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RIVERSIDE

Integral Bases for the Universal Enveloping Algebras of Map Algebras

A Dissertation submitted in partial satisfaction  
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Samuel Herron Chamberlin

June 2011

Dissertation Committee:

Professor Vyjayanthi Chari, Chairperson  
Professor Wee-Liang Gan  
Professor David Rush

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The Dissertation of Samuel Herron Chamberlin is approved:

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To the memory of my brother,

Max Madnick Chamberlin

Our time together was much too short but he taught me many valuable lessons.

I will always miss him.

## ABSTRACT OF THE DISSERTATION

Integral Bases for the Universal Enveloping Algebras of Map Algebras

by

Samuel Herron Chamberlin

Doctor of Philosophy, Graduate Program in Mathematics  
University of California, Riverside, June 2011  
Professor Vyjayanthi Chari, Chairperson

Given a finite-dimensional, simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  and  $A$ , a commutative, associative algebra with unity over  $\mathbb{C}$ , we exhibit an integral form for the universal enveloping algebra of the map algebra,  $\mathfrak{g} \otimes A$ , and an explicit  $\mathbb{Z}$ -basis for this integral form. We also produce explicit commutation formulas in the universal enveloping algebra of  $\mathfrak{sl}_2 \otimes A$  that allow us to write certain elements in Poincaré-Birkhoff-Witt order.

Finally we give some applications of these formulas to the representation theory of the map algebras for  $\mathfrak{sl}_2$ .

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# Introduction

Let  $\mathbb{Z}$  denote the integers. If  $A$  is an algebra, over a field  $\mathbb{F}$  of characteristic 0, define an integral form  $A_{\mathbb{Z}}$  of  $A$  to be a  $\mathbb{Z}$ -algebra such that  $A_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F} = A$ . An integral basis for  $A$  is a  $\mathbb{Z}$ -basis for  $A_{\mathbb{Z}}$ .

The theory of integral forms for finite-dimensional simple Lie algebras was first studied by Chevalley in 1955. His work led to the construction of Chevalley groups (of adjoint type). The representation theory of Chevalley groups relies on the existence of integral forms for the universal enveloping algebras associated to these simple finite-dimensional Lie algebras. In 1966, suitable integral forms were discovered by Cartier and Kostant independently. They obtained precise information about these integral forms through integral bases (bases whose  $\mathbb{Z}$ -span is the integral form). The construction of such bases relies heavily on straightening identities in the universal enveloping algebra, which allow one to write certain elements in Poincaré-Birkhoff-Witt (PBW) order. Cartier and Kostant's  $\mathbb{Z}$ -form led to the construction of Lie groups and Lie algebras over a field of positive characteristic, generalizing Chevalley's construction. This in turn led to the development of representation theory over a field of positive characteristic, [7].

Also in 1966, Serre showed that a finite dimensional Lie algebra can be presented by generators and relations determined solely by the Cartan matrix. With a generalized Cartan matrix one can use the Serre presentation to define the class of Kac-Moody Lie algebras. The most widely studied subclass of these algebras, the simple affine Lie algebras, have structure

and representation theories similar to those for simple finite dimensional Lie algebras. The best way to understand untwisted simple affine Lie algebras is as central extensions of loop algebras, [2]. For these affine Lie algebras Garland, in 1978, extended the theory of integral forms started by Chevalley and continued by Cartier and Kostant. Garland also gave explicit constructions of  $\mathbb{Z}$ -bases for these integral forms via a Chevalley-type basis for the affine Lie algebra. The complexity in formulating integral bases and straightening identities increased greatly in the affine case. These results were then extended to all simple affine Lie algebras by Mitzman, in 1983. In 2007, Jakelić and Moura used Garland and Mitzman's work on integral forms to study representations of affine Lie algebras over a field of positive characteristic, [8].

Recently there has been much interest in map algebras and their representations, [3], [10]. A map algebra is a Lie algebra  $\mathfrak{g} \otimes_{\mathbb{C}} A$ , where  $\mathfrak{g}$  is any finite-dimensional simple complex Lie algebra and  $A$  is any commutative associative complex algebra. The Lie bracket is given by

$$[z \otimes a, z' \otimes b] = [z, z'] \otimes ab, \quad z, z' \in \mathfrak{a}, \quad a, b \in A$$

These Lie algebras are so named because if  $X$  is an algebraic variety and  $A$  is its coordinate ring then  $\mathfrak{g} \otimes A$  can also be realized as the Lie algebra of regular maps  $X \rightarrow \mathfrak{g}$  with pointwise Lie bracket.

Map algebras are a generalization of the loop algebras, for which  $A$  is the Laurent polynomials. Therefore it is natural to wish to generalize the Garland's work to the map algebras. We formulate and prove straightening identities in the universal enveloping algebra of  $\mathfrak{sl}_2 \otimes A$ . The notational difficulties increase greatly when one moves to the general case.

Additionally the formula we have proved is much more general than the one proved by Garland in [6].

If  $A$  has a  $\mathbb{C}$ -basis which is closed under multiplication, these straightening identities lead to the construction of an integral form and integral basis for the universal enveloping algebra of  $\mathfrak{g} \otimes A$ .

A natural offshoot of this work will be to study representations for map algebras over a field of positive characteristic.

Part I

**Integral Bases for the Universal  
Enveloping Algebras of Map  
Algebras**

# Chapter 1

## Previous Work on Integral Bases

In this chapter we will review previous work of Cartier and Kostant on integral forms and their integral bases for the universal enveloping algebras of simple finite dimensional Lie algebras [7]. We will also review the work of Garland on integral forms and their integral bases for the universal enveloping algebras of loop algebras, [6]. As we will see our integral forms and bases for the map algebras are a generalization of Garland's integral forms and bases for the loop algebras.

### 1.1 Preliminaries

The following notation will be used throughout this manuscript:  $\mathbb{C}$  is the set of complex numbers and  $\mathbb{Z}_{\geq 0}$  is the set of non-negative integers. Given any Lie algebra  $\mathfrak{a}$ ,  $U(\mathfrak{a})$  is the universal enveloping algebra of  $\mathfrak{a}$ .

Let  $\mathfrak{g}$  be a finite-dimensional, complex simple Lie algebra of rank  $n$  where  $I = \{1, \dots, n\}$ . Fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and let  $R$  denote the corresponding set of

roots. Let  $\{\alpha_i\}_{i \in I}$  (respectively  $\{\omega_i\}_{i \in I}$ ) be a set of simple roots (respectively fundamental weights) and  $Q$  (respectively  $Q^+$ ),  $P$  (respectively  $P^+$ ) be the integer span (respectively  $\mathbb{Z}_{\geq 0}$ -span) of the simple roots and fundamental weights respectively. Denote by  $\leq$  the usual partial order on  $P$ ,

$$\lambda, \mu \in P, \lambda \leq \mu \text{ if and only if } \mu - \lambda \in Q^+.$$

Set  $R^+ = R \cap Q^+$  and fix an order on  $R^+ = \{\beta_1, \dots, \beta_m\}$ .

Let  $\{x_\alpha^\pm, h_i : \alpha \in R^+, i \in I\}$  be a Chevalley basis of  $\mathfrak{g}$  and set  $x_i^\pm = x_{\alpha_i}^\pm$ , and  $h_\alpha = [x_\alpha^+, x_\alpha^-]$ . Note that  $h_i = h_{\alpha_i}$ . For each  $\alpha \in R^+$ , the subalgebra of  $\mathfrak{g}$  spanned by  $\{x_\alpha^\pm, h_\alpha\}$  is isomorphic to  $\mathfrak{sl}_2$  (When  $\mathfrak{g} = \mathfrak{sl}_2$  we write the  $\mathbb{C}$ -basis as  $\{x^-, h, x^+\}$ ). Set

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathbb{C}x_\alpha^\pm,$$

and note that

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

By the Poincare Birkhoff Witt theorem, we know that if  $\mathfrak{b}$  and  $\mathfrak{c}$  are Lie subalgebras of  $\mathfrak{a}$  such that  $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{c}$  as vector spaces then

$$\mathbf{U}(\mathfrak{a}) \cong \mathbf{U}(\mathfrak{b}) \otimes \mathbf{U}(\mathfrak{c})$$

as vector spaces. So in particular we have a vector space isomorphism

$$\mathbf{U}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{n}^-) \otimes \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{n}^+)$$

Given any  $u \in \mathbf{U}(\mathfrak{a})$  and  $r \in \mathbb{Z}_{\geq 0}$  define  $u^{(r)} = \frac{u^r}{r!}$  and

$$\binom{u}{r} = \frac{u(u-1) \cdots (u-r+1)}{r!}.$$

**Definition.** Define  $T^0(\mathfrak{a}) = \mathbb{C}$ , and for all  $j \geq 1$ , define  $T^j(\mathfrak{a}) = \mathfrak{a}^{\otimes j}$ ,  $T(\mathfrak{a}) = \bigoplus_{j=0}^{\infty} T^j(\mathfrak{a})$ , and  $T_j(\mathfrak{a}) = \bigoplus_{k=0}^j T^k(\mathfrak{a})$ . Then set  $\mathbf{U}_j(\mathfrak{a})$  to be the image of  $T_j(\mathfrak{a})$  under the canonical surjection  $T(\mathfrak{a}) \rightarrow \mathbf{U}(\mathfrak{a})$ . Then for any  $u \in \mathbf{U}(\mathfrak{a})$  define the degree of  $u$  by

$$\deg u = \min_j \{u \in \mathbf{U}_j(\mathfrak{a})\}$$

## 1.2 The Classical Straightening Lemma and Integral Basis

The following classical straightening lemma, which is necessary for the proof of Cartier and Kostant's integral basis, can be easily proved by induction.

**Lemma 1** For all  $r, s \in \mathbb{Z}_{\geq 0}$ , we have, in  $\mathbf{U}(\mathfrak{sl}_2)$

$$(x^+)^{(r)}(x^-)^{(s)} = \sum_{k=0}^{\min(r,s)} (x^-)^{(s-k)} \binom{h-r-s+2k}{k} (x^+)^{(r-k)}$$

Define  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g})$  to be the  $\mathbb{Z}$  subalgebra of  $\mathbf{U}(\mathfrak{g})$  generated by  $\{(x_{\alpha}^{\pm})^{(k)} : \alpha \in R^+, k \in \mathbb{Z}_{\geq 0}\}$ .

Lemma 1 is a key ingredient in the proof of the following theorem, which gives Cartier and Kostant's integral form and basis

**Theorem 2** (B. Kostant, [7])  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g})$  is an integral form for  $\mathbf{U}(\mathfrak{g})$  and the set of all

$$\left(x_{\beta_1}^-\right)^{(r_1)} \cdots \left(x_{\beta_m}^-\right)^{(r_m)} \begin{pmatrix} h_1 \\ s_1 \end{pmatrix} \cdots \begin{pmatrix} h_n \\ s_n \end{pmatrix} \left(x_{\beta_1}^+\right)^{(t_1)} \cdots \left(x_{\beta_m}^+\right)^{(t_m)},$$

where  $(r_1, \dots, r_m), (t_1, \dots, t_m) \in \mathbb{Z}_{\geq 0}^{\times m}$  and  $(s_1, \dots, s_n) \in \mathbb{Z}_{\geq 0}^{\times n}$ , is a  $\mathbb{Z}$ -basis for  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g})$ .



### 1.3 Garland's Straightening Lemma and Integral Basis for the Loop Algebra

**Definition.** Let  $\mathfrak{a}$  be a Lie algebra, over  $\mathbb{C}$ , with Lie bracket  $[\cdot, \cdot]_{\mathfrak{a}}$ . The *loop algebra* of  $\mathfrak{a}$  is the vector space  $\mathfrak{a} \otimes \mathbb{C}[t, t^{-1}]$  with lie bracket determined by

$$[z \otimes f, z' \otimes g] = [z, z']_{\mathfrak{a}} \otimes fg, \quad \forall z, z' \in \mathfrak{g}, f, g \in \mathbb{C}[t, t^{-1}]$$

In order to extend Cartier and Kostant's integral forms and bases to the loop algebra case H. Garland needed a straightening identity, which allows one to write any product of the form  $(x^+ \otimes 1)^{(r)}(x^- \otimes t)^{(s)}$ , in  $\mathbf{U}(\mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}])$ , in PBW order, [6].

#### 1.3.1 Garland's $p_{\alpha}(r, s)$

The construction of Garland's identity requires one to generalize the binomial coefficients from Cartier and Kostant's straightening identity.

To that end given  $\alpha \in R^+$ , recursively define  $p_{\alpha} : \mathbb{Z}^+ \times \mathbb{Z} \rightarrow \mathbf{U}(h_{\alpha} \otimes \mathbb{C}[t^{\pm 1}])$  by  $p_{\alpha}(0, s) = 1$ , and for  $r > 0$  recursively define

$$p_{\alpha}(r, s) = -\frac{1}{r} \sum_{j=1}^r (h_{\alpha} \otimes t^{sj}) p_{\alpha}(r-j, s)$$

It is easy to show that  $p_{\alpha}(r, s)$  is the coefficient of  $u^r$  in the following power series,

$$\exp \left( - \sum_{r=1}^{\infty} \frac{h_{\alpha} \otimes t^{rs}}{r} u^r \right)$$

When  $\mathfrak{g} = \mathfrak{sl}_2$  the  $\alpha$  is omitted and when  $\alpha = \alpha_i$  we write  $p_i$  for  $p_{\alpha_i}$ .

### 1.3.2 Garland's Linear Maps and Straightening Identity

The final ingredients in the statement of Garland's straightening identity are the following linear maps.

Given an indeterminate  $\xi$ , and  $k \in \mathbb{Z}_{\geq 0}$  define linear maps  $D_k^\pm : \mathbb{C}[\xi] \rightarrow \mathbf{U}(\mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}])$  by

$$\begin{aligned} D_k^\pm(1) &= \delta_{k,0} \\ D_k^+(\xi) &= x^+ \otimes t^k \\ D_k^-(\xi) &= x^- \otimes t^{k+1} \\ D_k^\pm(\xi^m) &= \sum_{j=0}^k D_j^\pm(\xi) D_{k-j}^\pm(\xi^{m-1}) \end{aligned}$$

Given  $u \in \mathbf{U}(\mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}])$  define  $L_u$  to be left multiplication by  $u$ . Then define

$$\mathbb{D}_k = \sum_{j=0}^k L_{p(k-j,1)} D_j^+$$

Then we have Garland's straightening identity in  $\mathbf{U}(\mathfrak{sl}_2 \otimes \mathbb{C}[t^{\pm 1}])$ , [4], [6].

**Lemma 3** For all  $r, s \in \mathbb{Z}_{\geq 0}$ ,

$$(x^+ \otimes 1)^{(r)} (x^- \otimes t)^{(s)} = \sum_{j=0}^{\min(r,s)} (-1)^j \sum_{\substack{k+l=j \\ k,l \geq 0}} D_k^- \left( \xi^{(r-j)} \right) \mathbb{D}_l \left( \xi^{(s-j)} \right)$$

### 1.3.3 Garland's Integral Basis

In order to state Garland's integral basis theorem we will need to introduce multisets. They will also play a large role in our straightening lemma and integral basis.

**Definition.** Given any set  $S$  define a *multiset of elements of  $S$*  to be a multiplicity function

$\chi : S \rightarrow \mathbb{Z}_{\geq 0}$ . A multiset is finite if it has finite support. Define  $\mathcal{F}(S) = \{\chi : S \rightarrow \mathbb{Z}_{\geq 0} \mid \text{supp } \chi < \infty\}$ .

Define  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}])$  be the  $\mathbb{Z}$ -subalgebra of  $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}])$  generated by

$$\left\{ \left( x_{\alpha}^{\pm} \otimes t^k \right)^{(r)} \mid \alpha \in R^+, k \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0} \right\}$$

Define functions  $f_{\pm} : \mathcal{F}(\mathbb{Z})^{\times m} \rightarrow \mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}])$  and  $f_0 : \mathcal{F}(\mathbb{Z})^{\times n} \rightarrow \mathbf{U}(\mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}])$  by

$$f_{\pm}(\psi_1, \dots, \psi_m) = \prod_{k \in \mathbb{Z}} \left( x_{\beta_1}^{\pm} \otimes t^k \right)^{(\psi_1(k))} \cdots \prod_{k \in \mathbb{Z}} \left( x_{\beta_m}^{\pm} \otimes t^k \right)^{(\psi_m(k))}$$

$$f_0(\phi_1, \dots, \phi_n) = \prod_{k \in \mathbb{Z}} p_1(\phi_1(k), k) \cdots p_n(\phi_n(k), k)$$

Lemma 3 is basic for Garland's Integral Basis Theorem.

**Theorem 4** (*H. Garland, [6]*)  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}])$  is an integral form for  $\mathbf{U}(\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}])$  and the set of all

$$f_{-}(\psi_1, \dots, \psi_m) f_0(\phi_1, \dots, \phi_n) f_{+}(\psi'_1, \dots, \psi'_m)$$

is a  $\mathbb{Z}$ -basis for  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}])$ .

## Chapter 2

# Integral Bases for the Map Algebras

### 2.1 Preliminaries

#### 2.1.1 Map Algebras

**Definition.** Fix  $A$  a commutative associative algebra with unity over  $\mathbb{C}$ . Let  $\mathfrak{a}$  be a Lie algebra, over  $\mathbb{C}$ , with Lie bracket  $[\cdot, \cdot]_{\mathfrak{a}}$ . The *map algebra* of  $\mathfrak{a}$  is the vector space  $\mathfrak{a} \otimes A$ , with Lie bracket given by

$$[z \otimes a, z' \otimes b] = [z, z']_{\mathfrak{a}} \otimes ab, \quad z, z' \in \mathfrak{a}, \quad a, b \in A.$$

$\mathfrak{a}$  can be embedded in this Lie algebra as  $\mathfrak{a} \otimes 1$ .

Note that by the PBW Theorem we have a vector space isomorphism

$$\mathbf{U}(\mathfrak{g} \otimes A) \cong \mathbf{U}(\mathfrak{n}^+ \otimes A) \otimes \mathbf{U}(\mathfrak{h} \otimes A) \otimes \mathbf{U}(\mathfrak{n}^- \otimes A)$$

For each  $\alpha \in R^+$ , let  $\Omega_\alpha : \mathbf{U}(\mathfrak{sl}_2 \otimes A) \rightarrow \mathbf{U}(\mathfrak{g} \otimes A)$  be the algebra homomorphism defined by

$$x^\pm \otimes a \mapsto x_\alpha^\pm \otimes a \quad h \otimes a \mapsto h_\alpha \otimes a$$

### 2.1.2 Multisets

For  $\chi \in \mathcal{F}(S)$  define

$$|\chi| = \sum_{s \in S} \chi(s)$$

Notice that  $\mathcal{F}(S)$  is an abelian monoid under function addition. Define a partial order on  $\mathcal{F}(S)$  so that for  $\psi, \chi \in \mathcal{F}(S)$ ,  $\psi \leq \chi$  if  $\psi(s) \leq \chi(s)$  for all  $s \in S$ . Given pairs  $(\chi, \psi) \in \mathcal{F}(S) \times \mathcal{F}(S)$  with  $\chi \geq \psi$  for all  $s \in S$ , we define  $\chi - \psi$  by standard function subtraction. Define

$$\mathcal{F} = \{\chi : A \rightarrow \mathbb{Z}_{\geq 0} \mid \text{supp } \chi < \infty\}, \quad \mathcal{F}_k = \{\chi \in \mathcal{F} : |\chi| = k\}$$

and given  $\chi \in \mathcal{F}$  define

$$\mathcal{F}(\chi) = \{\psi \in \mathcal{F} : \psi \leq \chi\}, \quad \mathcal{F}_k(\chi) = \{\psi \in \mathcal{F}(\chi) : |\psi| = k\}$$

Also define a function  $\pi : \mathcal{F} \rightarrow A$  by

$$\psi \mapsto \prod_{a \in A} a^{\psi(a)}$$

## 2.2 An Integral Form and Integral Basis

In section we will define our integral form and give our integral basis for the map algebra,  $\mathfrak{g} \otimes A$ .

### 2.2.1 Definition of $p(\varphi, \chi)$

Given  $\varphi, \chi \in \mathcal{F}$ , Recursively define functions  $p : \mathcal{F}^2 \rightarrow \mathbf{U}(\mathfrak{h} \otimes A) \subset \mathbf{U}(\mathfrak{sl}_2 \otimes A)$  by  $p(0, 0) = 1$  and for  $\varphi, \chi \in \mathcal{F} - \{0\}$ ,

$$p(\varphi, \chi) = -\frac{\delta_{|\varphi|, |\chi|}}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \psi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\psi_1|!)^2 (h \otimes \pi(\psi_1)\pi(\psi_2)) p(\varphi - \psi_1, \chi - \psi_2)$$

Define  $p(\chi) = p(\chi, |\chi|\chi_1)$  and, for all  $\alpha \in R^+$ ,  $p_\alpha(\chi) = \Omega_\alpha(p(\chi)) \in \mathbf{U}(\mathfrak{h} \otimes A) \subset \mathbf{U}(\mathfrak{g} \otimes A)$ .

**Remark 5** *Note the following:*

(a)  $p(\varphi, \chi) = 0$  if  $|\varphi| \neq |\chi|$

(b) The  $p(\varphi, \chi)$  are a generalization of Garland's  $p(r, s)$  because  $p(r\chi_1, r\chi_{ts}) = p(r, s)$ , where  $\chi_a$  is the characteristic function on  $a \in A$ .

### 2.2.2 An Integral Form and Integral Basis

If  $A$  has a  $\mathbb{C}$ -basis,  $\mathbf{B}$ , which is closed under multiplication define the  $\mathbb{Z}$ -form of  $\mathfrak{g} \otimes A$ ,  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A)$ , to be the  $\mathbb{Z}$ -subalgebra of  $\mathbf{U}(\mathfrak{g} \otimes A)$  generated by

$$\left\{ (x_\alpha^\pm \otimes b)^{(r)} : \alpha \in R^+, b \in \mathbf{B}, r \in \mathbb{Z}_{\geq 0} \right\}$$

For all  $\alpha \in R_+$ , and  $\chi \in \mathcal{F}$ , define

$$x_\alpha^\pm(\chi) = \prod_{a \in A} (x_\alpha^\pm \otimes a)^{(\chi(a))}$$

Fix an order on  $R^+ = \{\beta_1, \dots, \beta_m\}$ . Then define functions  $f^\pm : \mathcal{F}(\mathbf{B})^{\times m} \rightarrow \mathbf{U}(\mathfrak{g} \otimes A)$  by

$$(\psi_1, \dots, \psi_m) \mapsto x_{\beta_1}^\pm(\psi_1)x_{\beta_2}^\pm(\psi_2) \cdots x_{\beta_m}^\pm(\psi_m)$$

and  $f^0 : \mathcal{F}(\mathbf{B})^{\times n} \rightarrow \mathbf{U}(\mathfrak{h} \otimes A)$  by

$$(\psi_1, \dots, \psi_n) \mapsto p_1(\psi_1)p_2(\psi_2) \cdots p_n(\psi_n)$$

Define  $\mathcal{B} = \{f^-(\underline{\psi})f^0(\underline{\chi})f^+(\underline{\psi}') \mid \underline{\psi}, \underline{\psi}' \in \mathcal{F}(\mathbf{B})^{\times m}, \underline{\chi} \in \mathcal{F}(\mathbf{B})^{\times n}\}$ . Define  $\mathcal{B}_\pm$  to be the set consisting of all  $f^\pm(\underline{\psi})$ , and  $\mathcal{B}_0$  to be the set consisting of all  $f^0(\underline{\chi})$ .

**Theorem 6**  $\mathcal{B}$  is a  $\mathbb{Z}$ -basis for  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A)$ .

Proposition 15 and the Poincaré-Birkhoff-Witt theorem easily give the following corollary this theorem.

**Corollary 7**  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A)$  is an integral form for  $\mathbf{U}(\mathfrak{g} \otimes A)$ .

The remainder of this chapter is devoted to the proof of Theorem 6. First we show that  $\mathcal{B} \subset \mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A)$ . This allows us to define  $\mathbb{Z}$ -subalgebras  $\mathbf{U}_{\mathbb{Z}}^+(\mathfrak{g} \otimes A)$ ,  $\mathbf{U}_{\mathbb{Z}}^-(\mathfrak{g} \otimes A)$ , and  $\mathbf{U}_{\mathbb{Z}}^0(\mathfrak{g} \otimes A)$ . Then we prove that  $\mathcal{B}_+$ ,  $\mathcal{B}_-$  and  $\mathcal{B}_0$  are  $\mathbb{Z}$ -bases of these subalgebras respectively. Finally, we prove a triangular decomposition  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A) = \mathbf{U}_{\mathbb{Z}}^-(\mathfrak{g} \otimes A)\mathbf{U}_{\mathbb{Z}}^0(\mathfrak{g} \otimes A)\mathbf{U}_{\mathbb{Z}}^+(\mathfrak{g} \otimes A)$ . This means that as  $\mathbb{Z}$ -modules  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A) \cong \mathbf{U}_{\mathbb{Z}}^-(\mathfrak{g} \otimes A) \otimes \mathbf{U}_{\mathbb{Z}}^0(\mathfrak{g} \otimes A) \otimes \mathbf{U}_{\mathbb{Z}}^+(\mathfrak{g} \otimes A)$ . The theorem follows.

## 2.3 The Subalgebras $\mathbf{U}_{\mathbb{Z}}^\pm(\mathfrak{g} \otimes A)$

Define  $\mathbf{U}_{\mathbb{Z}}^\pm(\mathfrak{g} \otimes A)$  to be the  $\mathbb{Z}$ -subalgebra of  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A)$  generated by

$$\left\{ (x_\alpha^\pm \otimes b)^{(r)} : \alpha \in R^+, b \in \mathbf{B}, r \in \mathbb{Z}_{\geq 0} \right\}.$$

### 2.3.1 $\mathcal{B}_\pm$ is a $\mathbb{Z}$ -Basis for $U_{\mathbb{Z}}^\pm(\mathfrak{g} \otimes A)$

**Lemma 8** *Let  $a, b \in \mathbf{B}$ ,  $r, s \in \mathbb{Z}_{\geq 0}$ , and  $\alpha, \beta \in R^+$  be given. Define*

*$R_{\alpha, \beta} = \{i\alpha + j\beta : i, j \in \mathbb{Z}\} \cap R$ . Then the following identities hold:*

(a) *If  $R_{\alpha, \beta}$  is of type  $A_2$*

$$(x_\alpha^\pm \otimes a)^{(r)} (x_\beta^\pm \otimes b)^{(s)} = \sum_{k=0}^{\min(r,s)} \varepsilon_k (x_\beta^\pm \otimes b)^{(s-k)} (x_{\alpha+\beta}^\pm \otimes ab)^{(k)} (x_\alpha^\pm \otimes a)^{(r-k)}$$

where  $\varepsilon_k \in \{1, -1\}$ , for all  $k$ .

