# TWO TREES

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**Abstract**

We solve a model with two "Lucas trees." Each tree has i.i.d. dividend growth. The investor has log utility and consumes the sum of the two trees' dividends. This model produces interesting asset-pricing dynamics, despite its simple ingredients. Investors want to rebalance their portfolios after any change in value. Since the size of the trees is fixed, however, prices must adjust to offset this desire. As a result, expected returns, excess returns, and return volatility all vary through time. Returns display serial correlation, and are predictable from price-dividend ratios in the time series and in the cross section. Return volatility can be greater than the volatility of cash flows, giving the appearance of "excess volatility." Returns can be cross-correlated even when the cash flows are independent, giving the appearance of "contagion" or "spurious comovement."

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# 1. INTRODUCTION

This paper solves an asset-pricing model with two "Lucas trees." Each tree's dividend stream follows a geometric Brownian motion. The representative investor has log utility and consumes the sum of the two trees' dividends. We obtain closed-form solutions for prices, expected returns, volatilities, correlations, and so forth.

Despite its simple ingredients, this model displays interesting dynamics, which we characterize. The intuition for these effects is as usual for endowment economies: prices adjust until investors are happy holding the endowments. For example, if the value of one asset increases, investors want to rebalance their portfolios. But, with fixed asset supply, investors cannot all rebalance, so subsequent expected returns change until investors are again happy to hold the asset in its now-larger proportion. We can also understand the asset-pricing dynamics by reference to aggregate consumption growth. As an asset gets larger, its dividend becomes a larger share of consumption. This changes the properties of aggregate consumption growth, and changes the covariance of the asset's return with consumption growth, both of which change the asset's subsequent expected return.

We can think of the two trees as any division of assets into broad sectors. They can represent industries or characteristic groups such as value vs. growth stocks. By dividing assets into stocks vs. bonds, or stocks and bonds vs. human capital or real estate, etc., our model can capture dynamics of the stock market as a whole induced by the presence of other assets and market clearing. The two trees can also represent two countries' asset markets, providing a natural benchmark for asset market dynamics in international finance.

The asset price and return dynamics of the two-tree model are qualitatively similar to those found in the empirical asset pricing literature. Underlying the dynamics, we find that expected returns of each tree are an increasing function of that tree's share of dividends for most parameter values. As a result, we find that a positive dividend shock, which increases current prices and returns, also typically raises subsequent expected returns. Thus, returns tend to display positive autocorrelation or "momentum," prices typically seem to "underreact" or not to "fully adjust" to dividend news, and to "drift" upwards for some time after that news. Interestingly, however, there are also parameters, horizons, and regions of the state space in which expected returns decline with the share, leading to "mean-reversion," price "overreaction," and "downward drift," with corresponding "excess volatility" of prices and returns.

When one asset has a positive dividend shock, this shock lowers the share of the other asset, so the expected return of the other asset typically declines. We see a negative cross-serial correlation, which Lo and MacKinlay (1990) and Lewellen (2002) argue is an important part of the momentum effect. We see movements in an asset's price even with no news about that asset's dividends, another source and form of apparent "excess volatility." Finally, we see that asset returns can be highly correlated with each other even when their underlying dividends are independent. When one asset has a good shock, the other asset's share, and thus its expected return, decline. The lower expected return raises the price of the other asset. Thus, the two prices move together even if their dividend growth rates are independent. A "common factor" or "contagion" emerges in returns even though there is no common factor in cash flows.

Since price-dividend ratios vary despite i.i.d. dividend growth, price-dividend ratios forecast returns in the time series and in the cross section. Thus, we see value and growth effects. In this way, simple market-clearing mechanics generate simultaneously short-run momentum (positive autocorrelation) and long-run mean reversion (valuation ratios forecast returns) in the assets.

This model is a contribution to an important emerging literature that models the cash flow processes of multiple assets. Our specification is simpler than other papers in this literature, including Santos and Veronesi (2001), Bansal, Dittmar, and Lundblad (2002), and Menzly, Santos, and Veronesi (2004). These papers include non-i.i.d. dividend processes and temporally nonseparable preferences in order to better fit some aspects of the data. Our goal is to provide instead the simple baseline textbook model. As the standard one-tree model, though unrealistic, delivers deep insights into key asset-pricing issues, so this simple two-tree model can isolate market-clearing effects that will be part of the story in more complex models, and allows us a clearer economic intuition for those effects.

The discussion so far seems to argue that models with multiple assets in non-zero net supply cannot have i.i.d. returns. This view contrasts with classic finance models such as the ICAPM of Merton (1973) and Cox, Ingersoll, and Ross (1985) that can imply i.i.d. returns for multiple assets in positive net-supply. The main difference with our endowment economy is that these models allow the supply of assets (number of shares) to vary. In economics language, they assume linear technologies – output is a linear function of capital – with no adjustment costs, diminishing returns, or irreversibilities, rather than assuming fixed endowments (trees). In such models, investors can and do collectively rebalance, leaving aggregate portfolio weights constant in the face of return shocks; investors instantly and costlessly transfer capital across production technologies or to and from consumption. Our model makes the opposite assumption that the capital invested in each technology is exogenously fixed which implies that portfolio weights keep changing. In turn, the moments of returns change to keep the investor fully invested at the optimum.

Which is the right assumption? In reality, market portfolio weights do change over time (for example, the capitalization of stocks vs. bonds). Thus, any realistic model, with at least short-run adjustment costs, irreversibilities, and other impediments to aggregate rebalancing, will contain some market-clearing induced dynamics of the sort we isolate and study in our simple two-tree exchange economy.

## 2. THE SINGLE-ASSET MODEL

To set the stage, we review the traditional single-asset model. The asset pays a dividend stream

$$\frac{dD}{D} = \mu\, dt + \sigma\, dZ, \tag{1}$$

with constant coefficients $\mu$ and $\sigma$, and where $Z$ is a standard Brownian motion. Unless necessary for clarity, we suppress time indices, e.g. $dD \equiv dD_t$, etc. The representative investor has log utility,

$$U_t = E_t\left[ \int_0^\infty e^{-\delta\tau} \ln\left(C_{t+\tau}\right)\, d\tau \right]. \tag{2}$$

This is an endowment economy, so prices adjust until consumption equals the dividend, $C = D$. The investor's first-order conditions imply that marginal utility is a discount factor that prices assets, i.e. they imply that we can use

$$M_t = \frac{e^{-\delta t}}{D_t}, \tag{3}$$

to find the price of the asset as

$$\frac{P_t}{D_t} = \frac{1}{D_t} E_t\left[ \int_0^\infty \frac{M_{t+\tau}}{M_t} D_{t+\tau}\, d\tau \right] = E_t\left[ \int_0^\infty e^{-\delta\tau} \frac{D_{t+\tau}}{D_{t+\tau}}\, d\tau \right] = \frac{1}{\delta}. \tag{4}$$

Since prices are proportional to dividends, price appreciation is the same as dividend growth, $dP/P = dD/D$, so the total instantaneous return $R_t$ is

$$R_t = \frac{dP_t}{P_t} + \frac{D_t}{P_t}\,dt = (\mu + \delta)\,dt + \sigma\,dZ. \tag{5}$$

The expected return and return variance are constant,

$$E_t[\,R_t\,] = (\mu + \delta)\,dt, \tag{6}$$

$$\mathrm{Var}_t[\,R_t\,] = \sigma^2\,dt. \tag{7}$$

Stock returns are i.i.d. through time in this model.

The instantaneous interest rate is given by

$$r = -E_t\left[\frac{dM_t}{M_t}\right] = \delta\,dt + E_t\left[\frac{dD_t}{D_t}\right] - \mathrm{Var}_t\left[\frac{dD_t}{D_t}\right] = (\delta + \mu - \sigma^2)\,dt. \tag{8}$$

The riskless asset is in zero net supply. We see the standard discount rate ($\delta$), consumption growth ($\mu$), and precautionary savings ($\sigma^2$) effects. Since the riskless rate is constant, the entire term structure is constant and flat.

## 3. THE TWO-ASSET MODEL

Now consider the same economy with two assets. As before, dividends follow simple geometric Brownian motions,

$$\frac{dD_i}{D_i} = \mu_i\,dt + \sigma_i\,dZ_i, \tag{9}$$

where $i = 1, 2$, and the correlation between $dZ_1$ and $dZ_2$ is $\rho\,dt$. Aggregate consumption is the sum of the two dividends $C = D_1 + D_2$. The investor has log utility as in Equation (2).

The dividend share,

$$s = \frac{D_1}{D_1 + D_2}, \tag{10}$$

is a natural state variable for the two-tree model. We first price the individual assets and find returns, expected returns, and variances of returns. We then find the interest rate and price the market portfolio from the aggregate consumption process. Finally, we derive the dynamics of the dividend share.

**3.1 Asset Prices.**

We focus on the first asset. The second asset is symmetric. From the usual Euler condition, the price $P_1$ of the first asset is given by

$$P_{1t} \; = \; E_t \left[ \int_0^\infty e^{-\delta\tau} \, \frac{C_t}{C_{t+\tau}} \, D_{t+\tau} \, d\tau \right]. \tag{11}$$

Recalling the definition of the dividend share from Equation (10), this result can be expressed as

$$\frac{P_{1t}}{C_t} \; = \; E_t \left[ \int_0^\infty e^{-\delta\tau} \, s_{t+\tau} \, d\tau \right]. \tag{12}$$

Valuing the individual asset is formally identical to the risk-neutral pricing of an asset that pays a cash flow equal to the dividend share $s$ with a discount rate $\delta$. The dividend share plays a similar role in many tractable models of long-lived cash flows, including Santos and Veronesi (2001), Bansal, Dittmar, and Lundblad (2002), Menzly, Santos, and Veronesi (2004), and Longstaff and Piazessi (2004).

We solve for the asset price from Equation (12) in two ways. First, we evaluate the double integral (expectation and time) directly, after changing the order of integration. Second, we derive the differential equation for the price-consumption ratio that results from the standard instantaneous pricing condition, and solve it. The two approaches are presented in Sections 1 and 2 of the Appendix, respectively.

The price of the first asset as a function of the dividend share $s$ is

$$
\begin{aligned}
\frac{P_{1t}}{C_t} &= \frac{1}{\psi(1-\gamma)} \left( \frac{s}{1-s} \right) F\left(1,\ 1-\gamma;\ 2-\gamma;\ \frac{s}{s-1}\right) \\
&+ \frac{1}{\psi\theta} F\left(1,\ \theta;\ 1+\theta;\ \frac{s-1}{s}\right),
\end{aligned} \tag{13}
$$

where

$$
\psi = \sqrt{\nu^2 + 2\delta\eta^2}, \qquad \gamma = \frac{\nu - \psi}{\eta^2}, \qquad \theta = \frac{\nu + \psi}{\eta^2},
$$

and

$$
\nu = \mu_2 - \mu_1 - \sigma_2^2/2 + \sigma_2^2/2, \qquad \eta^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.
$$

$F(\alpha, \beta; \gamma; z)$ is the standard hypergeometric function (see Abramowitz and Stegum (1970) Chapter 15). The hypergeometric function is defined by the power series

$$
\begin{aligned}
F(\alpha, \beta; \gamma; z) = 1 &+ \frac{\alpha \cdot \beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{\gamma(\gamma+1) \cdot 1 \cdot 2} z^2 \\
&+ \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2) \cdot 1 \cdot 2 \cdot 3} z^3 + \ldots
\end{aligned} \tag{14}
$$

The hypergeometric function has an integral representation, which can be used for numerical evaluation and as an analytic continuation beyond $\|z\| < 1$,

$$
F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 w^{\beta-1}(1-w)^{\gamma-\beta-1}(1-wz)^{-\alpha}\, dw, \tag{15}
$$

where $Re(\gamma) > Re(\beta) > 0$. The derivative of the hypergeometric function, needed for Itô's Lemma calculations, has the simple form

$$
\frac{d}{dz} F(\alpha, \beta; \gamma; z) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; z). \tag{16}
$$

This formula can be derived by differentiating the terms of the power series in Equation (14) (see also Gradshteyn and Ryzhik (2000), 9.100, 9.111). The price $P_2$ of the second asset is symmetric to that of the first,

$$\frac{P_{2t}}{C_t} = \frac{1}{\psi(1+\theta)} \left(\frac{1-s}{s}\right) F\left(1,\ 1+\theta;\ 2+\theta;\ \frac{s-1}{s}\right)$$
$$- \frac{1}{\psi\gamma}\ F\left(1,\ -\gamma;\ 1-\gamma;\ \frac{s}{s-1}\right). \tag{17}$$

### 3.2 Asset Returns.

Let $R_1$ denote the instantaneous return on the first asset. Given the explicit price function in Equation (13) and the functional form of its derivatives from Equation (16), $R_1$ is given by a direct application of Itô's Lemma,

$$R_1 = \left[\delta + \mu_1 s + \mu_2(1-s) + (\rho\sigma_1\sigma_2 - \sigma_2^2 + \eta^2 s)\ \Phi(s)\right]\ dt$$
$$+ \sigma_1[s + \Phi(s)]\ dZ_1 - \sigma_2[s - 1 + \Phi(s)]\ dZ_2, \tag{18}$$

where

$$\Phi(s) = \frac{A(s)}{B(s)},$$

$$A(s) = \frac{1}{1-\gamma}\ \left(\frac{s}{1-s}\right)\ F\left(1,\ 1-\gamma;\ 2-\gamma;\ \frac{s}{s-1}\right)$$
$$- \frac{1}{2-\gamma}\ \left(\frac{s}{1-s}\right)^2\ F\left(2,\ 2-\gamma; 3-\gamma;\ \frac{s}{s-1}\right)$$
$$+ \frac{1}{1+\theta}\ \left(\frac{1-s}{s}\right)\ F\left(2,\ 1+\theta;\ 2+\theta;\ \frac{s-1}{s}\right),$$

$$B(s) = \frac{1}{1-\gamma}\ \left(\frac{s}{1-s}\right)\ F\left(1,\ 1-\gamma;\ 2-\gamma,\ \frac{s}{s-1}\right)$$
$$+ \frac{1}{\theta}\ F\left(1,\ \theta;\ 1+\theta;\ \frac{s-1}{s}\right).$$

From this equation, it follows that both the mean return and return volatility vary with the state variable $s$:

$$E_t[\ R_1\ ] = \left[\delta + \mu_1 s + \mu_2(1-s) + (\rho\sigma_1\sigma_2 - \sigma_2^2 + \eta^2 s)\Phi(s)\right]\ dt, \tag{19}$$

$$\begin{aligned} \mathrm{Var}_t[\ R_1\ ] = \{ & \sigma_1^2[s+\Phi(s)]^2 + \sigma_2^2[s-1+\Phi(s)]^2 \\ & - \ 2\rho\sigma_1\sigma_2[s+\Phi(s)][s-1+\Phi(s)]\}\ dt. \end{aligned} \tag{20}$$

Section 3 of the Appendix shows that the limit of $\Phi(s)$ as $s \to 0$ is either 1 or $\theta$, depending on whether $\theta$ is greater than or less than one. Using this result, it follows that the expected excess return of the first asset need not converge to zero as $s \to 0$. Similarly, the volatility of the first asset's returns need not converge to the volatility of its cash flows as $s \to 0$. As $s \to 1$, however, the first asset becomes the market and its expected return and variance converge to the values for the market.

