UNIVERSITY OF CALIFORNIA,
IRVINE

Topics on Schrödinger Operators

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

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2016

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Acknowledgments

I would like to express the deepest appreciation to my advisor, Prof. Abel Klein, for his supervision. His guidance has helped me a lot throughout the whole period of my postgraduate study and on the preparation of this dissertation. Without his persistent help this dissertation would not have been possible.

Financial support was provided by the NSF under grant DMS-1301641.

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Abstract of the Dissertation

Topics on Schrödinger Operators

By

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Doctor of Philosophy in Mathematics
University of California, Irvine, 2016
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We study two topics in the theory of Schrödinger operators:

1. We establish bounds on the density of states measures for Schrödinger operators with singular potentials. We obtain log-Hölder continuity for the density of states outer-measure in one, two, and three dimensions for Schrödinger operators with singular potentials, results that hold for the density of states measure when it exists. To do this, we study the local behavior of solutions of the stationary Schrödinger equation with singular potentials, establishing a local decomposition into a homogeneous harmonic polynomial and a lower order term, and, we prove a quantitative unique continuation principle for Schrödinger operators with singular potentials.

2. We develop an eigensystem bootstrap multiscale analysis for proving localization for the Anderson model at high disorder. The eigensystem multiscale analysis studies finite volume eigensystems, not finite volume Green's functions. It yields pure point spectrum with exponentially decay-

ing eigenfunctions, and dynamical localization. The starting hypothesis for the eigensystem bootstrap multiscale analysis only requires the verification of polynomial decay of the finite volume eigenfunctions, at some sufficiently large scale, with some minimal probability independent of the scale. It yields exponential localization of finite volume eigenfunctions in boxes of side L, with the eigenvalues and eigenfunctions labeled by the sites of the box, with probability higher than $1 - \mathrm{e}^{-L^{\xi}}$, for any desired $0 < \xi < 1$.

Introduction

We study two topics in the theory of Schrödinger operators:

Bounds on the density of states of Schrödinger operators with singular potentials

In Chapter 1, we establish bounds on the density of states of Schrödinger operators $H = -\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$, where Δ is the Laplacian operator, and V is a singular real potential. Given $\Lambda = \Lambda_L(x) = x + (\frac{L}{2}, \frac{L}{2})^d \subset \mathbb{R}^d$, the open box of side $L > 0$ centered at $x \in \mathbb{R}^d$, we let H_Λ and Δ_Λ be the restriction of H and Δ to $\mathrm{L}^2(\Lambda)$ with Dirichlet boundary condition. The finite volume density of states measure is given by

$$\eta_\Lambda(B) := \frac{1}{|\Lambda|} \operatorname{tr}\{\chi_B(H_\Lambda)\} \quad \text{for Borel sets} \quad B \subset \mathbb{R}^d. \tag{0.0.1}$$

Note that for V satisfying appropriate conditions (as in Theorem 1.0.1) and all $E \in \mathbb{R}$ we have

$$\eta_\Lambda(B) \leq C_{d,V,E} < \infty \quad \text{for all Borel sets} \quad B \subset (-\infty, E]. \tag{0.0.2}$$

For periodic and ergodic Schrödinger operators, density of states measure η can be defined as weak limits of the finite volume density of states measure η_Λ for sequences of boxes $\Lambda \to \mathbb{R}^d$ in an appropriate sense. The infinite volume density of states measure cannot be defined for general Schrödinger operators, so we follow [BoKl] and study the density of states outer-measure, defined on Borel subsets B of $\mathbb{R}^d$ by

$$\eta^*(B) := \limsup_{L\to\infty} \eta_L^*(B), \quad \text{where} \quad \eta_L^*(B) := \sup_{x\in\mathbb{R}^d} \eta_{\Lambda_L(x)}(B), \tag{0.0.3}$$

always finite on bounded sets in view of (1.0.2).

We obtain log-Hölder continuity for the density of states outer-measure of Schrödinger operators with singular potentials in one, two, and three dimensions, extending [BoKl, Theorem 1.1].

To establish the bounds on the density of states for $d = 2, 3$, we follow the proof in [BoKl], consider a class of approximate eigenfunctions for which we have local upper bounds, and pick one for which we have a global lower bound. The local upper bounds will come from the local behavior of approximate solutions of the stationary Schrödinger equation, and the global lower bound will come from the quantitative unique continuation principle. We extend these theorems to singular potentials.

In Section 1.1, we study the local behavior of solutions of the stationary Schrödinger equation with singular potentials, establishing a local decomposition into a homogeneous harmonic polynomial and a lower order term. As a corollary, we obtain bounds on the local behavior of approximate solutions for these equations. Singular potentials introduce technical problems not present for bounded potentials. This can be seen by considering the Schrödinger operator $H = -\Delta + V$. If V is a bounded potential, i.e., $V \in \mathrm{L}^\infty$, we have $\mathcal{D}(H) = \mathcal{D}(-\Delta) \subset \mathrm{H}^2$. However, if V is a singular potential, say $V \in \mathrm{L}^p$, where $p \in (d, \infty)$, we only have $\mathcal{D}(H) \subset \mathrm{H}^1$. Thus we have to work with solutions in H^1, not solutions in H^2 as in [BoKl]. The results in this section are published in [KT2].

In Section 1.2, we prove a quantitative unique continuation principle for Schrödinger operators $H = -\Delta + V$ on $\mathrm{L}^2(\Omega)$, where Ω is an open subset of $\mathbb{R}^d$, Δ is the Laplacian operator, and V is a singular real potential: $V \in \mathrm{L}^\infty(\Omega) + \mathrm{L}^p(\Omega)$. Our results extend the original result of Bourgain and Kenig [BoK, Lemma 3.10], as well as subsequent versions [GK3, Theorem A.1] and [BoKl, Theorem 3.4], where V is a bounded potential: $V \in \mathrm{L}^\infty(\Omega)$. To prove the quantitative unique continuation principle for singular potentials we use Sobolev inequalities (not required for bounded potentials). Also, as an application, we derive a unique continuation principle for spectral projections of Schrödinger operators with singular potentials, extending the bounded potential results of [Kl2, Theorem 1.1] and [KN, Theorem B.1]. The results in this section are published in [KT1].

The proof for the bounds on the density of states of Schrödinger operators with singular potentials will be discussed in Section 1.3. The results in this section are published in [KT2].

Eigensystem bootstrap multiscale analysis for the Anderson model

The eigensystem multiscale analysis is a new approach for proving localization for the Anderson model introduced by Elgart and Klein [EK]. The usual proofs of localization for random Schrödinger operators are based on the study of finite volume Green's functions [FroS, FroMSS, Dr, DrK, Sp, CH, FK, GK1, Kl1, BoK, GK3, AiM, Ai, AiSFH, AiENSS]. In contrast to the usual strategy, the eigensystem multiscale analysis is based on finite volume eigensystems, not finite volume Green's functions. It treats all energies of the finite volume operator at the same time, establishing level spacing and localization of eigenfunctions in a fixed box with high probability. A new feature is the labeling of the eigenvalues and eigenfunctions by the sites of the box.

In Chapter 2, we use a bootstrap argument as in [GK1] to enhance the eigensystem multiscale analysis. It yields exponential localization of finite volume eigenfunctions in boxes of side L, with the eigenvalues and eigenfunctions labeled by the sites of the box, with probability higher than $1 - \mathrm{e}^{-L^{\xi}}$, for any $0 < \xi < 1$. The starting hypothesis for the eigensystem bootstrap multiscale analysis only requires the verification of polynomial decay of the finite volume eigenfunctions, at some sufficiently large scale, with some minimal probability independent of the scale. The advantage of the bootstrap multiscale analysis is that from the same starting hypothesis we get conclusions that are valid for any $0 < \xi < 1$. The results in this chapter are written in [KT3].

Chapter 1

Bounds on the density of states of Schrödinger operators with singular potentials

We establish bounds on the density of states of Schrödinger operators $H = -\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$, where now Δ is the Laplacian operator, and V is a singular real potential. Given $\Lambda = \Lambda_L(x) = x + (\frac{L}{2}, \frac{L}{2})^d \subset \mathbb{R}^d$, the open box of side $L > 0$ centered at $x \in \mathbb{R}^d$, we let H_Λ and Δ_Λ be the restriction of H and Δ to $\mathrm{L}^2(\Lambda)$ with Dirichlet boundary condition. The finite volume density of states measure is given by

$$\eta_\Lambda(B) := \frac{1}{|\Lambda|} \operatorname{tr}\{\chi_B(H_\Lambda)\} \quad \text{for Borel sets} \quad B \subset \mathbb{R}^d. \tag{1.0.1}$$

Recall that for V satisfying appropriate conditions (as in Theorem 1.0.1 below) and all $E \in \mathbb{R}$ we have

$$\eta_\Lambda(B) \le C_{d,V,E} < \infty \quad \text{for all Borel sets} \quad B \subset (-\infty, E]. \tag{1.0.2}$$

For periodic and ergodic Schrödinger operators, density of states measure η can be defined as weak limits of the finite volume density of states measure η_Λ for sequences of boxes $\Lambda \to \mathbb{R}^d$ in an appropriate sense. The infinite volume density of states measure cannot be defined for general Schrödinger operators, so we follow [BoKl] and study the density of states outer-measure, defined on Borel subsets B of $\mathbb{R}^d$ by

$$\eta^*(B) := \limsup_{L\to\infty} \eta_L^*(B), \quad \text{where} \quad \eta_L^*(B) := \sup_{x\in\mathbb{R}^d} \eta_{\Lambda_L(x)}(B), \tag{1.0.3}$$

always finite on bounded sets in view of (1.0.2).

We obtain log-Hölder continuity for the density of states outer-measure of Schrödinger operators with singular potentials in one, two, and three dimensions, extending [BoKl, Theorem 1.1].

Theorem 1.0.1. *Let $H = -\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$, where $d = 1, 2, 3$, and V is a real potential such that:*

(i) if $d = 1$, $\sup_{x\in\mathbb{R}} \int_{\{|x-y|\leq 1\}} |V(y)| dy < \infty$;

(ii) if $d = 2$, $V = V^{(1)} + V^{(2)}$, where $V^{(1)} \in \mathrm{L}^\infty(\mathbb{R}^d)$ and $V^{(2)} \in \mathrm{L}^p(\mathbb{R}^d)$ with $p > 2$;

(iii) if $d = 3$, $V = V^{(1)} + V^{(2)}$, where $V^{(1)} \in \mathrm{L}^\infty(\mathbb{R}^d)$ and $V^{(2)} \in \mathrm{L}^p(\mathbb{R}^d)$ with $p > 6$.

Then, given $E_0 \in \mathbb{R}$, for all $E \leq E_0$ and $0 < \varepsilon \leq \frac{1}{2}$, we have

$$\eta^*([E, E+\varepsilon]) \leq \frac{C_{d,p,V,E_0}}{\left(\log \frac{1}{\varepsilon}\right)^{\kappa_d}}, \quad \text{where } \kappa_1 = 1, \kappa_d = \tfrac{(4-d)p-2d}{8p-4d} \text{ for } d = 2, 3. \tag{1.0.4}$$

To prove Theorem 1.0.1 for $d = 2, 3$, we follow the proof in [BoKl], consider a class of approximate eigenfunctions for which we have local upper bounds, and pick one for which we have a global lower bound. The local upper bounds will come from the local behavior of approximate solutions of the stationary Schrödinger equation, and the global lower bound will come from the quantitative unique continuation principle. We extends these theorems to singular potentials.

1.1 Local behavior of solutions of the stationary Schrödinger equation

We study the local behavior of solutions of the stationary Schrödinger equation with singular potentials, establishing a local decomposition into a homogeneous harmonic polynomial and a lower order term. As a corollary, we obtain bounds on the local behavior of approximate solutions for these equations.

Singular potentials introduce technical problems not present for bounded potentials. This can be seen by considering the Schrödinger operator $H = -\Delta + V$. If V is a bounded potential, i.e., $V \in \mathrm{L}^\infty$, we have $\mathcal{D}(H) = \mathcal{D}(-\Delta) \subset \mathrm{H}^2$. However, if V is a singular potential, say $V \in \mathrm{L}^p$, where $p \in (d, \infty)$, we only have $\mathcal{D}(H) \subset \mathrm{H}^1$. Thus we have to work with solutions in H^1, not solutions in H^2 as in [BoKl].

Let $\Omega = B(x_0, r) = \{y \in \mathbb{R}^d : |y - x_0| < r\}$, the ball centered at $x_0 \in \mathbb{R}^d$ with radius $r > 0$, where $|x| := (\sum_{j=1}^d |x_j|^2)^{\frac{1}{2}}$ for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. Given a real potential $W \in \mathrm{L}^p(\Omega)$, where $p \in (d, \infty)$, we consider the stationary Schrödinger equation

$$-\Delta\phi + W\phi = 0 \quad \text{a.e. on} \quad \Omega. \tag{1.1.1}$$

We let $\mathcal{E}_0(\Omega)$ be the linear space of solutions $\phi \in \mathrm{H}^1(\Omega)$, and define linear subspaces

$$\mathcal{E}_N(\Omega) = \left\{ \phi \in \mathcal{E}_0(\Omega) : \limsup_{x \to x_0} \frac{|\phi(x)|}{|x - x_0|^N} < \infty \right\} \quad \text{for } N \in \mathbb{N}. \tag{1.1.2}$$

We have $\mathcal{E}_1(\Omega) = \{\phi \in \mathcal{E}_0(\Omega) : \phi(x_0) = 0\}$, and $\mathcal{E}_N(\Omega) \supset \mathcal{E}_{N+1}(\Omega)$ for all $N \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$. The following theorem is an extension of [BoKl, Lemma 3.2] to singular potentials. (See [B, HW] for previous results.)

For dimensions $d \geq 2$, let $\mathcal{H}_m^{(d)}$ denote the vector space of homogenous harmonic polynomials on $\mathbb{R}^d$ of degree $m \in \mathbb{N}_0$, and set $\mathcal{H}_{\leq N}^{(d)} = \bigoplus_{m=0}^{N} \mathcal{H}_m^{(d)}$. Recall that there exists a constant $\gamma_d > 0$ such that (e.g., [ABR])

$$\dim \mathcal{H}_{\leq N}^{(d)} = \sum_{m=0}^{N} \dim \mathcal{H}_m^{(d)} \leq \gamma_d N^{d-1} \quad \text{for all } N \in \mathbb{N}. \tag{1.1.3}$$

Constants such as $C_{a,b,\ldots}$ will always be finite and depending only on the parameters or quantities $a, b, \ldots$; they will be independent of other parameters or quantities in the equation. Note that $C_{a,b,\ldots}$ may stand for different constants in different sides of the same inequality.

Theorem 1.1.1. *Let $d = 2, 3, \ldots$, $\Omega = B(x_0, 3r_0)$ for some $x_0 \in \mathbb{R}^d$ and $r_0 > 0$. Fix a real potential $W \in \mathrm{L}^p(\Omega)$, where $p \in (d, \infty)$, and set $W_p = \|W\|_{\mathrm{L}^p(\Omega)}$. For all $N \in \mathbb{N}_0$ there exists a linear map $Y_N^{(\Omega)} : \mathcal{E}_N(\Omega) \to \mathcal{H}_N^{(d)}$ such that for all $\phi \in \mathcal{E}_N(\Omega)$ we have, for all $x \in \overline{B\left(x_0, \frac{r_0}{2}\right)}$, that*

$$\begin{aligned} &|\phi(x) - (Y_N^{(\Omega)}\phi)(x - x_0)| \\ &\quad \leq r_0^{-\frac{d}{2}} (C_{d,p,W_p,r_0})^{N+2} \left(\tfrac{16}{3}\right)^{\frac{(N+1)(N+2)}{2}} ((N+1)!)^{d-2} |x - x_0|^{N+1} \|\phi\|_{\mathrm{L}^2(\Omega)}. \end{aligned} \tag{1.1.4}$$

As a consequence, for all $N \in \mathbb{N}_0$ we have

$$\mathcal{E}_{N+1}(\Omega) = \ker Y_N^{(\Omega)} \text{ and } \dim \mathcal{E}_{N+1}(\Omega) \geq \dim \mathcal{E}_N(\Omega) - \dim \mathcal{H}_N^{(d)}. \quad (1.1.5)$$

In particular, if $\mathcal{J}$ is a vector subspace of $\mathcal{E}_0(\Omega)$ we have

$$\dim \mathcal{J} \cap \mathcal{E}_{N+1}(\Omega) \geq \dim \mathcal{J} - \gamma_d N^{d-1} \text{ for all } N \in \mathbb{N}, \quad (1.1.6)$$

where γ_d is the constant in (1.1.3).

As a corollary, we obtain bounds on the local behavior of approximate solutions of the stationary Schrödinger equation (1.1.1) with singular potentials, extending [BoKl, Theorem 3.1].

Corollary 1.1.2. *For $d = 2, 3, \ldots$, let $\Omega \subset \mathbb{R}^d$ be an open subset. Let $B(x_0, r_0) \subset \Omega$ for some $x_0 \in \mathbb{R}^d$ and $r_0 > 0$. Fix a real valued function $W \in \mathrm{L}^p(B(x_0, r_0))$ for some $p \in (d, \infty)$. Suppose $\mathcal{F}$ is a linear subspace of $\mathrm{H}^1(\Omega)$ such that for all $\psi \in \mathcal{F}$ we have $\Delta\psi \in \mathrm{L}^2(B(x_0, r_0))$ and*

$$\|(-\Delta + W)\psi\|_{\mathrm{L}^\infty(B(x_0,r_0))} \leq C_{\mathcal{F}} \|\psi\|_{\mathrm{L}^2(\Omega)}. \quad (1.1.7)$$

Then there exists $0 < r_1 = r_1(d, p, W_p) < r_0$, where $W_p = \|W\|_{\mathrm{L}^p(B(x_0,r_0))}$, with the property that for all $N \in \mathbb{N}$ there is a linear subspace $\mathcal{F}_N$ of $\mathcal{F}$, with

$$\dim \mathcal{F}_N \geq \dim \mathcal{F} - \gamma_d N^{d-1}, \quad (1.1.8)$$

where γ_d is the constant in (1.1.3), *such that for all $\psi \in \mathcal{F}_N$ we have*

$$|\psi(x)| \leq (C_{d,p,W_p,r_1}^{N^2} |x - x_0|^{N+1} + C_{\mathcal{F}}) \|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{for all } x \in B(x_0, r_1). \quad (1.1.9)$$

The fundamental solution to Laplace's equation is given by

$$\Phi(x) = \Phi_d(x) := \begin{cases} (d(d-2)\omega_d)^{-1} |x|^{-d+2} & \text{if} \quad d = 3, 4, \ldots \\ -\frac{1}{2\pi} \log |x| & \text{if} \quad d = 2 \end{cases}, \quad (1.1.10)$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Proof of Theorem 1.1.1. We start as in [BoKl, Proof of Lemma 3.2]. We take $d = 2, 3, \ldots$, and prove the lemma for $\Omega = B(0,3) \subset \mathbb{R}^d$; the general case then follows by translating and dilating. We set $\Omega' = B(0, \frac{3}{2})$, and write $\mathcal{E}_n = \mathcal{E}_n(\Omega)$. Since we only have $\mathcal{E}_0 \subset \mathrm{H}^1(\Omega)$, we must proceed differently from [BoKl, Proof of Lemma 3.2]. A function $\phi \in \mathrm{H}^1(\Omega)$ satisfies an elliptic regularity estimate [T, Theorem 5.1]:

$$\|\phi\|_{\mathrm{L}^\infty(\Omega')} \leq C_{d,p,W_p} \|\phi\|_{\mathrm{L}^2(\Omega)}, \tag{1.1.11}$$

but for $\phi \in \mathrm{H}^1(\Omega)$ we do not have a readily available estimate for $\|\nabla\phi\|_{\mathrm{L}^\infty(B(0,1))}$ as in [BoKl, Eq. (3.18)], where we had $\phi \in \mathrm{H}^2(\Omega)$, and thus we must modify the induction.

We fix $\phi \in \mathcal{E}_0$ and consider its Newtonian potential given by

$$\psi(x) = -\int_{\Omega'} W(y)\phi(y)\Phi(x-y)dy \quad \text{for } x \in \mathbb{R}^d. \tag{1.1.12}$$

Let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$, so $q < \frac{d}{d-1} < \frac{d}{d-2}$. Then $\Phi \in \mathrm{L}^q(\Omega)$, and it follows from (1.1.11) that

$$|\psi(x)| \leq W_p \|\phi\|_{\mathrm{L}^\infty(\Omega')} \|\Phi\|_{\mathrm{L}^q(\Omega)} \leq C_{d,p,W_p} W_p \|\phi\|_{\mathrm{L}^2(\Omega)} \quad \text{for all} \quad x \in \Omega'. \tag{1.1.13}$$

Setting $h = \phi - \psi$, we have $\Delta h = 0$ weakly in Ω', as $\Delta\psi = W\phi$ weakly in Ω'. It follows that h is a harmonic function in $\Omega' \supset \overline{B(0,1)}$, and, using [ABR, Corollary 5.34 and its proof]), we have that

$$h(x) = \sum_{m=0}^{\infty} p_m(x) \text{ for all } x \in B(0,1), \text{ where } p_m \in \mathcal{H}_m^{(d)} \text{ for } m = 0, 1, \ldots, \tag{1.1.14}$$

with

$$|p_m(x)| \leq C_d m^{d-2} |x|^m \sup_{y \in \partial B(0,1)} |h(y)| \text{ for all } x \in B(0,1). \tag{1.1.15}$$

It follows from the mean value property that for all $y \in \partial B(0,1)$ we have

$$|h(y)| \leq \frac{1}{|B\left(y, \frac{1}{2}\right)|} \int_{B(y,\frac{1}{2})} |h(y')| dy' \leq C_{d,p,W_p} \|\phi\|_{\mathrm{L}^2(\Omega)} \tag{1.1.16}$$

using (1.1.11) and (1.1.13). Thus, it follows from (1.1.15) that

$$|p_m(x)| \leq C_{d,p,W_p} m^{d-2} \|\phi\|_{\mathrm{L}^2(\Omega)} |x|^m \quad \text{for all } x \in B(0,1),\ m = 1, 2, \ldots. \tag{1.1.17}$$

Setting $h_N = \sum_{m=0}^{N} p_m(x) \in \mathcal{H}_{\leq N}^{(d)}$, it follows that

$$|h(x) - h_N(x)| \leq C_{d,p,W_p} \|\phi\|_{\mathrm{L}^2(\Omega)} (N+1)^{d-2} |x|^{N+1} \quad \text{for} \quad x \in \overline{B\left(0, \tfrac{1}{2}\right)}. \tag{1.1.18}$$

Given $y \in \mathbb{R}^d \backslash \{0\}$, we let $\Phi_y(x) = \Phi(x - y)$. Since Φ_y is a harmonic function on $\mathbb{R}^d \backslash \{y\}$, it is real analytic in $B(0, |y|)$, and we have (see [ABR])

$$\Phi(x - y) = \Phi_y(x) = \sum_{m=0}^{\infty} J_m(x, y) \quad \text{for all } x \in B(0, |y|), \tag{1.1.19}$$

where $J_m(\cdot, y) \in \mathcal{H}_m^{(d)}$ for all $m = 0, 1, \ldots$, and the series converges absolutely and uniformly on compact subsets of $B(0, |y|)$. Moreover, for all $y \in \mathbb{R}^d$ and $m = 1, 2, \ldots$ we have (see [ABR, Corollary 5.34 and its proof]) that

$$\begin{aligned} |J_m(x, y)| &\leq C_d m^{d-2} \left(\frac{4|x|}{3|y|}\right)^m \sup_{x' \in \partial B\left(0, \frac{3}{4}|y|\right)} |\Phi_y(x')| \\ &\leq C_d m^{d-2} \left(\frac{4|x|}{3|y|}\right)^m \Phi\left(\frac{y}{4}\right) \quad \text{for all} \quad x \in \mathbb{R}^d. \end{aligned} \tag{1.1.20}$$

Setting $\Phi_{y,N}(x) = \sum_{m=0}^{N} J_m(x, y) \in \mathcal{H}_{\leq N}^{(d)}$, it follows that for $x \in \overline{B\left(0, \frac{1}{2}|y|\right)}$ we have

$$|\Phi_y(x) - \Phi_{y,N}(x)| \leq C_d (N+1)^{d-2} \left(\frac{4|x|}{3|y|}\right)^{N+1} \Phi\left(\frac{y}{4}\right). \tag{1.1.21}$$

We now proceed by induction. We set $\mathcal{E}_{-1} = \mathcal{E}_0$ and $\mathcal{H}^{(d)}_{-1} = \{0\}$. We define $Y_{-1} : \mathcal{E}_{-1}(\Omega) \to \mathcal{H}^{(d)}_{-1}$ by $Y_{-1}\phi = 0$ for all $\phi \in \mathcal{E}_{-1}$. The theorem holds for $N = -1$ from the elliptic regularity estimate (1.1.11).

We now let $N \in \mathbb{N}_0$ and suppose that the lemma is valid for $N-1$. If $\phi \in \mathcal{E}_N$, it follows that $\phi \in \mathcal{E}_{N-1}$ with $Y_{N-1}\phi = 0$, so by the induction hypothesis

$$|\phi(x)| \leq C_N \|\phi(x)\|_{\mathrm{L}^2(\Omega)} |x|^N \quad \text{for all } \overline{B\left(0, \tfrac{1}{2}\right)}, \tag{1.1.22}$$

$$\text{where} \quad C_N = \tilde{C}^{N+1}_{d,p,W_p} \left(\tfrac{16}{3}\right)^{\frac{N(N+1)}{2}} (N!)^{d-2}. \tag{1.1.23}$$

Using (1.1.20) and (1.1.22), we define

$$\psi_N(x) = -\int_{\Omega'} W(y)\phi(y)\Phi_{y,N}(x)dy \in \mathcal{H}^{(d)}_{\leq N}. \tag{1.1.24}$$

We fix $x \in \overline{B\left(0, \frac{1}{2}\right)}$ and estimate

$$|\psi(x) - \psi_N(x)| \leq W_p \left(\int_{\Omega'} (|\phi(y)||\Phi_{y,>N}(x)|)^q dy \right)^{\frac{1}{q}}, \tag{1.1.25}$$

where $\Phi_{y,>N}(x) = \Phi_y(x) - \Phi_{y,N}(x)$. From (1.1.21) and (1.1.22), with $p > d$, we get

$$\begin{aligned}
&\left(\int_{\overline{B\left(0,\frac{1}{2}\right)} \setminus B(0,2|x|)} (|\phi(y)||\Phi_{y,>N}(x)|)^q dy \right)^{\frac{1}{q}} \qquad (1.1.26)\\
&\leq C_d C_N \|\phi\|_{\mathrm{L}^2(\Omega)} (N+1)^{d-2} \left(\tfrac{4}{3}\right)^{N+1} |x|^{N+1} \left(\int_{\overline{B\left(0,\frac{1}{2}\right)} \setminus B(0,2|x|)} \left(\tfrac{1}{|y|} \Phi\left(\tfrac{y}{4}\right) \right)^q dy \right)^{\frac{1}{q}}\\
&\leq C_{d,p} C_N \|\phi\|_{\mathrm{L}^2(\Omega)} (N+1)^{d-2} \left(\tfrac{4}{3}\right)^{N+1} |x|^{N+1}.
\end{aligned}$$

If $y \notin B(0, 2|x|) \cup \overline{B\left(0, \frac{1}{2}\right)}$ we have $y \geq 2|x|$ and $y \geq \frac{1}{2}$, and hence, using

(1.1.21),

$$\left(\int_{\Omega'\setminus\left(B(0,2|x|)\cup\overline{B\left(0,\frac{1}{2}\right)}\right)}(|\phi(y)||\Phi_{y,>N}(x)|)^q dy\right)^{\frac{1}{q}} \tag{1.1.27}$$
$$\leq C_d(N+1)^{d-2}\left(\tfrac{8}{3}\right)^{N+1}\Phi\left(\tfrac{1}{8}\right)|x|^{N+1}\left(\int_{\Omega'}|\phi(y)|^q\right)^{\frac{1}{q}}$$
$$\leq C_d(N+1)^{d-2}\left(\tfrac{8}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)}.$$

Using (1.1.20) and (1.1.22), we get

$$\left(\int_{B(0,2|x|)\cap\overline{B\left(0,\frac{1}{2}\right)}}(|\phi(y)||\Phi_{y,>N}(x)|)^q dy\right)^{\frac{1}{q}} \tag{1.1.28}$$
$$\leq C_N\|\phi\|_{\mathrm{L}^2(\Omega)}\left(\int_{B(0,2|x|)\cap\overline{B\left(0,\frac{1}{2}\right)}}(|y|^N|\Phi_{y,>N}(x)|)^q dy\right)^{\frac{1}{q}}$$
$$\leq C_N\|\phi\|_{\mathrm{L}^2(\Omega)}\left(\int_{B(0,2|x|)\cap\overline{B\left(0,\frac{1}{2}\right)}}(|y|^N|\Phi(x-y)|)^q dy\right)^{\frac{1}{q}}$$
$$+ C_dC_N\|\phi\|_{\mathrm{L}^2(\Omega)}\sum_{m=0}^{N}m^{d-2}\left(\tfrac{4}{3}|x|\right)^m\left(\int_{B(0,2|x|)\cap\overline{B\left(0,\frac{1}{2}\right)}}\left(|y|^{N-m}\left|\Phi\left(\tfrac{y}{4}\right)\right|\right)^q dy\right)^{\frac{1}{q}}$$
$$\leq C_dC_N\|\phi\|_{\mathrm{L}^2(\Omega)}\left(2^N+N^{d-2}\left(\tfrac{4}{3}\right)^{N+1}\right)|x|^{N+1},$$

where we used $\frac{3|x|}{|x-y|}\geq 1$ for $y\in B(0,2|x|)$. (Note that we get $|x|^{N+2-\frac{d}{p}}$ if

$d \geq 3$ and $|x|^{\left(N+2-\frac{d}{p}\right)-}$ if $d = 2$.) Also using (1.1.20), we get

$$\left(\int_{\Omega' \setminus \overline{B\left(0,\frac{1}{2}\right)}} (|\phi(y)||\Phi_{y,>N}(x)|)^q dy\right)^{\frac{1}{q}} \tag{1.1.29}$$
$$\leq \left(\int_{\Omega' \setminus \overline{B\left(0,\frac{1}{2}\right)}} (|\phi(y)||\Phi(x-y)|)^q dy\right)^{\frac{1}{q}}$$
$$+ C_d \sum_{m=0}^{N} m^{d-2} \left(\tfrac{4}{3}|x|\right)^m \left(\int_{\Omega' \setminus \overline{B\left(0,\frac{1}{2}\right)}} \left(|\phi(y)||y|^{-m} \left|\Phi\left(\tfrac{y}{4}\right)\right|\right)^q dy\right)^{\frac{1}{q}}$$
$$\leq C_{d,p,W_p} \|\phi\|_{\mathrm{L}^2(\Omega)} \left(1 + N^{d-2} \left(\tfrac{4}{3}\right)^{N+1}\right),$$

where we used $|x| \leq \frac{1}{2}$. Since $|x| > \frac{1}{4}$ if $y \in B(0, 2|x|) \setminus \overline{B\left(0, \frac{1}{2}\right)}$, we obtain

$$\left(\int_{(\Omega' \cap B(0,2|x|)) \setminus \overline{B\left(0,\frac{1}{2}\right)}} (|\phi(y)||\Phi_{y,>N}(x)|)^q dy\right)^{\frac{1}{q}} \tag{1.1.30}$$
$$\leq C_{d,p,W_p} \|\phi\|_{\mathrm{L}^2(\Omega)} \left(4^{N+1} + N^{d-2} \left(\tfrac{16}{3}\right)^{N+1}\right) |x|^{N+1}.$$

Combining (1.1.25), (1.1.26), (1.1.27), (1.1.28) and (1.1.30), we have ($C_N \geq 1$)

$$|\psi(x) - \psi_N(x)| \leq C_{d,p,W_p} C_N W_p (N+1)^{d-2} |x|^{N+1} \|\phi\|_{\mathrm{L}^2(\Omega)}, \tag{1.1.31}$$

for all $x \in \overline{B\left(0, \frac{1}{2}\right)}$.

Now let $Y_N \phi = h_N + \psi_N \in \mathcal{H}_N^{(d)}$. It follows from (1.1.18), (1.1.31) and (1.1.23), choosing the constant $\tilde{C}_{d,p,W_p}$ in (1.1.23) large enough, that for all

$x \in \overline{B\left(0, \frac{1}{2}\right)}$ we have

$$
\begin{aligned}
&|\phi(x) - (Y_N\phi)(x)| \le |h(x) - h_N(x)| + |\psi(x) - \psi_N(x)| \\
&\le (C_{d,p,W_p} + C_{d,p}W_pC_N)(N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)} \\
&\le \tilde{C}_{d,p,W_p}C_N(N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)} \\
&\le \tilde{C}_{d,p,W_p}\left(\tilde{C}^{N+1}_{d,p,W_p}\left(\tfrac{16}{3}\right)^{\frac{N(N+1)}{2}}(N!)^{d-2}\right)(N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)} \\
&\le \tilde{C}^{N+2}_{d,p,W_p}\left(\tfrac{16}{3}\right)^{\frac{(N+1)(N+2)}{2}}((N+1)!)^{d-2}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)}.
\end{aligned}
$$

This completes the induction.

Since (1.1.5) is a consequence of (1.1.4), and (1.1.6) follows from (1.1.5). the lemma is proven. □

Corollary 1.1.2 is an immediate consequence from the following corollary.

Corollary 1.1.3. *For $d = 2, 3, \ldots$, let $\Omega \subset \mathbb{R}^d$ be an open subset. Let $B(x_0, r_1) \subset \Omega$ for some $x_0 \in \mathbb{R}^d$ and $r_1 > 0$. Fix a real valued function $W \in \mathrm{L}^p(B(x_0, r_1))$ for some $p \in (d, \infty)$. Suppose $\mathcal{F}$ is a linear subspace of $\mathrm{H}^1(\Omega)$ such that for all $\psi \in \mathcal{F}$ we have $\Delta\psi \in \mathrm{L}^2(B(x_0, r_1))$ and*

$$
\|(-\Delta + W)\psi\|_{\mathrm{L}^\infty(B(x_0,r_1))} \le C_{\mathcal{F}}\|\psi\|_{\mathrm{L}^2(\Omega)}. \tag{1.1.32}
$$

Then there exists $0 < r_2 = r_2(d, p, W_p) < r_1$, where $W_p = \|W\|_{\mathrm{L}^p(B(x_0,r_1))}$, with the property that for all $r \in (0, r_2]$ there is a linear map $Z_r : \mathcal{F} \to \mathcal{E}_0(B(x_0, r))$ such that

$$
\|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B(x_0,r))} \le C_{d,r}C_{\mathcal{F}}\|\psi\|_{\mathrm{L}^2(\Omega)}, \quad \text{where } \lim_{r\to 0} C_{d,r} = 0. \tag{1.1.33}
$$

As a consequence, for all $N \in \mathbb{N}$ there is a vector subspace $\mathcal{F}_N$ of $\mathcal{F}$, with

$$
\dim \mathcal{F}_N \ge \dim \mathcal{F} - \gamma_d N^{d-1}, \tag{1.1.34}
$$

such that for all $\psi \in \mathcal{F}_N$ *we have*

$$|\psi(x)| \leq (C_{d,p,W_p,r_1}^{N^2}|x-x_0|^{N+1}+C_{\mathcal{F}})\|\psi\|_{\mathrm{L}^2(\Omega)} \quad \textit{for all}\, x \in \overline{B(x_0, \tfrac{r_2}{6})}. \tag{1.1.35}$$

Proof. We proceed as in [BoKl, Lemma 3.3]. It suffices to consider $x_0 = 0$. We set $B_r = B(0, r)$. Given $0 < r < r_1$ and $\psi \in \mathrm{H}^1(\Omega)$ with $\Delta\psi \in \mathrm{L}^2(B_r)$, we define $Z_r\psi \in \mathcal{E}_0(B_r)$ as the unique solution $\phi \in \mathrm{H}^1(B_r)$ to the Dirichlet problem on B_r given by

$$\begin{cases} -\Delta\phi + W\phi = 0 & \text{on } B_r, \\ \phi = \psi & \text{on } \partial B_r. \end{cases} \tag{1.1.36}$$

This map is well defined in view of [T, Theorem 3.2]. (Since $W \in \mathrm{L}^p(B_r)$ for some $p \in (d, \infty)$, $|W|$ is compactly bounded on $H_0^1(B_r)$ by [T, Lemma 1.4]. Moreover, for $\psi \in \mathrm{H}^1(\Omega)$ with $\Delta\psi \in \mathrm{L}^2(B_r)$ we have $\|\nabla\psi\|_{\mathrm{L}^2(B_r)}^2 + \int_{B_r} |W|\,|\psi|^2\,dx < \infty$ (see (1.2.36) and (1.2.61) for details). Therefore [T, Theorem 3.2] can be applied.) It is clearly a linear map.

