A_{m-1} and $\mathbb{B}^{\otimes n}$ the crystal on words of length n . We consider the tensor crystal elements in $\mathbb{B}^{\otimes n}$ as words in alphabet $\{\mathbf{1}, \dots, \mathbf{m}\}$.

Gessel and Reutenauer [20] gave a monomial expansion of $\mathrm{ch}(\mathcal{L}_\lambda(V))$ in terms of multisets of primitive necklaces, twenty years after Klyachko [31] gave a monomial expansion of $\mathrm{ch}(\mathcal{L}_n(V))$ in terms of primitive necklaces. A word in the alphabet $V = \{\mathbf{1}, \dots, \mathbf{m}\}$ is *primitive* if it is not a proper power of another word. For example, **121312** is primitive while **112112** is not. A *primitive necklace* is the set of all cyclic rotations of a primitive word. The character of the higher Lie module associated to $\lambda = \langle 1^{m_1}, 2^{m_2}, \dots \rangle \vdash n$ can be expressed in terms of multisets of primitive necklaces:

$$\mathrm{ch}(\mathcal{L}_\lambda(V)) = \sum_{\mathcal{M} \in \mathcal{M}_\lambda} x^{\mathcal{M}},$$

where $\mathcal{M}_\lambda$ is the collection of multisets containing m_k primitive necklaces of length k . In the monomial $x^{\mathcal{M}} = x_1^{a_1} \cdots x_m^{a_m}$, the exponent a_i is the number of occurrences of the letter $\mathbf{i}$ among all the necklaces in the multiset $\mathcal{M}$.

Using standardization of words, Gessel and Reutenauer describe a content-preserving bijection between $\mathcal{M}_\lambda$ and words of length n with standardization permutation of cycle type λ (see [20, Section 3] for the details of the bijection and [36] for discussion of unique factorization of a word

into Lyndon words). To standardize a word $w = w_1 \cdots w_n \in \mathbb{B}^{\otimes n}$, we label the letters using $\{1, \dots, n\}$ by first labeling all of the **1**'s from left to right, then all of the **2**'s, and so on. Let $\text{st}(w)$ be the *standardization* of w , and note that we can consider $\text{st}(w)$ as a permutation written in one-line notation.

EXAMPLE 3.3.1. If $w = \mathbf{1213}$, then $\text{st}(w) = 1324$. As a permutation in S_4 written in cycle notation, $\text{st}(w) = (1)(2, 3)(4)$ and thus $\text{type}(\text{st}(w)) = \langle 1^2, 2^1 \rangle$.

By [20, Theorem 3.6],

$$\text{ch}(\mathcal{L}_\lambda(V)) = \sum_{\mathcal{M} \in \mathcal{M}_\lambda} x^{\mathcal{M}} = \sum_{\substack{w \in \mathbb{B}^{\otimes n} \\ \text{type}(\text{st}(w)) = \lambda}} x^{\text{wt}(w)}.$$

Our objective is to realize the above monomial expansion as a crystal character so that we can count the multiplicity of $s_\mu(x_1, \dots, x_m)$ in the Schur expansion of $\text{ch}(\mathcal{L}_\lambda(V))$ by counting the number of highest weight elements of weight μ in the crystal.

3.3.2. Quasicrystals. Cain and Malheiro [11] define quasi-Kashiwara operators $\ddot{e}_i, \ddot{f}_i$ on $\mathbb{B}^{\otimes n}$ which map words that contain an i -inversion, instances in which $\mathbf{i}+1$ appears to the left of $\mathbf{i}$ in the word, to 0 and otherwise act as the usual crystal operators from Definition 1.2.5. They show that the quasi-Kashiwara operators preserve the standardization permutation and that connected components contain all words with the same standardization permutation.

PROPOSITION 3.3.2 ([11, Proposition 6.1, 6.4]). Let $w \in \mathcal{B}^{\otimes n}$. If $\ddot{e}_i(w)$ is defined, then $\text{st}(w) = \text{st}(\ddot{e}_i(w))$. If $\ddot{f}_i(w)$ is defined, then $\text{st}(w) = \text{st}(\ddot{f}_i(w))$. Furthermore, if $u \in \mathbb{B}^{\otimes n}$ with $\text{st}(u) = \text{st}(w)$, then there exists a sequence of quasi-Kashiwara operators $\ddot{e}_{i_1}^{k_1}, \ddot{f}_{i_2}^{k_2}, \dots, \ddot{e}_{i_{2m-1}}^{k_{2m-1}}, \ddot{f}_{i_{2m}}^{k_{2m}}$, with $k_i \geq 0$, such that $w = \ddot{e}_{i_1}^{k_1} \ddot{f}_{i_2}^{k_2} \cdots \ddot{e}_{i_{2m-1}}^{k_{2m-1}} \ddot{f}_{i_{2m}}^{k_{2m}}(u)$.

For $\pi \in S_n$, define St_π to be the set of all words in $\mathbb{B}^{\otimes n}$ which standardize to π . By Proposition 3.3.2, under the quasi-Kashiwara operators St_π is a connected component called a *quasicrystal* by Cain and Malheiro [11] and Maas–Gariépy [37].

PROPOSITION 3.3.3 ([37, Proposition 2.1]). *The quasicrystal character of St_π is the fundamental quasi-symmetric polynomial indexed by $D(\pi^{-1})$. That is,*

$$\sum_{w \in \text{St}_\pi} x^{\text{wt}(w)} = \sum_{\substack{1 \leq i_1 \leq \dots \leq i_n \leq m: \\ i_j < i_{j+1} \text{ if } j \in D(\pi^{-1})}} x_{i_1} \cdots x_{i_n} = F_{D(\pi^{-1})}(x_1, \dots, x_m).$$

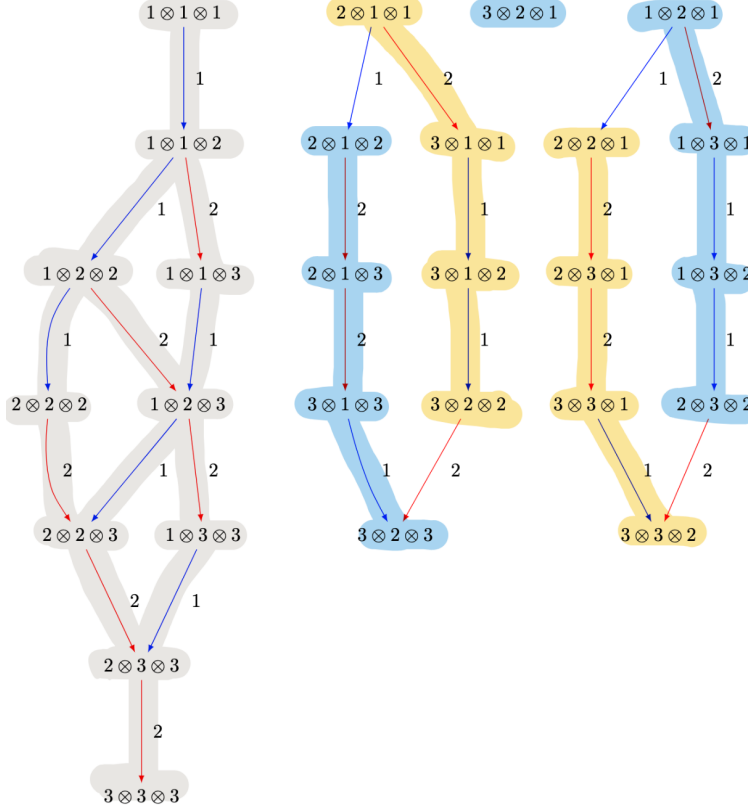


FIGURE 3.1. The quasicrystal decomposition of the type A_2 crystal on words of length 3, $\mathbb{B}^{\otimes 3}$.

EXAMPLE 3.3.4. *Figure 3.1 shows the type A_2 crystal on words of length 3, $\mathbb{B}^{\otimes 3}$, decomposed into its quasicrystal pieces. The quasicrystal decomposition is emphasized below, with permutations written in one-line notation:*



The quasicrystal character for St_{213} and St_{231} is $F_{\{1\}}(x_1, x_2, x_3)$,

$$\sum_{w \in \text{St}_{213}} x^{\text{wt}(w)} = \sum_{w \in \text{St}_{231}} x^{\text{wt}(w)} = x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2 = \sum_{1 \leq i_1 < i_2 \leq i_3 \leq 3} x_{i_1} x_{i_2} x_{i_3},$$

since $\{1\}$ is the descent set of 213 and 312 (the inverses of 213 and 231, respectively). Finally, the quasicrystals are colored based on the cycle type of the indexing standardization permutation,

$$\text{type}(123) = \langle 1^3 \rangle, \quad \text{type}(213) = \text{type}(321) = \text{type}(132) = \langle 1^1, 2^1 \rangle, \quad \text{type}(312) = \text{type}(231) = \langle 3^1 \rangle.$$

REMARK 3.3.5. Properties of the type A_{m-1} quasicrystal decomposition, called the crystal skeleton, are investigated in detail by Brauner, Corteel, Daugherty, and Schilling in [8].

From Example 3.3.4, we can see that the induced subgraph on $\bigcup_{\substack{\pi \in S_n \\ \text{type}(\pi) = \langle a^b \rangle}} \text{St}_\pi$ of $\mathbb{B}^{\otimes n}$ is not always a crystal graph. In particular, the connected components do not have Schur function crystal characters.

PROBLEM 3.3.6. Define rules describing how to stitch the quasicrystals $\{\text{St}_\pi \mid \pi \in S_n, \text{type}(\pi) = \langle a^b \rangle\}$ together to form complete type A_{m-1} connected components in which the cycle type of the standardization permutation is the same. In other words, define crystal operators on $\mathbb{B}^{\otimes n}$ that preserve the cycle type of standardization permutation.

Figure 3.2 demonstrates how to stitch the quasicrystals of the A_2 crystal on words of length 3 together to define crystals for the higher Lie modules $\mathcal{L}_{\langle 1^3 \rangle}(V)$, $\mathcal{L}_{\langle 1^1, 2^1 \rangle}(V)$, and $\mathcal{L}_3(V)$.

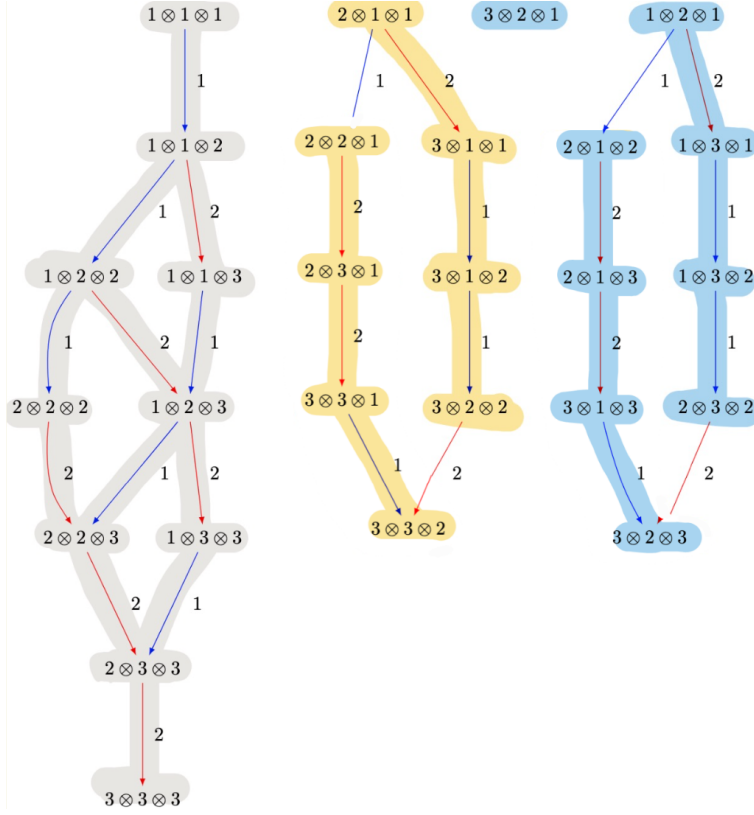


FIGURE 3.2. The stitched version of the type A_2 crystal on words of length 3, $\mathbb{B}^{\otimes 3}$, in which the cycle types of the standardization permutations contained in a connected component are the same.

REMARK 3.3.7. *All words which standardize to the identity permutation have cycle type $\langle 1^n \rangle$ and are exactly the crystal elements in $\mathcal{B}(\langle 1^n \rangle)$. Thus the crystal base approach to Thrall's problem recovers the known case:*

$$\text{ch}(\mathcal{L}_{\langle 1^n \rangle}(V)) = s_{(n)}(x_1, \dots, x_m) = \text{ch}(\mathcal{B}(\langle 1^n \rangle)).$$

3.3.3. The case of $\lambda = \langle 1^a, 2^b \rangle$. We solve Problem 3.3.6 for the case when $\lambda = \langle 1^a, 2^b \rangle$ using the Robinson–Schensted–Knuth (RSK) correspondence, for $a, b \geq 0$. The RSK insertion algorithm bijectively maps a word $w \in \mathbb{B}^{\otimes n}$ to a pair of tableaux $(P(w), Q(w))$ where $P(w)$ is the (semistandard) insertion tableau and $Q(w)$ is the (standard) recording tableau (for details of the RSK algorithm, see for example [60, Chapter 7.11]). As a result of RSK, the shape of $P(w)$ is the same as the shape of $Q(w)$, $Q(w)$ is standard, $\text{wt}(w) = \text{wt}(P(w))$, and $D(w) = D(Q(w))$, where

$D(Q(w)) := \{i \in [n-1] \mid i+1 \text{ appears in a row below } i\}$ is the descent set of the standard Young tableau $Q(w)$ (written in English notation).

The plactic monoid can be defined in terms of the $\mathrm{GL}_m(\mathbb{C})$ crystal $\mathbb{B}^{\otimes n}$, and as a result the RSK algorithm is involved in the structure of the crystal [10, Chapter 8]. We record the relevant results in the following theorem.

THEOREM 3.3.8 ([10, Theorem 8.6, Theorem 8.7]).

- (1) For $w \in \mathbb{B}^{\otimes n}$, if $P(w)$ has shape μ , then w is contained in a connected component of highest weight μ .
- (2) For $w, v \in \mathbb{B}^{\otimes n}$, $P(w) = P(v)$ if and only if w and v are plactically equivalent words, that is, there exists a sequence of elementary plactic relations,

$$bac \equiv bca \text{ for } a < b \leq c \quad \text{and} \quad acb \equiv cab \text{ for } a \leq b < c$$

for letters $a, b, c \in \mathbb{B}$, connecting w to v .

- (3) For $w, v \in \mathbb{B}^{\otimes n}$, if $P(w) = P(v)$ then there exists a sequence of raising operators $e_{i_1}, \dots, e_{i_k}$ such that $e_{i_1} \cdots e_{i_k}(w)$ is the highest weight element of the connected component containing w and $e_{i_1} \cdots e_{i_k}(v)$ is the highest weight element of the connected component containing v .
- (4) For $w, v \in \mathbb{B}^{\otimes n}$, $Q(w) = Q(v)$ if and only if w and v lie in the same connected component.

It is important to note that each connected component of highest weight μ in $\mathbb{B}^{\otimes n}$ is indexed by a standard Young tableau given by the common recording tableau obtained under RSK. Furthermore, if $\mathbb{B}$ is the type A_{n-1} standard crystal and $\mathcal{B}(\mu)$ is any connected component of highest weight μ in $\mathbb{B}^{\otimes n}$, the words of standard weight $(1, \dots, 1)$ in $\mathcal{B}(\mu)$ are in bijection with standard Young tableaux of shape μ , given by $w \mapsto P(w)$.

Finally, we record two well-known facts about the image of involutions under RSK. A permutation $\pi \in S_n$ with $\mathrm{type}(\pi) = \langle 1^a, 2^b \rangle$ is an *involution* (i.e. $\pi^{-1} = \pi$) with exactly a fixed points. For $\pi \in S_n$, $P(\pi) = Q(\pi)$ if and only if π is an involution, and in this case the number of columns of odd length in $P(\pi)$ is exactly the number of fixed points of π .

We now begin the construction of a type A_{m-1} crystal graph on quasicrystals indexed by permutations of cycle type $\langle 1^k, 2^{\frac{m-k}{2}} \rangle$, for $k \equiv m \pmod{2}$. The output of the construction will be a

crystal graph for the higher Lie module $\mathcal{L}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}(V)$, where $V = \{\mathbf{1}, \dots, \mathbf{m}\}$. Let $\mathcal{P}(m)_k$ denote the set of partitions of m with exactly k columns of odd length, and let $\mu \in \mathcal{P}(m)_k$. Let f^μ be the number of standard Young tableaux of shape μ . Consider all connected components $\mathcal{B}_\mu^{Q_1}, \dots, \mathcal{B}_\mu^{Q_{f^\mu}}$ in $\mathbb{B}^{\otimes m}$ which are isomorphic to $\mathcal{B}(\mu)$ and indexed by their recording tableau $Q_i \in \text{SYT}(\mu)$ for $1 \leq i \leq f^\mu$ by Theorem 3.3.8 (3). Each $\mathcal{B}_\mu^{Q_i}$ contains a unique word w_i which maps to (Q_i, Q_i) under RSK, and thus w_i is an involution with exactly k fixed points. Therefore, each $\mathcal{B}_\mu^{Q_i}$ contributes exactly one quasicrystal St_{w_i} to the “stitched” connected component of highest weight μ in the higher Lie crystal corresponding to $\langle 1^k, 2^{\frac{m-k}{2}} \rangle$. All together, $\mathcal{B}_\mu^{Q_1}, \dots, \mathcal{B}_\mu^{Q_{f^\mu}}$ contribute quasicrystals $\text{St}_{w_1}, \dots, \text{St}_{w_{f^\mu}}$. Since $P(w_i) \neq P(w_j)$ for all $i \neq j$ by Theorem 3.3.8 (2), there exists a natural way determined by the insertion tableaux under RSK to stitch all of these quasicrystals to complete an entire connected component of highest weight μ . We have

$$\sum_{i=1}^{f^\mu} \sum_{v \in \text{St}_{w_i}} x^{\text{wt}(v)} = \sum_{i=1}^{f^\mu} F_{D(Q_i)}(x_1, \dots, x_m) = \text{ch}(\mathcal{B}(\mu)).$$

By summing over all partitions in $\mathcal{P}(m)_k$, we achieve the higher Lie character for $\langle 1^k, 2^{\frac{m-k}{2}} \rangle$:

$$\begin{aligned} \sum_{\mu \in \mathcal{P}(m)_k} \text{ch}(\mathcal{B}(\mu)) &= \sum_{\mu \in \mathcal{P}(m)_k} \sum_{i=1}^{f^\mu} \sum_{v \in \text{St}_{w_i}} x^{\text{wt}(v)} \\ &= \sum_{\substack{w \in \mathbb{B}^{\otimes m} \\ \text{type}(\text{st}(w)) = \langle 1^k, 2^{\frac{m-k}{2}} \rangle}} x^{\text{wt}(w)} \\ &= \text{ch}(\mathcal{L}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}(V)). \end{aligned}$$

Stitching together the quasicrystals $\text{St}_{w_1}, \dots, \text{St}_{w_{f^\mu}}$ for each $\mu \in \mathcal{P}(m)_k$ as described above creates a type A_{m-1} crystal graph for the higher Lie module $\mathcal{L}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}(V)$, as summarized in the following construction.