### 3.3 Consumption Dynamics.

Aggregate consumption $C = D_1 + D_2$ follows

$$\begin{aligned} dC &= dD_1 + dD_2, \\ &= [\mu_1 D_1 + \mu_2 D_2\ ]\ dt + \sigma_1 D_1\ dZ_1 + \sigma_2 D_2\ dZ_2, \end{aligned} \tag{21}$$

so that

$$\frac{dC}{C} = \left[\mu_1 s + \mu_2(1-s)\right]\ dt + \sigma_1 s\ dZ_1 + \sigma_2(1-s)\ dZ_2. \tag{22}$$

Since the dynamics of consumption depend on the state variable $s$, consumption growth is no longer i.i.d. through time. Mean consumption growth,

$$E_t\left[\ \frac{dC}{C}\ \right] = [\mu_1 s + \mu_2(1-s)]\ dt, \tag{23}$$

is the share-weighted mean of the dividend growth rates. Consumption volatility,

$$\mathrm{Var}_t\left[\ \frac{dC}{C}\ \right] = [\sigma_1^2 s^2 + \sigma_2^2(1-s)^2 + 2\rho\sigma_1\sigma_2 s(1-s)]\ dt, \tag{24}$$

is a convex quadratic function of the share. Volatility is lower for intermediate values of the dividend share, as consumption is then diversified between the two dividends.

### 3.4 The Riskless Rate.

We find the instantaneous (zero net supply) interest rate as before

$$r \;=\; \delta \; dt \;+\; E_t \left[ \, \frac{dC}{C} \, \right] \;-\; \mathrm{Var}_t \left[ \, \frac{dC}{C} \, \right]. \tag{25}$$

Substituting the moments of the consumption dynamics into Equation (25) gives

$$r \;=\; \left[\delta \;+\; \mu_1 s \;+\; \mu_2(1-s) \;-\; \sigma_1^2 s^2 \;-\; \sigma_2^2(1-s)^2 \;-\; 2\rho\sigma_1\sigma_2 s(1-s)\right] \; dt. \tag{26}$$

Thus, the riskless rate varies over time, as a quadratic function of the state variable $s$. The riskless rate is higher for intermediate values of the dividend share because dividend diversification lowers consumption volatility, which lowers the precautionary savings motive. Since the interest rate is not constant, the term structure is not flat.

### 3.5 Market Price and Returns.

As is usual in log utility models, the price $P_m$ of the aggregate consumption stream $C = D_1 + D_2$ (the market portfolio) is given by the simple expression

$$\frac{P_{mt}}{C_t} \;=\; E_t \left[ \, \int_0^\infty e^{-\delta t} \, \frac{C_{t+\tau}}{C_{t+\tau}} \, d\tau \, \right] \;=\; \frac{1}{\delta}. \tag{27}$$

This calculation, which is the same as in the single-asset model, is valid for all consumption dynamics. As before, the price of the market is proportional to aggregate consumption so

$$\frac{dP_m}{P_m} = \frac{dC}{C}. \tag{28}$$

Since the total instantaneous return $R_M$ on the market equals price appreciation plus the dividend yield

$$R_m = \frac{dP_m}{P_m} + \frac{C}{P_m}dt = \frac{dC}{C} \ + \ \delta dt, \tag{29}$$

Equation (22) implies

$$R_m \ = \ [\delta \ + \ \mu_1 s \ + \ \mu_2(1-s)] \ dt \ + \ \sigma_1 s \ dZ_1 \ + \ \sigma_2(1-s) \ dZ_2. \tag{30}$$

The expected market return and variance are no longer constants,

$$E_t[\ R_m\ ] \ = \ [\delta \ + \ \mu_1 \ s \ + \ \mu_2 \ (1-s)] \ dt, \tag{31}$$

$$\text{Var}_t[\ R_m\ ] \ = \ \left[\sigma_1^2 \ s^2 \ + \ \sigma_2^2 \ (1-s)^2 \ + \ 2\rho\sigma_1\sigma_2 \ s \ (1-s)\right] \ dt. \tag{32}$$

The expected return equals the subjective discount rate $\delta$ plus expected consumption growth, which is the share-weighted average of the dividend growth rates $\mu_1$ and $\mu_2$. The variance of the market return equals the variance of consumption growth, and reflects diversification between the two assets' cash flows.

Finally, subtracting the expression for the riskless rate in Equation (26) from the expected return on the market in Equation (31) indicates that the equity premium equals the variance of the market,

$$E_t[\ R_m\ ] - r\ dt = \text{Var}_t[\ R_m\ ], \tag{33}$$

as usual for log utility models. From Equation (32), the variance of the market is a convex quadratic function of the dividend share. This means that the equity premium is also time varying, and generally increases as the market becomes more polarized.

### 3.6 Dividend Share Dynamics.

An application of Itô's Lemma to Equations (9) and (10) gives the dynamics of the dividend share,[1]

[1]The stochastic process for the share $s$ is a member of the important class of Wright-Fisher diffusions. In an interesting parallel to our two-asset model, these types of diffusions are often applied in genetic theory to characterize the evolution of genes in

$$
\begin{aligned}
ds = s(1-s)\left[\mu_1 - \mu_2 - s\sigma_1^2 + (1-s)\sigma_2^2 + (2s-1)\rho\sigma_1\sigma_2\right] dt \\
+ s(1-s)(\sigma_1 \, dZ_1 - \sigma_2 \, dZ_2).
\end{aligned} \tag{34}
$$

The drift of the dividend share process in Equation (34) is zero when $s = 0$, $\kappa$, or 1, where

$$
\kappa = \frac{\mu_1 - \mu_2 + \sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{35}
$$

When $\kappa$ lies between zero and one, the drift is positive from zero to $\kappa$, bringing the share up towards $\kappa$, and negative from $\kappa$ to one, bringing the share down towards $\kappa$. Thus, the drift term can induce patterns of mean reversion in the dividend share that are not present in the underlying dividend processes.[2] The diffusion coefficient in Equation (34) is quadratic, implying that changes in the dividend share are most volatile when $s = 1/2$.

This model does not possess a stationary share distribution. The share of one of the assets ultimately declines to the point that the other asset becomes dominant in the market.[3] However, this effect happens very slowly for reasonable parameter values. We discuss this and related issues at length below.

## 4. ASSET-PRICING IMPLICATIONS

In this section we characterize the asset-pricing implications of the model by plotting expected returns, price-dividend ratios, return volatility, etc. as functions of the dividend

a population of two genetic types. For example, Karlin and Taylor (1981) Ch. 15, pp. 184-188 present an example of the Wright-Fisher gene frequency diffusion model in which the fraction of genes in a population follows a process with drift and diffusion terms that are respectively third- and second-order polynomials in the fraction, just as in Equation (34). Also see Crow and Kimura (1970) for other examples and a discussion of the asymptotic properties of these models.

[2]The cubic drift of our share process is also closely related to that of the stochastic Ginzburg-Landau diffusion used in superconductivity physics to model phase transitions. See Kloeden and Platen (1992) and Katsoulakis and Kho (2001).

[3]This feature parallels the asymptotic properties of Wright-Fisher gene frequency models in which one of the two gene types ultimately becomes fixed in the population.

share. Since the dividend share varies over time as described by Equation (34), when a variable such as expected return moves as a function of the *share*, it also varies with *time*.

We specify baseline values of two percent mean dividend growth $\mu = 0.02$, 20 percent standard deviation of dividend growth $\sigma = 0.20$, a subjective discount factor $\delta = 0.10$, and we set the correlation between the dividend processes to zero. We start with the symmetric case in which both assets have identical parameter values. Then, holding fixed the parameters of the second asset at these baseline values, we set the first asset's mean and standard deviation to zero times the baseline, $\mu_1 = 0.00$, $\sigma_1 = 0.00$, and twice the baseline, $\mu_1 = 0.04$, $\sigma_1 = 0.40$. Although our model says nothing about what technologies to expect – which kind of trees will be planted – it seems reasonable to specify that trees with a higher mean will also have a higher standard deviation. (A more general model with investment is likely to lead to this result, as agents will only plant riskier trees if they have a higher mean). Also, in our investigations not much of interest results from separately varying the mean and standard deviation, so these cases illustrate compactly the model's behavior.

The "symmetric" case is the natural first case to examine in order to understand how the model works. This case can apply to a division of the market into two large primary sectors such as financials and industrials. It can also apply to two countries, so this case is the natural benchmark for international finance.[4]

When the first asset has zero mean and standard deviation, the first asset is a level perpetuity in nonzero net supply. This case is interesting theoretically, because the first asset's price is driven only by valuation effects with no cash flow news, and so it allows us to see valuation effects most clearly. We can view this case as representing the bond and stock market, so we refer to it as the "bond-stock" case.[5] With this interpretation of our two-tree model, we can address the question of whether dynamics in the aggregate stock market come from market-clearing in the overall capital market, dynamics that are not present when one identifies the stock market with the wealth portfolio.

The third case, in which the first asset is scaled by two, represents a market divided into a high risk and return sector and a safer but still risky sector. This case can represent the division between stocks or traded securities and less volatile assets

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[4]Recent examples of papers that use multi-tree frameworks to model countries include Pavlova and Rigobon (2003) and Hau and Rey (2004).

[5]We do this with some license. Corporate bonds and stocks are of course claims to the same underlying cash flows, so the cash flow properties of each are not invariant to changes in value. Since government bonds are claims to taxes, even stocks and government bonds are not perfect examples of this model.

including real estate, human capital, and so forth. It can also represent a smaller riskier set of stocks vs. the stock market as a whole. The last two parameterizations let us investigate dynamics for the aggregate stock market that flow from market-clearing effects, dynamics which do not occur under the conventional simplification that the stock market is the wealth portfolio claim to the entire consumption stream. We refer to this case as the "risky-asset" case.

**4.1 The Riskless Rate and Market Returns.**

Figure 1 plots the riskless rate, the expected return, the expected excess return, and return volatility for the market portfolio (consumption claim) as a function of the dividend share of the first asset. Since the share varies over time, Figure 1 documents time variation in these moments.

Consumption volatility is a quadratic function of the dividend share (see Equation (24)), with a minimum for intermediate values of the share, since aggregate consumption diversifies across the two trees. As a result, and consistent with Equation (26), the riskless rate in the top left panel of Figure 1 is also a quadratic function of the dividend share. The riskless rate is largest for intermediate shares, where consumption volatility and thus the precautionary savings motive are lowest. As the share increases, the interest rate is lower for the calibrations with higher-mean, higher-volatility dividend growth. Here the precautionary effect of higher consumption volatility dominates the intertemporal substitution effect of higher mean dividend growth. Since the market return is linear in consumption growth (see Equation (29)), the bottom right panel of Figure 1 shows that market return volatility is a quadratic function of the dividend share.

Consistent with Equation (31), the expected market return in the top right panel of Figure 1 is a linear function of the dividend share. The expected market return increases, decreases, or is constant as the mean dividend growth of the second asset is larger than, less than, or equal to that of the first asset. We can then understand the market expected excess return in the bottom left panel of Figure 1 as the mirror image of the quadratic riskless rate. Alternatively, the market expected excess return is equal to consumption (and market return) variance in any log utility model, as in Equation (33), and consumption volatility is lower for intermediate shares.

**4.2 Expected Returns and Excess Returns.**

Figure 2 plots expected returns for the individual assets. Figure 3 plots expected excess returns. As before, the dependence of these values on the dividend share implies that expected returns and excess returns vary through time.

Both the top and bottom panels of these (and most subsequent) graphs plot moments as a function of the first asset's dividend share. Thus, the second asset's share decreases from left to right, even in the bottom panels. The second asset's expected return is mostly rising in the second asset's share, and thus mostly declining in the first asset's share as plotted.