To prove (1.1.33), we use the Green's function $G_r(x, y)$ for the ball B_r (see [GiT, Section 2.5]),

$$G_r(x, y) = \begin{cases} \Phi(|x-y|) - \Phi(\frac{|y|}{r}|x - \frac{r^2}{|y|^2}y|) & \text{if } y \neq 0, \\ \Phi(|x|) - \Phi(r) & \text{if } y = 0. \end{cases} \tag{1.1.37}$$

Let $\psi \in \mathcal{F}$. Using Green's representation formula [GiT, Eq. (2.21)] for ψ and $Z_r\psi$, for all $x \in B_r$ we have

$$\psi(x) = -\int_{\partial B_r} \psi(\zeta)\partial_\nu G_r(x,\zeta)dS(\zeta) - \int_{B_r} W(y)\psi(y)G_r(x,y)dy \tag{1.1.38}$$

$$+ \int_{B_r} ((-\Delta + W)\psi)(y)\, G_r(x,y)dy, \tag{1.1.39}$$

$$(Z_r\psi)(x) = -\int_{\partial B_r} \psi(\zeta)\partial_\nu G_r(x,\zeta)dS(\zeta) - \int_{B_r} W(y)(Z_r\psi)(y)G_r(x,y)dy,$$

where dS denotes the surface measure and ∂_ν is the normal derivative. For all $x \in B_r$ an explicit calculation gives

$$\|G_r(x,\cdot)\|_{\mathrm{L}^1(B_r)} \le C'_d r^{\frac{d(\alpha_d-1)}{\alpha_d}} \|G_r(x,\cdot)\|_{\mathrm{L}^{\alpha_d}(B_r)} \le C_d r^{\frac{d(\alpha_d-1)}{\alpha_d}}, \tag{1.1.40}$$

$$\|G_r(x,\cdot)\|_{\mathrm{L}^q(B_r)} \le C'_d r^{\frac{d(\alpha_d-q)}{\alpha_d q}} \|G_r(x,\cdot)\|_{\mathrm{L}^{\alpha_d}(B_r)} \le C_d r^{\frac{d(\alpha_d-q)}{\alpha_d q}}, \tag{1.1.41}$$

where $\alpha_2 = 2$ and $\alpha_d = \frac{d-1}{d-2}$ for $d \ge 3$, and $\frac{1}{p} + \frac{1}{q} = 1$ ($q < \frac{d}{d-1} \le \alpha_d$ as $p > d$). We conclude that

$$\begin{aligned} &\|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B_r)} \\ &\quad \le C_d r^{\frac{d(\alpha_d-q)}{\alpha_d q}} W_p \|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B_r)} + C_d r^{\frac{d(\alpha_d-1)}{\alpha_d}} \|(-\Delta + W)\psi\|_{\mathrm{L}^\infty(B_r)}. \end{aligned} \tag{1.1.42}$$

Taking $r_2 \in (0, r_1)$ such that $C_d r^{\frac{d(\alpha_d-q)}{\alpha_d q}}(1 + W_p) \le \frac{1}{2}$, and using (1.1.32), we get (1.1.33).

Letting $\mathcal{J} = \operatorname{Ran} Z_{r_2}$, and setting $\mathcal{J}_N = \mathcal{J} \cap \mathcal{E}_{N+1}(B_{r_2})$, $\mathcal{F}_N = Z_{r_2}^{-1}(\mathcal{J}_N)$, the estimate (1.1.35) follows using the argument in [BoKl, Lemma 3.3]. □

1.2 Quantitative unique continuation principle

We prove a quantitative unique continuation principle for Schrödinger operators $H = -\Delta + V$ on $\mathrm{L}^2(\Omega)$, where Ω is an open subset of $\mathbb{R}^d$, Δ is the Laplacian operator, and V is a singular real potential: $V \in \mathrm{L}^\infty(\Omega) + \mathrm{L}^p(\Omega)$. Our results extend the original result of Bourgain and Kenig [BoK, Lemma 3.10], as well as subsequent versions [GK3, Theorem A.1] and [BoKl, Theorem 3.4], where V is a bounded potential: $V \in \mathrm{L}^\infty(\Omega)$.

As an application, we derive a unique continuation principle for spectral projections of Schrödinger operators with singular potentials, extending the bounded potential results of [Kl2, Theorem 1.1] and [KN, Theorem B.1].

To prove the quantitative unique continuation principle for singular potentials we use Sobolev inequalities (not required for bounded potentials). Since the Sobolev inequality we use in dimension $d = 2$ is expressed in terms of Orlicz norms, we review Orlicz spaces, following [RR]. A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is called a Young function if it is increasing, convex, $\varphi(0) = 0$, and $\lim_{t\to\infty} \varphi(t) = \infty$. Its complementary function, given by $\varphi^*(t) = \sup_{s\in\mathbb{R}^+}\{st - \varphi(s)\}$ for $t \in \mathbb{R}^+$, is also a Young function. Given a Young function φ and a σ-finite measure μ on a measurable space X, we define the Orlicz space

$$\mathrm{L}^{\varphi}(X) = \left\{ f : X \to \mathbb{R} \text{ measurable} \,\middle|\, \int_X \varphi(\alpha|f|)d\mu < \infty \text{ for some } \alpha > 0 \right\}, \tag{1.2.1}$$

a Banach space when equipped with the Orlicz norm

$$\|f\|_{\varphi} := \inf \left\{ k > 0 : \int_X \varphi\left(\tfrac{1}{k}|f|\right) d\mu \le 1 \right\}. \tag{1.2.2}$$

(A standard example is $\varphi(t) = t^p$ with $1 \le p < \infty$; in this case $\mathrm{L}^{\varphi}(X) = \mathrm{L}^p(X)$.) There is a Hölder's inequality for Orlicz spaces:

$$\int_X |fg|d\mu \le 2\|f\|_{\varphi}\|g\|_{\varphi^*} \quad \text{for all} \quad f \in \mathrm{L}^{\varphi}(X),\ g \in \mathrm{L}^{\varphi^*}(X). \tag{1.2.3}$$

We now state our main theorem, a quantitative unique continuation principle for Schrödinger operators with singular potentials. We fix the Young function

$$\varphi(t) = e^t - 1, \quad \text{so} \quad \varphi^*(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le 1 \\ t\log t - t + 1 & \text{if} \quad t > 1 \end{cases}. \tag{1.2.4}$$

Theorem 1.2.1. *Let Ω be an open subset of $\mathbb{R}^d$, $K = K_1 + K_2$ with $K_1, K_2 \geq 0$, and consider a real measurable function $V = V^{(1)} + V^{(2)}$ on Ω with $\|V^{(1)}\|_\infty \leq K_1$. Let $\psi \in \mathrm{L}^2(\Omega)$ be real valued with $\Delta\psi \in \mathrm{L}^2_{loc}(\Omega)$, and suppose*

$$\zeta = -\Delta\psi + V\psi \in \mathrm{L}^2(\Omega). \tag{1.2.5}$$

Fix a bounded measurable set $\Theta \subset \Omega$ where $\|\psi_\Theta\|_2 > 0$, and set

$$Q(x, \Theta) := \sup_{y\in\Theta} |y - x| \quad \textit{for} \quad x \in \Omega. \tag{1.2.6}$$

Consider $x_0 \in \Omega \backslash \overline{\Theta}$ such that

$$Q = Q(x_0, \Theta) \geq 1 \quad \textit{and} \quad B(x_0, 6Q + 2) \subset \Omega, \tag{1.2.7}$$

and take

$$0 < \delta \leq \min\{\mathrm{dist}(x_0, \Theta), \tfrac{1}{2}\}. \tag{1.2.8}$$

There is a constant $m_d > 0$, depending only on d, such that:

(i) If either $d \geq 3$ and $\|V^{(2)}\|_p \leq K_2$ with $p \geq d$, or $d = 2$ and $(\||V^{(2)}|^p\|_{\varphi^*})^{\frac{1}{p}} \leq K_2$ with $p \geq 2$, we have

$$\left(\frac{\delta}{Q}\right)^{m_d(1+K^{\frac{2p}{3p-2d}})(Q^{\frac{4p-2d}{3p-2d}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2})} \|\psi_\Theta\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2 + \delta^2\|\zeta_\Omega\|_2^2. \tag{1.2.9}$$

In particular, if $d = 2$ it suffices to require $\|V^{(2)}\|_p \leq K_2$ with $p > 2$ to obtain (1.2.9).

(ii) If $d = 1$ and $\|V^{(2)}\|_p \leq K_2$ with $p \geq 2$, we have

$$\left(\frac{\delta}{Q}\right)^{m_1(1+K^{\frac{2p}{3p-4}})(Q^{\frac{4p-4}{3p-4}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2})} \|\psi_\Theta\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2 + \delta^2\|\zeta_\Omega\|_2^2. \tag{1.2.10}$$

Letting $p \to \infty$ in Theorem 1.2.1 we recover [BoKl, Theorem 3.4]. The proof of Theorem 1.2.1, given in Section 1.2.1, relies on a Carleman estimate of Escauriaza and Vesella [EsV, Theorem 2], stated in Lemma 1.2.4. To control singular potentials we use all the terms in this estimate, including the the gradient term, and Sobolev's inequalities. In the proofs for bounded potentials [BoK, GK3, BoKl] it suffices to use a simpler version of this Carleman estimate without the the gradient term (see [BoK, Lemma 3.15]).

As an application of Theorem 1.2.1, we prove a unique continuation principle for spectral projections of Schrödinger operators with singular potentials, extending [Kl2, Theorem 1.1] (in the form given in [KN, Theorem B.1]) to Schrödinger operators with singular potentials. (See also [CHK1, Section 4], [CHK2, Theorem 2.1], [GK3, Theorem A.6], and [RoV, Theorem 2.1] for unique continuation principles for spectral projections of Schrödinger operators with bounded potentials.)

We consider rectangles in $\mathbb{R}^d$ of the form

$$\Lambda = \Lambda_{\mathbf{L}}(a) = a + \prod_{j=1}^{d} \left(-\tfrac{L_j}{2}, \tfrac{L_j}{2}\right) = \prod_{j=1}^{d} \left(a_j - \tfrac{L_j}{2}, a_j + \tfrac{L_j}{2}\right), \qquad (1.2.11)$$

where $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and $\mathbf{L} = (L_1, \dots, L_d) \in (0, \infty)^d$. (We write $\Lambda_L(a) = \Lambda_{\mathbf{L}}(a)$ in the special case $L_j = L$ for $j = 1, \dots, d$.) Given a Schrödinger operator $H = -\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$, by $H_\Lambda = -\Delta_\Lambda + V_\Lambda$ we denote the restriction of H to the rectangle Λ with either Dirichlet or periodic boundary condition: Δ_Λ is the Laplacian on Λ with either Dirichlet or periodic boundary condition, and V_Λ is the restriction of V to Λ.

Theorem 1.2.2. *Let $H = -\Delta + V$ be a Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^d)$, where $V = V^{(1)} + V^{(2)}$ with $\|V^{(1)}\|_\infty \le K_1 < \infty$ and $\|V^{(2)}\|_p \le K_2 < \infty$ with*

$p \geq d$ for $d \geq 3$, $p > 2$ for $d = 2$, and $p \geq 2$ for $d = 1$. Set $K = K_1 + K_2$. Fix $\delta \in (0, \frac{1}{2}]$, and let $\{y_k\}_{k \in \mathbb{Z}^d}$ be sites in $\mathbb{R}^d$ with $B(y_k, \delta) \subset \Lambda_1(k)$ for all $k \in \mathbb{Z}^d$. There exists a constant $M_d > 0$, depending only on d, such that, defining $\gamma = \gamma(d, p, K, \delta, E_0) > 0$ for $E_0 > 0$ by

$$\gamma^2 = \begin{cases} \frac{1}{2}\delta^{M_d\left(1+(K+E_0)^{\frac{4p^2}{(3p-2d)(2p-d)}}\right)} & \text{for} \quad d \geq 2 \\ \frac{1}{2}\delta^{M_d\left(1+(K+E_0)^{\frac{2p^2}{(3p-4)(p-1)}}\right)} & \text{for} \quad d = 1 \end{cases}, \tag{1.2.12}$$

then, given a rectangle Λ as in (1.2.11), where $a \in \mathbb{R}^d$ and $L_j \geq 114\sqrt{d}$ for $j = 1, \ldots, d$, and a closed interval $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma$, we have

$$\chi_I(H_\Lambda) W^{(\Lambda)} \chi_I(H_\Lambda) \geq \gamma^2 \chi_I(H_\Lambda), \tag{1.2.13}$$

where

$$W^{(\Lambda)} = \sum_{k \in \mathbb{Z}^d,\, \Lambda_1(k) \subset \Lambda} \chi_{B(y_k, \delta)}. \tag{1.2.14}$$

The proof of Theorem 1.2.2 is discussed in Section 1.2.2.

Remark 1.2.3. Using Theorem 1.2.2 we can prove optimal Wegner estimates for Anderson Hamiltonians with singular background potentials, extending the results of [Kl2].

1.2.1 The quantitative unique continuation principle

The proof of Theorem 1.2.1 is based on a Carleman estimate of Escauriaza and Vesella [EsV, Theorem 2], which we state in a ball of radius $\varrho > 0$.

Lemma 1.2.4. *Given $\varrho > 0$, the function $\omega_\varrho(x) = \phi(\frac{1}{\varrho}|x|)$ on $\mathbb{R}^d$, where $\phi(s) := s e^{-\int_0^s \frac{1-e^{-t}}{t} dt}$, is a strictly increasing continuous function on $[0, \infty)$,*

C^∞ *on* $(0,\infty)$*, satisfying*

$$\frac{1}{C_1\varrho}|x| \leq \omega_\varrho(x) \leq \frac{1}{\varrho}|x| \quad \textit{for} \quad x \in B(0,\varrho), \tag{1.2.15}$$

where $C_1 = \phi(1)^{-1} \in (2,3)$*. Moreover, there exist positive contants* C_2 *and* C_3*, depending only on* d*, such that for all* $\alpha \geq C_2$ *and all real valued functions* $f \in H^2(B(0,\varrho))$ *with* $\operatorname{supp} f \subset B(0,\varrho)\backslash\{0\}$ *we have*

$$\alpha^3 \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha} f^2 dx + \alpha\varrho^2 \int_{\mathbb{R}^d} \omega_\varrho^{1-2\alpha} |\nabla f|^2 dx \leq C_3 \varrho^4 \int_{\mathbb{R}^d} \omega_\varrho^{2-2\alpha} (\Delta f)^2 dx. \tag{1.2.16}$$

This estimate is given in the parabolic setting in [EsV], but the estimate in the elliptic setting as in the lemma follows immediately by the argument in [KSU, Proposition B.3]. In the proofs of the quantitative unique continuation principle for bounded potentials [BoK, GK3, BoKl] only the first term in the left hand side of (1.2.16) is used (see [BoK, Lemma 3.15]), but for singular potentials we also need to use the gradient term in the left hand side of (1.2.16) and Sobolev's inequalities.

Proof of Theorem 1.2.1. Let C_1, C_2, C_3 be the constants of Lemma 1.2.4, which depend only on d. Without loss of generality $C_2 > 1$. By C_j, $j = 4, 5, \ldots$, we will always denote an appropriate nonzero constant depending only on d.

We follow Bourgain and Klein's proof for bounded potentials [BoKl, Theorem 3.4]. Let $x_0 \in \Omega\backslash\overline{\Theta}$ be as in (1.2.7). Without loss of generality we take $x_0 = 0$, $\Theta \subset B(0, 2C_1Q)$, and $\Omega = B(0,\varrho)$, where $\varrho = 2C_1Q + 2$, and let δ be as in (1.2.8). Proceeding as in [BoKl, Theorem 3.4], we fix a function $\eta \in C_c^\infty(\mathbb{R}^d)$ given by $\eta(x) = \xi(|x|)$, where ξ is an even C^∞ function on $\mathbb{R}$,

$0 \le \xi \le 1$, such that

$$\xi(s) = 1 \quad \text{if} \quad \tfrac{3}{4}\delta \le |s| \le 2C_1Q, \quad \xi(s) = 0 \quad \text{if} \quad |s| \le \tfrac{1}{4}\delta \text{ or } |s| \ge 2C_1Q + 1,$$
$$|\xi^j(s)| \le \left(\tfrac{4}{\delta}\right)^j \quad \text{if} \quad |s| \le \tfrac{3}{4}\delta, \quad |\xi^j(s)| \le 2^j \quad \text{if} \quad |s| \ge 2C_1Q, j = 1, 2, \tag{1.2.17}$$

$$|\nabla\eta(x)| \le \sqrt{d}|\xi'(|x|)| \quad \text{and} \quad |\Delta\eta(x)| \le d|\xi''(|x|)|,$$
$$\operatorname{supp}\nabla\eta \subset \{\tfrac{\delta}{4} \le |x| \le \tfrac{3\delta}{4}\} \cup \{2C_1Q \le |x| \le 2C_1Q + 1\}.$$

Let $\alpha \ge C_2$. Applying Lemma 1.2.4 to the function $\eta\psi$ gives

$$\begin{aligned}
&\frac{\alpha^3}{3C_3\varrho^4}\int_{\mathbb{R}^d}\omega_\varrho^{-1-2\alpha}\eta^2\psi^2dx + \frac{\alpha}{3C_3\varrho^2}\int_{\mathbb{R}^d}\omega_\varrho^{1-2\alpha}|\nabla(\eta\psi)|^2dx \\
&\qquad \le \frac{1}{3}\int_{\mathbb{R}^d}\omega_\varrho^{2-2\alpha}(\Delta(\eta\psi))^2dx \le \int_{\mathbb{R}^d}\omega_\varrho^{2-2\alpha}\eta^2(\Delta\psi)^2dx \\
&\qquad\quad + 4\int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}|\nabla\eta|^2|\nabla\psi|^2dx + \int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}(\Delta\eta)^2\psi^2dx.
\end{aligned} \tag{1.2.18}$$

Using (1.2.5), $\|V^{(1)}\|_\infty \le K_1$, and $\omega_\varrho \le 1$ on $\operatorname{supp}\eta$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d}\omega_\varrho^{2-2\alpha}\eta^2(\Delta\psi)^2dx &\le 2\int_{\mathbb{R}^d}V^2\omega_\varrho^{2-2\alpha}\eta^2\psi^2dx + 2\int_{\mathbb{R}^d}\omega_\varrho^{2-2\alpha}\eta^2\zeta^2dx \\
&\le 4K_1^2\int_{\mathbb{R}^d}\omega_\varrho^{-1-2\alpha}\eta^2\psi^2dx + 4\int_{\mathbb{R}^d}(V^{(2)})^2\omega_\varrho^{2-2\alpha}\eta^2\psi^2dx + 2\int_{\mathbb{R}^d}\omega_\varrho^{2-2\alpha}\eta^2\zeta^2dx.
\end{aligned} \tag{1.2.19}$$

Given $M > 0$, we write $V^{(2)} = U_M + V_M$, where $U_M = V^{(2)}\chi_{\{|V^{(2)}| \le \sqrt{M}\}}$ and $W_M = V^{(2)}\chi_{\{|V^{(2)}| > \sqrt{M}\}}$. We have

$$\int_{\mathbb{R}^d}(V^{(2)})^2\omega_\varrho^{2-2\alpha}\eta^2\psi^2dx \le M\int_{\mathbb{R}^d}\omega_\varrho^{-1-2\alpha}\eta^2\psi^2dx + \int_{\mathbb{R}^d}W_M^2\omega_\varrho^{2-2\alpha}\eta^2\psi^2dx. \tag{1.2.20}$$

Combining (1.2.18), (1.2.19) and (1.2.20), we have

$$
\begin{aligned}
&\left(\frac{\alpha^3}{3C_3\varrho^4} - 4K_1^2 - 4M\right) \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha}\eta^2\psi^2 dx + \frac{\alpha}{3C_3\varrho^2} \int_{\mathbb{R}^d} \omega_\varrho^{1-2\alpha}|\nabla(\eta\psi)|^2 dx \\
&\quad \le 4\int_{\mathbb{R}^d} W_M^2 \omega_\varrho^{2-2\alpha}\eta^2\psi^2 dx + 2\int_{\mathbb{R}^d} \omega_\varrho^{2-2\alpha}\eta^2\zeta^2 dx \qquad (1.2.21)\\
&\qquad + 4\int_{\operatorname{supp}\nabla\eta} \omega_\varrho^{2-2\alpha}|\nabla\eta|^2|\nabla\psi|^2 dx + \int_{\operatorname{supp}\nabla\eta} \omega_\varrho^{2-2\alpha}(\Delta\eta)^2\psi^2 dx.
\end{aligned}
$$

Note that for $1 \le q \le p$ we have

$$
\|W_M\|_q \le M^{-\frac{p-q}{2q}} \|W_M\|_p^{\frac{p}{q}} \le M^{-\frac{p-q}{2q}} \|V^{(2)}\|_p^{\frac{p}{q}} \le M^{-\frac{p-q}{2q}} K_2^{\frac{p}{q}}. \qquad (1.2.22)
$$

We set $K = K_1 + K_2$ with $K_1, K_2 \ge 0$.

We consider three cases:

(a) $d \ge 3$: Let $\|V^{(2)}\|_p \le K_2$ with $p \ge d$. Using Hölder's inequality and (1.2.22) with $q = d$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d} W_M^2 \omega_\varrho^{2-2\alpha}\eta^2\psi^2 dx \le \|W_M^2\|_{\frac{d}{2}} \|\omega_\varrho^{2-2\alpha}\eta^2\psi^2\|_{\frac{d}{d-2}} \qquad (1.2.23)\\
&= \|W_M\|_d^2 \|\omega_\varrho^{1-\alpha}\eta\psi\|_{\frac{2d}{d-2}}^2 \le M^{-\frac{p-d}{d}} K_2^{\frac{2p}{d}} \|\omega_\varrho^{1-\alpha}\eta\psi\|_{\frac{2d}{d-2}}^2.
\end{aligned}
$$

Using Sobolev's inequality (e.g., [GiT, Theorem 7.10]), we get

$$
\begin{aligned}
\|\omega_\varrho^{1-\alpha}\eta\psi\|_{\frac{2d}{d-2}}^2 &\le C_4 \left(\int_{\mathbb{R}^d} |\nabla(\omega_\varrho^{1-\alpha}\eta\psi)|^2\right) \qquad (1.2.24)\\
&\le 2C_4 \int_{\mathbb{R}^d} |\nabla\omega_\varrho^{1-\alpha}|^2\eta^2\psi^2 dx + 2C_4 \int_{\mathbb{R}^d} \omega_\varrho^{1-2\alpha}|\nabla(\eta\psi)|^2 dx.
\end{aligned}
$$

Since

$$
|\nabla\omega_\varrho^{1-\alpha}|^2 = (1-\alpha)^2 \frac{\omega_\varrho^{2-2\alpha}}{|x|^2 \exp(\frac{2}{\varrho}|x|)} \le \frac{\alpha^2}{\varrho^2}\omega_\varrho^{-2\alpha}, \qquad (1.2.25)
$$

we have (recall $\omega_\varrho \le 1$ on $\operatorname{supp}\eta$)

$$
\int_{\mathbb{R}^d} |\nabla\omega_\varrho^{1-\alpha}|^2\eta^2\psi^2 dx \le \frac{\alpha^2}{\varrho^2} \int_{\mathbb{R}^d} \omega_\varrho^{-1-2\alpha}\eta^2\psi^2 dx. \qquad (1.2.26)
$$

Combining (1.2.21), (1.2.23), (1.2.24) and (1.2.26), we conclude that

$$\begin{aligned}\left(\frac{\alpha^3}{3C_3\varrho^4} - 4K_1^2 - 4M - 8C_4M^{-\frac{p-d}{d}}K_2^{\frac{2p}{d}}\frac{\alpha^2}{\varrho^2}\right)\int_{\mathbb{R}^d}\omega_\varrho^{-1-2\alpha}\eta^2\psi^2dx \\ + \left(\frac{\alpha}{3C_3\varrho^2} - 8C_4M^{-\frac{p-d}{d}}K_2^{\frac{2p}{d}}\right)\int_{\mathbb{R}^d}\omega_\varrho^{1-2\alpha}|\nabla(\eta\psi)|^2dx \\ \leq 4\int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}|\nabla\eta|^2|\nabla\psi|^2dx + \int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}(\Delta\eta)^2\psi^2dx \\ + 2\int_{\operatorname{supp}\eta}\omega_\varrho^{2-2\alpha}\eta^2\zeta^2dx.\end{aligned} \tag{1.2.27}$$

Assuming $\alpha \geq \varrho$ and setting $M = K_2^2\alpha^{\frac{2d}{p}}\varrho^{\frac{-2d}{p}}$, we have

$$\begin{aligned}4K_1^2 + 4M + 8C_4M^{-\frac{p-d}{d}}K_2^{\frac{2p}{d}}\alpha^2\varrho^{-2} &= 4K_1^2 + 4K_2^2(1+2C_4)\alpha^{\frac{2d}{p}}\varrho^{\frac{-2d}{p}} \\ &\leq (4K^2(1+2C_4))\alpha^{\frac{2d}{p}}\varrho^{\frac{-2d}{p}}.\end{aligned} \tag{1.2.28}$$

Taking

$$\alpha \geq C_5(1+K^{\frac{2p}{3p-2d}})\varrho^{\frac{4p-2d}{3p-2d}} \geq C_5(1+K^{\frac{2p}{3p-2d}})\varrho^{\frac{4}{3}}, \tag{1.2.29}$$

we can guarantee that $\alpha > C_2$,

$$\frac{\alpha^3}{3C_3\varrho^4} \geq 3(4K^2(1+2C_4)\alpha^{\frac{2d}{p}}\varrho^{\frac{-2d}{p}}), \tag{1.2.30}$$

and

$$\frac{\alpha}{3C_3\varrho^2} - 8C_4M^{-\frac{p-d}{d}}K_2^{\frac{2p}{d}} \geq 0. \tag{1.2.31}$$

Using (1.2.15) and recalling (1.2.6), we obtain

$$\int_{\mathbb{R}^d}\omega_\varrho^{-1-2\alpha}\eta^2\psi^2dx \geq \left(\frac{\varrho}{Q}\right)^{1+2\alpha}\|\psi_\Theta\|_2^2 \geq (2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2. \tag{1.2.32}$$

Combining (1.2.27), (1.2.30), (1.2.31) and (1.2.32), we conclude that

$$\begin{aligned}\frac{2\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \leq 4\int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}|\nabla\eta|^2|\nabla\psi|^2dx \\ + \int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}(\Delta\eta)^2\psi^2dx + 2\int_{\operatorname{supp}\eta}\omega_\varrho^{2-2\alpha}\eta^2\zeta^2dx.\end{aligned} \tag{1.2.33}$$

Let $f \in \mathcal{D}(\nabla)$. For arbitrary $M > 0$ we have

$$\left| \int_{\mathbb{R}^d} V f^2 dx \right| \leq (K_1 + M^{\frac{1}{2}}) \|f\|_2^2 + \int_{\mathbb{R}^d} |W_M| f^2 dx. \tag{1.2.34}$$

Using Hölder's inequality, (1.2.22) with $q = \frac{d}{2}$, and Sobolev's inequality, we get

$$\left| \int_{\mathbb{R}^d} V f^2 dx \right| \leq (K_1 + M^{\frac{1}{2}}) \|f\|_2^2 + C_4 M^{-\frac{2p-d}{2d}} K_2^{\frac{2p}{d}} \|\nabla f\|_2^2. \tag{1.2.35}$$

Taking $M = (2C_4 K_2^{\frac{2p}{d}})^{\frac{2d}{2p-d}}$ (we can require $C_4 \geq 1$), we get

$$\left| \int_{\mathbb{R}^d} V f^2 dx \right| \leq 2C_4 (1 + K^{\frac{2p}{2p-d}}) \|f\|_2^2 + \frac{1}{2} \|\nabla f\|_2^2. \tag{1.2.36}$$

We have

$$\begin{aligned} &\int_{\{2C_1 Q \leq |x| \leq 2C_1 Q + 1\}} \omega_\varrho^{2-2\alpha} (4|\nabla \eta|^2 |\nabla \psi|^2 + (\Delta \eta)^2 \psi^2) dx \qquad (1.2.37) \\ &\leq 16 d^2 \left(\frac{C_1 \varrho}{2C_1 Q} \right)^{2\alpha - 2} \int_{\{2C_1 Q \leq |x| \leq 2C_1 Q + 1\}} (4|\nabla \psi|^2 + \psi^2) dx \\ &\leq C_6 \left(\tfrac{5}{4} C_1 \right)^{2\alpha - 2} \int_{\{2C_1 Q - 1 \leq |x| \leq 2C_1 Q + 2\}} (\zeta^2 + (1 + K^{\frac{2p}{2p-d}}) \psi^2) dx \\ &\leq C_6 \left(\tfrac{5}{4} C_1 \right)^{2\alpha - 2} (\|\zeta_\Omega\|_2^2 + (1 + K^{\frac{2p}{2p-d}}) \|\psi_\Omega\|_2^2), \end{aligned}$$

where we used (1.2.36) and an interior estimate (e.g., [GK2, Lemma A.2]). Similarly,

$$\begin{aligned} &\int_{\{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\}} \omega_\varrho^{2-2\alpha} (4|\nabla \eta|^2 |\nabla \psi|^2 + (\Delta \eta)^2 \psi^2) dx \qquad (1.2.38) \\ &\leq 256 d^2 \delta^{-4} (4\delta^{-1} C_1 \varrho)^{2\alpha - 2} \int_{\{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\}} (4|\nabla \psi|^2 + \psi^2) dx \\ &\leq C_7 \delta^{-4} (4\delta^{-1} C_1 \varrho)^{2\alpha - 2} \int_{\{|x| \leq \delta\}} (\zeta^2 + (K^{\frac{2p}{2p-d}} + \delta^{-2}) \psi^2) dx \\ &\leq C_7 \delta^{-4} (16 \delta^{-1} C_1^2 Q)^{2\alpha - 2} (\|\zeta_\Omega\|_2^2 + (K^{\frac{2p}{2p-d}} + \delta^{-2}) \|\psi_{0,\delta}\|_2^2). \end{aligned}$$

In addition,

$$\int_{\operatorname{supp}\eta} \omega_\varrho^{2-2\alpha}\eta^2\zeta^2 dx \le (4\delta^{-1}C_1\varrho)^{2\alpha-2}\|\zeta_\Omega\|_2^2 \le (16\delta^{-1}C_1^2 Q)^{2\alpha-2}\|\zeta_\Omega\|_2^2. \tag{1.2.39}$$

If we have

$$\frac{\alpha^3}{\varrho^4}\left(\frac{8}{5}\right)^{2\alpha}\|\psi_\Theta\|_2^2 \ge C_8(1+K^{\frac{2p}{2p-d}})\|\psi_\Omega\|_2^2, \tag{1.2.40}$$

we obtain

$$C_6\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}(1+K^{\frac{2p}{2p-d}})\|\psi_\Omega\|_2^2 \le \frac{1}{2}\frac{2\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2, \tag{1.2.41}$$

so we conclude that

$$\begin{aligned}&\frac{\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \\ &\qquad \le C_9\delta^{-4}(16\delta^{-1}C_1^2Q)^{2\alpha-2}((K^{\frac{2p}{2p-d}}+\delta^{-2})\|\psi_{0,\delta}\|_2^2+\|\zeta_\Omega\|_2^2).\end{aligned} \tag{1.2.42}$$

Thus,

$$\frac{\alpha^3}{\varrho^4}Q^4((8C_1Q)^{-1}\delta)^{2\alpha+2}\|\psi_\Theta\|_2^2 \le C_{10}((K^{\frac{2p}{2p-d}}+\delta^{-2})\|\psi_{0,\delta}\|_2^2+\|\zeta_\Omega\|_2^2). \tag{1.2.43}$$

Since $(\frac{\delta}{Q})^5 \le (\frac{1}{2})^5 \le \frac{1}{8C_1}$ by (1.2.8), we have

$$\frac{\alpha^3}{\varrho^4}Q^6\left(\frac{\delta}{Q}\right)^{12\alpha+14}\|\psi_\Theta\|_2^2 \le C_{11}((1+K^{\frac{2p}{2p-d}})\|\psi_{0,\delta}\|_2^2+\delta^2\|\zeta_\Omega\|_2^2). \tag{1.2.44}$$

To satisfy (1.2.29) and (1.2.40), we choose

$$\alpha = C_{12}(1+K^{\frac{2p}{3p-2d}})\left(Q^{\frac{4p-2d}{3p-2d}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right), \tag{1.2.45}$$

Combining with (1.2.44), and recalling $Q \ge 1$, we get

$$\begin{aligned}&(1+K^{\frac{2p}{3p-2d}})^3\left(\frac{\delta}{Q}\right)^{C_{13}(1+K^{\frac{2p}{3p-2d}})\left(Q^{\frac{4p-2d}{3p-2d}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)}\|\psi_\Theta\|_2^2 \\ &\qquad \le C_{14}((1+K^{\frac{2p}{2p-d}})\|\psi_{0,\delta}\|_2^2+\delta^2\|\zeta_\Omega\|_2^2),\end{aligned} \tag{1.2.46}$$

and hence

$$\left(\frac{\delta}{Q}\right)^{m_d(1+K^{\frac{2p}{3p-2d}})(Q^{\frac{4p-2d}{3p-2d}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2})}\|\psi_\Theta\|_2^2 \le \|\psi_{x_0,\delta}\|_2^2 + \delta^2\|\zeta_\Omega\|_2^2, \quad (1.2.47)$$

where $m_d > 0$ is a constant depending only on d.