(b) *If  $R_{\alpha, \beta}$  is of type  $B_2$ , then*

$$(x_\alpha^\pm \otimes a)^{(r)} (x_\beta^\pm \otimes b)^{(s)} = \sum \varepsilon_{k_1, k_2} (x_\beta^\pm \otimes b)^{(s-k_1-k_2)} \prod_{j=1}^2 (x_{j\alpha+\beta}^\pm \otimes a^j b)^{(k_j)} \\ \times (x_\alpha^\pm \otimes a)^{(r-k_1-2k_2)}$$

where the sum is over all  $k_1, k_2 \in \mathbb{Z}_{\geq 0}$  such that  $k_1 + k_2 \leq s$  and  $k_1 + 2k_2 \leq r$ , and

$\varepsilon_{k_1, k_2} \in \{1, -1\}$ , for all  $k_1, k_2$ .

(c) *If  $R_{\alpha, \beta}$  is of type  $G_2$ , then*

$$(x_\alpha^\pm \otimes a)^{(r)} (x_\beta^\pm \otimes b)^{(s)} = \sum \varepsilon_{k_1, k_2, k_3, k_4} (x_\beta^\pm \otimes b)^{(s-\sum_{j=1}^3 k_j - 2k_4)} \prod_{j=1}^3 (x_{j\alpha+\beta}^\pm \otimes a^j b)^{(k_j)} \\ \times (x_{3\alpha+2\beta}^\pm \otimes a^3 b^2)^{(k_4)} (x_\alpha^\pm \otimes a)^{(r-\sum_{j=1}^3 jk_j - 3k_4)}$$

where the sum is over all  $k_1, k_2, k_3, k_4 \in \mathbb{Z}_{\geq 0}$  such that  $k_1 + k_2 + k_3 + 2k_4 \leq s$  and  $k_1 + 2k_2 +$

$3k_3 + 3k_4 \leq r$ , and  $\varepsilon_{k_1, k_2, k_3, k_4} \in \{1, -1\}$ , for all  $k_1, k_2, k_3, k_4$ .

**Remark 9** *In case (a)  $\varepsilon_k = (\pm m)^k$  where  $m$  is given by  $[x_\alpha^+, x_\beta^+] = mx_{\alpha+\beta}^+$ .*

*In case (b),  $\varepsilon_{k_1, k_2} = (\pm m_1)^{k_1} (m_1 m_2)^{k_2}$  where  $m_1, m_2$  are given by  $[x_\alpha^+, x_\beta^+] = m_1 x_{\alpha+\beta}^+$  and  $[x_\alpha^+, x_{\alpha+\beta}^+] = 2m_2 x_{2\alpha+\beta}^+$ .*



In case (c),  $\varepsilon_{k_1, k_2, k_3, k_4} = (\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + k_3 + k_4} m_3^{k_3} m_4^{k_4}$  where  $m_1, m_2$  are the same as in case (b), and  $m_3, m_4$  are given by  $[x_\alpha^+, x_{2\alpha+\beta}^+] = 3m_3 x_{3\alpha+\beta}^+$  and  $[x_{2\alpha+\beta}^+, x_{\alpha+\beta}^+] = 3m_4 x_{3\alpha+2\beta}^+$ .

Before proving the lemma we will state and prove the following corollary.

**Corollary 10**  $\mathcal{B}_\pm$  is a  $\mathbb{Z}$ -basis for  $\mathbf{U}_{\mathbb{Z}}^\pm(\mathfrak{g} \otimes A)$ .

**Proof.** It will suffice to show that any product of elements of the set  $\{(x_\alpha^\pm \otimes b) : \alpha \in R^+, b \in \mathbf{B}\}$  is in the  $\mathbb{Z}$ -span of  $\mathcal{B}_\pm$ . Clearly, for all  $\alpha \in R^+$  and  $b \in \mathbf{B}$ ,

$$(x_\alpha^\pm \otimes b)^{(r)} (x_\alpha^\pm \otimes b)^{(s)} = \binom{r+s}{r} (x_\alpha^\pm \otimes b)^{(r+s)}$$

So it will suffice to show if  $\alpha, \beta \in R^+$ ,  $a, b \in \mathbf{B}$  and  $r, s \in \mathbb{Z}_{\geq 0}$  then  $[(x_\alpha^\pm \otimes a)^{(r)}, (x_\beta^\pm \otimes b)^{(s)}]$  is in the  $\mathbb{Z}$ -span of  $\mathcal{B}_\pm$  and has degree less than  $r + s$ . If  $\alpha + \beta \notin R^+$  this claim is trivially true. If  $\alpha + \beta \in R^+$  this claim can be shown by induction on  $r + s$ . If  $r + s \leq 1$  the claim is trivially true. The inductive step is true by the lemma. ■

**Proof of Lemma 8.** In each case the lemma will be proved by induction on  $s$ . If  $s = 0$  the lemma is trivially true. If  $s = 1$  we will prove all three parts by induction on  $r$ . In all three parts the lemma is easily verified if  $s = 1$  and  $r \leq 3$ . Assume all three parts for  $s = 1$  and  $r \geq 3$ .

Then for part (a)

$$\begin{aligned}
(r+1)(x_\alpha^\pm \otimes a)^{(r+1)}(x_\beta^\pm \otimes b) &= (x_\alpha^\pm \otimes a)(x_\alpha^\pm \otimes a)^{(r)}(x_\beta^\pm \otimes b) \\
&= \sum_{k=0}^1 (\pm m)^k (x_\alpha^\pm \otimes a)(x_\beta^\pm \otimes b)^{(1-k)}(x_{\alpha+\beta}^\pm \otimes ab)^{(k)} \\
&\times (x_\alpha^\pm \otimes a)^{(r-k)} \quad (\text{By the induction hypothesis}) \\
&= (x_\alpha^\pm \otimes a)(x_\beta^\pm \otimes b)(x_\alpha^\pm \otimes a)^{(r)} \\
&\pm m(x_\alpha^\pm \otimes a)(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r-1)} \\
&= (r+1)(x_\beta^\pm \otimes b)(x_\alpha^\pm \otimes a)^{(r+1)} \\
&\pm m(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r)} \\
&\pm rm(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r)} \\
&= (r+1)\left((x_\beta^\pm \otimes b)(x_\alpha^\pm \otimes a)^{(r+1)}\right. \\
&\left. \pm m(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r)}\right)
\end{aligned}$$

For part (b)

$$\begin{aligned}
(r+1)(x_\alpha^\pm \otimes a)^{(r+1)}(x_\beta^\pm \otimes b) &= (x_\alpha^\pm \otimes a)(x_\alpha^\pm \otimes a)^{(r)}(x_\beta^\pm \otimes b) \\
&= \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq 1 \\ k_1 + 2k_2 \leq r}} (\pm m_1)^{k_1 + k_2} (\pm m_2)^{k_2} (x_\alpha^\pm \otimes a) \\
&\times (x_\beta^\pm \otimes b)^{(1 - k_1 - k_2)} (x_{\alpha+\beta}^\pm \otimes ab)^{(k_1)} (x_{2\alpha+\beta}^\pm \otimes a^2b)^{(k_2)} \\
&\times (x_\alpha^\pm \otimes a)^{(r - k_1 - 2k_2)} \quad (\text{By the induction hypothesis}) \\
&= (x_\alpha^\pm \otimes a)(x_\beta^\pm \otimes b)(x_\alpha^\pm \otimes a)^{(r)} \\
&\pm m_1(x_\alpha^\pm \otimes a)(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r-1)} \\
&+ m_1m_2(x_\alpha^\pm \otimes a)(x_{2\alpha+\beta}^\pm \otimes a^2b)(x_\alpha^\pm \otimes a)^{(r-2)} \\
&= (r+1)(x_\beta^\pm \otimes b)(x_\alpha^\pm \otimes a)^{(r+1)} \\
&\pm m_1(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r)} \\
&\pm rm_1(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r)} \\
&+ 2m_1m_2(x_{2\alpha+\beta}^\pm \otimes a^2b)(x_\alpha^\pm \otimes a)^{(r-1)} \\
&+ (r-1)m_1m_2(x_{2\alpha+\beta}^\pm \otimes a^2b)(x_\alpha^\pm \otimes a)^{(r-1)} \\
&= (r+1)\left((x_\beta^\pm \otimes b)(x_\alpha^\pm \otimes a)^{(r+1)}\right. \\
&\pm m_1(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r)} \\
&\left.+ m_1m_2(x_{2\alpha+\beta}^\pm \otimes a^2b)(x_\alpha^\pm \otimes a)^{(r-1)}\right)
\end{aligned}$$

For part (c),

$$\begin{aligned}
(r+1)(x_\alpha^\pm \otimes a)^{(r+1)}(x_\beta^\pm \otimes b) &= (x_\alpha^\pm \otimes a)(x_\alpha^\pm \otimes a)^{(r)}(x_\beta^\pm \otimes b) \\
&= \sum_{\substack{k_1, k_2, k_3 \geq 0 \\ k_1 + k_2 + k_3 \leq 1 \\ k_1 + 2k_2 + 3k_3 \leq r}} (\pm 1)^{k_1} m_1^{k_2} (\pm m_1 m_2)^{k_3} (x_\alpha^\pm \otimes a) \\
&\times (x_\beta^\pm \otimes b)^{(1-k_1-k_2-k_3)} \prod_{j=1}^3 m_j^{k_j} (x_{j\alpha+\beta}^\pm \otimes a^j b)^{(k_j)} \\
&\times (x_\alpha^\pm \otimes a)^{(r-k_1-2k_2-3k_3)} \\
&\quad \text{(By the induction hypothesis)} \\
&= (x_\alpha^\pm \otimes a)(x_\beta^\pm \otimes b)(x_\alpha^\pm \otimes a)^{(r)} \\
&\pm m_1(x_\alpha^\pm \otimes a)(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r-1)} \\
&+ m_1 m_2(x_\alpha^\pm \otimes a)(x_{2\alpha+\beta}^\pm \otimes a^2 b)(x_\alpha^\pm \otimes a)^{(r-2)} \\
&\pm m_1 m_2 m_3(x_\alpha^\pm \otimes a)(x_{3\alpha+\beta}^\pm \otimes a^3 b)(x_\alpha^\pm \otimes a)^{(r-3)} \\
&= (r+1)(x_\beta^\pm \otimes b)(x_\alpha^\pm \otimes a)^{(r+1)} \\
&\pm m_1(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r)} \\
&\pm r m_1(x_{\alpha+\beta}^\pm \otimes ab)(x_\alpha^\pm \otimes a)^{(r)} \\
&+ 2m_1 m_2(x_{2\alpha+\beta}^\pm \otimes a^2 b)(x_\alpha^\pm \otimes a)^{(r-1)} \\
&+ (r-1)m_1 m_2(x_{2\alpha+\beta}^\pm \otimes a^2 b)(x_\alpha^\pm \otimes a)^{(r-1)} \\
&\pm 3m_1 m_2 m_3(x_{3\alpha+\beta}^\pm \otimes a^3 b)(x_\alpha^\pm \otimes a)^{(r-2)} \\
&\pm (r-2)m_1 m_2 m_3(x_{3\alpha+\beta}^\pm \otimes a^3 b)(x_\alpha^\pm \otimes a)^{(r-2)} \\
&= (r+1)\left((x_\beta^\pm \otimes b)(x_\alpha^\pm \otimes a)^{(r+1)}\right. \\
&\left.+ \sum_{j=1}^3 (\pm 1)^j \prod_{k=1}^j m_k (x_{j\alpha+\beta}^\pm \otimes a^j b)(x_\alpha^\pm \otimes a)^{(r+1-j)}\right)
\end{aligned}$$

So the claim is true in each case for  $s = 1$ . Proceed by induction on  $s \geq 1$ . Assume the lemma for some  $s \geq 1$  then in the (a) case

$$\begin{aligned}
(s+1) (x_\alpha^\pm \otimes a)^{(r)} (x_\beta^\pm \otimes b)^{(s+1)} &= (x_\alpha^\pm \otimes a)^{(r)} (x_\beta^\pm \otimes b) (x_\beta^\pm \otimes b)^{(s)} \\
&= (x_\beta^\pm \otimes b) (x_\alpha^\pm \otimes a)^{(r)} (x_\beta^\pm \otimes b)^{(s)} \\
&\pm m (x_{\alpha+\beta}^\pm \otimes ab) (x_\alpha^\pm \otimes a)^{(r-1)} (x_\beta^\pm \otimes b)^{(s)} \\
&= \sum_{k=0}^{\min(r,s)} (\pm m)^k (s+1-k) (x_\beta^\pm \otimes b)^{(s+1-k)} \\
&\times (x_{\alpha+\beta}^\pm \otimes ab)^{(k)} (x_\alpha^\pm \otimes a)^{(r-k)} \\
&\pm \sum_{k=0}^{\min(r-1,s)} (\pm m)^k (k+1) (x_\beta^\pm \otimes b)^{(s-k)} \\
&\times (x_{\alpha+\beta}^\pm \otimes ab)^{(k+1)} (x_\alpha^\pm \otimes a)^{(r-1-k)} \\
&\text{(By the induction hypothesis)} \\
&= \sum_{k=0}^{\min(r,s)} (\pm m)^k (s+1-k) (x_\beta^\pm \otimes b)^{(s+1-k)} \\
&\times (x_{\alpha+\beta}^\pm \otimes ab)^{(k)} (x_\alpha^\pm \otimes a)^{(r-k)} \\
&+ \sum_{k=1}^{\min(r-1,s)+1} (\pm m)^k k (x_\beta^\pm \otimes b)^{(s+1-k)} \\
&\times (x_{\alpha+\beta}^\pm \otimes ab)^{(k)} (x_\alpha^\pm \otimes a)^{(r-k)} \\
&= (s+1) \sum_{k=0}^{\min(r,s+1)} (\pm m)^k (x_\beta^\pm \otimes b)^{(s+1-k)} \\
&\times (x_{\alpha+\beta}^\pm \otimes ab)^{(k)} (x_\alpha^\pm \otimes a)^{(r-k)}
\end{aligned}$$

In the (b) case we have

$$\begin{aligned}
(s+1) (x_\alpha^\pm \otimes a)^{(r)} (x_\beta^\pm \otimes b)^{(s+1)} &= (x_\alpha^\pm \otimes a)^{(r)} (x_\beta^\pm \otimes b) (x_\beta^\pm \otimes b)^{(s)} \\
&= (x_\beta^\pm \otimes b) (x_\alpha^\pm \otimes a)^{(r)} (x_\beta^\pm \otimes b)^{(s)} \\
&\pm m_1 (x_{\alpha+\beta}^\pm \otimes ab) (x_\alpha^\pm \otimes a)^{(r-1)} (x_\beta^\pm \otimes b)^{(s)} \\
&+ m_1 m_2 (x_{2\alpha+\beta}^\pm \otimes a^2 b) (x_\alpha^\pm \otimes a)^{(r-2)} (x_\beta^\pm \otimes b)^{(s)} \\
&= \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s \\ k_1 + 2k_2 \leq r}} (\pm m_1)^{k_1} (m_1 m_2)^{k_2} (s+1 - k_1 - k_2) \\
&\times (x_\beta^\pm \otimes b)^{(s+1-k_1-k_2)} (x_{\alpha+\beta}^\pm \otimes ab)^{(k_1)} \\
&\times (x_{2\alpha+\beta}^\pm \otimes a^2 b)^{(k_2)} (x_\alpha^\pm \otimes a)^{(r-k_1-2k_2)} \\
&+ \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s \\ k_1 + 2k_2 \leq r-1}} (\pm m_1)^{k_1+1} (m_1 m_2)^{k_2} (k_1 + 1) \\
&\times (x_\beta^\pm \otimes b)^{(s-k_1-k_2)} (x_{\alpha+\beta}^\pm \otimes ab)^{(k_1+1)} \\
&\times (x_{2\alpha+\beta}^\pm \otimes a^2 b)^{(k_2)} (x_\alpha^\pm \otimes a)^{(r-1-k_1-2k_2)} \\
&+ \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s \\ k_1 + 2k_2 \leq r-2}} (\pm m_1)^{k_1} (m_1 m_2)^{k_2+1} (k_2 + 1) \\
&\times (x_\beta^\pm \otimes b)^{(s-k_1-k_2)} (x_{\alpha+\beta}^\pm \otimes ab)^{(k_1)} \\
&\times (x_{2\alpha+\beta}^\pm \otimes a^2 b)^{(k_2+1)} (x_\alpha^\pm \otimes a)^{(r-2-k_1-2k_2)}
\end{aligned}$$

(by the induction hypothesis)

$$\begin{aligned}
&= \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s+1 \\ k_1 + 2k_2 \leq r}} (\pm m_1)^{k_1} (m_1 m_2)^{k_2} (s+1-k_1-k_2) \left(x_\beta^\pm \otimes b\right)^{(s+1-k_1-k_2)} \left(x_{\alpha+\beta}^\pm \otimes ab\right)^{(k_1)} \\
&\times \left(x_{2\alpha+\beta}^\pm \otimes a^2 b\right)^{(k_2)} \left(x_\alpha^\pm \otimes a\right)^{(r-k_1-2k_2)} \\
&+ \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s+1 \\ k_1 + 2k_2 \leq r}} (\pm m_1)^{k_1} (m_1 m_2)^{k_2} k_1 \left(x_\beta^\pm \otimes b\right)^{(s+1-k_1-k_2)} \left(x_{\alpha+\beta}^\pm \otimes ab\right)^{(k_1)} \\
&\times \left(x_{2\alpha+\beta}^\pm \otimes a^2 b\right)^{(k_2)} \left(x_\alpha^\pm \otimes a\right)^{(r-k_1-2k_2)} \\
&+ \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s+1 \\ k_1 + 2k_2 \leq r}} (\pm m_1)^{k_1} (m_1 m_2)^{k_2} k_2 \left(x_\beta^\pm \otimes b\right)^{(s+1-k_1-k_2)} \left(x_{\alpha+\beta}^\pm \otimes ab\right)^{(k_1)} \\
&\times \left(x_{2\alpha+\beta}^\pm \otimes a^2 b\right)^{(k_2)} \left(x_\alpha^\pm \otimes a\right)^{(r-k_1-2k_2)} \\
&= (s+1) \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s+1 \\ k_1 + 2k_2 \leq r}} (\pm m_1)^{k_1} (m_1 m_2)^{k_2} \left(x_\beta^\pm \otimes b\right)^{(s+1-k_1-k_2)} \left(x_{\alpha+\beta}^\pm \otimes ab\right)^{(k_1)} \\
&\times \left(x_{2\alpha+\beta}^\pm \otimes a^2 b\right)^{(k_2)} \left(x_\alpha^\pm \otimes a\right)^{(r-k_1-2k_2)}
\end{aligned}$$

Finally for the (c) case

$$\begin{aligned}
(s+1) \left(x_\alpha^\pm \otimes a\right)^{(r)} \left(x_\beta^\pm \otimes b\right)^{(s+1)} &= \left(x_\beta^\pm \otimes b\right) \left(x_\alpha^\pm \otimes a\right)^{(r)} \left(x_\beta^\pm \otimes b\right)^{(s)} \\
&\pm m_1 \left(x_{\alpha+\beta}^\pm \otimes ab\right) \left(x_\alpha^\pm \otimes a\right)^{(r-1)} \left(x_\beta^\pm \otimes b\right)^{(s)} \\
&+ m_1 m_2 \left(x_{2\alpha+\beta}^\pm \otimes a^2 b\right) \left(x_\alpha^\pm \otimes a\right)^{(r-2)} \left(x_\beta^\pm \otimes b\right)^{(s)} \\
&\pm m_1 m_2 m_3 \left(x_{3\alpha+\beta}^\pm \otimes a^3 b\right) \left(x_\alpha^\pm \otimes a\right)^{(r-3)} \left(x_\beta^\pm \otimes b\right)^{(s)}
\end{aligned}$$

So by the induction hypothesis we have

$$\begin{aligned}
& \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r}} \left( s + 1 - \sum_{j=1}^3 k_j - 2k_4 \right) (\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left( x_\beta^\pm \otimes b \right)^{(s+1-k_1-k_2-k_3-2k_4)} \prod_{j=1}^3 \left( x_{j\alpha+\beta}^\pm \otimes a^j b \right)^{(k_j)} \left( x_{3\alpha+2\beta}^\pm \otimes a^3 b^2 \right)^{(k_4)} \\
& \times \left( x_\alpha^\pm \otimes a \right)^{(r-k_1-2k_2-3k_3-3k_4)} \\
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-1}} (k_1 + 1) (\pm 1)^{k_1 + 1 + k_3} m_1^{k_1 + 1 + k_4} (m_1 m_2)^{k_2 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left( x_\beta^\pm \otimes b \right)^{(s-k_1-k_2-k_3-2k_4)} \left( x_{\alpha+\beta}^\pm \otimes ab \right)^{(k_1+1)} \left( x_{2\alpha+\beta}^\pm \otimes a^2 b \right)^{(k_2)} \left( x_{3\alpha+\beta}^\pm \otimes a^3 b \right)^{(k_3)} \\
& \times \left( x_{3\alpha+2\beta}^\pm \otimes a^3 b^2 \right)^{(k_4)} \left( x_\alpha^\pm \otimes a \right)^{(r-1-k_1-2k_2-3k_3-3k_4)} \\
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-2}} (\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + 1 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left( x_\beta^\pm \otimes b \right)^{(s-k_1-k_2-k_3-2k_4)} \left( x_{2\alpha+\beta}^\pm \otimes a^2 b \right) \\
& \times \left( x_{\alpha+\beta}^\pm \otimes ab \right)^{(k_1)} \prod_{j=2}^3 \left( x_{j\alpha+\beta}^\pm \otimes a^j b \right)^{(k_j)} \left( x_{3\alpha+2\beta}^\pm \otimes a^3 b^2 \right)^{(k_4)} \\
& \times \left( x_\alpha^\pm \otimes a \right)^{(r-2-k_1-2k_2-3k_3-3k_4)} \\
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-3}} (\pm 1)^{k_1 + k_3 + 1} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + 1 + k_3 + k_4} m_3^{k_3 + 1} m_4^{k_4} \left( x_{3\alpha+\beta}^\pm \otimes a^3 b \right) \\
& \times \left( x_\beta^\pm \otimes b \right)^{(s-k_1-k_2-k_3-2k_4)} \prod_{j=1}^3 \left( x_{j\alpha+\beta}^\pm \otimes a^j b \right)^{(k_j)} \left( x_{3\alpha+2\beta}^\pm \otimes a^3 b^2 \right)^{(k_4)} \\
& \times \left( x_\alpha^\pm \otimes a \right)^{(r-3-k_1-2k_2-3k_3-3k_4)}
\end{aligned}$$



This last expression gives

$$\begin{aligned}
& \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s+1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r}} \left( s + 1 - \sum_{j=1}^3 k_j - 2k_4 \right) (\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left( x_{\beta}^{\pm} \otimes b \right)^{(s+1-k_1-k_2-k_3-2k_4)} \prod_{j=1}^3 \left( x_{j\alpha+\beta}^{\pm} \otimes a^j b \right)^{(k_j)} \left( x_{3\alpha+2\beta}^{\pm} \otimes a^3 b^2 \right)^{(k_4)} \\
& \times \left( x_{\alpha}^{\pm} \otimes a \right)^{(r-k_1-2k_2-3k_3-3k_4)} \\
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-1}} (k_1 + 1) (\pm 1)^{k_1 + 1 + k_3} m_1^{k_1 + 1 + k_4} (m_1 m_2)^{k_2 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left( x_{\beta}^{\pm} \otimes b \right)^{(s-k_1-k_2-k_3-2k_4)} \left( x_{\alpha+\beta}^{\pm} \otimes ab \right)^{(k_1+1)} \left( x_{2\alpha+\beta}^{\pm} \otimes a^2 b \right)^{(k_2)} \left( x_{3\alpha+\beta}^{\pm} \otimes a^3 b \right)^{(k_3)} \\
& \times \left( x_{3\alpha+2\beta}^{\pm} \otimes a^3 b^2 \right)^{(k_4)} \left( x_{\alpha}^{\pm} \otimes a \right)^{(r-1-k_1-2k_2-3k_3-3k_4)} \\
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-2}} (k_2 + 1) (\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + 1 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left( x_{\beta}^{\pm} \otimes b \right)^{(s-k_1-k_2-k_3-2k_4)} \left( x_{\alpha+\beta}^{\pm} \otimes ab \right)^{(k_1)} \left( x_{2\alpha+\beta}^{\pm} \otimes a^2 b \right)^{(k_2+1)} \left( x_{3\alpha+\beta}^{\pm} \otimes a^3 b \right)^{(k_3)} \\
& \times \left( x_{3\alpha+2\beta}^{\pm} \otimes a^3 b^2 \right)^{(k_4)} \left( x_{\alpha}^{\pm} \otimes a \right)^{(r-2-k_1-2k_2-3k_3-3k_4)} \\
& + \sum_{\substack{k_1 \geq 1 \\ k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-2}} 3(k_4 + 1) (\pm 1)^{k_1 + k_3 + 1} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + 1 + k_3 + k_4} m_3^{k_3} m_4^{k_4 + 1} \\
& \times \left( x_{\beta}^{\pm} \otimes b \right)^{(s-k_1-k_2-k_3-2k_4)} \left( x_{\alpha+\beta}^{\pm} \otimes ab \right)^{(k_1-1)} \prod_{j=2}^3 \left( x_{j\alpha+\beta}^{\pm} \otimes a^j b \right)^{(k_j)} \\
& \times \left( x_{3\alpha+2\beta}^{\pm} \otimes a^3 b^2 \right)^{(k_4+1)} \left( x_{\alpha}^{\pm} \otimes a \right)^{(r-2-k_1-2k_2-3k_3-3k_4)}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-3}} (k_3 + 1)(\pm 1)^{k_1 + k_3 + 1} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + 1 + k_3 + k_4} m_3^{k_3 + 1} m_4^{k_4} \\
& \times \left(x_\beta^\pm \otimes b\right)^{(s - k_1 - k_2 - k_3 - 2k_4)} \prod_{j=1}^2 \left(x_{j\alpha + \beta}^\pm \otimes a^j b\right)^{(k_j)} \left(x_{3\alpha + \beta}^\pm \otimes a^3 b\right)^{(k_3 + 1)} \\
& \times \left(x_{3\alpha + 2\beta}^\pm \otimes a^3 b^2\right)^{(k_4)} \left(x_\alpha^\pm \otimes a\right)^{(r - 3 - k_1 - 2k_2 - 3k_3 - 3k_4)} \\
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s-1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-3}} (k_4 + 1)(\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + 1 + k_3 + k_4} m_3^{k_3 + 1} m_4^{k_4} m_5 \\
& \times \left(x_\beta^\pm \otimes b\right)^{(s - k_1 - k_2 - k_3 - 2k_4 - 1)} \prod_{j=1}^3 \left(x_{j\alpha + \beta}^\pm \otimes a^j b\right)^{(k_j)} \left(x_{3\alpha + 2\beta}^\pm \otimes a^3 b^2\right)^{(k_4 + 1)} \\
& \times \left(x_\alpha^\pm \otimes a\right)^{(r - 3 - k_1 - 2k_2 - 3k_3 - 3k_4)}
\end{aligned}$$

where  $\left[x_{3\alpha + \beta}^+, x_\beta^+\right] = m_5 x_{3\alpha + 2\beta}^+$ .