The individual-asset expected returns in Figure 2 show a roughly quadratic pattern. The expected return for the first asset is increasing in its dividend share over most of the range. This behavior of expected returns drives many of the results that follow. For example, a positive shock to the first asset's dividends has an immediate effect on its price. This shock also changes the dividend share, and hence affects expected returns. Where expected excess returns rise in their dividend share, we expect to see positive autocorrelation and momentum of returns. Where expected returns decline in the dividend share, we expect to see negative autocorrelation and mean-reversion. Changing expected returns take the form of further expected changes in prices, since dividend growth is i.i.d. Thus, dividend shocks may have long-lasting price and return effects. Where expected returns increase in the share, prices will seem initially to "underreact" and "slowly" incorporate dividend news. Where expected returns decline, prices will seem to "overreact."

A good part of the quadratic nature of expected returns in Figure 2 derives from the quadratic nature of the riskless rate, as shown in Figure 1. Figure 3 focuses on the potentially more interesting behavior of expected excess returns. Figure 3 shows that the expected excess returns for the individual assets are roughly linearly and monotonically increasing in some of the examples, but display very different behavior in others. To provide intuition for this behavior, we relate expected excess returns separately to "cash-flow betas" and "valuation betas" as follows. Recall that expected excess returns represent risk premia, reflecting the covariance of returns with the discount factor. With log utility,

$$E_t[\ R_1\ ]\ -\ r\ = \mathrm{Cov}_t\left[R_1, \frac{dC}{C}\right]. \tag{36}$$

Now, return shocks come from dividend growth shocks and shocks to the valuation of dividends,[6]

[6]To derive Equation (37), express $P_1$ as $D_1 \cdot (P_1/D_1)$ and apply Itô's Lemma. Since dividend growth is i.i.d., the first two terms on the right hand side of Equation (37) describe the effects on returns of current and expected future changes in cash flows. The remaining terms, and especially the third, capture discount rate effects, the effects on returns of changes in the discount rate applied to future cash flows. The first two terms also describe returns with no change in the state variable, and the $P_1/D_1$ terms

$$R_1 = \frac{dP_1}{P_1} \; + \; \frac{D_1}{P_1}\, dt = \frac{D_1}{P_1}\, dt + \frac{dD_1}{D_1} + \frac{d(P_1/D_1)}{P_1/D_1} + \frac{dD_1}{D_1}\frac{d(P_1/D_1)}{P_1/D_1}. \tag{37}$$

Thus, we can express the covariance of returns with consumption growth as

$$E_t[\, R_1 \,] \; - \; r \; = \text{Cov}_t\left[\frac{dD_1}{D_1}, \frac{dC}{C}\right] + \text{Cov}_t\left[\frac{d(P_1/D_1)}{P_1/D_1}, \frac{dC}{C}\right]. \tag{38}$$

Since consumption growth is the share-weighted sum of dividend growth rates, the first term is linear in the dividend share,

$$\text{Cov}_t\left[\frac{dD_1}{D_1}, \frac{dC}{C}\right] = s\ \sigma_1^2. \tag{39}$$

When the first asset has a share of zero, the covariance of its dividend growth with that of aggregate consumption – composed entirely of the uncorrelated dividends of the other asset in these examples – is zero. As the share increases, the covariance of the first asset's dividends with aggregate consumption increases linearly.

In this way, the approximate linearity of expected excess returns for some of the cases in Figure 3 is natural. It shows that "cash-flow betas" linear in $s$ dominate the covariance of returns with consumption growth in these cases. In the top panel of Figure 3, the expected excess returns of the first asset rise more quickly for the risky-asset case (dashed line) than for the symmetric (solid line) and bond-stock (dotted line) cases. We expect this behavior, following Equation (39). As the first asset comes to dominate the market, the market premium rises when the first asset has more volatile dividends.

The deviations from linearity seen in Figure 3 represent the additional effects of "valuation betas," expected returns corresponding to covariance of the change in the price-dividend ratio with aggregate consumption. These *deviations* from linearity are perhaps the most interesting part of the model.

The valuation effect is evident in the bond-stock case (dotted line, top panel of Figure 3). In this case, the first asset's cash flow is riskless. Therefore, the first asset's expected excess return is driven entirely by the valuation effect. Its expected excess return first rises as a function of the share. Here, as we show below, the first asset's valuation covaries positively with aggregate consumption growth. Then, the first asset's

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describe the effect of the changing state variable on returns.

expected excess return declines and even becomes negative, reflecting a negative covariance of its valuation with aggregate consumption growth. In this way, the expected excess long-term bond return – the term premium – can be both positive and negative, as found in bond data by Fama and Bliss (1987). As in reality, term premia are much smaller than expected excess stock returns, since there is no premium for cash-flow risk.

In the risky-asset calibration, the second asset also has a very important and non-linear valuation effect, shown in the dashed line of the bottom panel of Figure 3. In this case, the expected excess return does not go to zero as the second asset's share goes to zero on the right hand side of the graph. Even though this asset's dividend growth becomes uncorrelated with aggregate consumption, the valuation betas remain large. This case offers important caveats: There are parameterizations (here, a relatively safe ($\mu_2 = 0.02,\ \sigma_2 = 0.20$) asset in a market with a relatively risky one ($\mu_1 = 0.04,\ \sigma_1 = 0.40$) ) for which expected excess returns can be large even when the asset has a small share. In addition, expected excess returns can decline as a function of share for much of the state-space, leading to the opposite of the usual return and price dynamics.

All three cases make the point that important variation in expected returns and resulting dynamic asset pricing phenomena can occur from market-clearing mechanics. The symmetric calibration shows that we should expect asset price dynamics in international finance, comparing two country's asset markets that add up to a world portfolio. The bond-stock and risky-asset calibrations show that we expect to see a time-varying expected return and excess return of the stock market as a whole resulting from the fact that markets must clear and there are other assets (real estate, human capital) out there as well. The symmetric and risky asset calibrations show that we expect time varying expected returns in sectors, such as the "industry momentum" documented by Grinblatt and Moskowitz (1999).

### 4.3 Price-Dividend Ratios.

Figure 4 plots price-dividend ratios as a function of the dividend share. From Equation (27), the price-dividend ratio for the market is constant, and equals ten in each case. Price-dividend ratios for the individual assets vary, however, and are generally not monotonic in the share.

Since dividend growth is i.i.d., price-dividend ratios are driven entirely by expected returns. Because the share is highly autocorrelated, today's expected returns capture a great deal of the future expected returns that drive price-dividend ratios. Hence, the price-dividend ratios in Figure 4 are roughly the inverse of the expected returns of Figure 2, and can be understood as such.

Figure 4 shows that the price-dividend ratio for the first asset is typically largest when its share is small, declines until its value is less than that of the market, and then increases to equal the market's price-dividend ratio of ten when $s$ equals one. The bond-stock case shows variation in the bond price-dividend ratio (the inverse of the coupon yield) despite no cash flow uncertainty. The pattern (but not the magnitudes) of price-dividend ratios for the second asset is generally symmetric to that of the first asset.

The non-monotonic behavior of the price-dividend ratio is initially puzzling, but it is necessary. This logic is clearest in the symmetric case: If the price-dividend ratio of the first asset is greater than that of the market at a share of, say, 0.25, then the price-dividend ratio of the second asset must be less than that of the market at this point. But this observation means that the price-dividend ratio of the first asset must be less than that of the market at a share of 0.75, and must recover to equal that of the market at a share of one.

In each scenario shown in Figure 4, assets have lower expected returns and higher price-dividend ratios when their share of the total dividends approaches zero than otherwise. Assets are more highly valued when they represent a small share of total dividends, and hence, are more valuable from a diversification perspective. Thus, "small" sectors are "growth" sectors, in the sense of having high valuations and low expected returns.

The second asset in the risky-asset case, shown in the dashed line in the bottom panel of Figure 4, is particularly interesting, since the price-dividend ratio goes to infinity as the second asset's share goes to zero. This price-dividend ratio corresponds to a low, but positive, expected return in the bottom panel of Figure 2, and the substantial and somewhat puzzling expected excess return in the bottom panel of Figure 3.

To provide some intuition for why the price-dividend ratio can become infinite in this model, recall from Equation (27) that the price-consumption ratio is an exponentially-weighted average of the expected dividend share. Since $D_1 = sC$, the price-dividend ratio is simply $1/s$ times the price-consumption ratio, and we can write

$$\frac{P_1}{D_1} = E_t \left[ \int_0^\infty e^{-\delta\tau} \; \frac{s_{t+\tau}}{s_t} \; d\tau \right]. \tag{40}$$

Thus, the price-dividend ratio can be expressed as an exponentially-weighted average of expected dividend share growth rates. If the dividend share of an asset is expected to grow at a rate faster than $e^{\delta\tau}$, the integral in Equation (40), and hence the price-dividend ratio, diverges. This feature parallels the result from the classic Gordon growth model in which the price-dividend ratio for a stream of dividends can be infinite if the dividend growth rate exceeds the discount rate. Now, $s$ cannot grow faster than $e^{\delta\tau}$ forever, for any nonzero initial $s$. Hence, the price-dividend ratio is finite for any nonzero $s$. In the limit as $s \to 0$, we also have $E_t[s_{t+\tau}] \to 0$, but the latter occurs at a slower rate, so that the share is expected to grow sufficiently fast to give an infinite price-dividend ratio.[7]

The expected excess return of the second asset in the risky-asset case – the dashed line in the bottom panel of Figure 3 – was something of a puzzle. As an asset's share goes to zero, the volatility of that share goes to zero as well. Now, from Equation (38), the valuation beta is $\mathrm{Cov}_t[\frac{d(P/D)}{P/D}, \frac{dC}{C}]$, so one would think that as the share goes to zero and share volatility goes to zero, valuation betas would necessarily decline to zero as well. However, this argument assumes that the slope of the price-dividend ratio with respect to the share is finite. If that slope increases without bound, the covariance can stay non-zero even though the volatility of $s$ declines to zero. It is precisely this explosion of the price-dividend ratio in Figure 4 that allows the valuation beta and hence the expected excess returns of the second asset to remain large in the limit while its share goes to zero.

The shape of the price-dividend ratio curve for the bond in the stock-bond case (dotted line, top panel of Figure 4) also explains the changing sign of the bond risk premium we saw in Figure 3. Where the price-dividend ratio is a declining function of the share, from a share of zero to about 0.75, a shock to aggregate consumption, which is a shock to the second asset's dividend, will lower the share and raise the price of the first asset. Here, we have a positive (valuation) beta and a positive expected excess return. Where the price-dividend ratio starts to rise with the share, above a share of about 0.75, we have the opposite sign.

Since expected returns (Figures 2 and 3) and price-dividend ratios (Figure 4) both vary with dividend share, price-dividend ratios forecast returns in the time series and the cross section. In this way, our model generates value and growth effects in the cross section and it generates predictable stock market returns. Furthermore, since dividend growth is not forecastable, all variance in price-dividend ratios is due to changing return forecasts.

### 4.4 Return Volatility.

Figure 5 plots the standard deviation of individual-asset returns as a function of the dividend share. In the one-tree model, return volatility is constant and equals dividend volatility. In the two-tree model, however, return volatility can be different from dividend volatility. (Notice that the scale is different in the top and bottom panels of Figure 5. Thus, the behavior of the first asset in the symmetric case (solid line, top panel) is

[7]The price-dividend ratio increases at a rate less than or equal to $1/s$, since the share-weighted average of the price-dividend ratio must equal the market price-dividend ratio of ten, and the market value must decline to zero as $s \to 0$.

exactly the mirror image of the second asset (solid line, bottom panel)).

In general, return volatility is a nonlinear function of the dividend share. The volatility of each asset's return equals the dividend volatility when its share is zero, but then either increases or decreases as the share increases. The plots show that in each case there is a region of "excess volatility" where the volatility of asset returns exceeds the volatility of the underlying cash flows or dividends. This is particularly clear for the bond in the bond-stock case whose return volatility is generally greater than zero while its dividend volatility is always zero.

Excess volatility derives fundamentally from expected returns (Figure 2) in two ways. First, "excess volatility" tends to occur in the region in which the expected return is a declining function of the share, so prices "overreact" to dividends. A positive shock to an asset's dividends thus raises the price of the asset by more than the dividend shock. Second, shocks to one asset's dividends affect the price of the other asset even with no news about the other asset's dividends. Finally, comparing the expected excess returns in Figure 3 with the return volatilities in Figure 5, we see that Sharpe ratios vary considerably over time as the dividend share varies.

### 4.5 Market Betas.

With log utility, expected returns follow a conditional CAPM which in our case is also a consumption CAPM. Figure 6 plots the instantaneous betas. The dependence of betas on share in Figure 6 implies that betas will vary over time for individual assets. Of course, betas are implied by the expected excess return plots of Figure 3 together with the market excess return plots in Figure 1. However, it's still worthwhile to consider betas directly.

As shown, the beta for the first asset is zero when its share is zero for each of our cases. As we have seen, the return covariance is equal to the dividend covariance for a share of zero, and the first asset's dividends are uncorrelated with those of the second asset, which is now the entire market. As the share increases, however, the first asset contributes more to the total market return and its beta begins to increase correspondingly. In some cases, the beta eventually becomes greater than one (as a regression coefficient of $A$ on $A+B$ can exceed one).

As the share approaches one, however, the beta begins to decrease. In the symmetric and risky-asset cases, the beta of the first asset converges to one. This behavior is what we expect since the first asset becomes the entire market. In the bond-stock case, however, the beta for the bond converges to about $-0.25$ as the share goes to one. This happens since the volatility of the market also converges to zero and the bond's beta becomes an indeterminate form (zero over zero). The expected excess return on the bond still converges to the market premium (zero) as the share approaches one.

