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Essential dimension of the spin groups in characteristic 2

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The essential dimension of an algebraic group G is a measure of the number of parameters needed to describe all G-torsors over all fields. A major achievement of the subject was the calculation of the essential dimension of the spin groups over a field of characteristic not 2, started by Brosnan, Reichstein, and Vistoli, and completed by Chernousov and Merkurjev [3, 4], [17, Theorem 9.1].

In this paper, we determine the essential dimension of the spin groups $\mathrm{Spin}(n)$ for $n \geq 15$ over an arbitrary field (Theorem 2.1). We find that the answer is the same in all characteristics. In contrast, for the groups $O(n)$ and $SO(n)$, the essential dimension is smaller in characteristic 2, by Babic and Chernousov [1].

In characteristic not 2, the computation of essential dimension can be phrased to use a natural finite subgroup of $\mathrm{Spin}(2r+1)$, namely an extraspecial 2-group, a central extension of $(\mathbf{Z}/2)^{2r}$ by $\mathbf{Z}/2$. A distinctive feature of the argument in characteristic 2 is that the analogous subgroup is a finite group scheme, a central extension of $(\mathbf{Z}/2)^r \times (\mu_2)^r$ by μ_2, where μ_2 is the group scheme of square roots of unity.

In characteristic not 2, Garibaldi and Rost computed the essential dimension of $\mathrm{Spin}(n)$ for $n \leq 14$ [6, Table 23B], where case-by-case arguments seem to be needed. We show in Theorem 3.1 that for $n \leq 10$, the essential dimension of $\mathrm{Spin}(n)$ is the same in characteristic 2 as in characteristic not 2. It would be interesting to compute the essential dimension of $\mathrm{Spin}(n)$ in the remaining cases, $11 \leq n \leq 14$ in characteristic 2.

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1 Essential dimension

Let G be an affine group scheme of finite type over a field k. Write $H^1(k, G)$ for the set of isomorphism classes of G-torsors over k in the fppf topology. For G smooth over k, this is also the set of isomorphism classes of G-torsors over k in the etale topology.

Following Reichstein, the *essential dimension* $\mathrm{ed}(G)$ is the smallest natural number r such that for every G-torsor ξ over an extension field E of k, there is a subfield $k \subset F \subset E$ such that ξ is isomorphic to some G-torsor over F extended to E, and F has transcendence degree at most r over k. (It is essential that E is allowed to be any extension field of k, not just an algebraic extension field.) There are several survey articles on essential dimension, including [18, 16].

For example, let q_0 be a quadratic form of dimension n over a field k of characteristic not 2. Then $O(q_0)$-torsors can be identified with quadratic forms of dimension n, up to isomorphism. (For convenience, we sometimes write $O(n)$ for $O(q_0)$.)

Thus the essential dimension of $O(m)$ measures the number of parameters needed to describe all quadratic forms of dimension n. Indeed, every quadratic form of dimension n over a field of characteristic not 2 is isomorphic to a diagonal form $\langle a_1, \ldots, a_n\rangle$. It follows that the orthogonal group $O(n)$ in characteristic not 2 has essential dimension at most n; in fact, $O(n)$ has essential dimension equal to n, by one of the first computations of essential dimension [18, Example 2.5]. Reichstein also showed that the connected group $SO(n)$ in characteristic not 2 has essential dimension $n-1$ for $n \geq 3$ [18, Corollary 3.6].

For another example, for a positive integer n and any field k, the group scheme μ_n of nth roots of unity is smooth over k if and only if n is invertible in k. Independent of that, $H^1(k, \mu_n)$ is always isomorphic to $k^*/(k^*)^n$. From that description, it is immediate that μ_n has essential dimension at most 1 over k. It is not hard to check that the essential dimension is in fact equal to 1.

One simple bound is that for any generically free representation V of a group scheme G over k (meaning that G acts freely on a nonempty open subset of V), the essential dimension of G is at most $\dim(V) - \dim(G)$ [17, Proposition 5.1]. It follows, for example, that the essential dimension of any affine group scheme of finite type over k is finite.

For a prime number p, the p-essential dimension $\mathrm{ed}_p(G)$ is a simplified invariant, defined by "ignoring field extensions of degree prime to p". In more detail, for a G-torsor ξ over an extension field E of k, define the p-essential dimension $\mathrm{ed}_p(\xi)$ to be the smallest number r such that there is a finite extension E'/E of degree prime to p such that ξ over E' comes from a G-torsor over a subfield $k \subset F \subset E'$ of transcendence degree at most r over k. Then the *p-essential dimension* $\mathrm{ed}_p(G)$ is defined to be the supremum of the p-essential dimensions of all G-torsors over all extension fields of k.

It was a surprise when Brosnan, Reichstein, and Vistoli showed that the essential dimension of the spin group $\mathrm{Spin}(n)$ over a field k of characteristic not 2 is exponentially large, asymptotic to $2^{n/2}$ as n goes to infinity [3]. As an application, they showed that the number of "parameters" needed to describe all quadratic forms of dimension $2r$ in I^3 over all fields is asymptotic to 2^r.

We now turn to quadratic forms over a field which may have characteristic 2. Define a quadratic form (q, V) over a field k to be *nondegenerate* if the radical $V^\perp$ of the associated bilinear form is 0, and *nonsingular* if $V^\perp$ has dimension at most 1 and q is nonzero on any nonzero element of $V^\perp$. (In characteristic not 2, nonsingular and nondegenerate are the same.) The orthogonal group is defined as the automorphism group scheme of a nonsingular quadratic form [12, section VI.23]. For example, over a field k of characteristic 2, the quadratic form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}$$

is nonsingular of even dimension $2r$, while the form

$$x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r} + x_{2r+1}^2$$

is nonsingular of odd dimension $2r+1$, with $V^\perp$ of dimension 1. The *split* orthogonal group over k is the automorphism group of one of these particular quadratic forms.