CONSTRUCTION 3.3.9. *Let $W_k := \{\pi \in S_m \mid \text{type}(\pi) = \langle 1^k, 2^{\frac{m-k}{2}} \rangle\}$ for $k \equiv m \pmod{2}$ and let $W_k^\mu := \{\pi \in W_k \mid \text{shape}(P(\pi)) = \mu\}$. To construct a type A_{m-1} crystal graph for the higher Lie module $\mathcal{L}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}(V)$, place the quasicrystal indexed by the word $\pi \in W_k^\mu$ in its natural position determined by RSK in the unique crystal component of highest weight μ . (Note, this construction does not provide an explicit description of the crystal edges between different quasicrystals.)*

EXAMPLE 3.3.10. In Figure 3.2, the blue highlighted quasicrystals depict Construction 3.3.9 for W_1 in type A_2 .

We say that a word $w = w_1 \cdots w_n \in \mathbb{B}^{\otimes n}$ is a *Yamanouchi word* if there are at least as many $\mathbf{i}$'s as $\mathbf{i}+\mathbf{1}$'s in $\{w_j, \dots, w_{n-1}, w_n\}$ for $1 \leq j \leq n$ and $\mathbf{i}, \mathbf{i}+\mathbf{1} \in \{\mathbf{1}, \dots, \mathbf{m}\}$. By [10, Proposition 8.2], a word $w \in \mathbb{B}^{\otimes n}$ is a highest weight element if and only if w is Yamanouchi.

PROPOSITION 3.3.11. The highest weight elements of the crystal from Construction 3.3.9 are the Yamanouchi words in $\mathbb{B}^{\otimes n}$ which standardize to an involution with k fixed points. The multiplicity of the Schur polynomial $s_\mu(x_1, \dots, x_m)$ in the character for the higher Lie module $\mathcal{L}_{\langle 1^k, 2^{\frac{n-k}{2}} \rangle}(V)$ is counted by the number of Yamanouchi words of weight μ which standardize to an involution with k fixed points.

REMARK 3.3.12. This case of Thrall's problem was already taken care of by the known cases $\lambda = \langle 1^k \rangle$ and $\lambda = \langle 2^{\frac{m-k}{2}} \rangle$ using the Littlewood-Richardson rule:

$$\begin{aligned} \text{ch}(\mathcal{L}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}(V)) &= \text{ch}\left(\text{Sym}^k(V) \otimes \text{Sym}^{\frac{m-k}{2}}(\mathcal{L}_2(V))\right) \\ &= \sum_{\substack{\mu \vdash n-k \\ \text{even columns}}} s_{(k)}(x_1, \dots, x_m) s_\mu(x_1, \dots, x_m) \\ &= \sum_{\substack{\nu \vdash n \\ k \text{ odd columns}}} s_\nu(x_1, \dots, x_m). \end{aligned}$$

REMARK 3.3.13. Construction 3.3.9 relies heavily upon how RSK acts on involutions, and cannot be easily extended to the general rectangular case when $\lambda = \langle a^b \rangle$. In particular, it is not true in general that words indexing the quasicrystals within a connected component in the higher Lie crystal will all have the same shape under RSK. For example, when $\lambda = \langle 4^1 \rangle$, the quasicrystals are indexed by the permutations (written in one-line notation)

$$[3421], [2413], [4312], [4123], [3142], [2341],$$

which map to the following shapes under RSK, respectively:

$$(2, 1, 1), (2, 2), (2, 1, 1), (3, 1), (2, 2), (3, 1).$$

However, $\text{ch}(\mathcal{L}_4(V))(x_1, x_2, x_3, x_4) = s_{(3,1)}(x_1, x_2, x_3, x_4) + s_{(2,1,1)}(x_1, x_2, x_3, x_4)$, demonstrating that St_{2413} and St_{3142} will need to be stitched into a crystal component of a different highest weight than their original crystal component in $\mathbb{B}^{\otimes 4}$.

3.3.4. A dual equivalence graph on involutions. In this section, we define operators $\tilde{D}_i$ for $2 \leq i \leq m-1$ on involutions in S_m which preserve cycle type and RSK insertion shape, for $m \geq 3$. We show that this local rule on involutions gives the dual equivalence graph inherited from mapping involutions to their RSK insertion tableaux in the original dual equivalence graph on standard Young tableaux defined by Assaf [4], and therefore mirror the higher Lie module crystal created in Construction 3.3.9. By constructing a dual equivalence graph on permutations of cycle type $\langle 1^k, 2^{\frac{m-k}{2}} \rangle$ for $k \equiv m \pmod{2}$, called $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$, we give a slightly new combinatorial description for coefficients in the Schur expansion of the higher Lie character $\text{ch}(\mathcal{L}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}(V)) = \sum_{\mu \vdash m} d_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}^{\mu} s_{\mu}$, where $d_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}^{\mu}$ is the number of connected dual equivalence graphs of weight μ in $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$.

Assaf [4] defines dual equivalence graphs on combinatorial objects with descent sets by defining edges via elementary dual equivalence operators. Dual equivalence was first explicitly defined by Haiman [21] as a relation on tableaux dual to jeu de taquin equivalence under the Schensted correspondence. For the dual equivalence graph on standard Young tableaux and the axiomatic definition of dual equivalence graph, see Sections 2 and 3 of [4].

DEFINITION 3.3.14 ([21]). *The elementary dual equivalence operator D_i , for $2 \leq i \leq m-1$, on a permutation $\pi \in S_m$ is defined as follows. If i lies between $i-1$ and $i+1$ in the one-line notation of π , then $D_i(\pi) = \pi$. Otherwise, D_i interchanges i and whichever of $i \pm 1$ is further away from i .*

For $\pi, \sigma \in S_m$, there is an edge (π, σ) labeled i in the dual equivalence graph on S_m if $D_i(\pi) = \sigma$ and $\pi \neq \sigma$. Thus, connected subgraphs of the dual equivalence graph on S_m contain all words which are connected by a sequence of elementary dual equivalence operators. If two permutations are connected by a sequence of elementary dual equivalence operators, then they have the same shape under RSK [21]. Connected subgraphs of the dual equivalence graph on S_m are indexed by the common shape of their vertices under RSK. The vertex $\pi \in S_m$ in the dual equivalence graph on S_m is also labeled with the descent set of π^{-1} .

Now we define the operators $\tilde{D}_i$ for $2 \leq i \leq m-1$ on involutions in S_m that we will use to construct dual equivalence graphs for the higher Lie modules indexed by involutions.

DEFINITION 3.3.15. Let $\pi \in S_m$ be an involution and $2 \leq i \leq m-1$. If i lies between $i-1, i+1$ in the one-line notation of π , then $\tilde{D}_i(\pi) = \pi$. Else, let i' denote whichever of $i-1, i+1$ is further from i in the one-line notation of π . If $\pi(i) \neq i$ and $\pi(i') \neq i'$ (that is, i and i' are both in 2-cycles), then $\tilde{D}_i(\pi) = D_i(\pi)(i, i')$. If either $\pi(i) = i$ or $\pi(i') = i'$, then

$$\tilde{D}_i(\pi) = \begin{cases} D_i(\pi)(i, i') & \text{if } \pi(i) \neq i \pm 1 \\ D_i(\pi)(i, i+1) & \text{if } \pi(i) = i+1 \\ D_i(\pi)(i, i-1) & \text{if } \pi(i) = i-1. \end{cases}$$

REMARK 3.3.16. The elementary dual equivalence involution D_i can be described as left action by the transposition (i, i') on π , which interchanges the letters i, i' in the one-line notation of π . The new operator $\tilde{D}_i$ applies the right action of the transposition $(i, i-1)$ or $(i, i+1)$ to $D_i(\pi)$, which interchanges the letters in positions $i, i-1$ or $i, i+1$.

EXAMPLE 3.3.17. Applying $\tilde{D}_3$ on $\pi = [132546]$, we identify $i' = 4$, $\pi(3) = 2$ and $\pi(4) = 5$. Since $\pi(3) \neq 3$ and $\pi(4) \neq 4$, we have $\tilde{D}_3([132546]) = D_3([132546]) \cdot (3, 4)$, so

$$\tilde{D}_3([132546]) = [142536] \cdot (3, 4) = [145236].$$

Now applying $\tilde{D}_3$ on $\sigma = [145236]$, we identify $i' = 4$, $\sigma(3) = 5$ and $\sigma(4) = 2$. Since $\sigma(3) \neq 3$ and $\sigma(4) \neq 4$, we have $\tilde{D}_3([145236]) = D_3([145236]) \cdot (3, 4)$, so

$$\tilde{D}_3([145236]) = [135246] \cdot (3, 4) = [132546].$$

EXAMPLE 3.3.18. Applying $\tilde{D}_2$ on $\pi = [563412]$, we identify $i' = 3$, $\pi(2) = 6$ and $\pi(3) = 3$. Since $\pi(2) \neq 2 \pm 1$, we have $\tilde{D}_2([563412]) = D_2([563412]) \cdot (2, 3)$, so

$$\tilde{D}_2([563412]) = [562413] \cdot (2, 3) = [526413].$$

Now applying $\tilde{D}_2$ on $\sigma = [526413]$, we identify $i' = 3$, $\sigma(2) = 2$ and $\sigma(3) = 6$. Since $\sigma(2) \neq 2 \pm 1$, we have $\tilde{D}_2([526413]) = D_2([526413]) \cdot (2, 3)$, so

$$\tilde{D}_2([526413]) = [536412] \cdot (2, 3) = [563412].$$

EXAMPLE 3.3.19. Applying $\tilde{D}_3$ on $\pi = [124365]$, we identify $i' = 2$, $\pi(3) = 4$ and $\pi(2) = 2$. Since $\pi(3) = 3 + 1$, we have $\tilde{D}_3([124365]) = D_3([124365]) \cdot (3, 4)$, so

$$\tilde{D}_3([124365]) = [134265] \cdot (3, 4) = [132465].$$

Now applying $\tilde{D}_3$ on $\sigma = [132465]$, we identify $i' = 4$, $\sigma(3) = 2$ and $\sigma(4) = 4$. Since $\sigma(3) = 3 - 1$, we have $\tilde{D}_3([132465]) = D_3([132465]) \cdot (3, 2)$, so

$$\tilde{D}_3([132465]) = [142365] \cdot (3, 2) = [124365].$$

Next we show that $\tilde{D}_i$ is a well-defined involution on involutions in S_m .

PROPOSITION 3.3.20. Let $\pi \in S_m$ with cycle type $\langle 1^k, 2^{\frac{m-k}{2}} \rangle$ and let $2 \leq i \leq m-1$, for $m \geq 3$. Then the cycle type of $\tilde{D}_i(\pi)$ is $\langle 1^k, 2^{\frac{m-k}{2}} \rangle$ and $\tilde{D}_i(\tilde{D}_i(\pi)) = \pi$.

PROOF. We first show cycle type is preserved under $\tilde{D}_i$. Let $\pi \in S_m$ with cycle type $\langle 1^k, 2^{\frac{m-k}{2}} \rangle$. If i falls between $i-1, i+1$ in the one-line notation of π , then $\tilde{D}_i(\pi) = \pi$ and the result is immediate. Assume i does not fall between $i-1, i+1$ and let i' denote whichever of $i-1, i+1$ is further from i . We divide the proof into four cases depending on whether i, i' are in a 2-cycle or a 1-cycle in π .

(I) Suppose $\pi(i) \neq i$ and $\pi(i') \neq i'$. By assumption, $\pi(i) \neq i'$ and $\pi(i') \neq i$ because π is an involution and i' cannot be adjacent to i . We may write $\pi = (i, \pi(i))(i', \pi(i'))\sigma$ for some $\sigma \in S_{[m] \setminus \{i, i', \pi(i), \pi(i')\}}$ with $\text{type}(\sigma) = \langle 1^k, 2^{\frac{m-k-4}{2}} \rangle$. We have

$$\tilde{D}_i(\pi) = (i, i')(i, \pi(i))(i', \pi(i'))\sigma(i, i') = (i, \pi(i'))(i', \pi(i))\sigma$$

which demonstrates that $\text{type}(\tilde{D}_i(\pi)) = \langle 1^k, 2^{\frac{m-k}{2}} \rangle$. Since $\tilde{D}_i(\pi) = D_i(\pi)(i, i')$, the only difference between $\tilde{D}_i(\pi)$ and $D_i(\pi)$ is the letters in positions i, i' . Since $\pi(i) \neq i'$ and $\pi(i') \neq i$, i' remains the further distance from i in the one-line notation of $\tilde{D}_i(\pi)$, and thus

$$\tilde{D}_i(\tilde{D}_i(\pi)) = \tilde{D}_i((i, i')\pi(i, i')) = (i, i')((i, i')\pi(i, i'))(i, i') = \pi.$$

(II) If $\pi(i) = i$, then $\pi(i') = j$ for some $j \in [m] \setminus \{i, i'\}$. We may write $\pi = (i)(i', j)\sigma$ for some $\sigma \in S_{[m] \setminus \{i, i', j\}}$ with $\text{type}(\sigma) = \langle 1^{k-1}, 2^{\frac{m-k-2}{2}} \rangle$. We have

$$\tilde{D}_i(\pi) = (i, i')(i)(i', j)\sigma(i, i') = (i, j)(i')\sigma,$$

which demonstrates that $\text{type}(\tilde{D}_i(\pi)) = \langle 1^k, 2^{\frac{m-k}{2}} \rangle$. Acting by (i, i') on the right of $D_i(\pi)$ only interchanges the letter i' in position i with the letter j in position i' in the one-line notation of $\tilde{D}_i(\pi)$. Since $j \neq i \pm 1$, i' remains the further distance from i in the one-line notation of $\tilde{D}_i(\pi)$, and thus

$$\tilde{D}_i(\tilde{D}_i(\pi)) = \tilde{D}_i((i, i')\pi(i, i')) = (i, i')((i, i')\pi(i, i'))(i, i') = \pi.$$

(III) If $\pi(i') = i'$, then $\pi(i) = j$ for some $j \in [m] \setminus \{i, i'\}$. Suppose $j \neq i \pm 1$. We may write $(i')(i, j)\sigma$ for some $\sigma \in S_{[m] \setminus \{i, i', j\}}$ with $\text{type}(\sigma) = \langle 1^{k-1}, 2^{\frac{m-k-2}{2}} \rangle$. We have

$$\tilde{D}_i(\pi) = (i, i')(i')(i, j)\sigma(i, i') = (i)(i', j)\sigma,$$

which demonstrates that $\text{type}(\tilde{D}_i(\pi)) = \langle 1^k, 2^{\frac{m-k}{2}} \rangle$. Acting by (i, i') on the right of $D_i(\pi)$ only interchanges the letter i in position i' with the letter j in position i in the one-line notation of $\tilde{D}_i(\pi)$. Since $j \neq i \pm 1$, i' remains the further distance from i in the one-line notation of $\tilde{D}_i(\pi)$, and thus

$$\tilde{D}_i(\tilde{D}_i(\pi)) = \tilde{D}_i((i, i')\pi(i, i')) = (i, i')((i, i')\pi(i, i'))(i, i') = \pi.$$

(IV) Now suppose $i' = i - 1$, $\pi(i - 1) = i - 1$ and $\pi(i) = i + 1$. We may write $\pi = (i - 1)(i, i + 1)\sigma$ for some $\sigma \in S_{[m] \setminus \{i - 1, i, i + 1\}}$ with $\text{type}(\sigma) = \langle 1^{k-1}, 2^{\frac{m-k-2}{2}} \rangle$. We have

$$\tilde{D}_i(\pi) = (i, i - 1)(i - 1)(i, i + 1)\sigma(i, i + 1) = (i, i - 1)(i + 1)\sigma$$

which demonstrates that $\text{type}(\tilde{D}_i(\pi)) = \langle 1^k, 2^{\frac{m-k}{2}} \rangle$. The one-line notation of $\tilde{D}_i(\pi)$ contains $i i - 1 i + 1$, and so $i + 1$ is now further away from i and $\tilde{D}_i(\pi)(i) = i - 1$. Therefore,

$$\tilde{D}_i(\tilde{D}_i(\pi)) = (i, i + 1)((i, i - 1)(i - 1)(i, i + 1)\sigma(i, i + 1))(i, i - 1) = (i, i + 1)(i - 1)\sigma = \pi.$$

The case where $i' = i + 1$, $\pi(i + 1) = i + 1$, and $\pi(i) = i - 1$ is demonstrated with a similar argument. $\square$

By Proposition 3.3.20, we can now define a graph on involutions of fixed cycle type.

DEFINITION 3.3.21. For $m \geq 3$ and $k \equiv m \pmod{2}$, let $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle} = (V, \text{des}, E)$ be the labeled graph with the following properties:

- vertex set $V_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle} = \{\pi \in S_m \mid \text{type}(\pi) = \langle 1^k, 2^{\frac{m-k}{2}} \rangle\}$,
- vertex $\pi \in V_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$ is labeled by its descent set,
- and $(\pi, \sigma) \in V_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle} \times V_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$ is an edge labeled i if and only if $\tilde{D}_i(\pi) = \sigma$ and $\pi \neq \sigma$.

We next show that all vertices in a connected component in $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$ have the same shape under RSK. Furthermore, we show that a connected component of shape μ in $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$ is isomorphic to a dual equivalence graph of highest weight μ , as defined in Section 3.1 of [4]. Recall the elementary plactic relations stated in Theorem 3.3.8.

PROPOSITION 3.3.22. For an involution $\pi \in S_m$, $\tilde{D}_i(\pi)$ is plactically equivalent to $D_i(\pi)$ for $2 \leq i \leq m-1$.

PROOF. If i falls between $i-1, i+1$, then $D_i(\pi) = \pi = \tilde{D}_i(\pi)$ and the result is immediate. Now suppose $i-1$ is the further letter from i in the one-line notation of π . The case where $i+1$ is further from i follows from a similar argument.

First consider the case when $\pi(i-1) = i-1$ and $\pi(i) = i+1$, which means $\tilde{D}_i(\pi) = D_i(\pi)(i, i+1)$. By the elementary plactic relation $bca \equiv bac$ where $a < b \leq c$, we have (in two-line notation)

$$\begin{aligned}
 D_i(\pi) &= \begin{array}{cccccc} \cdots & i-1 & & i & & i+1 & \cdots \\ \cdots & & i & & i+1 & & i-1 & \cdots \end{array} \\
 &\equiv \begin{array}{cccccc} \cdots & i-1 & & i & & i+1 & \cdots \\ \cdots & & i & & i-1 & & i+1 & \cdots \end{array} \\
 &= D_i(\pi)(i, i+1) = \tilde{D}_i(\pi).
 \end{aligned}$$

In the remaining cases, $\tilde{D}_i(\pi) = D(\pi)(i-1, i)$ and so the only difference between $D_i(\pi)$ and $\tilde{D}_i(\pi)$ is the two letters interchanged in positions $i-1$ and i . Since π is an involution and $i+1$ falls between $i-1$ and i , either $\pi(i) < \pi(i+1) < \pi(i-1)$ or $\pi(i-1) < \pi(i+1) < \pi(i)$. We verify

that an application of the elementary plactic relation

$$acb \equiv cab \text{ for } a \leq b < c$$

interchanges these two letters in positions $i - 1$ and i , proving that $D_i(\pi) \equiv \tilde{D}_i(\pi)$. We divide the proof into three cases depending on whether $i - 1$ or i are in a 2-cycle or a 1-cycle in π .