(b) $d = 2$: Let $(\||V^{(2)}|^p\|_{\varphi^*})^{\frac{1}{p}} \le K_2$ with $p \ge 2$. Given $K_2 > 0$ and $M > 0$, we have

$$\int_{\mathbb{R}^2} \varphi^*\left(\frac{|W_M^2|}{M^{-\frac{p-2}{2}}K_2^p}\right)dx \le \int_{\mathbb{R}^2}\varphi^*\left(\frac{|V^{(2)}|^p}{K_2^p}\right)dx, \quad (1.2.48)$$

and hence, using $\||V^{(2)}|^p\|_{\varphi^*} \le K_2^p$, we get

$$\|W_M^2\|_{\varphi^*} \le M^{-\frac{p-2}{2}}K_2^p. \quad (1.2.49)$$

Using Hölder's inequality for Orlicz spaces (1.2.3), and (1.2.49), we get

$$\begin{aligned}\int_{\mathbb{R}^2} W_M^2\omega_\varrho^{2-2\alpha}\eta^2\psi^2dx &\le 2\|W_M^2\|_{\varphi^*}\|\omega_\varrho^{2-2\alpha}\eta^2\psi^2\|_\varphi \\ &\le 2M^{-\frac{p-2}{2}}K_2^p\|\omega_\varrho^{2-2\alpha}\eta^2\psi^2\|_\varphi. \quad (1.2.50)\end{aligned}$$

Using the Sobolev inequality given in [AT, Theorem 0.1], we obtain

$$\begin{aligned}\|\omega_\varrho^{2-2\alpha}\eta^2\psi^2\|_\varphi &\le C_4\left(\int_{\mathbb{R}^2}|\omega_\varrho^{1-\alpha}\eta\psi|^2dx + \int_{\mathbb{R}^2}|\nabla(\omega_\varrho^{1-\alpha}\eta\psi)|^2dx\right) \quad (1.2.51)\\ &\le C_4\int_{\mathbb{R}^2}|\omega_\varrho^{1-\alpha}\eta\psi|^2dx + 2C_4\int_{\mathbb{R}^2}|\nabla\omega_\varrho^{1-\alpha}|^2\eta^2\psi^2dx \\ &\quad + 2C_4\int_{\mathbb{R}^2}\omega_\varrho^{1-2\alpha}|\nabla(\eta\psi)|^2dx.\end{aligned}$$

Combining (1.2.21), (1.2.50), (1.2.51), and (1.2.26) with $d = 2$, we conclude that

$$\left(\frac{\alpha^3}{3C_3\varrho^4} - 4K_1^2 - 4M - 8C_4M^{-\frac{p-2}{2}}K_2^p - 16C_4M^{-\frac{p-2}{2}}K_2^p\frac{\alpha^2}{\varrho^2}\right)\int_{\mathbb{R}^2}\omega_\varrho^{-1-2\alpha}\eta^2\psi^2dx$$
$$+\left(\frac{\alpha}{3C_3\varrho^2} - 16C_4M^{-\frac{p-2}{2}}K_2^p\right)\int_{\mathbb{R}^2}\omega_\varrho^{1-2\alpha}|\nabla(\eta\psi)|^2dx$$
$$\leq 4\int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}|\nabla\eta|^2|\nabla\psi|^2dx + \int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}(\Delta\eta)^2\psi^2dx \tag{1.2.52}$$
$$+ 2\int_{\operatorname{supp}\eta}\omega_\varrho^{2-2\alpha}\eta^2\zeta^2dx.$$

Assuming $\alpha \geq \varrho$ and setting $M = K_2^2\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}}$, we have

$$4K_1^2 + 4M + 8C_4M^{-\frac{p-2}{2}}K_2^p + 16C_4M^{-\frac{p-2}{2}}K_2^p\frac{\alpha^2}{\varrho^2} \tag{1.2.53}$$
$$\leq 4K_1^2 + 4M + 24C_4M^{-\frac{p-2}{2}}K_2^p\frac{\alpha^2}{\varrho^2}$$
$$= 4K_1^2 + 4K_2^2(1+6C_4)\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}} \leq 4K^2(1+6C_4)\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}}.$$

Taking

$$\alpha \geq C_5(1+K^{\frac{2p}{3p-4}})\varrho^{\frac{4p-4}{3p-4}} \geq C_5(1+K^{\frac{2p}{3p-4}})\varrho^{\frac{4}{3}}, \tag{1.2.54}$$

we can guarantee that $\alpha > C_2$,

$$\frac{\alpha^3}{3C_3\varrho^4} \geq 3(4K^2(1+6C_4)\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}}), \tag{1.2.55}$$

and

$$\frac{\alpha}{3C_3\varrho^2} - 16C_4M^{-\frac{p-2}{2}}K_2^p \geq 0. \tag{1.2.56}$$

Using (1.2.15) and recalling (1.2.6), we obtain

$$\int_{\mathbb{R}^2}\omega_\varrho^{-1-2\alpha}\eta^2\psi^2dx \geq \left(\frac{\varrho}{Q}\right)^{1+2\alpha}\|\psi_\Theta\|_2^2 \geq (2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2. \tag{1.2.57}$$

Combining (1.2.52), (1.2.55), (1.2.56) and (1.2.57), we conclude that

$$\frac{2\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \leq 4\int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}|\nabla\eta|^2|\nabla\psi|^2dx$$
$$+\int_{\operatorname{supp}\nabla\eta}\omega_\varrho^{2-2\alpha}(\Delta\eta)^2\psi^2dx+2\int_{\operatorname{supp}\eta}\omega_\varrho^{2-2\alpha}\eta^2\zeta^2dx. \qquad (1.2.58)$$

Given $M>0$, we have

$$\int_{\mathbb{R}^2}\varphi^*\left(\frac{|W_M|}{M^{-\frac{p-1}{2}}K_2^p}\right)dx \leq \int_{\mathbb{R}^2}\varphi^*\left(\frac{|V^{(2)}|^p}{K_2^p}\right)dx, \qquad (1.2.59)$$

and hence, using $\||V^{(2)}|^p\|_{\varphi^*} \leq K_2^p$, we get $\|W_M\|_{\varphi^*} \leq M^{-\frac{p-1}{2}}K_2^p$. Let $f \in \mathcal{D}(\nabla)$. Then, using (1.2.34), Hölder's inequality for Orlicz spaces (1.2.3), and the Sobolev inequality in [AT, Theorem 0.1], we get

$$\left|\int_{\mathbb{R}^2}Vf^2dx\right| \leq (K_1+M^{\frac{1}{2}}+2C_4M^{-\frac{p-1}{2}}K_2^p)\|f\|_2^2+2C_4M^{-\frac{p-1}{2}}K_2^p\|\nabla f\|_2^2. \qquad (1.2.60)$$

Taking $M=(4C_4K_2^p)^{\frac{2}{p-1}}$ (we can require $C_4 \geq 1$), we get

$$\left|\int_{\mathbb{R}^2}Vf^2dx\right| \leq 4C_4(1+K^{\frac{p}{p-1}})\|f\|_2^2+\tfrac{1}{2}\|\nabla f\|_2^2. \qquad (1.2.61)$$

We have

$$\int_{\{2C_1Q\leq|x|\leq 2C_1Q+1\}}\omega_\varrho^{2-2\alpha}(4|\nabla\eta|^2|\nabla\psi|^2+(\Delta\eta)^2\psi^2)dx \qquad (1.2.62)$$
$$\leq 64\left(\frac{C_1\varrho}{2C_1Q}\right)^{2\alpha-2}\int_{\{2C_1Q\leq|x|\leq 2C_1Q+1\}}(4|\nabla\psi|^2+\psi^2)dx$$
$$\leq C_6\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}\int_{\{2C_1Q-1\leq|x|\leq 2C_1Q+2\}}(\zeta^2+(1+K^{\frac{p}{p-1}})\psi^2)dx$$
$$\leq C_6\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}(\|\zeta_\Omega\|_2^2+(1+K^{\frac{p}{p-1}})\|\psi_\Omega\|_2^2),$$

where we used (1.2.61) and an interior estimate. Similarly,

$$\begin{aligned}
&\int_{\{\frac{\delta}{4}\leq|x|\leq\frac{3\delta}{4}\}} \omega_\varrho^{2-2\alpha}(4|\nabla\eta|^2|\nabla\psi|^2+(\Delta\eta)^2\psi^2)dx \qquad (1.2.63)\\
&\quad\leq 1024\delta^{-4}(4\delta^{-1}C_1\varrho)^{2\alpha-2}\int_{\{\frac{\delta}{4}\leq|x|\leq\frac{3\delta}{4}\}}(4|\nabla\psi|^2+\psi^2)dx\\
&\quad\leq C_7\delta^{-4}(4\delta^{-1}C_1\varrho)^{2\alpha-2}\int_{\{|x|\leq\delta\}}(\zeta^2+(K^{\frac{p}{p-1}}+\delta^{-2})\psi^2)dx\\
&\quad\leq C_7\delta^{-4}(16\delta^{-1}C_1^2Q)^{2\alpha-2}(\|\zeta_\Omega\|_2^2+(K^{\frac{p}{p-1}}+\delta^{-2})\|\psi_{0,\delta}\|_2^2).
\end{aligned}$$

In addition,

$$\int_{\operatorname{supp}\eta}\omega_\varrho^{2-2\alpha}\eta^2\zeta^2dx\leq(4\delta^{-1}C_1\varrho)^{2\alpha-2}\|\zeta_\Omega\|_2^2\leq(16\delta^{-1}C_1^2Q)^{2\alpha-2}\|\zeta_\Omega\|_2^2. \qquad (1.2.64)$$

If we have

$$\frac{\alpha^3}{\varrho^4}\left(\frac{8}{5}\right)^{2\alpha}\|\psi_\Theta\|_2^2\geq C_8(1+K^{\frac{p}{p-1}})\|\psi_\Omega\|_2^2, \qquad (1.2.65)$$

we obtain

$$C_6\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}(1+K^{\frac{p}{p-1}})\|\psi_\Omega\|_2^2\leq\frac{1}{2}\frac{2\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2, \qquad (1.2.66)$$

so we conclude that

$$\begin{aligned}
&\frac{\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \qquad (1.2.67)\\
&\qquad\leq C_9\delta^{-4}(16\delta^{-1}C_1^2Q)^{2\alpha-2}((K^{\frac{p}{p-1}}+\delta^{-2})\|\psi_{0,\delta}\|_2^2+\|\zeta_\Omega\|_2^2).
\end{aligned}$$

Thus,

$$\frac{\alpha^3}{\varrho^4}Q^4((8C_1Q)^{-1}\delta)^{2\alpha+2}\|\psi_\Theta\|_2^2\leq C_{10}((K^{\frac{p}{p-1}}+\delta^{-2})\|\psi_{0,\delta}\|_2^2+\|\zeta_\Omega\|_2^2). \qquad (1.2.68)$$

Since $(\frac{\delta}{Q})^5\leq(\frac{1}{2})^5\leq\frac{1}{8C_1}$ by (1.2.8), we have

$$\frac{\alpha^3}{\varrho^4}Q^6\left(\frac{\delta}{Q}\right)^{12\alpha+14}\|\psi_\Theta\|_2^2\leq C_{11}((1+K^{\frac{p}{p-1}})\|\psi_{0,\delta}\|_2^2+\delta^2\|\zeta_\Omega\|_2^2). \qquad (1.2.69)$$

To satisfy (1.2.54) and (1.2.65), we choose

$$\alpha = C_{12}(1 + K^{\frac{2p}{3p-4}}) \left(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2} \right), \tag{1.2.70}$$

Combining with (1.2.69), and recalling $Q \geq 1$, we get

$$(1 + K^{\frac{2p}{3p-4}})^3 \left(\frac{\delta}{Q} \right)^{C_{13}(1+K^{\frac{2p}{3p-4}})\left(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2} \right)} \|\psi_\Theta\|_2^2$$
$$\leq C_{14}((1 + K^{\frac{p}{p-1}})\|\psi_{0,\delta}\|_2^2 + \delta^2 \|\zeta_\Omega\|_2^2), \tag{1.2.71}$$

and hence there exists $m > 0$ such that

$$\left(\frac{\delta}{Q} \right)^{m(1+K^{\frac{2p}{3p-4}})\left(Q^{\frac{4p-4}{3p-4}} + \log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2} \right)} \|\psi_\Theta\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2 + \delta^2 \|\zeta_\Omega\|_2^2. \tag{1.2.72}$$

If $\|V^{(2)}\|_p \leq K_2 < \infty$ for some $p > 2$, we have $(\| |V^{(2)}|^{p'} \|_{\varphi^*})^{\frac{1}{p'}} \leq K_2$ for any $p' \in [2, p)$ since

$$\int_{\mathbb{R}^2} \varphi^* \left(\frac{|V^{(2)}|^{p'}}{K_2^{p'}} \right) dx \leq \int_{\mathbb{R}^2} \left(\frac{|V^{(2)}|^{p'}}{K_2^{p'}} \right)^{\frac{p}{p'}} dx \leq \int_{\mathbb{R}^2} \frac{|V^{(2)}|^p}{K_2^p} dx \leq 1. \tag{1.2.73}$$

We conclude that (1.2.72) holds with p' substituted for p. Letting $p' \uparrow p$ we obtain (1.2.72) since K_2 is independent of p'.

(c) $d = 1$: Let $\|V^{(2)}\|_p \leq K_2$ with $p \geq 2$. Using Hölder's inequality and (1.2.22) with $q = 2$, we get

$$\int_{\mathbb{R}} W_M^2 \omega_\varrho^{2-2\alpha} \eta^2 \psi^2 dx \leq \|W_M\|_2^2 \|\omega_\varrho^{2-2\alpha} \eta^2 \psi^2\|_\infty \leq M^{-\frac{p-2}{2}} K_2^p \|\omega_\varrho^{2-2\alpha} \eta^2 \psi^2\|_\infty. \tag{1.2.74}$$

Applying Sobolev's inequality, we obtain

$$\|\omega_\varrho^{2-2\alpha} \eta^2 \psi^2\|_\infty \leq \int_{\mathbb{R}} |\omega_\varrho^{1-\alpha} \eta \psi|^2 dx + \int_{\mathbb{R}} |(\omega_\varrho^{1-\alpha} \eta \psi)'|^2 dx \tag{1.2.75}$$
$$\leq \int_{\mathbb{R}} |\omega_\varrho^{1-\alpha} \eta \psi|^2 dx + 2 \int_{\mathbb{R}} |(\omega_\varrho^{1-\alpha})'|^2 \eta^2 \psi^2 dx + 2 \int_{\mathbb{R}} \omega_\varrho^{1-2\alpha} |(\eta \psi)'|^2 dx.$$

Combining (1.2.21), (1.2.74), (1.2.75), and (1.2.26) with $d = 1$, we conclude that

$$\begin{aligned}
&\left(\frac{\alpha^3}{3C_3\varrho^4} - 4K_1^2 - 4M - 4M^{-\frac{p-2}{2}}K_2^p - 8C_4M^{-\frac{p-2}{2}}K_2^p\frac{\alpha^2}{\varrho^2}\right)\int_{\mathbb{R}}\omega_\varrho^{-1-2\alpha}\eta^2\psi^2dx \\
&\qquad + \left(\frac{\alpha}{3C_3\varrho^2} - 8M^{-\frac{p-2}{2}}K_2^p\right)\int_{\mathbb{R}}\omega_\varrho^{1-2\alpha}|(\eta\psi)'|^2dx \\
&\le 4\int_{\operatorname{supp}\eta'}\omega_\varrho^{2-2\alpha}|\eta'|^2|\psi'|^2dx + \int_{\operatorname{supp}\eta'}\omega_\varrho^{2-2\alpha}(\eta'')^2\psi^2dx \qquad (1.2.76)\\
&\qquad + 2\int_{\operatorname{supp}\eta}\omega_\varrho^{2-2\alpha}\eta^2\zeta^2dx.
\end{aligned}$$

Assuming $\alpha \ge \varrho$, and setting $M = K_2^2\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}}$, we have

$$\begin{aligned}
&4K_1^2 + 4M + 4M^{-\frac{p-2}{2}}K_2^p + 8M^{-\frac{p-2}{2}}K_2^p\frac{\alpha^2}{\varrho^2} \qquad (1.2.77)\\
&\quad \le 4K_1^2 + 4M + 12M^{-\frac{p-2}{2}}K_2^p\frac{\alpha^2}{\varrho^2} = 4K_1^2 + 16K_2^2\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}} \le 16K^2\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}}.
\end{aligned}$$

Taking

$$\alpha \ge C_5(1 + K^{\frac{2p}{3p-4}})\varrho^{\frac{4p-4}{3p-4}} \ge C_5(1 + K^{\frac{2p}{3p-4}})\varrho^{\frac{4}{3}}, \qquad (1.2.78)$$

we can guarantee that $\alpha > C_2$,

$$\frac{\alpha^3}{3C_3\varrho^4} \ge 3(16K^2\alpha^{\frac{4}{p}}\varrho^{-\frac{4}{p}}), \qquad (1.2.79)$$

and

$$\frac{\alpha}{3C_3\varrho^2} - 8M^{-\frac{p-2}{2}}K_2^p \ge 0. \qquad (1.2.80)$$

Using (1.2.15) and recalling (1.2.6), we obtain

$$\int_{\mathbb{R}}\omega_\varrho^{-1-2\alpha}\eta^2\psi^2dx \ge \left(\frac{\varrho}{Q}\right)^{1+2\alpha}\|\psi_\Theta\|_2^2 \ge (2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2. \qquad (1.2.81)$$

Combining (1.2.76), (1.2.79), (1.2.80) and (1.2.81), we conclude that

$$\frac{2\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \le 4\int_{\operatorname{supp}\eta'}\omega_\varrho^{2-2\alpha}|\eta'|^2|\psi'|^2dx$$

$$+\int_{\operatorname{supp}\eta'}\omega_\varrho^{2-2\alpha}(\eta'')^2\psi^2dx + 2\int_{\operatorname{supp}\eta}\omega_\varrho^{2-2\alpha}\eta^2\zeta^2dx \qquad (1.2.82)$$

Let $f \in \mathcal{D}(\nabla)$ and $M > 0$. Using (1.2.34), Hölder's inequality, (1.2.22) with $d = 1$, and Sobolev's inequality, we get

$$\left|\int_{\mathbb{R}} Vf^2dx\right| \le (K_1 + M^{\frac{1}{2}} + M^{-\frac{p-1}{2}}K_2^p)\|f\|_2^2 + M^{-\frac{p-1}{2}}K_2^p\|f'\|_2^2. \qquad (1.2.83)$$

Taking $M = (2K_2^p)^{\frac{2}{p-1}}$, we get

$$\left|\int_{\mathbb{R}} Vf^2dx\right| \le 2(1 + K^{\frac{p}{p-1}})\|f\|_2^2 + \frac{1}{2}\|f'\|_2^2. \qquad (1.2.84)$$

We have

$$\int_{\{2C_1Q\le|x|\le 2C_1Q+1\}}\omega_\varrho^{2-2\alpha}(4|\eta'|^2|\psi'|^2 + (\eta'')^2\psi^2)dx \qquad (1.2.85)$$

$$\le 64\left(\frac{C_1\varrho}{2C_1Q}\right)^{2\alpha-2}\int_{\{2C_1Q\le|x|\le 2C_1Q+1\}}(4|\psi'|^2+\psi^2)dx$$

$$\le C_6\left(\frac{5}{4}C_1\right)^{2\alpha-2}\int_{\{2C_1Q-1\le|x|\le 2C_1Q+2\}}(\zeta^2 + (1+K^{\frac{p}{p-1}})\psi^2)dx$$

$$\le C_6\left(\frac{5}{4}C_1\right)^{2\alpha-2}(\|\zeta_\Omega\|_2^2 + (1+K^{\frac{p}{p-1}})\|\psi_\Omega\|_2^2),$$

where we used (1.2.61) and an interior estimate. Similarly,

$$\int_{\{\frac{\delta}{4}\le|x|\le\frac{3\delta}{4}\}}\omega_\varrho^{2-2\alpha}(4|\eta'|^2|\psi'|^2 + (\eta'')^2\psi^2)dx \qquad (1.2.86)$$

$$\le 1024\delta^{-4}(4\delta^{-1}C_1\varrho)^{2\alpha-2}\int_{\{\frac{\delta}{4}\le|x|\le\frac{3\delta}{4}\}}(4|\psi'|^2+\psi^2)dx$$

$$\le C_7\delta^{-4}(4\delta^{-1}C_1\varrho)^{2\alpha-2}\int_{\{|x|\le\delta\}}(\zeta^2 + (K^{\frac{p}{p-1}}+\delta^{-2})\psi^2)dx$$

$$\le C_7\delta^{-4}(16\delta^{-1}C_1^2Q)^{2\alpha-2}(\|\zeta_\Omega\|_2^2 + (K^{\frac{p}{p-1}}+\delta^{-2})\|\psi_{0,\delta}\|_2^2).$$

In addition,

$$\int_{\operatorname{supp}\eta} \omega_\varrho^{2-2\alpha}\eta^2\zeta^2 dx \le (4\delta^{-1}C_1\varrho)^{2\alpha-2}\|\zeta_\Omega\|_2^2 \le (16\delta^{-1}C_1^2 Q)^{2\alpha-2}\|\zeta_\Omega\|_2^2. \tag{1.2.87}$$

If we have

$$\frac{\alpha^3}{\varrho^4}\left(\frac{8}{5}\right)^{2\alpha}\|\psi_\Theta\|_2^2 \ge C_8(1+K^{\frac{p}{p-1}})\|\psi_\Omega\|_2^2, \tag{1.2.88}$$

we obtain

$$C_6\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}(1+K^{\frac{p}{p-1}})\|\psi_\Omega\|_2^2 \le \frac{1}{2}\frac{2\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2, \tag{1.2.89}$$

so we conclude that

$$\begin{aligned}&\frac{\alpha^3}{9C_3\varrho^4}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \\ &\qquad \le C_9\delta^{-4}(16\delta^{-1}C_1^2Q)^{2\alpha-2}((K^{\frac{p}{p-1}}+\delta^{-2})\|\psi_{0,\delta}\|_2^2+\|\zeta_\Omega\|_2^2).\end{aligned} \tag{1.2.90}$$

Thus,

$$\frac{\alpha^3}{\varrho^4}Q^4((8C_1Q)^{-1}\delta)^{2\alpha+2}\|\psi_\Theta\|_2^2 \le C_{10}((K^{\frac{p}{p-1}}+\delta^{-2})\|\psi_{0,\delta}\|_2^2+\|\zeta_\Omega\|_2^2). \tag{1.2.91}$$

Since $(\frac{\delta}{Q})^5 \le (\frac{1}{2})^5 \le \frac{1}{8C_1}$ by (1.2.8), we have

$$\frac{\alpha^3}{\varrho^4}Q^6\left(\frac{\delta}{Q}\right)^{12\alpha+14}\|\psi_\Theta\|_2^2 \le C_{11}((1+K^{\frac{p}{p-1}})\|\psi_{0,\delta}\|_2^2+\delta^2\|\zeta_\Omega\|_2^2). \tag{1.2.92}$$

To satisfy (1.2.78) and (1.2.88), we choose

$$\alpha = C_{12}(1+K^{\frac{2p}{3p-4}})\left(Q^{\frac{4p-4}{3p-4}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right), \tag{1.2.93}$$

Combining with (1.2.92), and recalling $Q \ge 1$, we get

$$\begin{aligned}&(1+K^{\frac{2p}{3p-4}})^3\left(\frac{\delta}{Q}\right)^{C_{13}(1+K^{\frac{2p}{3p-4}})\left(Q^{\frac{4p-4}{3p-4}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)}\|\psi_\Theta\|_2^2 \\ &\qquad \le C_{14}((1+K^{\frac{p}{p-1}})\|\psi_{0,\delta}\|_2^2+\delta^2\|\zeta_\Omega\|_2^2),\end{aligned} \tag{1.2.94}$$

and hence there exists $m > 0$ such that

$$\left(\frac{\delta}{Q}\right)^{m(1+K^{\frac{2p}{3p-4}})\left(Q^{\frac{4p-4}{3p-4}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)} \|\psi_\Theta\|_2^2 \le \|\psi_{x_0,\delta}\|_2^2 + \delta^2\|\zeta_\Omega\|_2^2. \tag{1.2.95}$$

□

1.2.2 Unique continuation principle for spectral projections

The following theorem, a consequence of Theorem 1.2.1, is an extension of [KN, Theorem B.4] to Schrödinger operators with singular potentials. Theorem 1.2.2 follows from Theorem 1.2.5.

Theorem 1.2.5. *Let $H = -\Delta + V$ be a Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^d)$, where $V = V^{(1)} + V^{(2)}$ with $\|V^{(1)}\|_\infty \le K_1 < \infty$ and $\|V^{(2)}\|_p \le K_2 < \infty$ with $p \ge d$ for $d \ge 3$, $p > 2$ for $d = 2$, and $p \ge 2$ for $d = 1$. Set $K = K_1 + K_2$. Fix $\delta \in (0, \frac{1}{2}]$, let $\{y_k\}_{k\in\mathbb{Z}^d}$ be sites in $\mathbb{R}^d$ with $B(y_k, \delta) \subset \Lambda_1(k)$ for all $k \in \mathbb{Z}^d$. There exists a constant $M_d > 0$, such that given a rectangle Λ as in* (1.2.11)*, where $a \in \mathbb{R}^d$ and $L_j \ge 114\sqrt{d}$ for $j = 1, \ldots, d$, and a real-valued $\psi \in \mathcal{D}(H_\Lambda)$, we have*

$$\delta^{M_d\left(1+K^{\beta_{d,p}}\right)}\|\psi_\Lambda\|_2^2 \le \sum_{k\in\mathbb{Z}^d,\,\Lambda_1(k)\subset\Lambda} \|\psi_{y_k,\delta}\|_2^2 + \delta^2\|((-\Delta+V)\psi)_\Lambda\|_2^2, \tag{1.2.96}$$

where

$$\beta_{d,p} = \begin{cases} \frac{2p}{3p-2d} & \text{for} \quad d \ge 2 \\ \frac{2p}{3p-4} & \text{for} \quad d = 1 \end{cases}. \tag{1.2.97}$$

Proof of Theorem 1.2.5. Under the hypotheses of the theorem $V \in \mathrm{L}^2_{loc}(\mathbb{R}^d)$, which implies that $\mathcal{D}(\Delta_\Lambda) \cap \{\phi \in \mathrm{L}^2(\Lambda) : V\phi \in \mathrm{L}^2(\Lambda)\}$ is an operator core for H_Λ, so it suffices to prove the theorem for $\psi \in \mathcal{D}(\Delta_\Lambda)$ with $V\psi \in \mathrm{L}^2(\Lambda)$.

Using the notation in the proof of [KN, Theorem B.4], we have $\|\widehat{V^{(1)}}\|_\infty = \|V^{(1)}\|_\infty \leq K_1$ and $\|\widehat{V^{(2)}}_{\Lambda_{Y\tau}(\kappa)}\|_p \leq 3^d \|V_\Lambda^{(2)}\|_p \leq 3^d K_2$ for any $\kappa \in \Lambda$, since $\Lambda_{Y\tau}(\kappa) \subset \Lambda_{3\mathbf{L}}$ as $Y\tau_j < \frac{L_j}{2}, j = 1, 2, \ldots, d$. Using Theorem 1.2.1 and following the proof of [KN, Theorem B.4], we prove (1.2.96). □

Proof of Theorem 1.2.2. From (1.2.36), (1.2.61) and (1.2.84), there exists a constant $C_d > 0$ such that for all $f \in \mathcal{D}(\nabla)$

$$\left| \int_{\mathbb{R}^d} V f^2 dx \right| \leq \theta \|f\|_2^2 + \frac{1}{2} \|\nabla f\|_2^2 \tag{1.2.98}$$

where $\theta = C_d(1 + K^{\frac{2p}{2p-d}})$ for $d \geq 2$ and $\theta = C_1(1 + K^{\frac{p}{p-1}})$ for $d = 1$. Therefore $\sigma(H_\Lambda) \subset [-\theta, \infty)$, and hence it suffices to consider $E_0 \geq -\theta$ and $E \in [-\theta, E_0]$. We have $V - E = (V^{(1)} - E) + V^{(2)}$, where

$$\|V^{(1)} - E\|_\infty \leq \|V^{(1)}\|_\infty + \max\{E_0, \theta\} \leq K_1 + E_0 + \theta \tag{1.2.99}$$

and $\|V^{(2)}\|_p \leq K_2$. Applying Theorem 1.2.5 and following the proof of [KN, Theorem B.1], we prove (1.2.13). □

1.3 Bounds on the density of states

The proof of the Theorem 1.0.1 for $d = 1$ is almost the same as for bounded potentials. For $d = 2, 3$, we follow the proof in [BoKl], consider a class of approximate eigenfunctions for which we have local upper bounds, and pick one for which we have a global lower bound. The local upper bounds will come from Corollary 1.1.2, and the global lower bound will come from Theorem 1.2.1. Note that when applying Corollary 1.1.2 we use that $\mathrm{L}^\infty(\Omega) \subset \mathrm{L}^p(\Omega)$ for $\Omega \subset \mathbb{R}^d$ bounded, in which case $\mathrm{L}^\infty(\Omega) + \mathrm{L}^p(\Omega) = \mathrm{L}^p(\Omega)$.

1.3.1 One-dimensional Schrödinger operators

The case $d = 1$ of Theorem 1.0.1 is an immediate consequence of the following theorem.

Theorem 1.3.1. *Let* $H = -\Delta + V$ *on* $\mathrm{L}^2(\mathbb{R})$*, where* V *is a real potential such that*

$$\sup_{x\in\mathbb{R}} \int_{\{|x-y|\le 1\}} |V(y)| dy < \infty. \tag{1.3.1}$$

Given $E_0 \in \mathbb{R}$*, there exists* L_{V,E_0} *such that for all* $0 < \varepsilon \le \frac{1}{2}$*, open intervals* $\Lambda = \Lambda_L$ *with* $L \ge L_{V,E_0} \log \frac{1}{\varepsilon}$*, and* $E \le E_0$*, we have*

$$\eta_\Lambda([E, E+\varepsilon]) \le \frac{C_{V,E_0}}{\log \frac{1}{\varepsilon}}. \tag{1.3.2}$$

Proof. Proceeding as in [BoKl, Theorem 2.3], let $\Lambda = \Lambda_L = (a_0, a_0 + L)$, $E \in \mathbb{R}$, $\varepsilon \in (0, \frac{1}{2}]$ and

$$K = \sup_{x\in\mathbb{R}} \int_{\{|x-y|\le 1\}} |V(y)| dy < \infty. \tag{1.3.3}$$

Setting $P = \chi_{[E,E+\varepsilon]}(H_\Lambda)$, we have $\dim \operatorname{Ran} P \le \operatorname{tr} P < \infty$, $\operatorname{Ran} P \subset \mathcal{D}(H_\Lambda) \subset \mathrm{C}^1(\Lambda)$, and

$$\|(H_\Lambda - E)\psi\|_2 \le \varepsilon \|\psi\|_2 \quad \text{for all } \psi \in \operatorname{Ran} P. \tag{1.3.4}$$

Given $0 < R < L$, set $a_j = a_0 + jR$ for $j = 1, 2, \ldots, \lceil \frac{L}{R} \rceil - 1$, and consider the vector space

$$\mathcal{F}_R := \left\{ \psi \in \operatorname{Ran} P : \psi(a_j) = \psi'(a_j) = 0 \quad \text{for } j = 1, 2, \ldots, \lceil \tfrac{L}{R} \rceil - 1 \right\}. \tag{1.3.5}$$

Given $\psi \in \mathcal{F}_R$, set $\mathbf{\Psi} = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}$. We have

$$\mathbf{\Psi}' = \begin{pmatrix} \psi' \\ \psi'' \end{pmatrix} = \begin{pmatrix} \psi' \\ V\psi - H\psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ V - E & 0 \end{pmatrix} \mathbf{\Psi} + \begin{pmatrix} 0 \\ -\zeta \end{pmatrix} \tag{1.3.6}$$

where $\zeta = (H - E)\psi$. We have $\|\zeta\|_2 \le \varepsilon \|\psi\|_2$ from (1.3.4). For $j = 1, 2, \ldots, \lceil \frac{L}{R} \rceil - 1$ and $x \in (a_j - R, a_j + R) \cap \Lambda$, we have

$$\mathbf{\Psi}(x) = \int_{a_j}^{x} \begin{pmatrix} 0 & 1 \\ (V(y) - E) & 0 \end{pmatrix} \mathbf{\Psi}(y) dy + \int_{a_j}^{x} \begin{pmatrix} 0 \\ -\zeta(y) \end{pmatrix} dy \tag{1.3.7}$$

since $\psi(a_j) = \psi'(a_j) = 0$, and hence

$$|\mathbf{\Psi}(x)| \le \left| \int_{a_j}^{x} (1 + |E| + |V(y)|) |\mathbf{\Psi}(y)|) dy + \int_{a_j}^{x} |\zeta(y)| dy \right|. \tag{1.3.8}$$

By Gronwall's inequality (see [Ho]), we have

$$|\mathbf{\Psi}(x)| \le \left| \int_{a_j}^{x} \exp\left(\left| \int_{y}^{x} (1 + |E| + |V(z)|) dz \right| \right) |\zeta(y)| dy \right|. \tag{1.3.9}$$

We have

$$\begin{aligned} \left| \int_{y}^{x} (1 + |E| + |V(z)|) dz \right| &\le (1 + |E|)|x - y| + \left| \int_{y}^{x} |V(z) dz| \right| \\ &\le (1 + |E|)R + \left\lceil \tfrac{R}{2} \right\rceil K \le C \max\{R, 1\}, \end{aligned} \tag{1.3.10}$$

where $C = 1 + |E| + K$. Therefore

$$|\psi(x)| \le |\mathbf{\Psi}(x)| \le e^{C \max\{R,1\}} \sqrt{|x - a_j|} \|\zeta\|_2 \le e^{C \max\{R,1\}} \sqrt{R} \varepsilon \|\psi\|_2. \tag{1.3.11}$$

Since Λ is the union of these intervals, we conclude that

$$\|\psi\|_\infty \le e^{C \max\{R,1\}} \sqrt{R} \varepsilon \|\psi\|_2 \quad \text{for all } \psi \in \mathcal{F}_R. \tag{1.3.12}$$

We now assume that

$$\rho := \eta_{\Lambda_L}([E, E+\varepsilon]) = \tfrac{1}{L} \operatorname{tr} P > \tfrac{4}{L}, \tag{1.3.13}$$

since otherwise there is nothing to prove for large L. Taking $R = \frac{4}{\rho}$, it follows from (1.3.13) that

$$\dim \mathcal{F}_R \geq \rho L - 2\left(\left\lceil \tfrac{L}{R} \right\rceil - 1\right) \geq \rho L - 2\tfrac{L}{R} = \tfrac{1}{2}\rho L > 2. \tag{1.3.14}$$

Applying [BoKl, Lemma 2.1], we obtain $\psi_0 \in \mathcal{F}_R$, $\psi_0 \neq 0$, such that

$$\|\psi_0\|_\infty \geq \sqrt{\frac{\dim \mathcal{F}_R}{L}} \|\psi_0\|_2 \geq \sqrt{\tfrac{1}{2}\rho} \|\psi_0\|_2. \tag{1.3.15}$$

It follows from (1.3.12) and (1.3.15) that

$$\sqrt{\tfrac{1}{2}\rho} \leq e^{C \max\{R,1\}} \sqrt{R} \varepsilon = e^{C(\max\{\frac{4}{\rho},1\})} \sqrt{\tfrac{4}{\rho}} \varepsilon. \tag{1.3.16}$$

If $\rho \leq 4$, we have $\frac{4}{\rho} \geq 1$, and we get

$$\rho \leq \frac{8C}{\log \frac{1}{\varepsilon}}. \tag{1.3.17}$$

If $\rho > 4$, we have $\frac{4}{\rho} < 1$, and we get

$$\rho \leq 2\sqrt{2} e^C \varepsilon \leq \frac{2\sqrt{2}e^C}{\log \frac{1}{\varepsilon}}. \tag{1.3.18}$$

Since we have (1.3.13), we conclude that there exists $C_{K,E}$ such that

$$\rho \leq \frac{C_{K,E}}{\log \frac{1}{\varepsilon}} \quad \text{if} \quad L > \frac{4}{\rho} \geq \frac{4 \log \frac{1}{\varepsilon}}{C_{K,E}}. \tag{1.3.19}$$

Since H_Λ is semibounded (see [S]), there exists θ_V such that $\sigma(H_\Lambda) \subset [\theta_V, \infty)$. Thus we have $\eta_\Lambda([E, E+\varepsilon]) = 0$ unless $E \geq \theta_V - \frac{1}{2}$. Thus, given $E_0 \in \mathbb{R}$, there exists L_{V,E_0} such that, for all $0 < \varepsilon \leq \frac{1}{2}$, open intervals $\Lambda = \Lambda_L$ with $L \geq L_{V,E_0} \log \frac{1}{\varepsilon}$, and $E \leq E_0$, we have (1.3.2). □

1.3.2 Two and three dimensional Schrödinger operators

As noted in [GK3, Corollary A.2], when we apply Theorem 1.2.1 to approximate eigenfunction of Schrödinger operators defined on a box Λ with Dirichlet or periodic boundary condition, it can be extended to sites near the boundary of Λ as in the following corollary.

Corollary 1.3.2. *Let $d = 2, 3, \ldots$. Consider the Schrödinger operator $H_\Lambda := -\Delta_\Lambda + V$ on $\mathrm{L}^2(\Lambda)$, where $\Lambda = \Lambda_L(x_0)$ is the open box of side $L > 0$ centered at $x_0 \in \mathbb{R}^d$. Δ_Λ is the Laplacian with either Dirichlet or periodic boundary condition on Λ, and $V = V^{(1)} + V^{(2)}$ is a real potential on Λ with $\|V^{(1)}\|_\infty \leq K_1 < \infty$ and $\|V^{(2)}\|_p \leq K_2 < \infty$, with either $p \geq d$ if $d \geq 3$ or $p > 2$ if $d = 2$. Let $\psi \in \mathcal{D}(H_\Lambda)$ with $\Delta\psi \in \mathrm{L}^2(\Lambda)$ and fix a bounded measurable set $\Theta \subset \Lambda$ where $\|\psi_\Theta\|_2 > 0$. Set $Q(x, \Theta) := \sup_{y\in\Theta} |y - x|$ for $x \in \Lambda$, and consider $x_0 \in \Omega\backslash\overline{\Theta}$ such that $Q = Q(x_0, \Theta) \geq 1$. Then, given $0 < \delta \leq \min\{\operatorname{dist}(x_0, \Theta), \frac{1}{2}\}$, such that $B(x_0, \delta) \subset \Lambda$, we have*

$$\left(\frac{\delta}{Q}\right)^{m_d\left(1+K^{\frac{2p}{3p-2d}}\right)\left(Q^{\frac{4p-2d}{3p-2d}}+\log\frac{\|\psi\|_2}{\|\psi_\Theta\|_2}\right)} \|\psi_\Theta\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2 + \delta^2\|H_\Lambda\psi\|_2^2, \quad (1.3.20)$$

where $K = K_1 + K_2$ and $m_d > 0$ is a constant depending only on d.

This corollary is proved exactly as [GK3, Corollary A.2]. (Note that using the notation in the proof of [GK3, Corollary A.2], we have $\|\widehat{V^{(1)}}_{\Lambda_{L'}}\|_\infty = \|V^{(1)}_{\Lambda_L}\|_\infty$ and $\|\widehat{V^{(2)}}_{\Lambda_{L'}}\|_p \leq (2n+1)^d\|V^{(2)}_{\Lambda_L}\|_p$ if $L' = (2n+1))L$ for some $n \in \mathbb{N}$.)

The case $d = 2, 3$ of Theorem 1.0.1 is an immediate consequence of the following theorem.