In the asymmetric cases, the betas for the second asset display some interesting additional behavior. In the bond-stock case (dotted line, bottom panel of Figure 6), the beta increases without bound as the share of the second asset goes to zero (first asset goes to one). Again, since the market premium converges to zero as the share of the bond goes to one, the expected excess return of the stock remains bounded (and in fact converges to zero) despite its unbounded market beta. In the risky-asset case, the beta of the second asset remains bounded, but does not go to zero even though its share goes to zero. Again, this reflects the explosive slope in its price-dividend ratio, so that shocks to the first asset, which cause smaller and smaller changes to the share, still cause large changes to the second asset's valuation. This positive (valuation) beta is thus consistent with the positive expected excess return and the explosive price-dividend ratio.

**4.6 Serial Correlation of Returns.**

Figure 7 graphs the conditional serial correlation of one-year asset returns as a function of the initial dividend share, $\text{Corr}_t\left[R_{t,t+1}, R_{t+1,t+2} \mid s_t = s\right]$, where $R_{t,t+1}$ denotes the discrete-time return from time $t$ to time $t+1$. To calculate this correlation, we simulate 10,000,000 paths using the same random number generator seed for each initial value of the dividend share.

Asset returns are serially correlated in all cases. For the first asset, the serial correlations tend to be positive for small share values, but slightly negative for share values close to one in the symmetric and risky-asset cases. A symmetric pattern holds in each of the cases for the second asset. Thus, both "momentum" and "mean reversion" or "price-overreaction" are possible in this model. Positive momentum occurs over a larger portion of the state space, and where shares are low. As expected excess returns in Figure 3 are even more commonly positively sloped, positive momentum is even more prevalent among excess returns. In contrast, the serial correlation for the bond in the bond-stock case is uniformly large and negative in magnitude (dotted line, top panel of Figure 7). Bond yields and expected returns decline when prices rise, of course, since all shocks to bond prices come from expected returns. Although not shown, the serial correlation depends on horizon as well. The serial correlation of long-horizon returns can even have a different sign than the serial correlation of short-horizon returns.

Figure 8 graphs the cross-serial correlation $\text{Corr}_t(R_{1,t+1}, R_{2,t+2} \mid s_t = s)$ for the two assets. Again, these correlations can be positive for some share values, and negative for others.

The size as well as the signs of the autocorrelations and cross-serial correlations are about what one would expect from the expected return plots in Figure 2. For example, in the symmetric case the expected return rises about four percentage points as the share rises from zero to one. Now, a one percent dividend shock is about a one percent return shock, and it changes the share by $s(1-s) \times 1$ percentage points. Thus, in the middle, we expect to see a $4 \times s(1-s)/100 = 0.01$ autocorrelation coefficient. Where the slope of expected return as a function of share is lower or negative, we expect to see a corresponding autocorrelation, multiplied by the quadratic term $s(1-s)$. This reasoning well describes the shape and magnitude of the autocorrelation and cross-serial correlation plots.

The size of momentum effects reported in the literature requires autocorrelations and cross-serial correlations somewhat larger than these values. Typical estimates are consistent with a forecast $R^2$ of 0.01 and thus autocorrelations on the order of 0.10. For example, Fama and French (1996) report a 1.5 percent per month return from a strategy that goes long the past year winners and short the past year losers. The average return of the top decile of a normal distribution is 1.76 standard deviations above the mean. Therefore, using a 50 percent standard deviation of individual stock returns and a ten percent average stock return, the winning decile typically earned $10 + 1.76 \times 50 = 100$ percent in the past year, and the losing decile lost $10 - 1.76 \times 50 = -80$ percent. A serial correlation (or regression) coefficient of ten percent then implies that the long-short strategy is expected to earn $(100 + 80) \times 0.1 = 18$ percent in the following year, or 1.5 percent per month. Negative cross-serial correlation helps of course; a 0.05 own-serial correlation and 0.05 cross-serial correlation coefficient will have the same effect. Still, these numbers are a factor of five larger than what we see in our model. The model provides a qualitative but not a quantitative match to the data.

### 4.7 Return Correlations.

There can be significant correlation *between* the two asset returns, even when the underlying dividend processes are uncorrelated as in our scenarios. Figure 9 plots the conditional correlation coefficient between the instantaneous returns of the two assets as a function of the dividend share, $\text{Corr}_t\left[R_{1,t}, R_{2,t} \mid s_t = s\right]$. We calculate this correlation directly from the volatility terms in the instantaneous return formula, Equation (18).

In the symmetric case, return correlation is equal to zero when the share is near the endpoints. As we have seen above, return variation is dominated by dividend variation here, and the dividends are uncorrelated with each other. For intermediate values of the share, however, the return correlation rises to nearly 14 percent. A "common factor" or "contagion effect" appears in returns, even though there is none in cash flows.

The mechanism for correlation, as usual, traces back to expected returns as a function of the share in Figure 2. When the dividend of the second asset increases, the share of the first asset decreases, decreasing the expected return of the first asset. Lower expected return means higher prices, and a positive ex-post return on the first asset. Shocks to the first asset affect the second asset's returns in the same way, and the correlation represents the sum of the two effects.

The return correlation can be significantly higher. For example, in the risky-asset case, the correlation between asset returns is greater than 70 percent for large shares. For a large share of the volatile asset, consumption growth becomes very volatile. This large volatility gives large discount rate effects to the second asset's price, overwhelming the cash flow effects. A sector (the second asset) whose cash flows are much *less* volatile than those of the rest of the market will have returns that are much *more* influenced by market movements than those of other sectors.

In the bond-stock case, the first asset's dividends are the only shock, so returns are perfectly correlated. The sign of the correlation switches from $-1$ to $+1$ as the slope of the bond price-dividend ratio as a function of share, and thus the beta, switches sign from negative to positive. The difference in the pattern of correlation between the cases shows that the cross-sectional structure of the market, in terms of dividend shares and volatilities, can have a large effect on the joint distribution and factor structure of asset returns.

### 4.8 The Dividend Share.

Variation in the dividend share drives most of the results presented in this section. To provide some intuition about the dynamic behavior of the dividend share, Figure 10 plots the instantaneous drift and volatility of the share for the three cases (from Equations (34) and (35)).

The drift or expected change in the share begins at zero, becomes positive, then negative, and returns back to zero as the share ranges from zero to one. This demonstrates the local or instantaneous mean reverting nature of the share.

The volatility of changes in the share is greatest when $s = 1/2$ in each of the three cases. As shown, this volatility can be substantial, not only at the midpoint of the graph, but also for small values of the share (particularly when the volatility is calculated on a percentage basis). These graphs reinforce our point that dividend share dynamics play a major role in determining the nature of returns in this multi-asset framework.

#### *4.8.1 Limit Distribution of the Share.*

Our share process does not have a stationary distribution, as one asset always ends up dominating the other. For this reason we have plotted only *conditional* moments. Unconditional moments do not exist.

One might worry that the results are driven by this feature; that values depend crucially on this asymptotic behavior so that our results are a poor guide to the behavior of models with stationary share distributions. To demonstrate that this is not the case, first recall from Equation (12) that the price-consumption ratio is an exponentially weighted average of future share values,

$$\frac{P_{1t}}{C_t} = E_t \left[ \int_0^\infty e^{-\delta\tau} \; s_{t+\tau} \; d\tau \right]. \tag{41}$$

We see that the price depends on the *discounted* value of the *expected* share only. Whether that expectation is taken over a two-point distribution, with one asset always dominating the other, or whether it is taken over a smooth single-peaked distribution does not matter. Any behavior of the share that is hundreds of years in the future does not matter much, since the discount rate makes that event irrelevant to valuation.

To characterize the behavior of the expected share over a relevant horizon, Figure 11 plots the expected share value over the next 100 years for each of the three cases. In each plot, the initial value of the share ranges from 5 to 95 percent. As shown, the expected share is either almost constant over the next 100 years, or actually tends towards a more central value. Thus, the expected share over the next century appears indistinguishable from that which would be generated by a mean reverting process.

How much of the value of the asset comes from cash flows received more than 100 years in the future in the cases we consider? To evaluate this formally, we compare the price-consumption ratio in Equation (41) with the price-consumption ratio obtained by only allowing the integral to range from zero to 100 years (the integral is evaluated numerically). Although not shown, we find that the difference between the two ratios is less than one tenth of one percent in all cases. Thus, cash flows more the 100 years in the future clearly have little effect on asset prices. Taken together, these results demonstrate that the asymptotic nonstationary nature of the model does not drive the asset-pricing results presented in this section.

*4.8.2 Shares Near Zero and Multiple Assets.*

A natural question is, what happens as the number of assets (trees) increases? Without actually solving a model with multiple assets, of course, we can only speculate about the result. It is suggestive, however, that most of our dynamics disappear as the share goes to zero. Fundamentally, this behavior comes from the fact that share volatility goes to zero at the endpoints. Then, the variability of functions of the share (with bounded derivatives) such as expected returns also goes to zero. It is natural to speculate that as a result our dynamics disappear as the number of assets grows larger.

This is not necessarily the case, however. We should not take the limit as the number of assets grows while maintaining the assumption that dividend growth is uncorrelated across assets. In such a limit, aggregate consumption growth and the market portfolio return become constants. To preserve the variability of the market portfolio as the number of assets grows, we must instead add correlation between dividend growths of different firms, perhaps with a factor structure. For example, each firms' dividend growth could have an aggregate component, an industry component, perhaps a regional component, as well as idiosyncratic components. Now, if $N$ small trees have perfectly correlated dividend growth, they act in our model just like one larger tree. Following this logic, in this sensible kind of limit we are likely to see return dynamics that persist. We will of course continue to see dynamics of industry, country, asset class (stock vs. bond vs. real estate), and style aggregates.

*4.8.3 State Variables and Empirical Implications.*

Like the standard one-tree model, our two-tree model attempts to illustrate fundamental economic principles about asset pricing in a simple way. As such, our model abstracts from the type of model one would want to take literally and directly to the data in two important ways. First, we specify that shares are in fixed supply forever. In a realistic model, we would want to specify adjustment costs or other frictions that fix share supply in the short run, but we should eventually allow investors in aggregate to rebalance (invest or disinvest). Second, current dividends are the only shocks to our model. There is no room for news about future dividend growth to change the price-dividend ratio of a tree. Perhaps most importantly, there is no room for news about future aggregate consumption growth to affect the price of a tree through a discount rate effect. In our model, all news is revealed from the current dividend growth of the two trees, and dividend shares are completely-revealing state variables.

As a result of these abstractions, this model (like any model) makes predictions that can be swiftly rejected if taken literally. In particular, any model that produces prices (price-dividend ratios) that are functions of observable state variables can be quickly rejected. Price-dividend ratios do not lie exactly on any line – with no error term – as predicted in Figure 4. Similarly, ex-post returns in the model depend only on the realization of the two trees' dividend growth, so (appropriately nonlinear and state-dependent) regressions of returns on current and lagged dividend growth should yield 100% $R^2$ values. In the data, attempts to explain ex-post returns with observables always come up short.

The issue is, as usual, one of interpretation. We would like to believe that some predictions, such as the central return dynamics, will carry over to more complex models that avoid literal-minded easily-rejectable predictions like the above. In particular, it seems likely that an increase in *market value* of a firm or sector, even if not traceable to *current* dividends of that or any other firm in a more complex model, will lead to this sort of return dynamics, in order to choke off an investor's desire to rebalance. Thus, in thinking about data, we would view market values as state variables and changes in market values as the driving force, even though the model uses the level of the dividend share as a state variable for analytical convenience.

Similarly, the graph of average returns as a function of dividend share in the data is a flat line.[8] The familiar size effect means that the graph of average returns as a function of market value shares in the data slopes downward.[9] These findings in the data contrast with our plots of rising expected returns in Figures 2 and 3. Like the singularity relating price-dividend ratios to dividend shares, however, this literal prediction of the model strikes us as one that will be easily repaired in more realistic specifications that maintain the basic market-clearing mechanism and produce return dynamics. The *level* of the dividend or value share is a long-run variable, reflecting all the long-run investment, disinvestment, and rebalancing that our model rules out. It seems natural to conjecture that a model with short-run fixed shares but long-run adjustment will produce something like our return dynamics in response to short-run changes in market value, with no or differently-signed long-run relations between average returns (price-dividend ratios, volatility, betas, etc.) and dividend shares.

## 5. CONCLUDING REMARKS

We extend the classic single-asset "Lucas-tree" pure-exchange framework to the case of two assets and we solve the model in closed form. Our two-tree model has the simplest ingredients, log utility and i.i.d. normal dividend growth. Nonetheless, market-clearing logic and a fixed share supply generate interesting and complex patterns of time-varying asset prices, expected returns, risk premia, variances, covariances, and correlations. The patterns of these time-varying conditional moments are similar to many of those in the empirical asset pricing literature, including momentum, mean-reversion, price over-reaction and under-reaction to dividend news, excess volatility, value and growth effects (expected returns that vary across assets sorted on valuation ratios), return and excess return forecasts from price-dividend ratios, and return correlation despite uncorrelated cash flows.

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[8] We thank Ravi Bansal for pointing out this fact in a thoughtful discussion at the Spring 2004 NBER Asset Pricing conference.

[9] The two statements may seem contradictory, but as Berk (1996) points out, the size effect in returns occurs only in firms sorted by market equity, not in firms sorted by direct measures of size such as dividend shares.