So we have

$$\begin{aligned}
& \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s+1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r}} \left(s + 1 - \sum_{j=1}^3 k_j - 2k_4\right) (\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left(x_\beta^\pm \otimes b\right)^{(s + 1 - k_1 - k_2 - k_3 - 2k_4)} \prod_{j=1}^3 \left(x_{j\alpha + \beta}^\pm \otimes a^j b\right)^{(k_j)} \left(x_{3\alpha + 2\beta}^\pm \otimes a^3 b^2\right)^{(k_4)} \\
& \times \left(x_\alpha^\pm \otimes a\right)^{(r - k_1 - 2k_2 - 3k_3 - 3k_4)} \\
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s+1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r}} k_1 (\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left(x_\beta^\pm \otimes b\right)^{(s + 1 - k_1 - k_2 - k_3 - 2k_4)} \prod_{j=1}^3 \left(x_{j\alpha + \beta}^\pm \otimes a^j b\right)^{(k_j)} \left(x_{3\alpha + 2\beta}^\pm \otimes a^3 b^2\right)^{(k_4)} \\
& \times \left(x_\alpha^\pm \otimes a\right)^{(r - k_1 - 2k_2 - 3k_3 - 3k_4)}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s+1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r}} k_2 (\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left( x_\beta^\pm \otimes b \right)^{(s+1-k_1-k_2-k_3-2k_4)} \prod_{j=1}^3 \left( x_{j\alpha+\beta}^\pm \otimes a^j b \right)^{(k_j)} \left( x_{3\alpha+2\beta}^\pm \otimes a^3 b^2 \right)^{(k_4)} \\
& \times \left( x_\alpha^\pm \otimes a \right)^{(r-k_1-2k_2-3k_3-3k_4)} \\
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s-1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-3}} 3(k_4 + 1) (\pm 1)^{k_1 + k_3} m_1^{k_1 + 1 + k_4} (m_1 m_2)^{k_2 + 1 + k_3 + k_4} m_3^{k_3} m_4^{k_4 + 1} \\
& \times \left( x_\beta^\pm \otimes b \right)^{(s-k_1-k_2-k_3-2k_4-1)} \prod_{j=1}^3 \left( x_{j\alpha+\beta}^\pm \otimes a^j b \right)^{(k_j)} \left( x_{3\alpha+2\beta}^\pm \otimes a^3 b^2 \right)^{(k_4+1)} \\
& \times \left( x_\alpha^\pm \otimes a \right)^{(r-3-k_1-2k_2-3k_3-3k_4)} \\
& + \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s+1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r}} k_3 (\pm 1)^{k_1 + k_3} m_1^{k_1 + k_4} (m_1 m_2)^{k_2 + k_3 + k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left( x_\beta^\pm \otimes b \right)^{(s+1-k_1-k_2-k_3-2k_4)} \prod_{j=1}^3 \left( x_{j\alpha+\beta}^\pm \otimes a^j b \right)^{(k_j)} \left( x_{3\alpha+2\beta}^\pm \otimes a^3 b^2 \right)^{(k_4)} \\
& \times \left( x_\alpha^\pm \otimes a \right)^{(r-k_1-2k_2-3k_3-3k_4)} \\
& - \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s-1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r-3}} (k_4 + 1) (\pm 1)^{k_1 + k_3} m_1^{k_1 + 1 + k_4} (m_1 m_2)^{k_2 + 1 + k_3 + k_4} m_3^{k_3} m_4^{k_4 + 1} \\
& \times \left( x_\beta^\pm \otimes b \right)^{(s-k_1-k_2-k_3-2k_4-1)} \prod_{j=1}^3 \left( x_{j\alpha+\beta}^\pm \otimes a^j b \right)^{(k_j)} \left( x_{3\alpha+2\beta}^\pm \otimes a^3 b^2 \right)^{(k_4+1)} \\
& \times \left( x_\alpha^\pm \otimes a \right)^{(r-3-k_1-2k_2-3k_3-3k_4)}
\end{aligned}$$

because  $m_3 m_5 = -m_1 m_4$  by the algorithm in [11].

This in turn becomes

$$\begin{aligned}
& \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s+1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r}} (s+1-2k_4) (\pm 1)^{k_1+k_3} m_1^{k_1+k_4} (m_1 m_2)^{k_2+k_3+k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left(x_\beta^\pm \otimes b\right)^{(s+1-k_1-k_2-k_3-2k_4)} \prod_{j=1}^3 \left(x_{j\alpha+\beta}^\pm \otimes a^j b\right)^{(k_j)} \left(x_{3\alpha+2\beta}^\pm \otimes a^3 b^2\right)^{(k_4)} \\
& \times \left(x_\alpha^\pm \otimes a\right)^{(r-k_1-2k_2-3k_3-3k_4)} \\
& + 2 \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s+1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r}} k_4 (\pm 1)^{k_1+k_3} m_1^{k_1+k_4} (m_1 m_2)^{k_2+k_3+k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left(x_\beta^\pm \otimes b\right)^{(s+1-k_1-k_2-k_3-2k_4)} \\
& \times \prod_{j=1}^3 \left(x_{j\alpha+\beta}^\pm \otimes a^j b\right)^{(k_j)} \left(x_{3\alpha+2\beta}^\pm \otimes a^3 b^2\right)^{(k_4)} \\
& \times \left(x_\alpha^\pm \otimes a\right)^{(r-k_1-2k_2-3k_3-3k_4)} \\
& = (s+1) \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + 2k_4 \leq s+1 \\ k_1 + 2k_2 + 3k_3 + 3k_4 \leq r}} (\pm 1)^{k_1+k_3} m_1^{k_1+k_4} (m_1 m_2)^{k_2+k_3+k_4} m_3^{k_3} m_4^{k_4} \\
& \times \left(x_\beta^\pm \otimes b\right)^{(s+1-k_1-k_2-k_3-2k_4)} \prod_{j=1}^3 \left(x_{j\alpha+\beta}^\pm \otimes a^j b\right)^{(k_j)} \left(x_{3\alpha+2\beta}^\pm \otimes a^3 b^2\right)^{(k_4)} \\
& \times \left(x_\alpha^\pm \otimes a\right)^{(r-k_1-2k_2-3k_3-3k_4)}
\end{aligned}$$

■

## 2.4 Our Straightening Lemma

Define functions  $D^\pm : \mathcal{F}^3 \rightarrow \mathbf{U}(\mathfrak{sl}_2 \otimes A)$  by

$$\begin{aligned} D^\pm(\psi_1, \psi_2, 0) &= \delta_{|\psi_1|+|\psi_2|,0} \\ D^\pm(\psi_1, \psi_2, \chi_b) &= \delta_{|\psi_1|,|\psi_2|} (|\psi_1|!)^2 (x^\pm \otimes b\pi(\psi_1)\pi(\psi_2)) \\ D^\pm(\psi_1, \psi_2, \psi_3) &= \frac{1}{|\psi_3|} \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \sum_{b \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_b) D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_b) \end{aligned}$$

**Remark 11** *Note the following:*

(a)  $D^\pm(\psi_1, \psi_2, \psi_3) = 0$  if  $|\psi_1| \neq |\psi_2|$ .

(b) The  $D^\pm(\psi_1, \psi_2, \psi_3)$  are a generalization of Garland's  $D_k^\pm$  as

$$D^+(k\chi_t, k\chi_1, r\chi_1) = D_k^+(\xi^{(r)}) \text{ and } D^-(k\chi_t, k\chi_1, r\chi_t) = D_k^-(\xi^{(r)}).$$

### 2.4.1 The Degree of $D^\pm(\psi_1, \psi_2, \psi_3)$

The following proposition can easily be shown by induction on  $|\psi_3|$ .

**Proposition 12** *If  $\alpha \in R^+$ , and  $\psi_1, \psi_2, \psi_3 \in \mathcal{F}$  then  $D^\pm(\psi_1, \psi_2, \psi_3)$  is homogeneous of degree  $|\psi_3|$ .*

### 2.4.2 The Definition of $\mathbb{D}(\psi_1, \psi_2, \psi_3)$

Define  $\mathbb{D} : \mathcal{F}^3 \rightarrow \mathbf{U}(\mathfrak{sl}_2 \otimes A)$  by

$$\mathbb{D}(\psi_1, \psi_2, \psi_3) = \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} p(\phi_1, \phi_2) D^+(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3)$$

Using the previous two propositions it can be easily shown that, for all  $\psi_1, \psi_2, \psi_3 \in \mathcal{F}$ ,  $\mathbb{D}(\psi_1, \psi_2, \psi_3)$  has degree less than or equal to  $|\psi_3| + |\psi_1|$

Set

$$D_\alpha^\pm(\psi_1, \psi_2, \psi_3) = \Omega_\alpha(D^\pm(\psi_1, \psi_2, \psi_3))$$

$$\mathbb{D}_\alpha(\psi_1, \psi_2, \psi_3) = \Omega_\alpha(\mathbb{D}(\psi_1, \psi_2, \psi_3))$$

### 2.4.3 The Straightening Lemma

**Lemma 13** For all  $\varphi, \chi \in \mathcal{F}$ .

$$x^+(\varphi)x^-(\chi) = \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi) \\ \phi_1 + \phi_2 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi_2) \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2)$$

This lemma is proved in a later section. Before proving the lemma we will state and prove some corollaries and necessary lemmas.

**Corollary 14** For all  $\alpha \in R^+$ , and  $\varphi, \chi, \psi \in \mathcal{F}$

(i) $_\chi$

$$D_\alpha^+(\varphi, \chi, \psi), D_\alpha^-(\varphi, \chi, \psi) \in \mathbf{U}_\mathbb{Z}(\mathfrak{g} \otimes A)$$

(ii) $_\chi$

$$p_\alpha(\varphi, \chi) \in \mathbf{U}_\mathbb{Z}(\mathfrak{g} \otimes A)$$

**Proof.** *Proof of corollary.* We will prove both statements simultaneously by induction on  $k$  according to the scheme

$$(ii)^k \Rightarrow (i)^k \Rightarrow (ii)^{k+1},$$

where  $(i)^k$  is the statement that  $(i)_\chi$  holds for all  $\chi \in \mathcal{F}$  with  $|\chi| \leq k$ , and similarly for  $(ii)^k$ .  $(ii)^1$  is trivially true. For  $(i)_0$  we have, by induction on  $|\psi|$ ,  $D^\pm(0, 0, \psi) = x^\pm(\psi) \in \mathbf{U}_\mathbb{Z}(\mathfrak{g} \otimes A)$ . Assume that  $(i)^{k-1}$  and  $(ii)^k$  hold for some  $k \geq 1$ . Note that  $(i)_\chi$  is trivially

true if  $|\varphi| \neq |\chi|$  so assume that they have the same size. Then, by the previous lemma, we have for all  $\varphi, \chi \in \mathcal{F}(k)$  and all  $\psi \in \mathcal{F}$

$$\begin{aligned}
x^+(\varphi)x^-(\chi + \psi) &= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi + \psi) \\ \phi_1 + \phi_2 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi + \psi - \psi_1 - \psi_2) \\
&\times \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2) \\
&= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi + \psi) \\ \phi_1 + \phi_2 < \varphi \\ \psi_1 + \psi_2 \neq \chi}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi + \psi - \psi_1 - \psi_2) \\
&\times \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2) \\
&+ \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 = \chi \\ \phi_1 + \phi_2 = \varphi}} (-1)^{|\chi|} D^-(\phi_1, \psi_1, \psi) \mathbb{D}(\phi_2, \psi_2, 0) \\
&= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi + \psi) \\ \phi_1 + \phi_2 < \varphi \\ \psi_1 + \psi_2 \neq \chi}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi + \psi - \psi_1 - \psi_2) \\
&\times \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2) \\
&+ \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 = \chi \\ \phi_1 + \phi_2 = \varphi}} \sum_{\substack{\phi \in \mathcal{F}(\phi_2) \\ \phi' \in \mathcal{F}(\psi_2)}} (-1)^{|\chi|} D^-(\phi_1, \psi_1, \psi) p(\phi, \phi') \\
&\times D^+(\phi_2 - \phi, \psi_2 - \phi', 0) \\
&= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi + \psi) \\ \phi_1 + \phi_2 < \varphi \\ \psi_1 + \psi_2 \neq \chi}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi + \psi - \psi_1 - \psi_2) \\
&\times \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2) \\
&+ \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 = \chi \\ \phi_1 + \phi_2 = \varphi \\ \phi_1 < \varphi}} (-1)^{|\chi|} D^-(\phi_1, \psi_1, \psi) p(\phi_2, \psi_2) + (-1)^{|\chi|} D^-(\varphi, \chi, \psi)
\end{aligned}$$

The left side is clearly in  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_2 \otimes A)$ . The end sums are in  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_2 \otimes A)$  by  $(i)^{k-1}$  and  $(ii)^k$ . Thus  $D^-(\varphi, \chi, \psi) \in \mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_2 \otimes A)$ . Also by the lemma

$$\begin{aligned}
x^+(\varphi + \psi)x^-(\chi) &= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi) \\ \phi_1 + \phi_2 \in \mathcal{F}(\varphi + \psi)}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi_2) \\
&\times \mathbb{D}(\phi_2, \psi_2, \varphi + \psi - \phi_1 - \phi_2) \\
&= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 < \chi \\ \phi_1 + \phi_2 \in \mathcal{F}(\varphi + \psi) \\ \phi_1 + \phi_2 \neq \varphi}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi_2) \\
&\times \mathbb{D}(\phi_2, \psi_2, \varphi + \psi - \phi_1 - \phi_2) \\
&+ \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 = \chi \\ \phi_1 + \phi_2 = \varphi}} (-1)^{|\varphi|} D^-(\phi_1, \psi_1, 0) \mathbb{D}(\phi_2, \psi_2, \psi) \\
&= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 < \chi \\ \phi_1 + \phi_2 \in \mathcal{F}(\varphi + \psi) \\ \phi_1 + \phi_2 \neq \varphi}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi_2) \\
&\times \mathbb{D}(\phi_2, \psi_2, \varphi + \psi - \phi_1 - \phi_2) \\
&+ \sum_{\substack{\phi \in \mathcal{F}(\chi) \\ \phi' \in \mathcal{F}(\varphi)}} (-1)^{|\varphi|} p(\phi, \phi') D^+(\chi - \phi, \varphi - \phi', \psi) \\
&= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 < \chi \\ \phi_1 + \phi_2 \in \mathcal{F}(\varphi + \psi) \\ \phi_1 + \phi_2 \neq \varphi}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi_2) \\
&\times \mathbb{D}(\phi_2, \psi_2, \varphi + \psi - \phi_1 - \phi_2) \\
&+ \sum_{\substack{\phi \in \mathcal{F}(\chi) - \{0\} \\ \phi' \in \mathcal{F}(\varphi) - \{0\}}} (-1)^{|\varphi|} p(\phi, \phi') D^+(\chi - \phi, \varphi - \phi', \psi) + (-1)^{|\varphi|} D^+(\chi, \varphi, \psi)
\end{aligned}$$



Again the left side is clearly in  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_2 \otimes A)$ . The end sums are in  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_2 \otimes A)$  by  $(i)^{k-1}$  and  $(ii)^k$ . Thus  $D^{\pm}(\varphi, \chi, \psi) \in \mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_2 \otimes A)$ . Applying  $\Omega_{\alpha}$  we get  $(i)^k$ . Thus  $(ii)^k$  implies  $(i)^k$ . Assume that  $(i)^k$  and  $(ii)^k$  hold and let  $\varphi, \chi \in \mathcal{F}(k+1)$  be given. Then, again by the previous lemma

$$\begin{aligned}
x^+(\varphi)x^-(\chi) &= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi) \\ \phi_1 + \phi_2 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi_2) \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2) \\
&= \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 < \chi \\ \phi_1 + \phi_2 < \varphi}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi_2) \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2) \\
&\quad + (-1)^{|\chi|} p(\varphi, \chi)
\end{aligned}$$

Again the left side is clearly in  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_2 \otimes A)$ . The end sum is in  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_2 \otimes A)$  by  $(i)^k$  and  $(ii)^k$ . Thus  $p(\varphi, \chi) \in \mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_2 \otimes A)$ . Applying  $\Omega_{\alpha}$  we get  $(ii)^{k+1}$ . Thus  $(i)^k$  implies  $(ii)^{k+1}$ .

■

## 2.5 $\mathcal{B}_0$ is a $\mathbb{Z}$ -basis for $\mathbf{U}_{\mathbb{Z}}^0(\mathfrak{g} \otimes A)$

Define  $\mathbf{U}_{\mathbb{Z}}^0(\mathfrak{g} \otimes A)$  to be the  $\mathbb{Z}$ -subalgebra of  $\mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A)$  generated by the set  $\{p_i(\chi) : \chi \in \mathcal{F}(\mathbf{B}), i \in I\}$ .

For the remainder of this section  $\mathfrak{g} = \mathfrak{sl}_2$  unless stated otherwise.

We adapt Garland's proof of the corresponding fact in [6], section 9.

### 2.5.1 $\mathbf{U}(h \otimes A) = \mathbb{C} - \text{span } \mathcal{B}_0$

**Proposition 15** For all  $\chi \in \mathcal{F}$

$$p(\chi) = (-1)^{|\chi|} \prod_{a \in A} (h \otimes a)^{(\chi(a))} + \text{elements of } \mathbf{U}(h \otimes A) \text{ of degree less than } |\chi|$$

**Proof.** By definition  $p(0) = 1$  and it is easy to see that, for all  $a \in A$ ,  $p(\chi_a) = -(h \otimes a)$ .

Assume that for some  $k \geq 1$  the proposition holds for all  $\chi \in \mathcal{F}_k$ . Choose  $\chi \in \mathcal{F}_{k+1}$ . Then

$$\begin{aligned} p(\chi) &= -\frac{1}{|\chi|} \sum_{\psi \in \mathcal{F}(\chi) - \{0\}} |\psi|! (h \otimes \pi(\psi)) p(\chi - \psi) \\ &= -\frac{1}{|\chi|} \sum_{b \in \text{supp } \chi} (h \otimes b) p(\chi - \chi_b) - \frac{1}{|\chi|} \sum_{\substack{\psi \in \mathcal{F}(\chi) \\ |\psi| > 1}} |\psi|! (h \otimes \pi(\psi)) p(\chi - \psi) \\ &= -\frac{1}{|\chi|} \sum_{b \in \text{supp } \chi} (h \otimes b) (-1)^{|\chi| - 1} \prod_{a \in A} (h \otimes a)^{((\chi - \chi_b)(a))} + \text{lower degree terms} \\ &= (-1)^{|\chi|} \frac{1}{|\chi|} \sum_{b \in \text{supp } \chi} \chi(b) \prod_{a \in A} (h \otimes a)^{(\chi(a))} + \text{lower degree terms} \\ &= (-1)^{|\chi|} \prod_{a \in A} (h \otimes a)^{(\chi(a))} + \text{lower degree terms} \end{aligned}$$

■

**Lemma 16**

$$\mathbf{U}(h \otimes A) = \mathbb{C} - \text{span } \mathcal{B}_0$$

**Proof.** Clearly  $\mathbf{U}(h \otimes A) \subset \mathbb{C} - \text{span } \mathcal{B}_0$ . For the other inclusion it suffices to show that

$$\prod_{a \in A} (h \otimes a)^{(\chi(a))} \in \mathbb{C} - \text{span } \mathcal{B}_0$$

for all  $\chi \in \mathcal{F}$ . This can be shown by Proposition 15 and induction on  $|\chi|$ . ■

### 2.5.2 The Adjoint Action on the $\mathbb{Z}$ -Span of the Chevalley Basis

For this subsection let  $\mathfrak{g}$  be arbitrary. Define  $(\mathfrak{g} \otimes A)_{\mathbb{Z}}$  to be the  $\mathbb{Z}$ -span of the set  $S = \{(x_{\alpha}^{\pm} \otimes b), (h_i \otimes b) : b \in \mathbf{B}, i \in I, \alpha \in R^+\}$ .

**Lemma 17** *For all  $\alpha \in R^+$ ,  $r \in \mathbb{Z}_{\geq 0}$  and  $b \in \mathbf{B}$ ,*

$$\text{ad}(x_{\alpha}^{\pm} \otimes b)^{(r)}((\mathfrak{g} \otimes A)_{\mathbb{Z}}) \subset (\mathfrak{g} \otimes A)_{\mathbb{Z}}$$

**Proof.** It suffices to show that, for all  $s \in S$ ,

$$\text{ad}(x_{\alpha}^{\pm} \otimes b)^{(r)}(s) = (\text{ad } x_{\alpha}^{\pm} \otimes b)^{(r)}(s) \in (\mathfrak{g} \otimes A)_{\mathbb{Z}}$$

It can be easily shown by induction on  $r$  that, for all  $z, z' \in \mathfrak{g}$  and all  $a, b \in \mathbf{B}$ ,

$$(\text{ad } z \otimes b)^{(r)}(z' \otimes a) = (\text{ad } z)^{(r)}(z') \otimes b^r a$$

And it is well known that  $(\text{ad } x_{\alpha}^{\pm})^{(r)}$  leaves the  $\mathbb{Z}$ -span of the Chevalley basis invariant. ■

**Corollary 18** *For all  $C \in \mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A)$ ,*

$$\text{ad } C((\mathfrak{g} \otimes A)_{\mathbb{Z}}) \subset (\mathfrak{g} \otimes A)_{\mathbb{Z}}$$

### 2.5.3 The Adjoint Action on $((\mathfrak{g} \otimes A)_{\mathbb{Z}})^{\otimes r}$

We note the following simple lemma.

**Lemma 19** *Let  $V, W$  be  $\mathfrak{g} \otimes A$ -modules, with respective additive subgroups  $M, N$ . If  $M, N$  are preserved by  $(x_{\alpha}^{\pm} \otimes b)^{(r)}$ , for all  $\alpha \in R^+$ ,  $b \in \mathbf{B}$  and  $r \in \mathbb{Z}_{\geq 0}$ , then  $M \otimes N \subset V \otimes W$  is also preserved by these elements.*

Let  $\text{ad}^{(r)}$  denote the action of  $\mathfrak{g} \otimes A$  (and of  $\mathbf{U}(\mathfrak{g} \otimes A)$ ) induced by adjoint action, on  $(\mathfrak{g} \otimes A)^{\otimes r}$ . Then we have the following corollary to the previous two lemmas.