Theorem 1.3.3. *Let $H = -\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$, where $d = 2, 3$ and $V = V^{(1)} + V^{(2)}$ is a real potental with $V^{(1)} \in \mathrm{L}^\infty(\mathbb{R}^d)$ and $V^{(2)} \in \mathrm{L}^p(\mathbb{R}^d)$ with $p > \frac{2d}{4-d}$. Set $V_\infty = \|V\|_\infty$ and $V_p = \|V\|_p$. Given $E_0 \in \mathbb{R}$, there exists $L_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}$ such that for all $0 < \varepsilon \leq \frac{1}{2}$, open boxes $\Lambda = \Lambda_L$ with $L \geq L_{d,p,V_p,E_0} \left(\log \frac{1}{\varepsilon}\right)^{\frac{3p-2d}{8p-4d}}$, and $E \leq E_0$, we have*

$$\eta_\Lambda([E, E+\varepsilon]) \leq \frac{C_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}}{\left(\log \frac{1}{\varepsilon}\right)^{\frac{(4-d)p-2d}{8p-4d}}}. \tag{1.3.21}$$

Proof. We fix $\varepsilon \in (0, \frac{1}{2}]$, let $L \geq L_0(\varepsilon)$, where $L_0(\varepsilon) > 0$ will be specified later, and take a box $\Lambda = \Lambda_L$. There exists $\theta = \theta(d, p, V_\infty^{(1)}, V_p^{(2)}) \geq 0$ such that (see (1.2.36) and (1.2.61))

$$\left| \int_{\mathbb{R}^d} |V| \, |f|^2 \, dx \right| \leq \theta \|f\|_2^2 + \tfrac{1}{2} \|\nabla f\|_2^2 \quad \text{for all} \quad f \in \mathcal{D}(\nabla). \tag{1.3.22}$$

It follows that $\sigma(H_\Lambda) \subset [-\theta, \infty)$, and hence it suffices to consider $E_0 \geq -\theta - 1$ and $E \in [-\theta-1, E_0]$. We set $P = \chi_{[E,E+\varepsilon]}(H_\Lambda)$; note that $\operatorname{Ran} P \subset \mathcal{D}(H_\Lambda) \subset \mathrm{H}^1(\Lambda)$ and

$$\|(H_\Lambda - E)\psi\|_2 \leq \varepsilon \|\psi\|_2 \quad \text{for all } \psi \in \operatorname{Ran} P. \tag{1.3.23}$$

Recalling that for $t > 0$ we have

$$\begin{aligned} \|e^{-t(H_\Lambda+\theta)}\|_{\mathrm{L}^2(\Lambda)\to\mathrm{L}^\infty(\Lambda)} &\leq \|e^{\frac{1}{2}t\Delta_\Lambda}\|_{\mathrm{L}^2(\Lambda)\to\mathrm{L}^\infty(\Lambda)} \\ &\leq \|e^{\frac{1}{2}t\Delta}\|_{\mathrm{L}^2(\mathbb{R}^d)\to\mathrm{L}^\infty(\mathbb{R}^d)} < \infty, \end{aligned} \tag{1.3.24}$$

for $\psi \in \operatorname{Ran} P$ we get

$$\begin{aligned} \|\psi\|_\infty &= \|e^{-(H_\Lambda+\theta)} e^{(H_\Lambda+\theta)} \psi\|_\infty \\ &\leq \|e^{-(H_\Lambda+\theta)}\|_{\mathrm{L}^2(\Lambda)\to\mathrm{L}^\infty(\Lambda)} \|e^{(H_\Lambda+\theta)}\psi\|_2 \leq C_d e^{E_0+\theta+1} \|\psi\|_2. \end{aligned} \tag{1.3.25}$$

Since $P(H_\Lambda - E)\psi = (H_\Lambda - E)P\psi = (H_\Lambda - E)\psi$ for $\psi \in \operatorname{Ran} P$, we conclude that

$$\|(H_\Lambda - E)\psi\|_\infty \le \varepsilon C_{d,p,V_\infty^{(1)},V_p^{(2)},E_0} \|\psi\|_2 \quad \text{for all } \psi \in \operatorname{Ran} P. \tag{1.3.26}$$

Since $V \in \mathrm{L}^\infty(\mathbb{R}^d) + \mathrm{L}^p(\mathbb{R}^d)$ with $p > 2$, we have $V \in \mathrm{L}^2_{loc}(\mathbb{R}^d)$. Therefore $V\psi \in \mathrm{L}^2(\Lambda)$ as ψ is bounded. Thus we have $\Delta\psi = -H_\Lambda\psi + V\psi \in \mathrm{L}^2(\Lambda)$.

Let

$$\rho := \eta_{\Lambda_L}([E, E+\varepsilon]) = \tfrac{1}{L^d} \operatorname{tr} P. \tag{1.3.27}$$

We have the uniform upper bound (e.g., [GK2, Eq. (A.6)])

$$\rho \le \rho_{ub} := C_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}; \quad \text{without loss of generality} \quad \rho_{ub} \ge 1. \tag{1.3.28}$$

Let γ_d be the constant in Theorem 1.1.2; we assume $2^d\gamma_d \ge 1$ without loss of generality. We take

$$L^d > 2^{3d+1}\gamma_d \frac{\rho_{ub}}{\rho}; \tag{1.3.29}$$

otherwise there is nothing to prove for L large. Let R satisfy

$$2^{d+1}\gamma_d \frac{\rho_{ub}}{\rho} \le R^d < \left(\frac{L}{4}\right)^d; \tag{1.3.30}$$

we have

$$2 \le \rho R^d \text{ and } 2 \le R^d. \tag{1.3.31}$$

Using (1.3.28) and (1.3.30), we have

$$N := \left\lfloor \left(\frac{\rho}{2^{d+1}\gamma_d}\right)^{\frac{1}{d-1}} R^{\frac{d}{d-1}} \right\rfloor \ge \left\lfloor \rho_{ub}^{\frac{1}{d-1}} \right\rfloor \ge 1. \tag{1.3.32}$$

We now choose $\mathcal{G} \subset \Lambda$ such that

$$\overline{\Lambda} = \bigcup_{y \in \mathcal{G}} \overline{\Lambda_R}(y) \quad \text{and} \quad \sharp\mathcal{G} = \left(\left\lceil \tfrac{L}{R} \right\rceil\right)^d \in \left[\left(\tfrac{L}{R}\right)^d, \left(\tfrac{2L}{R}\right)^d\right] \cap \mathbb{N}. \tag{1.3.33}$$

Give $y_1 \in \mathcal{G}$, we apply Corollary 1.1.2 with $\Omega = \Lambda \supset B(y_1, 1)$, $W = V - E$, and $\mathcal{F} = \operatorname{Ran} P$. The hypothesis (1.1.7) follows from (1.3.26). We conclude that there exists a vector subspace $\mathcal{F}_{y_1,N}$ of $\operatorname{Ran} P$ and $r_0 = r_0(d, p, V_\infty^{(1)}, V_p^{(2)}, E_0) \in (0, 1)$ such that, using (1.3.32) and (1.3.30), we have

$$\dim \mathcal{F}_{y_1,N} \geq \rho L^d - \gamma_d N^{d-1} \geq 1, \tag{1.3.34}$$

and for all $\psi \in \mathcal{F}_{y_1,N}$ we have

$$|\psi(y_1 + x)| \leq (C^{N^2}_{d,p,V_\infty^{(1)},V_p^{(2)},E_0} |x|^{N+1} + \varepsilon C_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}) \|\psi\|_2 \quad \text{if} \quad |x| < r_0. \tag{1.3.35}$$

Picking $y_2 \in \mathcal{G}$, $y_2 \neq y_1$, and apply Theorem 1.1.2 with $\Omega = \Lambda \supset B(y_2, 1)$, $W = V - E$, and $\mathcal{F} = \mathcal{F}_{y_1,N}$, we obtain a vector subspace $\mathcal{F}_{y_1,y_2,N}$ of $\mathcal{F}_{y_1,N}$, and hence of $\operatorname{Ran} P$, such that

$$\dim \mathcal{F}_{y_1,y_2,N} \geq \dim \mathcal{F}_{y_1,N} - \gamma_d N^{d-1} \geq \rho L^d - 2\gamma_d N^{d-1} \geq 1, \tag{1.3.36}$$

and (1.3.35) holds for all $\psi \in \mathcal{F}_{y_1,y_2,N}$ also with y_2 substituted for y_1. Repeating this procedure until we exhaust the sites in $\mathcal{G}$, we conclude that there exists a vector subspace $\mathcal{F}_R$ of $\operatorname{Ran} P$ and $r_0 = r_0(d, p, V_\infty^{(1)}, V_p^{(2)}, E_0) \in (0, 1)$, such that

$$\dim \mathcal{F}_R \geq \rho L^d - \left(\tfrac{2L}{R}\right)^d \gamma_d N^{d-1} \geq \tfrac{1}{2}\rho L^d \geq 2^{3d}\gamma_d \rho_{ub} \geq 1, \tag{1.3.37}$$

where we used the assumption (1.3.29), and for all $\psi \in \mathcal{F}_R$ and $y \in \mathcal{G}$ we have

$$|\psi(y + x)| \leq (C^{N^2}_{d,p,V_\infty^{(1)},V_p^{(2)},E_0} |x|^{N+1} + \varepsilon C_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}) \|\psi\|_2 \quad \text{if } x < r_0. \tag{1.3.38}$$

We let Q_R denote the orthogonal projection onto $\mathcal{F}_R$. Since $\operatorname{tr} Q_R = \dim \mathcal{F}_R$, it follows from (1.3.37) by the argument in [BoKl, Eqs. (3.102)-(3.106)] that there exists $\psi_0 = Q_R \psi_0$ with $\|\psi_0\|_2 = 1$ such that

$$\gamma\rho \le \|\chi_{\Lambda_1}\psi_0\|_2 \le 1, \quad \text{where} \quad \gamma = \gamma_{d,p,V_\infty^{(1)},V_p^{(2)},E_0} > 0. \tag{1.3.39}$$

We pick $y_0 \in \mathcal{G}$ such that

$$\tfrac{1}{4} < \tfrac{1}{4}R \le \operatorname{dist}(y_0, \Lambda_1) \le 2\sqrt{d}R, \tag{1.3.40}$$

which can be done by our construction, and apply Corollary 1.3.2 with $x_0 = y_0$, $\Theta = \Lambda_1$, and potential $V - E$; note that

$$\tfrac{R}{4} + \sqrt{d} \le Q = Q(y_0, \Lambda_1) \le 2\sqrt{d}R + \sqrt{d} \le 3\sqrt{d}R. \tag{1.3.41}$$

Let $0 < \delta < \delta_0 := \min\left\{\frac{1}{2}, r_0\right\}$, where r_0 is as in (1.3.38). It follows from Corollary 1.3.2, using (1.3.23), that

$$\left(\frac{\delta}{3\sqrt{d}R}\right)^{m(1+K^{\frac{2p}{3p-2d}})(R^{\frac{4p-2d}{3p-2d}} - \log\|\psi_0\chi_{\Lambda_1}\|_2)} \|\psi_0\chi_{\Lambda_1}\|_2^2 \le \|\psi_0\chi_{B(y_0,\delta)}\|_2^2 + \varepsilon^2, \tag{1.3.42}$$

with a constant $m = m_d > 0$ and $K = V_\infty^{(1)} + V_p^{(2)} + |E|$. Using (1.3.38) and (1.3.39), we get

$$\begin{aligned}\left(\frac{\delta}{3\sqrt{d}R}\right)^{m(1+K^{\frac{2p}{3p-2d}})(R^{\frac{4p-2d}{3p-2d}} - \log(\gamma p))} & (\gamma p)^2 \\ & \le C_d C^{N^2}_{d,p,V_\infty^{(1)},V_p^{(2)},E_0} \delta^{2(N+1)+d} + C_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}\varepsilon^2.\end{aligned} \tag{1.3.43}$$

Since $\rho \ge 2R^{-d}$ and $\frac{\delta}{3\sqrt{d}R} < \frac{\delta}{3\sqrt{d}} < 1$ by (1.3.31), the inequality (1.3.43) implies the existence of strictly positive constants $\tilde{R} = \tilde{R}_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}$ and $M = M_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}$ such that

$$\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4p-2d}{3p-2d}}} \le C^{N^2}_{d,p,V_\infty^{(1)},V_p^{(2)},E_0} \delta^{2N} + C_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}\varepsilon^2 \quad \text{for } R \ge \tilde{R}. \tag{1.3.44}$$

We require

$$R > \widehat{R} = \max\{\tilde{R}, \delta_0^{-1}\}, \tag{1.3.45}$$

and choose δ by (note $C^N_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} \geq 1$)

$$\delta = (C^N_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} R)^{-1} < \delta_0, \text{ so } \tfrac{\delta}{R} = C^N_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} \delta^2 = (C^N_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} R^2)^{-1}, \tag{1.3.46}$$

obtaining

$$\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4p-2d}{3p-2d}}} \leq \left(\tfrac{\delta}{R}\right)^N + C_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} \varepsilon^2. \tag{1.3.47}$$

We now take $d = 2, 3$ and take R large enough so that

$$\left(\tfrac{\delta}{R}\right)^N \leq \tfrac{1}{2} \left(\tfrac{\delta}{R}\right)^{MR^{\frac{4p-2d}{3p-2d}}}, \quad \text{i.e., } (C^N_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} R^2)^{N - MR^{\frac{4p-2d}{3p-2d}}} \geq 2. \tag{1.3.48}$$

To see this, note that $\frac{4p-2d}{3p-2d} < \frac{d}{d-1}$ when $p > \frac{2d}{4-d}$ for $d = 2, 3$, so

$$MR^{\frac{4p-2d}{3p-2d}} < N = \left\lfloor \left(\frac{\rho}{2^{d+1}\gamma_d} \right)^{\frac{1}{d-1}} R^{\frac{d}{d-1}} \right\rfloor \text{ if } \rho > C''_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} R^{\frac{(d-4)p+2d}{3p-2d}}, \tag{1.3.49}$$

and hence

$$(C^N_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} R^2)^{N - MR^{\frac{4p-2d}{3p-2d}}} \geq 4^{N - MR^{\frac{4p-2d}{3p-2d}}} \geq 2 \quad \text{if } \rho > C'''_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} R^{\frac{(d-4)p+2d}{3p-2d}}. \tag{1.3.50}$$

We now choose R by

$$\rho = c_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} R^{\frac{(d-4)p+2d}{3p-2d}}, \tag{1.3.51}$$

where the constant $c_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0}$ is chosen large enough to ensure that, using (1.3.28), all the conditions (1.3.30), (1.3.45), (1.3.50), and (1.3.48) are satisfied. It follows from (1.3.47) and (1.3.48) that

$$\tfrac{1}{2} \left(\tfrac{\delta}{R}\right)^{MR^{\frac{4p-2d}{3p-2d}}} \leq C_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} \varepsilon^2, \quad \text{that is,} \tag{1.3.52}$$

$$(C^N_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} R^2)^{-MR^{\frac{4p-2d}{3p-2d}}} \leq 2C_{d,p,V^{(1)}_\infty,V^{(2)}_p,E_0} \varepsilon^2.$$

Using (1.3.32), and (1.3.51) with a sufficiently large constant $c_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}$, we get from (1.3.52) that

$$e^{-M'R^{\frac{8p-4d}{3p-2d}}} = e^{-M'R^{\frac{(d-4)p+2d}{(3p-2d)(d-1)}+\frac{d}{d+1}+\frac{8p-4d}{3p-2d}}} \le C_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}\varepsilon^2, \tag{1.3.53}$$

where $M' = M'_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}$. Thus

$$\log\frac{1}{\varepsilon} \le C_{d,p,V_\infty^{(1)},V_p^{(2)},E_0} R^{\frac{8p-4d}{3p-2d}} = \frac{\tilde{C}_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}}{\rho^{\frac{8p-4d}{(4-d)p-2d}}}, \tag{1.3.54}$$

and hence

$$\rho \le \tilde{C}_{d,p,V_\infty^{(1)},V_p^{(2)},E_0} \left(\log\tfrac{1}{\varepsilon}\right)^{-\frac{(4-d)p-2d}{8p-4d}}, \tag{1.3.55}$$

as long as L is large enough to satisfy (1.3.30) with the choice of R in (1.3.51), namely $L \ge L_{d,p,V_\infty^{(1)},V_p^{(2)},E_0}\left(\log\frac{1}{\varepsilon}\right)^{\frac{3p-2d}{8p-4d}}$. □

Chapter 2

Eigensystem bootstrap multiscale analysis for the Anderson model

The eigensystem multiscale analysis is a new approach for proving localization for the Anderson model introduced by Elgart and Klein [EK]. The usual proofs of localization for random Schrödinger operators are based on the study of finite volume Green's functions [FroS, FroMSS, Dr, DrK, Sp, CH, FK, GK1, Kl1, BoK, GK3, AiM, Ai, AiSFH, AiENSS]. In contrast to the usual strategy, the eigensystem multiscale analysis is based on finite volume eigensystems, not finite volume Green's functions. It treats all energies of the finite volume operator at the same time, establishing level spacing and localization of eigenfunctions in a fixed box with high probability. A new feature is the labeling of the eigenvalues and eigenfunctions by the sites of the box.

We use a bootstrap argument as in [GK1] to enhance the eigensystem multiscale analysis. It yields exponential localization of finite volume eigenfunctions in boxes of side L, with the eigenvalues and eigenfunctions labeled by the sites of the box, with probability higher than $1 - \mathrm{e}^{-L^{\xi}}$, for any $0 < \xi < 1$. The starting hypothesis for the eigensystem bootstrap multiscale analysis only requires the verification of polynomial decay of the finite volume eigenfunctions, at some sufficiently large scale, with some minimal probability independent of the scale. The advantage of the bootstrap multiscale analysis is that from the same starting hypothesis we get conclusions that are valid for any $0 < \xi < 1$.

We consider the Anderson model in the following form.

Definition 2.0.4. The Anderson model is the random Schrödinger operator

$$H_{\varepsilon,\omega} := -\varepsilon\Delta + V_{\omega} \quad \text{on} \quad \ell^2(\mathbb{Z}^d), \tag{2.0.1}$$

where $\varepsilon > 0$; Δ is the (centered) discrete Laplacian:

$$(\Delta\varphi)(x) := \sum_{y\in\mathbb{Z}^d, |y-x|=1} \varphi(y) \quad \text{for} \quad \varphi \in \ell^2(\mathbb{Z}^d); \tag{2.0.2}$$

$V_{\omega}(x) = \omega_x$ for $x \in \mathbb{Z}^d$, where $\omega = \{\omega_x\}_{x\in\mathbb{Z}^d}$ is a family of independent identically distributed random variables, with a non-degenerate probability distribution μ with bounded support and Hölder continuous of order $\alpha \in (\frac{1}{2}, 1]$:

$$S_{\mu}(t) \le K t^{\alpha} \quad \text{for all} \quad t \in [0,1], \tag{2.0.3}$$

with $S_{\mu}(t) := \sup_{a\in\mathbb{R}} \mu\{[a, a+t]\}$ the concentration function of the measure μ and K a constant.

Given $\Theta \subset \mathbb{Z}^d$, we let $T_\Theta = \chi_\Theta T \chi_\Theta$ be the restriction of the bounded operator T on $\ell^2(\mathbb{Z}^d)$ to $\ell^2(\Theta)$. If $\Phi \subset \Theta \subset \mathbb{Z}^d$, we identify $\ell^2(\Phi)$ with a subset of $\ell^2(\Theta)$ by extending functions on Φ to functions on Θ that are identically 0 on $\Theta \setminus \Phi$. We write $\varphi_\Phi = \chi_\Phi \varphi$ if φ is a function on Θ. We let $\|\varphi\| = \|\varphi\|_2$ and $\|\varphi\|_\infty = \max_{y \in \Theta} |\varphi(y)|$ for $\varphi \in \ell^2(\Theta)$.

For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ we set $\|x\| = |x|_\infty = \max_{j=1,2,\ldots,d} |x_j|$, $|x| = |x|_2 = \left(\sum_{j=1}^d x_j^2\right)^{\frac{1}{2}}$, and $|x|_1 = \sum_{j=1}^d |x_j|$. Given $\Xi \subset \mathbb{R}^d$, we let $\operatorname{diam} \Xi = \sup_{x,y \in \Xi} \|y - x\|$ denote its diameter, and set $\operatorname{dist}(x, \Xi) = \inf_{y \in \Xi} \|y - x\|$ for $x \in \mathbb{R}^d$.

We use boxes in $\mathbb{Z}^d$ centered at points in $\mathbb{R}^d$. The box in $\mathbb{Z}^d$ of side $L > 0$ centered at $x \in \mathbb{R}^d$ is given by

$$\Lambda_L(x) = \Lambda_L^{\mathbb{R}}(x) \cap \mathbb{Z}^d, \quad \text{where} \quad \Lambda_L^{\mathbb{R}}(x) = \left\{ y \in \mathbb{R}^d; \|y - x\| \le \tfrac{L}{2} \right\}. \tag{2.0.4}$$

We write Λ_L to denote a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$. We have $(L-2)^d < |\Lambda_L| \le (L+1)^d$ for $L \ge 2$, where for a set $\Theta \subset \mathbb{Z}^d$ we let $|\Theta|$ denote its cardinality.

The following definitions are for a fixed discrete Schrödinger operator H_ε. We omit ε from the notation (i.e., we write H for H_ε, H_Θ for $H_{\varepsilon,\Theta}$) when it does not lead to confusion. We always consider scales $L \ge 200$, and, for $\tau \in (0,1)$, set

$$L' = \left\lfloor \tfrac{L}{20} \right\rfloor \quad \text{and} \quad L_\tau = \lfloor L^\tau \rfloor. \tag{2.0.5}$$

For fixed $q > 0$, $\beta, \tau \in (0,1)$, we have the following definitions:

Definition 2.0.5. Let Λ_L be a box, $x \in \Lambda_L$, and $\varphi \in \ell^2(\Lambda_L)$ with $\|\varphi\| = 1$. Then:

(i) Given $\widetilde{\theta} > 0$, φ is said to be $(x, \widetilde{\theta})$-polynomially localized if

$$|\varphi(y)| \leq L^{-\widetilde{\theta}} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y - x\| \geq L'. \tag{2.0.6}$$

(ii) Given $\widetilde{s} \in (0,1)$, φ is said to be $(x, \widetilde{s})$-subexponentially localized if

$$|\varphi(y)| \leq \mathrm{e}^{-L^{\widetilde{s}}} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y - x\| \geq L'. \tag{2.0.7}$$

(iii) Given $m > 0$, φ is said to be (x, m)-localized if

$$|\varphi(y)| \leq \mathrm{e}^{-m\|y-x\|} \quad \text{for all} \quad y \in \Lambda_L \quad \text{with} \quad \|y - x\| \geq L_\tau. \tag{2.0.8}$$

Definition 2.0.6. Let $R > 0$, and $\Theta \subset \mathbb{Z}^d$ be a finite set such that all eigenvalues of H_Θ are simple (i.e., $|\sigma(H_\Theta)| = |\Theta|$). Then:

(i) Θ is called R-polynomially level spacing for H_Θ if $|\lambda - \lambda'| \geq R^{-q}$ for all $\lambda, \lambda' \in \sigma(H_\Theta), \lambda \neq \lambda'$.

(ii) Θ is called R-level spacing for H_Θ if $|\lambda - \lambda'| \geq \mathrm{e}^{-R^\beta}$ for all $\lambda, \lambda' \in \sigma(H_\Theta), \lambda \neq \lambda'$.

When $\Theta = \Lambda_L$, a box, and $R = L$, we will just say that Λ_L is polynomially level spacing for H_{Λ_L}, or Λ_L is level spacing for H_{Λ_L}.

Note that R-polynomially level spacing implies R-level spacing for sufficiently large R.

Given $\Theta \subset \mathbb{Z}^d$, (φ, λ) is called an eigenpair for H_Θ if $\varphi \in \ell^2(\Theta)$, $\lambda \in \mathbb{R}$ with $\|\varphi\| = 1$, and $H_\Theta \varphi = \lambda \varphi$ (i.e., λ is an eigenvalue for H_Θ with a corresponding normalized eigenfunction φ). A collection $\{(\varphi_j, \lambda_j)\}_{j \in J}$ of eigenpairs for H_Θ is called an eigensystem for H_Θ if $\{\varphi_j\}_{j \in J}$ is an orthonormal basis for $\ell^2(\Theta)$. We may rewrite the eigensystem as $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}$ if all eigenvalues of H_Θ are simple.

Definition 2.0.7. Let Λ_L be a box. Then:

(i) Given $\widetilde{\theta} > 0$, Λ_L will be called $\widetilde{\theta}$-polynomially localizing (PL) for H if the following holds:

   (a) Λ_L is polynomially level spacing for H_{Λ_L}.

   (b) There exists a $\widetilde{\theta}$-polynomially localized eigensystem for H_{Λ_L}, that is, an eigensystem $\{(\varphi_x, \lambda_x)\}_{x \in \Lambda_L}$ for H_{Λ_L} such that φ_x is $(x, \widetilde{\theta})$-polynomially localized for all $x \in \Lambda_L$.

(ii) Given $m^* > 0$, Λ_L will be called m^*-mix localizing (ML) for H if the following holds:

   (a) Λ_L is polynomially level spacing for H_{Λ_L}.

   (b) There exists an m^*-localized eigensystem for H_{Λ_L}, that is, an eigensystem $\{(\varphi_x, \lambda_x)\}_{x \in \Lambda_L}$ for H_{Λ_L} such that φ_x is (x, m^*)-localized for all $x \in \Lambda_L$.

(iii) Given $\widetilde{s} \in (0, 1)$, Λ_L will be called $\widetilde{s}$-subexponentially localizing (SEL) for H if the following holds:

   (a) Λ_L is level spacing for H_{Λ_L}.

   (b) There exists an $\widetilde{s}$-subexponentially localized eigensystem for H_{Λ_L}, that is, an eigensystem $\{(\varphi_x, \lambda_x)\}_{x \in \Lambda_L}$ for H_{Λ_L} such that φ_x is $(x, \widetilde{s})$-subexponentially localized for all $x \in \Lambda_L$.

(iv) Given $m > 0$, Λ_L will be called m-localizing (LOC) for H if the following holds:

(a) Λ_L is level spacing for H_{Λ_L}.

(b) There exists an m-localized eigensystem for H_{Λ_L}.

Remark 2.0.8. It follows immediately from the definition that given $\widetilde{s} \in (0,1)$,

$$\Lambda_L \text{ is } m^*\text{-mix localizing} \quad \Longrightarrow \quad \Lambda_L \text{ is } \left(1 - \frac{\log \frac{40}{m^*}}{\log L}\right)\text{-SEL} \Longrightarrow \quad \Lambda_L \text{ is } \widetilde{s}\text{-SEL}, \tag{2.0.9}$$

for sufficiently large L. (We consider $m^* < 40$.)

We now state the bootstrap multiscale analysis. We will use $C_{a,b,\ldots}$, $C'_{a,b,\ldots}$, $C(a,b,\ldots)$, etc., to denote a finite constant depending on the parameters $a,b,\ldots$. Note that $C_{a,b,\ldots}$ may denote different constants in different equations, and even in the same equation. By a constant we always mean a finite constant. We will omit the dependence on d and μ from the notation.

Given $\theta > \left(\frac{6}{2\alpha-1} + \frac{9}{2}\right) d$ and $0 < \xi < 1$, we introduce the following parameters:

- We fix q, p, γ_1 such that

$$\tfrac{3d}{2\alpha-1} < q < \tfrac{1}{2}\left(\theta - \tfrac{9}{2}d\right), \quad 0 < p < (2\alpha-1)q - 3d, \quad \text{and} \quad 1 < \gamma_1 < \min\left\{1 + \tfrac{p}{p+2d}, \tfrac{2\theta-4d}{5d+4q}\right\}, \tag{2.0.10}$$

  and note that

$$\theta > 2d + \gamma_1\left(\tfrac{5d}{2} + 2q\right) > \tfrac{9d}{2} + 2q \tag{2.0.11}$$

- We fix $\zeta, \beta, \gamma, \tau$ such that

$$0 < \xi < \zeta < \beta < \tfrac{1}{\gamma} < 1 < \gamma < \sqrt{\tfrac{\zeta}{\xi}} \text{ and } \max\left\{\tfrac{1+\gamma_1}{2\gamma_1}, \tfrac{1+\gamma\beta}{2}, \tfrac{(\gamma-1)\beta+1}{\gamma}\right\} < \tau < 1, \tag{2.0.12}$$

and note that

$$\tfrac{1}{\gamma_1} < 1 - \tau + \tfrac{1}{\gamma_1} < \tau, \quad \text{and} \tag{2.0.13}$$
$$0 < \xi < \xi\gamma^2 < \zeta < \beta < \tfrac{\tau}{\gamma} < \tfrac{1}{\gamma} < \tau < 1 < \tfrac{1-\beta}{\tau-\beta} < \gamma < \tfrac{\tau}{\beta}.$$

- We fix s such that

$$\max\left\{\gamma\beta, 1 - 2\gamma\left(\tau - \tfrac{1+\gamma\beta}{2}\right)\right\} < s < 1, \tag{2.0.14}$$

and note that

$$0 < \zeta < \beta < \gamma\beta < s < 1 \quad \text{and} \quad 1 - \tau + \tfrac{1-s}{\gamma} < \tau - \gamma\beta. \tag{2.0.15}$$

- We also let

$$\widetilde{\zeta} = \tfrac{\zeta+\beta}{2} \in (\zeta, \beta), \quad \widetilde{\tau} = \tfrac{1+\tau}{2} \in (\tau, 1) \quad \text{and} \quad L_{\widetilde{\tau}} = \lfloor L^{\widetilde{\tau}} \rfloor. \tag{2.0.16}$$

In what follows, given $\theta > \left(\frac{6}{2\alpha-1} + \frac{9}{2}\right) d$, we fix q, p, γ_1 as in (2.0.10), and then, given $0 < \xi < 1$, we fix $\zeta, \beta, \gamma, \tau$ as in (2.0.12). We use Definitions 2.0.5–2.0.7 with these fixed q, β, τ, which we omit from the dependence of the constants.

Theorem 2.0.9. *Let $\theta > \left(\frac{6}{2\alpha-1} + \frac{9}{2}\right) d$ and $\varepsilon_0 > 0$. There exists a finite scale $\mathcal{L}(\varepsilon_0, \theta)$ with the following property: Suppose for some $\varepsilon \in (0, \varepsilon_0]$, $L_0 \geq \mathcal{L}(\varepsilon_0, \theta)$, and $0 \leq P_0 < \frac{1}{2(800)^{2d}}$, we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P_0. \tag{2.0.17}$$

Then, given $0 < \xi < 1$, we can find a finite scale $\widetilde{L} = \widetilde{L}(\varepsilon_0, \theta, \xi, L_0)$ and $m_\xi = m(\xi, \widetilde{L}) > 0$ such that

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } m_\xi\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L^\xi} \text{ for all } L \geq \widetilde{L}. \tag{2.0.18}$$

The eigensystem bootstrap multiscale analysis, stated in Theorem 2.0.9, follows from a repeated use of a bootstrap argument, as in [GK1, Section 6], making successive use of Propositions 2.3.1, 2.3.3, 2.3.4, 2.3.6, 2.3.8, and 2.3.9. Propositions 2.3.1, 2.3.4, 2.3.6, and 2.3.9 are eigensystem multiscale analyses. But there is a difference in the procedure comparing with the Green's function bootstrap multiscale analysis of [GK1]. Unlike the definitions of good boxes for the Green's function multiscale analyses, the definitions of good (i.e., localizing) boxes for the eigensystem multiscale analyses, given in Definition 2.0.7, require intermediate scales, namely $\frac{L}{20}$ and L^{τ} in Definition 2.0.5. For this reason we only have the direct implications given in Remark 2.0.8. Thus the bootstrap between the eigensystem multiscale analyses requires some extra intermediate steps, given in Propositions 2.3.3 and 2.3.8.

In Section 2.4 we will prove that we can fulfill the hypotheses of Theorem 2.0.9, obtaining the following theorem.

Theorem 2.0.10. *There exists* $\varepsilon_0 > 0$ *such that, given* $0 < \xi < 1$*, we can find a finite scale* $\widetilde{L} = \widetilde{L}(\varepsilon_0, \xi)$ *and* $m_\xi = m(\xi, \widetilde{L}) > 0$ *such that for all* $0 < \varepsilon \le \varepsilon_0$ *we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } m_\xi\text{-localizing for } H_{\varepsilon,\omega}\} \ge 1 - \mathrm{e}^{-L^{\xi}} \text{ for } L \ge \widetilde{L}. \tag{2.0.19}$$

Theorem 2.0.10 yields all the usual forms of localization. To see this, we introduce some notation and definitions. We fix $\nu > \frac{d}{2}$, and set $\langle x\rangle = \sqrt{1 + \|x\|^2}$.

A function $\psi : \mathbb{Z}^d \to \mathbb{C}$ is called a ν-generalized eigenfunction for H_ε if ψ is a generalized eigenfunction (see (2.1.12)) and $0 < \|\langle x\rangle^{-\nu}\psi\| < \infty$. We let $\mathcal{V}_\varepsilon(\lambda)$ denote the collection of ν-generalized eigenfunctions for H_ε with generalized eigenvalue $\lambda \in \mathbb{R}$.

Given $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{Z}^d$, we set

$$W^{(a)}_{\varepsilon,\lambda}(b) := \begin{cases} \sup_{\psi \in \mathcal{V}_\varepsilon(\lambda)} \frac{|\psi(b)|}{\|\langle x-a\rangle^{-\nu}\psi\|} & \text{if} \quad \mathcal{V}_\varepsilon(\lambda) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}. \tag{2.0.20}$$

Theorem 2.0.10 yields the following theorem, from which one can derive Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) dynamical localization, and more, as in [EK, Corollary 1.8].

Theorem 2.0.11. *Let $H_{\varepsilon,\omega}$ be an Anderson model. There exists $\varepsilon_0 > 0$ such that, given $\xi \in (0,1)$, we can find a scale $\widehat{L} = \widehat{L}(\varepsilon_0, \xi)$ and $m_\xi = m(\xi, \widehat{L}) > 0$, such that for all $0 < \varepsilon \leq \varepsilon_0$, $L \geq \widehat{L}$ with $L \in 2\mathbb{N}$, and $a \in \mathbb{Z}^d$ there exists an event $\mathcal{Y}_{\varepsilon,L,a}$ with the following properties:*

(i) *$\mathcal{Y}_{\varepsilon,L,a}$ depends only on the random variables $\{\omega_x\}_{x\in\Lambda_{5L}(a)}$, and*

$$\mathbb{P}\{\mathcal{Y}_{\varepsilon,L,a}\} \geq 1 - C_{\varepsilon_0} \mathrm{e}^{-L^\xi}. \tag{2.0.21}$$

(ii) *For all $\omega \in \mathcal{Y}_{\varepsilon,L,a}$ and $\lambda \in \mathbb{R}$ we have, with*

$$\max_{b \in \Lambda_{\frac{\ell}{3}}(a)} W^{(a)}_{\varepsilon,\omega,\lambda}(b) > \mathrm{e}^{-\frac{1}{4}m_\xi L} \quad \Longrightarrow \quad \max_{y \in A_L(a)} W^{(a)}_{\varepsilon,\omega,\lambda}(y) \leq \mathrm{e}^{-\frac{7}{132}m_\xi \|y-a\|}, \tag{2.0.22}$$

where

$$A_L(a) := \left\{ y \in \mathbb{Z}^d; \tfrac{8}{7}L \leq \|y-a\| \leq \tfrac{33}{14}L \right\}. \tag{2.0.23}$$

In particular,

$$W^{(a)}_{\varepsilon,\omega,\lambda}(a) W^{(a)}_{\varepsilon,\omega,\lambda}(y) \leq \mathrm{e}^{-\frac{7}{132}m_\xi \|y-a\|} \quad \textit{for all} \quad y \in A_L(a). \tag{2.0.24}$$

Theorem 2.0.11 is proved as the same way as [EK, Theorem 1.7].

2.1 Preliminaries to the multiscale analysis

We consider a fixed discrete Schrödinger operator $H = -\varepsilon\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where $0 < \varepsilon \le \varepsilon_0$ for a fixed ε_0 and V is a bounded potential.

2.1.1 Some basic facts and definitions

Let $\Phi \subset \Theta \subset \mathbb{Z}^d$. We define the boundary, exterior boundary, and interior boundary of Φ relative to Θ, respectively, by

$$\begin{aligned}
\boldsymbol{\partial}^{\Theta}\Phi &= \{(u,v) \in \Phi \times (\Theta \setminus \Phi); |u - v| = 1\}, \qquad (2.1.1)\\
\partial^{\Theta}_{\text{ex}}\Phi &= \{v \in (\Theta \setminus \Phi); (u,v) \in \boldsymbol{\partial}^{\Theta}\Phi \quad \text{for some} \quad u \in \Phi\},\\
\partial^{\Theta}_{\text{in}}\Phi &= \{u \in \Phi; (u,v) \in \boldsymbol{\partial}^{\Theta}\Phi \quad \text{for some} \quad v \in \Theta \setminus \Phi\}.
\end{aligned}$$

We have

$$H_{\Theta} = H_{\Phi} \oplus H_{\Theta\setminus\Phi} + \varepsilon\Gamma_{\boldsymbol{\partial}^{\Theta}\Phi} \quad \text{on} \quad \ell^2(\Theta) = \ell^2(\Phi) \oplus \ell^2(\Theta \setminus \Phi), \qquad (2.1.2)$$

$$\text{where} \quad \Gamma_{\boldsymbol{\partial}^{\Theta}\Phi}(u,v) = \begin{cases} -1 & \text{if either} \quad (u,v) \text{ or } (v,u) \in \boldsymbol{\partial}^{\Theta}\Phi \\ 0 & \text{otherwise} \end{cases}. \qquad (2.1.3)$$

For $t \ge 1$ we set

$$\begin{aligned}
\Phi^{\Theta,t} &= \{y \in \Phi; \Lambda_{2t}(y) \cap \Theta \subset \Phi\} = \{y \in \Phi; \operatorname{dist}(y, \Theta \setminus \Phi) > \lfloor t \rfloor\}, \qquad (2.1.4)\\
\partial^{\Theta,t}_{\text{in}}\Phi &= \Phi \setminus \Phi^{\Theta,t} = \{y \in \Phi; \operatorname{dist}(y, \Theta \setminus \Phi) \le \lfloor t \rfloor\},\\
\partial^{\Theta,t}\Phi &= \partial^{\Theta,t}_{\text{in}}\Phi \cup \partial^{\Theta}_{\text{ex}}\Phi.
\end{aligned}$$

Given a box $\Lambda_L(x) \subset \Theta \subset \mathbb{Z}^d$ we write $\Lambda^{\Theta,t}_L(x)$ for $(\Lambda_L(x))^{\Theta,t}$.

For a box $\Lambda_L \subset \Theta \subset \mathbb{Z}^d$, there exists a unique $\hat{v} \in \partial^{\Lambda_L}_{\text{in}}\Theta$ for each $v \in \partial^{\Lambda_L}_{\text{ex}}\Theta$ such that $(\hat{v}, v) \in \partial_{\Lambda_L}\Theta$. Given $v \in \Theta$, we define $\hat{v}$ as above if $v \in \partial^{\Lambda_L}_{\text{ex}}\Theta$, and set $\hat{v} = v$ otherwise. Note that $|\partial_{\mathrm{ex}}^{\Lambda_L}\Theta| = |\boldsymbol{\partial}_{\Lambda_L}\Theta|$. If $L \geq 2$, we have

$$|\partial_{\mathrm{in}}^{\Theta}\Lambda_L| \leq |\partial_{\mathrm{ex}}^{\Theta}\Lambda_L| = |\boldsymbol{\partial}^{\Theta}\Lambda_L| \leq s_d L^{d-1}, \quad \text{where} \quad s_d = 2^d d. \tag{2.1.5}$$

To cover a box of side L by boxes of side $\ell < L$, we will use suitable covers as in [EK, Definition 3.10] (also see [GK3, Definition 3.12]).