## APPENDIX

### 1. Derivation of Asset Prices – The Integral Approach.

The price-consumption ratio of the first asset is given by

$$\frac{P_{1t}}{C_t} = E_t\left[\int_0^\infty e^{-\delta\tau}\frac{D_{t+\tau}}{C_{t+\tau}}\,d\tau\right] = E_t\left[\int_0^\infty e^{-\delta\tau}\frac{1}{1+\frac{D_{2,t+\tau}}{D_{1,t+\tau}}}\,d\tau\right] = E_t\left[\int_0^\infty e^{-\delta\tau}\frac{1}{1+qe^u}\,d\tau\right], \qquad (A1)$$

where $q$ is the initial dividend ratio $D_{2,t}/D_{1,t}$ and $u$ is a normally distributed random variate with mean $\nu\tau$ and variance $\eta^2\tau$, and where,

$$\nu = \mu_2 - \mu_1 - \sigma_2^2/2 + \sigma_1^2/2,$$

$$\eta^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

Note that $\nu\ dt = E[\ln(D_2/D_1)]$ and $\eta^2\ dt = \text{Var}[\ln(D_2/D_1)]$. Introducing the density for $u$ into the last integral gives

$$\frac{P_{1t}}{C_t} = \int_0^\infty\int_{-\infty}^\infty e^{-\delta\tau}\frac{1}{\sqrt{2\pi\eta^2\tau}}\frac{1}{1+qe^u}\exp\left(\frac{-(u-\nu\tau)^2}{2\eta^2\tau}\right)\,du\,d\tau. \qquad (A2)$$

Interchanging the order of integration and collecting terms in $\tau$ gives,

$$\frac{P_{1t}}{C_t} = \int_{-\infty}^\infty\frac{1}{\sqrt{2\pi\eta^2}}\frac{1}{1+qe^u}\exp\left(\frac{\nu u}{\eta^2}\right)\int_0^\infty\tau^{-1/2}\exp\left(-\frac{u^2}{2\eta^2}\frac{1}{\tau}-\frac{\nu^2+2\delta\eta^2}{2\eta^2}\tau\right)\,d\tau\,du. \qquad (A3)$$

From Equation (3.471.9) of Gradshteyn and Ryzhik (2000), this expression becomes,

$$\frac{P_{1t}}{C_t} = \int_{-\infty}^\infty\frac{2}{\sqrt{2\pi\eta^2}}\frac{1}{1+qe^u}\exp\left(\frac{\nu u}{\eta^2}\right)\left(\frac{u^2}{\nu^2+2\delta\eta^2}\right)^{1/4}K_{1/2}\left(2\sqrt{\frac{u^2(\nu^2+2\delta\eta^2)}{4\eta^4}}\right)\,du, \qquad (A4)$$

where $K_{1/2}(\cdot)$ is the modified Bessel function of order 1/2 (see Abramowitz and Stegum (1970) Chapter 9). From the identity relations for Bessel functions of order equal to an integer plus one half given in Gradshteyn and Ryzhik Equation (8.469.3), however, the above expression can be expressed as,

$$\frac{P_{1t}}{C_t} = \frac{1}{\psi}\int_{-\infty}^\infty\frac{1}{1+qe^u}\exp\left(\frac{\nu u}{\eta^2}\right)\exp\left(-\frac{\psi}{\eta^2}\mid u\mid\right)\,du, \qquad (A5)$$

where

$$\psi = \sqrt{\nu^2+2\delta\eta^2}.$$

In turn, Equation (A5) can be written

$$\frac{P_{1t}}{C_t} = \frac{1}{\psi} \int_0^{\infty} \frac{1}{1+qe^u} \ \exp(\gamma u) \ du \ + \frac{1}{\psi} \int_{-\infty}^{0} \frac{1}{1+qe^u} \ \exp(\theta u) \ du. \tag{A6}$$

where

$$\gamma = \frac{\nu - \psi}{\eta^2},$$

$$\theta = \frac{\nu + \psi}{\eta^2}.$$

Define $w = e^{-u}$. By a change of variables Equation (A6) can be written

$$\frac{P_{1t}}{C_t} = \frac{1}{q\psi} \int_0^1 \frac{1}{1+w/q} \ w^{-\gamma} \ dw \ + \frac{1}{\psi} \int_0^1 \frac{1}{1+qw} \ w^{\theta-1} \ dw, \tag{A7}$$

From Abramowitz and Stegum Equation (15.3.1), this expression becomes

$$\frac{P_{1t}}{C_t} = \frac{1}{q\psi(1-\gamma)} \ F(1, \ 1-\gamma; \ 2-\gamma; \ -1/q) \ + \ \frac{1}{\psi\theta} \ F(1, \ \theta; \ 1+\theta; \ -q). \tag{A8}$$

Finally, substituting $q = (1-s)/s$ into Equation (A8) gives the price of the first asset,

$$\frac{P_{1t}}{C_t} = \frac{1}{\psi(1-\gamma)} \left(\frac{s}{1-s}\right) F\left(1, \ 1-\gamma, \ 2-\gamma; \ \frac{s}{1-s}\right) + \frac{1}{\psi\theta} F\left(1, \ \theta; \ 1+\theta; \ \frac{s-1}{s}\right), \tag{A9}$$

which is Equation (13).

The same approach can be used to solve for the price of the second asset,

$$\frac{P_{2t}}{C_t} = \frac{1}{\psi(1+\theta)} \left(\frac{1-s}{s}\right) F\left(1, \ 1+\theta; \ 2+\theta; \ \frac{s-1}{s}\right) - \frac{1}{\psi\gamma} F\left(1, \ -\gamma; \ 1-\gamma; \ \frac{s}{s-1}\right). \tag{A10}$$

Applying the recurrence relations for contiguous hypergeometric functions presented in Abramowitz and Stegum (1970) Equations (15.2.18) and (15.2.20) gives the result

$$P_1 + P_2 = \frac{C}{\delta} = \frac{D_1 + D_2}{\delta} = P_m. \tag{A11}$$

**2. Derivation of Asset Prices – The Differential Equation Approach.**

For additional perspective, we provide an alternative derivation that parallels the well-known Feynman-Kac approach for solving differential equations. Rewrite the last term in Equation (A1) as,

$$E_t \left[ \int_0^{\infty} e^{-\delta\tau} \frac{1}{1+e^{-x_{t+\tau}}} d\tau \right], \tag{A12}$$

where the log dividend ratio $x = \ln(D_1/D_2)$ is now a state variable. To find the differential equation for the price-consumption ratio, we can either differentiate Equation (A12) explicitly or note the analogy to risk-neutral pricing of a security paying a dividend $(1 + e^{-x})^{-1}$, resulting in

$$0 = E_t \left[ d\left( \frac{P_{1t}}{C_t} \right) \right] + \frac{1}{1 + e^{-x_t}}\ dt - \delta\ \frac{P_{1t}}{C_t}\ dt. \tag{A13}$$

The price-consumption ratio is a function of the state variable,

$$y(x_t) \equiv \frac{P_{1t}}{C_t}(x_t). \tag{A14}$$

Applying Itô's Lemma to Equation (A14) and substituting into Equation (A13),

$$y'(x)\ E_t[dx]\ +\ \frac{1}{2}\ y''(x)\ E_t[dx^2]\ +\ \frac{1}{1 + e^{-x_t}}\ dt\ -\ \delta\ y(x)\ dt = 0. \tag{A15}$$

The dynamics of $x$ are given by

$$dx = d(\ln D_1 - \ln D_2) = \left[ \mu_1 - \mu_2 - \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right) \right] dt + \sigma_1 dZ_1 - \sigma_2 dZ_2, \tag{A16}$$

or, more simply by

$$dx = -\nu\ dt\ +\ \eta\ dZ. \tag{A17}$$

Using these dynamics in Equation (A15), we obtain a differential equation for $y(x)$,

$$\eta^2 y''(x)/2 - \nu y'(x) - \delta y(x) + \frac{1}{1 + e^{-x}} = 0. \tag{A18}$$

This is a linear second-order differential equation with an inhomogeneous term. The general solution to an inhomogeneous linear second-order equation,

$$y''(x) + ay'(x) + by(x) - h(x) = 0, \tag{A19}$$

is given by an application of the variation of parameters technique,

$$y = \left[ c_1 + \frac{1}{\lambda_2 - \lambda_1} \int_0^x h(\xi) e^{-\lambda_1 \xi} d\xi \right] e^{\lambda_1 x} + \left[ c_2 + \frac{1}{\lambda_1 - \lambda_2} \int_0^x h(\xi) e^{-\lambda_2 \xi} d\xi \right] e^{\lambda_2 x}, \tag{A20}$$

$$\lambda_i = \frac{-a \pm \sqrt{a^2 - 4b}}{2},$$

where $c_1$ and $c_2$ are constants and provided $\lambda_i$ are real and distinct.

Applying this result to our case, we have

$$\lambda_1 = \theta, \qquad \lambda_2 = \gamma, \tag{A21}$$

One root is positive and one is negative. Substituting into Equation (A20) gives,

$$y(x) = c_1 e^{\theta x} + c_2 e^{\gamma x} + \frac{e^{\gamma x}}{\psi} \int_0^x \frac{1}{1+e^{-\xi}} e^{-\gamma \xi}\, d\xi - \frac{e^{\theta x}}{\psi} \int_0^x \frac{1}{1+e^{-\xi}} e^{-\theta \xi}\, d\xi. \tag{A22}$$

Changing variables to $w = e^{-\xi}$ gives,

$$y(x) = c_1 e^{\theta x} + c_2 e^{\gamma x} + \frac{e^{\gamma x}}{\psi} \int_{e^{-x}}^1 \frac{1}{1+w}\, w^{\gamma-1}\, dw - \frac{e^{\theta x}}{\psi} \int_{e^{-x}}^1 \frac{1}{1+w}\, w^{\theta-1}\, dw. \tag{A23}$$

From Gradshteyn and Ryzhik (2000) Equations (3.194.2), (3.194.5) and (9.14.2) (and see also Equation (15)), the integrals in the above equation can be solved in terms of the hypergeometric function. This gives the general solution,

$$y(x) = d_1 e^{\theta x} + d_2 e^{\gamma x} + \frac{1}{\psi} \left[ \frac{e^x}{1-\gamma}\, F(1,\ 1-\gamma;\ 2-\gamma;\ -e^x) \ +\ \frac{1}{\theta}\, F(1,\ \theta;\ 1+\theta;\ -e^{-x}) \right], \tag{A24}$$

where $d_1$ and $d_2$ are new constants. Imposing the boundary condition that $y(x)$ be bounded as $x$ approaches zero and infinity implies $d_1 = d_2 = 0$. Finally, making a change of variables from $x$ to $s$ gives the price of the first asset Equation (13).

**3. Limits.**

In this section, we derive limits for price-dividend ratios as $s \to 0$ and $s \to 1$. Also, we derive limits for the function $\Phi(s)$ that figures prominently in the asset price dynamics in Equation (18). We focus on the first asset, as the second is symmetric. Start with the price-dividend ratio. By definition,

$$\frac{P_1}{D_1} = \frac{C}{D_1}\frac{P_1}{C} = \frac{1}{s}\frac{P_1}{C}. \tag{A25}$$

From Equations (13) and (A5), this implies,

$$\frac{P_1}{D_1} = \frac{1}{\psi(1-\gamma)} \left(\frac{1}{1-s}\right) F\left(1,\ 1-\gamma,\ 2-\gamma;\ \frac{s}{s-1}\right) \ +\ \frac{1}{\psi\theta} \left(\frac{1}{s}\right) F\left(1,\ \theta,\ 1+\theta;\ \frac{s-1}{s}\right). \tag{A26}$$

From the power series expression for the hypergeometric function, $F(a, b; c; 0) = 1$. Because of this result, it is useful to apply the linear transformation formula given in Abramowitz and Stegum Equation (15.3.7) so that the argument of the hypergeometric function goes to zero at the limit being evaluated,

$$\begin{aligned} F(\alpha, \beta; \gamma; z) = &\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-z)^{-\alpha} F(\alpha, 1-\gamma+\alpha; 1-\beta+\alpha; 1/z) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} (-z)^{-\beta} F(\beta, 1-\gamma+\beta; 1-\alpha+\beta; 1/z) \end{aligned} \tag{A27}$$

To obtain the limit of the price-dividend ratio as $s \to 0$, we use the linear transformation formula to rewrite Equation (A26) as,

$$\begin{aligned} \frac{P_1}{D_1} = &\frac{1}{\psi(1-\gamma)} \left(\frac{1}{1-s}\right) F\left(1,\ 1-\gamma,\ 2-\gamma;\ \frac{s}{s-1}\right) \\ &+ \frac{1}{\psi\theta} \left(\frac{\theta}{\theta-1}\right) \left(\frac{1}{1-s}\right) F\left(1,\ 1-\theta;\ 2-\theta;\ \frac{s}{s-1}\right) \\ &+ \frac{1}{\psi\theta} \left(\frac{1}{s}\right) \Gamma(\theta+1)\, \Gamma(1-\theta) \left(\frac{s}{1-s}\right)^{\theta} F\left(\theta,\ 0;\ \theta;\ \frac{s}{s-1}\right) \end{aligned} \tag{A28}$$

From this expression, it is readily seen that

$$\lim_{s \to 0} \frac{P_1}{D_1} = \begin{cases} \infty, & \text{if } \theta \leq 1; \\ \frac{1}{\delta + \nu - \eta^2/2}, & \text{if } \theta > 1. \end{cases} \tag{A29}$$

To obtain the limit of the price-dividend ratio as $s \to 1$, we again use the linear transformation formula and rewrite Equation (A26) as,

$$\begin{aligned} \frac{P_1}{D_1} &= -\frac{1}{\psi(1-\gamma)} \left(\frac{1-\gamma}{\gamma}\right) \left(\frac{1}{s}\right) F\left(1,\ \gamma;\ 1-\gamma;\ \frac{s-1}{s}\right) \\ &+ \frac{1}{\psi(1-\gamma)} \left(\frac{1}{1-s}\right) \Gamma(2-\gamma)\,\Gamma(\gamma) \left(\frac{1-s}{s}\right)^{1-\gamma} F\left(1-\gamma,\ 0;\ 1-\gamma;\ \frac{s-1}{s}\right) \\ &+ \frac{1}{\psi\theta} \left(\frac{1}{s}\right) F\left(1,\ \theta;\ 1+\theta;\ \frac{s-1}{s}\right). \end{aligned} \tag{A30}$$

From this, it follows immediately that

$$\lim_{s \to 1} \frac{P_1}{D_1} = \frac{1}{\delta}. \tag{A31}$$

A similar approach can be used to show that

$$\lim_{s \to 0} \frac{P_2}{D_2} = \frac{1}{\delta}, \tag{A32}$$

and that,

$$\lim_{s \to 1} \frac{P_2}{D_2} = \begin{cases} \frac{1}{\delta - \nu - \eta^2/2}, & \text{if } \gamma < -1; \\ \infty, & \text{if } \gamma \geq -1. \end{cases} \tag{A33}$$

Finally, the use of l'Hopital's rule and the repeated application of the linear transformation formula gives,

$$\lim_{s \to 0} \Phi(s) = \begin{cases} \theta, & \text{if } \theta \leq 1; \\ 1, & \text{if } \theta > 1. \end{cases} \tag{A34}$$

and

$$\lim_{s \to 1} \Phi(s) = 0. \tag{A35}$$

Substituting the limiting values of $\Phi(s)$ into the asset price dynamics in Equation (18) allows us to fully characterize the properties of these price dynamics as $s \to 0$ and $s \to 1$.