**Corollary 20** *For all  $Q \in \mathbf{U}_{\mathbb{Z}}(\mathfrak{g} \otimes A)$  and all  $r \in \mathbb{Z}_{\geq 0}$ ,*

$$\text{ad}^{(r)} Q \left( ((\mathfrak{g} \otimes A)_{\mathbb{Z}})^{\otimes r} \right) \subset ((\mathfrak{g} \otimes A)_{\mathbb{Z}})^{\otimes r}$$

**2.5.4**  $p(\chi)p(\chi') \in \mathbb{Z} - \text{span } \mathcal{B}_0$

**Lemma 21** *For all  $\chi, \chi', \psi \in \mathcal{F}(\mathbf{B})$ ,  $p(\chi)p(\chi') \in \mathbb{Z} - \text{span } \mathcal{B}_0$*

**Proof.** Choose  $\chi, \chi' \in \mathcal{F}(\mathbf{B})$ . Letting  $\mathfrak{g} = \mathfrak{sl}_3$ , applying  $\Omega_1$  and Corollary 14 we have

$$p_1(\chi)p_1(\chi') \in \mathbf{U}_{\mathbb{Z}}(\mathfrak{sl}_3 \otimes A)$$

Corollary 20 implies that, for all  $r \in \mathbb{Z}_{\geq 0}$ ,

$$\text{ad}_{p_1(\chi)p_1(\chi')}^{(r)} \left( \otimes^r (\mathfrak{sl}_3 \otimes A)_{\mathbb{Z}} \right) \subset \otimes^r (\mathfrak{sl}_3 \otimes A)_{\mathbb{Z}}$$

The previous lemma implies that

$$p(\chi)p(\chi') = \sum_{\psi \in \mathcal{F}(\mathbf{B})} a_{\psi} p(\psi)$$

for some  $a_{\psi} \in \mathbb{C}$ , with only finitely many nonzero. We want to show that  $a_{\psi} \in \mathbb{Z}$ , for all  $\psi \in \mathcal{F}(\mathbf{B})$ . It suffices to prove that if  $\sum_{\psi \in \mathcal{F}(\mathbf{B})} a_{\psi} p(\psi)$  is a finite sum such that, for all  $r \in \mathbb{Z}_{\geq 0}$ ,

$$\text{ad}^{(r)} \left( \sum_{\psi \in \mathcal{F}(\mathbf{B})} a_{\psi} p_1(\psi) \right) \left( \otimes^r (\mathfrak{sl}_3 \otimes A)_{\mathbb{Z}} \right) \subset \otimes^r (\mathfrak{sl}_3 \otimes A)_{\mathbb{Z}} \quad (2.5.1)$$

Then  $a_{\psi} \in \mathbb{Z}$ , for all  $\psi \in \mathcal{F}(\mathbf{B})$ . We shall prove this claim by induction on the number of nonzero  $a_{\psi}$ . For the base case if they are all zero there is nothing to prove. Assume that we

have proved (2.5.1) when there are  $k$  nonzero  $a_\psi$ . Now assume that there are  $k + 1$  nonzero  $a_\psi$  and choose  $\psi_0$ , of maximal size among the  $\psi \in \mathcal{F}(\mathbf{B})$  with  $a_\psi \neq 0$ . By Proposition 15

$$p(\psi_0) = \prod_{a \in \mathbf{B}} (h \otimes a)^{(\psi_0(a))} + \text{lower degree terms}$$

Define  $\rho = \otimes^{|\psi_0|} (x_{\alpha_2}^+ \otimes 1)$  and consider the  $\otimes_{a \in \mathbf{B}} (x_{\alpha_2}^+ \otimes a)^{\otimes \psi_0(a)}$ -component of

$$\text{ad}^{(|\psi_0|)} \left( \sum_{\psi \in \mathcal{F}(\mathbf{B})} a_\psi p_1(\psi) \right) (\rho)$$

From  $a_{\psi_0} p(\psi_0)$ , we get  $\pm a_{\psi_0} \otimes_{a \in \mathbf{B}} (x_{\alpha_2}^+ \otimes a)^{\otimes \psi_0(a)}$ . None of the other  $a_\psi p(\psi)$  contribute to this component, because they either have too small degree or they have the right degree but the wrong distribution of elements of  $\mathbf{B}$ . By assumption

$$\text{ad}^{(|\psi_0|)} \left( \sum_{\psi \in \mathcal{F}(\mathbf{B})} a_\psi p_1(\psi) \right) (\rho) \in \otimes^r (\mathfrak{sl}_3 \otimes A)_{\mathbb{Z}}$$

so  $a_{\psi_0} \in \mathbb{Z}$ .  $\sum_{\psi \in \mathcal{F}(\mathbf{B})} a_\psi p(\psi) - a_{\psi_0} p(\psi_0)$  has  $k$  nonzero  $a_\psi$  and still satisfies the assumption of (2.5.1) by Corollary 20 because  $a_{\psi_0} \in \mathbb{Z}$ . So by induction we have  $a_\psi \in \mathbb{Z}$  for all  $\psi \in \mathcal{F}(\mathbf{B})$ . ■

The following corollary follows directly

**Corollary 22**

$$p(\chi)p(\chi') - \prod_{a \in A} \binom{(\chi + \chi')(a)}{\chi(a)} p(\chi + \chi') \in \mathbb{Z} - \text{span } \mathcal{B}_0$$

**2.5.5  $\mathcal{B}_0$  is a  $\mathbb{Z}$ -basis for  $\mathbf{U}_{\mathbb{Z}}^0(\mathfrak{g} \otimes A)$**

**Lemma 23**  $\mathcal{B}_0$  is a  $\mathbb{Z}$ -basis for  $\mathbf{U}_{\mathbb{Z}}^0(\mathfrak{g} \otimes A)$ .

**Proof.** It will suffice to show that any product of elements of  $\{p_i(\chi) : \chi \in \mathcal{F}(\mathbf{B})\}$  is in the  $\mathbb{Z}$ -span of  $\mathbf{U}_{\mathbb{Z}}^0(\mathfrak{g} \otimes A)$ . To show this we will proceed by induction on the degree of such

a product. Since  $p_i(\chi)$  and  $p_j(\chi')$  commute it will suffice to apply  $\Omega_{\alpha_i}$  to the previous corollary because by Proposition 15

$$p(\chi)p(\chi') - \prod_{a \in A} \binom{(\chi + \chi')(a)}{\chi(a)} p(\chi + \chi')$$

has degree less than  $|\chi| + |\chi'|$ . ■

## 2.6 More Identities

We will state and prove some more necessary identities.

### 2.6.1 Identities for $(x_\alpha^+ \otimes b)p_i(\varphi, \chi)$ and $p_i(\varphi, \chi)(x_\alpha^- \otimes b)$

**Proposition 24** *For all  $\varphi, \chi \in \mathcal{F}$ ,  $\alpha \in R^+$ ,  $i \in I$  and  $b \in A$  with  $\alpha(h_i) \neq 0$  the following hold:*

(i)

$$(x_\alpha^+ \otimes b)p_i(\varphi, \chi) = \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) \\ \psi_2 \in \mathcal{F}(\chi) \\ |\psi_1| = |\psi_2|}} \binom{\alpha(h_i) + |\psi_1| - 1}{|\psi_1|} (|\psi_1|!)^2 p_i(\varphi - \psi_1, \chi - \psi_2) (x_\alpha^+ \otimes b\pi(\psi_1)\pi(\psi_2))$$

(ii)

$$p_i(\varphi, \chi)(x_\alpha^- \otimes b) = \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) \\ \psi_2 \in \mathcal{F}(\chi) \\ |\psi_1| = |\psi_2|}} \binom{\alpha(h_i) + |\psi_1| - 1}{|\psi_1|} (|\psi_1|!)^2 (x_\alpha^- \otimes b\pi(\psi_1)\pi(\psi_2)) p_i(\varphi - \psi_1, \chi - \psi_2)$$

**Proof.** The identities are trivially true if  $|\chi| \neq |\varphi|$ . So assume that  $|\chi| = |\varphi|$ . The proof of both parts will proceed by induction on  $|\chi| = |\varphi|$ . The identity is easy to check in the cases  $\varphi = \chi = 0$  and  $\varphi = \chi_d$ ,  $\chi = \chi_c$ . Choose some  $\chi \in \mathcal{F}$  with  $|\varphi| = |\chi| > 1$ . Assume the proposition for all pairs of elements of  $\mathcal{F}$  with size less than  $|\chi|$ . Then for (i)

$$\begin{aligned}
(x_\alpha^+ \otimes b)p_i(\varphi, \chi) &= -\frac{1}{|\varphi|} \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\phi_1|!)^2 (x_\alpha^+ \otimes b) (h_i \otimes \pi(\phi_1)\pi(\phi_2)) p_i(\varphi - \phi_1, \chi - \phi_2) \\
&= -\frac{1}{|\varphi|} \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\phi_1|!)^2 (h_i \otimes \pi(\phi_1)\pi(\phi_2)) (x_\alpha^+ \otimes b)p_i(\varphi - \phi_1, \chi - \phi_2) \\
&+ \frac{\alpha(h_i)}{|\varphi|} \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\phi_1|!)^2 (x_\alpha^+ \otimes b\pi(\phi_1)\pi(\phi_2)) p_i(\varphi - \phi_1, \chi - \phi_2) \\
&= -\frac{1}{|\varphi|} \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\phi_1|!)^2 (h_i \otimes \pi(\phi_1)\pi(\phi_2)) \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi - \phi_1) \\ \psi_2 \in \mathcal{F}(\chi - \phi_2) \\ |\psi_1| = |\psi_2|}} \\
&\times \binom{\alpha(h_i) + |\psi_1| - 1}{|\psi_1|} (|\psi_1|!)^2 p_i(\varphi - \phi_1 - \psi_1, \chi - \phi_2 - \psi_2) \\
&\times (x_\alpha^+ \otimes b\pi(\psi_1)\pi(\psi_2)) \\
&+ \frac{\alpha(h_i)}{|\varphi|} \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\phi_1|!)^2 \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi - \phi_1) \\ \psi_2 \in \mathcal{F}(\chi - \phi_2) \\ |\psi_1| = |\psi_2|}} \binom{\alpha(h_i) + |\psi_1| - 1}{|\psi_1|} (|\psi_1|!)^2 \\
&\times p_i(\varphi - \phi_1 - \psi_1, \chi - \phi_2 - \psi_2) (x_\alpha^+ \otimes b\pi(\phi_1)\pi(\phi_2)\pi(\psi_1)\pi(\psi_2))
\end{aligned}$$

(by the induction hypothesis)

$$\begin{aligned}
&= -\frac{1}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) \\ \psi_2 \in \mathcal{F}(\chi) \\ |\psi_1| = |\psi_2| < |\varphi|}} \binom{\alpha(h_i) + |\psi_1| - 1}{|\psi_1|} (|\psi_1|!)^2 \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi - \psi_1) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi - \psi_2) - \{0\}}} (|\phi_1|!)^2 \\
&\times (h_i \otimes \pi(\phi_1)\pi(\phi_2)) p_i(\varphi - \phi_1 - \psi_1, \chi - \phi_2 - \psi_2) (x_\alpha^+ \otimes b\pi(\psi_1)\pi(\psi_2)) \\
&+ \frac{\alpha(h_i)}{|\varphi|} \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\phi_1|!)^2 \sum_{\substack{\psi_1, \psi_2 \in \mathcal{F} \\ \phi_1 \leq \psi_1 \leq \varphi \\ \phi_2 \leq \psi_2 \leq \chi \\ |\psi_1| = |\psi_2|}} \binom{\alpha(h_i) + |\psi_1| - |\phi_1| - 1}{|\psi_1| - |\phi_1|} \\
&\times ((|\psi_1| - |\phi_1|)!)^2 \prod_{a \in A} \binom{\psi_1(a)}{\phi_1(a)} \binom{\psi_2(a)}{\phi_2(a)} p_i(\varphi - \psi_1, \chi - \psi_2) \\
&\times (x_\alpha^+ \otimes b\pi(\psi_1)\pi(\psi_2))
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|\varphi|} \sum_{\substack{\psi_1, \psi_2 \in \mathcal{F} \\ \psi_1 < \varphi \\ \psi_2 < \chi}} \binom{\alpha(h_i) + |\psi_1| - 1}{|\psi_1|} (|\psi_1|!)^2 (|\varphi| - |\psi_1|) p_i(\varphi - \psi_1, \chi - \psi_2) \\
&\times (x_\alpha^+ \otimes b\pi(\psi_1)\pi(\psi_2)) \\
&+ \frac{\alpha(h_i)}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) \\ \psi_2 \in \mathcal{F}(\chi) \\ |\psi_1| = |\psi_2| > 0}} \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) - \{0\} \\ \phi_2 \in \mathcal{F}(\psi_2) - \{0\}}} (|\phi_1|!)^2 \binom{\alpha(h_i) + |\psi_1| - |\phi_1| - 1}{|\psi_1| - |\phi_1|} ((|\psi_1| - |\phi_1|)!)^2 \\
&\times \prod_{a \in A} \binom{\psi_1(a)}{\phi_1(a)} \binom{\psi_2(a)}{\phi_2(a)} p_i(\varphi - \psi_1, \chi - \psi_2) (x_\alpha^+ \otimes b\pi(\psi_1)\pi(\psi_2)) \\
&= \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) \\ \psi_2 \in \mathcal{F}(\chi)}} \binom{\alpha(h_i) + |\psi_1| - 1}{|\psi_1|} (|\psi_1|!)^2 p_i(\varphi - \psi_1, \chi - \psi_2) (x_\alpha^+ \otimes b\pi(\psi_1)\pi(\psi_2)) \\
&+ \frac{1}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) \\ \psi_2 \in \mathcal{F}(\chi) \\ |\psi_1| = |\psi_2| > 0}} \binom{\alpha(h_i)}{|\psi_1|} \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) - \{0\} \\ \phi_2 \in \mathcal{F}(\psi_2) - \{0\}}} (|\phi_1|!)^2 \binom{\alpha(h_i) + |\psi_1| - |\phi_1| - 1}{|\psi_1| - |\phi_1|} ((|\psi_1| - |\phi_1|)!)^2 \\
&\times \prod_{a \in A} \binom{\psi_1(a)}{\phi_1(a)} \binom{\psi_2(a)}{\phi_2(a)} - \binom{\alpha(h_i) + |\psi_1| - 1}{|\psi_1|} (|\psi_1|!)^2 |\psi_1| \Big) p_i(\varphi - \psi_1, \chi - \psi_2) \\
&\times (x_\alpha^+ \otimes b\pi(\psi_1)\pi(\psi_2))
\end{aligned}$$

So it suffices to show that, for all  $\psi_1, \psi_2 \in \mathcal{F} - \{0\}$  with  $|\psi_1| = |\psi_2|$ ,

$$\begin{aligned}
\binom{\alpha(h_i) + |\psi_1| - 1}{|\psi_1|} (|\psi_1|!)^2 |\psi_1| &= \alpha(h_i) \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2) \\ |\phi_1| = |\phi_2| > 0}} (|\phi_1|!)^2 \binom{\alpha(h_i) + |\psi_1| - |\phi_1| - 1}{|\psi_1| - |\phi_1|} \\
&\times ((|\psi_1| - |\phi_1|)!)^2 \prod_{a \in A} \binom{\psi_1(a)}{\phi_1(a)} \binom{\psi_2(a)}{\phi_2(a)}
\end{aligned}$$

We will show this by induction on  $|\psi_1|$ . If  $|\psi_1| = 1$  it is easily checked. Assume that for some  $k$  the previous equation holds for all  $\psi_1, \psi_2 \in \mathcal{F}_k$  and let  $\psi_1, \psi_2 \in \mathcal{F}_{k+1}$  be given. Then



$$\begin{aligned}
& \alpha(h_i) \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2) \\ |\phi_1| = |\phi_2| > 0}} (|\phi_1|!)^2 \binom{\alpha(h_i) + k - |\phi_1|}{k + 1 - |\phi_1|} ((k + 1 - |\phi_1|)!)^2 \prod_{a \in A} \binom{\psi_1(a)}{\phi_1(a)} \binom{\psi_2(a)}{\phi_2(a)} \\
&= \alpha(h_i) \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2) \\ |\phi_1| = |\phi_2| > 0}} \sum_{\substack{c \in \text{supp } \phi_1 \\ d \in \text{supp } \phi_2}} \phi_1(c) \phi_2(d) ((|\phi_1| - 1)!)^2 \binom{\alpha(h_i) + k - |\phi_1|}{k + 1 - |\phi_1|} \\
&\times ((k + 1 - |\phi_1|)!)^2 \prod_{a \in A} \binom{\psi_1(a)}{\phi_1(a)} \binom{\psi_2(a)}{\phi_2(a)} \\
&= \alpha(h_i) \sum_{\substack{c \in \text{supp } \psi_1 \\ d \in \text{supp } \psi_2}} \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1 - \chi_c) \\ \phi_2 \in \mathcal{F}(\psi_2 - \chi_d) \\ |\phi_1| = |\phi_2|}} (\phi_1(c) + 1)(\phi_2(d) + 1)(|\phi_1|!)^2 \binom{\alpha(h_i) + k - |\phi_1| - 1}{k - |\phi_1|} \\
&\times ((k - |\phi_1|)!)^2 \prod_{a \in A} \binom{\psi_1(a)}{(\phi_1 + \chi_c)(a)} \binom{\psi_2(a)}{(\phi_2 + \chi_d)(a)} \\
&= \alpha(h_i) \sum_{\substack{c \in \text{supp } \psi_1 \\ d \in \text{supp } \psi_2}} \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1 - \chi_c) \\ \phi_2 \in \mathcal{F}(\psi_2 - \chi_d) \\ |\phi_1| = |\phi_2|}} \psi_1(c) \psi_2(d) (|\phi_1|!)^2 \binom{\alpha(h_i) + k - |\phi_1| - 1}{k - |\phi_1|} \\
&\times ((k - |\phi_1|)!)^2 \prod_{a \in A} \binom{(\psi_1 - \chi_c)(a)}{\phi_1(a)} \binom{(\psi_2 - \chi_d)(a)}{\phi_2(a)} \\
&= \alpha(h_i) \sum_{\substack{c \in \text{supp } \psi_1 \\ d \in \text{supp } \psi_2}} \psi_1(c) \psi_2(d) \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1 - \chi_c) \\ \phi_2 \in \mathcal{F}(\psi_2 - \chi_d) \\ |\phi_1| = |\phi_2| > 0}} (|\phi_1|!)^2 \binom{\alpha(h_i) + k - |\phi_1| - 1}{k - |\phi_1|} \\
&\times ((k - |\phi_1|)!)^2 \prod_{a \in A} \binom{(\psi_1 - \chi_c)(a)}{\phi_1(a)} \binom{(\psi_2 - \chi_d)(a)}{\phi_2(a)} \\
&+ \alpha(h_i) \sum_{\substack{c \in \text{supp } \psi_1 \\ d \in \text{supp } \psi_2}} \psi_1(c) \psi_2(d) \binom{\alpha(h_i) + k - 1}{k} (k!)^2 \\
&= (k + 1)^2 \binom{\alpha(h_i) + k - 1}{k} (k!)^2 k + \alpha(h_i) (k + 1)^2 \binom{\alpha(h_i) + k - 1}{k} (k!)^2 \\
&= (k + 1)^2 \binom{\alpha(h_i) + k - 1}{k} (k!)^2 (k + \alpha(h_i)) \\
&= \binom{\alpha(h_i) + k - 1}{k} (k + 1)^2 (k + 1)
\end{aligned}$$

Thus (i) is proved by induction on  $|\chi| = |\varphi|$ . The proof of (ii) is almost exactly the same and hence is omitted. ■

### 2.6.2 Identities for $(x_\alpha^+ \otimes b)^{(r)} p_i(\chi)$ and $p_i(\chi)(x_\alpha^- \otimes b)^{(s)}$

**Definition.** Given  $\chi \in \mathcal{F}$  define

$$\mathcal{P}(\chi) = \left\{ \psi \in \mathcal{F}(\mathcal{F}) : \sum_{\phi \in \mathcal{F}} \psi(\phi) \phi = \chi \right\}$$

$$\mathcal{S}(\chi) = \left\{ \psi \in \mathcal{F}(\mathcal{F}) : \sum_{\phi \in \mathcal{F}} \psi(\phi) \phi \leq \chi \right\}$$

A *partition* of  $\chi$  is an element of  $\mathcal{P}(\chi)$ . Define  $\mathcal{P}_k(\chi) = \mathcal{P}(\chi) \cap \mathcal{F}_k(\mathcal{F})$  and  $\mathcal{S}_k(\chi) = \mathcal{S}(\chi) \cap \mathcal{F}_k(\mathcal{F})$ .

**Proposition 25** For all  $\chi \in \mathcal{F}$ ,  $\alpha \in R^+$ ,  $i \in I$  and  $b \in A$  with  $\alpha(h_i) \neq 0$  and  $r, s \in \mathbb{Z}_{\geq 0}$  the following hold: (i)

$$\begin{aligned} (x_\alpha^+ \otimes b)^{(r)} p_i(\chi) &= \sum_{\psi \in \mathcal{S}_r(\chi)} p_i \left( \chi - \sum_{\phi \in \mathcal{F}} \psi(\phi) \phi \right) \prod_{\phi \in \mathcal{F}} \left( \binom{\alpha(h_i) + |\phi| - 1}{|\phi|} |\phi|! \right)^{\psi(\phi)} \\ &\times (x_\alpha^+ \otimes b\pi(\phi))^{\psi(\phi)} \end{aligned}$$

(ii)

$$\begin{aligned} p_i(\chi)(x_\alpha^- \otimes b)^{(r)} &= \sum_{\psi \in \mathcal{S}_r(\chi)} \prod_{\phi \in \mathcal{F}} \left( \binom{\alpha(h_i) + |\phi| - 1}{|\phi|} |\phi|! \right)^{\psi(\phi)} (x_\alpha^- \otimes b\pi(\phi))^{\psi(\phi)} \\ &\times p_i \left( \chi - \sum_{\phi \in \mathcal{F}} \psi(\phi) \phi \right) \end{aligned}$$

**Proof.** The proof will proceed by induction on  $r$ . The case  $r = 1$  is Proposition 24. Assume the proposition for some  $r \geq 1$ . Then for (i)

$$\begin{aligned}
(r+1)(x_\alpha^+ \otimes b)^{(r+1)} p_i(\chi) &= (x_\alpha^+ \otimes b)^{(r)} (x_\alpha^+ \otimes b) p_i(\chi) \\
&= \sum_{\phi' \in \mathcal{F}(\chi)} (x_\alpha^+ \otimes b)^{(r)} p_i(\chi - \phi') \binom{\alpha(h_i) + |\phi'| - 1}{|\phi'|} |\phi'|! \\
&\times (x_\alpha^+ \otimes b\pi(\phi')) \quad (\text{by Proposition 24(i)}) \\
&= \sum_{\phi' \in \mathcal{F}(\chi)} \sum_{\psi \in \mathcal{S}_r(\chi - \phi')} p_i \left( \chi - \phi' - \sum_{\phi \in \mathcal{F}} \psi(\phi)\phi \right) \\
&\times \prod_{\phi \in \mathcal{F}} \left( \binom{\alpha(h_i) + |\phi| - 1}{|\phi|} |\phi|! \right)^{\psi(\phi)} (x_\alpha^+ \otimes b\pi(\phi))^{\psi(\phi)} \\
&\times \binom{\alpha(h_i) + |\phi'| - 1}{|\phi'|} |\phi'|! (x_\alpha^+ \otimes b\pi(\phi')) \\
&(\text{by the induction hypothesis}) \\
&= \sum_{\phi' \in \mathcal{F}(\chi)} \sum_{\psi \in \mathcal{S}_r(\chi - \phi')} (\psi(\phi') + 1) p_i \left( \chi - \sum_{\phi \in \mathcal{F}} (\psi + \chi_{\phi'}) (\phi)\phi \right) \\
&\times \prod_{\phi \in \mathcal{F}} \left( \binom{\alpha(h_i) + |\phi| - 1}{|\phi|} |\phi|! \right)^{(\psi + \chi_{\phi'}) (\phi)} \\
&\times (x_\alpha^+ \otimes b\pi(\phi))^{\psi(\phi)} \\
&= \sum_{\psi \in \mathcal{S}_{r+1}(\chi)} \sum_{\phi' \in \mathcal{F}} \psi(\phi') p_i \left( \chi - \sum_{\phi \in \mathcal{F}} \psi(\phi)\phi \right) \\
&\times \prod_{\phi \in \mathcal{F}} \left( \binom{\alpha(h_i) + |\phi| - 1}{|\phi|} |\phi|! \right)^{\psi(\phi)} (x_\alpha^+ \otimes b\pi(\phi))^{\psi(\phi)} \\
&= (r+1) \sum_{\psi \in \mathcal{S}_{r+1}(\chi)} p_i \left( \chi - \sum_{\phi \in \mathcal{F}} \psi(\phi)\phi \right) \\
&\times \prod_{\phi \in \mathcal{F}} \left( \binom{\alpha(h_i) + |\phi| - 1}{|\phi|} |\phi|! \right)^{\psi(\phi)} (x_\alpha^+ \otimes b\pi(\phi))^{\psi(\phi)}
\end{aligned}$$

Thus (i) is proved by induction on  $r$ . The proof of (ii) is almost exactly the same and hence is omitted. ■

### 2.6.3 An Explicit Formula for $D^\pm(\psi, |\psi|\chi_b, k\chi_c)$

**Proposition 26** For all  $\psi \in \mathcal{F}$ ,  $k \in \mathbb{Z}_{\geq 0}$  with  $k \geq 1$ , and  $b, c \in A$ .

$$D^\pm(\psi, |\psi|\chi_b, k\chi_c) = \sum_{\chi \in \mathcal{P}_k(\psi)} \prod_{\phi \in \mathcal{F}} (|\phi|!)^{\chi(\phi)} \left( x^\pm \otimes cb^{|\phi|} \pi(\phi) \right)^{(\chi(\phi))}$$

**Proof.** The proof will proceed by induction on  $k$ . If  $k = 1$  the proposition is clear by definition. Assume the proposition for some  $k \geq 1$ . Then by definition

$$\begin{aligned} D^\pm(\psi, |\psi|\chi_b, (k+1)\chi_c) &= \frac{1}{k+1} \sum_{\phi' \in \mathcal{F}(\psi)} |\phi'|! \left( x^\pm \otimes cb^{|\phi'|} \pi(\phi') \right) \\ &\times D^\pm(\psi - \phi', (|\psi| - |\phi'|)\chi_b, k\chi_c) \\ &= \frac{1}{k+1} \sum_{\phi' \in \mathcal{F}(\psi)} |\phi'|! \left( x^\pm \otimes cb^{|\phi'|} \pi(\phi') \right) \\ &\times \sum_{\chi \in \mathcal{P}_k(\psi - \phi')} \prod_{\phi \in \mathcal{F}} (|\phi|!)^{\chi(\phi)} \left( x^\pm \otimes cb^{|\phi|} \pi(\phi) \right)^{(\chi(\phi))} \\ &\text{(by the induction hypothesis)} \\ &= \frac{1}{k+1} \sum_{\phi' \in \mathcal{F}(\psi)} \sum_{\chi \in \mathcal{P}_k(\psi - \phi')} (\chi(\phi') + 1) \prod_{\phi \in \mathcal{F}} (|\phi|!)^{(\chi + \chi_{\phi'}) (\phi)} \\ &\times \left( x^\pm \otimes cb^{|\phi|} \pi(\phi) \right)^{((\chi + \chi_{\phi'}) (\phi))} \\ &= \frac{1}{k+1} \sum_{\phi' \in \mathcal{F}(\psi)} \sum_{\chi + \chi_{\phi'} \in \mathcal{P}_{k+1}(\psi)} (\chi(\phi') + 1) \prod_{\phi \in \mathcal{F}} (|\phi|!)^{(\chi + \chi_{\phi'}) (\phi)} \\ &\times \left( x^\pm \otimes cb^{|\phi|} \pi(\phi) \right)^{((\chi + \chi_{\phi'}) (\phi))} \\ &= \frac{1}{k+1} \sum_{\chi \in \mathcal{P}_{k+1}(\psi)} \sum_{\phi' \in \mathcal{F}} \chi(\phi') \prod_{\phi \in \mathcal{F}} (|\phi|!)^{\chi(\phi)} \left( x^\pm \otimes cb^{|\phi|} \pi(\phi) \right)^{(\chi(\phi))} \\ &= \sum_{\chi \in \mathcal{P}_{k+1}(\psi)} \prod_{\phi \in \mathcal{F}} (|\phi|!)^{\chi(\phi)} \left( x^\pm \otimes cb^{|\phi|} \pi(\phi) \right)^{(\chi(\phi))} \end{aligned}$$

■

## 2.7 A Triangular Decomposition of $U_{\mathbb{Z}}(\mathfrak{g} \otimes A)$

This section is devoted to the proof of the following lemma.