Babic and Chernousov computed the essential dimension of $O(n)$ and the smooth connected subgroup $O^+(n)$ over an infinite field k of characteristic 2 [1]. (We also write $SO(n)$ for $O^+(n)$ by analogy with the case of characteristic not 2, even though the whole group $O(2r)$ is contained in $SL(2r)$ in characteristic 2.) The answer is smaller than in characteristic not 2. Namely, $O(2r)$ has essential dimension $r+1$ (not $2r$) over k. Also, $O^+(2r)$ has essential dimension $r+1$ for r even, and either r or $r+1$ for r odd, not $2r-1$. Finally, the group scheme $O(2r+1)$ has essential dimension $r+2$ over k, and $O^+(2r+1)$ has essential dimension $r+1$. The lower bounds here are difficult, while the upper bounds are straightforward. For example, to show that $O(2r)$ has essential dimension at most $r+1$ in characteristic 2, write any quadratic form of dimension $2r$ as a direct sum of 2-dimensional forms, thus reducing the structure group to $(\mathbf{Z}/2)^r \times (\mu_2)^r$, and then use that the group $(\mathbf{Z}/2)^r$ has essential dimension only 1 over an infinite field of characteristic 2 [1, proof of Proposition 13.1].

In this paper, we determine the essential dimension of $\mathrm{Spin}(n)$ in characteristic 2 for $n \leq 10$ or $n \geq 15$. Surprisingly, in view of what happens for $O(n)$ and $O^+(n)$, the results for spin groups are the same in characteristic 2 as in characteristic not 2. For $n \leq 10$, the lower bound for the essential dimension is proved by constructing suitable cohomological invariants. It is not known whether a similar approach is possible for $n \geq 15$, either in characteristic 2 or in characteristic not 2.

2 Main result

Theorem 2.1. *Let k be a field. For every integer n at least 15, the essential dimension of the split group $\mathrm{Spin}(n)$ over k is given by:*

$$\mathrm{ed}_2(\mathrm{Spin}(n)) = \mathrm{ed}(\mathrm{Spin}(n)) = \begin{cases} 2^{n-1} - n(n-1)/2 & \text{if } n \text{ is odd;} \\ 2^{(n-2)/2} - n(n-1)/2 & \text{if } n \equiv 2 \pmod 4; \\ 2^{(n-2)/2} + 2^m - n(n-1)/2 & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

where 2^m is the largest power of 2 dividing n.

Proof. For k of characteristic 0, this was proved by Chernousov and Merkurjev, sharpening the results of Brosnan, Reichstein, and Vistoli [4, Theorem 2.2]. Their argument works in any characteristic not 2, using the results of Garibaldi and Guralnick for the upper bounds [7]. Namely, Garibaldi and Guralnick showed that for any field k and any n at least 15, $\mathrm{Spin}(n)$ acts generically freely on the spin representation for n odd, on each of the two half-spin representations if $n \equiv 2 \pmod 4$, and on the direct sum of a half-spin representation and the standard representation if $n \equiv 0 \pmod 4$. Moreover, for n at least 20 with $n \equiv 0 \pmod 4$, $\mathrm{HSpin}(n) = \mathrm{Spin}(n)/\mu_2$ (the quotient different from $O^+(n)$) acts generically freely on a half-spin representation [7, Theorem 1.1].

It remains to consider a field k of characteristic 2. Garibaldi and Guralnick's result gives the desired upper bound in most cases. Namely, for n odd and at least 15, the spin representation has dimension $2^{(n-1)/2}$, and so $\mathrm{ed}(\mathrm{Spin}(n)) \leq 2^{(n-1)/2} - \dim(\mathrm{Spin}(n)) = 2^{(n-1)/2} - n(n-1)/2$. For $n \equiv 2 \pmod 4$, the half-spin representations have dimension $2^{(n-2)/2}$, and so $\mathrm{ed}(\mathrm{Spin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2$. For $n = 16$, since the spin group acts generically freely on the direct sum of a half-spin representation and the standard representation, $\mathrm{ed}(\mathrm{Spin}(n)) \leq 2^{(n-2)/2} + n - n(n-1)/2$ $(= 24)$.

For n at least 20 and divisible by 4, the optimal upper bound requires more effort. The following argument is modeled on Chernousov and Merkurjev's characteristic zero argument [4, Theorem 2.2]. Namely, consider the map of exact sequences of k-group schemes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Spin}(n) & \longrightarrow & \mathrm{HSpin}(n) & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & O^+(n) & \longrightarrow & \mathrm{PGO}^+(n) & \longrightarrow & 1. \end{array}$$

Since $\mathrm{HSpin}(n)$ acts generically freely on a half-spin representation, which has dimension $2^{(n-2)/2}$, we have $\mathrm{ed}(\mathrm{HSpin}(n)) \leq 2^{(n-2)/2} - n(n-1)/2$.

By Chernousov-Merkurjev or independently Lötscher, for any normal subgroup scheme C of an affine group scheme G over a field k,

$$\mathrm{ed}(G) \leq \mathrm{ed}(G/C) + \max\ \mathrm{ed}\,[E/G],$$

where the maximum runs over all field extensions F of k and all G/C-torsors E over F [4, Proposition 2.1], [14, Example 3.4]. Thus $[E/G]$ is a gerbe over F banded by C.

Identifying $H^2(K, \mu_p)$ with the p-torsion in the Brauer group of K, we can talk about the index of an element of $H^2(K, \mu_p)$, meaning the degree of the corresponding division algebra over K. For a prime number p and a nonzero element E of $H^2(K, \mu_p)$ over a field K, the essential dimension (or also the p-essential dimension) of the corresponding μ_p-gerbe over K is equal to the index of E, by Karpenko and Merkurjev [11, Theorems 2.1 and 3.1].