Definition 2.1.1. Let $\Lambda_L = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$ be a box in $\mathbb{Z}^d$, and let $\ell < L$. A suitable ℓ-cover of Λ_L is the collection of boxes

$$\mathcal{C}_{L,\ell}(x_0) = \{\Lambda_\ell(a)\}_{a \in \Xi_{L,\ell}}, \tag{2.1.6}$$

where

$$\Xi_{L,\ell} := \{x_0 + \rho\ell\mathbb{Z}^d\} \cap \Lambda_L^{\mathbb{R}} \quad \text{with} \quad \rho \in [\tfrac{3}{5}, \tfrac{4}{5}] \cap \left\{\tfrac{L-\ell}{2\ell k}; k \in \mathbb{N}\right\}. \tag{2.1.7}$$

We call $\mathcal{C}_{L,\ell}(x_0)$ the suitable ℓ-cover of Λ_L if $\rho = \rho_{L,\ell} := \max\left\{[\tfrac{3}{5}, \tfrac{4}{5}] \cap \left\{\tfrac{L-\ell}{2\ell k}; k \in \mathbb{N}\right\}\right\}$.

Note that $[\tfrac{3}{5}, \tfrac{4}{5}] \cap \left\{\tfrac{L-\ell}{2\ell k}; k \in \mathbb{N}\right\} \neq \emptyset$ if $\ell \leq \frac{L}{6}$. For a suitable ℓ-cover $\mathcal{C}_{L,\ell}(x_0)$, we have (see [EK, Lemma 3.11])

$$\Lambda_L = \bigcup_{a \in \Xi_{L,\ell}} \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a); \tag{2.1.8}$$

$$\left(\tfrac{L}{\ell}\right)^d \leq \#\Xi_{L,\ell} = \left(\tfrac{L-\ell}{\rho\ell} + 1\right)^d \leq \left(\tfrac{2L}{\ell}\right)^d. \tag{2.1.9}$$

2.1.2 Lemmas about eigenpairs

Given $\Theta \subset \mathbb{Z}^d$ and an eigensystem $\{(\varphi_j, \lambda_j)\}_{j \in J}$ for H_Θ. We have

$$\delta_y = \sum_{j \in J} \overline{\varphi_j(y)}\varphi_j \quad \text{for all} \quad y \in \Theta, \tag{2.1.10}$$

$$\psi(y) = \langle \delta_y, \psi \rangle = \sum_{j \in J} \varphi_j(y)\langle \varphi_j, \psi \rangle \quad \text{for all} \quad \psi \in \ell^2(\Theta) \quad \text{and} \quad y \in \Theta.$$

Given $\Theta \subset \mathbb{Z}^d$, a function $\psi : \Theta \to \mathbb{C}$ is called a generalized eigenfunction for H_Θ with generalized eigenvalue $\lambda \in \mathbb{R}$ if ψ is not identically zero and

$$- \varepsilon \sum_{y \in \Theta, |y-x|=1} \psi(y) + (V(x) - \lambda)\psi(x) = 0 \quad \text{for all} \quad x \in \Theta, \tag{2.1.11}$$

or, equivalently,

$$\langle (H_\Theta - \lambda)\varphi, \psi \rangle = 0 \quad \text{for all} \quad \varphi \in \ell^2(\Theta) \quad \text{with finite support.} \tag{2.1.12}$$

If $\psi \in \ell^2(\Theta)$, ψ is an eigenfunction for H_Θ with eigenvalue λ. We do not require generalized eigenfunctions to be in $\ell^2(\Theta)$, we only require the pointwise equality in (2.1.12). If Θ is finite there is no difference between generalized eigenfunctions and eigenfunctions.

Lemma 2.1.2. *Let a box $\Lambda_L \subset \Theta \subset \mathbb{Z}^d$, and suppose (φ, λ) is an eigenpair for H_{Λ_L}. Then:*

(i) *Given $\widetilde{\theta} > 0$, if φ is $(x, \widetilde{\theta})$-polynomially localized for some $x \in \Lambda_L^{\Theta, L'}$, we have*

$$\operatorname{dist}(\lambda, \sigma(H_\Theta)) \leq \|(H_\Theta - \lambda)\varphi\| \leq C_{d,\varepsilon_0} L^{-\left(\widetilde{\theta} - \frac{d-1}{2}\right)}. \tag{2.1.13}$$

(ii) *Given $\widetilde{s} \in (0,1)$, if φ is $(x, \widetilde{s})$-subexponentially localized for some $x \in \Lambda_L^{\Theta, L'}$, we have*

$$\operatorname{dist}(\lambda, \sigma(H_\Theta)) \leq \|(H_\Theta - \lambda)\varphi\| \leq \mathrm{e}^{-c_1 L^{\widetilde{s}}}, \tag{2.1.14}$$

$$\textit{where} \quad c_1 = c_1(L) \geq 1 - C_{d,\varepsilon_0} \tfrac{\log L}{L^{\widetilde{s}}}. \tag{2.1.15}$$

(iii) *Given* $m > 0$ *and* $\tau \in (0,1)$, *if* φ *is* (x,m) *localized for some* $x \in \Lambda_L^{\Theta, L_\tau}$, *we have*

$$\operatorname{dist}(\lambda, \sigma(H_\Theta)) \le \|(H_\Theta - \lambda)\varphi\| \le \mathrm{e}^{-m_1 L_\tau}, \tag{2.1.16}$$

$$\textit{where} \quad m_1 = m_1(L) \ge m - C_{d,\varepsilon_0} \tfrac{\log L}{L_\tau}. \tag{2.1.17}$$

Proof. We prove part (i), the proofs of (ii) and (iii) are similar. If $x \in \Lambda_L^{\Theta, L'}$, we have $\operatorname{dist}(x, \partial_{\text{in}}^{\Theta} \Lambda_L) \ge L'$, thus it follows from [EK, Lemma 3.2] that

$$\begin{aligned} \|(H_\Theta - \lambda)\varphi\| &\le \varepsilon \sqrt{s_d} L^{\frac{d-1}{2}} \|\varphi_{\partial_{\text{in}}^{\Theta} \Lambda_L}\|_\infty \le \varepsilon \sqrt{s_d} L^{\frac{d-1}{2}} L^{-\widetilde{\theta}} \\ &\le \varepsilon_0 \sqrt{s_d} L^{-\left(\widetilde{\theta} - \frac{d-1}{2}\right)}. \end{aligned} \tag{2.1.18}$$

□

For the following lemmas in this and next subsections, we fix $\theta > \left(\frac{6}{2\alpha - 1} + \frac{9}{2}\right) d$ and $0 < \xi < 1$ (so $q, p, \gamma_1, \zeta, \beta, \gamma, \tau, s$ are fixed). Also, when we consider Λ_ℓ to be a $\sharp$ box, where $\sharp$ stands for θ-PL, m^*-ML, s-SEL or m-LOC, with $m^* \ge m^*_-(\ell) > 0$ and $m \ge m_-(\ell) > 0$, we let:

$$L = L_\sharp = \begin{cases} Y\ell \text{ or } \ell^{\gamma_1} & \text{if } \sharp \text{ is } \theta\text{-PL} \\ \ell^{\gamma_1} & \text{if } \sharp \text{ is } m^*\text{-ML} \\ Y\ell \text{ or } \ell^{\gamma} & \text{if } \sharp \text{ is } s\text{-SEL} \\ \ell^{\gamma} & \text{if } \sharp \text{ is } m\text{-LOC} \end{cases} \quad \text{and} \quad \ell_\sharp = \begin{cases} \ell' & \text{if } \sharp \text{ is } \theta\text{-PL or } s\text{-SEL} \\ \ell_\tau & \text{if } \sharp \text{ is } m^*\text{-ML or } m\text{-LOC} \end{cases}, \tag{2.1.19}$$

where $Y \ge 1$. We will omit the dependence on θ, ξ and Y from the notation.

We prove most of the lemmas only for $\sharp$ being θ-PL. The proofs of other cases are similar.

Lemma 2.1.3. *Given $\Theta \subset \mathbb{Z}^d$, let $\psi : \Theta \to \mathbb{C}$ be a generalized eigenfunction for H_Θ with generalized eigenvalue $\lambda \in \mathbb{R}$. Consider a $\sharp$ box $\Lambda_\ell \subset \Theta$ with a corresponding eigensystem $\{(\varphi_u, \nu_u)\}_{u \in \Lambda_\ell}$, and suppose for all $u \in \Lambda_\ell^{\Theta,\ell_\sharp}$ we have*

$$|\lambda - \nu_u| \geq \begin{cases} \frac{1}{2}L^{-q} & \text{if } \sharp \text{ is } \theta\text{-PL or } m^*\text{-ML} \\ \frac{1}{2}\mathrm{e}^{-L^\beta} & \text{if } \sharp \text{ is } s\text{-SEL or } m\text{-LOC} \end{cases}. \tag{2.1.20}$$

Then the following holds for sufficiently large ℓ:

(i) *Let $y \in \Lambda_\ell^{\Theta,2\ell_\sharp}$. Then:*

(a) *If $\sharp$ is θ-PL, we have*

$$|\psi(y)| \leq C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} |\psi(y_1)| \quad \text{for some} \quad y_1 \in \partial^{\Theta,2\ell'}\Lambda_\ell. \tag{2.1.21}$$

(b) *If $\sharp$ is s-SEL, we have*

$$|\psi(y)| \leq \mathrm{e}^{-c_2 \ell^s} |\psi(y_1)| \quad \text{for some} \quad y_1 \in \partial^{\Theta,2\ell'}\Lambda_\ell, \tag{2.1.22}$$

$$\text{where} \quad c_2 = c_2(\ell) \geq 1 - C_{d,\varepsilon_0} L^\beta \ell^{-s}. \tag{2.1.23}$$

(c) *If $\sharp$ is m^*-ML, we have*

$$|\psi(y)| \leq \mathrm{e}^{-m_2^* \ell_\tau} |\psi(y_1)| \quad \text{for some} \quad y_1 \in \partial^{\Theta,2\ell_\tau}\Lambda_\ell, \tag{2.1.24}$$

$$\text{where} \quad m_2^* = m_2^*(\ell) \geq m^* - C_{d,\varepsilon_0} \gamma_1 q \tfrac{\log \ell}{\ell_\tau}. \tag{2.1.25}$$

(d) *If $\sharp$ is m-LOC, we have*

$$|\psi(y)| \leq \mathrm{e}^{-m_2 \ell_\tau} |\psi(y_1)| \quad \text{for some} \quad y_1 \in \partial^{\Theta,2\ell_\tau}\Lambda_\ell, \tag{2.1.26}$$

$$\text{where} \quad m_2 = m_2(\ell) \geq m - C_{d,\varepsilon_0} \ell^{\gamma\beta-\tau}. \tag{2.1.27}$$

(ii) *Let* $y \in \Lambda_\ell^{\Theta, 2\ell_{\tilde{\tau}}}$. *Then:*

(a) *If* $\sharp$ *is* m^**-ML, we have*

$$|\psi(y)| \le \mathrm{e}^{-m_3^* \|y_2 - y\|} |\psi(y_2)| \quad \textit{for some} \quad y_2 \in \partial^{\Theta, \ell_{\tilde{\tau}}} \Lambda_\ell, \tag{2.1.28}$$

$$\textit{where} \quad m_3^* = m_3^*(\ell) \ge m^* \left(1 - 4\ell^{\frac{\tau - 1}{2}}\right) - C_{d,\varepsilon_0} \gamma_1 q \frac{\log \ell}{\ell_{\tilde{\tau}}}. \tag{2.1.29}$$

(b) *If* $\sharp$ *is* m*-LOC, we have*

$$|\psi(y)| \le \mathrm{e}^{-m_3 \|y_2 - y\|} |\psi(y_2)| \quad \textit{for some} \quad y_2 \in \partial^{\Theta, \ell_{\tilde{\tau}}} \Lambda_\ell, \tag{2.1.30}$$

$$\textit{where} \quad m_3 = m_3(\ell) \ge m \left(1 - 4\ell^{\frac{\tau - 1}{2}}\right) - C_{d,\varepsilon_0} \ell^{\gamma\beta - \tilde{\tau}}. \tag{2.1.31}$$

Proof. Let $y \in \Lambda_\ell$, we have (see (2.1.10))

$$\psi(y) = \sum_{u \in \Lambda_\ell} \varphi_u(y) \langle \varphi_u, \psi \rangle = \sum_{u \in \Lambda_\ell^{\Theta, \ell'}} \varphi_u(y) \langle \varphi_u, \psi \rangle + \sum_{u \in \partial_{\mathrm{in}}^{\Theta, \ell'} \Lambda_\ell} \varphi_u(y) \langle \varphi_u, \psi \rangle. \tag{2.1.32}$$

If $u \in \Lambda_\ell^{\Theta, \ell'}$, we have $|\lambda - \nu_u| \ge \frac{1}{2} L^{-q}$ by (2.1.20). Using (2.1.12), we get

$$\langle \varphi_u, \psi \rangle = (\lambda - \nu_u)^{-1} \langle \varphi_u, (H_\Theta - \nu_u) \psi \rangle = (\lambda - \nu_u)^{-1} \langle (H_\Theta - \nu_u) \varphi_u, \psi \rangle. \tag{2.1.33}$$

It follows from [EK, Lemma 3.2] that

$$|\varphi_u(y) \langle \varphi_u, \psi \rangle| \le 2 L^q \varepsilon \sum_{v \in \partial_{\mathrm{ex}}^{\Theta} \Lambda_\ell} |\varphi_u(y) \varphi_u(\hat{v})| |\psi(v)|. \tag{2.1.34}$$

If $v' \in \partial_{\mathrm{in}}^{\Theta} \Lambda_\ell$, we have $\|v' - u\| \ge \ell'$, so (2.0.6) gives $|\varphi_u(v')| \le \ell^{-\theta}$. It follows from (2.1.34) and $\|\varphi_u\| = 1$ that

$$|\varphi_u(y) \langle \varphi_u, \psi \rangle| \le 2\varepsilon L^q \ell^{-\theta} \sum_{v \in \partial_{\mathrm{ex}}^{\Theta} \Lambda_\ell} |\psi(v)| \le 2\varepsilon s_d L^q \ell^{-(\theta - d + 1)} |\psi(v_1)| \tag{2.1.35}$$

for some $v_1 \in \partial_{\text{ex}}^{\Theta}\Lambda_\ell$. Therefore

$$\left| \sum_{u \in \Lambda_\ell^{\Theta,\ell'}} \varphi_u(y)\langle \varphi_u, \psi \rangle \right| \leq 2\varepsilon s_d L^q \ell^{-(\theta-2d+1)} |\psi(v_2)| \tag{2.1.36}$$

for some $v_2 \in \partial_{\text{ex}}^{\Theta}\Lambda_\ell$.

Let $y \in \Lambda_\ell^{\Theta,2\ell'}$. If $u \in \partial_{\text{in}}^{\Theta,\ell'}\Lambda_\ell$, we have $\|u - y\| \geq 2\ell' - \ell' = \ell'$, thus (2.0.6) gives $|\varphi_u(y)| \leq \ell^{-\theta}$, and hence

$$\left| \sum_{u \in \partial_{\text{in}}^{\Theta,\ell'}\Lambda_\ell} \varphi_u(y)\langle \varphi_u, \psi \rangle \right| \leq \ell^{-(\theta-d)} \|\psi \chi_{\Lambda_\ell}\| \leq \ell^{-\left(\theta - \frac{3d}{2}\right)} |\psi(v_3)| \tag{2.1.37}$$

for some $v_3 \in \Lambda_\ell$. Combining (2.1.32), (2.1.36) and (2.1.37), we conclude that

$$|\psi(y)| \leq (1 + 2\varepsilon_0 s_d) L^q \ell^{-(\theta-2d)} |\psi(y_1)| \tag{2.1.38}$$

for some $y_1 \in \Lambda_\ell \cup \partial_{\text{ex}}^{\Theta}\Lambda_\ell$. If $y_1 \notin \partial^{\Theta,2\ell'}\Lambda_\ell$ we repeat the procedure to estimate $|\psi(y_1)|$. Since we can suppose $\psi(y) \neq 0$ without loss of generality, the procedure must stop after finitely many times, and at that time we must have (2.1.21).

We prove part (ii) only for $\sharp$ being m^*-ML. The proof for $\sharp$ being m-LOC is similar. Let $y \in \Lambda_\ell^{\Theta,\ell_{\widetilde{\tau}}}$, then $\|y - v'\| \geq \ell_{\widetilde{\tau}}$ for $v' \in \partial_{\text{in}}^{\Theta}\Lambda_\ell$. Thus for $u \in \Lambda_\ell^{\Theta,\ell_\tau}$ and $v' \in \partial_{\text{in}}^{\Theta}\Lambda_\ell$ we have

$$|\varphi_u(y)\varphi_u(v')| \leq \begin{cases} \mathrm{e}^{-m^*(\|y-u\|+\|v'-u\|)} \leq \mathrm{e}^{-m^*\|v'-y\|} & \text{if} \quad \|y-u\| \geq \ell_\tau \\ \mathrm{e}^{-m^*\|v'-u\|} \leq \mathrm{e}^{-m_1'\|v'-y\|} & \text{if} \quad \|y-u\| < \ell_\tau \end{cases}, \tag{2.1.39}$$

where

$$m_1' \geq m^*\left(1 - 2\ell^{\tau-\widetilde{\tau}}\right) = m^*\left(1 - 2\ell^{\frac{\tau-1}{2}}\right), \tag{2.1.40}$$

since for $\|y-u\| < \ell_\tau$, we have

$$\|v'-u\| \geq \|v'-y\| - \|y-u\| \geq \|v'-y\| - \ell_\tau \geq \|v'-y\| \left(1 - \tfrac{\ell_\tau}{\ell_{\tilde{\tau}}}\right). \tag{2.1.41}$$

Combining (2.1.34) and (2.1.39), we conclude that

$$\begin{aligned} |\varphi_u(y)\langle\varphi_u,\psi\rangle| &\leq 2\varepsilon L^q \sum_{v\in\partial_{\mathrm{ex}}^{\Theta}\Lambda_\ell} \mathrm{e}^{-m_1'(\|v-y\|-1)}|\psi(v)| \\ &\leq 2\varepsilon s_d \ell^{\gamma_1 q+d-1}\mathrm{e}^{-m_1'(\|v_1-y\|-1)}|\psi(v_1)| \leq \mathrm{e}^{-m_2'\|v_1-y\|}|\psi(v_1)| \end{aligned} \tag{2.1.42}$$

for some $v_1 \in \partial_{\mathrm{ex}}^{\Theta}\Lambda_\ell$, where we used $\|v_1-y\| \geq \ell_{\tilde{\tau}}$ and took

$$m_2' \geq m_1'\left(1-2\ell^{\tilde{\tau}}\right) - C_{d,\varepsilon_0}\gamma_1 q\tfrac{\log\ell}{\ell_{\tilde{\tau}}} \geq m^*\left(1-4\ell^{\frac{\tau-1}{2}}\right) - C_{d,\varepsilon_0}\gamma_1 q\tfrac{\log\ell}{\ell_{\tilde{\tau}}}. \tag{2.1.43}$$

Therefore

$$\left|\sum_{u\in\Lambda_\ell^{\Theta,\ell_\tau}} \varphi_u(y)\langle\varphi_u,\psi\rangle\right| \leq \ell^d\mathrm{e}^{-m_2'\|v_2-y\|}|\psi(v_2)| \leq \mathrm{e}^{-m_3'\|v_2-y\|}|\psi(v_2)| \tag{2.1.44}$$

for some $v_2 \in \partial_{\mathrm{ex}}^{\Theta}\Lambda_\ell$, where

$$m_3' \geq m_2' - C_d\tfrac{\log\ell}{\ell_{\tilde{\tau}}} \geq m^*\left(1-4\ell^{\frac{\tau-1}{2}}\right) - C_{d,\varepsilon_0}\gamma_1 q\tfrac{\log\ell}{\ell_{\tilde{\tau}}}. \tag{2.1.45}$$

If $u \in \partial_{\mathrm{in}}^{\Theta,\ell_\tau}\Lambda_\ell$ we have $\|u-y\| \geq \ell_{\tilde{\tau}} - \ell_\tau > \frac{1}{2}\ell_{\tilde{\tau}}$, thus (2.0.8) gives $|\varphi_u(y)| \leq \mathrm{e}^{-m^*\|u-y\|}$. Also, (2.0.8) implies

$$|\varphi_u(v)| \leq \mathrm{e}^{m^*\ell_\tau}\mathrm{e}^{-m^*\|v-u\|} \quad \text{for all} \quad v\in\Lambda_\ell. \tag{2.1.46}$$

Therefore

$$|\langle\varphi_u,\psi\rangle| = \left|\sum_{v\in\Lambda_\ell}\varphi_u(v)\psi(v)\right| \leq \sum_{v\in\Lambda_\ell}\mathrm{e}^{-m^*(\|v-u\|-\ell_\tau)}|\psi(v)|, \tag{2.1.47}$$

so we get

$$|\varphi_u(y)\langle\varphi_u,\psi\rangle| \le \sum_{v\in\Lambda_\ell} \mathrm{e}^{-m^*(\|u-y\|-\ell_\tau+\|v-u\|)}|\psi(v)| \tag{2.1.48}$$
$$\le (\ell+1)^d \mathrm{e}^{-m^*(\|u-y\|-\ell_\tau)-m^*\|v_3-u\|}|\psi(v_3)|$$
$$\le \mathrm{e}^{-m_4'\|u-y\|-m^*\|v_3-u\|}|\psi(v_3)|$$
$$\le \mathrm{e}^{-m_4'\max\{\|v_3-y\|,\|u-y\|\}}|\psi(v_3)| \le \mathrm{e}^{-m_4'\max\{\|v_3-y\|,\frac{1}{2}\ell_{\widetilde{\tau}}\}}|\psi(v_3)|$$

for some $v_3 \in \Lambda_\ell$, where we used $\|u-y\| \ge \frac{1}{2}\ell_{\widetilde{\tau}}$ and took

$$m_4' \ge m^*\left(1-4\ell^{\frac{\tau-1}{2}}\right) - C_d\frac{\log \ell}{\ell_{\widetilde{\tau}}}. \tag{2.1.49}$$

Therefore

$$\left|\sum_{u\in\partial_{\mathrm{in}}^{\Theta,\ell_\tau}\Lambda_\ell} \varphi_u(y)\langle\varphi_u,\psi\rangle\right| \le \ell^d \mathrm{e}^{-m_4'\max\{\|v_3-y\|,\frac{1}{2}\ell_{\widetilde{\tau}}\}}|\psi(v_3)| \tag{2.1.50}$$
$$\le \mathrm{e}^{-m_5'\max\{\|v_3-y\|,\frac{1}{2}\ell_{\widetilde{\tau}}\}}|\psi(v_3)|$$

for some $v_3 \in \Lambda_\ell$, where

$$m_5' \ge m_4^{*\prime} - C_d\frac{\log \ell}{\ell_{\widetilde{\tau}}} \ge m^*\left(1-4\ell^{\frac{\tau-1}{2}}\right) - C_d\frac{\log \ell}{\ell_{\widetilde{\tau}}}. \tag{2.1.51}$$

Combining (2.1.32), (2.1.44), and (2.1.50), we conclude that

$$|\psi(y)| \le \mathrm{e}^{-m_3^*\max\{\|y_1-y\|,\frac{1}{2}\ell_{\widetilde{\tau}}\}}|\psi(y_1)| \quad \text{for some} \quad y_1 \in \Lambda_\ell \cup \partial_{\mathrm{ex}}^{\Theta}\Lambda_\ell, \tag{2.1.52}$$

where m_3^* is given in (2.1.29). If $y_1 \notin \partial^{\Theta,\ell_{\widetilde{\tau}}}\Lambda_\ell$ we repeat the procedure to estimate $|\psi(y_1)|$. Since we can suppose $\psi(y) \ne 0$ without loss of generality, the procedure must stop after finitely many times, and at that time we must have

$$|\psi(y)| \le \mathrm{e}^{-m_3^*\max\{\|\widetilde{y}-y\|,\frac{1}{2}\ell_{\widetilde{\tau}}\}}|\psi(\widetilde{y})| \quad \text{for some} \quad \widetilde{y} \in \partial^{\Theta,\ell_{\widetilde{\tau}}}\Lambda_\ell. \tag{2.1.53}$$

If $y \in \Lambda_\ell^{\Theta,2\ell_{\widetilde{\tau}}}$, (2.1.28) follows immediately from (2.1.53). □

Lemma 2.1.4. *Given a finite set* $\Theta \subset \mathbb{Z}^d$*, let* $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_\Theta)}$ *be an eigensystem for* H_Θ*.*

Then the following holds for sufficiently large ℓ*:*

(i) *Let* $\Lambda_\ell(a) \subset \Theta$*, where* $a \in \mathbb{R}^d$*, be a* $\sharp$*-localizing box with a corresponding eigensystem* $\left\{(\varphi_x^{(a)}, \lambda_x^{(a)})\right\}_{x \in \Lambda_\ell(a)}$*, and let* Θ *be L-polynomially level spacing for H if* $\sharp$ *is* θ*-PL or* m^**-ML, L-level spacing for H if* $\sharp$ *is s-SEL or m-LOC.*

(a) *There exists an injection*

$$x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a) \mapsto \widetilde{\lambda}_x^{(a)} \in \sigma(H_\Theta), \tag{2.1.54}$$

such that for all $x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a)$*:*

i. *If* $\sharp$ *is* θ*-PL, we have*

$$\left|\widetilde{\lambda}_x^{(a)} - \lambda_x^{(a)}\right| \leq C_{d,\varepsilon_0} \ell^{-\left(\theta - \frac{d-1}{2}\right)}, \tag{2.1.55}$$

and, multiplying each $\varphi_x^{(a)}$ *by a suitable phase factor,*

$$\left\|\psi_{\widetilde{\lambda}_x^{(a)}} - \varphi_x^{(a)}\right\| \leq 2C_{d,\varepsilon_0} L^q \ell^{-\left(\theta - \frac{d-1}{2}\right)}. \tag{2.1.56}$$

ii. *If* $\sharp$ *is s-SEL, we have*

$$\left|\widetilde{\lambda}_x^{(a)} - \lambda_x^{(a)}\right| \leq \mathrm{e}^{-c_1 \ell^s}, \text{ with } c_1 = c_1(\ell) \text{ as in (2.1.15)}, \tag{2.1.57}$$

and, multiplying each $\varphi_x^{(a)}$ *by a suitable phase factor,*

$$\left\|\psi_{\widetilde{\lambda}_x^{(a)}} - \varphi_x^{(a)}\right\| \leq 2\mathrm{e}^{-c_1 \ell^s} \mathrm{e}^{L^\beta}. \tag{2.1.58}$$

iii. If $\sharp$ is m^*-ML, we have

$$\left|\widetilde{\lambda}_x^{(a)} - \lambda_x^{(a)}\right| \leq \mathrm{e}^{-m_1^* \ell_\tau}, \text{ with } m_1^* = m_1^*(\ell) \text{ as in (2.1.17)}, \tag{2.1.59}$$

and, multiplying each $\varphi_x^{(a)}$ by a suitable phase factor,

$$\left\|\psi_{\widetilde{\lambda}_x^{(a)}} - \varphi_x^{(a)}\right\| \leq 2\mathrm{e}^{-m_1^* \ell_\tau} L^q. \tag{2.1.60}$$

iv. If $\sharp$ is m-LOC, we have

$$\left|\widetilde{\lambda}_x^{(a)} - \lambda_x^{(a)}\right| \leq \mathrm{e}^{-m_1 \ell_\tau}, \text{ with } m_1 = m_1(\ell) \text{ as in (2.1.17)}, \tag{2.1.61}$$

and, multiplying each $\varphi_x^{(a)}$ by a suitable phase factor,

$$\left\|\psi_{\widetilde{\lambda}_x^{(a)}} - \varphi_x^{(a)}\right\| \leq 2\mathrm{e}^{-m_1 \ell_\tau} \mathrm{e}^{L^\beta}. \tag{2.1.62}$$

(b) *Set*

$$\sigma_{\{a\}}(H_\Theta) := \left\{\widetilde{\lambda}_x^{(a)}; x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a)\right\}. \tag{2.1.63}$$

Then if $\lambda \in \sigma_{\{a\}}(H_\Theta)$, for all $y \in \Theta \setminus \Lambda_\ell(a)$ we have

$$|\psi_\lambda(y)| \leq \begin{cases} 2C_{d,\varepsilon_0} L^q \ell^{-\left(\theta - \frac{d-1}{2}\right)} & \text{if } \sharp \text{ is } \theta\text{-PL} \\ 2\mathrm{e}^{-c_1 \ell^s} \mathrm{e}^{L^\beta} & \text{if } \sharp \text{ is } s\text{-SEL} \\ 2\mathrm{e}^{-m_1^* \ell_\tau} L^q & \text{if } \sharp \text{ is } m^*\text{-ML} \\ 2\mathrm{e}^{-m_1 \ell_\tau} \mathrm{e}^{L^\beta} & \text{if } \sharp \text{ is } m\text{-LOC} \end{cases}. \tag{2.1.64}$$

(c) *If $\lambda \in \sigma(H_\Theta) \setminus \sigma_{\{a\}}(H_\Theta)$, for all $x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a)$ we have*

$$\left|\lambda - \lambda_x^{(a)}\right| \geq \begin{cases} \frac{1}{2} L^{-q} & \text{if } \sharp \text{ is } \theta\text{-PL or } m^*\text{-ML} \\ \frac{1}{2}\mathrm{e}^{-L^\beta} & \text{if } \sharp \text{ is } s\text{-SEL or } m\text{-LOC} \end{cases}, \tag{2.1.65}$$

and for all $y \in \Lambda_\ell^{\Theta,2\ell_\sharp}(a)$,

$$|\psi_\lambda(y)| \leq \begin{cases} C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} |\psi_\lambda(y_1)| & \text{if } \sharp \text{ is } \theta\text{-PL} \\ \mathrm{e}^{-c_2 \ell^s} |\psi_\lambda(y_1)| & \text{if } \sharp \text{ is } s\text{-SEL} \\ \mathrm{e}^{-m_2^* \ell_\tau} |\psi_\lambda(y_1)| & \text{if } \sharp \text{ is } m^*\text{-ML} \\ \mathrm{e}^{-m_2 \ell_\tau} |\psi_\lambda(y_1)| & \text{if } \sharp \text{ is } m\text{-LOC} \end{cases} \tag{2.1.66}$$

for some $y_1 \in \partial^{\Theta,2\ell_\sharp}\Lambda_\ell(a)$, *where* $c_2 = c_2(\ell)$ *as in* (2.1.23), $m_2^* = m_2^*(\ell)$ *as in* (2.1.25), $m_2 = m_2(\ell)$ *as in* (2.1.27). *Moreover, for all* $y \in \Lambda_\ell^{\Theta,2\ell_{\widetilde{\tau}}}(a)$,

$$|\psi_\lambda(y)| \leq \begin{cases} \mathrm{e}^{-m_3^* \|y_2 - y\|} |\psi_\lambda(y_2)| & \text{if } \sharp \text{ is } m^*\text{-ML} \\ \mathrm{e}^{-m_3 \|y_2 - y\|} |\psi_\lambda(y_2)| & \text{if } \sharp \text{ is } m\text{-LOC} \end{cases} \tag{2.1.67}$$

for some $y_2 \in \partial^{\Theta,\ell_{\widetilde{\tau}}}\Lambda_\ell(a)$, *where* $m_3^* = m_3^*(\ell)$ *as in* (2.1.29), $m_3 = m_3(\ell)$ *as in* (2.1.31).

(ii) *Let* $\{\Lambda_\ell(a)\}_{a\in\mathcal{G}}$, *where* $\mathcal{G} \subset \mathbb{R}^d$ *such that* $\Lambda_\ell(a) \subset \Theta$ *for all* $a \in \mathcal{G}$, *be a collection of* $\sharp$ *boxes with corresponding eigensystems* $\left\{(\varphi_x^{(a)}, \lambda_x^{(a)})\right\}_{x\in\Lambda_\ell(a)}$ *and let* Θ *be L-polynomially level spacing for H if* $\sharp$ *is* θ*-PL or* m^**-ML, L-level spacing for H if* $\sharp$ *is s-SEL or m-LOC. Set*

$$\mathcal{E}_\mathcal{G}^\Theta(\lambda) = \left\{\lambda_x^{(a)}; a \in \mathcal{G}, x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a), \widetilde{\lambda}_x^{(a)} = \lambda\right\} \text{ for } \lambda \in \sigma(H_\Theta),$$

$$\tag{2.1.68}$$

$$\sigma_\mathcal{G}(H_\Theta) = \left\{\lambda \in \sigma(H_\Theta); \mathcal{E}_\mathcal{G}^\Theta(\lambda) \neq \emptyset\right\} = \bigcup_{;a\in\mathcal{G}} \sigma_{\{a\}}(H_\Theta).$$

(a) *For* $a, b \in \mathcal{G}$, $a \neq b$, *if* $x \in \Lambda_\ell^{\Theta,\ell_\sharp}(a)$ *and* $y \in \Lambda_\ell^{\Theta,\ell_\sharp}(b)$,

$$\lambda_x^{(a)}, \lambda_x^{(b)} \in \mathcal{E}_\mathcal{G}^\Theta(\lambda) \Longrightarrow \|x - y\| < 2\ell_\sharp. \tag{2.1.69}$$

As a consequence,

$$\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset \Longrightarrow \sigma_{\{a\}}(H_\Theta) \cap \sigma_{\{b\}}(H_\Theta) = \emptyset. \tag{2.1.70}$$

(b) *If $\lambda \in \sigma_{\mathcal{G}}(H_\Theta)$, we have for all $y \in \Theta \backslash \Theta_{\mathcal{G}}$, where $\Theta_{\mathcal{G}} := \bigcup_{a\in\mathcal{G}} \Lambda_\ell(a)$,*

$$|\psi_\lambda(y)| \le \begin{cases} 2C_{d,\varepsilon_0} L^q \ell^{-\left(\theta - \frac{d-1}{2}\right)} & \text{if } \sharp \text{ is } \theta\text{-PL} \\ 2\mathrm{e}^{-c_1 \ell^s} \mathrm{e}^{L^\beta} & \text{if } \sharp \text{ is } s\text{-SEL} \\ 2\mathrm{e}^{-m_1^* \ell_\tau} L^q & \text{if } \sharp \text{ is } m^*\text{-ML} \\ 2\mathrm{e}^{-m_1 \ell_\tau} \mathrm{e}^{L^\beta} & \text{if } \sharp \text{ is } m\text{-LOC} \end{cases}. \tag{2.1.71}$$

(c) *If $\lambda \in \sigma(H_\Theta) \setminus \sigma_{\mathcal{G}}(H_\Theta)$, we have for all $y \in \Theta'_{\mathcal{G}} := \bigcup_{a\in\mathcal{G}} \Lambda_\ell^{\Theta, 2\ell_\sharp}(a)$,*

$$|\psi_\lambda(y)| \le \begin{cases} C_{d,\varepsilon_0} L^q \ell^{-(\theta - 2d)} & \text{if } \sharp \text{ is } \theta\text{-PL} \\ \mathrm{e}^{-c_2 \ell^s} & \text{if } \sharp \text{ is } s\text{-SEL} \\ \mathrm{e}^{-m_2^* \ell_\tau} & \text{if } \sharp \text{ is } m^*\text{-ML} \\ \mathrm{e}^{-m_2 \ell_\tau} & \text{if } \sharp \text{ is } m\text{-LOC} \end{cases}. \tag{2.1.72}$$

(d) *If $|\Theta| \le (L+1)^d$, we have*

$$|\Theta'_{\mathcal{G}}| \le |\sigma_{\mathcal{G}}(H_\Theta)| \le |\Theta_{\mathcal{G}}|. \tag{2.1.73}$$

Proof. Let $\Lambda_\ell(a) \subset \Theta$, where $a \in \mathbb{R}^d$, be a θ-polynomially localizing box with a corresponding eigensystem $\left\{(\varphi_x^{(a)}, \lambda_x^{(a)})\right\}_{x\in\Lambda_\ell(a)}$. It follows from Lemma 2.1.2 that there exists $\widetilde{\lambda}_x^{(a)} \in \sigma(H_\Theta)$ satisfying (2.1.55) for $x \in \Lambda_\ell^{\Theta,\ell'}(a)$. $\widetilde{\lambda}_x^{(a)}$ is unique since Θ is L-polynomially level spacing for H_Θ and $q < \gamma_1 q < \theta - \frac{d-1}{2}$. Moreover, we have $\widetilde{\lambda}_x^{(a)} \neq \widetilde{\lambda}_y^{(a)}$ if $x, y \in \Lambda_\ell^{\Theta,\ell'}(a)$, $x \neq y$, since

$$\begin{aligned} \left|\widetilde{\lambda}_x^{(a)} - \widetilde{\lambda}_y^{(a)}\right| &\ge \left|\lambda_x^{(a)} - \lambda_y^{(a)}\right| - \left|\widetilde{\lambda}_x^{(a)} - \lambda_x^{(a)}\right| - \left|\widetilde{\lambda}_y^{(a)} - \lambda_y^{(a)}\right| \\ &\ge \ell^{-q} - 2C_{d,\varepsilon_0} \ell^{-\left(\theta - \frac{d-1}{2}\right)} \ge \tfrac{1}{2}\ell^{-q}, \end{aligned} \tag{2.1.74}$$

$\Lambda_\ell(a)$ is polynomially level spacing for $H_{\Lambda_\ell(a)}$, and $q < \theta - \frac{d-1}{2}$. (2.1.56) follows from [EK, Lemma 3.3].