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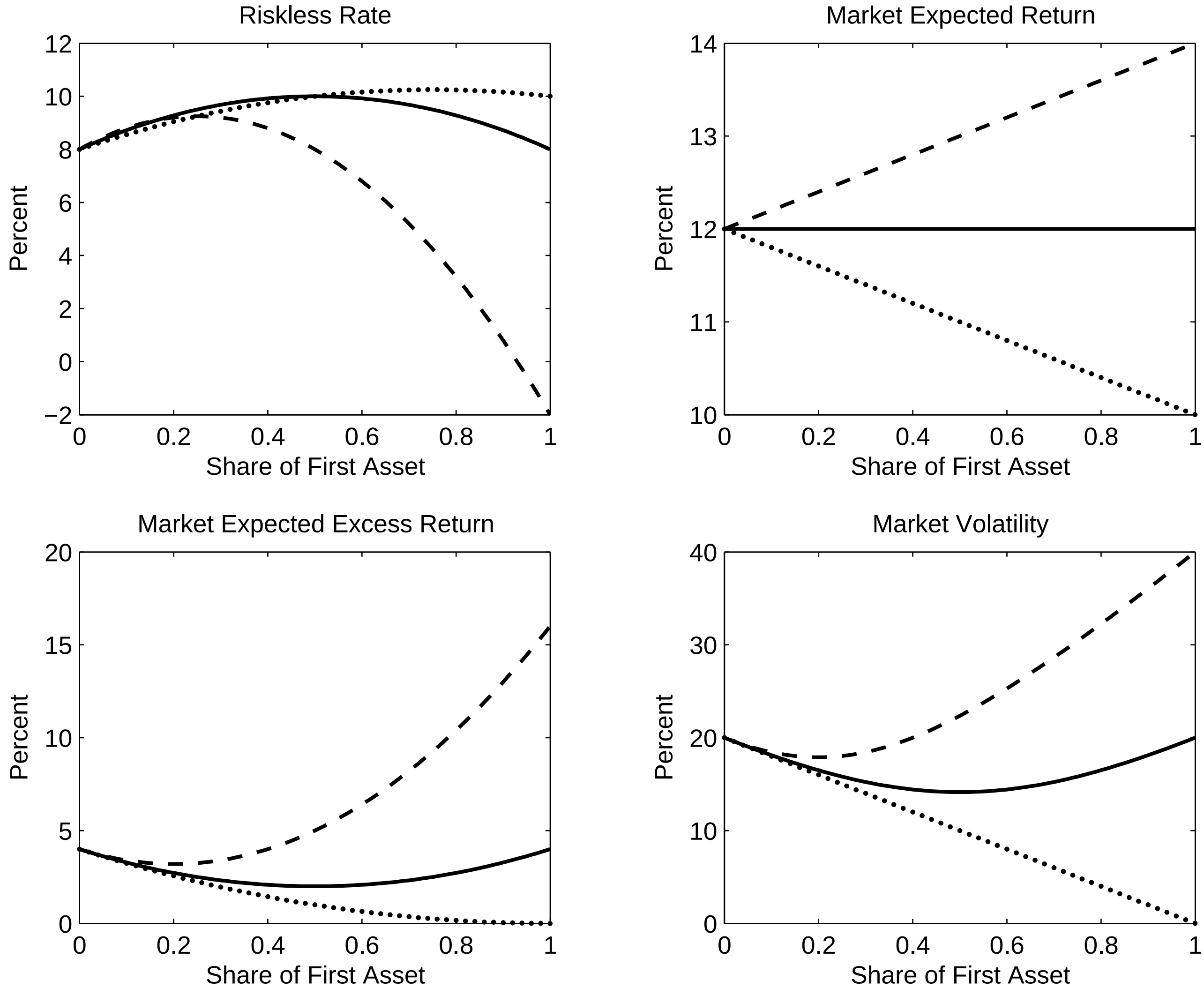
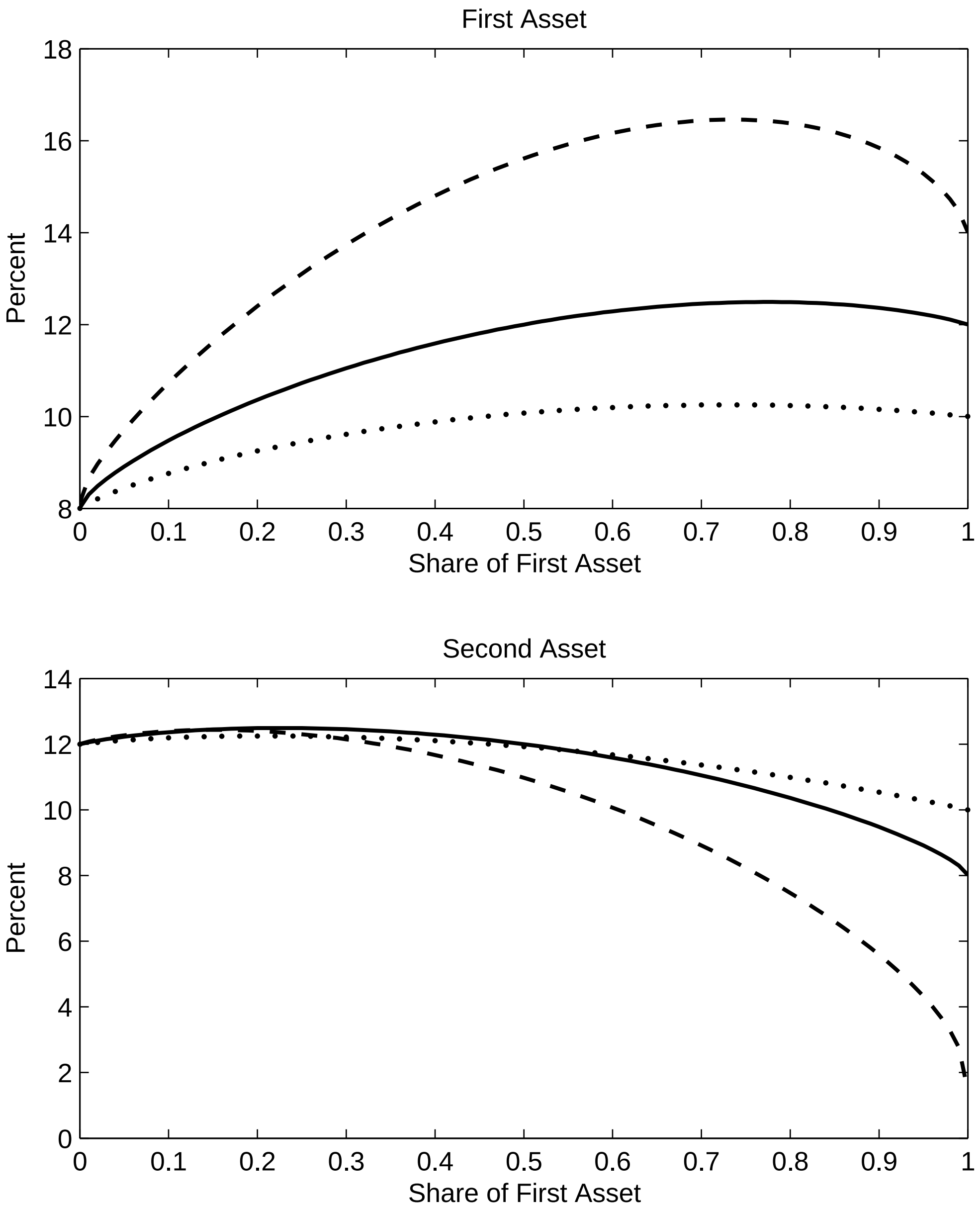
**Figure 1: The Riskless Rate and Market Return.**

This figure plots the indicated riskless rate and market moments as functions of the dividend share of the first asset. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line $(\cdots)$ is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line $(--)$ is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$.



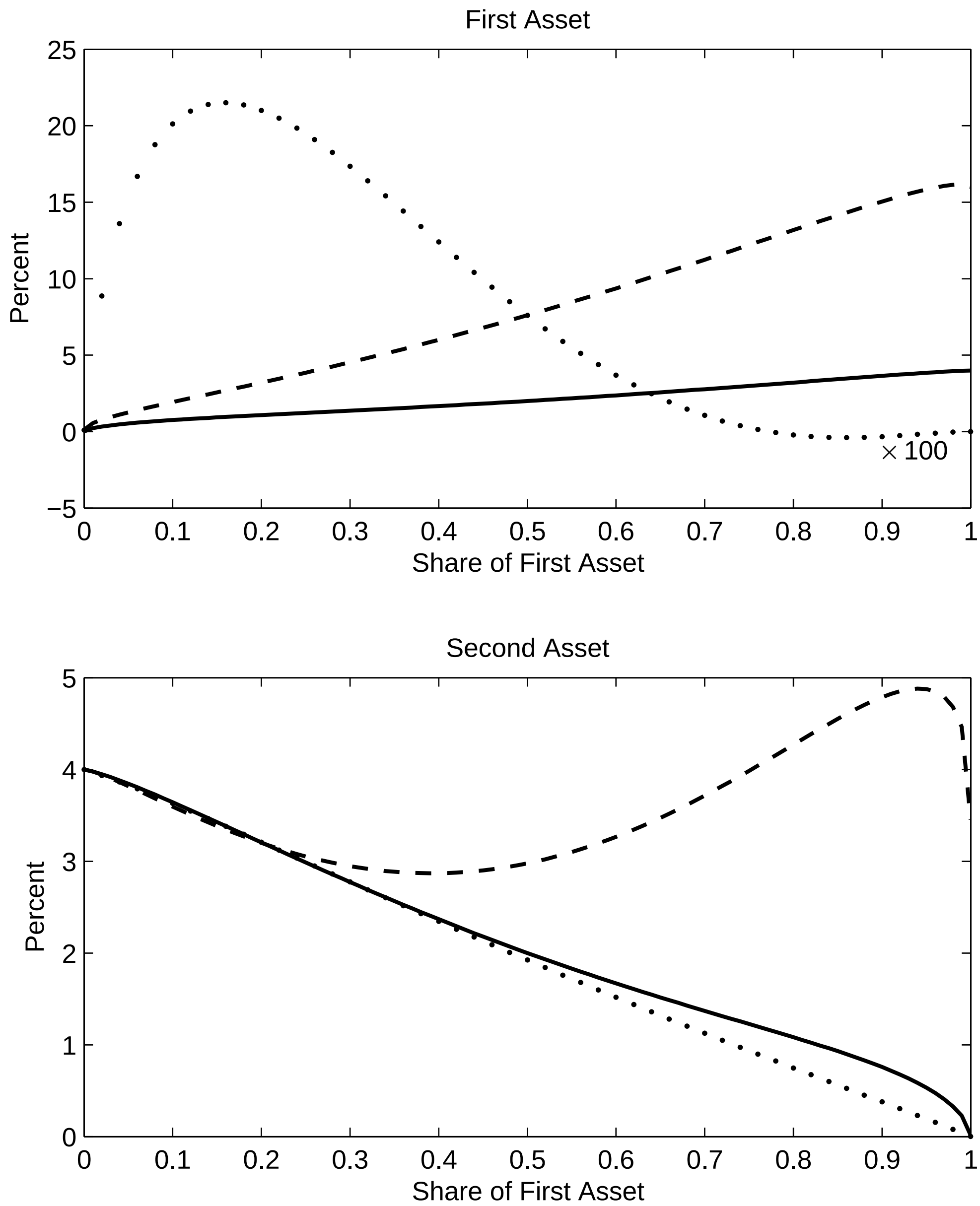
## Figure 2: Expected Returns.

This figure plots the expected returns of the two assets as functions of the dividend share of the first asset. The top panel is for the first asset; the bottom panel is for the second asset. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line ($\cdots$) is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line ($--$) is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$.



## Figure 3: Expected Excess Returns.

This figure plots the expected excess returns of the two assets as functions of the dividend share of the first asset. The top panel is for the first asset; the bottom panel is for the second asset. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line $(\cdots)$ is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line $(--)$ is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$. The expected excess return of the first asset in the bond-stock case is multiplied by one hundred.