By the diagram above, for any field F over k, the image of the connecting map

$$H^1(F, \mathrm{HSpin}(n)) \to H^2(F, \mu_2) \subset \mathrm{Br}(F)$$

is contained in the image of the other connecting map

$$H^1(F, PGO^+(n)) \to H^2(F, \mu_2) \subset \mathrm{Br}(F).$$

In the terminology of the Book of Involutions, the image of the latter map consists of the classes $[A]$ of all central simple F-algebras A of degree n with a quadratic pair (σ, f) of trivial discriminant [12, section 29.F]. Any torsor for $PGO^+(n)$ is split by a field extension of degree a power of 2, by reducing to the corresponding fact about quadratic forms. So $\mathrm{ind}(A)$ must be a power of 2, but it also divides n, and so $\mathrm{ind}(A) \leq 2^m$, where 2^m is the largest power of 2 dividing n. We conclude that

$$\begin{aligned} \mathrm{ed}(\mathrm{Spin}(n)) &\leq \mathrm{ed}(\mathrm{HSpin}(n)) + 2^m \\ &\leq 2^{(n-2)/2} - n(n-1)/2 + 2^m. \end{aligned}$$

This completes the proof of the upper bound in Theorem 2.1.

We now prove the corresponding lower bound for the 2-essential dimension of the spin group over a field k of characteristic 2. Since $\mathrm{ed}_2(\mathrm{Spin}(n)) \leq \mathrm{ed}(\mathrm{Spin}(n))$, this will imply that the 2-essential dimension and the essential dimension are both equal to the number given in Theorem 2.1.

Write $O(2r)$ for the orthogonal group of the quadratic form $x_1x_2 + x_3x_4 + \cdots + x_{2r-1}x_{2r}$ over k, and $O(2r+1)$ for the orthogonal group of $x_1x_2 + x_3x_4 + \cdots +$

$x_{2r-1}x_{2r} + x_{2r+1}^2$. Then we have an inclusion $O(2r) \subset O(2r+1)$. Note that $O(2r)$ is smooth over k, with $O(2r)/O^+(2r) \cong \mathbf{Z}/2$. The group scheme $O(2r+1)$ is not smooth over k, but it contains a smooth connected subgroup $O^+(2r+1)$ with $O(2r+1) \cong O^+(2r+1) \times \mu_2$. It follows that $O(2r)$ is contained in $O^+(2r+1)$. Using the subgroup $\mathbf{Z}/2 \times \mu_2$ of $O(2)$, we have a k-subgroup scheme $K := (\mathbf{Z}/2 \times \mu_2)^r \subset O(2r) \subset O^+(2r+1)$. Let G be the inverse image of K in the double cover $\mathrm{Spin}(2r+1)$ of $O^+(2r+1)$. Thus G is a central extension

$$1 \to \mu_2 \to G \to (\mathbf{Z}/2)^r \times (\mu_2)^r \to 1.$$

Let V be a representation of G over k on which μ_2 acts by its standard representation. I claim that V has dimension a multiple of 2^r. This bound is optimal, since the spin representation W of $\mathrm{Spin}(2r+1)$ has dimension 2^r over k, and the center μ_2 acts faithfully by scalars on W.

To prove that V has dimension a multiple of 2^r, note that the identity component G^0 of G is isomorphic to $(\mu_2)^{r+1}$. Choose such an isomorphism with the center μ_2 of G corresponding to the 0th copy of μ_2, with the copies of μ_2 numbered as $0, 1, \ldots, r$. As a representation of G^0, V is a direct sum $V = \oplus_I V_I$, where I runs over all subsets of $\{0, 1, \ldots, r\}$. (On V_I, the ith copy of μ_2 acts trivially if $i \not\in I$ and by the standard representation of μ_2 if $i \in I$.) The quotient group G/G^0 is isomorphic to $(\mathbf{Z}/2)^r$. For $1 \leq i \leq r$, the ith copy of $\mathbf{Z}/2$ sends V_I to itself if $0 \not\in I$, and to $V_{I\Delta\{i\}}$ if $0 \in I$ (using that the 0th copy of μ_2 in G^0 is central in G). Here $I \Delta J$ denotes the symmetric difference $(I - J) \cup (J - I)$. As a result, $(\mathbf{Z}/2)^r$ acts simply transitively on the set of subspaces V_I with $0 \in I$. It follows that 2^r divides the dimension of $\oplus_{0 \in I} V_I$. As a result, if the center μ_2 acts on V by its standard representation, then 2^r divides the dimension of V, as we want.

We use the following result of Merkurjev's [15, Theorem 5.2], [11, Remark 4.5]:

Theorem 2.2. *Let k be a field and p be a prime number. Let $1 \to \mu_p \to G \to Q \to 1$ be a central extension of affine group schemes over k. For a field extension K of k, let $\partial_K \colon H^1(K, Q) \to H^2(K, \mu_p)$ be the boundary homomorphism in fppf cohomology. Then the maximal value of the index of $\partial_K(E)$, as K ranges over all field extensions of k and E ranges over all Q-torsors over K, is equal to the greatest common divisor of the dimensions of all representations of G on which μ_p acts by its standard representation.*

As mentioned above, for a prime number p and a nonzero element E of $H^2(K, \mu_p)$ over a field K, the essential dimension (or also the p-essential dimension) of the corresponding μ_p-gerbe over K is equal to the index of E.

Finally, consider a central extension $1 \to \mu_p \to G \to Q \to 1$ of finite group schemes over a field k. Generalizing an argument of Brosnan-Reichstein-Vistoli, Karpenko and Merkurjev showed that the p-essential dimension of G (and hence the essential dimension of G) is at least the p-essential dimension of the μ_p-gerbe over K associated to any Q-torsor over any field K/k [11, Theorem 4.2]. So the analysis above of representations of the finite subgroup scheme G of $\mathrm{Spin}(2r+1)$ over a field k of characteristic 2, we find that $\mathrm{ed}_2(G) \geq 2^r$. For a closed subgroup scheme G of a group scheme L over a field k and any prime number p, we have $\mathrm{ed}_p(L) + \dim(L) \geq \mathrm{ed}_p(G) + \dim(G)$ [16, Corollary 4.3]. Applying this to the subgroup scheme G of $\mathrm{Spin}(2r)$, we conclude that $\mathrm{ed}_2(\mathrm{Spin}(2r+1)) \geq 2^r - \dim(\mathrm{Spin}(2r+1)) = 2^r - r(2r+1)$.

Combining this with the upper bound discussed above, we have

$$\mathrm{ed}(\mathrm{Spin}(2r+1)) = \mathrm{ed}_2(\mathrm{Spin}(2r+1)) = 2^r - r(2r+1)$$

for $r \geq 7$.