(I) Suppose $\pi(i - 1) \neq i - 1$ and $\pi(i) \neq i$. If $\pi(i) \neq i + 1$, then we have

$$\begin{aligned} D_i(\pi) &= \begin{array}{ccccccccc} & & & i-1 & & i & & i+1 & & \\ \cdots & & & & & & & & & \cdots \\ & & & \pi(i-1) & & \pi(i) & & \pi(i+1) & & \cdots \end{array} \\ &\equiv_K \begin{array}{ccccccccc} & & & i-1 & & i & & i+1 & & \\ \cdots & & & & & & & & & \cdots \\ & & & \pi(i) & & \pi(i-1) & & \pi(i+1) & & \cdots \end{array} \\ &= D_i(\pi)(i-1, i) = \tilde{D}_i(\pi). \end{aligned}$$

If $\pi(i) = i + 1$, then $\pi(i - 1) < i - 1$ and we have

$$\begin{aligned} D_i(\pi) &= \begin{array}{ccccccccc} & & & i-1 & & i & & i+1 & & \\ \cdots & & & & & & & & & \cdots \\ & & & \pi(i-1) & & i+1 & & i-1 & & \cdots \end{array} \\ &\equiv_K \begin{array}{ccccccccc} & & & i-1 & & i & & i+1 & & \\ \cdots & & & & & & & & & \cdots \\ & & & i+1 & & \pi(i-1) & & i-1 & & \cdots \end{array} \\ &= D_i(\pi)(i-1, i) = \tilde{D}_i(\pi). \end{aligned}$$

(II) Suppose $\pi(i) = i$. We have

$$\begin{aligned} D_i(\pi) &= \begin{array}{ccccccccc} & & & i-1 & & i & & i+1 & & \\ \cdots & & & & & & & & & \cdots \\ & & & \pi(i-1) & & i-1 & & \pi(i+1) & & \cdots \end{array} \\ &\equiv_K \begin{array}{ccccccccc} & & & i-1 & & i & & i+1 & & \\ \cdots & & & & & & & & & \cdots \\ & & & i-1 & & \pi(i-1) & & \pi(i+1) & & \cdots \end{array} \\ &= D_i(\pi)(i-1, i) = \tilde{D}_i(\pi). \end{aligned}$$

(III) Suppose $\pi(i-1) = i-1$ and $\pi(i) \neq i+1$. We have

$$\begin{aligned}
D_i(\pi) &= \begin{array}{cccccc} \cdots & i-1 & i & i+1 & \cdots \\ \cdots & i & \pi(i) & \pi(i+1) & \cdots \end{array} \\
&\equiv_K \begin{array}{cccccc} \cdots & i-1 & i & i+1 & \cdots \\ \cdots & \pi(i) & i & \pi(i+1) & \cdots \end{array} \\
&= D_i(\pi)(i-1, i) = \tilde{D}_i(\pi).
\end{aligned}$$

□

Since plactically equivalent words have the same RSK insertion tableau (Theorem 3.3.8), Proposition 3.3.22 implies that $P(\tilde{D}_i(\pi)) = P(D_i(\pi))$ for any involution $\pi \in S_m$.

COROLLARY 3.3.23. *For an involution $\pi \in S_m$, the operators $\tilde{D}_i$ for $2 \leq i \leq m-1$ preserve RSK insertion shape.*

PROOF. This follows from the fact that the shape of $P(\pi)$ is equal to the shape of $P(D_i(\pi))$. □

COROLLARY 3.3.24. *For an involution $\pi \in S_m$, the descent set of $\tilde{D}_i(\pi)$ is equal to the descent set of the inverse of $D_i(\pi)$. Thus both $\tilde{D}_i(\pi)$ and $D_i(\pi)$ are labeled by the same set in their respective graphs.*

PROOF. This follows from the fact that for an involution $\sigma \in S_m$, the descent set of σ is equal to the descent set of $P(\sigma)$, and for any permutation $\omega \in S_m$, the descent set ω^{-1} is equal to the descent set of $P(\omega)$. □

THEOREM 3.3.25. *For $m \geq 3$ and $k \equiv m \pmod{2}$, the labeled graph $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$ is a dual equivalence graph. Specifically, if $\pi \in V_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$ with the shape of $P(\pi)$ equal to μ , then π is a vertex in a connected component of $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$ which is isomorphic to the dual equivalence graph of highest weight μ that contains π and is generated by the elementary dual equivalence operators.*

PROOF. For $\pi \in V_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$, let G_π be the connected component of $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$ containing π . Consider the map $\phi_\pi : G_\pi \rightarrow S_m$ generated by $\phi_\pi(\pi) = \pi$, meaning for any sequence of operators

$\tilde{D}_{i_1}, \dots, \tilde{D}_{i_\ell}$, we have

$$\phi_\pi(\tilde{D}_{i_\ell} \cdots \tilde{D}_{i_1}(\pi)) := D_{i_\ell} \cdots D_{i_1}(\pi).$$

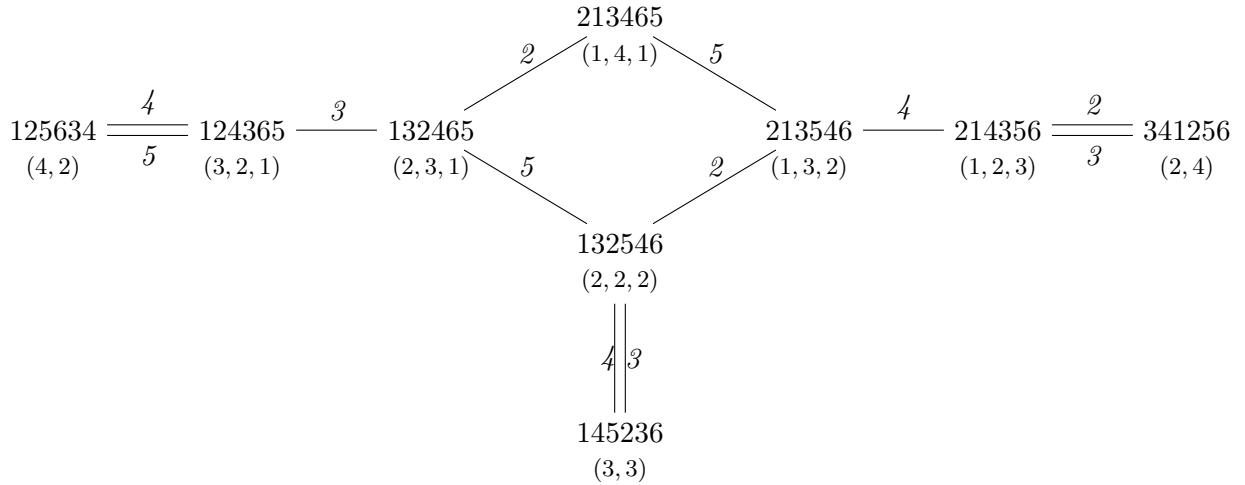
By Proposition 3.3.22 and Corollaries 3.3.23 and 3.3.24, ϕ_π is an injective dual equivalence graph morphism (Definition 3.5 [4]). $\square$

REMARK 3.3.26. Theorem 3.3.25 implies that each involution of type $\langle 1^k, 2^{\frac{m-k}{2}} \rangle$ in S_m is placed in its natural position in $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$. This shows that the dual equivalence graphs created in Definition 3.3.21 match the higher Lie crystals for involutions created in Construction 3.3.9.

In conclusion, the coefficients in the Schur expansion of the higher Lie character indexed by the involution $\langle 1^k, 2^{\frac{m-k}{2}} \rangle$ can be described by the number of connected components in the dual equivalence graph $\mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}$ of a given weight. That is, $\text{ch}(\mathcal{L}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}(V)) = \sum_{\mu \vdash m} d_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}^\mu s_\mu$, where

$$d_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}^\mu = \#\{\text{connected dual equivalence graphs of weight } \mu \text{ in } \mathcal{G}_{\langle 1^k, 2^{\frac{m-k}{2}} \rangle}\}.$$

EXAMPLE 3.3.27. Two connected components of $\mathcal{G}_{(1^2, 2^2)}$ are shown below, which are isomorphic to the dual equivalence graphs on standard Young tableaux of shape $(4, 2)$ and $(2, 2, 2)$. The edges come from the (nontrivial) action of the operators $\tilde{D}_2, \tilde{D}_3, \tilde{D}_4, \tilde{D}_5$. Underneath each vertex is the descent composition (see the correspondence in equation 3.1 to get to the descent sets).



$$\begin{array}{ccccc}
563412 & \xlongequal[3]{2} & 526413 & & \\
(2, 2, 2) & & (1, 2, 1, 2) & & \\
5 & \parallel & 4 & & 5 \\
& & & & | \\
463152 & \xlongequal[2]{} & 426153 & \xlongequal[4]{3} & 321654 \\
(2, 1, 2, 1) & & (1, 2, 2, 1) & & (1, 1, 2, 1, 1)
\end{array}$$

CHAPTER 4

The immersion poset on partitions

This chapter is based on work in collaboration with Lisa Johnston, David Kenepp, Evuilynn Nguyen, Digjoy Paul, Anne Schilling, and Regina Zhou in [25]. My specific contributions include investigations of maximal elements in Section 4.2.1 (specifically Proposition 4.2.11 and Proposition 4.2.12) and the study of Schur-positive interval power sums in Section 4.3.

4.1. Background and Definitions

We define a new poset on partitions to model the following representation theoretic phenomenon observed in polynomial representations of $\mathrm{GL}_N(\mathbb{C})$. Given two finite-dimensional representations $\pi_1: G \rightarrow \mathrm{GL}(W_1)$ and $\pi_2: G \rightarrow \mathrm{GL}(W_2)$ of a group G , we say that the representation π_1 is *immersed* in the representation π_2 if the eigenvalues of $\pi_1(g)$, counting multiplicities, are contained in the eigenvalues of $\pi_2(g)$ for all $g \in G$. In this case, we call (W_1, W_2) an *immersion pair* denoted by $W_1 \leqslant_I W_2$. Note that, if π_1 is a subrepresentation of π_2 , then $W_1 \leqslant_I W_2$, but the converse is not true.

QUESTION 4.1.1 (Prasad and Raghunathan [50]). *Classify immersion of representations $W_1 \leqslant_I W_2$ for a given group.*

Recently, some progress was made on the above problem for symmetric groups [44] and alternating groups [43]. In this chapter, we study immersion pairs for finite-dimensional irreducible polynomial representations of the general linear group $\mathrm{GL}_N(\mathbb{C})$.

4.1.1. Immersion of Weyl modules: the immersion poset. For a partition λ of n with length $\ell(\lambda) \leqslant N$, let

$$\rho_\lambda: \mathrm{GL}_N(\mathbb{C}) \longrightarrow \mathrm{GL}(W_\lambda(\mathbb{C}^N))$$

be the irreducible polynomial representation of degree n of highest weight λ . It is known that if $g \in \mathrm{GL}_N(\mathbb{C})$ has the eigenvalues $x_1, \dots, x_N$, then the eigenvalues of $\rho_\lambda(g)$ are the monomials appearing in the Schur polynomial $s_\lambda(x_1, \dots, x_N)$ (see, for example, [60, Chapter 7]). Thus, given

two partitions λ, μ of n with $\ell(\lambda), \ell(\mu) \leq N$, the Weyl module $W_\lambda(\mathbb{C}^N)$ is immersed in $W_\mu(\mathbb{C}^N)$ if and only if $s_\mu(x_1, \dots, x_N) - s_\lambda(x_1, \dots, x_N)$ is monomial-positive. Hence studying the immersion of Weyl modules is equivalent to studying monomial positivity of the difference of Schur polynomials.

DEFINITION 4.1.2. *We define a partial order on $\mathcal{P}(n)$ as follows. For $\lambda, \mu \in \mathcal{P}(n)$ and $N \geq n$, we define $\lambda \leq_I \mu$ if $s_\mu(x_1, \dots, x_N) - s_\lambda(x_1, \dots, x_N)$ is monomial-positive. We call the poset $(\mathcal{P}(n), \leq_I)$ the **immersion poset**.*

Recall the monomial expansion of the Schur polynomial s_λ for $\lambda \vdash n$,

$$(4.1) \quad s_\lambda(x_1, \dots, x_N) = \sum_{\mu \vdash n} K_{\lambda, \mu} m_\mu(x_1, \dots, x_N),$$

where $K_{\lambda, \mu}$ are the **Kostka numbers** which count the number of semistandard Young tableaux of shape λ and content μ . Note that with this definition the Schur polynomials are zero unless $N \geq \ell(\lambda)$, that is, the number of variables needs to be at least as large as the number of parts in λ .

LEMMA 4.1.3. *For $\lambda, \mu \in \mathcal{P}(n)$, $\lambda \leq_I \mu$ if $K_{\lambda, \alpha} \leq K_{\mu, \alpha}$ for all $\alpha \in \mathcal{P}(n)$.*

PROOF. By Definition 4.1.2, two partitions $\lambda, \mu \in \mathcal{P}(n)$ are comparable in the immersion poset $\lambda \leq_I \mu$ if

$$s_\mu(x_1, \dots, x_N) - s_\lambda(x_1, \dots, x_N)$$

is monomial-positive. Using (4.1), this can be restated as saying $\lambda \leq_I \mu$ if $K_{\lambda, \alpha} \leq K_{\mu, \alpha}$ for all $\alpha \in \mathcal{P}(n)$. $\square$

To summarize, immersions of irreducible polynomial representations of $\mathrm{GL}_N(\mathbb{C})$ are recorded in the immersion poset on partitions and the enumeration of semistandard Young tableaux. We next turn to our results about structure within the immersion poset. Later we apply the immersion poset to a question in representation theory of the symmetric group.

4.2. The Immersion Poset

In Section 4.2.1, we classify certain maximal elements in the immersion poset by defining a refinement called the standard immersion poset. In Section 4.2.2, we identify covers and relations in the immersion poset by constructing explicit weight preserving injections between sets of semistandard Young tableaux and analyzing induced subposets of the immersion poset.

4.2.1. The standard immersion poset. Lemma 4.1.3 implies that a necessary condition for $\lambda \leq_I \mu$ is that $K_{\lambda, (1^n)} \leq K_{\mu, (1^n)}$, which count the standard Young tableaux of shape λ and μ , respectively. Note that $f^\lambda := K_{\lambda, (1^n)}$ is also the dimension of the Specht module V_λ (the irreducible representation of S_n) indexed by λ .

The Kostka matrix $(K_{\lambda, \alpha})_{\lambda, \alpha \in \mathcal{P}(n)}$ is unit upper-triangular with respect to dominance order, that is, $K_{\lambda, \lambda} = 1$ and $K_{\lambda, \alpha} = 0$ unless $\alpha \leq_D \lambda$. This implies another necessary condition for $\lambda \leq_I \mu$, namely $\lambda \leq_D \mu$. This motivates the definition of the standard immersion poset.

DEFINITION 4.2.1. *On $\mathcal{P}(n)$, define $\lambda \leq_{std} \mu$ if $\lambda \leq_D \mu$ in dominance order and $f^\lambda \leq f^\mu$. We call this poset the **standard immersion poset**.*

As argued above, the standard immersion poset is a refinement of the immersion poset, that is, $\lambda \leq_I \mu$ implies that $\lambda \leq_{std} \mu$. The converse is not always true. For $n \geq 12$, there are examples of $\lambda \leq_{std} \mu$, which do not satisfy $\lambda \leq_I \mu$. For example $(5, 3, 1, 1, 1, 1)$ covers $(4, 2, 2, 2, 1, 1)$ in the standard immersion poset for $n = 12$, but not in the immersion poset.

EXAMPLE 4.2.2. *The immersion poset for $n = 8$ is given in Figure 4.1. It is equal to the standard immersion poset. The first time the immersion and standard immersion poset differ is for $n = 12$.*

4.2.1.1. Properties of the standard immersion poset. We now state and prove properties of the standard immersion poset. Our main tool is the **hook length formula** for $\lambda \in \mathcal{P}(n)$

$$(4.2) \quad f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

where $h(u)$ is the hook length of the cell u in λ which counts the cells weakly to the right of u and strictly below u (in English notation for partitions).

We write $\lambda <_{std} \mu$ if μ covers λ in the standard immersion poset. More precisely, $\lambda <_{std} \mu$ if $\lambda \leq_{std} \mu$ and there does not exist any ν such that $\lambda <_{std} \nu <_{std} \mu$.

LEMMA 4.2.3. *The partition (1^n) is the unique minimal element in the standard immersion poset.*

PROOF. The partition (1^n) is the unique minimal element in dominance order. Furthermore, $f^{(1^n)} = 1 \leq f^\lambda$ for all $\lambda \in \mathcal{P}(n)$. This proves the claim. $\square$

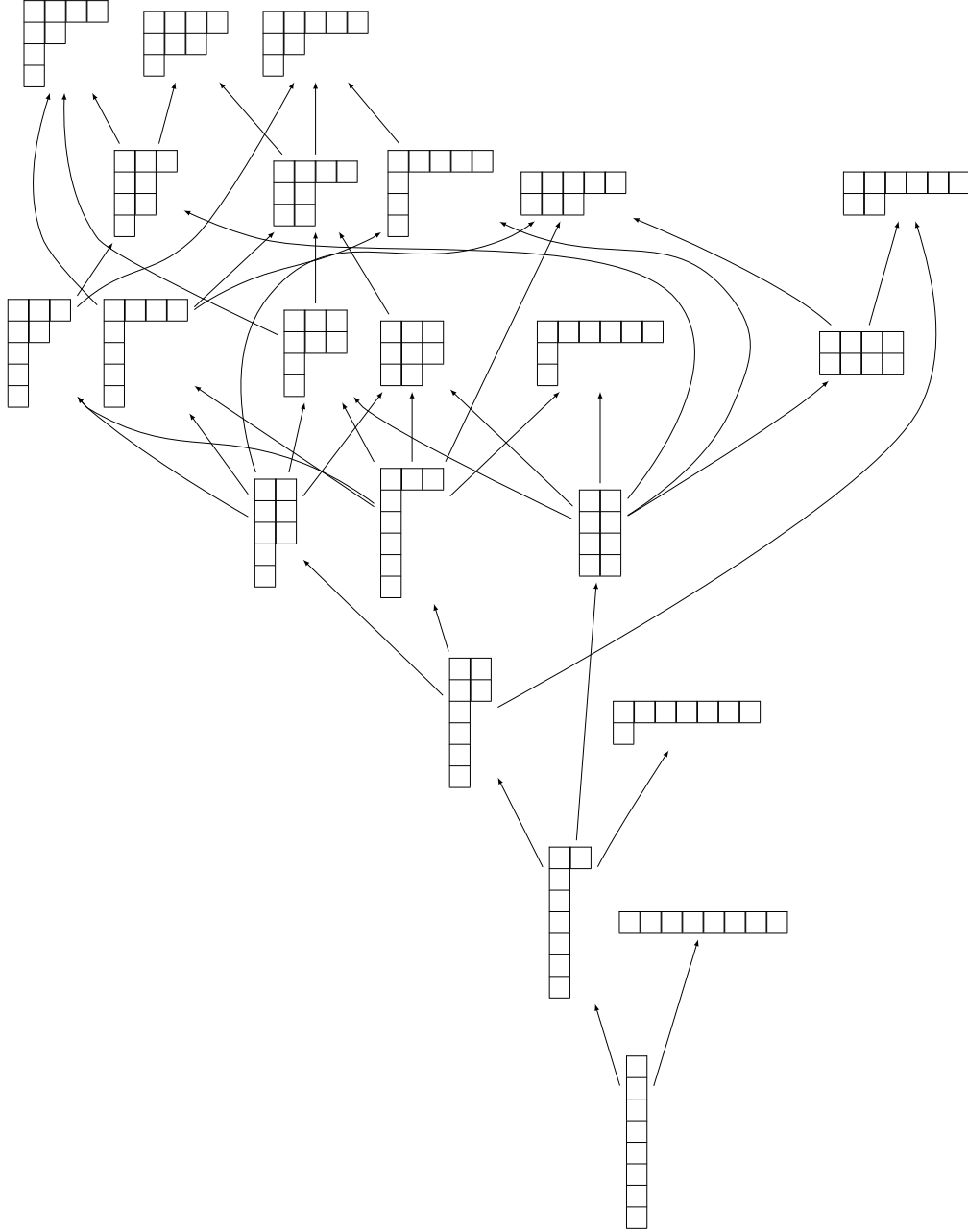


FIGURE 4.1. The (standard) immersion poset for $n = 8$.