## Figure 4: Price-Dividend Ratios.

This figure plots the price-dividend ratios of the two assets as functions of the dividend share of the first asset. The top panel is for the first asset; the bottom panel is for the second asset. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line $(\cdots)$ is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line $(- \, -)$ is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$.

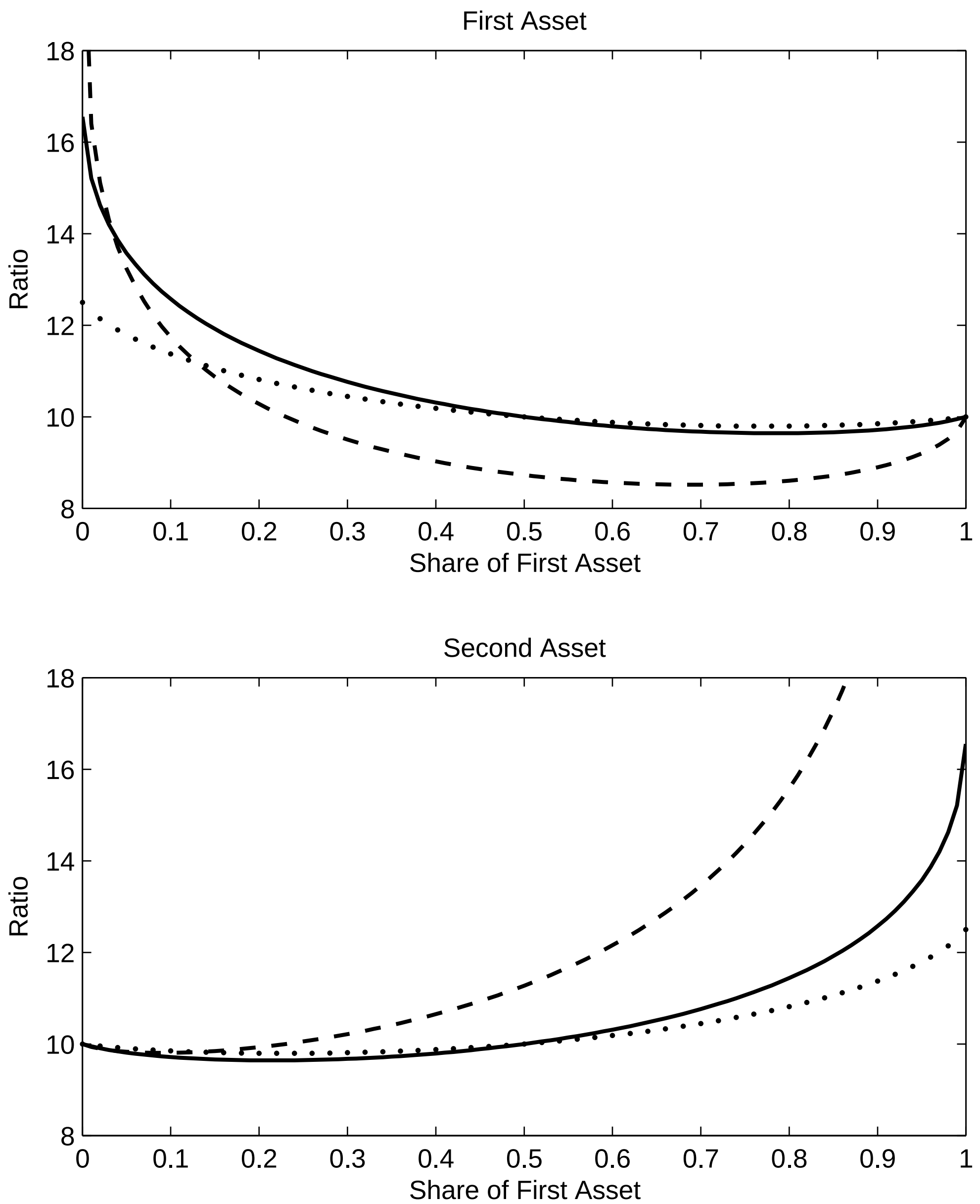
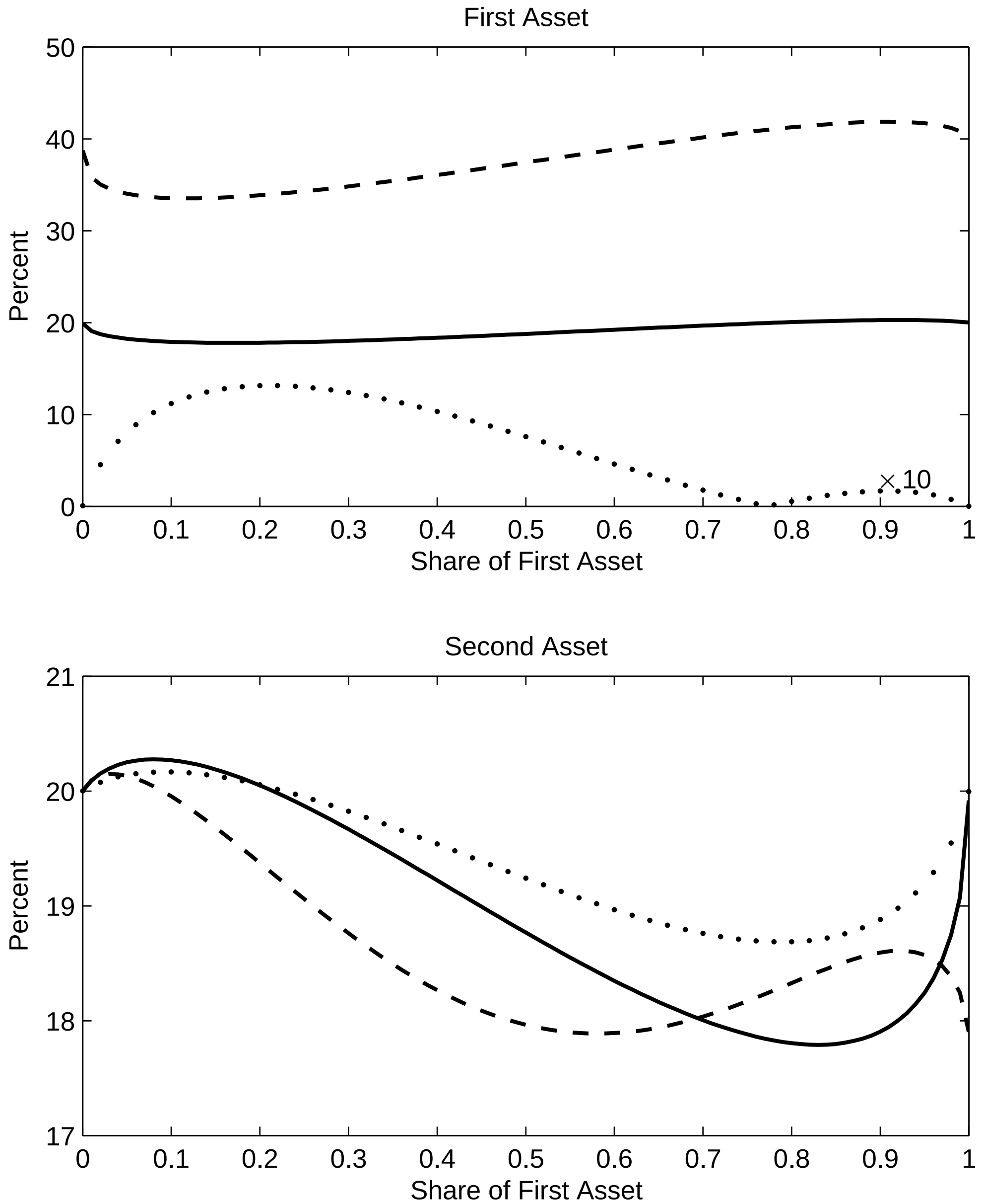


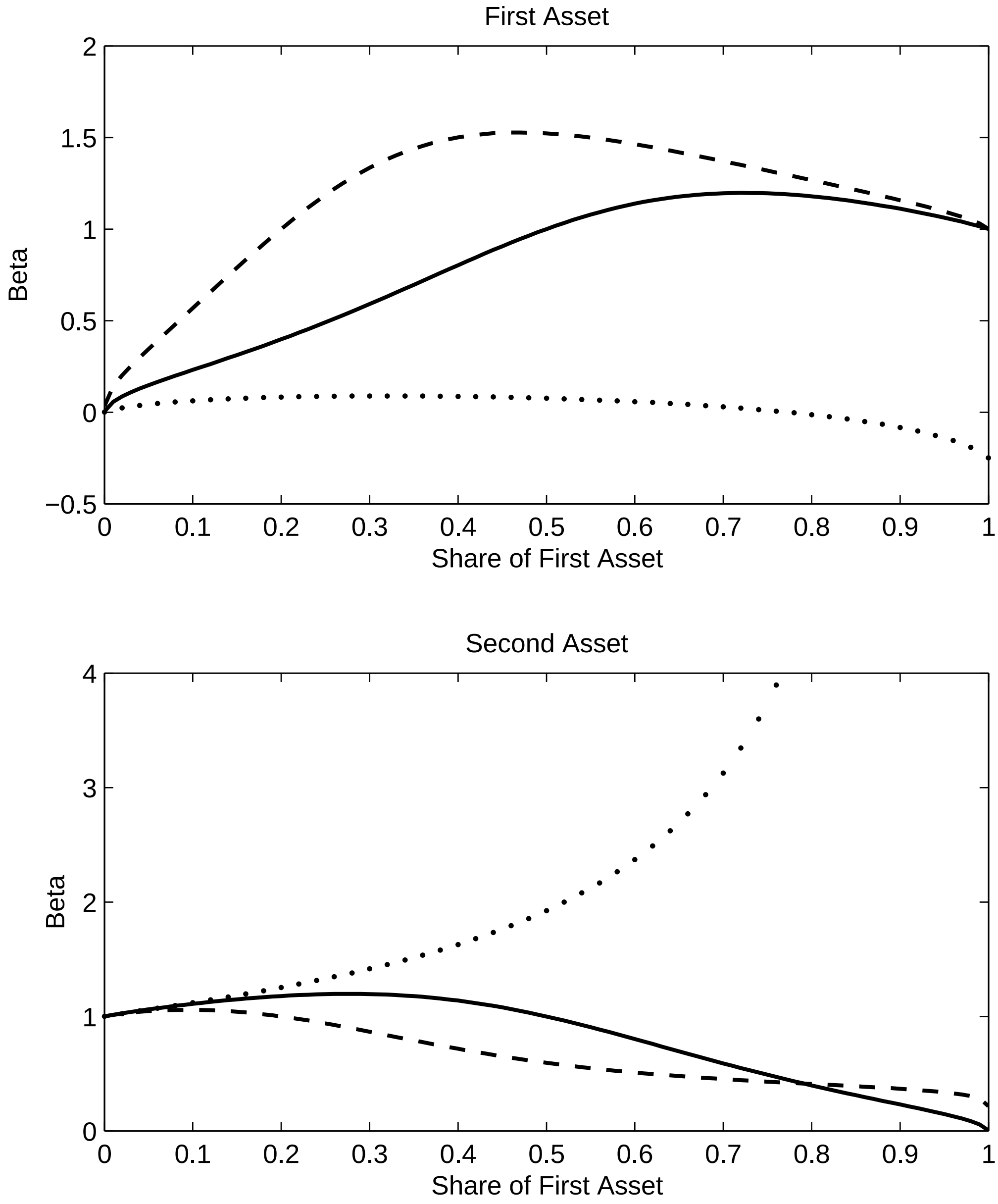
**Figure 5: Volatilities.**

This figure plots the standard deviation of returns of the two assets as functions of the dividend share of the first asset. The top panel is for the first asset; the bottom panel is for the second asset. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line $(\cdots)$ is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line $(--)$ is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$. The volatility of the first asset in the bond-stock case is multiplied by ten.