The proof of the lower bound for $\mathrm{ed}_2(\mathrm{Spin}(2r))$ when r is odd is similar. The intersection of the subgroup $K = (\mu_2 \times \mathbf{Z}/2)^r \subset O(2r)$ with $O^+(2r)$ is $K_1 \cong (\mu_2)^r \times (\mathbf{Z}/2)^{r-1}$, where $(\mathbf{Z}/2)^{r-1}$ denotes the kernel of the sum $(\mathbf{Z}/2)^r \to \mathbf{Z}/2$. As a result, the double cover $\mathrm{Spin}(2r)$ contains a subgroup G_1 which is a central extension

$$1 \to \mu_2 \to G_1 \to (\mathbf{Z}/2)^{r-1} \times (\mu_2)^r \to 1.$$

In this case, an argument analogous to the one for G shows that every representation of G_1 on which the center μ_2 acts by its standard representation has dimension a multiple of 2^{r-1} (rather than 2^r). The argument is otherwise identical to the argument for $\mathrm{Spin}(2r+1)$, and we find that $\mathrm{ed}_2(\mathrm{Spin}(2r)) \geq 2^{r-1} - r(2r-1)$. For r odd at least 9, this agrees with the lower bound found earlier, which proves the theorem on $\mathrm{Spin}(n)$ for $n \equiv 0 \pmod 4$.

It remains to show that for n a multiple of 4, with 2^m the largest power of 2 dividing n, we have

$$\mathrm{ed}_2(\mathrm{Spin}(n)) \geq 2^{(n-2)/2} + 2^m - n(n-1)/2.$$

The argument follows that of Merkurjev in characteristic not 2 [16, Theorem 4.9].

Namely, for n a multiple of 4, the center C of $G := \mathrm{Spin}(n)$ is isomorphic to $\mu_2 \times \mu_2$, and $H := G/C$ is the group $PGO^+(n)$. An H-torsor over a field L over k is equivalent to a central simple algebra A of degree n over L with a quadratic pair (σ, f) and with trivialized discriminant, meaning an isomorphism from the center of the Clifford algebra $C(A, \sigma, f)$ to $L \times L$ [12, section 29.F]. The image of the homomorphism from $C^* \cong (\mathbf{Z}/2)^2$ to the Brauer group of L is equal to $\{0, [A], [C^+], [C^-]\}$, where C^+ and C^- are the simple components of the Clifford algebra; each is a central simple algebra of degree $2^{(n-2)/2}$ over L. By Merkurjev, there is a field L over k and an H-torsor E over L such that $\mathrm{ind}(C^+) = \mathrm{ind}(C^-) = 2^{(n-2)/2}$ and $\mathrm{ind}(A) = 2^m$ [15, section 4.4 and Theorem 5.2]. We use the following result [16, Example 3.7]:

Lemma 2.3. *Let L be a field, p a prime number, and r a natural number. Let C be the group scheme $(\mu_p)^r$, and let Y be a C-gerbe over L. Then the p-essential dimension of Y, and also the essential dimension of Y, is the minimum, over all bases $u_1, \ldots, u_r$ for C^*, of $\sum_{i=1}^r \mathrm{ind}(u_i(Y))$.*

It follows that the 2-essential dimension of the $(\mu_2)^2$-gerbe E/G over L associated to the H-torsor E above is

$$\mathrm{ed}_2(E/G) = \mathrm{ind}(A) + \mathrm{ind}(C^+) = 2^{(n-2)/2} + 2^m.$$

It follows that

$$\begin{aligned} \mathrm{ed}(\mathrm{Spin}(n)) &\geq \mathrm{ed}_2(\mathrm{Spin}(n)) \\ &\geq \mathrm{ed}_2(E/G) - \dim(G/C) \\ &= 2^{(n-2)/2} + 2^m - n(n-1)/2. \end{aligned}$$

∎

3 Low-dimensional spin groups

Garibaldi and Rost determined the essential dimension of the spin groups $\mathrm{Spin}(n)$ with $n \leq 14$ in characteristic not 2 [6, Table 23B]. It should be possible to compute the essential dimension of low-dimensional spin groups in characteristic 2 as well. The following section carries this out for $\mathrm{Spin}(n)$ with $n \leq 10$. We find that in this range (as for $n \geq 15$), the essential dimension of the spin group is the same in characteristic 2 as in characteristic not 2, unlike what happens for $O(n)$ and $SO(n)$.

For $n \leq 10$, we give group-theoretic proofs which work almost the same way in any characteristic, despite the distinctive features of quadratic forms in characteristic 2.

Theorem 3.1. *For $n \leq 10$, the essential dimension, as well as the 2-essential dimension, of the split group $\mathrm{Spin}(n)$ over a field k of any characteristic is given by:*

n	$\mathrm{ed}(\mathrm{Spin}(n))$
≤ 6	0
7	4
8	5
9	5
10	4

Proof. As discussed above, it suffices to consider the case of a field k of characteristic 2. For $n \leq 6$, every $\mathrm{Spin}(n)$-torsor over a field is trivial, for example by the exceptional isomorphisms $\mathrm{Spin}(3) \cong SL(2)$, $\mathrm{Spin}(4) \cong SL(2) \times SL(2)$, $\mathrm{Spin}(5) \cong Sp(4)$, and $\mathrm{Spin}(6) \cong SL(4)$. It follows that $\mathrm{ed}(\mathrm{Spin}(n)) = 0$ for $n \leq 6$.