LEMMA 4.2.4. *We have*

- (1) $(1^n) \leq_{std} (n)$ for all n and
- (2) $(2, 1^{n-2}) \leq_{std} (n-1, 1)$ for all $n > 3$ with $n \neq 6$.

PROOF. We have $(1^n) <_D (n)$ and $f^{(1^n)} = f^{(n)} = 1$. There is no other partition λ with $f^\lambda = 1$. This implies $(1^n) \leq_{std} (n)$. Similarly, $(2, 1^{n-2}) <_D (n-1, 1)$ and $f^{(2, 1^{n-2})} = f^{(n-1, 1)} =$

$n - 1$. There is no other partition λ with $f^\lambda = n - 1$ for n in the specified range. This implies $(2, 1^{n-2}) \prec_{std} (n - 1, 1)$. $\square$

REMARK 4.2.5.

- (1) Let $\lambda \prec_{std} \mu$. If μ covers λ in dominance order, then μ covers λ with respect to $\prec_{std}$. The converse is not true. Take $\lambda = (1^n)$ and $\mu = (n)$.
- (2) For a given partition λ with transpose λ^t , if $\lambda \prec_D \lambda^t$, then $\lambda \prec_{std} \lambda^t$ as both representations have the same dimension, that is, $f^\lambda = f^{\lambda^t}$. In general, λ^t does not cover λ .

Given a partition λ such that $\lambda \prec_D \lambda^t$, it would be interesting to find all partitions $\lambda \prec_D \nu \prec_D \lambda^t$ satisfying $f^\lambda = f^\nu$. This would help to understand when the transpose of λ covers λ in the immersion poset.

LEMMA 4.2.6. Let $\lambda = (2^a, 1^b)$ and $\mu = (2^{a+1}, 1^{b-2})$. Then $\lambda \prec_{std} \mu$ if and only if $\frac{b(b-1)}{2} > a$.

PROOF. We have $\lambda \prec_D \mu$. Hence by Remark 4.2.5(1), it suffices to show that $\lambda \prec_{std} \mu$. By the hook length formula, this is true if $\frac{f^\lambda}{f^\mu} = \frac{(b+1)(a+1)}{(b-1)(a+b+1)} \leq 1$, which is equivalent to the condition $\frac{b(b-1)}{2} > a$. $\square$

4.2.1.2. *Classifying maximal elements.* In this section, we study the maximal elements of the standard immersion poset. Recall that the standard immersion poset is a refinement of the immersion poset. This implies that if a partition is maximal in the standard immersion poset, then it is also maximal in the immersion poset.

PROPOSITION 4.2.7. The partition $(a + b, a)$ is a maximal element in the standard immersion poset if and only if $\frac{b(b+3)}{2} \geq a$.

PROOF. Let $\lambda = (a + b, a)$. Any partition ν which dominates λ , that is, $\nu \succ_D \lambda$, must have the form $\nu = \nu^{(i)} = (a + b + i, a - i)$ for some $i \geq 1$. Note that $\frac{f^{\nu^{(1)}}}{f^\lambda} = \frac{a(b+3)}{(b+1)(a+b+2)}$. Hence $f^{\nu^{(1)}} < f^\lambda$ if and only if $\frac{b(b+3)}{a-1} > 2$ (which is equivalent to $\frac{b(b+3)}{2} \geq a$). Thus, the condition is necessary.

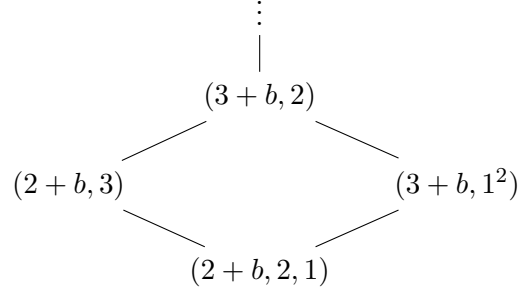
To prove that the condition $\frac{b(b+3)}{a-1} > 2$ is sufficient, note that

$$\frac{f^{\nu^{(i+1)}}}{f^{\nu^{(i)}}} = \frac{(a - i)(b + 3 + 2i)}{(b + 1 + 2i)(a + b + i + 2)} < \frac{a(b + 3)}{(b + 1)(a + b + 2)} = \frac{f^{\nu^{(1)}}}{f^\lambda}.$$

Since $f^{\nu^{(1)}} < f^\lambda$ when $\frac{b(b+3)}{a-1} > 2$, we must have $\frac{f^{\nu^{(i+1)}}}{f^{\nu^{(i)}}} < 1$. This is true for each i . Hence λ is a maximal element. $\square$

PROPOSITION 4.2.8. *Let $\lambda = (a + b, a, 1)$ where $a \geq 2$. Then λ is maximal in the standard immersion poset if and only if $a \leq \frac{(b+1)(b+2)}{2}$.*

PROOF. We first prove the reverse direction by inducting on a . For our base case, let $a = 2$ and $2 \leq \frac{(b+1)(b+2)}{2}$. To prove that $\lambda = (2 + b, 2, 1)$ is maximal, we show that there exists no partition ν such that $\lambda <_D \nu$ and $f^\lambda < f^\nu$. We start by classifying all partitions ν such that $\lambda <_D \nu$. It is known that $\lambda <_D \nu$ if and only if the Young diagram of ν can be obtained from the Young diagram of λ by moving a single box in row k to row $k - 1$ or by moving a single box in column k to column $k + 1$. This means that the partition $(2 + b, 2, 1)$ has exactly two covers: $(2 + b, 3)$ and $(3 + b, 1^2)$. The former is obtained by moving the box in row 3 to row 2, and the latter is obtained by moving the box at the end of row 2 to row 1. Furthermore, $(2 + b, 3)$ and $(3 + b, 1^2)$ are only covered by $(3 + b, 2)$. Below is the Hasse diagram in dominance order summarizing the specific covering relations:



Let ν be any partition such that $\lambda <_D \nu$. By our covering relations, we have that either $\nu = (2 + b, 3)$ or ν is contained in some chain $\lambda <_D (3 + b, 1^2) <_D \cdots <_D \nu$.

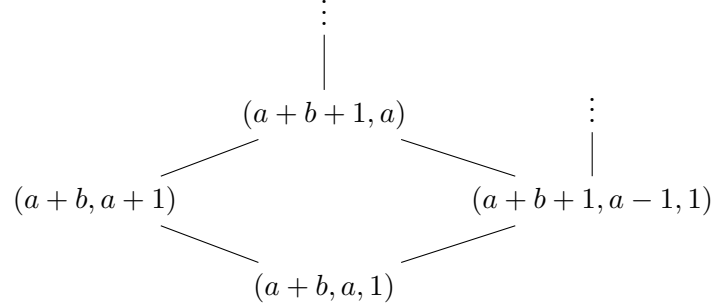
Now, we will show that $f^\lambda > f^\nu$ for all ν such that $\lambda <_D \nu$.

By Proposition 4.2.11, we know that $(3 + b, 1^2)$ is maximal in the standard immersion poset. That is, if $(3 + b, 1^2) <_D \nu$ then $f^{(3+b,1^2)} > f^\nu$. Note that our assumption that $2 \leq \frac{(b+1)(b+2)}{2}$ implies $b \geq 1$. By this fact and the hook length formula,

$$\frac{f^{(2+b,3)}}{f^\lambda} = \frac{b(b+4)}{2(b+1)(b+3)} < 1 \quad \text{and} \quad \frac{f^{(3+b,1^2)}}{f^\lambda} = \frac{3(b+4)}{2(b+1)(b+5)} < 1.$$

Since $f^\lambda > f^{(2+b,3)}$ and $f^\lambda > f^{(3+b,1^2)} > f^\nu$, we have shown that $f^\lambda > f^\nu$ for all ν such that $\lambda <_D \nu$.

Now, let $\lambda = (a+b, a, 1)$ where $a \leq \frac{(b+1)(b+2)}{2}$ and suppose that for some $a \geq 2$, the partition $(c+b, c, 1)$ is maximal when $c < a \leq \frac{(b+1)(b+2)}{2}$. We follow a similar argument as the base case and show that $f^\lambda > f^\nu$ for $\lambda <_D \nu$. Observe that the Hasse diagram in dominance order around λ looks as follows:



We first consider the partition $(a+b, a+1)$. Then by the hook length formula,

$$\frac{f^{(a+b, a+1)}}{f^\lambda} = \frac{(a+b+2)(b)}{(a+b+1)(a)(b+1)} < \frac{(a+b+2)}{(a+b+1)(a)} < 1$$

where the last inequality follows since $a \geq 2$.

Next, consider the partition $(a+b+1, a-1, 1)$. By our inductive hypothesis, $(a+b+1, a-1, 1) = ((a-1) + (b+2), a-1, 1)$ is maximal since $a-1 < a \leq \frac{(b+1)(b+2)}{2} < \frac{(b+3)(b+4)}{2}$. Suppose that a is the upper bound of our inequality $a \leq \frac{(b+1)(b+2)}{2}$, that is, $a = \frac{(b+1)(b+2)}{2}$. Then by the hook length formula,

$$\begin{aligned} (4.3) \quad \frac{f^{(a+b+1, a-1, 1)}}{f^\lambda} &= \frac{(a+b+2)(a+1)(a-1)(b+3)}{(a+b+3)(a+b+1)(a)(b+1)} \\ &= \frac{((b+1)(b+2) + 2b + 4)((b+1)(b+2) + 2)((b+1)(b+1-2))(2b+6)}{((b+1)(b+2) + 2b + 6)((b+1)(b+2) + 2b + 2)((b+1)(b+2))(2b+2)} \\ &= \frac{b^7 + 14b^6 + 82b^5 + 260b^4 + 477b^3 + 486b^2 + 216b}{b^7 + 14b^6 + 82b^5 + 260b^4 + 477b^3 + 502b^2 + 280b + 64} < 1. \end{aligned}$$

It follows that if $a < \frac{(b+1)(b+2)}{2}$ then $f^{(a+b+1, a-1, 1)} < f^\lambda$ because for fixed b , Equation (4.3) decreases as a decreases. To see this, we examine the effect of decreasing a on $\frac{a+b+2}{a+b+3}$, $\frac{a+1}{a+b+1}$, and $\frac{a-1}{a}$ individually. Each of these factors is of the form $\frac{x}{x+d}$ for fixed $d > 0$. Notice that $g(x) = \frac{x}{x+d}$ is a strictly increasing function for $x > 0$. Therefore, each of the above factors decreases as a decreases.

Thus, we have shown that for all ν such that $\lambda <_D \nu$, $f^\lambda > f^{(a+b, a+1)}$ and $f^\lambda > f^{(a+b+1, a-1, 1)} > f^\nu$. Hence, $(a+b, a, 1)$ is maximal whenever $a \leq \frac{(b+1)(b+2)}{2}$.

Now in the reverse direction, if $a > \frac{(b+1)(b+2)}{2}$, then $f^{(a+b+1, a-1, 1)} > f^\lambda$. To see this it suffices to consider $a = \frac{(b+1)(b+2)}{2} + 1$ since Equation (4.3) increases as a increases for the same reason as above. If $a = \frac{(b+1)(b+2)}{2} + 1$, then

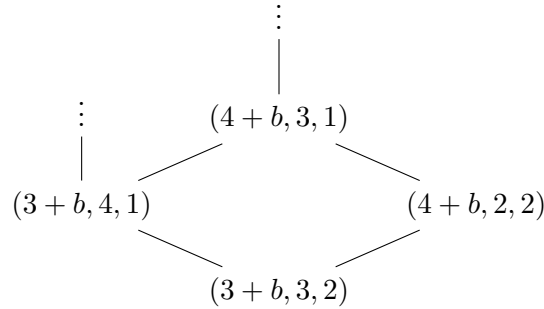
$$\frac{f^{(a+b+1, a-1, 1)}}{f^\lambda} = \frac{b^7 + 14b^6 + 88b^5 + 322b^4 + 739b^3 + 1056b^2 + 852b + 288}{b^7 + 14b^6 + 88b^5 + 318b^4 + 707b^3 + 964b^2 + 740b + 240} > 1.$$

Therefore, λ is maximal if only if $a \leq \frac{(b+1)(b+2)}{2}$. $\square$

PROPOSITION 4.2.9. *Let $\lambda = (a+b, a, 2)$ where $a \geq 3$. Then λ is maximal in the standard immersion poset if and only if $a \leq \frac{(b+1)(b+2)}{2}$.*

PROOF. We first prove the reverse direction by inducting on a . For our base case, let $a = 3$ and $3 \leq \frac{(b+1)(b+2)}{2}$. To prove that $\lambda = (3+b, 3, 2)$ is maximal, we follow a similar argument to Proposition 4.2.8. We first classify all partitions ν such that $\lambda <_D \nu$ and then show that $f^\lambda > f^\nu$ for all such ν by finding chains in the dominance order that contain maximal elements from the standard immersion poset. Our assumption that $3 \leq \frac{(b+1)(b+2)}{2}$ implies that $b \geq 1$. Hence it suffices to show that $\lambda = (3+b, 3, 2)$ is maximal for all $b \geq 1$. We consider the cases $b = 1, b = 2$, and $b \geq 3$ separately. It can be checked explicitly (for example using SAGEMATH [66]) that $(4, 3, 2)$ and $(5, 3, 2)$ are maximal in the standard immersion poset.

For $b \geq 3$, the Hasse diagram in dominance order around $\lambda = (3+b, 3, 2)$ looks as follows:



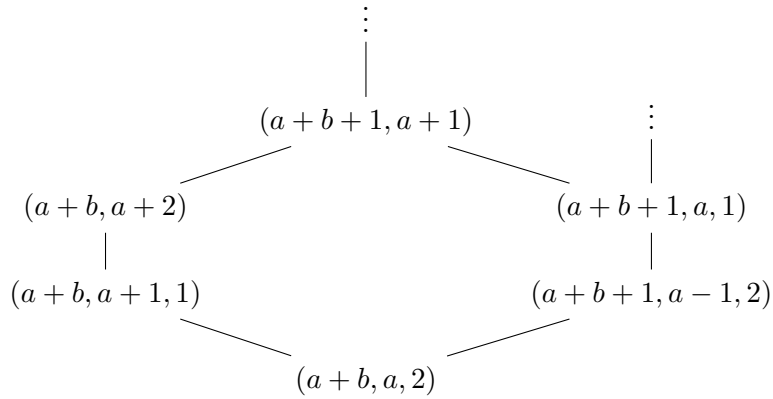
If $\lambda <_D \nu$ then $\nu = (3+b, 4, 1)$, $(4+b, 2, 2)$, or ν is contained in some chain $\lambda <_D (3+b, 4, 1) <_D \nu$. By Proposition 4.2.8, $(3+b, 4, 1)$ is maximal in the standard immersion poset so it suffices to show

that $f^\lambda > f^\nu$ for $\nu = (3+b, 4, 1)$ and $(4+b, 2, 2)$. By the hook length formula,

$$\frac{f^{(3+b,4,1)}}{f^\lambda} = \frac{4(b)(b+4)}{5(b+1)(b+3)} = \frac{4(b^2+4b)}{5(b^2+4b+3)} < 1 \quad \text{and} \quad \frac{f^{(4+b,2,2)}}{f^\lambda} = \frac{4(b+4)}{2(b+1)(b+6)} < 1.$$

Hence, for $b \geq 3$, $(3 + b, 3, 1)$ is maximal in the standard immersion poset, so we have shown that $(3 + b, 3, 1)$ is maximal for all $b \geq 1$.

Now, let $\lambda = (a + b, a, 2)$ where $a \leq \frac{(b+1)(b+2)}{2}$ and suppose that for some $a \geq 3$, the partition $(c + b, c, 2)$ is maximal when $c < a \leq \frac{(b+1)(b+2)}{2}$. Again, we show that $f^\lambda > f^\nu$ for $\lambda <_D \nu$. Observe that the Hasse diagram in dominance order around λ looks as follows:



By the Hasse diagram, if ν is a partition such that $\lambda <_D \nu$, then $\nu = (a + b, a + 1, 1), (a + b, a + 2), (a + b + 1, a - 1, 2)$, or ν is contained in some chain $\lambda <_D (a + b + 1, a - 1, 2) <_D \nu$. Observe that $(a + b + 1, a - 1, 2) = ((a - 1) + (b + 2), a - 1, 2)$ is maximal by our inductive hypothesis since $a - 1 < a \leq \frac{(b+1)(b+2)}{2} \leq \frac{(b+3)(b+4)}{2}$. Therefore, it suffices to check that $f^\lambda > f^\nu$ for $\nu = (a + b, a + 1, 1), (a + b, a + 2)$, and $(a + b + 1, a - 1, 2)$.

For $\nu = (a + b, a + 1, 1)$, we have that

$$(4.4) \quad \frac{f^{(a+b, a+1, 1)}}{f^\lambda} = \frac{2b(a+b+1)(a+1)}{(a+b)(a-1)(a+2)(b+1)}.$$

Since

$$\frac{d}{da} \frac{f^{(a+b, a+1, 1)}}{f^\lambda} = - \frac{(2b(a^4 + 2a^3b + 4a^3 + a^2b^2 + 5a^2b + 7a^2 + 2ab^2 + 8ab + 2a + 3b^2 + 3b - 2))}{((a-1)^2(a+2)^2(b+1)(a+b)^2)},$$

we have that Equation (4.4) decreases as a increases. Therefore, it suffices to consider $a = 3$ which we have done in our base case. Hence, $f^{(a+b, a+1, 1)} < f^\lambda$.

For $\nu = (a + b, a + 2)$, we have that

$$\frac{f^{(a+b, a+2)}}{f^\lambda} = \frac{2(b-1)(a+b+2)}{(a-1)(b+1)(a+b)(a+2)} \leq \frac{a+b+2}{(a+b)(a+2)}$$

since $a \geq 3$. As $(a+b)(a+2) = a^2 + 2a + ab + 2b \geq a + b + 2$, we have that $f^{(a+b, a+2)} < f^\lambda$.

Lastly, for $\nu = (a + b + 1, a - 1, 2)$, we first consider when $a = \frac{(b+1)(b+2)}{2}$. By the hook length formula, we have

$$\begin{aligned} (4.5) \quad \frac{f^{(a+b+1, a-1, 2)}}{f^\lambda} &= \frac{(b+3)(a+b+1)(a-2)(a+1)}{(b+1)(a+b+3)(a-1)(a+b)} \\ &= \frac{(2b+6)((b+1)(b+2) + 2b+2)((b+1)(b+2) - 4)((b+1)(b+2) + 2)}{((b+1)(b+2) + 2b)((2b+2)((b+1)(b+2) + 2b+6)((b+1)(b+2) - 2)} \\ &= \frac{b^7 + 14b^6 + 78b^5 + 220b^4 + 321b^3 + 182b^2 - 80b - 96}{b^7 + 14b^6 + 78b^5 + 220b^4 + 321b^3 + 214b^2 + 48b} \\ &< 1. \end{aligned}$$

Following a similar argument as in Proposition 4.2.8 for Equation (4.3), we can see that for fixed b , Equation (4.5) decreases as a decreases by considering $\frac{a+b+1}{a+b+3}$, $\frac{a-2}{a-1}$, and $\frac{a+1}{a+b}$.