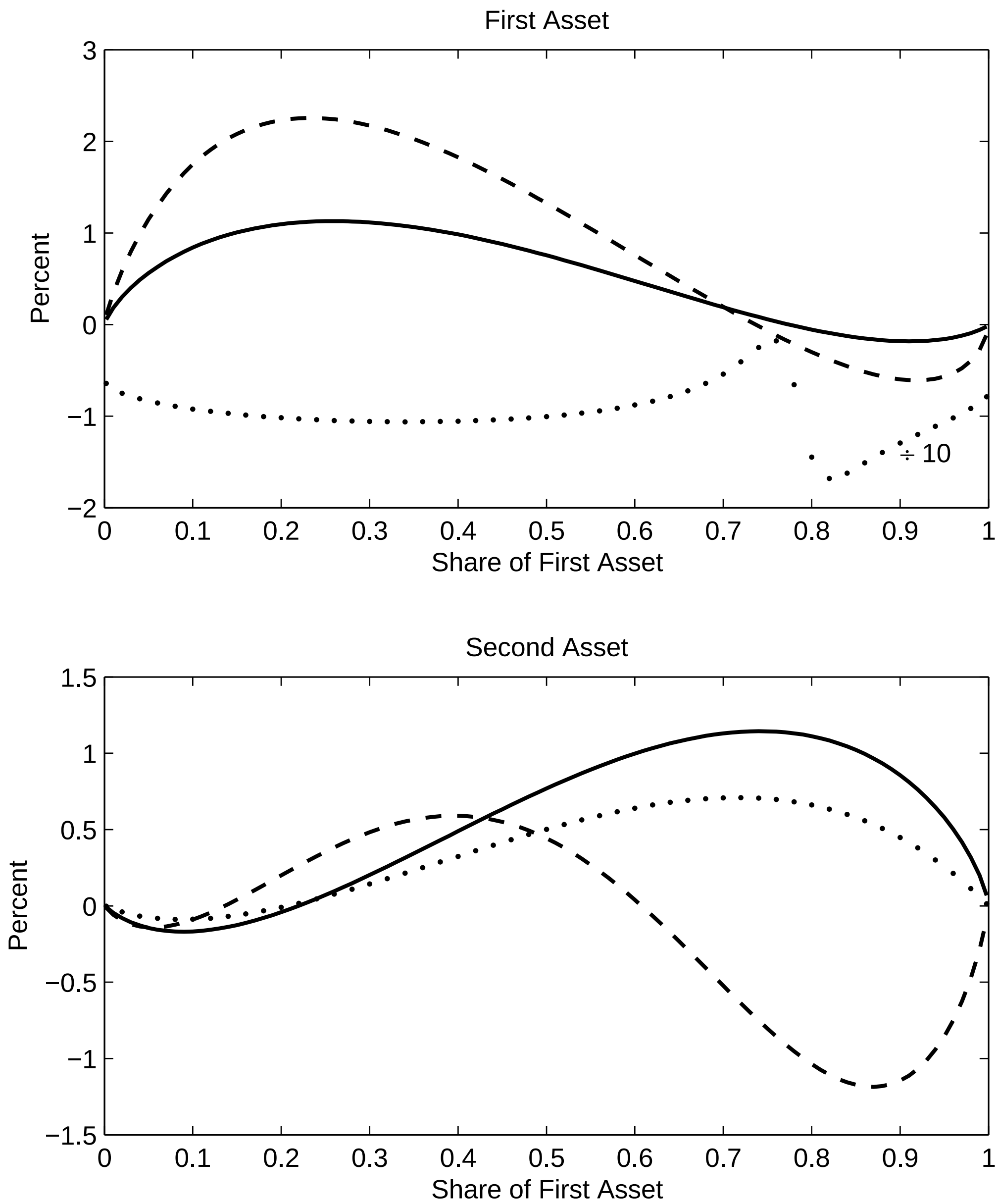
## Figure 6: Betas.

This figure plots the betas of the two assets as functions of the dividend share of the first asset. The top panel is for the first asset; the bottom panel is for the second asset. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line $(\cdots)$ is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line $(--)$ is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$.



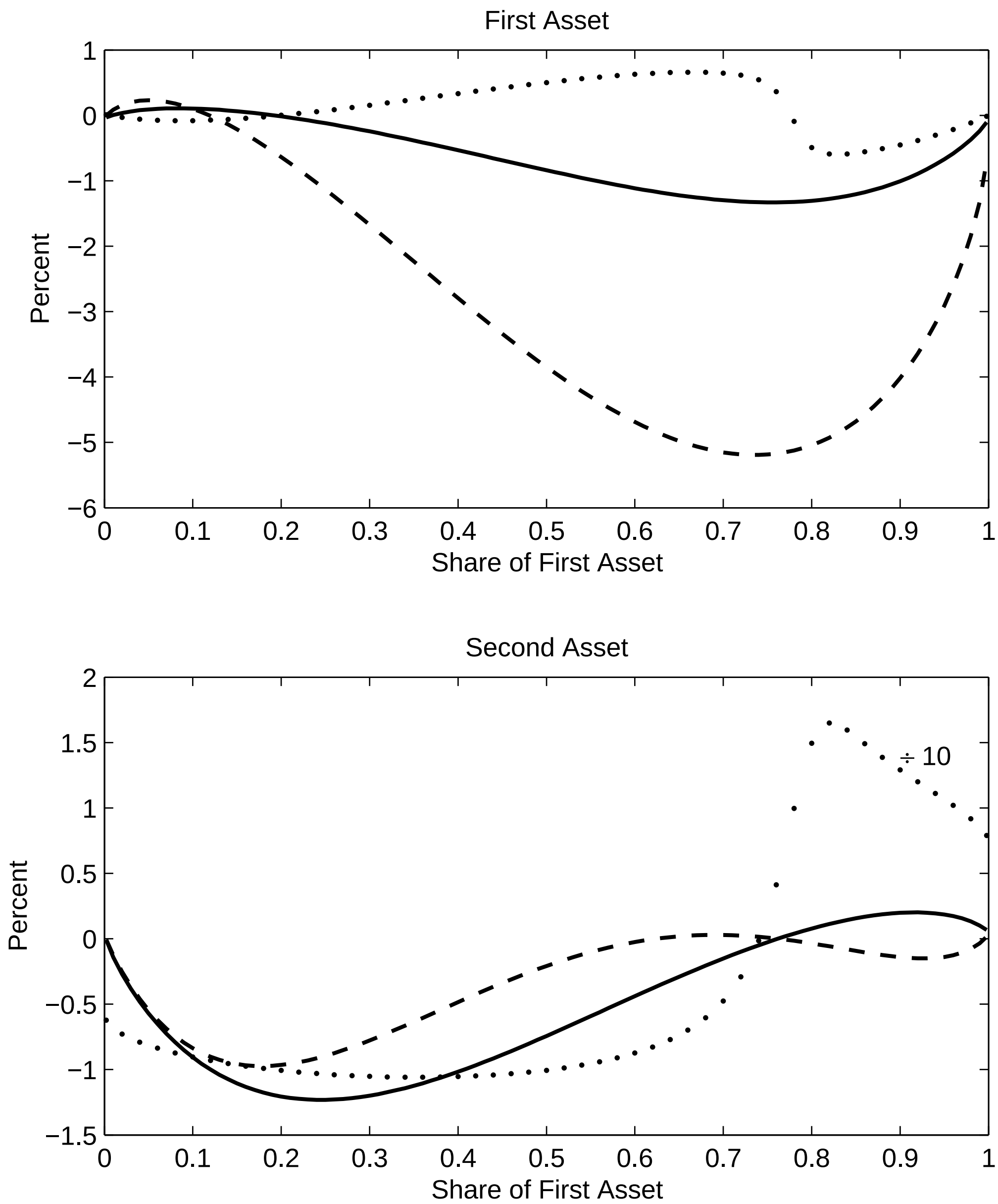
## Figure 7: Own Serial Correlation.

This figure plots the own serial correlation of the two assets' one-year returns as functions of the dividend share of the first asset. The top panel is for the first asset; the bottom panel is for the second asset. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line ($\cdots$) is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line ($--$) is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$. The own serial correlation of the first asset's return in the bond-stock case is divided by ten.



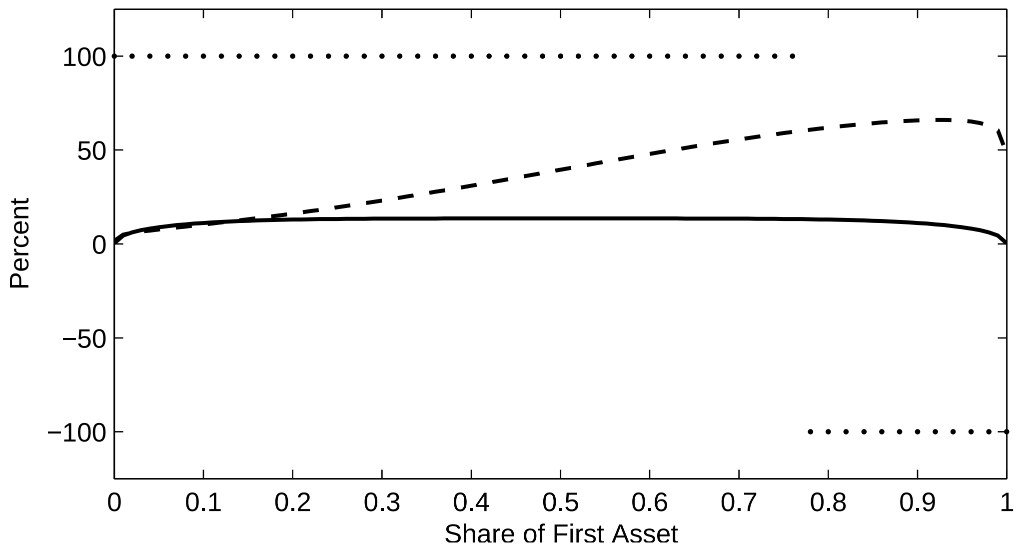
## Figure 8: Cross-Serial Correlation.

This figure plots the cross serial correlation of the two assets' one-year returns as functions of the dividend share of the first asset. The top panel shows the correlation between the first asset's return and the second asset's return in the subsequent year; the bottom panel shows the correlation between the second asset's return and the first asset's return in the subsequent year. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line $(\cdots)$ is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line $(--)$ is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$. The correlation between the second asset's return and the first asset's return in the subsequent year in the bond-stock case is divided by ten.



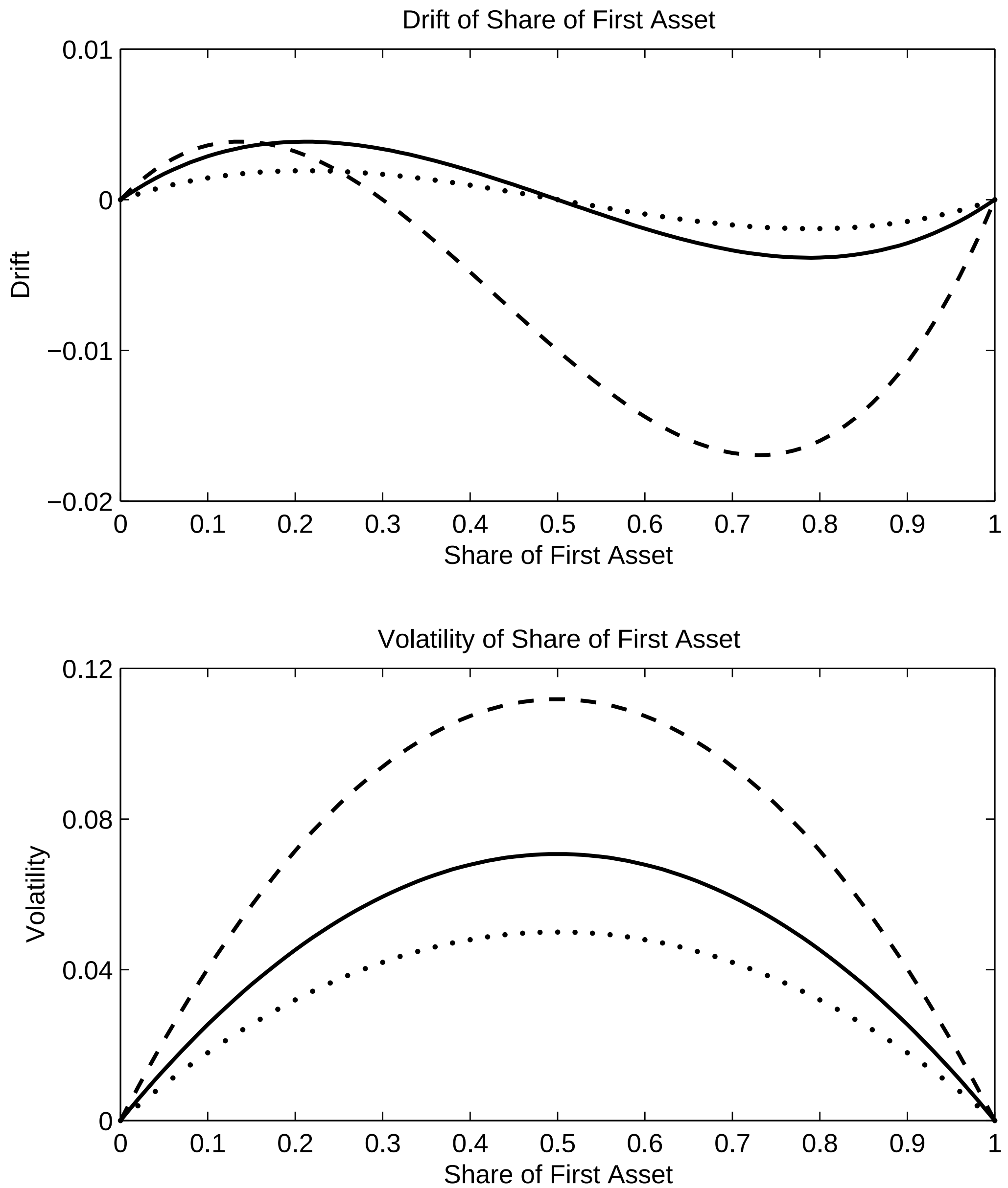
## Figure 9: Correlation.

This figure plots the correlation between the two assets' instantaneous returns as a function of the dividend share of the first asset. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line ($\cdots$) is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line ($--$) is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$.



## Figure 10: Drift and Volatility of Share Process.

This figure plots the instantaneous drift and the volatility of the process followed by the share of the first asset as functions of the share. The solid line (—) is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the dotted line $(\cdots)$ is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and the dashed line $(--)$ is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$.



## Figure 11: Expected Share Values.

This figure plots the expected value of the dividend share of the first asset for horizons up to 100 years, conditional on the indicated share values at time zero of 0.05, 0.25, 0.50, 0.75, and 0.95. The top panel is for the symmetric case, $\mu_1 = 0.02$ and $\sigma_1 = 0.2$; the middle panel is for the bond-stock case $\mu_1 = 0$ and $\sigma_1 = 0$; and bottom panel is for the risky-asset case, $\mu_1 = 0.04$ and $\sigma_1 = 0.4$. In all cases, $\mu_2 = 0.02$ and $\sigma_2 = 0.2$.