We first recall some general definitions. For a field k of characteristic $p > 0$, let $H^{i,j}(k)$ be the etale motivic cohomology group $H^i_{\text{et}}(k, \mathbf{Z}/p(j))$, or equivalently

$$H^i_{\text{et}}(k, \mathbf{Z}/p(j)) \cong H^{i-j}_{\text{et}}(k, \Omega^j_{\log}),$$

where $\Omega^j_{\log}$ is the subgroup of the group Ω^j of differential forms on the separable closure k_s over $\mathbf{F}_p$ spanned by products $(da_1/a_1) \wedge \cdots \wedge (da_j/a_j)$ with $a_1, \ldots, a_j \in k_s^*$ [9]. The group $H^{i,j}(k)$ is zero except when i equals j or $j+1$, because k has p-cohomological dimension at most 1 [19, section II.2.2]. The symbol $\{a_1, \ldots, a_{n-1}, b]$ denotes the element of $H^{n,n-1}(k)$ which is the product of the elements $a_i \in k^*/(k^*)^p \cong H^{1,1}(k)$ and $b \in k/\{a^p - a : a \in k\} \cong H^{1,0}(k)$.

Also, for a field k of characteristic 2, let $W(k)$ denote the Witt ring of symmetric bilinear forms over k, and let $I_q(k)$ be the Witt group of nondegenerate quadratic forms over k. (By the conventions in section 1, $I_q(k)$ consists only of even-dimensional forms.) Then $I_q(k)$ is a module over $W(k)$ via tensor product [5, Lemma 8.16]. Let I be the kernel of the homomorphism rank: $W(k) \to \mathbf{Z}/2$, and let

$$I^m_q(k) = I^{m-1} \cdot I_q(k),$$

following [5, p. 53]. To motivate the notation, observe that the class of an m-fold quadratic Pfister form $\langle\langle a_1, \ldots, a_{m-1}, b]]$ lies in $I^m_q(k)$. By definition, for $a_1, \ldots, a_{m-1}$ in k^* and b in k, $\langle\langle a_1, \ldots, a_{m-1}, b]]$ is the quadratic form $\langle\langle a_1\rangle\rangle_b \otimes \cdots \otimes \langle\langle a_{m-1}\rangle\rangle_b \otimes \langle\langle b]]$ of dimension 2^m, where $\langle\langle a\rangle\rangle_b$ is the bilinear form $\langle 1, a\rangle$ and $\langle\langle b]]$ is the quadratic form $[1, b] = x^2 + xy + by^2$.

In analogy with the Milnor conjecture, Kato proved the isomorphism

$$I_q^m(F)/I_q^{m+1} \cong H^{m,m-1}(F)$$

for every field F of characteristic 2 [5, Fact 16.2]. The isomorphism takes the quadratic Pfister form $\langle\langle a_1, \ldots, a_{m-1}, b]]$ to the symbol $\{a_1, \ldots, a_{m-1}, b]$. (For this paper, it would suffice to have Kato's homomorphism, without knowing that it is an isomorphism.)

We will use the following standard approach to bounding the essential dimension of a group.

Lemma 3.2. *Let G be an affine group scheme of finite type over a field k. Suppose that G acts on a k-scheme Y with a nonempty open orbit U. Suppose that for every G-torsor E over an infinite field F over k, the twisted form $(E \times Y)/G$ of Y over F has a Zariski-dense set of F-points. Finally, suppose that U has a k-point x, and let N be the stabilizer k-group scheme of x in G. Then*

$$H^1(F, N) \to H^1(F, G)$$

is surjective for every infinite field F over k (or for every field F over k, if G is smooth and connected over k). As a result, $\mathrm{ed}_k(G) \leq \mathrm{ed}_k(N)$.

The proof is short, the same as that of [6, Theorem 9.3]. (Note that even if k is finite, we get the stated upper bound for the essential dimension of G: a G-torsor over a finite extension field F of k causes no problem, because F has transcendence degree 0 over k.) If G is smooth and connected over k, then $H^1(F, G)$ is in fact trivial for every finite field F over k by Lang [13]; that implies the statement in the theorem that $H^1(F, N) \to H^1(F, G)$ is surjective for *every* field F over k.

The assumption about Zariski-dense sets of rational points holds, for example, if Y is a linear representation V of G, or if Y is the associated projective space $P(V)$ to a representation, or (as we use later) a product $P(V) \times P(W)$.

We use Garibaldi and Guralnick's calculation of the stabilizer group scheme of a general k-point in the spin (for n odd) or a half-spin (for n even) representation W of the split group $\mathrm{Spin}(n)$, listed in Table 1 here p[7, Table 1]. Here $\mathrm{Spin}(n)$ has an open orbit on the projective space $P(W)$ of lines in W if $n \leq 12$ or $n = 14$, and an open orbit on W if $n = 10$. (To be precise, we will use that even if k is finite, there is a k-point in the open orbit for which the stabilizer k-group scheme is the *split* group listed in the table.)

We now begin to compute the essential dimension of the split group $G = \mathrm{Spin}(7)$ over a field k of characteristic 2. Let W be the 8-dimensional spin representation of G. Then G has an open orbit on the projective space $P(W)$ of lines in W. The stabilizer k-group scheme of a suitable k-point in W whose image in $P(W)$ is in this open orbit is conjugate to the exceptional group G_2, according to Table 1. Since G preserves a quadratic form on W, the stabilizer H of the corresponding k-point in $P(W)$ is at most $G_2 \times \mu_2$. In fact, H is equal to $G_2 \times \mu_2$, because the center μ_2 of G acts trivially on $P(W)$.

By Lemma 3.2, the inclusion $G_2 \times \mu_2 \hookrightarrow G$ induces a surjection

$$H^1(F, G_2 \times \mu_2) \to H^1(F, G)$$

n	char $k \neq 2$	char $k = 2$
6	$SL(3) \cdot (G_a)^3$	same
7	G_2	same
8	Spin(7)	same
9	Spin(7)	same
10	$\text{Spin}(7) \cdot (G_a)^8$	same
11	$SL(5)$	$\mathbf{Z}/2 \ltimes SL(5)$
12	$SL(6)$	$\mathbf{Z}/2 \ltimes SL(6)$
13	$SL(3) \times SL(3)$	$\mathbf{Z}/2 \ltimes (SL(3) \times SL(3))$
14	$G_2 \times G_2$	$\mathbf{Z}/2 \ltimes (G_2 \times G_2)$

Table 1: Generic stabilizer of spin (or half-spin) representation of Spin(n)

for every field F over k. Over any field F, G_2-torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms $\langle\langle a_1, a_2, b]]$ (with $a_1, a_2 \in F^*$ and $b \in F$), and so G_2 has essential dimension 3 [19, Théorème 11]. Since μ_2 has essential dimension 1, the surjectivity above implies that $G = \text{Spin}(7)$ has essential dimension at most 4.