We have thus shown that when $a \geq 3$, $\lambda = (a + b, a, 2)$ is maximal if $a \leq \frac{(b+1)(b+2)}{2}$. For the reverse direction, consider Equation (4.5) when $a = \frac{(b+1)(b+2)}{2} + 1$. We have that

$$\frac{f^{(a+b+1, a-1, 2)}}{f^\lambda} = \frac{b^6 + 12b^5 + 60b^4 + 162b^3 + 243b^2 + 162b}{b^6 + 12b^5 + 60b^4 + 158b^3 + 219b^2 + 150b + 40} > 1.$$

Since Equation (4.5) increases as a increases, $\lambda \leq_{std} (a + b + 1, a - 1, 2)$ when $a > \frac{(b+1)(b+2)}{2}$.

Therefore, λ is maximal if and only if $a \leq \frac{(b+1)(b+2)}{2}$. $\square$

REMARK 4.2.10. We may translate the results of Propositions 4.2.7, 4.2.8, and 4.2.9 into statements about partitions of the form (α, β) , $(\alpha, \beta, 1)$, and $(\alpha, \beta, 2)$. Table 4.1 summarizes our maximality conditions.

We next classify all maximal hook shape partitions. As noted in Lemma 4.2.4, $(1^n) \leq_{std} (n)$ and so the single column shape is only maximal when $n = 1$. By Lemma 4.2.6, $(2, 1^b) \leq_{std} (2^2, 1^{b-2})$ whenever $b \geq 3$. Since $(2, 1, 1) \leq_{std} (3, 1)$, the only maximal hook shape with arm length 2 is $(2, 1)$. In the following proposition, we investigate all hook shape partitions with arm length greater than 2.

Value for α	$\lambda = (\alpha, \beta)$	$\lambda = (\alpha, \beta, 1)$	$\lambda = (\alpha, \beta, 2)$
$\alpha \geq 2$	$(\alpha, 1)$		
$\alpha \geq 3$	$(\alpha, 2)$	$(\alpha, 2, 1)$	
$\alpha \geq 4$		$(\alpha, 3, 1)$	$(\alpha, 3, 2)$
$\alpha \geq 5$	$(\alpha, 3)$		
$\alpha \geq 6$	$(\alpha, 4)$	$(\alpha, 4, 1)$	$(\alpha, 4, 2)$
$\alpha \geq 7$	$(\alpha, 5)$	$(\alpha, 5, 1)$	$(\alpha, 5, 2)$
$\alpha \geq 8$		$(\alpha, 6, 1)$	$(\alpha, 6, 2)$
$\alpha \geq 9$	$(\alpha, 6)$		
$\alpha \geq 10$	$(\alpha, 7)$	$(\alpha, 7, 1)$	$(\alpha, 7, 2)$
$\alpha \geq 11$	$(\alpha, 8)$	$(\alpha, 8, 1)$	$(\alpha, 8, 2)$
$\alpha \geq 12$	$(\alpha, 9)$	$(\alpha, 9, 1)$	$(\alpha, 9, 2)$
$\alpha \geq 13$		$(\alpha, 10, 1)$	$(\alpha, 10, 2)$
$\alpha \geq 14$	$(\alpha, 10)$		
$\alpha \geq 15$	$(\alpha, 11)$	$(\alpha, 11, 1)$	$(\alpha, 11, 2)$
$\alpha \geq 16$	$(\alpha, 12)$	$(\alpha, 12, 1)$	$(\alpha, 12, 2)$
$\alpha \geq 17$	$(\alpha, 13)$	$(\alpha, 13, 1)$	$(\alpha, 13, 2)$
$\alpha \geq 18$	$(\alpha, 14)$	$(\alpha, 14, 1)$	$(\alpha, 14, 2)$
$\alpha \geq 19$		$(\alpha, 15, 1)$	$(\alpha, 15, 2)$
$\alpha \geq 20$	$(\alpha, 15)$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 4.1. Necessary and sufficient conditions for maximality of a partition λ .

PROPOSITION 4.2.11. *Let $\lambda = (a, 1^b)$ be a hook shape partition such that $a > 2$. Then λ is a maximal element in the standard immersion poset if and only if $b \leq 2$.*

PROOF. When $b = 1$, the only partition that dominates $(a, 1)$ is $(a+1)$ and $f^{(a,1)} = (a+1) - 1 > 1 = f^{(a+1)}$. Thus, $(a, 1)$ is maximal. When $b = 2$, the only partitions that dominate $(a, 1^2)$ are $(a+2)$, $(a+1, 1)$, and $(a, 2)$. By the hook length formula,

$$\frac{f^{(a+1,1)}}{f^{(a,1^2)}} = \frac{2}{a} < 1 \quad \text{and} \quad \frac{f^{(a,2)}}{f^{(a,1^2)}} = \frac{(a+2)(a-1)}{(a+1)a} = \frac{a^2 + a - 2}{a^2 + a} < 1.$$

Therefore, no partition dominates $(a, 1^2)$ and has more standard Young tableaux, so $(a, 1^2)$ is maximal.

When $b \geq 3$, $(a, 1^b) <_{std} (a, 2, 1^{b-2})$, by the hook length formula:

$$\frac{f^{(a, 1^b)}}{f^{(a, 2, 1^{b-2})}} = \frac{a(a+b-1)}{(a+b)(a-1)(b-1)} \leq 1$$

since

$$(a+b)(a-1)(b-1) \geq 2(a-1)(a+b) \geq a(a+b-1).$$

Therefore $f^{(a, 1^b)} \leq f^{(a, 2, 1^{b-2})}$ and $(a, 1^b) <_D (a, 2, 1^{b-2})$, so we have $(a, 1^b) <_{std} (a, 2, 1^{b-2})$ whenever $b \geq 3$. $\square$

PROPOSITION 4.2.12. *If λ is a maximal element in the standard immersion poset, then $\lambda_1 > \lambda_2$.*

PROOF. Suppose by contradiction that $\lambda = (a^b, \lambda_{b+1}, \dots)$ with $a > \lambda_{b+1}$ and $b \geq 2$. Let $\mu = (a+1, a^{b-2}, a-1, \lambda_{b+1}, \dots)$ and denote by $\text{SYT}(\lambda)$ the set of all standard Young tableaux of shape λ . The map

$$\varphi: \text{SYT}(\lambda) \rightarrow \text{SYT}(\mu),$$

where $\varphi(T)$ is the standard Young tableau obtained from T by moving the box in position (b, a) to position $(1, a+1)$, is an injection. Therefore, $\lambda <_D \mu$ and $f^\lambda \leq f^\mu$, which implies $\lambda \leq_I \mu$ and thus demonstrates that λ is not a maximal element in the standard immersion poset. $\square$

In fact, the injection φ used in the proof of Proposition 4.2.12 remains an injection when the domain and codomain are extended to semistandard Young tableaux of content ν , for any $\nu \vdash |\lambda|$. Injection arguments between sets of semistandard Young tableaux are expanded on in Section 4.2.2.2. In particular, this result is extended to the immersion poset in Corollary 4.2.20.

REMARK 4.2.13. *Note that partitions λ for which f^λ is maximal among all partitions of n are maximal elements in the standard immersion poset. These partitions and their asymptotics have been studied in [29, 39, 70, 71, 72]. See also [55, Theorem 1.24].*

4.2.2. The immersion poset. In this section we turn to the immersion poset. In Section 4.2.2.1, we study basic properties of the immersion poset. In Section 4.2.2.2, we provide explicit injections between certain sets of semistandard Young tableaux, which are used to determine statements about maximal elements and cover relations in the immersion poset. In Sections 4.2.2.3 and 4.2.2.4, we study the immersion poset restricted to hook partitions and two column partitions,

respectively. We conclude in Section 4.3.2 with conjectures about certain lower intervals in the immersion poset and prove that the conjectured intervals give Schur-positive sums of power sum symmetric functions.

4.2.2.1. *Properties of the immersion poset.* We begin by specifying the minimal element.

LEMMA 4.2.14. *The partition (1^n) is the unique minimal element in the immersion poset $(\mathcal{P}(n), \leq_I)$.*

PROOF. We have $f^{(1^n)} = 1 \leq f^\lambda$ for all $\lambda \in \mathcal{P}(n)$. Furthermore $K_{(1^n), \alpha} = 0 \leq K_{\lambda, \alpha}$ for all $\alpha \neq (1^n)$ and $\lambda \in \mathcal{P}(n)$. By Lemma 4.1.3 this proves the claim. $\square$

Analogously to Lemma 4.2.4, we prove the following result.

LEMMA 4.2.15. *We have*

- (1) $(1^n) \leq_I (n)$ for all n and
- (2) $(2, 1^{n-2}) \leq_I (n-1, 1)$ for all $n > 3$ with $n \neq 6$.

PROOF. By Lemma 4.2.14, we have $(1^n) <_I (n)$. By Lemma 4.2.4, $(1^n) \leq_{std} (n)$. Since in the immersion poset there are fewer order relations than in the standard immersion poset, the first part of the lemma follows.

We have $(2, 1^{n-2}) <_I (n-1, 1)$ since $s_{(2, 1^{n-2})} = (n-1)m_{(1^n)} + m_{(2, 1^{n-2})}$ and

$$s_{(n-1, 1)} = (n-1)m_{(1^n)} + (n-2)m_{(2, 1^{n-2})} + \sum_{\mu \neq (1^n), (2, 1^{n-2})} K_{(n-1, 1), \mu} m_\mu.$$

Again, since by Lemma 4.2.4 we have $(2, 1^{n-2}) \leq_{std} (n-1, 1)$, the second part of the lemma follows. $\square$

Unlike in the standard immersion poset, where λ and λ^t are always comparable as long as they are comparable in dominance order (see Remark 4.2.5), this is not always true in the immersion poset. For example $\lambda = (4, 4, 2, 1, 1)$ and λ^t are not comparable in the immersion poset since $K_{(4, 4, 2, 1, 1), (4, 4, 1, 1, 1)} > K_{(5, 3, 2, 2), (4, 4, 1, 1, 1)}$. For hook partitions, it is however true that $\lambda <_I \lambda^t$ if $\lambda <_D \lambda^t$ (see Corollary 4.2.31).

We prove the analog of Lemma 4.2.6 in the next section using injections on semistandard Young tableaux. See Corollaries 4.2.19, 4.2.20, and 4.2.21.

4.2.2.2. *Explicit injections.* Recall from Lemma 4.1.3 that $\lambda \leq_I \mu$ if and only if $K_{\lambda,\nu} \leq K_{\mu,\nu}$ for all $\nu \in \mathcal{P}(n)$. The Kostka number $K_{\lambda,\nu}$ is the cardinality of the set of semistandard Young tableaux $\text{SSYT}(\lambda, \nu)$ of shape λ and content ν . Hence we can analyze the order relations $\lambda \leq_I \mu$ by constructing explicit injections

$$(4.6) \quad \varphi: \text{SSYT}(\lambda, \nu) \rightarrow \text{SSYT}(\mu, \nu)$$

for all $\nu \in \mathcal{P}(n)$.

To this end, we present one such injection, where μ differs from λ by moving a single cell from the c -th column to the $(c+1)$ -th column, and λ has a bound on the relative size of the two columns. In [25], we go on to refine this first injection to obtain more precise bounds on the relative size of the columns (excluded here). We partially characterize what elements cannot be maximal in the immersion poset, similar to those given in Section 4.2.1.2 for the standard immersion poset.

Let

$$(4.7) \quad \begin{aligned} \lambda &= (\lambda_1, \dots, \lambda_\alpha, c^\beta, \lambda_{\alpha+\beta+1}, \dots), \\ \mu &= (\lambda_1, \dots, \lambda_\alpha, c+1, c^{\beta-2}, c-1, \lambda_{\alpha+\beta+1}, \dots), \end{aligned}$$

such that either $\alpha > 0$ and $\lambda_{\beta+\alpha+1} < c < \lambda_\alpha$, or $\alpha = 0$ and $\lambda_{\beta+\alpha+1} < c$. In particular, $\lambda_{\beta+\alpha+1}$ can be 0. We define a map

$$\varphi_0: \text{SSYT}(\lambda, \nu) \rightarrow \text{YT}(\mu, \nu),$$

where $\text{YT}(\mu, \nu)$ is the set of all tableaux of shape μ and content ν , not necessarily semistandard. We will show in Proposition 4.2.18 that when $\beta \geq \alpha + 2$, the image of φ_0 will be contained in $\text{SSYT}(\mu, \nu)$, so φ_0 will be as in (4.6).

For $T \in \text{SSYT}(\lambda, \nu)$, we define $\varphi_0(T)$ as follows. Suppose the entries in the c -th column of T in increasing order are $x_{\beta+\alpha}, x_{\beta+\alpha-1}, \dots, x_1$ and the entries in the $(c+1)$ -th column of T in increasing order are $y_\alpha, y_{\alpha-1}, \dots, y_1$. Let i be the smallest index such that $x_i > y_i$. If no such index exists, let $i = \alpha + 1$. Then $\varphi_0(T)$ is the tableau such that the entries in the c -th column of $\varphi_0(T)$ are

$$x_{\beta+\alpha}, x_{\beta+\alpha-1}, \dots, x_{i+1}, y_{i-1}, y_{i-2}, \dots, y_1,$$

the entries in the $(c + 1)$ -th column of $\varphi_0(T)$ are

$$y_\alpha, y_{\alpha-1}, \dots, y_i, x_i, x_{i-1}, \dots, x_1,$$

and all other entries are the same as those in T . In other words, φ_0 moves the cell containing x_1 to the $(\alpha + 1)$ -th row of the $(c + 1)$ -th column, and swaps each x_j with y_{j-1} for all $2 \leq j \leq i$.

More concretely, the c -th and $(c + 1)$ -th column in T and $\varphi_0(T)$ look as follows:

$$(4.8) \quad T : \begin{array}{|c|c|} \hline x_{\beta+\alpha} & y_\alpha \\ \hline \vdots & \vdots \\ \hline x_{\beta+i} & y_i \\ \hline x_{\beta+i-1} & y_{i-1} \\ \hline \vdots & \vdots \\ \hline x_{\beta+1} & y_1 \\ \hline x_\beta & \\ \hline \vdots & \\ \hline x_{i+1} & \\ \hline x_i & \\ \hline \vdots & \\ \hline x_2 & \\ \hline x_1 & \\ \hline \end{array} \quad \varphi_0(T) : \begin{array}{|c|c|} \hline x_{\beta+\alpha} & y_\alpha \\ \hline \vdots & \vdots \\ \hline x_{\beta+i} & y_i \\ \hline x_{\beta+i-1} & x_i \\ \hline \vdots & \vdots \\ \hline x_{\beta+1} & x_2 \\ \hline x_\beta & x_1 \\ \hline \vdots & \\ \hline x_{i+1} & \\ \hline y_{i-1} & \\ \hline \vdots & \\ \hline y_1 & \\ \hline \end{array}$$

The cells marked in green contain the entries that move from the c -th column to the $(c + 1)$ -th column, and the cells marked in yellow are the entries that move from $(c + 1)$ -th column to the c -th column. We continue to use this convention for all subsequent examples of φ_0 .

REMARK 4.2.16. *Observe that by our choice of i , both $x_{i+1} < x_{i-1} \leq y_{i-1}$ and $y_i < x_i$, so the columns of $\varphi_0(T)$ are strictly increasing by construction.*

EXAMPLE 4.2.17. *For $\lambda = (3, 2, 1^4)$ and $\mu = (3, 2, 2, 1^2, 0)$, we have $c = 1$, $\alpha = 2$ and $\beta = 4$. Here are some examples of the injection φ_0 on various tableaux of shape λ :*

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline 5 & & \\ \hline 6 & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \hline 3 & 6 & \\ \hline 4 & & \\ \hline 5 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 7 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline 5 & & \\ \hline 6 & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 5 & \\ \hline 3 & 6 & \\ \hline 4 & & \\ \hline 7 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 6 & 9 \\ \hline 2 & 8 & \\ \hline 3 & & \\ \hline 4 & & \\ \hline 5 & & \\ \hline 7 & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 4 & 9 \\ \hline 2 & 5 & \\ \hline 3 & 7 & \\ \hline 6 & & \\ \hline 8 & & \\ \hline \end{array}$$

PROPOSITION 4.2.18. *Let λ and μ be as in (4.7) with $\beta \geq \alpha + 2$. Then φ_0 as defined above is an injection*

$$\varphi_0: \text{SSYT}(\lambda, \nu) \rightarrow \text{SSYT}(\mu, \nu).$$

PROOF. Let $T \in \text{SSYT}(\lambda, \nu)$. Note that the content does not change under φ_0 . We need to check that the c -th and $(c+1)$ -th columns of $\varphi_0(T)$ are strictly increasing, and that the $(\alpha-i+2)$ -th through $(\alpha+\beta-1)$ -th rows of $\varphi_0(T)$ are weakly increasing, since all other entries are identical to those in T . (It may be helpful to consult (4.8).) The columns are strictly increasing by Remark 4.2.16.

For rows, we first consider the $(\alpha-i+2)$ -th through $(\alpha+1)$ -th rows. Due to the bound $\beta \geq \alpha+2$, in the c -th column, these rows contain $x_{\beta+i-1}, \dots, x_\beta$. In the $(c+1)$ -th column, irrespective of the bound on α and β , these rows contain $x_i, \dots, x_1$. In particular, the bound $\beta \geq \alpha+2$ makes it so that there is no y_j entry in these rows, so there is no “overlap” of y_j and x_k for $1 \leq k \leq i$. The rows are thus strictly increasing because $x_j > x_k$ for all $j < k$, so the x_j entries in the $(c+1)$ -th column are greater than the entries to their left in the c -th column; and $x_j < x_{j-1} \leq y_{j-1}$ for $2 \leq j \leq i$, so the x_j entries in the $(c+1)$ -th column are less than any entries to their right, originally from T . (Such entries on the right do not necessarily exist. In particular, x_1 never has any cell to its right.)

Now consider the $(\alpha+2)$ -th through $(\alpha+\beta-1)$ -th rows. In the c -th column, these rows contain

$$x_{\beta-1}, \dots, x_{i+1}, y_{i-1}, \dots, y_1,$$

and in the $(c+1)$ -th column, these rows contain no cells. They are weakly increasing because $y_{j-1} \geq x_{j-1} > x_j$ for $2 \leq j \leq i$, so the y_{j-1} entries in the c -th column are greater than the entries to their left, originally from T , and they have no cells to their right.

To show injectivity, we define an explicit inverse ψ_0 . Let $T' \in \varphi_0(\text{SSYT}(\lambda, \nu))$. Suppose the entries in the c -th column of T' in increasing order are $x'_{\beta+\alpha-1}, x'_{\beta+\alpha-2}, \dots, x'_1$, and the entries in the $(c+1)$ -th column of T' in increasing order are $y'_{\alpha+1}, y'_\alpha, \dots, y'_1$. Let i' be the smallest index such that $y'_{i'} > x'_{i'}$. This i' will be equal to the i from the definition of φ_0 , because $x_i > x_{i+1}$ and $x_j \leq y_j$ for $1 \leq j \leq i-1$ in T .