**Lemma 27**

$$U_{\mathbb{Z}}(\mathfrak{g} \otimes A) = U_{\mathbb{Z}}^{-}(\mathfrak{g} \otimes A)U_{\mathbb{Z}}^0(\mathfrak{g} \otimes A)U_{\mathbb{Z}}^{+}(\mathfrak{g} \otimes A)$$

**Definition.** Define a *monomial* in  $U_{\mathbb{Z}}(\mathfrak{g} \otimes A)$  to be any product of elements in the set

$$\{(x_{\alpha}^{\pm} \otimes b)^{(r)}, q_i(\chi) : \alpha \in R^+, b \in \mathbf{B}, r \in \mathbb{Z}_{\geq 0}, i \in I, \chi \in \mathcal{F}(\mathbf{B})\}$$

By induction on the degree of monomials in  $U_{\mathbb{Z}}(\mathfrak{g} \otimes A)$  Lemma 27 will follow from the next lemma.

### 2.7.1 Brackets in $U_{\mathbb{Z}}(\mathfrak{g} \otimes A)$

**Lemma 28** For all  $\alpha \in R^+$ ,  $i \in I$ ,  $a, b \in \mathbf{B}$ ,  $\chi \in \mathcal{F}(\mathbf{B})$  and  $r, s \in \mathbb{Z}_{\geq 0}$  the following hold

- (a)  $\left[ (x_{\alpha}^{+} \otimes a)^{(r)}, (x_{\alpha}^{-} \otimes b)^{(s)} \right] \in \mathbb{Z} - \text{span } \mathcal{B}$  and has degree less than  $r + s$ .
- (b)  $\left[ (x_{\alpha}^{+} \otimes a)^{(r)}, p_i(\chi) \right] \in \mathbb{Z} - \text{span } \mathcal{B}$  and has degree less than  $r + |\chi|$ .
- (c)  $\left[ p_i(\chi), (x_{\alpha}^{-} \otimes a)^{(s)} \right] \in \mathbb{Z} - \text{span } \mathcal{B}$  and has degree less than  $s + |\chi|$ .

**Proof.** We start by proving (a). By Lemma 13, for all  $a, b \in \mathbf{B}$  and  $r, s \in \mathbb{Z}_{\geq 0}$ ,

$$\begin{aligned} x^{+}(r\chi_a)x^{-}(s\chi_b) &= \sum_{\substack{j, k \in \mathbb{Z}_{\geq 0} \\ j+k \leq \min(r, s)}} (-1)^{j+k} D^{-}(j\chi_a, j\chi_b, (s-j-k)\chi_b) \\ &\times \mathbb{D}(k\chi_a, k\chi_b, (r-j-k)\chi_a) \\ &= \sum_{\substack{j, k \in \mathbb{Z}_{\geq 0} \\ j+k \leq \min(r, s)}} \sum_{l=0}^k (-1)^{j+k} D^{-}(j\chi_a, j\chi_b, (s-j-k)\chi_b) p(l\chi_a, l\chi_b) \\ &\times D^{+}((k-l)\chi_a, (k-l)\chi_b, (r-j-k)\chi_a) \end{aligned}$$

It can be easily shown by induction on  $l$  that  $q(l\chi_a, l\chi_b) = q(l\chi_{ab})$ . So we have

$$\begin{aligned} x^+(r\chi_a)x^-(s\chi_b) &= \sum_{\substack{j,k \in \mathbb{Z}_{\geq 0} \\ 0 < j+k \leq \min(r,s)}} \sum_{l=0}^k (-1)^{j+k} D^-(j\chi_a, j\chi_b, (s-j-k)\chi_b) p(l\chi_{ab}) \\ &\times D^+((k-l)\chi_a, (k-l)\chi_b, (r-j-k)\chi_a) + x^-(s\chi_b)x^+(r\chi_a) \end{aligned}$$

$D^-(j\chi_a, j\chi_b, (s-j-k)\chi_b) \in \mathbb{Z}$ -span  $\mathcal{B}_-$  and  $D^+((k-l)\chi_a, (k-l)\chi_b, (r-j-k)\chi_a) \in \mathbb{Z}$ -span  $\mathcal{B}_+$  by Proposition 26 and the fact that, for all  $\phi \in \mathcal{F}$ ,

$$\frac{|\phi|!}{\prod_{a \in A} \phi(a)!} \in \mathbb{Z} \quad (2.7.1)$$

(2.7.1) can be easily shown by induction on  $|\phi|$  using the fact that, for all  $b \in \text{supp } \phi$ ,

$$\frac{|\phi|!}{\prod_{a \in A} \phi(a)!} = \binom{|\phi|}{\phi(b)} \frac{(|\phi| - \phi(b))!}{\prod_{a \neq b} \phi(a)!}$$

Also we clearly see that the degree of this commutator is less than  $r+s$ . Thus (a) is proved.

(b) and (c) hold by (2.7.1) and Proposition 25. ■

The theorem is now proved once we prove Lemma 13. The next section are dedicated to the proof of Lemma 13.

## 2.8 Proof of Lemma 13

We will adapt and extend the proof of the corresponding result in [4]. We begin with some necessary identities.

### 2.8.1 Identity for $p(\varphi, \chi)(x^+ \otimes b)$

**Lemma 29** For all  $\varphi, \chi \in \mathcal{F}$ , and  $b \in A$ .

$$\begin{aligned} p(\varphi, \chi)(x^+ \otimes b) &= (x^+ \otimes b)p(\varphi, \chi) - 2 \sum_{\substack{c \in \text{supp } \varphi \\ d \in \text{supp } \chi}} (x^+ \otimes bcd)p(\varphi - \chi_c, \chi - \chi_d) \\ &\quad + 4 \sum_{\substack{\phi_1 \in \mathcal{F}_2(\varphi) \\ \phi_2 \in \mathcal{F}_2(\chi)}} (x^+ \otimes b\pi(\psi_1)\pi(\psi_2)) p(\varphi - \psi_1, \chi - \psi_2) \end{aligned}$$

**Proof.** If  $|\varphi| \neq |\chi|$  then both sides are zero. So assume that  $|\chi| = |\varphi|$ . The proof will proceed by induction on  $|\chi| = |\varphi|$ . The identity is easy to check in the cases  $\varphi = \chi = 0$  and  $\varphi = \chi_a, \chi = \chi_b$ . Assume that  $|\varphi| = |\chi| > 1$ . Then

$$\begin{aligned} p(\varphi, \chi)(x^+ \otimes b) &= -\frac{1}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \psi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\psi_1|!)^2 (h \otimes \pi(\psi_1)\pi(\psi_2)) p(\varphi - \psi_1, \chi - \psi_2)(x^+ \otimes b) \\ &= -\frac{1}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \psi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\psi_1|!)^2 (h \otimes \pi(\psi_1)\pi(\psi_2)) \left( (x^+ \otimes b)p(\varphi - \psi_1, \chi - \psi_2) \right. \\ &\quad \left. - 2 \sum_{\substack{c \in \text{supp}(\varphi - \psi_1) \\ d \in \text{supp}(\chi - \psi_2)}} (x^+ \otimes bcd)p(\varphi - \psi_1 - \chi_c, \chi - \psi_2 - \chi_d) \right. \\ &\quad \left. + 4 \sum_{\substack{\phi_1 \in \mathcal{F}_2(\varphi - \psi_1) \\ \phi_2 \in \mathcal{F}_2(\chi - \psi_2)}} (x^+ \otimes b\pi(\phi_1)\pi(\phi_2)) p(\varphi - \psi_1 - \phi_1, \chi - \psi_2 - \phi_2) \right) \\ &\quad \text{(by the induction hypothesis)} \end{aligned}$$

$$\begin{aligned}
&= (x^+ \otimes b)p(\varphi, \chi) - \frac{2}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) - \{0\} \\ 0 < \psi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\psi_1|!)^2 (x^+ \otimes b\pi(\psi_1)\pi(\psi_2)) p(\varphi - \psi_1, \chi - \psi_2) \\
&- \frac{1}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \psi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\psi_1|!)^2 \left( -2 \sum_{\substack{c \in \text{supp}(\varphi - \psi_1) \\ d \in \text{supp}(\chi - \psi_2)}} (x^+ \otimes bcd) (h \otimes \pi(\psi_1)\pi(\psi_2)) \right. \\
&\times p(\varphi - \psi_1 - \chi_c, \chi - \psi_2 - \chi_d) \\
&+ 4 \sum_{\substack{\phi_1 \in \mathcal{F}_2(\varphi - \psi_1) \\ \phi_2 \in \mathcal{F}_2(\chi - \psi_2)}} (x^+ \otimes b\pi(\phi_1)\pi(\phi_2)) (h \otimes \pi(\psi_1)\pi(\psi_2)) p(\varphi - \psi_1 - \phi_1, \chi - \psi_2 - \phi_2) \\
&- 4 \sum_{\substack{c \in \text{supp}(\varphi - \psi_1) \\ d \in \text{supp}(\chi - \psi_2)}} (x^+ \otimes bcd\pi(\psi_1)\pi(\psi_2)) p(\varphi - \psi_1 - \chi_c, \chi - \psi_2 - \chi_d) \\
&+ 8 \sum_{\substack{\phi_1 \in \mathcal{F}_2(\varphi - \psi_1) \\ \phi_2 \in \mathcal{F}_2(\chi - \psi_2)}} (x^+ \otimes b\pi(\phi_1)\pi(\phi_2)\pi(\psi_1)\pi(\psi_2)) p(\varphi - \psi_1 - \phi_1, \chi - \psi_2 - \phi_2) \Big) \\
&= (x^+ \otimes b)p(\varphi, \chi) - \frac{2}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) - \{0\} \\ 0 < \psi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\psi_1|!)^2 (x^+ \otimes b\pi(\psi_1)\pi(\psi_2)) p(\varphi - \psi_1, \chi - \psi_2) \\
&+ \frac{2}{|\varphi|} \sum_{\substack{c \in \text{supp} \varphi \\ d \in \text{supp} \chi}} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi - \chi_c) - \{0\} \\ \psi_2 \in \mathcal{F}(\chi - \chi_d) - \{0\}}} (|\psi_1|!)^2 (x^+ \otimes bcd) (h \otimes \pi(\psi_1)\pi(\psi_2)) \\
&\times p(\varphi - \psi_1 - \chi_c, \chi - \psi_2 - \chi_d) \\
&- \frac{4}{|\varphi|} \sum_{\substack{\phi_1, \phi_2 \in \mathcal{F}_2 \\ \phi_1 < \varphi \\ \phi_2 < \chi}} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi - \phi_1) - \{0\} \\ \psi_2 \in \mathcal{F}(\chi - \phi_2) - \{0\}}} (|\psi_1|!)^2 (x^+ \otimes b\pi(\phi_1)\pi(\phi_2)) (h \otimes \pi(\psi_1)\pi(\psi_2)) \\
&\times p(\varphi - \psi_1 - \phi_1, \chi - \psi_2 - \phi_2) \\
&- \frac{8}{|\varphi|} \sum_{\substack{\psi'_1 \in \mathcal{F}(\varphi) \\ \psi'_2 \in \mathcal{F}(\chi) \\ |\psi'_1| = |\psi'_2| > 2}} \sum_{\substack{\phi_1 \in \mathcal{F}_2(\psi'_1) \\ \phi_2 \in \mathcal{F}_2(\psi'_2)}} ((|\psi'_1| - 2)!)^2 \prod_{a \in A} \begin{pmatrix} \psi'_1(a) \\ \phi_1(a) \end{pmatrix} \begin{pmatrix} \psi'_2(a) \\ \phi_2(a) \end{pmatrix} (x^+ \otimes b\pi(\psi'_1)\pi(\psi'_2)) \\
&\times p(\varphi - \psi'_1, \chi - \psi'_2) \\
&+ \frac{4}{|\varphi|} \sum_{\substack{\psi'_1 \in \mathcal{F}(\varphi) \\ \psi'_2 \in \mathcal{F}(\chi) \\ |\psi'_1| = |\psi'_2| > 1}} \sum_{\substack{c \in \text{supp} \psi'_1 \\ d \in \text{supp} \psi'_2}} ((|\psi'_1| - 1)!)^2 \psi'_1(c) \psi'_2(d) (x^+ \otimes b\pi(\psi'_1)\pi(\psi'_2)) p(\varphi - \psi'_1, \chi - \psi'_2)
\end{aligned}$$



$$\begin{aligned}
&= (x^+ \otimes b)p(\varphi, \chi) - \frac{2}{|\varphi|} \sum_{\substack{\psi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \psi_2 \in \mathcal{F}(\chi) - \{0\}}} (|\psi_1|!)^2 (x^+ \otimes b\pi(\psi_1)\pi(\psi_2)) p(\varphi - \psi_1, \chi - \psi_2) \\
&- \frac{2(|\varphi| - 1)}{|\varphi|} \sum_{\substack{c \in \text{supp } \varphi \\ d \in \text{supp } \chi}} (x^+ \otimes bcd)p(\varphi - \chi_c, \chi - \chi_d) \\
&+ \frac{4(|\varphi| - 2)}{|\varphi|} \sum_{\substack{\phi_1 \in \mathcal{F}_2(\varphi) \\ \phi_2 \in \mathcal{F}_2(\chi)}} (x^+ \otimes b\pi(\phi_1)\pi(\phi_2)) p(\varphi - \phi_1, \chi - \phi_2) \\
&- \frac{2}{|\varphi|} \sum_{\substack{\psi'_1 \in \mathcal{F}(\varphi) \\ \psi'_2 \in \mathcal{F}(\chi) \\ |\psi'_1| = |\psi'_2| > 2}} (|\psi'_1|!)^2 (x^+ \otimes b\pi(\psi'_1)\pi(\psi'_2)) p(\varphi - \psi'_1, \chi - \psi'_2) \\
&+ \frac{4}{|\varphi|} \sum_{\substack{\psi'_1 \in \mathcal{F}(\varphi) \\ \psi'_2 \in \mathcal{F}(\chi) \\ |\psi'_1| = |\psi'_2| > 1}} (|\psi'_1|!)^2 (x^+ \otimes b\pi(\psi'_1)\pi(\psi'_2)) p(\varphi - \psi'_1, \chi - \psi'_2) \\
&= (x^+ \otimes b)p(\varphi, \chi) - 2 \sum_{\substack{c \in \text{supp } \varphi \\ d \in \text{supp } \chi}} (x^+ \otimes bcd)p(\varphi - \chi_c, \chi - \chi_d) \\
&+ 4 \sum_{\substack{\phi_1 \in \mathcal{F}_2(\varphi) \\ \phi_2 \in \mathcal{F}_2(\chi)}} (x^+ \otimes b\pi(\phi_1)\pi(\phi_2)) p(\varphi - \phi_1, \chi - \phi_2)
\end{aligned}$$

The second to last equation holds because

$$\sum_{\phi_1 \in \mathcal{F}_2(\psi'_1), \phi_2 \in \mathcal{F}_2(\psi'_2)} \prod_{a \in A} \binom{\psi'_1(a)}{\phi_1(a)} \binom{\psi'_2(a)}{\phi_2(a)} = \frac{(|\psi'_1|(|\psi'_1| - 1))^2}{4} \quad (2.8.1)$$

which can be easily shown by considering cases for  $\phi_1, \phi_2$ . ■

### 2.8.2 Identities Involving $D^\pm(\psi_1, \psi_2, \psi_3)$

**Proposition 30** *Let  $b \in A$ , and  $\psi_1, \psi_2, \psi_3 \in \mathcal{F}$  with  $|\psi_3| \geq 1$  be given. Then*

(i)

$$\begin{aligned}
\psi_2(b)D^\pm(\psi_1, \psi_2, \psi_3) &= \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \sum_{c \in \text{supp } \psi_3} \phi_2(b)D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c)
\end{aligned}$$

(ii)

$$\begin{aligned}
(|\psi_2| + |\psi_3|)D^\pm(\psi_1, \psi_2, \psi_3) &= \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \sum_{c \in \text{supp } \psi_3} (|\phi_1| + 1)D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c)
\end{aligned}$$

**Proof.** The proofs will proceed by induction on  $|\psi_3|$ . The case  $|\psi_3| = 1$  is trivial. Assume that  $|\psi_3| > 1$  then for (i) we have

$$\begin{aligned}
|\psi_3|\psi_2(b)D^\pm(\psi_1, \psi_2, \psi_3) &= \psi_2(b) \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c)(\psi_2(b) - \phi_2(b)) \\
&\times D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \phi_2(b) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) \sum_{\substack{\phi'_1 \in \mathcal{F}(\psi_1 - \phi_1) \\ \phi'_2 \in \mathcal{F}(\psi_2 - \phi_2)}} \sum_{d \in \text{supp}(\psi_3 - \chi_c)} \\
&\times D^\pm(\phi'_1, \phi'_2, \chi_d)D^\pm(\psi_1 - \phi_1 - \phi'_1, \psi_2 - \phi_2 - \phi'_2, \psi_3 - \chi_c - \chi_d) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \phi_2(b) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c)
\end{aligned}$$

(by the induction hypothesis)

$$\begin{aligned}
&= \sum_{\substack{\phi'_1 \in \mathcal{F}(\psi_1) \\ \phi'_2 \in \mathcal{F}(\psi_2)}} \sum_{d \in \text{supp } \psi_3} \phi'_2(b) D^\pm(\phi'_1, \phi'_2, \chi_d) \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1 - \phi'_1) \\ \phi_2 \in \mathcal{F}(\psi_2 - \phi'_2)}} \sum_{c \in \text{supp}(\psi_3 - \chi_d)} D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times D^\pm(\psi_1 - \phi_1 - \phi'_1, \psi_2 - \phi_2 - \phi'_2, \psi_3 - \chi_c - \chi_d) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \phi_2(b) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&= (|\psi_3| - 1) \sum_{\substack{\phi'_1 \in \mathcal{F}(\psi_1) \\ \phi'_2 \in \mathcal{F}(\psi_2)}} \phi'_2(b) \sum_{d \in \text{supp } \psi_3} D^\pm(\phi'_1, \phi'_2, \chi_d) D^\pm(\psi_1 - \phi'_1, \psi_2 - \phi'_2, \psi_3 - \chi_d) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \phi_2(b) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&= |\psi_3| \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \phi_2(b) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c)
\end{aligned}$$

For (ii) we have

$$\begin{aligned}
|\psi_3|(|\psi_2| + |\psi_3|) D^\pm(\psi_1, \psi_2, \psi_3) &= (|\psi_2| + |\psi_3|) \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times (|\psi_2| - |\phi_2| + |\psi_3| - 1) D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} (|\phi_2| + 1) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) \sum_{b \in \text{supp}(\psi_3 - \chi_c)} \sum_{\substack{\phi'_1 \in \mathcal{F}(\psi_1 - \phi_1) \\ \phi'_2 \in \mathcal{F}(\psi_2 - \phi_2)}} (|\phi'_1| + 1) D^\pm(\phi'_1, \phi'_2, \chi_b) \\
&\times D^\pm(\psi_1 - \phi_1 - \phi'_1, \psi_2 - \phi_2 - \phi'_2, \psi_3 - \chi_c - \chi_b) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} (|\phi_2| + 1) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&\quad \text{(by the induction hypothesis)} \\
&= \sum_{\substack{\phi'_1 \in \mathcal{F}(\psi_1) \\ \phi'_2 \in \mathcal{F}(\psi_2)}} \sum_{b \in \text{supp } \psi_3} (|\phi'_1| + 1) D^\pm(\phi'_1, \phi'_2, \chi_b) \sum_{c \in \text{supp}(\psi_3 - \chi_b)} \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1 - \phi'_1) \\ \phi_2 \in \mathcal{F}(\psi_2 - \phi'_2)}} D^\pm(\phi_1, \phi_2, \chi_c) \\
&\times D^\pm(\psi_1 - \phi_1 - \phi'_1, \psi_2 - \phi_2 - \phi'_2, \psi_3 - \chi_c - \chi_b) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} (|\phi_2| + 1) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&= (|\psi_3| - 1) \sum_{\substack{\phi'_1 \in \mathcal{F}(\psi_1) \\ \phi'_2 \in \mathcal{F}(\psi_2)}} \sum_{b \in \text{supp } \psi_3} (|\phi'_1| + 1) D^\pm(\phi'_1, \phi'_2, \chi_b) D^\pm(\psi_1 - \phi'_1, \psi_2 - \phi'_2, \psi_3 - \chi_b) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} (|\phi_2| + 1) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c) \\
&= |\psi_3| \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2)}} (|\phi_2| + 1) \sum_{c \in \text{supp } \psi_3} D^\pm(\phi_1, \phi_2, \chi_c) D^\pm(\psi_1 - \phi_1, \psi_2 - \phi_2, \psi_3 - \chi_c)
\end{aligned}$$

■

The following lemma is necessary for the proof of Lemma 13.

### 2.8.3 Identity for $\sum_{\phi \in \mathcal{F}_{|\chi|}} \mathbb{D}(\phi, \chi, \varphi - \phi)(x^- \otimes b)$

**Lemma 31** For all  $b \in A$  and  $\varphi, \chi \in \mathcal{F}$ ,

$$\begin{aligned} \sum_{\phi \in \mathcal{F}(\varphi)} \mathbb{D}(\phi, \chi, \varphi - \phi)(x^- \otimes b) &= -(\chi(b) + 1) \sum_{\phi' \in \mathcal{F}(\varphi)} \mathbb{D}(\phi', \chi + \chi b, \varphi - \phi') \\ &+ \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) D^-(\phi_1, \phi_2, \chi b) \\ &\times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi) \end{aligned}$$

### 2.8.4 Proof of Lemma 13

The following is the proof of Lemma 13 using Lemma 31 and induction on  $|\chi|$ .

**Proof.** It can easily be shown by induction on  $|\varphi|$  that

$$\mathbb{D}(0, 0, \varphi) = x^+(\varphi)$$

Thus Lemma 13 is true for  $\chi = 0$ . Assume that  $\chi \in \mathcal{F} - \{0\}$  then

$$\begin{aligned} |\chi| x^+(\varphi) x^-(\chi) &= \sum_{b \in \text{supp } \chi} \chi(b) x^+(\varphi) x^-(\chi) \\ &= \sum_{b \in \text{supp } \chi} x^+(\varphi) x^-(\chi - \chi b)(x^- \otimes b) \\ &= \sum_{b \in \text{supp } \chi} \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi - \chi b) \\ \phi_1 + \phi_2 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \chi b - \psi_1 - \psi_2) \\ &\times \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2)(x^- \otimes b) \quad (\text{by the induction hypothesis}) \\ &= \sum_{b \in \text{supp } \chi} \sum_{\substack{\psi_1, \psi_2, \phi_1 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi - \chi b) \\ \phi_1 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \chi b - \psi_1 - \psi_2) \\ &\times \sum_{\phi_2 \in \mathcal{F}(\varphi - \phi_1)} \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2)(x^- \otimes b) \end{aligned}$$

$$\begin{aligned}
&= \sum_{b \in \text{supp } \chi} \sum_{\substack{\psi_1, \psi_2, \phi_1 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi - \chi_b) \\ \phi_1 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \chi_b - \psi_1 - \psi_2) \\
&\times \left( -(\psi_2(b) + 1) \sum_{\phi'_2 \in \mathcal{F}(\varphi - \phi_1)} \mathbb{D}(\phi'_2, \psi_2 + \chi_b, \varphi - \phi_1 - \phi'_2) \right. \\
&+ \left. \sum_{\phi_2 \in \mathcal{F}(\varphi - \phi_1)} \sum_{\substack{\tau \in \mathcal{F}(\phi_2) \\ \tau' \in \mathcal{F}(\psi_2)}} (|\tau| + 1) D^-(\tau, \tau', \chi_b) \mathbb{D}(\phi_2 - \tau, \psi_2 - \tau', \varphi - \phi_1 - \phi_2) \right) \\
&\text{(by Lemma 31)} \\
&= \sum_{b \in \text{supp } \chi} \sum_{\substack{\psi_1, \psi_2, \phi_1 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi - \chi_b) \\ \phi_1 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi_2| + 1} D^-(\phi_1, \psi_1, \chi - \chi_b - \psi_1 - \psi_2) (\psi_2(b) + 1) \\
&\times \sum_{\phi'_2 \in \mathcal{F}(\varphi - \phi_1)} \mathbb{D}(\phi'_2, \psi_2 + \chi_b, \varphi - \phi_1 - \phi'_2) \\
&+ \sum_{b \in \text{supp } \chi} \sum_{\substack{\psi_1, \psi_2, \phi_1 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi - \chi_b) \\ \phi_1 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \chi_b - \psi_1 - \psi_2) \sum_{\phi_2 \in \mathcal{F}(\varphi - \phi_1)} \\
&\times \sum_{\substack{\tau \in \mathcal{F}(\phi_2) \\ \tau' \in \mathcal{F}(\psi_2)}} (|\tau| + 1) D^-(\tau, \tau', \chi_b) \mathbb{D}(\phi_2 - \tau, \psi_2 - \tau', \varphi - \phi_1 - \phi_2) \\
&= \sum_{b \in \text{supp } \chi} \sum_{\substack{\psi_1, \psi'_2, \phi_1 \in \mathcal{F} \\ \psi_1 + \psi'_2 \in \mathcal{F}(\chi) \\ \phi_1 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi'_2|} D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi'_2) \\
&\times \psi'_2(b) \sum_{\phi'_2 \in \mathcal{F}(\varphi - \phi_1)} \mathbb{D}(\phi'_2, \psi'_2, \varphi - \phi_1 - \phi'_2) \\
&+ \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \in \mathcal{F}(\chi) \\ \phi_1 + \phi_2 \in \mathcal{F}(\varphi)}} \sum_{b \in \text{supp}(\chi - \psi_1 - \psi_2)} \sum_{\substack{\tau \in \mathcal{F}(\phi_2) \\ \tau' \in \mathcal{F}(\psi_2)}} (|\tau| + 1) (-1)^{|\psi_1| + |\psi_2|} \\
&\times D^-(\phi_1, \psi_1, \chi - \chi_b - \psi_1 - \psi_2) D^-(\tau, \tau', \chi_b) \mathbb{D}(\phi_2 - \tau, \psi_2 - \tau', \varphi - \phi_1 - \phi_2)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\psi_1, \psi'_2, \phi_1, \phi'_2 \in \mathcal{F} \\ \psi_1 + \psi'_2 \in \mathcal{F}(\chi) \\ \phi_1 + \phi'_2 \in \mathcal{F}(\varphi)}} (-1)^{|\psi_1| + |\psi'_2|} |\psi'_2| D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi'_2) \mathbb{D}(\phi'_2, \psi'_2, \varphi - \phi_1 - \phi'_2) \\
&+ \sum_{\substack{\psi'_1, \psi'_2, \phi'_1, \phi'_2 \in \mathcal{F} \\ \psi'_1 + \psi'_2 \in \mathcal{F}(\chi) \\ \phi'_1 + \phi'_2 \in \mathcal{F}(\varphi)}} (-1)^{|\psi'_1| + |\psi'_2|} \sum_{\substack{\tau \in \mathcal{F}(\phi'_1) \\ \tau' \in \mathcal{F}(\psi'_1)}} \sum_{b \in \text{supp}(\chi - \psi'_1 - \psi'_2)} (|\tau| + 1) \\
&\times D^-(\tau, \tau', \chi_b) D^-(\phi'_1 - \tau, \psi'_1 - \tau', \chi - \chi_b - \psi'_1 - \psi'_2) \mathbb{D}(\phi'_2, \psi'_2, \varphi - \phi_1 - \phi_2) \\
&= \sum_{\substack{\psi_1, \psi'_2, \phi_1, \phi'_2 \in \mathcal{F} \\ \psi_1 + \psi'_2 \leq \chi \\ \phi_1 + \phi'_2 \leq \varphi}} (-1)^{|\psi_1| + |\psi'_2|} |\psi'_2| D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi'_2) \mathbb{D}(\phi'_2, \psi'_2, \varphi - \phi_1 - \phi'_2) \\
&+ \sum_{\substack{\psi'_1, \psi'_2, \phi'_1, \phi'_2 \in \mathcal{F} \\ \psi'_1 + \psi'_2 \leq \chi \\ \phi'_1 + \phi'_2 \leq \varphi}} (-1)^{|\psi'_1| + |\psi'_2|} (|\chi| - |\psi'_2|) D^-(\phi'_1, \psi'_1, \chi - \psi'_1 - \psi'_2) \mathbb{D}(\phi'_2, \psi'_2, \varphi - \phi_1 - \phi_2) \\
&\text{(by Proposition 30(ii))} \\
&= |\chi| \sum_{\substack{\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{F} \\ \psi_1 + \psi_2 \leq \chi \\ \phi_1 + \phi_2 \leq \varphi}} (-1)^{|\psi_1| + |\psi_2|} D^-(\phi_1, \psi_1, \chi - \psi_1 - \psi_2) \mathbb{D}(\phi_2, \psi_2, \varphi - \phi_1 - \phi_2)
\end{aligned}$$

■

So all that remains to prove for Lemma 13 and Theorem 6 is Lemma 31.