Next, a G-torsor determines two quadratic forms of dimension 8. Besides the obvious homomorphism $\chi_1 \colon G \hookrightarrow \text{Spin}(8) \to SO(8)$ (which is trivial on the center μ_2 of G), we have the spin representation $\chi_2 \colon G \to SO(8)$, on which μ_2 acts faithfully by scalars. Thus a G-torsor u over a field F over k determines two quadratic forms of dimension 8 over F, which we call q_1 and q_2.

To describe these quadratic forms in more detail, use that every G-torsor comes from a torsor for $G_2 \times \mu_2$. The two homomorphisms $G_2 \hookrightarrow G \to SO(8)$ (via χ_1 and χ_2) are both conjugate to the standard inclusion. Also, χ_1 is trivial on the μ_2 factor, while χ_2 acts faithfully by scalars on the μ_2 factor. It follows that q_1 is a quadratic Pfister form, $\langle\langle a, b, c]]$ (the form associated to a G_2-torsor), while q_2 is a scalar multiple of that form, $d\langle\langle a, b, c]]$.

Therefore, a G-torsor u canonically determines a 4-fold quadratic Pfister form,

$$q_1 + q_2 = \langle\langle d, a, b, c]].$$

Define $f_4(u)$ to be the associated element of $H^{4,3}(F)$,

$$f_4(u) = \{d, a, b, c].$$

By construction, this is well-defined and an invariant of u. This invariant is normalized (zero on the trivial G-torsor) and not zero. (By considering the subgroup $G_2 \times \mu_2 \subset \text{Spin}(7)$, where there is a $G_2 \times \mu_2$-torsor associated to any elements a, b, d in F^* and c in F, we see that a, b, c, d can be chosen arbitrarily. By taking F to be the rational function field $k(a, b, c, d)$, we see that the element $f_4(u) = \{d.a, b, c]$ of $H^{4,3}(F)$ can be nonzero. For that, one can use the computation of $H^{n,n-1}$ of a rational function field by Izhboldin [10].)

Therefore, $G = \text{Spin}(7)$ has essential dimension at least 4. The opposite inequality was proved above, and so Spin(7) has essential dimension equal to 4. Since the lower bound is proved by constructing a mod 2 cohomological invariant, this argument also shows that Spin(7) has 2-essential dimension equal to 4. For the same reason, the computations of essential dimension below (for $\mathrm{Spin}(n)$ with $8 \leq n \leq 10$) also give the 2-essential dimension.

Next, we turn to $\mathrm{Spin}(8)$. At first, let $G = \mathrm{Spin}(2r)$ for a positive integer r over a field k of characteristic 2. Let V be the standard $2r$-dimensional representation of G. Then G has an open orbit in the projective space $P(V)$ of lines in V. The stabilizer k-group scheme H of a general k-point in $P(V)$ is conjugate to $\mathrm{Spin}(2r-1)\cdot Z$, where Z is the center of $\mathrm{Spin}(2r)$, with $\mathrm{Spin}(2r-1) \cap Z = \mu_2$. (In more detail, a general line in V is spanned by a vector x with $q(x) \neq 0$, where q is the quadratic form on V. Then the stabilizer of x in $SO(V)$ is isomorphic to $SO(S)$, where $S := x^\perp$ is a hyperplane in V on which q restricts to a nonsingular quadratic form of dimension $2r-1$, with $S^\perp$ equal to the line $k \cdot x \subset S$.) Here

$$Z \cong \begin{cases} \mu_2 \times \mu_2 & \text{if } r \text{ is even} \\ \mu_4 & \text{if } r \text{ is odd.} \end{cases}$$

In particular, if r is even, then $H \cong \mathrm{Spin}(2r-1) \times \mu_2$. Thus, for r even, the inclusion $\mathrm{Spin}(2r-1) \times \mu_2 \hookrightarrow G$ induces a surjection

$$H^1(F, \mathrm{Spin}(2r-1) \times \mu_2) \to H^1(F, G)$$

for every field F over k, by Lemma 3.2.

It follows that, for r even, the essential dimension of $\mathrm{Spin}(2r)$ is at most 1 plus the essential dimension of $\mathrm{Spin}(2r-1)$. Since $\mathrm{Spin}(7)$ has essential dimension 4, $G = \mathrm{Spin}(8)$ has essential dimension at most 5.

Before proving that equality holds, let us analyze G-torsors in more detail. We know that $H^1(F, \mathrm{Spin}(7) \times \mu_2) \to H^1(F, G)$ is onto, for all fields F over k. Also, we showed earlier that $H^1(F, G_2 \times \mu_2) \to H^1(F, \mathrm{Spin}(7))$ is surjective. Therefore,

$$H^1(F, G_2 \times \mu_2 \times \mu_2) \to H^1(F, G)$$

is surjective for all fields F over k, where $Z = \mu_2 \times \mu_2$ is the center of G. As discussed earlier, G_2-torsors up to isomorphism can be identified with 3-fold quadratic Pfister forms. It follows that every G-torsor is associated to some 3-fold quadratic Pfister form $\langle\langle a, b, c]]$ and some elements d, e in F^*, which yield elements of $H^1(F, \mu_2) = F^*/(F^*)^2$.

Next, observe that a G-torsor determines several quadratic forms. Besides the obvious double covering $\chi_1 \colon G \to SO(8)$, the two half-spin representations of G give two other homomorphisms $\chi_2, \chi_3 \colon G \to SO(8)$. (These three homomorphisms can be viewed as the quotients of G by the three k-subgroup schemes of order 2 in Z. They are permuted by the group S_3 of "triality" automorphisms of G.) Thus a G-torsor u over a field F over k determines three quadratic forms of dimension 8, which we call q_1, q_2, q_3.