If $\lambda \in \sigma_{\{a\}}(H_\Theta)$, we have $\lambda = \widetilde{\lambda}_x^{(a)}$ for some $x \in \Lambda_\ell^{\Theta,\ell'}(a)$, thus (2.1.64) follows from (2.1.56) as $\varphi_x^{(a)}(y) = 0$ for all $y \in \Theta \setminus \Lambda_\ell(a)$.

If $\lambda \in \sigma(H_\Theta) \setminus \sigma_{\{a\}}(H_\Theta)$, for all $x \in \Lambda_\ell^{\Theta,\ell'}(a)$ we have

$$\left|\lambda - \lambda_x^{(a)}\right| \geq \left|\lambda - \widetilde{\lambda}_x^{(a)}\right| - \left|\widetilde{\lambda}_x^{(a)} - \lambda_x^{(a)}\right| \geq L^{-q} - C_{d,\varepsilon_0}\ell^{-\left(\theta - \frac{d-1}{2}\right)} \geq \tfrac{1}{2}L^{-q}, \tag{2.1.75}$$

since Θ is L-polynomially level spacing for H_Θ, we have (2.1.55), and $q < \gamma_1 q < \theta - \frac{d-1}{2}$. Therefore (2.1.66) follows from Lemma 2.1.3(i). (Note that (2.1.67) follows from Lemma 2.1.3(ii).)

Now let $\{\Lambda_\ell(a)\}_{a\in\mathcal{G}}$, where $\mathcal{G} \subset \mathbb{R}^d$ such that $\Lambda_\ell(a) \subset \Theta$ for all $a \in \mathcal{G}$, be a collection of θ-polynomially localizing boxes with corresponding eigensystems $\left\{(\varphi_x^{(a)}, \lambda_x^{(a)})\right\}_{x\in\Lambda_\ell(a)}$. Let $\lambda \in \sigma(H_\Theta)$, $a, b \in \mathcal{G}$, $a \neq b$, $x \in \Lambda_\ell^{\Theta,\ell'}(a)$ and $y \in \Lambda_\ell^{\Theta,\ell'}(b)$. Assume $\lambda_x^{(a)}, \lambda_x^{(b)} \in \mathcal{E}_{\mathcal{G}}^{\Theta}(\lambda)$, then it follows from (2.1.56) that

$$\left\|\varphi_x^{(a)} - \varphi_y^{(b)}\right\| \leq 4C_{d,\varepsilon_0}L^q\ell^{-\left(\theta - \frac{d-1}{2}\right)}, \tag{2.1.76}$$

thus

$$\left|\left\langle\varphi_x^{(a)}, \varphi_y^{(b)}\right\rangle\right| \geq \Re\left\langle\varphi_x^{(a)}, \varphi_y^{(b)}\right\rangle \geq 1 - 8C_{d,\varepsilon_0}^2 L^{2q}\ell^{-2\left(\theta - \frac{d-1}{2}\right)}. \tag{2.1.77}$$

On the other hand, (2.0.6) gives

$$\|x - y\| \geq 2\ell' \Longrightarrow \left|\left\langle\varphi_x^{(a)}, \varphi_y^{(b)}\right\rangle\right| \leq (\ell+1)^d\ell^{-\theta}. \tag{2.1.78}$$

Combining (2.1.77) and (2.1.78), we conclude that

$$\lambda_x^{(a)}, \lambda_x^{(b)} \in \mathcal{E}_{\mathcal{G}}^{\Theta}(\lambda) \Longrightarrow \|x - y\| < 2\ell'. \tag{2.1.79}$$

To prove (2.1.70), let $a, b \in \mathcal{G}$, $a \neq b$. Assume $\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset$, then

$$x \in \Lambda_\ell^{\Theta,\ell'}(a) \quad \text{and} \quad y \in \Lambda_\ell^{\Theta,\ell'}(b) \Longrightarrow \|x - y\| \geq 2\ell', \tag{2.1.80}$$

thus it follows from (2.1.69) that $\sigma_{\{a\}}(H_\Theta) \cap \sigma_{\{b\}}(H_\Theta) = \emptyset$.

Parts (ii)(b) and (ii)(c) follow immediately from parts (i)(b) and (i)(c) respectively. To prove part (ii)(d), we let $P_{\mathcal{G}}$ be the orthogonal projection onto the span of $\{\psi_\lambda; \lambda \in \sigma_{\mathcal{G}}(H_\Theta)\}$. (2.1.72) gives

$$\|(1 - P_{\mathcal{G}})\delta_y\| \leq C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} |\Theta|^{\frac{1}{2}} \quad \text{for all} \quad y \in \Theta'_{\mathcal{G}}, \tag{2.1.81}$$

thus

$$\|(1 - P_{\mathcal{G}})\chi_{\Theta'_{\mathcal{G}}}\| \leq |\Theta'_{\mathcal{G}}|^{\frac{1}{2}} |\Theta|^{\frac{1}{2}} C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} \leq |\Theta| C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)}. \tag{2.1.82}$$

If $|\Theta| \leq (L+1)^d$, we have

$$\|(1 - P_{\mathcal{G}})\chi_{\Theta'_{\mathcal{G}}}\| \leq (L+1)^d C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} < 1 \tag{2.1.83}$$

since $d + q < \gamma_1(d+q) < \theta - 2d$, so it follows from [EK, Lemma A.1] that

$$|\Theta'_{\mathcal{G}}| = \operatorname{tr} \chi_{\Theta'_{\mathcal{G}}} \leq \operatorname{tr} P_{\mathcal{G}} = |\sigma_{\mathcal{G}}(H_\Theta)|. \tag{2.1.84}$$

Using a similar argument and (2.1.71), we can prove $|\sigma_{\mathcal{G}}(H_\Theta)| \leq |\Theta_{\mathcal{G}}|$. □

2.1.3 Buffered subsets

For boxes $\Lambda_\ell \subset \Lambda_L$ that are not ♯ for H, we will surround them with a buffer of ♯ boxes and study eigensystems for the augmented subset.

Definition 2.1.5. Let $\Lambda_L = \Lambda_L(x_0)$ and $x_0 \in \mathbb{R}^d$. $\Upsilon \subset \Lambda_L$ is called a ♯-buffered subset of Λ_L, where ♯ stands for θ-PL, s-SEL, m^*-ML or m-LOC, if the following holds:

(i) Υ is a connected set in $\mathbb{Z}^d$ of the form

$$\Upsilon = \bigcup_{j=1}^{J} \Lambda_{R_j}(a_j) \cap \Lambda_L, \tag{2.1.85}$$

where $J \in \mathbb{N}$, $a_1, a_2, \ldots, a_J \in \Lambda_L^{\mathbb{R}}$, and $\ell \le R_j \le L$ for $j = 1, 2, \ldots, J$.

(ii) Υ is L-polynomially level spacing for H if $\sharp$ is θ-PL or m^*-ML, L-level spacing for H if $\sharp$ is s-SEL or m-LOC.

(iii) There exists $\mathcal{G}_\Upsilon \subset \Lambda_L^{\mathbb{R}}$ such that:

(a) For all $a \in \mathcal{G}_\Upsilon$ we have $\Lambda_\ell(a) \subset \Upsilon$, $\Lambda_\ell(a)$ is a $\sharp$ box for H.

(b) For all $y \in \partial_{\text{in}}^{\Lambda_L} \Upsilon$ there exists $a_y \in \mathcal{G}_\Upsilon$ such that $y \in \Lambda_\ell^{\Upsilon, 2\ell_\sharp}(a_y)$.

In this case we set

$$\breve{\Upsilon} = \bigcup_{a \in \mathcal{G}_\Upsilon} \Lambda_\ell(a), \quad \breve{\Upsilon}' = \bigcup_{a \in \mathcal{G}_\Upsilon} \Lambda_\ell^{\Upsilon, 2\ell_\sharp}(a), \quad \widehat{\Upsilon} = \Upsilon \setminus \breve{\Upsilon}, \quad \text{and} \quad \widehat{\Upsilon}' = \Upsilon \setminus \breve{\Upsilon}'. \tag{2.1.86}$$

($\breve{\Upsilon} = \Upsilon_{\mathcal{G}_\Upsilon}$ and $\breve{\Upsilon}' = \Upsilon'_{\mathcal{G}_\Upsilon}$ in the notation of Lemma 2.1.4.)

Lemma 2.1.6. *Given a $\sharp$-buffered subset Υ of Λ_L, let $\{(\psi_\nu, \nu)\}_{\nu \in \sigma(H_\Upsilon)}$ be an eigensystem for H_Υ. Let $\mathcal{G} = \mathcal{G}_\Upsilon$ and set*

$$\sigma_{\mathcal{B}}(H_\Upsilon) = \sigma(H_\Upsilon) \setminus \sigma_{\mathcal{G}}(H_\Upsilon), \tag{2.1.87}$$

where $\sigma_{\mathcal{G}}(H_\Upsilon)$ is as in (2.1.68)*. Then the following holds for sufficiently large ℓ:*

(i) *If* $\nu \in \sigma_{\mathcal{B}}(H_{\Upsilon})$ *we have for all* $y \in \breve{\Upsilon}'$*:*

$$|\psi_\lambda(y)| \le \begin{cases} C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} & \text{if } \sharp \text{ is } \theta\text{-PL} \\ \mathrm{e}^{-c_2 \ell^s}, \text{ with } c_2 = c_2(\ell) \text{ as in } (2.1.23) & \text{if } \sharp \text{ is } s\text{-SEL} \\ \mathrm{e}^{-m_2^* \ell_\tau}, \text{ with } m_2^* = m_2^*(\ell) \text{ as in } (2.1.25) & \text{if } \sharp \text{ is } m^*\text{-ML} \\ \mathrm{e}^{-m_2 \ell_\tau}, \text{ with } m_2 = m_2(\ell) \text{ as in } (2.1.27) & \text{if } \sharp \text{ is } m\text{-LOC} \end{cases}, \tag{2.1.88}$$

and

$$\left|\widehat{\Upsilon}\right| \le |\sigma_{\mathcal{B}}(H_{\Upsilon})| \le \left|\widehat{\Upsilon}'\right|. \tag{2.1.89}$$

(ii) *Let* Λ_L *be polynomially level spacing for* H *if* $\sharp$ *is* θ*-PL or* m^**-ML, level spacing for* H *if* $\sharp$ *is* s*-SEL or* m*-LOC, and let* $\{(\phi_\lambda, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})}$ *be an eigensystem for* H_{Λ_L}*. There exists an injection*

$$\nu \in \sigma_{\mathcal{B}}(H_{\Upsilon}) \mapsto \widetilde{\nu} \in \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}), \tag{2.1.90}$$

such that for all $\nu \in \sigma_{\mathcal{B}}(H_{\Upsilon})$*:*

(a) *If* $\sharp$ *is* θ*-PL, we have*

$$|\widetilde{\nu} - \nu| \le C_{d,\varepsilon_0} L^{\frac{d}{2}+q} \ell^{-(\theta-2d)}, \tag{2.1.91}$$

and, multiplying each ψ_ν *by a suitable phase factor,*

$$\|\phi_{\widetilde{\nu}} - \psi_\nu\| \le 2C_{d,\varepsilon_0} L^{\frac{d}{2}+2q} \ell^{-(\theta-2d)}. \tag{2.1.92}$$

(b) *If* $\sharp$ *is* s*-SEL, we have*

$$|\widetilde{\nu} - \nu| \le \mathrm{e}^{-c_3 \ell^s}, \text{ where } c_3 = c_3(\ell) \ge 1 - C_{d,\varepsilon_0} L^\beta \ell^{-s}, \tag{2.1.93}$$

and, multiplying each ψ_ν *by a suitable phase factor,*

$$\|\phi_{\widetilde{\nu}} - \psi_\nu\| \le 2\mathrm{e}^{-c_3 \ell^s} \mathrm{e}^{L^\beta}. \tag{2.1.94}$$

(c) *If $\sharp$ is m^*-ML, we have*

$$|\widetilde{\nu}-\nu| \le \mathrm{e}^{-m_4^*\ell_\tau}, \text{ where } m_4^* = m_4^*(\ell) \ge m^* - C_{d,\varepsilon_0}\gamma_1 q \frac{\log \ell}{\ell_\tau}, \tag{2.1.95}$$

and, multiplying each ψ_ν by a suitable phase factor,

$$\|\phi_{\widetilde{\nu}} - \psi_\nu\| \le 2\mathrm{e}^{-m_4^*\ell_\tau} L^q. \tag{2.1.96}$$

(d) *If $\sharp$ is m-LOC, we have*

$$|\widetilde{\nu}-\nu| \le \mathrm{e}^{-m_4\ell_\tau}, \text{ where } m_4 = m_4(\ell) \ge m - C_{d,\varepsilon_0}\ell^{\gamma\beta-\tau}, \tag{2.1.97}$$

and, multiplying each ψ_ν by a suitable phase factor,

$$\|\phi_{\widetilde{\nu}} - \psi_\nu\| \le 2\mathrm{e}^{-m_4\ell_\tau}\mathrm{e}^{L^\beta}. \tag{2.1.98}$$

Proof. Part (i) follows immediately from Lemma 2.1.4(ii)(c) and (ii)(d).

Let Λ_L be polynomially level spacing, and let $\{(\phi_\lambda, \lambda)\}_{\lambda\in\sigma(H_{\Lambda_L})}$ be an eigensystem for H_{Λ_L}. It follows from [EK, Lemma 3.2] that for $\nu \in \sigma_{\mathcal{B}}(H_\Upsilon)$ we have

$$\begin{aligned}\|(H_{\Lambda_L}-\nu)\psi_\nu\| &\le (2d-1)\varepsilon|\partial_{\mathrm{ex}}^{\Lambda_L}\Upsilon|^{\frac{1}{2}}\left\|\varphi_{\partial_{\mathrm{in}}^{\Lambda_L}\Upsilon}\right\|_\infty \le (2d-1)\varepsilon L^{\frac{d}{2}}C_{d,\varepsilon_0}L^q\ell^{-(\theta-2d)} \\ &\le C_{d,\varepsilon_0}L^{\frac{d}{2}+q}\ell^{-(\theta-2d)},\end{aligned} \tag{2.1.99}$$

where we used $\partial_{\mathrm{in}}^{\Lambda_L}\Upsilon \subset \check{\Upsilon}'$ and (2.1.88). The map in (2.1.90) is a well defined injection into $\sigma(H_{\Lambda_L})$ since Λ_L and Υ are L-polynomially level spacing for H, and (2.1.92) follows from (2.1.91) and [EK, Lemma 3.3].

To show $\widetilde{\nu} \notin \sigma_{\mathcal{G}}(H_{\Lambda_L})$ for all $\nu \in \sigma_{\mathcal{B}}(H_\Upsilon)$, we assume $\widetilde{\nu_1} \in \sigma_{\mathcal{G}}(H_{\Lambda_L})$ for some $\nu_1 \in \sigma_{\mathcal{B}}(H_\Upsilon)$. Then there is $a \in \mathcal{G}$ and $x \in \Lambda_\ell^{\Lambda_L,\ell'}(a)$ such that

$\lambda_x^{(a)} \in \mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\widetilde{\nu_1})$. On the other hand, $\lambda_x^{(a)} \in \mathcal{E}_{\mathcal{G}}^{\Upsilon}(\lambda_1)$ for some $\lambda_1 \in \sigma_{\mathcal{G}}(H_\Upsilon)$ by Lemma 2.1.4(i)(a). We conclude from (2.1.56) and (2.1.92) that

$$\begin{aligned}\sqrt{2} = \|\psi_{\lambda_1} - \psi_{\nu_1}\| &\leq \left\|\psi_{\lambda_1} - \varphi_x^{(a)}\right\| + \left\|\varphi_x^{(a)} - \phi_{\widetilde{\nu_1}}\right\| + \|\phi_{\widetilde{\nu_1}} - \psi_{\nu_1}\| \quad (2.1.100)\\ &\leq 4C_{d,\varepsilon_0} L^q \ell^{-\left(\theta - \frac{d-1}{2}\right)} + 2C_{d,\varepsilon_0} L^{\frac{d}{2}+2q} \ell^{-(\theta-2d)} < 1,\end{aligned}$$

a contradiction. □

Lemma 2.1.7. *Given $\Lambda_L = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$, let Υ be a $\sharp$-buffered subset of Λ_L. Let $\mathcal{G} = \mathcal{G}_\Upsilon$ and set*

$$\mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\nu) = \left\{\lambda_x^{(a)}; a \in \mathcal{G}, x \in \Lambda_\ell^{\Lambda_L,\ell_\sharp}(a), \widetilde{\lambda}_x^{(a)} = \nu\right\} \subset \mathcal{E}_{\mathcal{G}}^{\Upsilon}(\nu) \text{ for } \nu \in \sigma(H_\Upsilon), \quad (2.1.101)$$

$$\sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon) = \left\{\nu \in \sigma(H_\Upsilon); \mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\lambda) \neq \emptyset\right\} \subset \sigma_{\mathcal{G}}(H_\Upsilon).$$

The following holds for sufficiently large ℓ:

(i) *Let (ψ, λ) be an eigenpair for H_{Λ_L} such that for all $\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon) \cup \sigma_{\mathcal{B}}(H_\Upsilon)$,*

$$|\lambda - \nu| \geq \begin{cases} \frac{1}{2}L^{-q} & \text{if } \sharp \text{ is } \theta\text{-PL or } m^*\text{-ML} \\ \frac{1}{2}\mathrm{e}^{-L^\beta} & \text{if } \sharp \text{ is } s\text{-SEL or } m\text{-LOC} \end{cases}. \quad (2.1.102)$$

Then for all $y \in \Upsilon^{\Lambda_L,2\ell_\sharp}$:

(a) *If $\sharp$ is θ-PL, we have*

$$|\psi(y)| \leq C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} |\psi(v)| \quad \text{for some} \quad v \in \partial^{\Lambda_L,2\ell'}\Upsilon. \quad (2.1.103)$$

(b) *If $\sharp$ is s-SEL, we have*

$$|\psi(y)| \leq \mathrm{e}^{-c_4 \ell^s} |\psi(v)| \quad \textit{for some} \quad v \in \partial^{\Lambda_L, 2\ell'} \Upsilon, \tag{2.1.104}$$

$$\textit{where} \quad c_4 = c_4(\ell) \geq 1 - C_{d,\varepsilon_0} L^{\beta} \ell^{-s}. \tag{2.1.105}$$

(c) *If $\sharp$ is m^*-ML, we have*

$$|\psi(y)| \leq \mathrm{e}^{-m_5^* \ell_\tau} |\psi(v)| \quad \textit{for some} \quad v \in \partial^{\Lambda_L, 2\ell_\tau} \Upsilon, \tag{2.1.106}$$

$$\textit{where} \quad m_5^* = m_5^*(\ell) \geq m^* - C_{d,\varepsilon_0} \gamma_1 q \tfrac{\log \ell}{\ell_\tau}. \tag{2.1.107}$$

(d) *If $\sharp$ is m-LOC, we have*

$$|\psi(y)| \leq \mathrm{e}^{-m_5 \ell_\tau} |\psi(v)| \quad \textit{for some} \quad v \in \partial^{\Lambda_L, 2\ell_\tau} \Upsilon, \tag{2.1.108}$$

$$\textit{where} \quad m_5 = m_5(\ell) \geq m - C_{d,\varepsilon_0} \ell^{\gamma\beta - \tau}. \tag{2.1.109}$$

(ii) *Let Λ_L be polynomially level spacing for H if $\sharp$ is θ-PL or m^*-ML, level spacing for H if $\sharp$ is s-SEL or m-LOC. Let $\{(\psi_\lambda, \lambda)\}_{\lambda \in \sigma(H_{\Lambda_L})}$ be an eigensystem for H_{Λ_L}, and set (recalling* (2.1.90)*)*

$$\sigma_\Upsilon(H_{\Lambda_L}) = \{\widetilde{\nu}; \nu \in \sigma_{\mathcal{B}}(H_\Upsilon)\} \subset \sigma(H_{\Lambda_L}) \setminus \sigma_{\mathcal{G}}(H_{\Lambda_L}). \tag{2.1.110}$$

Then the condition (2.1.102) *is satisfied for all $\lambda \in \sigma(H_{\Lambda_L}) \setminus (\sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_\Upsilon(H_{\Lambda_L}))$, so for all $y \in \Upsilon^{\Lambda_L, 2\ell_\sharp}$*

$$|\psi_\lambda(y)| \leq \begin{cases} C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta - 2d)} |\psi(v)| & \textit{if } \sharp \textit{ is } \theta\textit{-PL} \\ \mathrm{e}^{-c_4 \ell^s} |\psi(v)| & \textit{if } \sharp \textit{ is s-SEL} \\ \mathrm{e}^{-m_5^* \ell_\tau} |\psi(v)| & \textit{if } \sharp \textit{ is } m^*\textit{-ML} \\ \mathrm{e}^{-m_5 \ell_\tau} |\psi(v)| & \textit{if } \sharp \textit{ is m-LOC} \end{cases} \tag{2.1.111}$$

for some $v \in \partial^{\Lambda_L, 2\ell_\sharp}\Upsilon$.

Proof. Let $\{(\vartheta_\nu, \nu)\}_{\nu\in\sigma(H_\Upsilon)}$ be an eigensystem for H_Υ. For $\nu \in \sigma_{\mathcal{G}}(H_\Upsilon)$ we fix $\lambda_{x_\nu}^{(a_\nu)} \in \mathcal{E}_{\mathcal{G}}^{\Upsilon}(\nu)$, where $a_\nu \in \mathcal{G}$, $x_\nu \in \Lambda_\ell^{\Upsilon,\ell'}(a_\nu)$. If $\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)$, we choose $\lambda_{x_\nu}^{(a_\nu)} \in \mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\nu)$, thus $x_\nu \in \Lambda_\ell^{\Lambda_L,\ell'}(a_\nu)$. If $\nu \in \sigma_{\mathcal{G}}(H_\Upsilon) \setminus \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)$ we have $x_\nu \in \Lambda_\ell^{\Upsilon,\ell'}(a_\nu) \setminus \Lambda_\ell^{\Lambda_L,\ell'}(a_\nu)$.

Given $y \in \Upsilon$, we have (see (2.1.10))

$$\begin{aligned}\psi(y) &= \sum_{\nu\in\sigma(\Upsilon)} \vartheta_\nu(y)\langle\vartheta_\nu, \psi\rangle \\ &= \sum_{\nu\in\sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)\cup\sigma_{\mathcal{B}}(H_\Upsilon)} \vartheta_\nu(y)\langle\vartheta_\nu, \psi\rangle + \sum_{\nu\in\sigma_{\mathcal{G}}(H_\Upsilon)\setminus\sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)} \vartheta_\nu(y)\langle\vartheta_\nu, \psi\rangle.\end{aligned} \tag{2.1.112}$$

Let (ψ, λ) be an eigenpair for H_{Λ_L} satisfying (2.1.102). If $\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon) \cup \sigma_{\mathcal{B}}(H_\Upsilon)$, we have

$$\langle\vartheta_\nu, \psi\rangle = (\lambda-\nu)^{-1}\langle\vartheta_\nu, (H_{\Lambda_L}-\nu)\psi\rangle = (\lambda-\nu)^{-1}\langle(H_{\Lambda_L}-\nu)\vartheta_u, \psi\rangle. \tag{2.1.113}$$

It follows from (2.1.102) and [EK, Lemma 3.2] that

$$\begin{aligned}|\vartheta_\nu(y)\langle\vartheta_\nu, \psi\rangle| &\le 2L^q\varepsilon|\vartheta_\nu(y)| \sum_{v\in\partial_{\mathrm{ex}}^{\Lambda_L}\Upsilon} \left(\sum_{v'\in\partial_{\mathrm{in}}^{\Lambda_L}\Upsilon, |v'-v|=1} |\vartheta_\nu(v')| \right) |\psi(v)| \\ &\le 2\varepsilon L^{q+d} \left(2d \max_{u\in\partial_{\mathrm{in}}^{\Lambda_L}\Upsilon} |\vartheta_\nu(u)| \right) |\psi(v_1)| \quad \text{for some} \quad v_1 \in \partial_{\mathrm{ex}}^{\Lambda_L}\Upsilon.\end{aligned} \tag{2.1.114}$$

If $\nu \in \sigma_{\mathcal{B}}(H_\Upsilon)$, (2.1.88) gives

$$\max_{u\in\partial_{\mathrm{in}}^{\Lambda_L}\Upsilon} |\vartheta_\nu(u)| \le C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)}. \tag{2.1.115}$$

If $\nu \in \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)$, it follows from (2.1.56) and (2.0.6), that

$$\begin{aligned}\max_{u\in\partial_{\text{in}}^{\Lambda_L}\Upsilon} |\vartheta_\nu(u)| &\le \max_{u\in\partial_{\text{in}}^{\Lambda_L}} \left(\left|\vartheta_\nu(u) - \varphi_{x_\nu}^{(a_\nu)}\right| + \left|\varphi_{x_\nu}^{(a_\nu)}\right|\right) \\ &\le 2C_{d,\varepsilon_0} L^q \ell^{-\left(\theta - \frac{d-1}{2}\right)} + \ell^{-\theta} \le 3C_{d,\varepsilon_0} L^q \ell^{-\left(\theta-\frac{d-1}{2}\right)} \le C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)}.\end{aligned} \tag{2.1.116}$$

Therefore (recalling (2.1.38)),

$$\begin{aligned}\left|\sum_{\nu\in\sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)\cup\sigma_{\mathcal{B}}(H_\Upsilon)} \vartheta_\nu(y)\langle\vartheta_\nu,\psi\rangle\right| &\le 4d\varepsilon L^{2d+q}\left(C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)}\right)|\psi(v_2)| \\ &\le C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} |\psi(v_2)|,\end{aligned} \tag{2.1.117}$$

for some $v_2 \in \partial_{\text{ex}}^{\Lambda_L}\Upsilon$.

If $\nu \in \sigma_{\mathcal{G}}(H_\Upsilon) \setminus \sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)$, we have $x_\nu \in \Lambda_\ell^{\Upsilon,\ell'}(a_\nu) \setminus \Lambda_\ell^{\Lambda_L,\ell'}(a_\nu)$, thus

$$\operatorname{dist}(x_\nu, \Upsilon \setminus \Lambda_\ell(a_\nu)) > \ell' \quad \text{and} \quad \operatorname{dist}(x_\nu, \Lambda_L \setminus \Lambda_\ell(a_\nu)) \le \ell', \tag{2.1.118}$$

and hence there is $u_0 \in \Lambda_L \setminus \Upsilon$ such that $\|x_\nu - u_0\| \le \ell'$. We suppose $y \in \Upsilon^{\Lambda_L,2\ell'}$, then $\|y - u_0\| > 2\ell'$. Therefore

$$\|x_\nu - y\| \ge \|y-u_0\| - \|x_\nu - u_0\| > 2\ell' - \ell' = \ell'. \tag{2.1.119}$$

Thus it follows from (2.1.56) and (2.0.6) that

$$\begin{aligned}|\vartheta_\nu(u)| &\le \left|\vartheta_\nu(u) - \varphi_{x_\nu}^{(a_\nu)}\right| + \left|\varphi_{x_\nu}^{(a_\nu)}\right| \le 2C_{d,\varepsilon_0} L^q \ell^{-\left(\theta-\frac{d-1}{2}\right)} + \ell^{-\theta} \\ &\le 3C_{d,\varepsilon_0} L^q \ell^{-\left(\theta-\frac{d-1}{2}\right)}.\end{aligned} \tag{2.1.120}$$

Therefore

$$\left|\sum_{\nu\in\sigma_{\mathcal{G}}(H_\Upsilon)\setminus\sigma_{\mathcal{G}}^{\Lambda_L}(H_\Upsilon)} \vartheta_\nu(y)\langle\vartheta_\nu,\psi\rangle\right| \le 3C_{d,\varepsilon_0} L^q (L+1)^{\frac{3d}{2}} \ell^{-\left(\theta-\frac{d-1}{2}\right)} |\psi(v_3)|, \tag{2.1.121}$$

for some $v_3 \in \Upsilon$.

Combining (2.1.112), (2.1.117) and (2.1.121), we conclude that for all $y \in \Upsilon^{\Lambda_L, 2\ell'}$,

$$|\psi(y)| \leq C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} |\psi(v_4)|, \tag{2.1.122}$$

for some $v_4 \in \Upsilon \cup \partial_{\text{ex}}^{\Lambda_L} \Upsilon$. If $v_4 \in \Upsilon^{\Lambda_L, 2\ell'}$ we repeat the procedure to estimate $|\psi(v_4)|$. Since we can suppose $\psi(y) \neq 0$ without loss of generality, the procedure must stop after finitely many times, and at that time we must have (2.1.103).

Now let Λ_L be polynomially level spacing. If $\lambda \notin \sigma_{\mathcal{G}}(H_{\Lambda_L})$, it follows from Lemma 2.1.4(i)(c) that (2.1.65) holds for all $a \in \mathcal{G}$. If $\lambda \notin \sigma_\Upsilon(H_{\Lambda_L})$, using the argument in (2.1.75), with (2.1.91) instead of (2.1.55), we get $|\lambda - \nu| \geq \frac{1}{2} L^{-q}$ for all $\nu \in \sigma_{\mathcal{B}}(H_\Upsilon)$. Therefore we have (2.1.102), which implies (2.1.103). $\square$

2.2 Probability estimates

The following lemma gives the probability estimates for polynomially level spacing and level spacing.

Lemma 2.2.1. *Let $H_{\varepsilon,\omega}$ be the Anderson model. Let $\Theta \subset \mathbb{Z}^d$ and $L > 1$. Then, for all $\varepsilon \leq \varepsilon_0$,*

$$\mathbb{P}\{\Theta \textit{ is } L\textit{-polynomially level spacing for } H\} \geq 1 - Y_{\varepsilon_0} L^{-(2\alpha-1)q} |\Theta|^2, \tag{2.2.1}$$

and

$$\mathbb{P}\{\Theta \textit{ is } L\textit{-level spacing for } H\} \geq 1 - Y_{\varepsilon_0} \mathrm{e}^{-(2\alpha-1)L^\beta} |\Theta|^2, \tag{2.2.2}$$

where

$$Y_{\varepsilon_0} = 2^{2\alpha-1} \widetilde{K}^2 (\operatorname{diam} \operatorname{supp} \mu + 2d\varepsilon_0 + 1), \tag{2.2.3}$$

with $\widetilde{K} = K$ *if* $\alpha = 1$ *and* $\widetilde{K} = 8K$ *if* $\alpha \in \left(\frac{1}{2}, 1\right)$.

Lemma 2.2.1 follows from [EK, Lemma 2.1] and its proof. (Also see [KM, Lemma 2].)

2.3 Bootstrap multiscale analysis

In this section, we fix $\theta > \left(\frac{6}{2\alpha-1} + \frac{9}{2}\right) d$ and $0 < \xi < 1$. (Note that Proposition 2.3.1 is independent to ξ.) We will omit the dependence on θ and ξ from the notation. We denote the complementary event of an event $\mathcal{E}$ by $\mathcal{E}^c$.

2.3.1 The first multiscale analysis

Proposition 2.3.1. *Fix* $\varepsilon_0 > 0$, $Y \geq 400$, *and* $P_0 < \frac{1}{2}(2Y)^{-2d}$. *There exists a finite scale* $\mathcal{L}(\varepsilon_0, Y)$ *with the following property: Suppose for some scale* $L_0 \geq \mathcal{L}(\varepsilon_0, Y)$, *and* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P_0. \tag{2.3.1}$$

Then, setting $L_{k+1} = Y L_k$ *for* $k = 0, 1, \ldots$, *there exists* $K_0 = K_0(Y, L_0, P_0) \in \mathbb{N}$ *such that*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - L_k^{-p} \text{ for } k \geq K_0. \tag{2.3.2}$$

Proposition 2.3.1 follows from the following induction step for the multiscale analysis.