Then $\psi_0(T')$ is the tableau of shape λ such that the entries in the c -th column of $\psi_0(T')$ are

$$x'_{\beta+\alpha-1}, x'_{\beta+\alpha-2}, \dots, x'_{\beta-1}, x'_{\beta-2}, x'_{\beta-3}, \dots, x'_{i'}, y'_{i'}, y'_{i'-1}, \dots, y'_1,$$

the entries in the $(c + 1)$ -th column of $\psi_0(T')$ are

$$y'_{\alpha+1}, y'_\alpha, \dots, y'_{i'+1}, x'_{i'-1}, x'_{i'-2}, \dots, x'_1,$$

and all other entries are the same as those in T' . In other words, ψ_0 moves the cell containing y'_1 to the $(\alpha + \beta)$ -th position in the c -th column, and swaps each $y'_{j'}$ with $x'_{j'-1}$ for all $2 \leq j' \leq i'$. Concretely:

$$(4.9) \quad T' : \begin{array}{|c|c|} \hline x'_{\beta+\alpha-1} & y'_{\alpha+1} \\ \hline \vdots & \vdots \\ \hline x'_{\beta+i'-1} & y'_{i'+1} \\ \hline x'_{\beta+i'-2} & y'_{i'} \\ \hline \vdots & \vdots \\ \hline x'_\beta & y'_2 \\ \hline x'_{\beta-1} & y'_1 \\ \hline x'_{\beta-2} & \\ \hline \vdots & \\ \hline x'_{i'} & \\ \hline x'_{i'-1} & \\ \hline \vdots & \\ \hline x'_1 & \\ \hline \end{array} \quad \psi_0(T') : \begin{array}{|c|c|} \hline x'_{\beta+\alpha-1} & y'_{\alpha+1} \\ \hline \vdots & \vdots \\ \hline x'_{\beta+i'-1} & y'_{i'+1} \\ \hline x'_{\beta+i'-2} & x'_{i'-1} \\ \hline \vdots & \vdots \\ \hline x'_\beta & x'_1 \\ \hline x'_{\beta-1} & \\ \hline x'_{\beta-2} & \\ \hline \vdots & \\ \hline x'_{i'} & \\ \hline y'_{i'} & \\ \hline \vdots & \\ \hline y'_2 & \\ \hline y'_1 & \\ \hline \end{array}$$

Since $i' = i$, ψ_0 moves back exactly the entries in T' that were originally moved by φ_0 in T , so ψ_0 is the inverse of φ_0 . $\square$

As a corollary, the injection describes a class of cover relations in the immersion poset. As a specific example, it can partially address the two column case, which was completely addressed by Lemma 4.2.6 for the standard immersion poset.

COROLLARY 4.2.19. *The partitions λ and μ as in (4.7) with $\beta \geq \alpha + 2$ form a cover in the immersion poset. In particular, $\lambda = (2^\alpha, 1^\beta)$ and $\mu = (2^{\alpha+1}, 1^{\beta-2})$ form a cover.*

PROOF. The partition μ covers λ in dominance order, and the injection shows that μ is greater than λ in the immersion poset, so μ must also cover λ in the immersion poset. $\square$

The injection also gives a few conditions on which partitions cannot be maximal.

COROLLARY 4.2.20. *If $\lambda = (a^\beta, b, \dots)$, where $a > b$, and $\beta \geq 2$, then λ is not maximal.*

PROOF. We have $\beta \geq 2$ with $\alpha = 0$, so we can apply the injection. $\square$

COROLLARY 4.2.21. *If $\lambda = (a, b^\beta, c, \dots)$, where $a > b > c$, and $\beta \geq 3$, then λ is not maximal. In particular, $\lambda = (a, 1^\beta)$ is not maximal for $a \geq 2$, $\beta \geq 3$.*

PROOF. We have $\beta \geq 3$ with $\alpha = 1$, so we can apply the injection. $\square$

Note that Corollary 4.2.20 and Corollary 4.2.21 repeat the results from Proposition 4.2.12 and the forward direction of Proposition 4.2.11 concerning nonmaximal elements in the standard immersion poset.

COROLLARY 4.2.22. *If $\lambda = (a, b, c, d)$ is maximal in the immersion poset, then it has no more than two identical nonzero parts.*

PROOF. If λ has three or more identical parts, then λ is one of (a^4) , (a^3, d) , or (a, b^3) , so we can apply the injection. $\square$

4.2.2.3. *Immersion poset on hook partitions.* For this section, set $\lambda^i = (i, 1^{n-i}) \vdash n$ and let $S = \{\lambda^i \mid 1 \leq i \leq n\}$ be the set of all hook partitions. We study the immersion poset restricted to S .

PROPOSITION 4.2.23. *Let $1 \leq i \leq n$ and $\alpha = (\alpha_1, \dots, \alpha_k) \vdash n$ such that $\alpha \leq_D \lambda^i$. Then*

$$K_{\lambda^i, \alpha} = \binom{k-1}{n-i}.$$

PROOF. Since λ^i dominates α , we know that $K_{\lambda^i, \alpha} \geq 1$. To form a semistandard Young tableau of shape λ^i and content α , the α_1 entries 1 must be placed leftmost in the first row of λ^i . The remaining $n - i$ positions in the first column of λ^i can be filled with distinct values from the set $\{2, 3, \dots, k\}$. This gives $\binom{k-1}{n-i}$ choices. Once these are placed, there is only one way to fill the remainder of the first row so that the resulting tableau is semistandard. $\square$

Recall from Lemma 4.1.3 that $\mu \leq_I \lambda$ if and only if $K_{\mu, \alpha} \leq K_{\lambda, \alpha}$ for all $\alpha \vdash n$. Hence with Proposition 4.2.23, we are now ready to describe all the relations between hook partitions $\lambda^i \in S$ in the immersion poset. To illustrate what the proposition implies, we form a matrix of values in the following way:

- The j -th column is indexed by the content α^j , where α^j is any content that has j parts.

- The i -th row is indexed by the shape λ^i .
- The (i, j) entry of this matrix is the value $T_{i,j} := K_{\lambda^i, \alpha^j} = \binom{j-1}{n-i}$ for $1 \leq i, j \leq n$.

EXAMPLE 4.2.24. We give the matrix for $n = 7$:

<i>Partition</i> \ # of parts	1	2	3	4	5	6	7
(1^7)	0	0	0	0	0	0	1
$(2, 1^5)$	0	0	0	0	0	1	6
$(3, 1^4)$	0	0	0	0	1	5	15
$(4, 1^3)$	0	0	0	1	4	10	20
$(5, 1^2)$	0	0	1	3	6	10	15
$(6, 1)$	0	1	2	3	4	5	6
(7)	1	1	1	1	1	1	1

REMARK 4.2.25. In this context, $\lambda^i \geq_I \lambda^j$ if and only if $T_{i,m} \geq T_{j,m}$ for all m . Equivalently, since $T_{j,m} = 0$ when $n - j \geq m$ and λ^i dominates λ^j when $i > j$, we need only show $T_{i,m} \geq T_{j,m}$ for all $m > n - j$ when $i > j$.

The following lemma is used to prove the structure of the immersion poset restricted to hook partitions.

LEMMA 4.2.26. Suppose $\binom{n-1}{n-i} \geq \binom{n-1}{n-j}$ and $i > j$ (note that this implies $j \leq \frac{n}{2}$). Then for all $0 \leq p \leq j - 1$, we have

$$\binom{n-1-p}{n-i} \geq \binom{n-1-p}{n-j}.$$

The proof follows from basic properties of binomial coefficients, and is omitted here.

COROLLARY 4.2.27. If $T_{i,n} \geq T_{j,n}$ for $i > j$, then $\lambda^i \geq_I \lambda^j$.

PROOF. By Proposition 4.2.23, $T_{i,n-p} = K_{\lambda^i, \alpha^{n-p}} = \binom{(n-p)-1}{n-i}$. Hence if $T_{i,n} \geq T_{j,n}$, by Lemma 4.2.26, we also have $T_{i,n-p} \geq T_{j,n-p}$ for $0 \leq p \leq j-1$. By Remark 4.2.25, this implies $\lambda^i \geq_I \lambda^j$. $\square$

EXAMPLE 4.2.28. Take the rows corresponding to the partitions $(5, 1^2)$ and $(3, 1^4)$ in Example 4.2.24. Since the last column entries give $T_{5,7} = 15 \geq 15 = T_{3,7}$, then by Corollary 4.2.27 we also have $T_{5,7-p} \geq T_{3,7-p}$ for $1 \leq p \leq 2$: $T_{5,6} = 10 \geq 5 = T_{3,6}$, $T_{5,5} = 6 \geq 1 = T_{3,5}$.

We now describe the relations in the immersion poset on S depending upon whether n is even or odd.

PROPOSITION 4.2.29. Let $n = 2k + 1$ be odd, then:

- (1) $\lambda^{\ell+1} \geq_I \lambda^\ell$ for all $1 \leq \ell \leq k$.
- (2) $(\lambda^{k+1-\ell})^t = \lambda^{k+1+\ell} \geq_I \lambda^{k+1-\ell}$ for all $1 \leq \ell \leq k$.
- (3) For any $1 < i \leq k+1$, λ^i is incomparable to λ^j for all $j > n-i+1$.
- (4) For any $k+2 \leq i < n$, λ^i is incomparable to λ^j for all $j > i$.

These describe all relations in the immersion poset restricted to hook partitions S .

PROOF. Let us first prove (1). Fix an ℓ with $1 \leq \ell \leq k$. Then by Corollary 4.2.27, $\lambda^{\ell+1} \geq_I \lambda^\ell$ if and only if $T_{\ell+1,n} \geq T_{\ell,n}$. Note that $T_{\ell+1,n} = \binom{n-1}{n-\ell-1} = \binom{n-1}{\ell}$ and $T_{\ell,n} = \binom{n-1}{n-\ell} = \binom{n-1}{\ell-1}$. Since $1 \leq \ell \leq k$, we have $\binom{n-1}{\ell} \geq \binom{n-1}{\ell-1}$ and the result follows.

To prove (2), note that $\lambda^{k+1+\ell} \geq_I \lambda^{k+1-\ell}$ if and only if $T_{k+1+\ell,n} \geq T_{k+1-\ell,n}$. Since $T_{k+1+\ell,n} = \binom{n-1}{k-\ell} = \binom{2k}{k-\ell} = \binom{2k}{k+\ell} = \binom{n-1}{k+\ell} = T_{k+1-\ell,n}$, the result follows.

To prove (3) we show for any $1 < i \leq k+1$ that λ^i is incomparable to λ^j for all $j > n-i+1$. Since λ^j dominates λ^i , we need only show there exists some α such that $K_{\lambda^i, \alpha} > K_{\lambda^j, \alpha}$. Choose $\alpha = (1^n)$. Then $K_{\lambda^i, \alpha} = \binom{n-1}{n-i} = \binom{n-1}{i-1} > \binom{n-1}{n-j} = K_{\lambda^j, \alpha}$ because $i-1 > n-j$ and $1 < i \leq k+1$.

Lastly, to prove (4) we follow the same strategy as (3). Since λ^j dominates λ^i , we can let $\alpha = (1^n)$, and since $k+1 < i < j$ we get $K_{\lambda^i, \alpha} = \binom{n-1}{n-i} > \binom{n-1}{n-j} = K_{\lambda^j, \alpha}$, and the result follows. $\square$

PROPOSITION 4.2.30. Let $n = 2k$ be even, then:

- (1) $\lambda^{\ell+1} \geq_I \lambda^\ell$ for all $1 \leq \ell < k$.
- (2) $(\lambda^{k-\ell})^t = \lambda^{k+1+\ell} \geq_I \lambda^{k-\ell}$ for all $0 \leq \ell \leq k-1$.
- (3) For any $1 < i \leq k$, λ^i is incomparable to λ^j for all $j > n-i+1$.

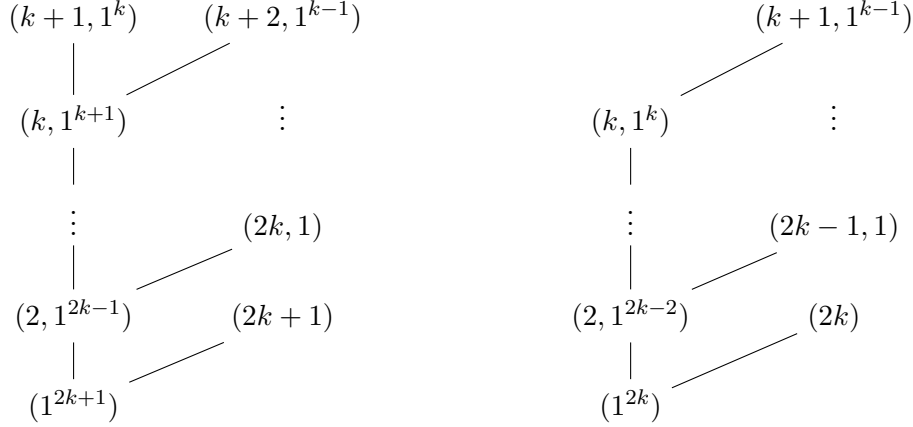


FIGURE 4.2. Immersion poset restricted to hook partitions for $n = 2k + 1$ (left) and $n = 2k$ (right).

(4) For any $k + 1 \leq i < n$, λ^i is incomparable to λ^j for all $j > i$.

These describe all relations in the immersion poset restricted to hook partitions S .

The proof of the even case is similar to the odd case.

The Hasse diagram of the immersion poset restricted to hook partitions is given in Figure 4.2. Notice that item (1) in Propositions 4.2.29 and 4.2.30 proves the string of covers on the left going up each Hasse diagram, while (2) proves the covers going up the right side which are the transposes.

COROLLARY 4.2.31. Let $\lambda \in S$ be a hook partition such that $\lambda \leq_D \lambda^t$. Then $\lambda \leq_I \lambda^t$.

The *rank* of a poset is the length of the longest chain of elements of the poset.

COROLLARY 4.2.32. The rank of the immersion poset $(\mathcal{P}(n), \leq_I)$ is at least $\lfloor n/2 \rfloor$.

4.2.2.4. *Immersion poset on two column partitions.* Now let S be the set partitions with at most two columns, that is, $S = \{\lambda \mid \lambda_1 \leq 2\}$. If $n = 2k$, then for this section we define $\lambda^j = (2^{k-j}, 1^{2j})$ for $0 \leq j \leq k$. Similarly, if $n = 2k + 1$, then $\lambda^j = (2^{k-j}, 1^{2j+1})$ for $0 \leq j \leq k$. In this section, we study the immersion poset restricted to S .

REMARK 4.2.33. Note that $K_{\lambda, \mu} = 0$ if $\lambda \in S$ and $\mu \notin S$. Hence there does not exist an immersion pair $\mu \leq_I \lambda$ with $\lambda \in S$ and $\mu \notin S$. This implies that if λ^i is a cover for λ^j in the subposet restricted to S , then λ^i is a cover for λ^j in the immersion poset.

This remark implies that we only need to consider $K_{\lambda,\mu}$ for $\lambda, \mu \in S$ when determining the immersion relations for this subset. Recall that f^λ is the number of standard Young tableaux of shape λ .

PROPOSITION 4.2.34.

- (1) Let $\lambda^j = (2^{k-j}, 1^{2j}) \vdash 2k$ for $0 \leq j \leq k$. Then $K_{\lambda^i, \lambda^j} = f^{(j+i, j-i)}$ when $i \leq j$ and $K_{\lambda^i, \lambda^j} = 0$ when $i > j$.
- (2) Let $\lambda^j = (2^{k-j}, 1^{2j+1}) \vdash 2k+1$ for $0 \leq j \leq k$. Then $K_{\lambda^i, \lambda^j} = f^{(j+i+1, j-i)}$ when $i \leq j$ and $K_{\lambda^i, \lambda^j} = 0$ when $i > j$.

PROOF. For (1), if $i > j$, λ^j dominates λ^i and hence $K_{\lambda^i, \lambda^j} = 0$. If $i = j$, clearly $K_{\lambda^i, \lambda^i} = f^{(2i)} = 1$. Suppose $j > i$. Then the first $k-j$ of the $k-i$ two length rows of any tableau $T \in \text{SSYT}(\lambda^i, \lambda^j)$ are fixed by the content. Hence, there is a bijection $\text{SSYT}(\lambda^i, \lambda^j) \rightarrow \text{SSYT}((2^{j-i}, 1^{2i}), (1^{2j}))$ by removing the first $k-j$ rows. Note that $K_{(2^{j-i}, 1^{2i}), (1^{2j})} = f^{(2^{j-i}, 1^{2i})}$, which is also equal to the number of standard tableaux of the transpose of $(2^{j-i}, 1^{2i})$. The result follows.

The proof of part (2) is similar. □

Using the hook length formula with Proposition 4.2.34, we can describe K_{λ^i, λ^j} . We present this in the form of a matrix. More explicitly, suppose $n = 2k$ or $n = 2k + 1$. Then for all $0 \leq i, j \leq k$, the i -th row and j -th column entry of the matrix $T = (T_{i,j})$ is $T_{i,j} = K_{\lambda^i, \lambda^j}$. Note that the indexing starts with 0.

EXAMPLE 4.2.35. Below are the matrices in tabular form for cases $n = 14, 15$.

The case when $n = 14$:

<i>Content</i> <i>Shape</i>	(2^7)	$(2^6, 1^2)$	$(2^5, 1^4)$	$(2^4, 1^6)$	$(2^3, 1^8)$	$(2^2, 1^{10})$	$(2, 1^{12})$	(1^{14})
(2^7)	1	1	2	5	14	42	132	429
$(2^6, 1^2)$	0	1	3	9	28	90	297	1001
$(2^5, 1^4)$	0	0	1	5	20	75	275	1001
$(2^4, 1^6)$	0	0	0	1	7	35	154	637
$(2^3, 1^8)$	0	0	0	0	1	9	54	273
$(2^2, 1^{10})$	0	0	0	0	0	1	11	77
$(2, 1^{12})$	0	0	0	0	0	0	1	13
(1^{14})	0	0	0	0	0	0	0	1

The case when $n = 15$:

<i>Content</i> <i>Shape</i>	$(2^7, 1)$	$(2^6, 1^3)$	$(2^5, 1^5)$	$(2^4, 1^7)$	$(2^3, 1^9)$	$(2^2, 1^{11})$	$(2, 1^{13})$	(1^{15})
$(2^7, 1)$	1	2	5	14	42	132	429	1430
$(2^6, 1^3)$	0	1	4	14	48	165	572	2002
$(2^5, 1^5)$	0	0	1	6	27	110	429	1638
$(2^4, 1^7)$	0	0	0	1	8	44	208	910
$(2^3, 1^9)$	0	0	0	0	1	10	65	350
$(2^2, 1^{11})$	0	0	0	0	0	1	12	90
$(2, 1^{13})$	0	0	0	0	0	0	1	14
(1^{15})	0	0	0	0	0	0	0	1

Since the columns and rows are decreasing in dominance order, for any $i < j$ we have $\lambda^i \geq_I \lambda^j$ if $T_{i,m} \geq T_{j,m}$ for all $0 \leq m \leq k$. In the following lemma, we prove some properties of the matrix T that will show that this statement is equivalent to only comparing values in the last column of the matrix. That is, if $i < j$ and $T_{i,k} \geq T_{j,k}$, then $\lambda^i \geq_I \lambda^j$. The reader can verify this in Example 4.2.35.