To describe how these three quadratic forms are related, use that every G-torsor comes from a torsor for $G_2 \times \mu_2 \times \mu_2$. The three homomorphisms $G_2 \to G \to SO(8)$ (via χ_1, χ_2, and χ_3) are all conjugate to the standard inclusion, whereas the three homomorphisms send $\mu_2 \times \mu_2$ to the center $\mu_2 \subset SO(8)$ by the three possible surjections. It follows that the three quadratic forms can be written as $q_1 = d\langle\langle a, b, c]]$, $q_2 = e\langle\langle a, b, c]]$, and $q_3 = de\langle\langle a, b, c]]$.

Note that a scalar multiple of a quadratic Pfister form, $q = d\langle\langle a_1, \ldots, a_{m-1}, b]]$ (as a quadratic form up to isomorphism), uniquely determines the associated quadratic

Pfister form $q_0 = \langle\langle a_1, \ldots, a_{m-1}, b]]$ up to isomorphism. (Proof: it suffices to show that if q and r are m-fold quadratic Pfister forms over F with $aq \cong r$ for some a in F^*, then $q \cong r$. Since r takes value 1, so does aq, and so q takes value a^{-1}. But then $a^{-1}q \cong q$ by the multiplicativity of quadratic Pfister forms [5, Corollary 9.9]. Therefore, $r \cong aq \cong q$.)

We now define an invariant for $G = \mathrm{Spin}(8)$ over k with values in $H^{5,4}$. Given a G-torsor u over a field F over k, consider the three associated quadratic forms q_1, q_2, q_3 as above. By the previous paragraph, $q_1 = d\langle\langle a, b, c]]$ determines the quadratic Pfister form $q_0 = \langle\langle a, b, c]]$. So u determines the 5-fold Pfister form

$$q_0 + q_1 + q_2 + q_3 = \langle\langle d, e, a, b, c]].$$

The associated class

$$f_5(u) = \{d, e, a, b, c] \in H^{5,4}(F)$$

is therefore an invariant of u.

The invariant f_5 is normalized and not 0, as shown by considering the subgroup $G_2 \times Z \subset G = \mathrm{Spin}(8)$, where $Z = \mu_2 \times \mu_2$: there is a $G_2 \times Z$-torsor associated to any elements a, b, d, e in F^* and c in F, and f_5 of the associated G-torsor is $\{d, e, a, b, c]$ in $H^{5,4}(F)$. Therefore, G has essential dimension at least 5. Since the opposite inequality was proved above, $G = \mathrm{Spin}(8)$ has essential dimension over k equal to 5.

Next, let $G = \mathrm{Spin}(9)$ over a field k of characteristic 2. Let W be the spin representation of G, of dimension 16, corresponding to a homomorphism $G \to SO(16)$. (A reference for the fact that this self-dual representation is orthogonal in characteristic 2, as in other characteristics, is [8, Theorem 9.2.2].) By Table 1, G has an open orbit on the space of lines $P(W)$ in W, and the stabilizer in G of a general k-point in W is conjugate to $\mathrm{Spin}(7)$. (This is not the standard inclusion of $\mathrm{Spin}(7)$ in $\mathrm{Spin}(9)$, but rather a lift of the spin representation $\chi_2\colon \mathrm{Spin}(7) \to SO(8)$ to $\mathrm{Spin}(8)$ followed by the standard inclusion $\mathrm{Spin}(8) \hookrightarrow \mathrm{Spin}(9)$. In particular, the image of $\mathrm{Spin}(7)$ does not contain the center μ_2 of $G = \mathrm{Spin}(9)$.) Since G preserves a quadratic form on W, it follows that the stabilizer in G of a general k-point in $P(W)$ is conjugate to $\mathrm{Spin}(7) \times \mu_2$, where μ_2 is the center of $\mathrm{Spin}(9)$ (which acts faithfully by scalars on W). Therefore, by Lemma 3.2, the inclusion of $\mathrm{Spin}(7) \times \mu_2$ in $G = \mathrm{Spin}(9)$ induces a surjection

$$H^1(F, \mathrm{Spin}(7) \times \mu_2) \to H^1(F, G)$$

for every field F over k.

Since $\mathrm{Spin}(7)$ has essential dimension 4 over k as shown above, $G = \mathrm{Spin}(9)$ has essential dimension at most $4 + 1 = 5$.

Next, a G-torsor determines several quadratic forms. Besides the obvious homomorphism $R\colon G \hookrightarrow \mathrm{Spin}(10) \to SO(10)$, we have the spin representation $S\colon G \to SO(16)$. Thus a G-torsor over a field F over k determines a quadratic form r of dimension 10 and a quadratic form s of dimension 16.

To describe how these forms are related, use that every G-torsor comes from a torsor for the subgroup $\mathrm{Spin}(7) \times \mu_2$ described above. The restriction of R to the given subgroup $\mathrm{Spin}(7)$ is the composition of the spin representation $\chi_2\colon \mathrm{Spin}(7) \to SO(8)$ with the obvious inclusion $SO(8) \hookrightarrow SO(10)$. The restriction of S to the given subgroup Spin(7) is the direct sum of the standard representation $\chi_1\colon \mathrm{Spin}(7) \to SO(8)$ and the spin representation $\chi_2\colon \mathrm{Spin}(7) \to SO(8)$. Finally, R is trivial on the second factor μ_2 (the center of G), whereas S acts faithfully by scalars on S.

Now, let (u_1, e) be a $\mathrm{Spin}(7)\times\mu_2$-torsor over k, where u_1 is a Spin(7)-torsor and e is in $H^1(F, \mu_2) = F^*/(F^*)^2$, which we lift to an element e of F^*. By the earlier analysis of the quadratic forms associated to a Spin(7)-torsor, the quadratic form associated to u_1 via the standard representation $\chi_1\colon \mathrm{Spin}(7) \to SO(8)$ is a 3-fold quadratic Pfister form $\langle\langle a, b, c]]$, while the quadratic form associated to u_1 via the spin representation $\chi_2\colon \mathrm{Spin}(7) \to SO(8)$ is a multiple of the same form, $d\langle\langle a, b, c]]$.