Lemma 2.3.2. *Fix $\varepsilon_0 > 0$, $Y \geq 400$, and $P \leq 1$. Suppose for some scale ℓ and $0 < \varepsilon \leq \varepsilon_0$ we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P. \tag{2.3.3}$$

Then, if ℓ is sufficiently large, for $L = Y\ell$ we have

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - \left((2Y)^{2d}P^2 + \tfrac{1}{2}L^{-p}\right). \tag{2.3.4}$$

Proof. We fix $0 < \varepsilon \leq \varepsilon_0$ and suppose (2.3.3) for some scale ℓ. Let $\Lambda_L = \Lambda_L(x_0)$, where $x_0 \in \mathbb{R}^d$, and let $\mathcal{C}_{L,\ell} = \mathcal{C}_{L,\ell}(x_0)$ be the suitable ℓ-cover of Λ_L. For $N \in \mathbb{N}$, let $\mathcal{B}_N$ denote the event that there exist at most N disjoint boxes in $\mathcal{C}_{L,\ell}$ that are not θ-PL for $H_{\varepsilon,\omega}$. Using (2.3.3), (2.1.9) and the fact that events on disjoint boxes are independent, if $N = 1$ we have

$$\mathbb{P}\{\mathcal{B}_N^{\mathrm{c}}\} \leq \left(\tfrac{2L}{\ell}\right)^{(N+1)d} P^{N+1} = (2Y)^{(N+1)d}P^{N+1} = (2Y)^{2d}P^2. \tag{2.3.5}$$

We now fix $\omega \in \mathcal{B}_N$. There exists $\mathcal{A}_N = \mathcal{A}_N(\omega) \in \Xi_{L,\ell} = \Xi_{L,\ell}(x_0)$, with $|\mathcal{A}_N| \leq N$ and $\|a - b\| \geq 2\rho\ell$ (i.e., $\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset$) if $a, b \in \mathcal{A}_N$, $a \neq b$, such that for all $a \in \Xi_{L,\ell}$ with $\operatorname{dist}(a, \mathcal{A}_N) \geq 2\rho\ell$ (i.e., $\Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset$ for all $b \in \mathcal{A}_N$), $\Lambda_\ell(a)$ is a $\sharp$ box for $H_{\varepsilon,\omega}$ ($\sharp$ stands for θ-PL). In other words,

$$a \in \Xi_{L,\ell} \setminus \bigcup_{b\in\mathcal{A}_N} \Lambda^{\mathbb{R}}_{(2\rho+1)\ell}(a_0) \Longrightarrow \Lambda_\ell(a) \text{ is a } \sharp \text{ box for } H_{\varepsilon,\omega}. \tag{2.3.6}$$

To embed the box $\{\Lambda_\ell(b)\}_{b\in\mathcal{A}_N}$ into $\sharp$-buffered subsets of Λ_L, we consider graphs $\mathbb{G}_i = (\Xi_{L,\ell}, \mathbb{E}_i)$, $i = 1, 2$, both having $\Xi_{L,\ell}$ as the set of vertices, with sets of edges given by

$$\begin{aligned}\mathbb{E}_1 &= \{\{a,b\} \in \Xi_{L,\ell}^2; \|a-b\| = \rho\ell\} \\ &= \{\{a,b\} \in \Xi_{L,\ell}^2; a \neq b \text{ and } \Lambda_\ell(a) \cap \Lambda_\ell(b) \neq \emptyset\}, \\ \mathbb{E}_2 &= \{\{a,b\} \in \Xi_{L,\ell}^2; \text{ either } \|a-b\| = 2\rho\ell \text{ or } \|a-b\| = 3\rho\ell\} \\ &= \{\{a,b\} \in \Xi_{L,\ell}^2; \Lambda_\ell(a) \cap \Lambda_\ell(b) = \emptyset \text{ and } \Lambda_{(2\rho+1)\ell}(a) \cap \Lambda_{(2\rho+1)\ell}(b) \neq \emptyset\}.\end{aligned} \tag{2.3.7}$$

Let $\{\Phi_r\}_{r=1}^R = \{\Phi_r(\omega)\}_{r=1}^R$ denote the $\mathbb{G}_2$-connected components of $\mathcal{A}_N$ (i.e., connected in the graph $\mathcal{G}_2$). Note that

$$R \in \{1, 2, \ldots, N\}, \quad \sum_{r=1}^R |\Phi_r| = |\mathcal{A}_N| \leq N, \quad \text{and} \quad \operatorname{diam} \Phi_r \leq 3\rho\ell(|\Phi_r| - 1). \tag{2.3.8}$$

Set

$$\widetilde{\Phi}_r = \Xi_{L,\ell} \cap \bigcup_{a \in \Phi_r} \Lambda^{\mathbb{R}}_{(2\rho+1)\ell}(a) = \{a \in \Xi_{L,\ell}; \operatorname{dist}(a, \Phi_r) \leq \rho\ell\}, \tag{2.3.9}$$

and note that $\left\{\widetilde{\Phi}_r\right\}_{r=1}^R$ is a collection of disjoint, $\mathbb{G}_1$-connected subsets of $\Xi_{L,\ell}$, such that

$$\operatorname{diam} \widetilde{\Phi}_r \leq \operatorname{diam} \Phi_r + 2\rho\ell \leq \rho\ell(3|\Phi_r| - 1) \text{ and } \operatorname{dist}(\widetilde{\Phi}_r, \widetilde{\Phi}_{\widetilde{r}}) \geq 2\rho\ell, \ r \neq \widetilde{r}. \tag{2.3.10}$$

Moreover, (2.3.6) gives

$$a \in \mathcal{G} = \mathcal{G}(\omega) = \Xi_{L,\ell} \setminus \bigcup_{r=1}^R \widetilde{\Phi}_r \quad \Longrightarrow \quad \Lambda_\ell(a) \text{ is a } \sharp \text{ box for } H_{\varepsilon,\omega}. \tag{2.3.11}$$

For $\Psi \subset \Xi_{L,\ell}$, we define the exterior boundary of Ψ in the graph $\mathbb{G}_1$ by

$$\partial_{\text{ex}}^{\mathbb{G}_1} \Psi = \{a \in \Xi_{L,\ell}; \operatorname{dist}(a, \Psi) = \rho\ell\}. \tag{2.3.12}$$

It follows from (2.3.11) that $\Lambda_\ell(a)$ is $\sharp$ for $H_{\varepsilon,\omega}$ for all $a \in \partial_{\mathrm{ex}}^{\mathbb{G}_1}\widetilde{\Phi}_r$, $r = 1, 2, \ldots, R$. Set $\overline{\Psi} = \Psi \cup \partial_{\mathrm{ex}}^{\mathbb{G}_1}\Psi$, and set, for $r = 1, 2, \ldots, R$,

$$\Upsilon_r^{(0)} = \Upsilon_r^{(0)}(\omega) = \bigcup_{a \in \widetilde{\Phi}_r} \Lambda_\ell(a), \tag{2.3.13}$$

$$\Upsilon_r = \Upsilon_r(\omega) = \Upsilon_r^{(0)} \cup \bigcup_{a \in \partial_{\mathrm{ex}}^{\mathbb{G}_1}\widetilde{\Phi}_r} \Lambda_\ell(a) = \bigcup_{a \in \overline{\widetilde{\Phi}_r}} \Lambda_\ell(a).$$

Each Υ_r, $r = 1, 2, \ldots, R$, satisfies all the requirements to be a θ-PL-buffered subset of Λ_L with $\mathcal{G}_{\Upsilon_r} = \partial_{\mathrm{ex}}^{\mathbb{G}_1}\widetilde{\Phi}_r$ (see Definition 2.1.5), except that we do not know if Υ_r is L-polynomially level spacing for $H_{\varepsilon,\omega}$. (Note that the sets $\{\Upsilon_r^{(0)}\}_{r=1}^R$ are disjoint, but the sets $\{\Upsilon_r\}_{r=1}^R$ are not necessarily disjoint.) Note also that

$$\operatorname{diam} \overline{\widetilde{\Phi}}_r \le \operatorname{diam} \widetilde{\Phi}_r + 2\rho\ell \le \rho\ell(3|\Phi_r| + 1), \tag{2.3.14}$$

and hence

$$\operatorname{diam} \Upsilon_r \le \operatorname{diam} \overline{\widetilde{\Phi}}_r + \ell \le \rho\ell(3|\Phi_r| + 1) + \ell \le 5\ell|\Phi_r|, \tag{2.3.15}$$

thus

$$\sum_{r=1}^{R} \operatorname{diam} \Upsilon_r \le 5\ell N. \tag{2.3.16}$$

We can arrange for $\{\Upsilon_r\}_{r=1}^R$ to be a collection of θ-PL-buffered subsets of Λ_L as follows. It follows from Lemma 2.2.1 that for any $\Theta \subset \Lambda_L$ we have

$$\mathbb{P}\{\Theta \text{ is } L\text{-polynomially level spacing for } H_{\varepsilon,\omega}\} \ge 1 - Y_{\varepsilon_0} \mathrm{e}^{-(2\alpha-1)L^\beta}(L+1)^{2d}. \tag{2.3.17}$$

Given a $\mathbb{G}_2$-connected subset Φ of $\Xi_{L,\ell}$, let $\Upsilon(\Phi) \subset \Lambda_L$ be constructed from

Φ as in (2.3.13). Set

$$\mathcal{F}_N = \bigcup_{r=1}^{N} \mathcal{F}(r), \text{ where } \mathcal{F}(r) = \{\Phi \subset \Xi_{L,\ell}; \Phi \text{ is } \mathbb{G}_2\text{-connected and } |\Phi| = r\}. \tag{2.3.18}$$

Let $\mathcal{F}(r,a) = \{\Phi \in \mathcal{F}_r; a \in \Phi\}$ for $a \in \Xi_{L,\ell}$, and note that each vertex in the graph $\mathbb{G}_2$ has less than $d(3^{d-1} + 4^{d-1}) \leq d4^d$ nearest neighbors , we have

$$|\mathcal{F}(r,a)| \leq (r-1)!(d4^d)^{r-1} \quad \Longrightarrow \quad |\mathcal{F}(r)| \leq (L+1)^d (r-1)!(d4^d)^{r-1} \tag{2.3.19}$$

$$\Longrightarrow \quad |\mathcal{F}_N| \leq (L+1)^d N!(d4^d)^{N-1}.$$

Let $\mathcal{S}_N$ denote the event that the box Λ_L and the subsets $\{\Upsilon(\Phi)\}_{\Phi \in \mathcal{F}_N}$ are all L-polynomially level spacing for $H_{\varepsilon,\omega}$, using (2.3.17) and (2.3.19), if $N = 1$ we have

$$\mathbb{P}\{\mathcal{S}_N^c\} \leq Y_{\varepsilon_0} \left(1 + (L+1)^d N!(d4^d)^{N-1}\right) (L+1)^{2d} (L+1)^{2d} L^{-(2\alpha-1)q} < \tfrac{1}{2} L^{-p} \tag{2.3.20}$$

for sufficiently large L since $p < (2\alpha - 1)q - 3d$.

Let $\mathcal{E}_N = \mathcal{B}_N \cap \mathcal{S}_N$. Combining (2.3.5) and (2.3.20), we conclude that if $N = 1$,

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - \left((2Y)^{2d} P^2 + \tfrac{1}{2} L^{-p}\right). \tag{2.3.21}$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box Λ_L is θ-PL for $H_{\varepsilon,\omega}$.

We fix $\omega \in \mathcal{E}_N$. Then we have (2.3.11), Λ_L is polynomially level spacing for $H_{\varepsilon,\omega}$, and the subsets $\{\Upsilon_r\}_{r=1}^R$ constructed in (2.3.13) are θ-PL-buffered subsets of Λ_L for $H_{\varepsilon,\omega}$. It follows from (2.1.8) and Definition 2.1.5(iii) that

$$\Lambda_L = \left\{ \bigcup_{a\in\mathcal{G}} \Lambda_\ell^{\Lambda_L,\frac{\ell}{10}}(a) \right\} \cup \left\{ \bigcup_{r=1}^{R} \Upsilon_r^{\Lambda_L,\frac{\ell}{10}} \right\}. \tag{2.3.22}$$

We omit ε and ω from the notation since they are now fixed. Let $\{(\psi_\lambda,\lambda)\}_{\lambda\in\sigma(H_{\Lambda_L})}$ be an eigensystem for H_{Λ_L}. For $a\in\mathcal{G}$, let $\left\{(\varphi_x^{(a)},\lambda_x^{(a)})\right\}_{x\in\Lambda_\ell(a)}$ be a θ-polynomially localized eigensystem for $\Lambda_\ell(a)$. For $r=1,2,\ldots,R$, let $\left\{(\phi_{\nu^{(r)}},\nu^{(r)})\right\}_{\nu^{(r)}\in\sigma(H_{\Upsilon_r})}$ be an eigensystem for H_{Υ_r}, and set

$$\sigma_{\Upsilon_r} = \left\{\widetilde{\nu}^{(r)};\nu^{(r)}\in\sigma_{\mathcal{B}}(H_{\Upsilon_r})\right\} \subset \sigma(H_{\Lambda_L})\setminus\sigma_{\mathcal{G}}(H_{\Lambda_L}), \tag{2.3.23}$$

where $\widetilde{\nu}^{(r)}$ is given in (2.1.90), which also gives $\sigma_{\Upsilon_r}(H_{\Lambda_L})\subset\sigma(H_{\Lambda_L})\setminus\sigma_{\mathcal{G}_{\Upsilon_r}}(H_{\Lambda_L})$, but the argument actually shows $\sigma_{\Upsilon_r}(H_{\Lambda_L})\subset\sigma(H_{\Lambda_L})\setminus\sigma_{\mathcal{G}}(H_{\Lambda_L})$. We also set

$$\sigma_{\mathcal{B}}(H_{\Lambda_L}) = \bigcup_{r=1}^{R}\sigma_{\Upsilon_r}(H_{\Lambda_L})\subset\sigma(H_{\Lambda_L})\setminus\sigma_{\mathcal{G}}(H_{\Lambda_L}). \tag{2.3.24}$$

We claim

$$\sigma(H_{\Lambda_L}) = \sigma_{\mathcal{G}}(H_{\Lambda_L})\cup\sigma_{\mathcal{B}}(H_{\Lambda_L}). \tag{2.3.25}$$

To do this, we assume $\lambda\in\sigma_{\mathcal{G}}\setminus(\sigma_{\mathcal{G}}(H_{\Lambda_L})\cup\sigma_{\mathcal{B}}(H_{\Lambda_L}))$. Since Λ_L is polynomially level spacing for H, Lemma 2.1.4(ii)(c) gives

$$|\psi_\lambda(y)|\le C_{d,\varepsilon_0}L^q\ell^{-(\theta-2d)} \quad \text{for all} \quad y\in\bigcup_{a\in\mathcal{G}}\Lambda_\ell^{\Lambda_L,2\ell'}(a), \tag{2.3.26}$$

and Lemma 2.1.7(ii) gives

$$|\psi_\lambda(y)|\le C_{d,\varepsilon_0}L^{2d+2q}\ell^{-(\theta-2d)} \quad \text{for all} \quad y\in\bigcup_{r=1}^{R}\Upsilon_r^{\Lambda_L,2\ell'}. \tag{2.3.27}$$

Using (2.3.22) and $\theta-2d>\gamma_1\left(\frac{5d}{2}+2q\right)>\frac{5d}{2}+2q$, we conclude that

$$1=\|\psi_\lambda(y)\|\le C_{d,\varepsilon_0}L^{2d+2q}\ell^{-(\theta-2d)}(L+1)^{\frac{d}{2}}<1 \tag{2.3.28}$$

for sufficiently large ℓ, a contradiction. This establishes the claim.

We now index the eigenvalues and eigenvectors of H_{Λ_L} by sites in Λ_L using Hall's Marriage Theorem, which states a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph. (See [EK, Appendix C] and [BuDM, Chapter 2].) We consider the bipartite graph $\mathbb{G} = (\Lambda_L, \sigma(H_{\Lambda_L}); \mathbb{E})$, where the edge set $\mathbb{E} \subset \Lambda_L \times \sigma(H_{\Lambda_L})$ is defined as follows. For each $\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L})$ we fix $\lambda_{x_\lambda}^{(a_\lambda)} \in \mathcal{E}_{\mathcal{G}}^{\Lambda_L}(\lambda)$, and set (recall (2.1.86) and (2.1.19))

$$\mathcal{N}_0(x) = \begin{cases} \{\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}); \|x_\lambda - x\| < \ell_\sharp\} & \text{for} \quad x \in \Lambda_L \setminus \bigcup_{r=1}^{R} \widehat{\Upsilon}_r \\ \emptyset & \text{for} \quad x \in \bigcup_{r=1}^{R} \widehat{\Upsilon}_r \end{cases}. \tag{2.3.29}$$

We define

$$\mathcal{N}(x) = \begin{cases} \mathcal{N}_0(x) & \text{for} \quad x \in \Lambda_L \setminus \bigcup_{r=1}^{R} \widehat{\Upsilon}'_r \\ \sigma_\Upsilon(H_{\Lambda_L}) & \text{for} \quad x \in \widehat{\Upsilon}_r,\ r = 1, 2, \ldots, R \\ \mathcal{N}_0(x) \cup \sigma_\Upsilon(H_{\Lambda_L}) & \text{for} \quad x \in \widehat{\Upsilon}'_r, \setminus \widehat{\Upsilon}_r,\ r = 1, 2, \ldots, R \end{cases}, \tag{2.3.30}$$

and let $\mathbb{E} = \{(x, \lambda) \in \Lambda_L \times \sigma(H_{\Lambda_L}); \lambda \in \mathcal{N}(x)\}$.

$\mathcal{N}(x)$ was defined to ensure $|\psi_\lambda(x)| \ll 1$ for $\lambda \notin \mathcal{N}(x)$. This can be seen as follows:

- If $x \in \Lambda_L$ and $\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}) \setminus \mathcal{N}_0(x)$, we have $\lambda = \widetilde{\lambda}_{x_\lambda}^{(a_\lambda)}$ with $\|x_\lambda - x\| \geq \ell'$, so, using (2.0.6) and (2.1.56),

$$|\psi_\lambda(x)| \leq \left|\varphi_{x_\lambda}^{(a_\lambda)}(x)\right| + \left\|\varphi_{x_\lambda}^{(a_\lambda)} - \psi_\lambda\right\| \leq \ell^{-\Theta} + 2C_{d,\varepsilon_0} L^q \ell^{-\left(\theta - \frac{d-1}{2}\right)} \tag{2.3.31}$$

$$\leq 3C_{d,\varepsilon_0} L^q \ell^{-\left(\theta - \frac{d-1}{2}\right)}.$$

- If $x \in \Lambda_L \setminus \widehat{\Upsilon'}_r$ and $\lambda \in \sigma_{\Upsilon_r}(H_{\Lambda_L})$, then $\lambda = \widetilde{\nu}^{(r)}$ for some $\nu^{(r)} \in \sigma_{\mathcal{B}}(H_\Upsilon)$, and, using (2.1.88) and (2.1.92), (Note $\phi_{\nu^{(r)}}(x) = 0$ if $x \notin \Upsilon_r$.)

$$|\psi_\lambda(x)| \le |\phi_{\nu^{(r)}}(x)| + \|\phi_{\nu^{(r)}}(x) - \psi_\lambda\| \le C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} + 2C_{d,\varepsilon_0} L^{\frac{d}{2}+2q} \ell^{-(\theta-2d)} \tag{2.3.32}$$

$$\le 3C_{d,\varepsilon_0} L^{\frac{d}{2}+2q} \ell^{-(\theta-2d)}.$$

Therefore for all $x \in \Lambda_L$ and $\lambda \in \sigma(H_{\Lambda_L}) \setminus \mathcal{N}(x)$ we have

$$|\psi_\lambda(x)| \le C_{d,\varepsilon_0} L^{\frac{d}{2}+2q} \ell^{-(\theta-2d)}. \tag{2.3.33}$$

Since $|\Lambda_L| = |\sigma(H_{\Lambda_L})|$, to apply Hall's Marriage Theorem we only need to verify $|\Theta| \le |\mathcal{N}(\Theta)|$, where $\mathcal{N}(\Theta) = \bigcup_{x\in\Theta} \mathcal{N}(x)$ for $\Theta \subset \Lambda_L$. For $\Theta \subset \Lambda_L$, let Q_Θ be the orthogonal projection onto the span of $\{\psi_\lambda; \lambda \in \mathcal{N}(\Theta)\}$. If $\lambda \notin \mathcal{N}(\Theta)$, for all $x \in \Theta$ we have (2.3.33), thus

$$\begin{aligned} \|(1 - Q_\Theta)\chi_\Theta\| &\le |\Lambda_L|^{\frac{1}{2}} |\Theta|^{\frac{1}{2}} C_{d,\varepsilon_0} L^{\frac{d}{2}+2q} \ell^{-(\theta-2d)} \\ &\le (L+1)^d C_{d,\varepsilon_0} L^{\frac{d}{2}+2q} \ell^{-(\theta-2d)} < 1, \end{aligned} \tag{2.3.34}$$

for sufficiently large ℓ since $\theta - 2d > \gamma_1 \left(\frac{5d}{2} + 2q\right) > \frac{5}{2}d + 2q$, so it follows from [EK, Lemma A.1] that

$$|\Theta| = \operatorname{tr} \chi_\Theta \le \operatorname{tr} Q_\Theta = |\mathcal{N}(\Theta)|. \tag{2.3.35}$$

Using Hall's Marriage Theorem, we conclude that there exists a bijection

$$x \in \Lambda_L \mapsto \lambda_x \in \sigma(H_{\Lambda_L}), \quad \text{where} \quad \lambda_x \in \mathcal{N}(x). \tag{2.3.36}$$

We set $\psi_x = \psi_{\lambda_x}$ for all $x \in \Lambda_L$.

To finish the proof we need to show that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is a θ-polynomially localized eigensystem for Λ_L. We fix $N = 1$, $x \in \Lambda_L$, take $y \in \Lambda_L$, and consider several cases:

(i) Suppose $\lambda_x \in \sigma_{\mathcal{G}}(\Lambda_L)$. Then $x \in \Lambda_\ell(a_{\lambda_x})$ with $a_{\lambda_x} \in \mathcal{G}$, and $\lambda_x \in \sigma_{\{a_{\lambda_x}\}}(H_{\Lambda_L})$. In view of (2.3.22) we consider two cases:

(a) If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$ and $\|y - x\| \geq 2\ell$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Lambda_\ell(a) = \emptyset$, so it follows from (2.1.70) that $\lambda_x \notin \sigma_{\{a\}}(H_{\Lambda_L})$, and (2.1.66) gives

$$|\psi_x| \leq C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} |\psi_x(y_1)| \text{ for some } y_1 \in \partial^{\Theta,2\ell'} \Lambda_\ell(a). \quad (2.3.37)$$

(b) If $y \in \Upsilon_1^{\Lambda_L, \frac{\ell}{10}}$, and $\|y - x\| \geq \ell + \operatorname{diam} \Upsilon_1$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Upsilon_1 = \emptyset$, so it follows from (2.1.70) that $\lambda_x \notin \sigma_{\mathcal{G}_{\Upsilon_1}}(H_{\Lambda_L})$, and clearly $\lambda_x \notin \sigma_{\Upsilon_1}(H_{\Lambda_L})$ in view of (2.3.23). Thus Lemma 2.1.7(ii) gives

$$|\psi_x(y)| \leq C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} |\psi_x(v)| \quad \text{for some} \quad v \in \partial^{\Lambda_L, 2\ell'} \Upsilon_1. \quad (2.3.38)$$

(ii) Suppose $\lambda_x \notin \sigma_{\mathcal{G}}(\Lambda_L)$. Then it follows from (2.3.25) that we must have $\lambda_x \in \sigma_{\Upsilon_1}(H_{\Lambda_L})$. If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$, and $\|y - x\| \geq \ell + \operatorname{diam} \Upsilon_1$, we must have $\Lambda_\ell(a) \cap \Upsilon_1 = \emptyset$, and (2.1.66) gives (2.3.37).

Now we fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L'$. Suppose $|\psi_x(y)| > 0$ without loss of generality. We estimate $|\psi_x(y)|$ using either (2.3.37) or (2.3.38) repeatedly, as appropriate, stopping when we get too close to x so we are not in any case described above. (Note that this must happen since $|\psi_x(y)| > 0$.) We accumulate decay only when using (2.3.37), and just use $C_{d,\varepsilon_0} L^{2d+2q} \ell^{-(\theta-2d)} < 1$ when using (2.3.38), then recalling $L = Y\ell$, we get

$$|\psi_x(y)| \leq \left(C_{d,\varepsilon_0} L^q \ell^{-(\theta-2d)} \right)^{n(Y)}, \quad (2.3.39)$$

where $n(Y)$ is the number of times we used (2.3.37). We have

$$n(Y)(\ell+1) + \operatorname{diam}\Upsilon_1 + 2\ell \geq L'. \tag{2.3.40}$$

Thus, using (2.3.16), we have

$$n(Y) \geq \tfrac{1}{\ell+1}(L' - 5\ell - 2\ell) \geq \tfrac{\ell}{\ell+1}\left(\tfrac{Y}{40} - 7\right) \geq 2. \tag{2.3.41}$$

for sufficiently large ℓ since $Y \geq 400$. It follows from (2.3.39),

$$|\psi_x(y)| \leq \left(C_{d,\varepsilon_0} Y^q \ell^{-(\theta-2d-q)}\right)^2 \leq L^{-\theta}, \tag{2.3.42}$$

for sufficiently large ℓ since $2(\theta - 2d - q) = \theta + (\theta - 4d - 2q) > \theta$.

We conclude that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is a θ-polynomially localized eigensystem for Λ_L, so the box Λ_L is θ-polynomially localizing for $H_{\varepsilon,\omega}$.

□

Proof of Proposition 2.3.1. We assume (2.3.1) and set $L_{k+1} = YL_k$ for $k = 0, 1, \ldots$. We set

$$P_k = \sup_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is not } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \text{ for } k = 1, 2, \ldots. \tag{2.3.43}$$

Then by Lemma 2.3.2, we have

$$P_{k+1} \leq (2Y)^{2d} P_k^2 + \tfrac{1}{2} L_{k+1}^{-p} \quad \text{for} \quad k = 0, 1, \ldots \tag{2.3.44}$$

If $P_k \leq L_k^{-p}$ for some $k \geq 0$, we have

$$P_{k+1} \leq (2Y)^{2d} L_k^{-2p} + \tfrac{1}{2} L_{k+1}^{-p} \leq (2Y)^{2d+2p} L_{k+1}^{-2p} + \tfrac{1}{2} L_{k+1}^{-p} \leq L_{k+1}^{-p} \tag{2.3.45}$$

for L_0 sufficiently large. Therefore to finish the proof, we need to show that

$$K_0 = \inf\{k \in \mathbb{N}; P_k \leq L_k^{-p}\} < \infty. \tag{2.3.46}$$

It follows from (2.3.44) that for any $1 \le k < K_0$,

$$P_k \le (2Y)^{2d} P_{k-1}^2 + \tfrac{1}{2} L_k^{-p} < (2Y)^{2d} P_{k-1}^2 + \tfrac{1}{2} P_k, \tag{2.3.47}$$

so

$$2(2Y)^{2d} P_k < \left(2(2Y)^{2d} P_{k-1}\right)^2. \tag{2.3.48}$$

Therefore for $1 \le k < K_0$, we have

$$2^{2d+1} Y^{-(kp-2d)} L_0^{-p} = 2(2Y)^{2d} L_k^{-p} < 2(2Y)^{2d} P_k < \left(2(2Y)^{2d} P_0\right)^{2^k}. \tag{2.3.49}$$

Since $2(2Y)^{2d} P_0 < 1$, (2.3.49) cannot be satisfied for large k. We conclude that $K_0 < \infty$. □

2.3.2 The first intermediate step

Proposition 2.3.3. *Fix $\varepsilon_0 > 0$. Suppose for some scale ℓ and $0 < \varepsilon \le \varepsilon_0$ we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \ge 1 - \ell^{-p}. \tag{2.3.50}$$

Then, if ℓ is sufficiently large, for $L = \ell^{\gamma_1}$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } m_0^*\text{-mix localizing for } H_{\varepsilon,\omega}\} \ge 1 - L^{-p}, \tag{2.3.51}$$

where

$$m_0^* \ge \tfrac{1}{8}\left(\tfrac{5d}{2} + q\right) L^{-(1-\tau+\frac{1}{\gamma_1})} \log L. \tag{2.3.52}$$

Proof. We follow the proof of Lemma 2.3.2. For $N \in \mathbb{N}$, let $\mathcal{B}_N$, $\mathcal{S}_N$ and $\mathcal{E}_N$ as in the proof of Lemma 2.3.2. Using (2.3.50), (2.1.9) and the fact that events on disjoint boxes are independent, if $N = 1$ we have,

$$\mathbb{P}\{\mathcal{B}_N^c\} \le \left(\tfrac{2L}{\ell}\right)^{2d} \ell^{-2p} = 2^{2d} \ell^{-2p-2d(\gamma_1-1)} < \tfrac{1}{2} \ell^{-\gamma_1 p} = \tfrac{1}{2} L^{-p} \tag{2.3.53}$$

for all ℓ sufficiently large since $1 < \gamma_1 < 1 + \frac{p}{p+2d}$. Also, using (2.3.17) and (2.3.19), if $N = 1$ we have,

$$\mathbb{P}\{\mathcal{S}_N^c\} \leq \left(1 + (L+1)^d\right) Y_{\varepsilon_0} (L+1)^{2d} L^{-(2\alpha-1)q} < \tfrac{1}{2} L^{-p} \tag{2.3.54}$$

for sufficiently large L, since $p < (2\alpha - 1)q - 3d$. Combining (2.3.53) and (2.3.54), we conclude that

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - L^{-p}. \tag{2.3.55}$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box Λ_L is m_0^*-mix localizing for $H_{\varepsilon,\omega}$, where m_0^* is given in (2.3.52). Following the proof of Lemma 2.3.2, we get (2.3.25) and obtain an eigensystem $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ for H_{Λ_L} using Hall's Marriage Theorem. To finish the proof we need to show that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is an m_0^*-localized eigensystem for Λ_L. We proceed as in the proof of Lemma 2.3.2. We fix $N = 1$, $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L_\tau$, we have

$$n(\ell)(\ell + 1) + \operatorname{diam} \Upsilon_1 + 2\ell \geq L_\tau. \tag{2.3.56}$$

where $n(\ell)$ is the number of times we used (2.3.37). Thus, using (2.3.16), we have

$$n(\ell) \geq \tfrac{1}{\ell+1}(L_\tau - 5\ell - 2\ell) \geq \tfrac{\ell}{\ell+1}\left(\tfrac{1}{2}\ell^{\gamma_1\tau-1} - 7\right) \geq \tfrac{1}{4}\ell^{\gamma_1\tau-1}. \tag{2.3.57}$$

for sufficiently large ℓ. It follows from (2.3.39),

$$\begin{aligned} |\psi_x(y)| &\leq \left(C_{d,\varepsilon_0}\ell^{-(\theta-2d-\gamma_1 q)}\right)^{\frac{1}{4}\ell^{\gamma_1\tau-1}} \\ &\leq \mathrm{e}^{-\frac{1}{8}\left(\frac{5d}{2}+q\right)L^{-(1-\tau+\frac{1}{\gamma_1})}(\log L)\|y-x\|}, \end{aligned} \tag{2.3.58}$$

for sufficiently large ℓ.

We conclude that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an m_0^*-localized eigensystem for Λ_L, where m_0^* is given in (2.3.52), so the box Λ_L is m_0^*-mix localizing for $H_{\varepsilon,\omega}$.

□

2.3.3 The second multiscale analysis

Proposition 2.3.4. *Fix $\varepsilon_0 > 0$. There exists a finite scale $\mathcal{L}(\varepsilon_0)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}(\varepsilon_0)$, $0 < \varepsilon \leq \varepsilon_0$, and $m_0^* \geq L_0^{-\kappa}$ where $0 < \kappa < \tau$, we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \text{ is } m_0^*\text{-mix localizing for } H_{\varepsilon,\omega}\} \geq 1 - L_0^{-p}. \tag{2.3.59}$$

Then, setting $L_{k+1} = L_k^{\gamma_1}$ for $k = 0, 1, \ldots$, we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \tfrac{m_0^*}{2}\text{-mix localizing for } H_{\varepsilon,\omega}\} \geq 1 - L_k^{-p} \text{ for } k = 0, 1, \ldots. \tag{2.3.60}$$

Proposition 2.3.4 follows from the following induction step for the multiscale analysis.

Lemma 2.3.5. *Fix $\varepsilon_0 > 0$. Suppose for some scale ℓ, $0 < \varepsilon \leq \varepsilon_0$, and $m^* \geq \ell^{-\kappa}$, where $0 < \kappa < \tau$, we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } m^*\text{-mix localizing for } H_{\varepsilon,\omega}\} \geq 1 - \ell^{-p}. \tag{2.3.61}$$

Then, if ℓ is sufficiently large, for $L = \ell^{\gamma_1}$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } M^*\text{-mix localizing for } H_{\varepsilon,\omega}\} \geq 1 - L^{-p}, \tag{2.3.62}$$

where

$$M^* \geq m^* \left(1 - C_{d,\varepsilon_0} \gamma_1 q \ell^{-\min\left\{\frac{1-\tau}{2}, \gamma_1 \tau - 1, \tau - \kappa\right\}}\right) \geq L^{-\kappa}. \tag{2.3.63}$$

Proof. We follow the proof of Lemma 2.3.2. For $N \in \mathbb{N}$, let $\mathcal{B}_N$ denote the event that there do not exist two disjoint boxes in $\mathcal{C}_{L,\ell}$ that are not m^*-mix localizing for $H_{\varepsilon,\omega}$. Using (2.3.61), (2.1.9) and the fact that events on disjoint boxes are independent, if $N = 1$ we have

$$\mathbb{P}\{\mathcal{B}_N^c\} \leq \left(\tfrac{2L}{\ell}\right)^{(N+1)d} \ell^{-(N+1)p} = 2^{2d}\ell^{-(2p-2d(\gamma_1-1))} < \tfrac{1}{2}\ell^{-\gamma_1 p} = \tfrac{1}{2}L^{-p} \quad (2.3.64)$$

for all ℓ sufficiently large since $1 < \gamma_1 < 1 + \frac{p}{p+2d}$.

We now fix $\omega \in \mathcal{B}_N$, and proceed as in the proof of Lemma 2.3.2 with $\sharp$ being m^*-ML. Then we have Υ_r, $r = 1, 2, \ldots, R$ such that each Υ_r satisfies all the requirements to be an m^*-ML-buffered subset of Λ_L with $\mathcal{G}_{\Upsilon_r} = \partial_{\mathrm{ex}}^{\mathbb{G}_1}\widetilde{\Phi}_r$, except we do not know if Υ_r is L-polynomially level spacing for $H_{\varepsilon,\omega}$.

Given a $\mathbb{G}_2$-connected subset Φ of $\Xi_{L,\ell}$, let $\Upsilon(\Phi) \subset \Lambda_L$ be constructed from Φ as in (2.3.13) with $\sharp$ being m^*-ML. Let $\mathcal{S}_N$ denote the event that the box Λ_L and the subsets $\{\Upsilon(\Phi)\}_{\Phi\in\mathcal{F}_N}$ are all L-polynomially level spacing for $H_{\varepsilon,\omega}$. Using (2.3.17) and (2.3.19), if $N = 1$ we have

$$\mathbb{P}\{\mathcal{S}^c\} \leq \left(1 + \left(\tfrac{2L}{\ell}\right)^d\right) Y_{\varepsilon_0}(L+1)^{2d}L^{-(2\alpha-1)q} < \tfrac{1}{2}L^{-p} \quad (2.3.65)$$

for sufficiently large L, since $p < (2\alpha - 1)q - 3d$.

Let $\mathcal{E}_N = \mathcal{B}_N \cap \mathcal{S}_N$. Combining (2.3.64) and (2.3.65), we conclude that if $N = 1$,

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - L^{-p}. \quad (2.3.66)$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box Λ_L is M^*-mix localizing for $H_{\varepsilon,\omega}$, where M^* is given in (2.3.63).

We fix $\omega \in \mathcal{E}_N$. Then we have (2.3.11), Λ_L is polynomially level spacing for $H_{\varepsilon,\omega}$, and the subsets $\{\Upsilon_r\}_{r=1}^R$ constructed in (2.3.13) are m^*-ML-buffered subset of Λ_L for $H_{\varepsilon,\omega}$. We proceed as in the proof of Lemma 2.3.2. To claim (2.3.25), we assume $\lambda \in \sigma_{\mathcal{G}} \setminus (\sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L}))$. Since Λ_L is polynomially level spacing for H, Lemma 2.1.4(ii)(c) gives

$$|\psi_\lambda(y)| \leq \mathrm{e}^{-m_2^* \ell_\tau} \quad \text{for all} \quad y \in \bigcup_{a \in \mathcal{G}} \Lambda_\ell^{\Lambda_L, 2\ell_\tau}(a), \tag{2.3.67}$$

and Lemma 2.1.7(ii) gives

$$|\psi_\lambda(y)| \leq \mathrm{e}^{-m_5^* \ell_\tau} \quad \text{for all} \quad y \in \bigcup_{r=1}^{R} \Upsilon_r^{\Lambda_L, 2\ell_\tau}. \tag{2.3.68}$$

Using (2.3.22), we conclude that (note $m_5^* \leq m_2^*$)

$$1 = \|\psi_\lambda(y)\| \leq \mathrm{e}^{-m_5^* \ell_\tau}(L+1)^{\frac{d}{2}} < 1, \tag{2.3.69}$$

a contradiction. This establishes the claim.

To index the eigenvalues and eigenvectors of H_{Λ_L} by sites in Λ_L, we define $\mathcal{N}(x)$ as in (2.3.30) and proceed as in the proof of Lemma 2.3.2. We have:

- If $x \in \Lambda_L$ and $\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}) \setminus \mathcal{N}_0(x)$, we have $\lambda = \widetilde{\lambda}_{x_\lambda}^{(a_\lambda)}$ with $\|x_\lambda - x\| \geq \ell_\tau$, so, using (2.0.8) and (2.1.60),

$$|\psi_\lambda(x)| \leq \left|\varphi_{x_\lambda}^{(a_\lambda)}(x)\right| + \left\|\varphi_{x_\lambda}^{(a_\lambda)} - \psi_\lambda\right\| \leq \mathrm{e}^{-m^* \ell_\tau} + 2\mathrm{e}^{-m_1^* \ell_\tau} L^q \leq 3\mathrm{e}^{-m_1 \ell_\tau} L^q. \tag{2.3.70}$$

- If $x \in \Lambda_L \setminus \widehat{\Upsilon}'_r$ and $\lambda \in \sigma_{\Upsilon_r}(H_{\Lambda_L})$, then $\lambda = \widetilde{\nu}^{(r)}$ for some $\nu^{(r)} \in \sigma_{\mathcal{B}}(H_{\Upsilon_r})$, and, using (2.1.88) and (2.1.96), (Note $\phi_{\nu^{(r)}}(x) = 0$ if $x \notin \Upsilon_r$.)

$$|\psi_\lambda(x)| \leq |\phi_{\nu^{(r)}}(x)| + \|\phi_{\nu^{(r)}}(x) - \psi_\lambda\| \leq \mathrm{e}^{-m_2^* \ell_\tau} + 2\mathrm{e}^{-m_4^* \ell_\tau} L^q \leq 3\mathrm{e}^{-m_4^* \ell_\tau} L^q. \tag{2.3.71}$$

Therefore for all $x \in \Lambda_L$ and $\lambda \in \sigma(H_{\Lambda_L}) \setminus \mathcal{N}(x)$ we have

$$|\psi_\lambda(x)| \leq 3\mathrm{e}^{-m_4^* \ell_\tau} L^q \leq \mathrm{e}^{-\frac{1}{2} m_4^* \ell_\tau}. \tag{2.3.72}$$

If $\lambda \notin \mathcal{N}(\Theta)$, for all $x \in \Theta$ we have (2.3.72), thus

$$\|(1 - Q_\Theta)\chi_\Theta\| \leq |\Lambda_L|^{\frac{1}{2}} |\Theta|^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{2} m_4^* \ell_\tau} \leq (L+1)^d \mathrm{e}^{-\frac{1}{2} m_4^* \ell_\tau} < 1. \tag{2.3.73}$$

Following the proof of Lemma 2.3.2, we can apply Hall's Marriage Theorem to obtain an eigensystem $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ for H_{Λ_L}.

To finish the proof we need to show that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an M^*-localized eigensystem for Λ_L, where M^* is given in (2.3.63). We fix $N = 1$, $x \in \Lambda_L$, take $y \in \Lambda_L$, and consider several cases:

(i) Suppose $\lambda_x \in \sigma_{\mathcal{G}}(\Lambda_L)$. Then $x \in \Lambda_\ell(a_{\lambda_x})$ with $a_{\lambda_x} \in \mathcal{G}$, and $\lambda_x \in \sigma_{\{a_{\lambda_x}\}}(H_{\Lambda_L})$. In view of (2.3.22) we consider two cases:

(a) If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$ and $\|y - x\| \geq 2\ell$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Lambda_\ell(a) = \emptyset$, so it follows from (2.1.70) that $\lambda_x \notin \sigma_{\{a\}}(H_{\Lambda_L})$, and (2.1.67) gives

$$|\psi_x| \leq \mathrm{e}^{-m_3^* \|y_1 - y\|} |\psi_x(y_1)| \text{ for some } y_1 \in \partial^{\Theta, \ell_{\tilde{\tau}}} \Lambda_\ell(a). \tag{2.3.74}$$

(b) If $y \in \Upsilon_1^{\Lambda_L, \frac{\ell}{10}}$, and $\|y - x\| \geq \ell + \operatorname{diam} \Upsilon_1$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Upsilon_1 = \emptyset$, so it follows from (2.1.70) that $\lambda_x \notin \sigma_{\mathcal{G}_{\Upsilon_1}}(H_{\Lambda_L})$, and clearly $\lambda_x \notin \sigma_{\Upsilon_1}(H_{\Lambda_L})$ in view of (2.3.23). Thus Lemma 2.1.7(ii) gives

$$|\psi_x(y)| \leq \mathrm{e}^{-m_5^* \ell_\tau} |\psi_x(v)| \quad \text{for some} \quad v \in \partial^{\Lambda_L, 2\ell_\tau} \Upsilon_1. \tag{2.3.75}$$

(i) Suppose $\lambda_x \notin \sigma_{\mathcal{G}}(\Lambda_L)$. Then it follows from (2.3.25) that we must have $\lambda_x \in \sigma_{\Upsilon_1}(H_{\Lambda_L})$. If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$, and $\|y - x\| \geq \ell + \operatorname{diam} \Upsilon_1$, we must have $\Lambda_\ell(a) \cap \Upsilon_1 = \emptyset$, and (2.1.67) gives (2.3.74).