### 2.8.5 Proof of Lemma 31

For all  $b \in A$  and  $\varphi, \chi \in \mathcal{F}$ ,

$$\begin{aligned}
\sum_{\phi \in \mathcal{F}(\varphi)} \mathbb{D}(\phi, \chi, \varphi - \phi)(x^- \otimes b) &= -(\chi(b) + 1) \sum_{\phi' \in \mathcal{F}(\varphi)} \mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi') \\
&+ \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) D^-(\phi_1, \phi_2, \chi_b) \\
&\times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi)
\end{aligned}$$

**Proof.** If  $|\phi| > |\varphi|$  we have 0 on both sides. So the identity is trivially true in this case.

Assume that  $|\varphi| \geq |\phi|$  and proceed by induction on  $|\varphi| - |\phi|$ . In the case  $|\phi| = |\varphi|$  Lemma

31 becomes

$$p(\varphi, \chi)(x^- \otimes b) = \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1)! |\phi_1|! (x^- \otimes b \pi(\phi_1) \pi(\phi_2)) p(\varphi - \phi_1, \chi - \phi_2)$$

which is just Lemma 24(ii). So Assume that  $|\varphi| - |\phi| > 0$  then

$$\begin{aligned} (|\varphi| - |\phi|) \mathbb{D}(\phi, \chi, \varphi - \phi) &= (|\varphi| - |\phi|) \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} p(\phi_1, \phi_2) D^+(\phi - \phi_1, \chi - \phi_2, \varphi - \phi) \\ &= \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi - \phi_2)}} \sum_{c \in \text{supp}(\varphi - \phi)} p(\phi_1, \phi_2) D^+(\phi'_1, \phi'_2, \chi_c) \\ &\times D^+(\phi - \phi_1 - \phi'_1, \chi - \phi_2 - \phi'_2, \varphi - \phi - \chi_c) \\ &= \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \phi'_1) \\ \phi_2 \in \mathcal{F}(\chi - \phi'_2)}} \sum_{c \in \text{supp}(\varphi - \phi)} (|\phi'_1|!)^2 p(\phi_1, \phi_2) \\ &\times (x^+ \otimes c \pi(\phi'_1) \pi(\phi'_2)) D^+(\phi - \phi_1 - \phi'_1, \chi - \phi_2 - \phi'_2, \varphi - \phi - \chi_c) \\ &= \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \phi'_1) \\ \phi_2 \in \mathcal{F}(\chi - \phi'_2)}} \sum_{c \in \text{supp}(\varphi - \phi)} (|\phi'_1|!)^2 (x^+ \otimes c \pi(\phi'_1) \pi(\phi'_2)) \\ &\times p(\phi_1, \phi_2) D^+(\phi - \phi_1 - \phi'_1, \chi - \phi_2 - \phi'_2, \varphi - \phi - \chi_c) \\ &+ \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \phi'_1) \\ \phi_2 \in \mathcal{F}(\chi - \phi'_2)}} \sum_{c \in \text{supp}(\varphi - \phi)} (|\phi'_1|!)^2 \\ &\times [p(\phi_1, \phi_2), (x^+ \otimes c \pi(\phi'_1) \pi(\phi'_2))] \\ &\times D^+(\phi - \phi_1 - \phi'_1, \chi - \phi_2 - \phi'_2, \varphi - \phi - \chi_c) \end{aligned}$$



$$\begin{aligned}
&= \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} \sum_{c \in \text{supp}(\varphi - \phi)} (|\phi'_1|!)^2 (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \phi'_1) \\ \phi_2 \in \mathcal{F}(\chi - \phi'_2)}} p(\phi_1, \phi_2) \\
&\times D^+(\phi - \phi_1 - \phi'_1, \chi - \phi_2 - \phi'_2, \varphi - \phi - \chi_c) \\
&+ \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} \sum_{\phi_1 \in \mathcal{F}(\phi - \phi'_1)} \sum_{\phi_2 \in \mathcal{F}(\chi - \phi'_2)} \sum_{c \in \text{supp}(\varphi - \phi)} (|\phi'_1|!)^2 \\
&\times \left( -2 \sum_{\substack{c' \in \text{supp} \phi_1 \\ d \in \text{supp} \phi_2}} (x^+ \otimes cc'd\pi(\phi'_1)\pi(\phi'_2)) p(\phi_1 - \chi_{c'}, \phi_2 - \chi_d) \right. \\
&+ \left. 4 \sum_{\psi_1 \in \mathcal{F}(\phi_1), \psi_2 \in \mathcal{F}(\phi_2)} (x^+ \otimes c\pi(\psi_1)\pi(\psi_2)\pi(\phi'_1)\pi(\phi'_2)) p(\phi_1 - \psi_1, \phi_2 - \psi_2) \right) \\
&\times D^+(\phi - \phi_1 - \phi'_1, \chi - \phi_2 - \phi'_2, \varphi - \phi - \chi_c) \quad (\text{by Lemma 29}) \\
&= \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} \sum_{c \in \text{supp}(\varphi - \phi)} (|\phi'_1|!)^2 (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) \mathbb{D}(\phi - \phi'_1, \chi - \phi'_2, \varphi - \phi - \chi_c) \\
&- 2 \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} \sum_{\phi_1 \in \mathcal{F}(\phi - \phi'_1)} \sum_{\phi_2 \in \mathcal{F}(\chi - \phi'_2)} \sum_{c \in \text{supp}(\varphi - \phi)} (|\phi'_1|!)^2 \sum_{\substack{c' \in \text{supp} \phi_1 \\ d \in \text{supp} \phi_2}} (x^+ \otimes cc'd\pi(\phi'_1)\pi(\phi'_2)) \\
&\times p(\phi_1 - \chi_{c'}, \phi_2 - \chi_d) D^+(\phi - \phi_1 - \phi'_1, \chi - \phi_2 - \phi'_2, \varphi - \phi - \chi_c) \\
&+ 4 \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} \sum_{\phi_1 \in \mathcal{F}(\phi - \phi'_1)} \sum_{\phi_2 \in \mathcal{F}(\chi - \phi'_2)} \sum_{c \in \text{supp}(\varphi - \phi)} (|\phi'_1|!)^2 \sum_{\substack{\psi_1 \in \mathcal{F}(\phi_1) \\ \psi_2 \in \mathcal{F}(\phi_2)}} (x^+ \otimes c\pi(\psi_1)\pi(\psi_2)\pi(\phi'_1)\pi(\phi'_2)) \\
&\times p(\phi_1 - \psi_1, \phi_2 - \psi_2) D^+(\phi - \phi_1 - \phi'_1, \chi - \phi_2 - \phi'_2, \varphi - \phi - \chi_c)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} \sum_{c \in \text{supp}(\varphi - \phi)} (|\phi'_1|!)^2 (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) \mathbb{D}(\phi - \phi'_1, \chi - \phi'_2, \varphi - \phi - \chi_c) \\
&- 2 \sum_{\substack{\psi'_1 \in \mathcal{F}(\phi) \\ \psi'_2 \in \mathcal{F}(\chi) \\ |\psi'_1| = |\psi'_2| > 0}} \sum_{\psi_1 \in \mathcal{F}(\phi - \psi'_1)} \sum_{\psi_2 \in \mathcal{F}(\chi - \psi'_2)} \sum_{c \in \text{supp}(\varphi - \phi)} (|\psi'_1|!)^2 (x^+ \otimes c\pi(\psi'_1)\pi(\psi'_2)) p(\psi_1, \psi_2) \\
&\times D^+(\phi - \psi_1 - \psi'_1, \chi - \psi_2 - \psi'_2, \varphi - \phi - \chi_c) \\
&+ \sum_{\substack{\psi'_1 \in \mathcal{F}(\phi) \\ \psi'_2 \in \mathcal{F}(\chi) \\ |\psi'_1| = |\psi'_2| > 1}} \sum_{\psi_1 \in \mathcal{F}(\phi - \psi'_1)} \sum_{\psi_2 \in \mathcal{F}(\chi - \psi'_2)} \sum_{c \in \text{supp}(\varphi - \phi)} (|\psi'_1|!)^2 (x^+ \otimes c\pi(\psi'_1)\pi(\psi'_2)) p(\psi_1, \psi_2) \\
&\times D^+(\phi - \psi_1 - \psi'_1, \chi - \psi_2 - \psi'_2, \varphi - \phi - \chi_c) \quad (\text{by 2.8.1}) \\
&= \sum_{c \in \text{supp}(\varphi - \phi)} \left( \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1|!)^2 (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) \mathbb{D}(\phi - \phi'_1, \chi - \phi'_2, \varphi - \phi - \chi_c) \right. \\
&- 2 \sum_{\substack{\psi'_1 \in \mathcal{F}(\phi) \\ \psi'_2 \in \mathcal{F}(\chi) \\ |\psi'_1| = |\psi'_2| > 0}} (|\psi'_1|!)^2 (x^+ \otimes c\pi(\psi'_1)\pi(\psi'_2)) \mathbb{D}(\phi - \psi'_1, \chi - \psi'_2, \varphi - \phi - \chi_c) \\
&+ \left. \sum_{\substack{\psi'_1 \in \mathcal{F}(\phi) \\ \psi'_2 \in \mathcal{F}(\chi) \\ |\psi'_1| = |\psi'_2| > 1}} (|\psi'_1|!)^2 (x^+ \otimes c\pi(\psi'_1)\pi(\psi'_2)) \mathbb{D}(\phi - \psi'_1, \chi - \psi'_2, \varphi - \phi - \chi_c) \right) \\
&= \sum_{c \in \text{supp}(\varphi - \phi)} \left( (x^+ \otimes c) \mathbb{D}(\phi, \chi, \varphi - \phi - \chi_c) \right. \\
&- \left. \sum_{\substack{d \in \text{supp} \phi \\ d' \in \text{supp} \chi}} (x^+ \otimes cdd') \mathbb{D}(\phi - \chi_d, \chi - \chi_{d'}, \varphi - \phi - \chi_c) \right)
\end{aligned}$$

Thus

$$\begin{aligned}
(|\varphi| - |\phi|)\mathbb{D}(\phi, \chi, \varphi - \phi) &= \sum_{c \in \text{supp}(\varphi - \phi)} \left( (x^+ \otimes c)\mathbb{D}(\phi, \chi, \varphi - \phi - \chi_c) \right. \\
&\quad \left. - \sum_{\substack{d \in \text{supp} \phi \\ d' \in \text{supp} \chi}} (x^+ \otimes cdd')\mathbb{D}(\phi - \chi_d, \chi - \chi_{d'}, \varphi - \phi - \chi_c) \right) \quad (2.8.2)
\end{aligned}$$

Hence

$$\begin{aligned}
(|\varphi| - |\phi|) \sum_{\phi \in \mathcal{F}(\varphi)} \mathbb{D}(\phi, \chi, \varphi - \phi)(x^- \otimes b) &= \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \left( (x^+ \otimes c) \right. \\
&\quad \times \mathbb{D}(\phi, \chi, \varphi - \phi - \chi_c) \\
&\quad - \sum_{\substack{d \in \text{supp} \phi \\ d' \in \text{supp} \chi}} (x^+ \otimes cdd') \\
&\quad \times \mathbb{D}(\phi - \chi_d, \chi - \chi_{d'}, \varphi - \phi - \chi_c) \left. \right) (x^- \otimes b) \\
&= \sum_{c \in \text{supp} \varphi} (x^+ \otimes c) \\
&\quad \times \sum_{\phi \in \mathcal{F}(\varphi - \chi_c)} \mathbb{D}(\phi, \chi, \varphi - \phi - \chi_c)(x^- \otimes b) \\
&\quad - \sum_{\substack{c \in \text{supp} \varphi \\ d \in \text{supp}(\varphi - \chi_c) \\ d' \in \text{supp} \chi}} (x^+ \otimes cdd') \\
&\quad \times \sum_{\phi' \in \mathcal{F}(\varphi - \chi_c - \chi_d)} \mathbb{D}(\phi', \chi - \chi_{d'}, \varphi - \phi' - \chi_d - \chi_c) \\
&\quad \times (x^- \otimes b)
\end{aligned}$$

$$\begin{aligned}
&= -(\chi(b) + 1) \sum_{c \in \text{supp } \varphi} (x^+ \otimes c) \sum_{\phi' \in \mathcal{F}(\varphi - \chi_c)} \mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi' - \chi_c) \\
&+ \sum_{c \in \text{supp } \varphi} \sum_{\phi \in \mathcal{F}(\varphi - \chi_c)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1)(x^+ \otimes c) D^-(\phi_1, \phi_2, \chi_b) \\
&\times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi_c) \\
&+ \sum_{\substack{d' \in \text{supp } \chi \\ c \in \text{supp } \varphi \\ d \in \text{supp}(\varphi - \chi_c)}} \sum_{\phi' \in \mathcal{F}(\varphi - \chi_c - \chi_d)} (\chi(b) - \chi_{d'}(b) + 1)(x^+ \otimes c d d') \\
&\times \mathbb{D}(\phi', \chi - \chi_{d'} + \chi_b, \varphi - \phi' - \chi_d - \chi_c) \\
&- \sum_{\substack{d' \in \text{supp } \chi \\ c \in \text{supp } \varphi \\ d \in \text{supp}(\varphi - \chi_c)}} \sum_{\phi \in \mathcal{F}(\varphi - \chi_c - \chi_d)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi - \chi_{d'})}} (|\phi_1| + 1)(x^+ \otimes c d d') D^-(\phi_1, \phi_2, \chi_b) \\
&\times \mathbb{D}(\phi - \phi_1, \chi - \chi_{d'} - \phi_2, \varphi - \phi - \chi_d - \chi_c) \quad (\text{by the induction hypothesis}) \\
&= -(\chi(b) + 1) \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi')} (x^+ \otimes c) \mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi' - \chi_c) \\
&+ \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1)(x^+ \otimes c) D^-(\phi_1, \phi_2, \chi_b) \\
&\times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi_c) \\
&+ \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{d \in \text{supp } \phi \\ d' \in \text{supp } \chi}} (\chi(b) - \chi_{d'}(b) + 1)(x^+ \otimes c d d') \\
&\times \mathbb{D}(\phi - \chi_d, \chi - \chi_{d'} + \chi_b, \varphi - \phi - \chi_c) \\
&- \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{d \in \text{supp } \phi \\ d' \in \text{supp } \chi}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \chi_d) \\ \phi_2 \in \mathcal{F}(\chi - \chi_{d'})}} (|\phi_1| + 1) \\
&\times (x^+ \otimes c d d') D^-(\phi_1, \phi_2, \chi_b) \mathbb{D}(\phi - \chi_d - \phi_1, \chi - \chi_{d'} - \phi_2, \varphi - \phi - \chi_c)
\end{aligned}$$

On the other hand

$$\begin{aligned}
& (|\varphi| - |\phi|) \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) D^-(\phi_1, \phi_2, \chi_b) \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi) \\
&= \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) D^-(\phi_1, \phi_2, \chi_b) \sum_{c \in \text{supp}(\varphi - \phi)} \left( (x^+ \otimes c) \right. \\
&\times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi_c) \\
&- \left. \sum_{\substack{d \in \text{supp} \phi - \phi_1 \\ d' \in \text{supp} \chi - \phi_2}} (x^+ \otimes c d d') \mathbb{D}(\phi - \phi_1 - \chi_d, \chi - \phi_2 - \chi_{d'}, \varphi - \phi - \chi_c) \right) \quad (\text{by 2.8.2}) \\
&= \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) D^-(\phi_1, \phi_2, \chi_b) \sum_{c \in \text{supp}(\varphi - \phi)} (x^+ \otimes c) \\
&\times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi_c) \\
&- \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) D^-(\phi_1, \phi_2, \chi_b) \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{d \in \text{supp} \phi - \phi_1 \\ d' \in \text{supp} \chi - \phi_2}} (x^+ \otimes c d d') \\
&\times \mathbb{D}(\phi - \phi_1 - \chi_d, \chi - \phi_2 - \chi_{d'}, \varphi - \phi - \chi_c)
\end{aligned}$$

So it suffices to show that

$$\begin{aligned}
0 &= -(\chi(b) + 1) \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi')} (x^+ \otimes c) \mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi' - \chi_c) \\
&+ \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) (x^+ \otimes c) D^-(\phi_1, \phi_2, \chi_b) \\
&\times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi_c) \\
&+ \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{d \in \text{supp} \phi' \\ d' \in \text{supp} \chi}} (\chi(b) - \chi_{d'}(b) + 1) (x^+ \otimes c d d') \\
&\times \mathbb{D}(\phi' - \chi_d, \chi - \chi_{d'} + \chi_b, \varphi - \phi' - \chi_c) \\
&- \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{d \in \text{supp} \phi \\ d' \in \text{supp} \chi}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \chi_d) \\ \phi_2 \in \mathcal{F}(\chi - \chi_{d'})}} (|\phi_1| + 1) (x^+ \otimes c d d') D^-(\phi_1, \phi_2, \chi_b) \\
&\times \mathbb{D}(\phi - \chi_d - \phi_1, \chi - \chi_{d'} - \phi_2, \varphi - \phi - \chi_c) \\
&+ (|\varphi| - |\phi|)(\chi(b) + 1) \sum_{\phi' \in \mathcal{F}(\varphi)} \mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi') \\
&- \sum_{\substack{\phi \in \mathcal{F}(\varphi) \\ \phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) D^-(\phi_1, \phi_2, \chi_b) \sum_{c \in \text{supp}(\varphi - \phi)} (x^+ \otimes c) \\
&\times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi_c) \\
&+ \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) D^-(\phi_1, \phi_2, \chi_b) \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{d \in \text{supp} \phi - \phi_1 \\ d' \in \text{supp} \chi - \phi_2}} (x^+ \otimes c d d') \\
&\times \mathbb{D}(\phi - \phi_1 - \chi_d, \chi - \phi_2 - \chi_{d'}, \varphi - \phi - \chi_c)
\end{aligned}$$

or equivalently that

$$\begin{aligned}
0 &= -(\chi(b) + 1) \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi')} (x^+ \otimes c) \mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi' - \chi_c) \\
&+ \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) [(x^+ \otimes c), D^-(\phi_1, \phi_2, \chi_b)] \\
&\times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi_c) \\
&+ \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{d \in \text{supp} \phi' \\ d' \in \text{supp} \chi}} (\chi(b) - \chi_{d'}(b) + 1)(x^+ \otimes cdd') \\
&\times \mathbb{D}(\phi' - \chi_d, \chi - \chi_{d'} + \chi_b, \varphi - \phi' - \chi_c) \\
&- \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{d \in \text{supp} \phi \\ d' \in \text{supp} \chi}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \chi_d) \\ \phi_2 \in \mathcal{F}(\chi - \chi_{d'})}} (|\phi_1| + 1) [(x^+ \otimes cdd'), D^-(\phi_1, \phi_2, \chi_b)] \\
&\times \mathbb{D}(\phi - \chi_d - \phi_1, \chi - \chi_{d'} - \phi_2, \varphi - \phi - \chi_c) \\
&+ (|\varphi| - |\phi|)(\chi(b) + 1) \sum_{\phi' \in \mathcal{F}(\varphi)} \mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi')
\end{aligned}$$

Concentrating on the terms with Lie brackets we obtain

$$\begin{aligned}
& \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) [(x^+ \otimes c), D^-(\phi_1, \phi_2, \chi b)] \\
& \times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi c) \\
& - \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{d \in \text{supp} \phi \\ d' \in \text{supp} \chi}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \chi d) \\ \phi_2 \in \mathcal{F}(\chi - \chi d')}} (|\phi_1| + 1) [(x^+ \otimes c d d'), D^-(\phi_1, \phi_2, \chi b)] \\
& \times \mathbb{D}(\phi - \chi d - \phi_1, \chi - \chi d' - \phi_2, \varphi - \phi - \chi c) \\
& = \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi) \\ |\phi_1| = |\phi_2|}} (|\phi_1| + 1) |\phi_1|! (h \otimes b c \pi(\phi_1) \pi(\phi_2)) \\
& \times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi c) \\
& - \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{d \in \text{supp} \phi \\ d' \in \text{supp} \chi}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \chi d) \\ \phi_2 \in \mathcal{F}(\chi - \chi d')}} (|\phi_1| + 1) |\phi_1|! (h \otimes b c d d' \pi(\phi_1) \pi(\phi_2)) \\
& \times \mathbb{D}(\phi - \chi d - \phi_1, \chi - \chi d' - \phi_2, \varphi - \phi - \chi c) \\
& = \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi) \\ |\phi_1| = |\phi_2|}} (|\phi_1| + 1) |\phi_1|! (h \otimes b c \pi(\phi_1) \pi(\phi_2)) \\
& \times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi c) \\
& - \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1|!)^2 |\phi'_1| (h \otimes b c \pi(\phi'_1) \pi(\phi'_2)) \mathbb{D}(\phi - \phi'_1, \chi - \phi'_2, \varphi - \phi - \chi c) \\
& = \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi) \\ |\phi_1| = |\phi_2|}} (|\phi_1|!)^2 (h \otimes b c \pi(\phi_1) \pi(\phi_2)) \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi c) \\
& = \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi) \\ |\phi_1| = |\phi_2|}} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi - \phi_2) \\ |\phi_1| = |\phi_2|}} (|\phi_1|!)^2 (h \otimes b c \pi(\phi_1) \pi(\phi_2)) p(\phi'_1, \phi'_2) \\
& \times D^+(\phi - \phi_1 - \phi'_1, \chi - \phi_2 - \phi'_2, \varphi - \phi - \chi c)
\end{aligned}$$



$$\begin{aligned}
&= \sum_{c \in \text{supp } \varphi} \sum_{\phi \in \mathcal{F}(\varphi - \chi_c)} \sum_{\substack{\psi_1 \in \mathcal{F}(\phi) \\ \psi_2 \in \mathcal{F}(\chi)}} \sum_{\substack{\phi_1 \in \mathcal{F}(\psi_1) \\ \phi_2 \in \mathcal{F}(\psi_2) \\ |\phi_1| = |\phi_2|}} (|\phi_1|!)^2 (h \otimes bc\pi(\phi_1)\pi(\phi_2)) p(\psi_1 - \phi_1, \psi_2 - \phi_2) \\
&\times D^+(\phi - \psi_1, \chi - \psi_2, \varphi - \phi - \chi_c) \\
&= \sum_{\substack{\phi' \in \mathcal{F}(\varphi) \\ \psi_2 \in \mathcal{F}(\chi)}} \sum_{\psi'_1 \in \mathcal{F}(\phi')} \sum_{c \in \text{supp } \psi'_1} \sum_{\substack{\phi_1 \in \mathcal{F}(\psi'_1 - \chi_c) \\ \phi_2 \in \mathcal{F}(\psi_2) \\ |\phi_1| = |\phi_2|}} (|\phi_1|!)^2 (h \otimes bc\pi(\phi_1)\pi(\phi_2)) \\
&\times p(\psi'_1 - \chi_c - \phi_1, \psi_2 - \phi_2) D^+(\phi' - \psi'_1, \chi - \psi_2, \varphi - \phi')
\end{aligned}$$