By the analysis of representations two paragraphs back, it follows that the quadratic form associated to (u_1, e) via the representation $R\colon G \to SO(10)$ is $r = H + d\langle\langle a, b, c]]$, where H is the hyperbolic plane. Also, the quadratic form associated to (u_1, e) via the representation $S\colon G \to SO(16)$ is $s = e\langle\langle a, b, c]] + de\langle\langle a, b, c]]$.

Next, r determines the quadratic form $r_0 = d\langle\langle a, b, c]]$ by Witt cancellation [5, Theorem 8.4], and that in turn determines the quadratic Pfister form $q_0 = \langle\langle a, b, c]]$ as shown above. Therefore, a G-torsor u determines the 5-fold quadratic Pfister form

$$q_0 + r_0 + s = \langle\langle d, e, a, b, c]]$$

up to isomorphism.

Therefore, defining

$$f_5(u) = \{d, e, a, b, c]$$

in $H^{5,4}(F)$ yields an invariant of u. By our earlier description of Spin(7)-torsors, we can take a, b, d, e to be any elements of F^* and c any element of F. Therefore, f_5 is a nonzero normalized invariant of G over k with values in $H^{5,4}$. It follows that G has essential dimension at least 5. Since the opposite inequality was proved earlier, $G = \mathrm{Spin}(9)$ over k has essential dimension equal to 5.

Finally, let $G = \mathrm{Spin}(10)$ over a field k of characteristic 2. Let V be the 10-dimensional standard representation of G, corresponding to the double covering $G \to SO(10)$, and let W be one of the 16-dimensional half-spin representations of G, corresponding to a homomorphism $G \to SL(16)$. (The other half-spin representation of G is the dual W^*.)

As discussed above for any group $\mathrm{Spin}(2r)$, $G = \mathrm{Spin}(10)$ has an open orbit on $P(V)$, with generic stabilizer $\mathrm{Spin}(9)\cdot\mu_4$. (Here μ_4 is the center of G, which contains the center μ_2 of Spin(9).) Consider the action of G on $P(V) \times P(W) \cong \mathbf{P}^9 \times \mathbf{P}^{15}$. As discussed above, Spin(9) (and hence $\mathrm{Spin}(9)\cdot\mu_4$) has an open orbit on $P(W)$. As a result, G has an open orbit on $P(V)\times P(W)$. Moreover, the generic stabilizer of Spin(9) on $P(W)$ is $\mathrm{Spin}(7)\times\mu_2$, where the inclusion $\mathrm{Spin}(7) \hookrightarrow \mathrm{Spin}(9)$ is the composition of the spin representation $\mathrm{Spin}(7) \hookrightarrow \mathrm{Spin}(8)$ with the standard inclusion into Spin(9); in particular, the image does not contain the center μ_2 of Spin(9). Therefore, the generic stabilizer of $\mathrm{Spin}(9)\cdot\mu_4 \subset \mathrm{Spin}(10)$ on $P(W)$ is $\mathrm{Spin}(7)\times\mu_4$. We conclude that G has an open orbit on $P(V) \times P(W)$, with generic stabilizer $\mathrm{Spin}(7)\times\mu_4$. It follows that

$$H^1(F, \mathrm{Spin}(7)\times\mu_4) \to H^1(F, G)$$

is surjective for every field F over k, by Lemma 3.2.

The image H_2 of the subgroup $H = \mathrm{Spin}(7)\times\mu_4 \subset G$ in $SO(10)$ is $\mathrm{Spin}(7)\times\mu_2$, where Spin(7) is contained in $SO(8)$ (and contains the center μ_2 of $SO(8)$) and μ_2

is the center of $SO(10)$. In terms of the subgroup $SO(8) \times SO(2)$ of $SO(10)$, we can also describe H_2 as $\mathrm{Spin}(7) \times \mu_2$, where $\mathrm{Spin}(7)$ is contained in $SO(8)$ and μ_2 is contained in $SO(2)$. Thus H_2 is contained in $\mathrm{Spin}(7) \times SO(2)$. Therefore, H is contained in $\mathrm{Spin}(7) \times G_m \subset G = \mathrm{Spin}(10)$, where the multiplicative group G_m is the inverse image in G of $SO(2) \subset SO(10)$. It follows that

$$H^1(F, \mathrm{Spin}(7) \times G_m) \to H^1(F, G)$$

is surjective for every field F over k. Since every G_m-torsor over a field is trivial,

$$H^1(F, \mathrm{Spin}(7)) \to H^1(F, G)$$

is surjective for every field F over k.

Here $\mathrm{Spin}(7)$ maps into $\mathrm{Spin}(8)$ by the spin representation, and then $\mathrm{Spin}(8) \hookrightarrow G = \mathrm{Spin}(10)$ by the standard inclusion. By the description above of the 8-dimensional quadratic form associated to a $\mathrm{Spin}(7)$-torsor by the spin representation, it follows that the quadratic form associated to a G-torsor is of the form $H + d\langle\langle a, b, c]]$.

Every 10-dimensional quadratic form in I_q^3 over a field is associated to some G-torsor. So we have given another proof that every 10-dimensional quadratic form in I_q^3 is isotropic. This was proved in characteristic not 2 by Pfister, and it was extended to characteristic 2 by Baeza and Tits, independently [2, pp. 129-130], [20, Theorem 4.4.1(ii)].

Since $\mathrm{Spin}(7)$ has essential dimension 4, the surjectivity above implies that $G = \mathrm{Spin}(10)$ has essential dimension at most 4. To prove equality, we define a nonzero normalized invariant for G with values in $H^{4,3}$ by the same argument used for $\mathrm{Spin}(7)$. Namely, a G-torsor u over a field F over k determines a 4-fold quadratic Pfister form

$$\langle\langle d, a, b, c]]$$

up to isomorphism, and hence the element

$$f_4(u) = \{d, a, b, c]$$

in $H^{4,3}(F)$. This completes the proof that $G = \mathrm{Spin}(10)$ over k has essential dimension equal to 4. As in the previous cases, since the lower bound is proved using a mod 2 cohomological invariant, G also has 2-essential dimension equal to 4. ☐

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