Now we fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L_\tau$. Suppose $|\psi_x(y)| > 0$ without loss of generality. We estimate $|\psi_x(y)|$ using either (2.3.74) or (2.3.75) repeatedly, as appropriate, stopping when we get too close to x so we are not in any case described above. (Note that this must happen since $|\psi_x(y)| > 0$.) We accumulate decay only when using (2.3.74), and just use $\mathrm{e}^{-m_5^* \ell_\tau} < 1$ when using (2.3.75), then we get

$$
\begin{aligned}
|\psi_x(y)| &\leq \mathrm{e}^{-m_3^*(\|y-x\| - \operatorname{diam}\Upsilon - 2\ell)} \leq \mathrm{e}^{-m_3^*(\|y-x\| - 7\ell)} \\
&\leq \mathrm{e}^{-m_3^* \|y-x\| \left(1 - 7\ell^{1-\gamma_1 \tau}\right)} \leq \mathrm{e}^{M\|y-x\|},
\end{aligned}
\tag{2.3.76}
$$

where we used (2.3.16) and took

$$
\begin{aligned}
M^* = m_3^* \left(1 - 7\ell^{1-\gamma_1\tau}\right) &\geq \left(m^* \left(1 - 4\ell^{\frac{\tau-1}{2}}\right) - C_{d,\varepsilon_0}\gamma_1 q \frac{\log \ell}{\ell_{\widetilde{\tau}}}\right)\left(1 - 7\ell^{1-\gamma_1\tau}\right) \qquad (2.3.77)\\
&\geq m^* \left(1 - 4\ell^{\frac{\tau-1}{2}} - C_{d,\varepsilon_0}\gamma_1 q \ell^{\kappa - \tau}\right)\left(1 - 7\ell^{1-\gamma_1\tau}\right) \\
&\geq m^* \left(1 - C_{d,\varepsilon_0}\gamma_1 q \ell^{-\min\{\frac{1-\tau}{2}, \gamma_1\tau - 1, \tau - \kappa\}}\right) \\
&\geq \tfrac{1}{2}\ell^{-\kappa} \geq \ell^{-\gamma_1\kappa} = L^{-\kappa}
\end{aligned}
$$

for ℓ sufficiently large, where we used (2.1.29) and $m^* \geq \ell^{-\kappa}$.

We conclude that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an M^*-localized eigensystem for Λ_L, where M^* is given in (2.3.63), so the box Λ_L is M^*-mix localizing for $H_{\varepsilon,\omega}$.

□

Proof of Proposition 2.3.4. We assume (2.3.59) and set $L_{k+1} = L_k^{\gamma_1}$ for $k = 0, 1, \ldots$. If L_0 is sufficiently large it follows from Lemma 2.3.5 by an induction argument that

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } m_k^*\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - L_k^{-p} \text{ for } k = 0, 1, \ldots, \tag{2.3.78}$$

where for $k = 1, 2, \ldots$ we have

$$m_k^* \geq m_{k-1}^* \left(1 - C_{d,\varepsilon_0}\gamma_1 q L_{k-1}^{-\varrho}\right), \text{ with } \varrho = \min\left\{\tfrac{1-\tau}{2}, \gamma_1\tau - 1, \tau - \kappa\right\}. \tag{2.3.79}$$

Thus for all $k = 1, 2, \ldots$, taking L_0 sufficiently large we get

$$m_k^* \geq m_0^* \prod_{j=0}^{k-1} \left(1 - C_{d,\varepsilon_0}\gamma_1 q L_0^{-\varrho\gamma^j}\right) \geq m_0^* \prod_{j=0}^{\infty} \left(1 - C_{d,\varepsilon_0}\gamma_1 q L_0^{-\varrho\gamma_1^j}\right) \geq \tfrac{m_0^*}{2}, \tag{2.3.80}$$

finishing the proof of Proposition 2.3.4. □

2.3.4 The third multiscale analysis

Proposition 2.3.6. *Fix* $\varepsilon_0 > 0$, $Y \geq 400^{\frac{1}{1-s}}$, *and* $\widetilde{P}_0 < \left(2(2Y)^{(\lfloor Y^s\rfloor+1)d}\right)^{-\frac{1}{\lfloor Y^s\rfloor}}$. *There exists a finite scale* $\mathcal{L}(\varepsilon_0, Y)$ *with the following property: Suppose for some scale* $L_0 \geq \mathcal{L}(\varepsilon_0, Y)$ *and* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \textit{ is s-SEL for } H_{\varepsilon,\omega}\} \geq 1 - \widetilde{P}_0. \tag{2.3.81}$$

Then, setting $L_{k+1} = YL_k$ *for* $k = 0, 1, \ldots$, *there exists* $K_0 = K_0(Y, L_0, \widetilde{P}_0) \in \mathbb{N}$ *such that*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \textit{ is s-SEL for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L_k^{\zeta}} \textit{ for } k \geq K_0. \tag{2.3.82}$$

Proposition 2.3.6 follows from the following induction step for the multi-scale analysis.

Lemma 2.3.7. *Fix* $\varepsilon_0 > 0$, $Y \geq 400^{\frac{1}{1-s}}$ *and* $P \leq 1$. *Suppose for some scale* ℓ *and* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } s\text{-SEL for } H_{\varepsilon,\omega}\} \geq 1 - P. \tag{2.3.83}$$

Then, if ℓ *is sufficiently large, for* $L = Y\ell$ *we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } s\text{-SEL for } H_{\varepsilon,\omega}\} \geq 1 - \left((2Y)^{(\lfloor Y^s \rfloor + 1)d} P^{\lfloor Y^s \rfloor + 1} + \tfrac{1}{2} \mathrm{e}^{-L^\varsigma} \right). \tag{2.3.84}$$

Proof. We follow the proof of Lemma 2.3.2. For $N \in \mathbb{N}$, let $\mathcal{B}_N$ denote the event that there exist at most N disjoint boxes in $\mathcal{C}_{L,\ell}$ that are not s-SEL for $H_{\varepsilon,\omega}$. Using (2.3.83), (2.1.9) and the fact that events on disjoint boxes are independent, if $N = \lfloor Y^s \rfloor$ we have

$$\mathbb{P}\{\mathcal{B}^c\} \leq \left(\tfrac{2L}{\ell}\right)^{(N+1)d} P^{N+1} = (2Y)^{(\lfloor Y^s \rfloor + 1)d} P^{\lfloor Y^s \rfloor + 1}. \tag{2.3.85}$$

We now fix $\omega \in \mathcal{B}_N$, and proceed as in the proof of Lemma 2.3.2 with $\sharp$ being s-SEL. Then we have Υ_r, $r = 1, 2, \ldots, R$ such that each Υ_r satisfies all the requirements to be an s-SEL-buffered subset of Λ_L with $\mathcal{G}_{\Upsilon_r} = \partial_{\mathrm{ex}}^{\mathbb{G}_1} \widetilde{\Phi}_r$, except we do not know if Υ_r is L-level spacing for $H_{\varepsilon,\omega}$.

It follows from Lemma 2.2.1 that for any $\Theta \subset \Lambda_L$ we have

$$\mathbb{P}\{\Theta \text{ is } L\text{-level spacing for } H_{\varepsilon,\omega}\} \geq 1 - Y_{\varepsilon_0} \mathrm{e}^{-(2\alpha - 1)L^\beta} (L+1)^{2d}. \tag{2.3.86}$$

Given a $\mathbb{G}_2$-connected subset Φ of $\Xi_{L,\ell}$, let $\Upsilon(\Phi) \subset \Lambda_L$ be constructed from Φ as in (2.3.13) with $\sharp$ being s-SEL. Let $\mathcal{S}_N$ denote the event that the box

Λ_L and the subsets the subsets $\{\Upsilon(\Phi)\}_{\Phi\in\mathcal{F}_N}$ are all L-level spacing for $H_{\varepsilon,\omega}$. Using (2.3.86) and (2.3.19), if $N = \lfloor Y^s \rfloor$ we have

$$\mathbb{P}\{\mathcal{S}_N^c\} \leq Y_{\varepsilon_0}\left(1 + (L+1)^d N!(d4^d)^{N-1}\right)(L+1)^{2d}\mathrm{e}^{-(2\alpha-1)L^\beta} < \tfrac{1}{2}\mathrm{e}^{-L^\zeta} \tag{2.3.87}$$

for sufficiently large L, since $\zeta < \beta$.

Let $\mathcal{E}_\mathcal{N} = \mathcal{B}_\mathcal{N} \cap \mathcal{S}_\mathcal{N}$. Combining (2.3.85) and (2.3.87), we conclude that

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - \left((2Y)^{(\lfloor Y^s\rfloor+1)d} P^{\lfloor Y^s\rfloor+1} + \tfrac{1}{2}\mathrm{e}^{-L^\zeta}\right). \tag{2.3.88}$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box Λ_L is s-SEL for $H_{\varepsilon,\omega}$.

We fix $\omega \in \mathcal{E}_N$. Then we have (2.3.11), Λ_L is level spacing for $H_{\varepsilon,\omega}$, and the subsets $\{\Upsilon_r\}_{r=1}^R$ constructed in (2.3.13) are s-SEL-buffered subsets of Λ_L for $H_{\varepsilon,\omega}$. We proceed as in the proof of Lemma 2.3.2. To claim (2.3.25), we assume $\lambda \in \sigma_\mathcal{G} \setminus (\sigma_\mathcal{G}(H_{\Lambda_L}) \cup \sigma_\mathcal{B}(H_{\Lambda_L}))$. Since Λ_L is level spacing for H, Lemma 2.1.4(ii)(c) gives

$$|\psi_\lambda(y)| \leq \mathrm{e}^{-c_2\ell^s} \quad \text{for all} \quad y \in \bigcup_{a\in\mathcal{G}} \Lambda_\ell^{\Lambda_L,2\ell'}(a), \tag{2.3.89}$$

and Lemma 2.1.7(ii) gives

$$|\psi_\lambda(y)| \leq \mathrm{e}^{-c_4\ell^s} \quad \text{for all} \quad y \in \bigcup_{r=1}^R \Upsilon_r^{\Lambda_L,2\ell'}. \tag{2.3.90}$$

Using (2.3.22), we conclude that (note $c_4 \leq c_2$)

$$1 = \|\psi_\lambda(y)\| \leq \mathrm{e}^{-c_4\ell^s}(L+1)^{\frac{d}{2}} < 1, \tag{2.3.91}$$

a contradiction. This establishes the claim.

To index the eigenvalues and eigenvectors of H_{Λ_L} by sites in Λ_L, we define $\mathcal{N}(x)$ as in (2.3.30) proceed as in the proof of Lemma 2.3.2. We have:

- If $x \in \Lambda_L$ and $\lambda \in \sigma_{\mathcal{G}}(H_{\Lambda_L}) \setminus \mathcal{N}_0(x)$, we have $\lambda = \widetilde{\lambda}_{x_\lambda}^{(a_\lambda)}$ with $\|x_\lambda - x\| \geq \ell'$, so, using (2.0.7) and (2.1.58),

$$|\psi_\lambda(x)| \leq \left|\varphi_{x_\lambda}^{(a_\lambda)}(x)\right| + \left\|\varphi_{x_\lambda}^{(a_\lambda)} - \psi_\lambda\right\| \leq \mathrm{e}^{-\ell^s} + 2\mathrm{e}^{-c_1 \ell^s}\mathrm{e}^{L^\beta} \leq 3\mathrm{e}^{-c_1 \ell^s}\mathrm{e}^{L^\beta}. \tag{2.3.92}$$

- If $x \in \Lambda_L \setminus \widehat{\Upsilon}'_r$ and $\lambda \in \sigma_{\Upsilon_r}(H_{\Lambda_L})$, then $\lambda = \widetilde{\nu}^{(r)}$ for some $\nu^{(r)} \in \sigma_{\mathcal{B}}(H_{\Upsilon_r})$, and, using (2.1.88) and (2.1.94), (Note $\phi_{\nu^{(r)}}(x) = 0$ if $x \notin \Upsilon_r$.)

$$|\psi_\lambda(x)| \leq |\phi_\nu(x)| + \|\phi_\nu(x) - \psi_\lambda\| \leq \mathrm{e}^{-c_2 \ell^s} + 2\mathrm{e}^{-c_3 \ell^s}\mathrm{e}^{L^\beta} \leq 3\mathrm{e}^{-c_3 \ell^s}\mathrm{e}^{L^\beta}. \tag{2.3.93}$$

Therefore for all $x \in \Lambda_L$ and $\lambda \in \sigma(H_{\Lambda_L}) \setminus \mathcal{N}(x)$ we have

$$|\psi_\lambda(x)| \leq 3\mathrm{e}^{-c_3 \ell^s}\mathrm{e}^{L^\beta} \leq \mathrm{e}^{-\frac{1}{2}c_3 \ell^s}. \tag{2.3.94}$$

If $\lambda \notin \mathcal{N}(\Theta)$, for all $x \in \Theta$ we have (2.3.94), thus

$$\|(1 - Q_\Theta)\chi_\Theta\| \leq |\Lambda_L|^{\frac{1}{2}} |\Theta|^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}c_3 \ell^s} \leq (L+1)^d \mathrm{e}^{-\frac{1}{2}c_3 \ell^s} < 1. \tag{2.3.95}$$

Following the proof of Lemma 2.3.2, we can apply Hall's Marriage Theorem to obtain an eigensystem $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ for H_{Λ_L}.

To finish the proof we need to show that $\{(\psi_x, \lambda_x)\}_{x \in \Lambda_L}$ is an s-subexponentially localized eigensystem for Λ_L. We fix $N = \lfloor Y^s \rfloor$, $x \in \Lambda_L$, take $y \in \Lambda_L$, and consider several cases:

(i) Suppose $\lambda_x \in \sigma_{\mathcal{G}}(\Lambda_L)$. Then $x \in \Lambda_\ell(a_{\lambda_x})$ with $a_{\lambda_x} \in \mathcal{G}$, and $\lambda_x \in \sigma_{\{a_{\lambda_x}\}}(H_{\Lambda_L})$. In view of (2.3.22) we consider two cases:

  (a) If $y \in \Lambda_\ell^{\Lambda_L, \frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$ and $\|y - x\| \geq 2\ell$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Lambda_\ell(a) = \emptyset$, so it follows from (2.1.70) that $\lambda_x \notin$

$\sigma_{\{a\}}(H_{\Lambda_L})$, and (2.1.66) gives

$$|\psi_x| \le \mathrm{e}^{-c_2\ell^s}|\psi_x(y_1)| \text{ for some } y_1 \in \partial^{\Theta,2\ell'}\Lambda_\ell(a). \tag{2.3.96}$$

(b) If $y \in \Upsilon_r^{\Lambda_L,\frac{\ell}{10}}$ for some $r \in \{1,2,\dots,R\}$, and $\|y-x\| \ge \ell + \operatorname{diam}\Upsilon_r$, we must have $\Lambda_\ell(a_{\lambda_x}) \cap \Upsilon_r = \emptyset$, so it follows from (2.1.70) that $\lambda_x \notin \sigma_{\mathcal{G}_{\Upsilon_r}}(H_{\Lambda_L})$, and clearly $\lambda_x \notin \sigma_{\Upsilon_r}(H_{\Lambda_L})$ in view of (2.3.23). Thus Lemma 2.1.7(ii) gives

$$|\psi_x(y)| \le \mathrm{e}^{-c_4\ell^s}|\psi_x(v)| \quad \text{for some} \quad v \in \partial^{\Lambda_L,2\ell'}\Upsilon_r. \tag{2.3.97}$$

(ii) Suppose $\lambda_x \notin \sigma_{\mathcal{G}}(\Lambda_L)$. Then it follows from (2.3.25) that we must have $\lambda_x \in \sigma_{\Upsilon_{\widetilde{r}}}(H_{\Lambda_L})$ for some $\widetilde{r} \in \{1,2,\dots,R\}$. In view of (2.3.22) we consider two cases:

(a) If $y \in \Lambda_\ell^{\Lambda_L,\frac{\ell}{10}}(a)$ for some $a \in \mathcal{G}$, and $\|y-x\| \ge \ell + \operatorname{diam}\Upsilon_{\widetilde{r}}$, we must have $\Lambda_\ell(a) \cap \Upsilon_{\widetilde{r}} = \emptyset$, and (2.1.66) gives (2.3.96).

(b) If $y \in \Upsilon_r^{\Lambda_L,\frac{\ell}{10}}$ for some $r \in \{1,2,\dots,R\}$, and $\|y-x\| \ge \operatorname{diam}\Upsilon_{\widetilde{r}} + \operatorname{diam}\Upsilon_r$, we must have $r \neq \widetilde{r}$. Thus Lemma 2.1.7(ii) gives (2.3.97).

Now we fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y-x\| \ge L'$. Suppose $|\psi_x(y)| > 0$ without loss of generality. We estimate $|\psi_x(y)|$ using either (2.3.96) or (2.3.97) repeatedly, as appropriate, stopping when we get too close to x so we are not in any case described above. (Note that this must happen since $|\psi_x(y)| > 0$.) We accumulate decay only when we use (2.3.96), and just use $\mathrm{e}^{-c_4\ell^s} < 1$ when using (2.3.97), recalling $L = Y\ell$, then we get

$$|\psi_x(y)| \le \left(\mathrm{e}^{-c_2\ell^s}\right)^{n(Y)}, \tag{2.3.98}$$

where $n(Y)$ is the number of times we used (2.3.96). We have

$$n(Y)(\ell+1)+\sum_{r=1}^{R}\operatorname{diam}\Upsilon_r+2\ell\geq L'. \tag{2.3.99}$$

Thus, using (2.3.16), we have

$$n(Y)\geq \tfrac{1}{\ell+1}(L'-5\ell\lfloor Y^s\rfloor-2\ell)\geq \tfrac{\ell}{\ell+1}\left(\tfrac{Y}{40}-5Y^s-2\right)\geq 2Y^s. \tag{2.3.100}$$

for sufficiently large ℓ since $Y\geq 400^{\frac{1}{1-s}}$. It follows from (2.3.98),

$$|\psi_x(y)|\leq\left(\mathrm{e}^{-c_2\ell^s}\right)^{2Y^s}\leq \mathrm{e}^{-L^s}, \tag{2.3.101}$$

for sufficiently large ℓ.

We conclude that $\{(\psi_x,\lambda_x)\}_{x\in\Lambda_L}$ is an s-subexponentially localized eigensystem for Λ_L, so the box Λ_L is s-SEL for $H_{\varepsilon,\omega}$.

□

Proof of Proposition 2.3.6. We assume (2.3.81) and set $L_{k+1}=YL_k$ for $k=0,1,\ldots$. We set

$$\widetilde{P}_k=\sup_{x\in\mathbb{R}^d}\mathbb{P}\{\Lambda_{L_k}(x)\text{ is not }s\text{-SEL for }H_{\varepsilon,\omega}\}\text{ for }k=1,2,\ldots. \tag{2.3.102}$$

Then by Lemma 2.3.7, we have

$$\widetilde{P}_{k+1}\leq(2Y)^{(\lfloor Y^s\rfloor+1)d}\widetilde{P}_k^{\lfloor Y^s\rfloor+1}+\tfrac{1}{2}\mathrm{e}^{-L_{k+1}^{\zeta}}\quad\text{for}\quad k=0,1,\ldots \tag{2.3.103}$$

If $\widetilde{P}_k\leq \mathrm{e}^{-L_k^{\zeta}}$ for some $k\geq 0$, we have

$$\begin{aligned}\widetilde{P}_{k+1}&\leq(2Y)^{(\lfloor Y^s\rfloor+1)d}\left(\mathrm{e}^{-L_k^{\zeta}}\right)^{\lfloor Y^s\rfloor+1}+\tfrac{1}{2}\mathrm{e}^{-L_{k+1}^{\zeta}}\\&\leq(2Y)^{(\lfloor Y^s\rfloor+1)d}\mathrm{e}^{-\frac{\lfloor Y^s\rfloor+1}{Y^{\zeta}}L_{k+1}^{\zeta}}+\tfrac{1}{2}\mathrm{e}^{-L_{k+1}^{\zeta}}\leq \mathrm{e}^{-L_{k+1}^{\zeta}}\end{aligned} \tag{2.3.104}$$

for L_0 sufficiently large, since $\zeta < s$. Therefore to finish the proof, we need to show that

$$K_0 = \inf\{k \in \mathbb{N}; \widetilde{P}_k \le \mathrm{e}^{-L_k^\zeta}\} < \infty. \tag{2.3.105}$$

It follows from (2.3.103) that for any $1 \le k < K_0$,

$$\widetilde{P}_k \le (2Y)^{(\lfloor Y^s \rfloor +1)d} \widetilde{P}_{k-1}^{\lfloor Y^s \rfloor +1} + \tfrac{1}{2}\mathrm{e}^{-L_k+\zeta} < (2Y)^{(\lfloor Y^s \rfloor +1)d} \widetilde{P}_{k-1}^{\lfloor Y^s \rfloor +1} + \tfrac{1}{2}\widetilde{P}_k, \tag{2.3.106}$$

so

$$\left(2(2Y)^{(\lfloor Y^s \rfloor +1)d}\right)^{\frac{1}{\lfloor Y^s \rfloor}} \widetilde{P}_k < \left(\left(2(2Y)^{(N+1)d}\right)^{\frac{1}{\lfloor Y^s \rfloor}} \widetilde{P}_{k-1}\right)^{\lfloor Y^s \rfloor +1}. \tag{2.3.107}$$

For $1 \le k < K_0$, since $\left(2(2Y)^{(\lfloor Y^s \rfloor +1)d}\right)^{\frac{1}{\lfloor Y^s \rfloor}} \widetilde{P}_0 < 1$, we have

$$\begin{aligned} \left(2(2Y)^{(\lfloor Y^s \rfloor +1)d}\right)^{\frac{1}{\lfloor Y^s \rfloor}} \mathrm{e}^{-Y^{k\zeta} L_0^\zeta} &= \left(2(2Y)^{(\lfloor Y^s \rfloor +1)d}\right)^{\frac{1}{\lfloor Y^s \rfloor}} \mathrm{e}^{-L_k^\zeta} \\ < \left(2(2Y)^{(\lfloor Y^s \rfloor +1)d}\right)^{\frac{1}{\lfloor Y^s \rfloor}} \widetilde{P}_k &< \left(\left(2(2Y)^{(\lfloor Y^s \rfloor +1)d}\right)^{\frac{1}{\lfloor Y^s \rfloor}} \widetilde{P}_0\right)^{(\lfloor Y^s \rfloor +1)^k} \\ &\le \left(\left(2(2Y)^{(\lfloor Y^s \rfloor +1)d}\right)^{\frac{1}{\lfloor Y^s \rfloor}} \widetilde{P}_0\right)^{Y^{ks}}. \end{aligned} \tag{2.3.108}$$

Since $\zeta < s$, $\left(2(2Y)^{(\lfloor Y^s \rfloor +1)d}\right)^{\frac{1}{\lfloor Y^s \rfloor}} \widetilde{P}_0 < 1$, (2.3.108) cannot be satisfied for large k. We conclude that $K_0 < \infty$. □

2.3.5 The second intermediate step

Proposition 2.3.8. *Fix $\varepsilon_0 > 0$. Suppose for some scale ℓ and $0 < \varepsilon \le \varepsilon_0$ we have*

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_\ell(x) \text{ is } s\text{-SEL for } H_{\varepsilon,\omega}\} \ge 1 - \mathrm{e}^{-\ell^\zeta}. \tag{2.3.109}$$

Then, if ℓ is sufficiently large, for $L = \ell^\gamma$ we have

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } m_0\text{-localizing for } H_{\varepsilon,\omega}\} \ge 1 - \mathrm{e}^{-L^\zeta}, \tag{2.3.110}$$

where

$$m_0 \geq \tfrac{1}{8} L^{-\left(1-\tau+\frac{1-s}{\gamma}\right)}. \tag{2.3.111}$$

Proof. We let $\mathcal{B}_N$, $\mathcal{S}_N$ and $\mathcal{E}_N$ as in the proof of Lemma 2.3.7. We proceed as in the proof of Lemma 2.3.7. Using (2.3.109), (2.1.9) and the fact that events on disjoint boxes are independent, we have

$$\begin{aligned}\mathbb{P}\{\mathcal{B}^c\} &\leq \left(\tfrac{2L}{\ell}\right)^{(N+1)d} \mathrm{e}^{-(N+1)\ell^{\zeta}} = 2^{(N+1)d} \ell^{(\gamma-1)(N+1)d} \mathrm{e}^{-(N+1)\ell^{\zeta}} \\ &< \tfrac{1}{2}\mathrm{e}^{-\ell^{\gamma\zeta}} = \tfrac{1}{2}\mathrm{e}^{-L^{\zeta}},\end{aligned} \tag{2.3.112}$$

if $N+1 > \ell^{(\gamma-1)\zeta}$ and ℓ is sufficiently large. For this reason we take

$$N = N_\ell = \left\lfloor \ell^{(\gamma-1)\widetilde{\zeta}} \right\rfloor \Longrightarrow \mathbb{P}\{\mathcal{B}^c_{N_\ell}\} \leq \tfrac{1}{2}\mathrm{e}^{-L^{\zeta}} \quad \text{for all} \quad \ell \quad \text{sufficiently large.} \tag{2.3.113}$$

Also, using (2.3.86) and (2.3.19), we have,

$$\mathbb{P}\{\mathcal{S}^c_N\} \leq Y_{\varepsilon_0} \left(1 + (L+1)^d N_\ell! (d4^d)^{N|\ell-1}\right) (L+1)^{2d} \mathrm{e}^{-(2\alpha-1)L^{\beta}} < \tfrac{1}{2}\mathrm{e}^{-L^{\zeta}} \tag{2.3.114}$$

for sufficiently large L, since $(\gamma-1)\widetilde{\zeta} < (\gamma-1)\beta < \gamma\beta$ and $\zeta < \beta$. Combining (2.3.112) and (2.3.114), we conclude that

$$\mathbb{P}\{\mathcal{E}_N\} > 1 - \mathrm{e}^{-L^{\zeta}}. \tag{2.3.115}$$

To finish the proof we need to show that for all $\omega \in \mathcal{E}_N$ the box Λ_L is m_0-localizing for $H_{\varepsilon,\omega}$, where m_0 is given in (2.3.111). Following the proof of Lemma 2.3.7, we get $\sigma(H_{\Lambda_L}) = \sigma_{\mathcal{G}}(H_{\Lambda_L}) \cup \sigma_{\mathcal{B}}(H_{\Lambda_L})$ and obtain an eigensystem $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ for H_{Λ_L}. To finish the proof we need to show that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is an m_0-localized eigensystem for Λ_L. We proceed as in the proof of Lemma 2.3.7. We fix $x \in \Lambda_L$, and take $y \in \Lambda_L$ such that $\|y - x\| \geq L_\tau$, we have

$$n(\ell)(\ell+1) + \sum_{r=1}^{R} \operatorname{diam} \Upsilon_r + 2\ell \geq L_\tau. \tag{2.3.116}$$

where $n(\ell)$ is the number of times we used (2.3.96). Thus, recalling $N = \lfloor \ell^{(\gamma-1)\widetilde{\zeta}} \rfloor$ and using (2.3.16), we have

$$n(\ell) \geq \tfrac{1}{\ell+1}(L_\tau - 5\ell\lfloor \ell^{(\gamma-1)\widetilde{\zeta}} \rfloor - 2\ell) \geq \tfrac{\ell}{\ell+1}\left(\tfrac{1}{2}\ell^{\gamma\tau-1} - 5\ell^{(\gamma-1)\widetilde{\zeta}} - 2\right) \geq \tfrac{1}{4}\ell^{\gamma\tau-1}. \tag{2.3.117}$$

for sufficiently large ℓ since $(\gamma-1)\widetilde{\zeta} + 1 < \gamma\tau$. It follows from (2.3.98),

$$\begin{aligned} |\psi_x(y)| &\leq \left(\mathrm{e}^{-c_2\ell^s}\right)^{\frac{1}{4}\ell^{\gamma\tau-1}} \\ &\leq \mathrm{e}^{-\frac{1}{8}L^{-\left(1-\tau+\frac{1-s}{\gamma}\right)}\|y-x\|} \end{aligned} \tag{2.3.118}$$

for sufficiently large ℓ.

We conclude that $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ is an m_0-localized eigensystem for Λ_L, where m_0 is given in (2.3.111), so the box Λ_L is m_0-localizing for $H_{\varepsilon,\omega}$. □

2.3.6 The fourth multiscale analysis

Proposition 2.3.9. *Fix $\varepsilon_0 > 0$. There exists a finite scale $\mathcal{L}(\varepsilon_0)$ with the following property: Suppose for some scale $L_0 \geq \mathcal{L}(\varepsilon_0)$, $0 < \varepsilon \leq \varepsilon_0$ and $m_0 \geq L_0^{-\kappa}$ where $0 < \kappa < \tau - \gamma\beta$, we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_0}(x) \textit{ is } m_0\textit{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L_0^{\zeta}}. \tag{2.3.119}$$

Then, setting $L_{k+1} = L_k^{\gamma}$ for $k = 0, 1, \ldots$, we have

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \textit{ is } \tfrac{m_0}{2}\textit{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L_k^{\zeta}} \textit{ for } k = 0, 1, \ldots. \tag{2.3.120}$$

Moreover, we have

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L_k}(x) \text{ is } \tfrac{m_0}{4}\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L_k^{\xi}} \text{ for all } L \geq L_0^{\gamma}. \tag{2.3.121}$$

Lemma 2.3.10. *Fix $\varepsilon_0 > 0$. Suppose for some scale ℓ, $0 < \varepsilon \leq \varepsilon_0$, and $m \geq \ell^{-\kappa}$, where $0 < \kappa < \tau - \gamma\beta$, we have*

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{\ell}(x) \text{ is } m\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-\ell^{\zeta}}. \tag{2.3.122}$$

Then, if ℓ is sufficiently large, for $L = \ell^{\gamma}$ we have

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_{L}(x) \text{ is } M\text{-localizing for } H_{\varepsilon,\omega}\} \geq 1 - \mathrm{e}^{-L^{\zeta}}, \tag{2.3.123}$$

where

$$M \geq m\left(1 - C_{d,\varepsilon_0}\ell^{-\min\left\{\frac{1-\tau}{2},\gamma\tau-(\gamma-1)\widetilde{\zeta}-1,\tau-\gamma\beta-\kappa\right\}}\right) \geq \tfrac{1}{L^{\kappa}}. \tag{2.3.124}$$

Lemma (2.3.10) and Proposition (2.3.9) follow from [EK, Lemma 4.5], [EK, Proposition 4.3], and [EK, Section 4.3]. (Note that in [EK], they assume $m \geq m_-$ for a fixed m_-. However, all the results still hold when $m \geq \ell^{-\kappa}, 0 < \kappa < \tau - \gamma\beta$. (See the Lemmas for $\sharp$ being LOC in Sections 2.1.2 and 2.1.3.))

2.3.7 The proof of the bootstrap multiscale analysis

To prove Theorem 2.0.9, first we assume (2.0.18), which is the same as (2.3.1) with letting $Y = 400$, for some length scales. We apply Proposition 2.3.1, obtaining a sequence of length scales satisfying (2.3.2). Therefore (2.3.50) is satisfied for some length scales. Applying Proposition 2.3.3, we get a length scale satisfying (2.3.51). It follows that (2.3.59) is satisfied since $0 < 1 - \tau +$

$\frac{1}{\gamma_1} < \tau$. We apply Proposition 2.3.4, obtaining a sequence of length scales satisfying (2.3.60). Therefore, In view of Remark 2.0.8, (2.3.81) is satisfied with letting $Y = 400^{\frac{1}{1-s}}$. We apply Proposition 2.3.6, obtaining a sequence of length scales satisfying (2.3.82). Therefore (2.3.109) is satisfied for some length scales. Applying Proposition 2.3.8, we get a length scale satisfying (2.3.110). It follows that (2.3.119) is satisfied since $0 < 1-\tau+\frac{1-s}{\gamma} < \tau-\gamma\beta$. We apply Proposition 2.3.9, getting (2.3.121), so (2.0.18) holds.

2.4 The initial step for the bootstrap multi-scale analysis

Theorem 2.0.10 is an immediate consequence of Theorem 2.0.9 and Proposition 2.4.1.

Proposition 2.4.1. *Given $q > \frac{2d}{\alpha}$ and $\varepsilon > 0$, set*

$$\theta_{\varepsilon,L} = \frac{\left\lfloor \frac{L}{20} \right\rfloor}{\log L} \log\left(1 + \frac{L^{-q}}{2d\varepsilon}\right). \tag{2.4.1}$$

Then

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } \theta_{\varepsilon,L}\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \tag{2.4.2}$$
$$\geq 1 - \tfrac{1}{2}K(L+1)^{2d}\left(8d\varepsilon + 2L^{-q}\right)^{\alpha}.$$

In particular, given $\theta > 0$ and $P_0 > 0$, there exists a finite scale $\mathcal{L}(q,\theta,P_0)$ such that for all $L \geq \mathcal{L}(q,\theta,P_0)$ and $0 < \varepsilon \leq \frac{1}{4d}L^{-q}$ we have

$$\inf_{x\in\mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P_0. \tag{2.4.3}$$

Proposition 2.4.1 shows that the starting hypothesis for the bootstrap multiscale analysis can be fulfilled for $\varepsilon \ll 1$.

To prove Proposition 2.4.1, we will use the following lemma given in [EK, Lemma 4.4].

Lemma 2.4.2 ([EK, Lemma 4.4]). *Let $H_\varepsilon = -\varepsilon\Delta + V$ on $\ell^2(\mathbb{Z}^d)$, where V is a bounded potential and $\varepsilon > 0$. Let $\Theta \subset \mathbb{Z}^d$, and suppose there is $\eta > 0$ such that*

$$|V(x) - V(y)| \geq \eta \quad \textit{for all} \quad x, y \in \Theta, x \neq y. \tag{2.4.4}$$

Then for $\varepsilon < \frac{\eta}{4d}$ the operator $H_{\varepsilon,\Theta}$ has an eigensystem $\{(\psi_x, \lambda_x)\}_{x\in\Theta}$ such that

$$|\lambda_x - \lambda_y| \geq \eta - 4d\varepsilon > 0 \quad \textit{for all} \quad x, y \in \Theta, x \neq y, \tag{2.4.5}$$

and for all $y \in \Theta$ we have

$$|\psi_y(x)| \leq \left(\tfrac{2d\varepsilon}{\eta - 2d\varepsilon}\right)^{|x-y|_1} \quad \textit{for all} \quad x \in \Theta. \tag{2.4.6}$$

Proof of Proposition 2.4.1. Let $\varepsilon > 0$ and $\Lambda_L = \Lambda_L(x_0)$ for some $x_0 \in \mathbb{R}^d$. Let $\eta = 4d\varepsilon + L^{-q}$ and suppose

$$|V(x) - V(y)| \geq \eta \quad \text{for all} \quad x, y \in \Theta, x \neq y. \tag{2.4.7}$$

It follows from Lemma 2.4.2 that $H_{\varepsilon,\Lambda_L}$ has an eigensystem $\{(\psi_x, \lambda_x)\}_{x\in\Lambda_L}$ satisfying (2.4.5) and (2.4.6). We conclude from (2.4.5) that Λ_L is polynomially level spacing for H_ε. Moreover, using (2.4.6) and $\|x\| \leq |x|_1$, for all $y, x \in \Lambda_L$ with $\|x - y\| \geq L'$ we have

$$\begin{aligned} |\psi_y(x)| \leq \left(\tfrac{2d\varepsilon}{\eta - 2d\varepsilon}\right)^{\|x-y\|} &= L^{-\frac{\|x-y\|}{\log L}\log\left(\frac{\eta-2d\varepsilon}{2d\varepsilon}\right)} \\ &= L^{-\frac{\|x-y\|}{\log L}\log\left(1+\frac{L^{-q}}{2d\varepsilon}\right)} \leq L^{-\theta_{\varepsilon,L}} \end{aligned} \tag{2.4.8}$$

with $\theta_{\varepsilon,L}$ as in (2.4.1). Therefore $\Lambda_L(x)$ is θ-polynomially localizing.

We have

$$\mathbb{P}\{\Lambda_L \text{ is not } \theta_{\varepsilon,L}\text{-polynomially localizing}\} \leq \mathbb{P}\{(2.4.7) \text{ does not hold}\} \tag{2.4.9}$$

$$\leq \tfrac{(L+1)^{2d}}{2} S_\mu\left(2\left(4d\varepsilon + L^{-q}\right)\right) \leq \tfrac{1}{2} K (L+1)^{2d}\left(8d\varepsilon + 2L^{-q}\right)^\alpha,$$

which yields (2.4.2). (We assumed $8d\varepsilon + 2\mathrm{L}^{-q} \leq 1$; if not (2.4.2) holds trivially.)

If $0 < \varepsilon \leq \frac{1}{4d} L^{-q}$, for sufficiently large L we have $\theta_{\varepsilon,L} \geq \theta$, and

$$\inf_{x \in \mathbb{R}^d} \mathbb{P}\{\Lambda_L(x) \text{ is } \theta\text{-polynomially localizing for } H_{\varepsilon,\omega}\} \geq 1 - P_0, \tag{2.4.10}$$

since $\alpha q - 2d > 0$. □

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