The following formula will be necessary. Suppose that  $b \in A$  and  $\varphi, \chi \in \mathcal{F}$  then

$$\begin{aligned}
-(\chi(b) + 1)p(\varphi, \chi + \chi b) &= \sum_{c \in \text{supp } \varphi} \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi - \chi_c) \\ \phi_2 \in \mathcal{F}(\chi) \\ |\phi_1| = |\phi_2|}} (|\phi_1|!)^2 \\
&\times (h \otimes bc\pi(\phi_1)\pi(\phi_2)) p(\varphi - \chi_c - \phi_1, \chi - \phi_2)
\end{aligned} \tag{2.8.3}$$

This formula will be proved after we finish the proof of Lemma 31. From above

$$\begin{aligned}
& \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi) \\ \phi_2 \in \mathcal{F}(\chi)}} (|\phi_1| + 1) [(x^+ \otimes c), D^-(\phi_1, \phi_2, \chi_b)] \\
& \times \mathbb{D}(\phi - \phi_1, \chi - \phi_2, \varphi - \phi - \chi_c) \\
& - \sum_{\phi \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi)} \sum_{\substack{d \in \text{supp} \phi \\ d' \in \text{supp} \chi}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi - \chi_d) \\ \phi_2 \in \mathcal{F}(\chi - \chi_{d'})}} (|\phi_1| + 1) [(x^+ \otimes cdd'), D^-(\phi_1, \phi_2, \chi_b)] \\
& \times \mathbb{D}(\phi - \chi_d - \phi_1, \chi - \chi_{d'} - \phi_2, \varphi - \phi - \chi_c) \\
& = \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{\substack{\psi'_1 \in \mathcal{F}(\phi') \\ \psi_2 \in \mathcal{F}(\chi)}} \sum_{c \in \text{supp} \psi'_1} \sum_{\substack{\phi_1 \in \mathcal{F}(\psi'_1 - \chi_c) \\ \phi_2 \in \mathcal{F}(\psi_2) \\ |\phi_1| = |\phi_2|}} (|\phi_1|!)^2 (h \otimes bc\pi(\phi_1)\pi(\phi_2)) \\
& \times p(\psi'_1 - \chi_c - \phi_1, \psi_2 - \phi_2) D^+(\phi' - \psi'_1, \chi - \psi_2, \varphi - \phi') \\
& = - \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{\substack{\psi'_1 \in \mathcal{F}(\phi') \\ \psi_2 \in \mathcal{F}(\chi)}} (\psi_2(b) + 1) p(\psi'_1, \psi_2 + \chi_b) D^+(\phi' - \psi'_1, \chi - \psi_2, \varphi - \phi') \quad (\text{by (2.8.3)}) \\
& = - \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{\substack{\psi'_1 \in \mathcal{F}(\phi') \\ \psi'_2 \in \mathcal{F}(\chi + \chi_b)}} \psi'_2(b) p(\psi'_1, \psi'_2) D^+(\phi' - \psi'_1, \chi + \chi_b - \psi'_2, \varphi - \phi')
\end{aligned}$$

So the proof of Lemma 31 is reduced to proving (2.8.3) and showing that

$$\begin{aligned}
0 & = - \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{\substack{\psi'_1 \in \mathcal{F}(\phi') \\ \psi'_2 \in \mathcal{F}(\chi + \chi_b)}} \psi'_2(b) p(\psi'_1, \psi'_2) D^+(\phi' - \psi'_1, \chi + \chi_b - \psi'_2, \varphi - \phi') \\
& - (\chi(b) + 1) \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi')} (x^+ \otimes c) \mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi' - \chi_c) \\
& + \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{d \in \text{supp} \phi' \\ d' \in \text{supp} \chi}} (\chi(b) - \chi_{d'}(b) + 1) (x^+ \otimes cdd') \\
& \times \mathbb{D}(\phi' - \chi_d, \chi - \chi_{d'} + \chi_b, \varphi - \phi' - \chi_c) \\
& + (|\varphi| - |\phi|)(\chi(b) + 1) \sum_{\phi' \in \mathcal{F}(\varphi)} \mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi') \tag{2.8.4}
\end{aligned}$$

Using Proposition 24(i) we have

$$\begin{aligned}
(x^+ \otimes c)\mathbb{D}(\phi', \chi + \chi_b, \varphi - \phi' - \chi_c) &= \sum_{\substack{\phi_1 \in \mathcal{F}(\phi') \\ \phi_2 \in \mathcal{F}(\chi + \chi_b)}} (x^+ \otimes c)p(\phi_1, \phi_2) \\
&\times D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi' - \chi_c) \\
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\phi') \\ \phi_2 \in \mathcal{F}(\chi + \chi_b)}} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi_1) \\ \phi'_2 \in \mathcal{F}(\phi_2) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1| + 1)! |\phi'_1|! \\
&\times p(\phi_1 - \phi'_1, \phi_2 - \phi'_2) (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) \\
&\times D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi' - \chi_c) \\
&\times \text{(by Proposition 24(i))} \\
&= \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi') \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b) \\ |\phi'_1| = |\phi'_2|}} \sum_{\substack{\varphi_1 \in \mathcal{F}(\phi' - \phi'_1) \\ \varphi_2 \in \mathcal{F}(\chi + \chi_b - \phi'_2)}} (|\phi'_1| + 1)! |\phi'_1|! p(\varphi_1, \varphi_2) \\
&\times (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) \\
&\times D^+(\phi' - \phi'_1 - \varphi_1, \chi + \chi_b - \phi'_2 - \varphi_2, \varphi - \phi' - \chi_c) \\
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\phi') \\ \phi_2 \in \mathcal{F}(\chi + \chi_b)}} p(\phi_1, \phi_2) \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1| + 1)! |\phi'_1|! \\
&\times (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) \\
&\times D^+(\phi' - \phi'_1 - \phi_1, \chi + \chi_b - \phi'_2 - \phi_2, \varphi - \phi' - \chi_c)
\end{aligned}$$

Similarly

$$\begin{aligned}
& (x^+ \otimes cdd')\mathbb{D}(\phi' - \chi_d, \chi + \chi_b - \chi_{d'}, \varphi - \phi' - \chi_c) \\
&= \sum_{\substack{\varphi_1 \in \mathcal{F}(\phi' - \chi_d) \\ \varphi_2 \in \mathcal{F}(\chi + \chi_b - \chi_{d'})}} (x^+ \otimes cdd')p(\varphi_1, \varphi_2)D^+(\phi' - \chi_d - \varphi_1, \chi + \chi_b - \chi_{d'} - \varphi_2, \varphi - \phi' - \chi_c) \\
&= \sum_{\substack{\varphi_1 \in \mathcal{F}(\phi' - \chi_d) \\ \varphi_2 \in \mathcal{F}(\chi + \chi_b - \chi_{d'})}} \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi_1) \\ \phi'_2 \in \mathcal{F}(\varphi_2) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1| + 1)!|\phi'_1|!p(\varphi_1 - \phi'_1, \varphi_2 - \phi'_2) (x^+ \otimes cdd'\pi(\phi'_1)\pi(\phi'_2)) \\
&\times D^+(\phi' - \chi_d - \varphi_1, \chi + \chi_b - \chi_{d'} - \varphi_2, \varphi - \phi' - \chi_c) \quad (\text{by Proposition 24}(i)) \\
&= \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \chi_d) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \chi_{d'}) \\ |\phi'_1| = |\phi'_2|}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi' - \chi_d - \phi'_1) \\ \phi_2 \in \mathcal{F}(\chi + \chi_b - \chi_{d'} - \phi'_2) \\ |\phi_1| = |\phi_2|}} (|\phi'_1| + 1)!|\phi'_1|!p(\phi_1, \phi_2) (x^+ \otimes cdd'\pi(\phi'_1)\pi(\phi'_2)) \\
&\times D^+(\phi' - \chi_d - \phi_1 - \phi'_1, \chi + \chi_b - \chi_{d'} - \phi_2 - \phi'_2, \varphi - \phi' - \chi_c) \\
&= \sum_{\substack{\varphi_1 \in \mathcal{F}(\phi') \\ \varphi_2 \in \mathcal{F}(\chi + \chi_b) \\ |\varphi_1| = |\varphi_2|}} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi' - \varphi_1) \\ \phi_2 \in \mathcal{F}(\chi + \chi_b - \varphi_2) \\ |\phi_1| = |\phi_2|}} (|\varphi_1| - 1)!|\varphi_1|!\varphi_1(d)\varphi_2(d')p(\phi_1, \phi_2) (x^+ \otimes c\pi(\varphi_1)\pi(\varphi_2)) \\
&\times D^+(\phi' - \phi_1 - \varphi_1, \chi + \chi_b - \phi_2 - \varphi_2, \varphi - \phi' - \chi_c) \\
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\phi') \\ \phi_2 \in \mathcal{F}(\chi + \chi_b)}} p(\phi_1, \phi_2) \sum_{\substack{\varphi_1 \in \mathcal{F}(\phi' - \phi_1) \\ \varphi_2 \in \mathcal{F}(\chi + \chi_b - \phi_2) \\ |\varphi_1| = |\varphi_2|}} (|\varphi_1| - 1)!|\varphi_1|!\varphi_1(d)\varphi_2(d') \\
&\times (x^+ \otimes c\pi(\varphi_1)\pi(\varphi_2))D^+(\phi' - \phi_1 - \varphi_1, \chi + \chi_b - \phi_2 - \varphi_2, \varphi - \phi' - \chi_c)
\end{aligned}$$

Using these identities (2.8.4) becomes

$$\begin{aligned}
& \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi') \\ \phi_2 \in \mathcal{F}(\chi + \chi_b)}} p(\phi_1, \phi_2) \phi_2(b) D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi') \\
&= \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi') \\ \phi_2 \in \mathcal{F}(\chi + \chi_b)}} p(\phi_1, \phi_2) (-\chi(b) + 1) \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1| + 1)! |\phi'_1|! \\
&\times (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) D^+(\phi' - \phi'_1 - \phi_1, \chi + \chi_b - \phi'_2 - \phi_2, \varphi - \phi' - \chi_c) \\
&+ \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi') \\ \phi_2 \in \mathcal{F}(\chi + \chi_b)}} p(\phi_1, \phi_2) \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{d \in \text{supp} \phi' \\ d' \in \text{supp} \chi}} (\chi(b) - \chi_{d'}(b) + 1) \\
&\times \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1| - 1)! |\phi'_1|! \phi'_1(d) \phi'_2(d') (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) \\
&\times D^+(\phi' - \phi_1 - \phi'_1, \chi + \chi_b - \phi_2 - \phi'_2, \varphi - \phi' - \chi_c) \\
&+ \sum_{\phi' \in \mathcal{F}(\varphi)} \sum_{\substack{\phi_1 \in \mathcal{F}(\phi') \\ \phi_2 \in \mathcal{F}(\chi + \chi_b)}} p(\phi_1, \phi_2) (|\varphi| - |\chi|) (\chi(b) + 1) D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi')
\end{aligned}$$

So it suffices to show (2.8.3) and for  $\phi', \phi_1, \phi_2 \in \mathcal{F}$  with  $\phi_1 \leq \phi', \phi_2 \leq \chi + \chi_b$ , and  $|\phi_1| = |\phi_2|$

$$\begin{aligned}
& \phi_2(b) D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi') \\
&= -(\chi(b) + 1) \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1| + 1)! |\phi'_1|! (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) \\
&\times D^+(\phi' - \phi'_1 - \phi_1, \chi + \chi_b - \phi'_2 - \phi_2, \varphi - \phi' - \chi_c) \\
&+ \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{d \in \text{supp} \phi' \\ d' \in \text{supp} \chi}} (\chi(b) - \chi_{d'}(b) + 1) \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1| - 1)! |\phi'_1|! \phi'_1(d) \phi'_2(d') \\
&\times (x^+ \otimes c\pi(\phi'_1)\pi(\phi'_2)) D^+(\phi' - \phi_1 - \phi'_1, \chi + \chi_b - \phi_2 - \phi'_2, \varphi - \phi' - \chi_c) \\
&+ (|\varphi| - |\chi|) (\chi(b) + 1) D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi')
\end{aligned}$$

$$\begin{aligned}
&= \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2)}} \left( \sum_{d' \in \text{supp} \chi} (\chi(b) - \chi_{d'}(b) + 1) \phi'_2(d') - (\chi(b) + 1)(|\phi'_1| + 1) \right) \\
&\times D^+(\phi'_1, \phi'_2, \chi_c) D^+(\phi' - \phi_1 - \phi'_1, \chi + \chi_b - \phi_2 - \phi'_2, \varphi - \phi' - \chi_c) \\
&+ (|\varphi| - |\chi|)(\chi(b) + 1) D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi') \\
&= \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2)}} \left( \chi(b) \phi'_2(b) + (\chi(b) + 1)(|\phi'_2| - \phi'_2(b)) - (|\phi'_2| + 1) \right. \\
&\times \left. (\chi(b) + 1) \right) D^+(\phi'_1, \phi'_2, \chi_c) D^+(\phi' - \phi_1 - \phi'_1, \chi + \chi_b - \phi_2 - \phi'_2, \varphi - \phi' - \chi_c) \\
&+ (|\varphi| - |\chi|)(\chi(b) + 1) D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi') \\
&= - \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2)}} \phi'_2(b) D^+(\phi'_1, \phi'_2, \chi_c) \\
&\times D^+(\phi' - \phi_1 - \phi'_1, \chi + \chi_b - \phi_2 - \phi'_2, \varphi - \phi' - \chi_c) \\
&- (\chi(b) + 1) \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2)}} D^+(\phi'_1, \phi'_2, \chi_c) \\
&\times D^+(\phi' - \phi_1 - \phi'_1, \chi + \chi_b - \phi_2 - \phi'_2, \varphi - \phi' - \chi_c) \\
&+ (\chi(b) + 1)(|\varphi| - |\chi|) D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi') \\
&= - \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2)}} \phi'_2(b) D^+(\phi'_1, \phi'_2, \chi_c) \\
&\times D^+(\phi' - \phi_1 - \phi'_1, \chi + \chi_b - \phi_2 - \phi'_2, \varphi - \phi' - \chi_c) \\
&+ (\chi(b) + 1) D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi')
\end{aligned}$$

So the proof is reduced to proving (2.8.3) and showing that

$$\begin{aligned}
& (\chi(b) + 1 - \phi_2(b))D^+(\phi' - \phi_1, \chi + \chi_b - \phi_2, \varphi - \phi') \\
&= \sum_{c \in \text{supp}(\varphi - \phi')} \sum_{\substack{\phi'_1 \in \mathcal{F}(\phi' - \phi_1) \\ \phi'_2 \in \mathcal{F}(\chi + \chi_b - \phi_2)}} \phi'_2(b)D^+(\phi'_1, \phi'_2, \chi_c) \\
&\times D^+(\phi' - \phi_1 - \phi'_1, \chi + \chi_b - \phi_2 - \phi'_2, \varphi - \phi' - \chi_c)
\end{aligned}$$

but this is true by Proposition 30(i). So it suffices to give the

**Proof of (2.8.3).** In the case  $|\varphi| \neq |\chi| + 1$  both sides are 0. So assume that  $|\varphi| = |\chi| + 1$ .

The proof will proceed by induction on  $|\chi| + 1 = |\varphi| \geq 1$ . The case  $|\varphi| = 1$  is easy. So assume that  $|\varphi| > 1$ . Then

$$\begin{aligned}
-|\varphi|(\chi(b) + 1)p(\varphi, \chi + \chi_b) &= (\chi(b) + 1) \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi + \chi_b) - \{0\}}} (|\phi_1|!)^2 (h \otimes \pi(\phi_1)\pi(\phi_2)) \\
&\times p(\varphi - \phi_1, \chi + \chi_b - \phi_2)
\end{aligned}$$

$$\begin{aligned}
&= (\chi(b) + 1) \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi + \chi_b) \\ \phi_2(b) = \chi(b) + 1}} (|\phi_1|!)^2 (h \otimes \pi(\phi_1)\pi(\phi_2)) p(\varphi - \phi_1, \chi + \chi_b - \phi_2) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi + \chi_b) \\ 0 < \phi_2(b) \leq \chi(b)}} (|\phi_1|!)^2 \phi_2(b) (h \otimes \pi(\phi_1)\pi(\phi_2)) p(\varphi - \phi_1, \chi + \chi_b - \phi_2) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi + \chi_b) \\ 0 < \phi_2(b) \leq \chi(b)}} (|\phi_1|!)^2 (\chi(b) - \phi_2(b) + 1) (h \otimes \pi(\phi_1)\pi(\phi_2)) p(\varphi - \phi_1, \chi + \chi_b - \phi_2) \\
&+ (\chi(b) + 1) \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi + \chi_b) - \{0\} \\ \phi_2(b) = 0}} (|\phi_1|!)^2 (h \otimes \pi(\phi_1)\pi(\phi_2)) p(\varphi - \phi_1, \chi + \chi_b - \phi_2) \\
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ \phi'_2(b) = \chi(b)}} (|\phi_1|!)^2 (h \otimes b\pi(\phi_1)\pi(\phi'_2)) p(\varphi - \phi_1, \chi - \phi'_2) \\
&+ \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) \\ \phi'_2 \in \mathcal{F}(\chi) \\ \phi'_2(b) < \chi(b)}} (|\phi_1|!)^2 (h \otimes b\pi(\phi_1)\pi(\phi'_2)) p(\varphi - \phi_1, \chi - \phi'_2) \\
&- \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi) \\ 0 < \phi_2(b) \leq \chi(b)}} (|\phi_1|!)^2 (h \otimes \pi(\phi_1)\pi(\phi_2)) \sum_{c \in \text{supp}(\varphi - \phi_1)} \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi - \phi_1 - \chi_c) \\ \phi'_2 \in \mathcal{F}(\chi - \phi_2) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1|!)^2 \\
&\times (h \otimes bc\pi(\phi'_1)\pi(\phi'_2)) p(\varphi - \phi_1 - \chi_c - \phi'_1, \chi - \phi_2 - \phi'_2) \\
&- \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) - \{0\} \\ \phi_2 \in \mathcal{F}(\chi + \chi_b) - \{0\} \\ \phi_2(b) = 0}} (|\phi_1|!)^2 (h \otimes \pi(\phi_1)\pi(\phi_2)) \sum_{c \in \text{supp}(\varphi - \phi_1)} \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi - \phi_1 - \chi_c) \\ \phi'_2 \in \mathcal{F}(\chi - \phi_2) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1|!)^2 \\
&\times (h \otimes bc\pi(\phi'_1)\pi(\phi'_2)) p(\varphi - \phi_1 - \chi_c - \phi'_1, \chi - \phi_2 - \phi'_2) \quad (\text{by the induction hypothesis}) \\
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) \\ \phi'_2 \in \mathcal{F}(\chi)}} (|\phi_1|!)^2 (h \otimes b\pi(\phi_1)\pi(\phi'_2)) p(\varphi - \phi_1, \chi - \phi'_2) \\
&- \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) \\ \phi_2 \in \mathcal{F}(\chi) \\ |\phi_1| = |\phi_2| > 0}} (|\phi_1|!)^2 (h \otimes \pi(\phi_1)\pi(\phi_2)) \sum_{c \in \text{supp}(\varphi - \phi_1)} \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi - \phi_1 - \chi_c) \\ \phi'_2 \in \mathcal{F}(\chi - \phi_2)}} (|\phi'_1|!)^2 \\
&\times (h \otimes bc\pi(\phi'_1)\pi(\phi'_2)) p(\varphi - \phi_1 - \chi_c - \phi'_1, \chi - \phi_2 - \phi'_2)
\end{aligned}$$



$$\begin{aligned}
&= \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi) \\ \phi'_2 \in \mathcal{F}(\chi)}} \sum_{c \in \text{supp } \phi_1} |\phi_1|!(|\phi_1| - 1)!\phi_1(c)(h \otimes b\pi(\phi_1)\pi(\phi'_2))p(\varphi - \phi_1, \chi - \phi'_2) \\
&- \sum_{c \in \text{supp } \varphi} \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi - \chi_c) \\ \phi_2 \in \mathcal{F}(\chi) \\ |\phi_1| = |\phi_2| > 0}} (|\phi_1|!)^2(h \otimes \pi(\phi_1)\pi(\phi_2)) \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi - \phi_1 - \chi_c) \\ \phi'_2 \in \mathcal{F}(\chi - \phi_2)}} (|\phi'_1|!)^2 \\
&\times (h \otimes bc\pi(\phi'_1)\pi(\phi'_2))p(\varphi - \phi_1 - \chi_c - \phi'_1, \chi - \phi_2 - \phi'_2) \\
&= \sum_{c \in \text{supp } \varphi} \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi - \chi_c) \\ \phi'_2 \in \mathcal{F}(\chi)}} (|\phi'_1| + 1)!|\phi'_1|!(h \otimes bc\pi(\phi'_1)\pi(\phi'_2))p(\varphi - \chi_c - \phi'_1, \chi - \phi'_2) \\
&- \sum_{c \in \text{supp } \varphi} \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi - \chi_c) \\ \phi'_2 \in \mathcal{F}(\chi)}} (|\phi'_1|!)^2(h \otimes bc\pi(\phi'_1)\pi(\phi'_2)) \sum_{\substack{\phi_1 \in \mathcal{F}(\varphi - \phi'_1 - \chi_c) \\ \phi_2 \in \mathcal{F}(\chi - \phi'_2) \\ |\phi_1| = |\phi_2| > 0}} (|\phi_1|!)^2(h \otimes \pi(\phi_1)\pi(\phi_2)) \\
&\times p(\varphi - \phi_1 - \chi_c - \phi'_1, \chi - \phi_2 - \phi'_2) \\
&= \sum_{c \in \text{supp } \varphi} \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi - \chi_c) \\ \phi'_2 \in \mathcal{F}(\chi)}} (|\phi'_1| + 1)!|\phi'_1|!(h \otimes bc\pi(\phi'_1)\pi(\phi'_2))p(\varphi - \chi_c - \phi'_1, \chi - \phi'_2) \\
&+ \sum_{c \in \text{supp } \varphi} \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi - \chi_c) \\ \phi'_2 \in \mathcal{F}(\chi)}} (|\phi'_1|!)^2(|\varphi| - 1 - |\phi'_1|)(h \otimes bc\pi(\phi'_1)\pi(\phi'_2))p(\varphi - \chi_c - \phi'_1, \chi - \phi'_2) \\
&= |\varphi| \sum_{c \in \text{supp } \varphi} \sum_{\substack{\phi'_1 \in \mathcal{F}(\varphi - \chi_c) \\ \phi'_2 \in \mathcal{F}(\chi) \\ |\phi'_1| = |\phi'_2|}} (|\phi'_1|!)^2(h \otimes bc\pi(\phi'_1)\pi(\phi'_2))p(\varphi - \chi_c - \phi'_1, \chi - \phi'_2)
\end{aligned}$$

So (2.8.3) is proved by induction on  $|\varphi|$  and hence We have completed the proofs of Lemma 13 and Theorem 6.

■

## Chapter 3

# Applications to Representation

## Theory

In this chapter we will discuss an application of ingredients of the straightening lemma to the representation theory of  $\mathbf{U}(\mathfrak{sl}_2 \otimes A)$ .

### 3.1 A $\mathbb{C}$ -Basis for the Global Weyl Module for $\mathfrak{sl}_2$

**Definition.** Define  $\omega \in \mathfrak{h}^*$  by  $\omega(h) = 1$ . For  $k \in \mathbb{Z}_{\geq 0}$ , define  $W_A(k\omega)$  to be the  $\mathfrak{sl}_2 \otimes A$ -module generated by a vector  $w_{k\omega}$  with defining relations

$$(x^+ \otimes A)w_{k\omega} = 0, \quad hw_{k\omega} = kw_{k\omega}, \quad (x^-)^{k+1}w_{k\omega} = 0$$

Then  $W_A(k\omega)$  is the *global Weyl module* with highest weight  $k\omega$ .

Define, for all  $\psi \in \mathcal{F}$  and  $k \in \mathbb{Z}_{\geq 0}$ ,

$$D^\pm(\psi, \chi) = D^\pm(\psi, |\psi|\chi_1, \chi)$$

Define a function  $q : \mathcal{F}^2 \rightarrow \mathbf{U}(\mathfrak{sl}_2 \otimes A)$  by

$$q(\varphi, \chi) = (-1)^{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} D^-(\psi, \varphi) p(\chi - \psi)$$

The goal of this chapter is to prove the following theorem.

### 3.1.1 The Basis Theorem

**Theorem 32** *Given a  $\mathbb{C}$ -basis  $\mathbf{B}$  for  $A$  the set*

$$\left\{ q(\varphi, \chi) \cdot w_{k\omega} \mid \varphi, \chi \in \mathcal{F}(\mathbf{B}), |\chi| + |\varphi| = k \right\}$$

*is a  $\mathbb{C}$ -basis for  $W_A(k\omega)$ .*

This theorem will follow from a theorem of B. Feigin and S. Loktev, [5], and two lemmas.

### 3.1.2 A Theorem of B. Feigin and S. Loktev

There is a natural action of  $S_k$  on  $T^k(V)$  given by extending

$$\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$$

linearly.

Define a subspace  $S^k(V) \subset T^k(V)$  to be the span of

$$\left\{ \sum_{\sigma \in S_k} \sigma(v_1 \otimes \cdots \otimes v_k) \mid v_1, \dots, v_k \in V \right\}$$

Define

$$\Delta^{k-1} : \mathbf{U}(\mathfrak{sl}_2 \otimes A) \rightarrow T^k(\mathbf{U}(\mathfrak{sl}_2 \otimes A))$$

by extending the map

$$\begin{aligned} \mathfrak{sl}_2 \otimes A &\rightarrow T^k(\mathfrak{sl}_2 \otimes A) \\ z &\mapsto \sum_{j=0}^k 1^{\otimes j} \otimes z \otimes 1^{\otimes k-j} \end{aligned}$$

to an algebra homomorphism.