LEMMA 4.2.36. *The matrix $(T_{i,j}) = (K_{\lambda^i, \lambda^j})$ defined above with $0 \leq i, j \leq k$ has the following properties:*

- (1) *The entries weakly increase within each row.*
- (2) *The entries within each column are unimodal.*
- (3) *The rate of change of entries within a row increases as the row number increases. In particular, for any fixed i and j with $0 \leq i < j \leq k$, we have for all $j \leq r < k$:*

$$\frac{T_{i,r+1}}{T_{i,r}} < \frac{T_{j,r+1}}{T_{j,r}}.$$

- (4) *For any fixed i and j with $i < j$, if $T_{i,k} \geq T_{j,k}$, then $T_{i,m} \geq T_{j,m}$ for all $0 \leq m \leq k$.*

PROOF. We begin by proving (1). Let $n = 2k$ be even. Then for a fixed row i , given any $i \leq j < k$, we need to show that $T_{i,j+1} \geq T_{i,j}$. Using Proposition 4.2.34, we have:

$$\frac{T_{i,j+1}}{T_{i,j}} = \frac{f^{(j+1+i, j+1-i)}}{f^{(j+i, j-i)}} = \frac{(2j+2)(2j+1)}{(j+i+2)(j+1-i)} \geq 1$$

because $j \geq i$ implies

$$2j+2 \geq j+i+2 \quad \text{and} \quad 2j+1 \geq j+1-i.$$

Now let $n = 2k+1$ be odd. Using the same strategy as the even case we have:

$$\frac{T_{i,j+1}}{T_{i,j}} = \frac{f^{(j+2+i, j+1-i)}}{f^{(j+i+1, j-i)}} = \frac{(2j+3)(2j+2)}{(j+i+3)(j+1-i)} \geq 1$$

because $j \geq i$ implies

$$2j+3 \geq j+i+3 \quad \text{and} \quad 2j+2 \geq j+1-i.$$

Next we prove statement (2). Let $n = 2k$ be even. Since statement (2) holds trivially if there is only one nonzero entry in the column, we focus on columns with more than one nonzero entry. Fix

a $2 \leq j \leq k$. To determine when the column is increasing and decreasing we consider the fraction:

$$\frac{T_{i+1,j}}{T_{i,j}} = \frac{f^{(j+i+1,j-i-1)}}{f^{(j+i,j-i)}} = \frac{\frac{(2j)!(2i+3)}{(j+i+2)!(j-i-1)!}}{\frac{(2j)!(2i+1)}{(j+i+1)!(j-i)!}} = \frac{(2i+3)(j-i)}{(2i+1)(j+i+2)}.$$

Analyzing the following inequalities gives:

$$(4.10) \quad \begin{aligned} \frac{T_{i+1,j}}{T_{i,j}} &> 1 && \iff && 2i^2 + 4i + 1 < j, \\ \frac{T_{i+1,j}}{T_{i,j}} &= 1 && \iff && 2i^2 + 4i + 1 = j, \\ \frac{T_{i+1,j}}{T_{i,j}} &< 1 && \iff && 2i^2 + 4i + 1 > j. \end{aligned}$$

Thus, for values of i such that $2i^2 + 4i + 1 < j$ the column entries are increasing, and when the values of i satisfy $2i^2 + 4i + 1 > j$ the column entries are decreasing. This proves (2) for the even case.

Now let $n = 2k + 1$ be odd. Fix a $2 \leq j \leq k$. Similar to the even case we have:

$$\frac{T_{i+1,j}}{T_{i,j}} = \frac{f^{(j+i+2,j-i-1)}}{f^{(j+i+1,j-i)}} = \frac{\frac{(2j+1)!(2i+4)}{(j+i+3)!(j-i-1)!}}{\frac{(2j+1)!(2i+2)}{(j+i+2)!(j-i)!}} = \frac{(2i+4)(j-i)}{(2i+2)(j+i+3)}.$$

Analyzing the following inequalities gives:

$$(4.11) \quad \begin{aligned} \frac{T_{i+1,j}}{T_{i,j}} &> 1 && \iff && 2i^2 + 6i + 3 < j, \\ \frac{T_{i+1,j}}{T_{i,j}} &= 1 && \iff && 2i^2 + 6i + 3 = j, \\ \frac{T_{i+1,j}}{T_{i,j}} &< 1 && \iff && 2i^2 + 6i + 3 > j. \end{aligned}$$

Again, we notice that for values of i such that $2i^2 + 6i + 3 < j$ the column entries are increasing, and when the values of i satisfy $2i^2 + 6i + 3 > j$ the column entries are decreasing. This concludes the proof of (2).

To prove statement (3), we use Proposition 4.2.34 and the hook length formula to get the following equivalences:

$$\begin{aligned}
\frac{T_{i,r+1}}{T_{i,r}} < \frac{T_{j,r+1}}{T_{j,r}} &\iff \frac{K_{\lambda^i, \lambda^{r+1}}}{K_{\lambda^i, \lambda^r}} < \frac{K_{\lambda^j, \lambda^{r+1}}}{K_{\lambda^j, \lambda^r}} \\
&\iff \frac{(r+j+2)(r+1-j)}{(r+i+2)(r+1-i)} < 1 \\
&\iff i^2 + i < j^2 + j.
\end{aligned}$$

The last inequality is always true since $0 \leq i < j$, thus proving (3).

To prove (4), fix i and j with $i < j$ where $T_{i,k} \geq T_{j,k}$. Then by statement (3) it directly follows that $T_{i,m} \geq T_{j,m}$ for all $j \leq m \leq k$. Because $T_{j,m} = 0$ for all $0 \leq m < j$, it trivially follows that $T_{i,m} \geq T_{j,m}$ for these values of m , this finishes the proof of (4). $\square$

The beauty of Lemma 4.2.36, in particular statement (4), is that we can now reduce much of the work in determining the immersion relations between partitions in S to just comparing the numbers of standard Young tableaux, as is done in the next proposition.

PROPOSITION 4.2.37. *For $n = 2k$ even or $n = 2k + 1$ odd, the last (k -th) column of T can be used to completely determine relations in the immersion poset restricted to the subset S . In particular:*

- (1) $\lambda^i \geq_I \lambda^j$ if and only if $i < j$ and $T_{i,k} \geq T_{j,k}$,
- (2) For $i < j$, λ^i and λ^j are incomparable if and only if $T_{j,k} > T_{i,k}$.

PROOF. To prove (1), by definition $\lambda^i \geq_I \lambda^j$ if and only if $\lambda^i >_D \lambda^j$ and $T_{i,\ell} \geq T_{j,\ell}$ for all $0 \leq \ell \leq k$. But λ^i dominates λ^j if and only if $i < j$, and by (4) of Lemma 4.2.36, $T_{i,\ell} \geq T_{j,\ell}$ for all $0 \leq \ell \leq k$ if and only if $T_{i,k} \geq T_{j,k}$ (when $i < j$).

To prove (2), let $i < j$. If λ^i and λ^j are incomparable, then there exists some ℓ such that $T_{j,\ell} > T_{i,\ell}$. By (3) of Lemma 4.2.36, we have:

$$\frac{T_{i,r+1}}{T_{i,r}} \leq \frac{T_{j,r+1}}{T_{j,r}}$$

for all $\ell \leq r < k$, which guarantees that $T_{j,k} > T_{i,k}$. $\square$

As a consequence we obtain the following immediate corollary.

COROLLARY 4.2.38. *The cover relations for the immersion poset of the set S are exactly the same as those in the standard immersion poset.*

We can now explain the cover relations of the immersion poset restricted to the set S .

PROPOSITION 4.2.39. *Let $n = 2k$ be even or $n = 2k + 1$ be odd, then:*

- (1) $\lambda^i \succ_I \lambda^{i+1}$ when $2i^2 + 4i + 2 > k$ for n even and $2i^2 + 6i + 4 > k$ for n odd. This also coincides with Lemma 4.2.6, taking $a = k - i - 1$ and $b = 2i + 2$ (n even) or $b = 2i + 3$ (n odd).
- (2) λ^i and λ^j are incomparable in the immersion poset for all $0 \leq i, j \leq i_{\max}$ with $i \neq j$ and $i_{\max}$ being the largest i value not satisfying (1).
- (3) Fix i with $0 \leq i \leq i_{\max}$ and let $m > i_{\max} - i$ be smallest such that $T_{i,k} \geq T_{i+m,k}$. Then $\lambda^i \succ_I \lambda^{i+m}$.

PROOF. If $T_{i,k} \geq T_{i+1,k}$, then by Proposition 4.2.37 and Remark 4.2.33, we have that $\lambda^i \succ_I \lambda^{i+1}$ is a cover. We determine the values for i such that $T_{i,k} \geq T_{i+1,k}$ by using the middle equation and bottom inequality of (4.10) (for n even) and (4.11) (for n odd), where we replace j with k . Specifically, for n even:

$$2i^2 + 4i + 1 \geq k \implies 2i^2 + 4i + 2 > k,$$

and for n odd:

$$2i^2 + 6i + 3 \geq k \implies 2i^2 + 6i + 4 > k.$$

To prove (2), notice that since $i_{\max}$ is the number of the row containing the first maximum, by the increasing nature of the column up to the maximum value given by (2) of Lemma 4.2.36, then for any $0 \leq i < j \leq i_{\max}$ we have $T_{i,k} < T_{j,k}$. Hence by Proposition 4.2.37 (2), λ^i and λ^j are incomparable.

To prove (3) notice that by Proposition 4.2.37 statement (1), since m is the smallest value it must be a cover. □

EXAMPLE 4.2.40. *Suppose $n = 14$, so that $k = 7$. By (1) of Proposition 4.2.39, the inequality holds for $1 \leq i \leq k = 7$ so we obtain:*

$$\lambda^7 \prec_I \lambda^6 \prec_I \lambda^5 \prec_I \lambda^4 \prec_I \lambda^3 \prec_I \lambda^2 \prec_I \lambda^1.$$

Applying (3) of Proposition 4.2.39 to λ^0 , with $i = 0$ we find that $m = 4$:

$$T_{0,7} = 429 \geq 273 = T_{4,7}.$$

Notice that $m = 3$ does not satisfy the inequality:

$$T_{0,7} = 429 \not\geq 637 = T_{3,7}.$$

So our final cover relation for the poset is $\lambda^4 <_I \lambda^0$.

4.3. Schur-Positivity of Interval Power Sums

In this section we explore an application of the immersion poset to a question posed by Sundaram about the Schur-positivity of certain sums of power sum symmetric functions. In Section 4.3.1 we give the background and state relevant definitions. In Section 4.3.2 we examine lower intervals in the immersion poset.

4.3.1. Representation theory of symmetric groups. The irreducible representations as well as the conjugacy classes of the symmetric group S_n are indexed by partitions of n . Let $\chi^\lambda(\mu)$ denote the character value of the irreducible character χ^λ evaluated at an element of cycle type μ . The character table of S_n is a square matrix encoding character values, whose rows are indexed by irreducible characters χ^λ and whose columns are indexed by conjugacy classes C_μ . The character values of S_n are all integers. Solomon [59] proved that all row sums of the character table of S_n are nonnegative integers. Finding a combinatorial interpretation of the row sums is still an open problem (see [60, Exercise 7.71]).

Schur-positivity is intimately tied to representation theory. Namely, the symmetric function f is Schur-positive if it is the character of a representation W of $\mathrm{GL}_N(\mathbb{C})$ which admits the decomposition into irreducibles $W \cong \bigoplus_{\lambda \vdash n} W_\lambda(\mathbb{C}^N)^{\oplus c_\lambda}$.

The *Frobenius characteristic map* is a bridge between characters of the symmetric group and symmetric polynomials. The irreducible character χ^λ maps to s_λ under the Frobenius characteristic map. Via the Frobenius map, Schur-positivity of f implies that there exists a representation V of S_n such that $V \cong \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus c_\lambda}$, where V_λ is the irreducible representation of S_n indexed by λ .

4.3.1.1. *Power sum symmetric polynomials and restricted row sums of character table.* Given a subset A_n of partitions of n , consider the sum of power sum symmetric polynomials

$$(4.12) \quad p_{A_n} := \sum_{\mu \in A_n} p_\mu.$$

By the Murnaghan–Nakayama rule [60, Corollary 7.17.4], p_μ can be expressed in the basis of Schur polynomials as

$$p_\mu = \sum_{\lambda \vdash n} \chi^\lambda(\mu) s_\lambda.$$

Observe that the coefficient of s_λ in the expansion of p_{A_n} is $\sum_{\mu \in A_n} \chi^\lambda(\mu)$. This is precisely the restricted row sum (ignoring the columns not in A_n) of the character table of S_n . These values need not always be nonnegative integers, that is, p_{A_n} need not be Schur-positive. For example, if $A_4 = \{(1^4), (2, 1, 1), (4)\}$, then one can deduce from the character table of S_4 that $p_{A_4} = 3s_{(4)} + 3s_{(3,1)} + 2s_{(2,2)} + 3s_{(2,1,1)} - s_{(1^4)}$ is not Schur-positive.

QUESTION 4.3.1 (Sundaram [64]). *For which choices of A_n is the symmetric polynomial p_{A_n} Schur-positive? In other words, which subsets A_n of columns in the character table of S_n result in nonnegative row sums?*

Sundaram identifies certain A_n coming from plethystic generating functions associated to Foulkes characters that give Schur-positivity [64], and also shows that $\sum_{(1^n) \leq_{lex} \lambda \leq_{lex} \mu} p_\lambda$ is Schur-positive when $\mu \leq_{lex} (3, 1^{n-3})$ and when $\mu \geq_{lex} (n-4, 1^4)$ [65].

Inspired by Sundaram’s question, we explore the immersion poset in detail, hence understanding the immersion of polynomial representations for $\mathrm{GL}_N(\mathbb{C})$. Given a partition μ of n , consider the interval in the immersion poset $[(1^n), \mu] := \{\lambda \mid (1^n) \leq_I \lambda \leq_I \mu\}$. One may ask for what choices of μ , the symmetric polynomial $p_{[(1^n), \mu]}$ defined in Equation (4.12) is Schur-positive. We conjecture that $[(1^n), (n-2, 2)] = A_{(n-2, 2)}$ and $[(1^n), (n-2, 1, 1)] = A_{(n-2, 1, 1)}$, where $A_{(n-2, 2)} = \{(1^n), (2, 1^{n-2}), (2, 2, 1^{n-4}), (n-2, 2)\}$ and $A_{(n-2, 1, 1)} = \{(1^n), (2, 1^{n-2}), (2, 2, 1^{n-4}), (3, 1^{n-3}), (n-2, 1, 1)\}$ for n sufficiently large (see Conjectures 4.3.2 and 4.3.5). Furthermore, we prove that (for large enough n):

- (1) $p_{A_{(n-2, 1, 1)}}$ is Schur-positive;
- (2) $p_{A_{(n-2, 2)}}$ is Schur-positive for $n \neq 7$.

For precise statements, see Propositions 4.3.4 and 4.3.7.

One natural question which arises from the Schur-positivity of the above symmetric functions is to explore the representation theory behind it. It would be interesting to construct a natural representation V of the symmetric group such that its character maps to the symmetric polynomial $p_{[(1^n), \mu]}$ under the Frobenius map, when $\mu = (n-2, 1, 1)$ or $(n-2, 2)$.

4.3.2. Lower intervals and Schur-positivity of interval power sums. In this section, we make conjectures about certain lower intervals $[(1^n), \mu] := \{\lambda \mid (1^n) \leq_I \lambda \leq_I \mu\}$ in the immersion poset. Determining intervals will

- (1) enhance our understanding of the immersion of polynomial representations for $\mathrm{GL}_N(\mathbb{C})$ and
- (2) allow us to investigate when $p_{[(1^n), \mu]}$ of Equation (4.12) is Schur-positive, as asked in Question 4.3.1. We call $p_{[(1^n), \mu]}$ an interval power sum. It also helps towards constructing a natural corresponding representation of the symmetric group.

In this section, we prove that $p_{[(1^n), \mu]}$ is Schur-positive for the conjectured intervals.

CONJECTURE 4.3.2. *For $n = 5$ and $n \geq 9$, the interval $[(1^n), (n-2, 2)]$ is given by*

$$\{(1^n), (2, 1^{n-2}), (2, 2, 1^{n-4}), (n-2, 2)\}.$$

REMARK 4.3.3. *The cover relations $(1^n) \leq_I (2, 1^{n-2}) \leq_I (2, 2, 1^{n-4})$ are consequences of Corollary 4.2.19. The map*

$$\mathrm{SSYT}((2, 2, 1^{n-4}), \nu) \longrightarrow \mathrm{SSYT}((n-2, 2), \nu),$$

which is the transpose if $\nu_1 = 1$ and which moves the boxes in positions $(3, 1), \dots, (n-2, 1)$ to positions $(1, 3), \dots, (1, n-2)$ if $\nu_1 = 2$ is an injection. This shows that $(2, 2, 1^{n-4}) <_I (n-2, 2)$. Therefore, we have

$$\{(1^n), (2, 1^{n-2}), (2, 2, 1^{n-4}), (n-2, 2)\} \subseteq [(1^n), (n-2, 2)].$$

However, we have not proven the cover relation $(2, 2, 1^{n-4}) \leq_I (n-2, 2)$. One strategy to show the reverse containment is to argue that for all partitions λ such that λ and λ^t are not included

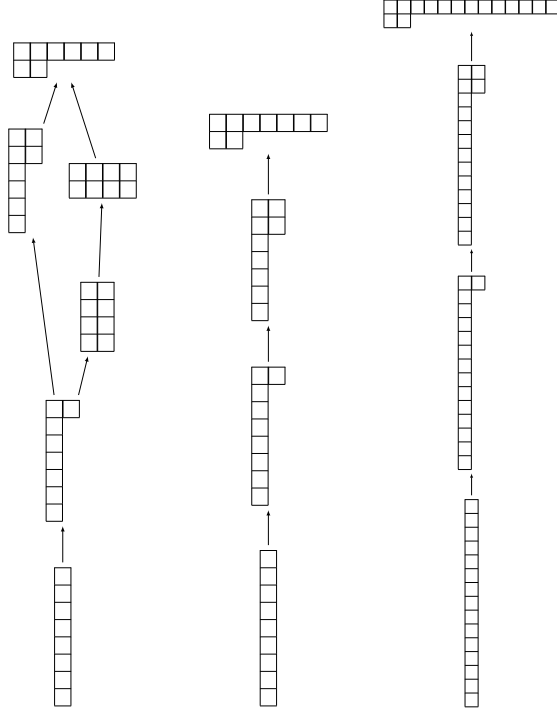


FIGURE 4.3. Subposet of the immersion poset only containing partitions in $[(1^n), (n-2, 2)]$ for $n = 8$ (left), $n = 9$ (middle), and $n = 15$ (right).

in the above list, we have $f^\lambda > \frac{n(n-3)}{2} = f^{(n-2, 2)}$. This would prove that $\lambda \not\prec_I (n-2, 2)$, hence $\lambda \notin [(1^n), (n-2, 2)]$. We have confirmed the conjecture up to $n = 18$. See Figure 4.3.

PROPOSITION 4.3.4.

- (1) For $n < 7$ and $n = 8$, $p_{[(1^n), (n-2, 2)]}$ is Schur-positive.
- (2) For $n \geq 9$, $p_{(1^n)} + p_{(2, 1^{n-2})} + p_{(2, 2, 1^{n-4})} + p_{(n-2, 2)}$ is Schur-positive.

PROOF. Part (1) can be checked explicitly by SAGEMATH [66]. For part (2), let

$$(4.13) \quad p_{(1^n)} + p_{(2, 1^{n-2})} + p_{(2, 2, 1^{n-4})} + p_{(n-2, 2)} = \sum_{\lambda \vdash n} c_\lambda s_\lambda.$$

We prove that $c_\lambda \geq 0$ for all $\lambda \vdash n$ by proving that all partial sums $p_{(1^n)}$, $p_{(1^n)} + p_{(2, 1^{n-2})}$, $p_{(1^n)} + p_{(2, 1^{n-2})} + p_{(2, 2, 1^{n-4})}$, $p_{(1^n)} + p_{(2, 1^{n-2})} + p_{(2, 2, 1^{n-4})} + p_{(n-2, 2)}$ are Schur-positive. We employ the combinatorial Murnaghan–Nakayama rule involving ribbon tableaux (see for example [60, Chapter 7.17])

$$p_\mu = \sum_{\lambda \vdash n} \chi^\lambda(\mu) s_\lambda \quad \text{where} \quad \chi^\lambda(\mu) = \sum_{T \in R(\lambda, \mu)} (-1)^{\text{ht}(T)}$$

and $R(\lambda, \mu)$ is the set of all ribbon tableaux of shape λ and type μ and $\text{ht}(T)$ is equal to the sum of the heights of all ribbons in T . We show that each subset of ribbon tableaux that contributes a negative term to c_λ is in bijection with a distinct subset of ribbon tableaux that contributes a positive number to c_λ , ensuring that $c_\lambda \geq 0$. We examine each partial sum of power sum symmetric functions, and demonstrate Schur-positivity at each step through these bijections.

(1) It is well-known that $p_{(1^n)} = \sum_{\lambda \vdash n} f^\lambda s_\lambda$ since $R(\lambda, (1^n))$ is the set of all standard Young tableaux of shape λ .

(2) For $T \in R(\lambda, (2, 1^{n-2}))$, T has either a horizontal or a vertical 2-ribbon and the remaining are single box ribbons. If T has a horizontal 2-ribbon, then $\text{ht}(T) = 0$ and T contributes $+1$ to $\chi^\lambda((2, 1^{n-2}))$. There are $f^{\lambda/(2)}$ such ribbon tableaux in $R(\lambda, (2, 1^{n-2}))$, where $f^{\lambda/\mu}$ is the cardinality of $\text{SYT}(\lambda/\mu)$, the set of standard Young tableaux of skew shape λ/μ . If T has a vertical 2-ribbon, then $\text{ht}(T) = 1$ and T contributes -1 to $\chi^\lambda((2, 1^{n-2}))$. There are $f^{\lambda/(1,1)}$ such ribbon tableaux in $R(\lambda, (2, 1^{n-2}))$. Therefore, the coefficient of s_λ in $p_{(2, 1^{n-2})}$ is $f^{\lambda/(2)} - f^{\lambda/(1,1)}$. If $(1, 1) \subseteq \lambda$, then c_λ includes $-f^{\lambda/(1,1)}$. The natural bijection

$$\text{SYT}(\lambda/(1, 1)) \rightarrow \{T \in \text{SYT}(\lambda) \mid T_{1,1} = 1 \text{ and } T_{2,1} = 2\}$$

demonstrates that $f^\lambda - f^{\lambda/(1,1)} \geq 0$. Hence $p_{(1^n)} + p_{(2, 1^{n-2})}$ is Schur-positive.

(3) For any $T \in R(\lambda, (2, 2, 1^{n-4}))$, there are six possible ways to arrange two 2-ribbons.

1	1	1	2	1	1	2	2	1	1	2	2	1	1
2	2	1	2					1		1		2	
								2				2	
								2					

$$\text{ht}(T) = 0 \quad \text{ht}(T) = 2 \quad \text{ht}(T) = 0 \quad \text{ht}(T) = 2 \quad \text{ht}(T) = 1 \quad \text{ht}(T) = 1$$

The remaining $n - 4$ ribbons in T are single boxes. Therefore, the coefficient of s_λ in $p_{(2, 2, 1^{n-4})}$ is

$$2f^{\lambda/(2,2)} + f^{\lambda/(4)} + f^{\lambda/(1^4)} - f^{\lambda/(3,1)} - f^{\lambda/(2,1,1)}.$$

If $(3, 1) \subseteq \lambda$, then c_λ includes $-f^{\lambda/(3,1)}$. Consider the bijection

$$\text{SYT}(\lambda/(3, 1)) \rightarrow \{T \in \text{SYT}(\lambda/(2)) \mid T_{1,3} = 1, \text{ and } T_{2,1} = 2\}.$$

If $(2, 1, 1) \subseteq \lambda$, then c_λ includes $-f^{\lambda/(2,1,1)}$. Consider the bijection

$$\text{SYT}(\lambda/(2, 1, 1)) \rightarrow \{T \in \text{SYT}(\lambda/(2)) \mid T_{2,1} = 1 \text{ and } T_{3,1} = 2\}.$$

Hence $f^{\lambda/(2)}$ from (2) and the terms $-f^{\lambda/(3,1)} - f^{\lambda/(2,1,1)}$ from (3) satisfy $f^{\lambda/(2)} - f^{\lambda/(3,1)} - f^{\lambda/(2,1,1)} \geq 0$. So far, we have shown that $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})}$ is Schur-positive.

(4) For $T \in R(\lambda, (n-2, 2))$, the possible ways of arranging one $(n-2)$ -ribbon and one 2-ribbon in T are the following. Note that 1^0 appearing in λ means that there are no parts of size 1 in λ .

$$\begin{array}{ccc} \lambda^{(1)} = (a, 1^b) & \lambda^{(2)} = (a, 1^b) & \lambda^{(3)} = (a, 3, 1^b) \\ 0 \leq b \leq n-3 & 2 \leq b \leq n-1 & 0 \leq b \leq n-6 \\ \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & \cdots & 1 & 2 & 2 \\ \hline 1 & & & & & \\ \hline \vdots & & & & & \\ \hline 1 & & & & & \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 1 & & & \\ \hline \vdots & & & \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline 2 & & & \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 1 & 2 & 2 & \\ \hline \vdots & & & \\ \hline 1 & & & \\ \hline \end{array} \end{array}$$

$$\text{ht}(T) = b \quad \text{ht}(T) = b - 2 + 1 \quad \text{ht}(T) = b + 1$$

$$\begin{array}{ccc} \lambda^{(4)} = (a, 2, 2, 1^b) & \lambda^{(5)} = (2, 2, 1^{n-4}) & \lambda^{(6)} = (n-2, 2) \\ 0 \leq b \leq n-6 & & \\ \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 1 & 2 & & \\ \hline 1 & 2 & & \\ \hline \vdots & & & \\ \hline 1 & & & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & 2 \\ \hline \vdots & \\ \hline 1 & \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline 1 & 1 & \cdots & 1 \\ \hline 2 & 2 & & \\ \hline \end{array} \end{array}$$

$$\text{ht}(T) = b + 2 + 1 \quad \text{ht}(T) = n - 3 + 1 \quad \text{ht}(T) = 0$$

If $\lambda = (a, 1^b)$ with $2 \leq b \leq n-3$, then cases $\lambda^{(1)}$ and $\lambda^{(2)}$ both apply. Since $\text{ht}(\lambda^{(1)})$ and $\text{ht}(\lambda^{(2)})$ have opposite parity, $\chi^{(a,1^b)}((n-2, 2)) = 0$. For $\lambda = (2, 1^{n-2})$, only $\lambda^{(2)}$ applies and $\chi^{(2,1^{n-2})}((n-2, 2)) = (-1)^{n-3}$. For $\lambda = (1^n)$, only $\lambda^{(2)}$ applies and $\chi^{(1^n)}((n-2, 2)) = (-1)^{n-1}$. Since (1^4) is contained in both $\lambda = (2, 1^{n-2})$, (1^n) , c_λ also includes $f^{\lambda/(1^4)} \geq 1$. For $\lambda = (n-1, 1)$, only $\lambda^{(1)}$ applies and $\chi^{(n-1,1)}((n-2, 2)) = -1$. In this case, $(4) \subseteq \lambda$, and thus c_λ also includes

$f^{\lambda/(4)} \geq 1$. For $\lambda = (n)$, the height of any ribbon tableaux is 0, so there are no negatives to worry about.

If λ is of the form $\lambda^{(3)}$, $\lambda^{(4)}$, or $\lambda^{(5)}$, then it is possible that $c_{\lambda^{(i)}}$ includes -1 from the unique $T \in R(\lambda^{(i)}, (n-2, 2))$ for $i = 3, 4, 5$. In any of these disjoint cases, $(2, 2) \subseteq \lambda^{(i)}$, which means $c_{\lambda^{(i)}}$ also includes $f^{\lambda^{(i)}/(2,2)} \geq 1$. Hence $f^{\lambda/(1^4)}$, $f^{\lambda/(4)}$, and $f^{\lambda/(2,2)}$ from (3) and $\chi^\lambda((n-2, 2))$ from (4) satisfy $f^{\lambda/(1^4)} + f^{\lambda/(4)} + f^{\lambda/(2,2)} + \chi^\lambda((n-2, 2)) \geq 0$. We have shown that $p_{(1^n)} + p_{(2, 1^{n-2})} + p_{(2, 2, 1^{n-4})} + p_{(n-2, 2)}$ is Schur-positive. $\square$

CONJECTURE 4.3.5. For $n \geq 9$, the interval $[(1^n), (n-2, 1, 1)]$ is given by

$$\{(1^n), (2, 1^{n-2}), (2, 2, 1^{n-4}), (3, 1^{n-3}), (n-2, 1, 1)\}.$$

REMARK 4.3.6. The cover relations $(1^n) \leq_I (2, 1^{n-2}) \leq_I (2, 2, 1^{n-4})$ are consequences of Corollary 4.2.19. By Proposition 4.2.18,

$$\varphi_0: \text{SSYT}((2, 2, 1^{n-4}), \nu) \rightarrow \text{SSYT}((3, 1^{n-3}), \nu)$$

is an injection (with $\alpha = 0, \beta = 2$). Since $(2, 2, 1^{n-4}) <_D (3, 1^{n-3})$, this implies $(2, 2, 1^{n-4}) \leq_I (3, 1^{n-3})$. By Corollary 4.2.31, we know $(3, 1^{n-3}) <_I (n-2, 1, 1)$ because $(3, 1^{n-3}) <_D (3, 1^{n-3})^t = (n-2, 1, 1)$. This implies

$$\{(1^n), (2, 1^{n-2}), (2, 2, 1^{n-4}), (3, 1^{n-3}), (n-2, 1, 1)\} \subseteq [(1^n), (n-2, 1, 1)].$$

However, we have not proven the cover relation $(3, 1^{n-3}) \leq_I (n-2, 1, 1)$. To show that $[(1^n), (n-2, 1, 1)]$ is contained in the above set, one could show that for all partitions λ such that λ and λ^t are not included in the above list, we have $f^\lambda > \frac{(n-1)(n-2)}{2} = f^{(n-2, 1, 1)}$. This would prove that $\lambda \notin [(1^n), (n-2, 1, 1)]$. We have confirmed the conjecture up to $n = 18$.

PROPOSITION 4.3.7.

- (1) For $n < 9$, $p_{[(1^n), (n-2, 1, 1)]}$ is Schur-positive.
- (2) For $n \geq 9$, $p_{(1^n)} + p_{(2, 1^{n-2})} + p_{(2, 2, 1^{n-4})} + p_{(3, 1^{n-3})} + p_{(n-2, 1, 1)}$ is Schur-positive.

PROOF. Part (1) can be checked explicitly using SAGEMATH. For part (2), as shown in the proof of Proposition 4.3.4, $p_{(1^n)} + p_{(2, 1^{n-2})} + p_{(2, 2, 1^{n-4})}$ is Schur-positive. We next show $p_{(1^n)} +$

$p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(3,1^{n-3})}$ is Schur-positive. For $T \in R(\lambda, (3, 1^{n-3}))$, there are three possible ways of arranging one 3-ribbon in T .

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array}$$

$$\text{ht}(T) = 0 \quad \text{ht}(T) = 1 \quad \text{ht}(T) = 2$$

Therefore, the coefficient of s_λ in $p_{(3,1^{n-3})}$ is $f^{\lambda/(3)} - f^{\lambda/(2,1)} + f^{\lambda/(1,1,1)}$.

If $(2, 1) \subseteq \lambda$, then c_λ includes $-f^{\lambda/(2,1)}$. Consider the bijection

$$\text{SYT}(\lambda/(2, 1)) \rightarrow \{T \in \text{SYT}(\lambda) \mid T_{1,1} = 1, T_{1,2} = 2, \text{ and } T_{2,1} = 3\}.$$

Note, the above subset of standard Young tableaux of shape λ in the term f^λ in $p_{(1^n)}$ was not used in the bijections in the proof of Proposition 4.3.4. This shows that $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(3,1^{n-3})}$ is Schur-positive.

We now examine the Schur expansion of $p_{(n-2,1,1)}$. There are a few specific shapes λ where $R(\lambda, (n-2, 1, 1))$ is nonempty. Note that for a ribbon tableau T of type $(n-2, 1, 1)$, $\text{ht}(T) = \text{ht}(R_1)$, where R_1 is the $(n-2)$ -ribbon of 1's in T . In Case 1, we examine all hook shapes $\lambda = (a, 1^b)$. In Case 2, we examine all shapes $\lambda = (a, 2, 1^b)$. In Case 3, we examine the remaining two shapes $(a, 3, 1^b)$ and $(a, 2, 2, 1^b)$.

Case 1a: $\lambda = (a, 1^b)$ with $4 \leq a \leq n-3$ and $3 \leq b \leq n-4$. These conditions on a, b require that the $(n-2)$ -ribbon forms a hook with nontrivial arm and nontrivial leg.

$$\begin{array}{|c|c|c|c|c|} \hline 1 & \cdots & 1 & 2 & 3 \\ \hline \vdots & & & & \\ \hline 1 & & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & \cdots & 1 & \\ \hline \vdots & & & \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline 3 & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 1 & \cdots & 1 & 2 & \\ \hline \vdots & & & & \\ \hline 1 & & & & \\ \hline 3 & & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 1 & \cdots & 1 & 3 & \\ \hline \vdots & & & & \\ \hline 1 & & & & \\ \hline 2 & & & & \\ \hline \end{array}$$

$$\text{ht}(T) = b \quad \text{ht}(T) = b-2 \quad \text{ht}(T) = b-1 \quad \text{ht}(T) = b-1$$

Since $b, b-2$ have opposite parity to $b-1$, $\chi^\lambda((n-2, 1, 1)) = 0$ for $\lambda = (a, 1^b)$ with $4 \leq a \leq n-3$ and $3 \leq b \leq n-4$.

Case 1b: $\lambda = (3, 1^{n-3})$.

1	2	3
:		
1		

1	1	2
:		
1		
3		

1	1	3
:		
1		
2		

1	1	1
:		
1		
2		
3		

$$\text{ht}(T) = n - 3 \quad \text{ht}(T) = n - 4 \quad \text{ht}(T) = n - 4 \quad \text{ht}(T) = n - 5$$

Since $n-3, n-5$ have opposite parity to $n-4$, $\chi^\lambda((n-2, 1, 1)) = 0$ for $\lambda = (3, 1^{n-3})$.

Case 1c: $\lambda = (2, 1^{n-2})$.

1	2
:	
1	
3	

1	3
:	
1	
2	

1	1
:	
1	
2	
3	

$$\text{ht}(T) = n - 3 \quad \text{ht}(T) = n - 3 \quad \text{ht}(T) = n - 4$$

Since $n - 3$ and $n - 4$ have opposite parity, $\chi^\lambda((n - 2, 1, 1)) = (-1)^{n-3}$ for $\lambda = (2, 1^{n-2})$. Since $n \geq 5$, $(1^4) \subseteq \lambda$ and $f^{\lambda/(1^4)} \geq 1$, the coefficient of s_λ in $p_{(2, 2, 1^{n-4})}$ will cancel this potential negative.

Case 1d: $\lambda = (1^n)$. The unique ribbon tableau $T \in \mathbf{R}((1^n), (n-2, 1, 1))$ has height $n-3$, hence $\chi^\lambda((n-2, 1, 1)) = (-1)^{n-3}$ for $\lambda = (1^n)$. Since $n \geq 5$, $(1^4) \subseteq \lambda$ and $f^{\lambda/(1^4)} \geq 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this potential negative.

Case 1e: $\lambda = (n - 2, 1, 1)$.

1	...	1
2		
3		

1	...	1	2
1			
3			

1	...	1	3
1			
2			

1	...	1	2	3
1				
1				

$$\text{ht}(T) = 0 \quad \text{ht}(T) = 1 \quad \text{ht}(T) = 1 \quad \text{ht}(T) = 2$$

Thus, $\chi^\lambda((n-2, 1, 1)) = 0$ for $\lambda = (n-2, 1, 1)$.

Case 1f: $\lambda = (n-1, 1)$.

1	...	1	2
3			

1	...	1	3
2			

1	...	1	2	3
1				

$$\text{ht}(T) = 0 \quad \text{ht}(T) = 0 \quad \text{ht}(T) = 1$$

Thus, $\chi^\lambda((n-2, 1, 1)) = 1$ for $\lambda = (n-1, 1)$.

Case 1g: $\lambda = (n)$. The unique ribbon tableau $T \in R((n), (n-2, 1, 1))$ has height 0, hence $\chi^\lambda((n-2, 1, 1)) = 1$ for $\lambda = (n)$.

Case 2a: $\lambda = (a, 2, 1^b)$ with $3 \leq a \leq n-3$ and $1 \leq b \leq n-5$.

1	1	...	1	3
1	2			
⋮				
1				

1	1	...	1	2
1	3			
⋮				
1				

1	1	...	1
1	2		
⋮			
1			
3			

1	1	...	1
1	3		
⋮			
1			
2			

$$\text{ht}(T) = b+1 \quad \text{ht}(T) = b+1 \quad \text{ht}(T) = b \quad \text{ht}(T) = b$$

Since b and $b+1$ have opposite parity, $\chi^\lambda((n-2, 1, 1)) = 0$ for $\lambda = (a, 2, 1^b)$ with $3 \leq a \leq n-3$ and $1 \leq b \leq n-5$.

Case 2b: $\lambda = (n-2, 2)$.

1	1	...	1
2	3		

1	1	...	1	3
1	2			

1	1	...	1	2
1	3			

$$\text{ht}(T) = 0 \quad \text{ht}(T) = 1 \quad \text{ht}(T) = 1$$

Thus, $\chi^\lambda((n-2, 1, 1)) = -1$ for $\lambda = (n-2, 2)$. Since $(2, 2) \subseteq \lambda$ and $f^{\lambda/(2,2)} \geq 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this negative.

Case 2c: $\lambda = (2, 2, 1^{n-4})$.

1	1
1	2
$\vdots$	
1	
3	

$$\text{ht}(T) = n - 3 \quad \text{ht}(T) = n - 4 \quad \text{ht}(T) = n - 4$$

Since $n-3$ and $n-4$ have opposite parity, $\chi^\lambda((n-2, 1, 1)) = (-1)^{n-4}$ for $\lambda = (2, 2, 1^{n-4})$. Since $(2, 2) \subseteq \lambda$ and $f^{\lambda/(2,2)} \geq 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this potential negative.

Case 3a: $\lambda = (n - (3 + b), 3, 1^b)$ for $0 \leq b \leq n - 6$. The unique ribbon tableau $T \in R((n - (3 + b), 3, 1^b), (n - 2, 1, 1))$ has height $b + 1$, hence $\chi^\lambda((n - 2, 1, 1)) = (-1)^{b+1}$ for $\lambda = (n - (3 + b), 3, 1^b)$. Since $(2, 2) \subseteq \lambda$ and $f^{\lambda/(2,2)} \geq 1$, the coefficient of s_λ in $p_{(2,2), 1^{n-4}}$ will cancel this potential negative.

Case 3b: $\lambda = (n - (4 + b), 2, 2, 1^b)$ for $0 \leq b \leq n - 6$. The unique ribbon tableau $T \in R((n - (4 + b), 2, 2, 1^b), (n - 2, 1, 1))$ has height $b + 2$, hence $\chi^\lambda((n - 2, 1, 1)) = (-1)^{b+2}$ for $\lambda = (n - (4 + b), 2, 2, 1^b)$. Since $(2, 2) \subseteq \lambda$ and $f^{\lambda/(2,2)} \geq 1$, the coefficient of s_λ in $p_{(2,2,1^{n-4})}$ will cancel this potential negative. $\square$

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