Given a  $\mathbf{U}(\mathfrak{sl}_2 \otimes A)$ -module  $W$  define

$$\rho : T^k(\mathbf{U}(\mathfrak{sl}_2 \otimes A)) \rightarrow \text{End } T^k(W)$$

to be the coordinate-wise module action. Then  $T^k(W)$  is a left  $\mathbf{U}(\mathfrak{sl}_2 \otimes A)$ -module via the map  $\rho \circ \Delta^{k-1}$ . Moreover  $S^k(W)$  is a submodule under this action.

Define  $V(\omega)$  to be the 2-dimensional  $\mathfrak{sl}_2$  module with  $\mathbb{C}$ -basis  $\{v_-, v_+\}$  and module action given by

$$x_{\pm}.v_{\pm} = 0 \quad x_{\mp}.v_{\pm} = v_{\pm} \quad h.v_{\pm} = \pm v_{\pm}$$

**Theorem 33** (*B. Feigin, S. Loktev, [5]*) *The assignment  $w_{k\omega} \mapsto (v_+ \otimes 1)^{\otimes k}$  extends to a  $\mathfrak{sl}_2 \otimes A$ -module isomorphism  $W_A(k\omega) \rightarrow S^k(V(\omega) \otimes A)$ .*

### 3.2 The Proof of Theorem 32

Using Theorem 33 we only need to show that the set

$$\left\{ q(\varphi, \chi).(v_+ \otimes 1)^{\otimes k} : \varphi, \chi \in \mathcal{F}(\mathbf{B}) \quad |\varphi| + |\chi| = k \right\}$$

is a  $\mathbb{C}$ -basis for  $S^k(V(\omega) \otimes A)$ . This will follow from the next two lemmas.

### 3.2.1 A Basis for $S^k(V(\omega) \otimes A)$

Given a complex vector space  $V$  and  $v \in V$ , and  $k \in \mathbb{Z}_{\geq 0}$  define

$$v^{(\otimes k)} = \frac{1}{k!} v^{\otimes k}$$

Let  $\varphi, \chi \in \mathcal{F}$  be given. Define  $v(\varphi, \chi) \in S^{|\varphi|+|\chi|}(V(\omega) \otimes A)$  by

$$v(\varphi, \chi) = \sum_{\sigma \in S^{|\varphi|+|\chi|}} \sigma \left( \bigotimes_{b \in \text{supp } \varphi} (v_- \otimes b)^{(\otimes \varphi(b))} \bigotimes_{a \in \text{supp } \chi} (v_+ \otimes a)^{(\otimes \chi(a))} \right)$$

**Lemma 34** *Let  $\mathbf{B}$  be a  $\mathbb{C}$ -basis of  $A$ . Then  $\mathcal{B}_k = \left\{ v(\varphi, \chi) \mid \varphi, \chi \in \mathcal{F}(\mathbf{B}), |\chi| + |\varphi| = k \right\}$  is a  $\mathbb{C}$ -basis for  $S^k(V(\omega) \otimes A)$ .*

**Proof.** The set of all

$$\sigma \left( \bigotimes_{b \in \text{supp } \varphi} (v_- \otimes b)^{(\otimes \varphi(b))} \bigotimes_{a \in \text{supp } \chi} (v_+ \otimes a)^{(\otimes \chi(a))} \right)$$

where  $\varphi, \chi \in \mathcal{F}(\mathbf{B})$ ,  $|\varphi| + |\chi| = k$  and  $\sigma \in S_k$  is a  $\mathbb{C}$ -basis for  $T^k(V(\omega) \otimes A)$ . Thus  $\mathcal{B}_k$  spans  $S^k(V(\omega) \otimes A)$ . Now assume that  $v(\varphi_1, \chi_1), \dots, v(\varphi_k, \chi_k) \in \mathcal{B}$  and  $\gamma_1, \dots, \gamma_k \in \mathbb{C}$  are given with

$$\sum_{j=1}^k \gamma_j v(\varphi_j, \chi_j) = 0$$

Then

$$\sum_{\sigma \in S_k} \sum_{j=1}^k \gamma_j \sigma \left( \bigotimes_{b \in \text{supp } \varphi_j} (v_- \otimes b)^{(\otimes \varphi_j(b))} \bigotimes_{a \in \text{supp } \chi_j} (v_+ \otimes a)^{(\otimes \chi_j(a))} \right) = 0$$

Hence  $\gamma_1 = \dots = \gamma_k = 0$ . ■

### 3.2.2 The Action of $q(\varphi, \chi)$ on $(v_+ \otimes 1)^{\otimes k}$

**Lemma 35** *Under the module action of  $\mathbf{U}(\mathfrak{sl}_2 \otimes A)$  on  $S^k(V(\omega) \otimes A)$ , for all  $\varphi, \chi \in \mathcal{F}$  with  $|\varphi| + |\chi| = k$*

$$q(\varphi, \chi) \cdot (v_+ \otimes 1)^{\otimes k} = (-1)^k v(\varphi, \chi)$$

Lemma 34, Lemma 35 and Theorem 33 immediately imply Theorem 32. So to prove Theorem 32 it suffices to prove Lemma 35.

To prove Lemma 35 we need to study the action of  $\Delta^k$  on  $q(\varphi, \chi)$ .

### 3.2.3 Action of $\Delta^k$ on $q(\varphi, \chi)$

Given  $\chi \in \mathcal{F}$  define

$$\text{comp}_k(\chi) = \left\{ \underline{\chi} : \{1, \dots, k\} \rightarrow (\mathcal{F}(\chi))^{\times k} \mid \sum_{j=1}^k \underline{\chi}(j) = \chi \right\}$$

**Proposition 36** *Let  $\chi, \varphi \in \mathcal{F}$  be given. Then*

$$\Delta^{k-1}(q(\varphi, \chi)) = \sum_{\substack{\underline{\varphi} \in \text{comp}_k(\varphi) \\ \underline{\chi} \in \text{comp}_k(\chi)}} q(\underline{\varphi}(1), \underline{\chi}(1)) \otimes \cdots \otimes q(\underline{\varphi}(k), \underline{\chi}(k))$$

**Proof.** The proof will proceed by induction on  $k \geq 1$ . The case  $k = 1$  is trivial. It will suffice to show the case  $k = 2$  because assuming that case the remainder of the proof is as follows. An easy induction shows that

$$\Delta^l = (1^{\otimes(l-1)} \otimes \Delta^1) \circ \Delta^{l-1}$$

So

$$\begin{aligned}
\Delta^k(q(\varphi, \chi)) &= (1^{\otimes k-1} \otimes \Delta^1) \circ \Delta^{k-1}(q(\varphi, \chi)) \\
&= (1^{\otimes k-1} \otimes \Delta^1) \left( \sum_{\substack{\varphi \in \text{comp}_k(\varphi) \\ \chi \in \text{comp}_k(\chi)}} q(\underline{\varphi}(1), \underline{\chi}(1)) \otimes \cdots \otimes q(\underline{\varphi}(k), \underline{\chi}(k)) \right) \\
&= \sum_{\substack{\varphi \in \text{comp}_k(\varphi) \\ \chi \in \text{comp}_k(\chi)}} q(\underline{\varphi}(1), \underline{\chi}(1)) \otimes \cdots \otimes q(\underline{\varphi}(k-1), \underline{\chi}(k-1)) \\
&\otimes \Delta^1(q(\underline{\varphi}(k), \underline{\chi}(k))) \\
&= \sum_{\substack{\varphi \in \text{comp}_k(\varphi) \\ \chi \in \text{comp}_k(\chi)}} q(\underline{\varphi}(1), \underline{\chi}(1)) \otimes \cdots \otimes q(\underline{\varphi}(k-1), \underline{\chi}(k-1)) \\
&\otimes \left( \sum_{\substack{\phi \in \text{comp}_2(\varphi(k)) \\ \psi \in \text{comp}_2(\chi(k))}} q(\underline{\phi}(1), \underline{\psi}(1)) \otimes q(\underline{\phi}(2), \underline{\psi}(2)) \right) \\
&= \sum_{\substack{\varphi \in \text{comp}_k(\varphi) \\ \chi \in \text{comp}_k(\chi)}} \sum_{\substack{\phi \in \text{comp}_2(\varphi(k)) \\ \psi \in \text{comp}_2(\chi(k))}} q(\underline{\varphi}(1), \underline{\chi}(1)) \otimes \cdots \otimes q(\underline{\varphi}(k-1), \underline{\chi}(k-1)) \\
&\otimes q(\underline{\phi}(1), \underline{\psi}(1)) \otimes q(\underline{\phi}(2), \underline{\psi}(2)) \\
&= \sum_{\substack{\varphi \in \text{comp}_{k+1}(\varphi) \\ \chi \in \text{comp}_{k+1}(\chi)}} q(\underline{\varphi}(1), \underline{\chi}(1)) \otimes \cdots \otimes q(\underline{\varphi}(k), \underline{\chi}(k)) q(\underline{\varphi}(k+1), \underline{\chi}(k+1))
\end{aligned}$$

Thus the proposition is proved by induction if we assume the  $k = 2$  case. It remains to show the  $k = 2$  case. This case will have two subcases one where  $\varphi = 0$  and one where  $\varphi \neq 0$ . The subcase  $\varphi = 0$  will proceed by induction on  $|\chi|$ . For the base case the proposition is easily checked for  $\chi \equiv 0$ . Assume that there exists  $l \geq 0$  such that the proposition is true

for all  $\chi' \in \mathcal{F}$  with  $|\chi'| \leq l$ . Fix  $\chi \in \mathcal{F}_{l+1}$ . Then

$$\begin{aligned}
\Delta^1(q(0, \chi)) &= \Delta^1(p(\chi)) \\
&= -\frac{1}{|\chi|} \sum_{\psi \in \mathcal{F}(\chi) - \{0\}} |\psi|! \Delta^1(h \otimes \pi(\psi)) \Delta^1(p(\chi - \psi)) \\
&= -\frac{1}{|\chi|} \sum_{\psi \in \mathcal{F}(\chi) - \{0\}} |\psi|! ((h \otimes \pi(\psi)) \otimes 1 + 1 \otimes (h \otimes \pi(\psi))) \\
&\quad \times \left( \sum_{\underline{\phi} \in \text{comp}_2(\chi - \psi)} p(\underline{\phi}(1)) \otimes p(\underline{\phi}(2)) \right) \quad (\text{By the induction hypothesis}) \\
&= -\frac{1}{|\chi|} \sum_{\psi \in \mathcal{F}(\chi) - \{0\}} |\psi|! ((h \otimes \pi(\psi)) \otimes 1 + 1 \otimes (h \otimes \pi(\psi))) \\
&\quad \times \left( \sum_{\psi' \in \mathcal{F}(\chi - \psi)} p(\psi') \otimes p(\chi - \psi - \psi') \right) \\
&= -\frac{1}{|\chi|} \sum_{\psi \in \mathcal{F}(\chi) - \{0\}} \sum_{\psi' \in \mathcal{F}(\chi - \psi)} |\psi|! ((h \otimes \pi(\psi)) \otimes 1 + 1 \otimes (h \otimes \pi(\psi))) \\
&\quad \times (p(\psi') \otimes p(\chi - \psi - \psi')) \\
&= -\frac{1}{|\chi|} \sum_{\psi \in \mathcal{F}(\chi) - \{0\}} \sum_{\psi' \in \mathcal{F}(\chi - \psi)} |\psi|! (h \otimes \pi(\psi)) p(\psi') \otimes p(\chi - \psi - \psi') \\
&\quad - \frac{1}{|\chi|} \sum_{\psi \in \mathcal{F}(\chi) - \{0\}} \sum_{\psi' \in \mathcal{F}(\chi - \psi)} |\psi|! p(\psi') \otimes (h \otimes \pi(\psi)) p(\chi - \psi - \psi') \\
&= -\frac{1}{|\chi|} \sum_{\chi' \in \mathcal{F}(\chi) - \{0\}} \sum_{\psi \in \mathcal{F}(\chi') - \{0\}} |\psi|! (h \otimes \pi(\psi)) p(\chi' - \psi) \otimes p(\chi - \chi') \\
&\quad - \frac{1}{|\chi|} \sum_{\substack{\chi' \in \mathcal{F}(\chi) \\ \chi' < \chi}} p(\chi') \otimes \sum_{\psi \in \mathcal{F}(\chi - \chi') - \{0\}} |\psi|! (h \otimes \pi(\psi)) p(\chi - \chi' - \psi) \\
&= \sum_{\chi' \in \mathcal{F}(\chi) - \{0\}} \frac{|\chi'|}{|\chi|} p(\chi') \otimes p(\chi - \chi') + \sum_{\substack{\chi' \in \mathcal{F}(\chi) \\ \chi' < \chi}} \frac{|\chi - \chi'|}{|\chi|} p(\chi') \otimes p(\chi - \chi') \\
&= \sum_{\chi' \in \mathcal{F}(\chi)} p(\chi') \otimes p(\chi - \chi') \\
&= \sum_{\underline{\chi} \in \text{comp}_2(\chi)} q(0, \underline{\chi}(1)) \otimes q(0, \underline{\chi}(2))
\end{aligned}$$



So by induction on  $|\chi|$  the proposition is proved for  $k = 2$  and  $\varphi = 0$ . The subcase  $\varphi \neq 0$  will proceed by induction on  $|\chi| + |\varphi|$ . For the base case the proposition is easily checked for  $|\chi| + |\varphi| = 1$ . Assume that there exists  $l \geq 1$  such that the proposition is true for pairs of elements of  $\mathcal{F}$  with  $|\chi| + |\varphi| \leq l$ . Fix  $\chi, \varphi \in \mathcal{F}$  with  $|\chi| + |\varphi| = l + 1$ . In this case we have by definition

$$\begin{aligned}
q(\varphi, \chi) &= (-1)^{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} D^-(\psi, \varphi) p(\chi - \psi) \\
&= \frac{(-1)^{|\varphi|}}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} \sum_{\phi \in \mathcal{F}(\psi)} |\phi|! (x^- \otimes d\pi(\phi)) D^-(\psi - \phi, \varphi - \chi_d) p(\chi - \psi) \\
&= \frac{(-1)^{|\varphi|}}{|\varphi|} \sum_{\phi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} \sum_{\phi' \in \mathcal{F}(\chi - \phi)} |\phi|! (x^- \otimes d\pi(\phi)) D^-(\phi', \varphi - \chi_d) p(\chi - \phi - \phi') \\
&= -\frac{1}{|\varphi|} \sum_{\phi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} |\phi|! (x^- \otimes d\pi(\phi)) q(\varphi - \chi_d, \chi - \phi)
\end{aligned}$$

So we have

$$q(\varphi, \chi) = -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} |\psi|! (x^- \otimes d\pi(\psi)) q(\varphi - \chi_d, \chi - \psi) \quad (3.2.1)$$

Then

$$\begin{aligned}
\Delta^1(q(\varphi, \chi)) &= -\frac{1}{|\varphi|} \sum_{d \in \text{supp } \varphi} \sum_{\psi \in \mathcal{F}(\chi)} |\psi|! \Delta^1((x^- \otimes d\pi(\psi))) \Delta^1(q(\varphi - \chi_d, \chi - \psi)) \\
&= -\frac{1}{|\varphi|} \sum_{d \in \text{supp } \varphi} \sum_{\psi \in \mathcal{F}(\chi)} |\psi|! ((x^- \otimes d\pi(\psi)) \otimes 1 + 1 \otimes (x^- \otimes d\pi(\psi))) \\
&\quad \times \sum_{\underline{\xi} \in \text{comp}_2(\varphi - \chi_d)} \sum_{\underline{\phi} \in \text{comp}_2(\chi - \psi)} (q(\underline{\xi}(1), \underline{\phi}(1)) \otimes q(\underline{\xi}(2), \underline{\phi}(2))) \\
&\quad \text{(By the induction hypothesis)} \\
&= -\frac{1}{|\varphi|} \sum_{d \in \text{supp } \varphi} \sum_{\psi \in \mathcal{F}(\chi)} |\psi|! ((x^- \otimes d\pi(\psi)) \otimes 1 + 1 \otimes (x^- \otimes d\pi(\psi))) \\
&\quad \times \sum_{\xi \in \mathcal{F}(\varphi - \chi_d)} \sum_{\psi' \in \mathcal{F}(\chi - \psi)} (q(\xi, \psi') \otimes q(\varphi - \chi_d - \xi, \chi - \psi - \psi'))
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{|\varphi|} \sum_{d \in \text{supp } \varphi} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{\xi \in \mathcal{F}(\varphi - \chi_d)} \sum_{\psi' \in \mathcal{F}(\chi - \psi)} |\psi|! ((x^- \otimes d\pi(\psi)) \otimes 1 \\
&+ 1 \otimes (x^- \otimes d\pi(\psi))) (q(\xi, \psi') \otimes q(\varphi - \chi_d - \xi, \chi - \psi - \psi')) \\
&= -\frac{1}{|\varphi|} \sum_{d \in \text{supp } \varphi} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{\xi \in \mathcal{F}(\varphi - \chi_d)} \sum_{\psi' \in \mathcal{F}(\chi - \psi)} |\psi|! (x^- \otimes d\pi(\psi)) q(\xi, \psi') \\
&\otimes q(\varphi - \chi_d - \xi, \chi - \psi - \psi') \\
&- \frac{1}{|\varphi|} \sum_{d \in \text{supp } \varphi} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{\xi \in \mathcal{F}(\varphi - \chi_d)} \sum_{\psi' \in \mathcal{F}(\chi - \psi)} |\psi|! q(\xi, \psi') \\
&\otimes (x^- \otimes d\pi(\psi)) q(\varphi - \chi_d - \xi, \chi - \psi - \psi') \\
&= -\frac{1}{|\varphi|} \sum_{\chi' \in \mathcal{F}(\chi)} \sum_{\psi \in \mathcal{F}(\chi')} \sum_{\varphi' \in \mathcal{F}(\varphi) - \{0\}} \sum_{d \in \text{supp } \varphi'} |\psi|! (x^- \otimes d\pi(\psi)) q(\varphi' - \chi_d, \chi' - \psi) \\
&\otimes q(\varphi - \varphi', \chi - \chi') \\
&- \frac{1}{|\varphi|} \sum_{\substack{\varphi' \in \mathcal{F}(\varphi) \\ \varphi' < \varphi}} \sum_{\chi' \in \mathcal{F}(\chi)} q(\varphi', \chi') \otimes \sum_{d \in \text{supp}(\varphi - \varphi')} \sum_{\psi \in \mathcal{F}(\chi - \chi')} |\psi|! (x^- \otimes d\pi(\psi)) \\
&\times q(\varphi - \varphi' - \chi_d, \chi - \chi' - \psi) \\
&= \frac{|\varphi'|}{|\varphi|} \sum_{\varphi' \in \mathcal{F}(\varphi) - \{0\}} \sum_{\chi' \in \mathcal{F}(\chi)} q(\varphi', \chi') \otimes q(\varphi - \varphi', \chi - \chi') \\
&+ \frac{|\varphi - \varphi'|}{|\varphi|} \sum_{\substack{\varphi' \in \mathcal{F}(\varphi) \\ \varphi' < \varphi}} \sum_{\chi' \in \mathcal{F}(\chi)} q(\varphi', \chi') \otimes q(\varphi - \varphi', \chi - \chi') \quad (\text{By (3.2.1)}) \\
&= \sum_{\varphi' \in \mathcal{F}(\varphi)} \sum_{\chi' \in \mathcal{F}(\chi)} q(\varphi', \chi') \otimes q(\varphi - \varphi', \chi - \chi') \\
&= \sum_{\substack{\underline{\varphi} \in \text{comp}_2(\varphi) \\ \underline{\chi} \in \text{comp}_2(\chi)}} q(\underline{\varphi}(1), \underline{\chi}(1)) \otimes q(\underline{\varphi}(2), \underline{\chi}(2))
\end{aligned}$$

Thus the proposition is proved in all cases. ■

### 3.2.4 The Proof of Lemma 35

**Proof.** The module action is given by  $\rho \circ \Delta^{k-1}$ . So

$$\begin{aligned}
q(\varphi, \chi).(v_+ \otimes 1)^{\otimes k} &= \rho \circ \Delta^{k-1}(q(\varphi, \chi))(v_+ \otimes 1)^{\otimes k} \\
&= \rho \left( \sum_{\substack{\underline{\varphi} \in \text{comp}_k(\varphi) \\ \underline{\chi} \in \text{comp}_k(\chi)}} q(\underline{\varphi}(1), \underline{\chi}(1)) \otimes \cdots \otimes q(\underline{\varphi}(k), \underline{\chi}(k)) \right) (v_+ \otimes 1)^{\otimes k} \\
&= \sum_{\substack{\underline{\varphi} \in \text{comp}_k(\varphi) \\ \underline{\chi} \in \text{comp}_k(\chi)}} q(\underline{\varphi}(1), \underline{\chi}(1)).(v_+ \otimes 1) \otimes \cdots \otimes q(\underline{\varphi}(k), \underline{\chi}(k)).(v_+ \otimes 1)
\end{aligned}$$

It is easy to see that each term of  $q(\varphi, \chi)$  has  $x^-$  appearing  $|\varphi|$  times. Hence  $q(\varphi, \chi).(v_+ \otimes 1) = 0$  if  $|\varphi| \geq 2$ . Claim that, for any  $\chi \in \mathcal{F}$  with  $|\chi| \geq 2$ ,  $q(0, \chi).(v_+ \otimes 1) = 0$ . It is easy to check this for  $\chi \in \mathcal{F}_2$ . To prove this claim by induction on  $|\chi|$  assume that, for some  $k \geq 2$ ,  $q(0, \chi).(v_+ \otimes 1) = 0$  for all  $\chi \in \mathcal{F}$  with  $|\chi| \leq k$ . Let  $\chi \in \mathcal{F}_{k+1}$  be given. Then

$$\begin{aligned}
q(0, \chi).(v_+ \otimes 1) &= -\frac{1}{|\chi|} \sum_{\psi \in \mathcal{F}(\chi) - \{0\}} |\psi|!(h \otimes \pi(\psi))q(0, \chi - \psi).(v_+ \otimes 1) \\
&= 0
\end{aligned}$$

So by induction on  $|\chi|$   $q(0, \chi).(v_+ \otimes 1) = 0$  for all  $\chi \in \mathcal{F}$  with  $|\chi| \geq 2$ . Thus for  $|\chi| > 0$

$$\begin{aligned}
q(\chi_d, \chi).(v_+ \otimes 1) &= \sum_{\psi \in \mathcal{F}(\chi)} |\psi|! (x^- \otimes d\pi(\psi)) q(0, \chi - \psi).(v_+ \otimes 1) \\
&= \sum_{j=0}^{|\chi|} j! \sum_{\psi \in \mathcal{F}_j(\chi)} (x^- \otimes d\pi(\psi)) q(0, \chi - \psi).(v_+ \otimes 1) \\
&= \sum_{j=|\chi|-1}^{|\chi|} j! \sum_{\psi \in \mathcal{F}_j(\chi)} (x^- \otimes d\pi(\psi)) q(0, \chi - \psi).(v_+ \otimes 1) \\
&= (|\chi| - 1)! \sum_{b \in \text{supp } \chi} (x^- \otimes d\pi(\chi - \chi_b)) (h \otimes b).(v_+ \otimes 1) \\
&\quad - |\chi|! (x^- \otimes d\pi(\chi)).(v_+ \otimes 1) \\
&= (|\chi| - 1)! \sum_{b \in \text{supp } \chi} (x^- \otimes d\pi(\chi - \chi_b)).(v_+ \otimes b) - |\chi|! (v_- \otimes d\pi(\chi)) \\
&= (|\chi| - 1)! \sum_{b \in \text{supp } \chi} (v_- \otimes b d\pi(\chi - \chi_b)) - |\chi|! (v_- \otimes d\pi(\chi)) \\
&= (|\chi| - 1)! \sum_{b \in \text{supp } \chi} \chi(b) (v_- \otimes d\pi(\chi)) - |\chi|! (v_- \otimes d\pi(\chi)) \\
&= |\chi|! (v_- \otimes d\pi(\chi)) - |\chi|! (v_- \otimes d\pi(\chi)) \\
&= 0
\end{aligned}$$

Thus  $q(\varphi, \chi).(v_+ \otimes 1) = 0$  if  $|\varphi| + |\chi| > 1$ . Thus in the equation

$$q(\varphi, \chi).(v_+ \otimes 1)^{\otimes k} = \sum_{\substack{\underline{\varphi} \in \text{comp}_k(\varphi) \\ \underline{\chi} \in \text{comp}_k(\chi)}} q(\underline{\varphi}(1), \underline{\chi}(1)).(v_+ \otimes 1) \otimes \cdots \otimes q(\underline{\varphi}(k), \underline{\chi}(k)).(v_+ \otimes 1)$$

$q(\underline{\varphi}(j), \underline{\chi}(j)).(v_+ \otimes 1) = 0$  when  $|\underline{\varphi}(j)| + |\underline{\chi}(j)| > 1$ .

$$|\chi| + |\varphi| = k = \sum_{j=1}^k (|\underline{\varphi}(j)| + |\underline{\chi}(j)|)$$

So in every potentially nonzero term of the expansion above, for all  $j \in \{1, \dots, k\}$ ,  $|\underline{\varphi}(j)| + |\underline{\chi}(j)| = 1$ . In other words there exists  $b_j, d_j \in A$  so that  $\underline{\varphi}(j) = \chi_{d_j}$  and  $\underline{\chi}(j) = 0$  or

$\underline{\varphi}(j) = 0$  and  $\underline{\chi}(j) = \chi_{b_j}$ . In the either case we have one of

$$q(\varphi_j, 0) \cdot (v_+ \otimes 1) = -(x^- \otimes d_j) \cdot (v_+ \otimes 1) = -v_- \otimes d_j$$

$$q(0, \chi_j) \cdot (v_+ \otimes 1) = -(h \otimes b_j) \cdot (v_+ \otimes 1) = -v_+ \otimes b_j$$

So these are the only possibilities for factors in the tensor product above. Thus  $q(\varphi, \chi) \cdot (v_+ \otimes 1)^{\otimes k} = (-1)^k v(\varphi, \chi)$ . ■

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