University of California

Los Angeles

Lipschitz Maps in Metric Spaces

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Guy David

2014

Abstract of the Dissertation

Lipschitz Maps in Metric Spaces

by

Guy David

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2014

Professor Mario Bonk, Chair

In this dissertation we study Lipschitz and bi-Lipschitz mappings on abstract, non-smooth metric measure spaces. The dissertation consists of two separate parts.

The first part considers a well-known class of questions that ask the following: If X and Y are metric measure spaces and $f\colon X \to Y$ is a Lipschitz mapping whose image has positive measure, then must f have large pieces on which it is bi-Lipschitz? Building on methods of David (who is not the present author) and Semmes, we answer this question in the affirmative for Lipschitz mappings between certain types of Ahlfors regular topological manifolds. In general, these manifolds need not admit bi-Lipschitz embeddings into any Euclidean space. To prove the result, we use some facts on the Gromov-Hausdorff convergence of manifolds and a topological theorem of Bonk and Kleiner. This also yields a new proof of the uniform rectifiability of some metric manifolds.

In the second part, we study the class of "Lipschitz differentiability spaces" introduced by Cheeger. These are spaces on which an appropriate version of Rademacher's theorem holds. We show that if an Ahlfors regular Lipschitz differentiability space has a differentiable structure of maximal dimension, then at almost every point all its tangents are uniformly rectifiable. In particular, it admits Euclidean tangents at almost every point. Conversely, we show that if the dimension of the differentiable structure is not extremal, then the space is strongly unrectifiable, in the sense of Ambrosio-Kirchheim. In proving these results,

we generalize some results of Cheeger from the setting of doubling spaces with Poincaré inequalities (PI spaces) to general doubling Lipschitz differentiability spaces. The starting point is a result of Bate on the local structure of these spaces.

The dissertation of Guy David is approved.

Peter Petersen

John B. Garnett

Michael Gutperle

Mario Bonk, Committee Chair

University of California, Los Angeles

2014

To my family

Table of Contents

Acknowledgments

This dissertation would not exist without the support of a number of people. First and foremost, I must thank my advisor, Mario Bonk, who for four years has been incredibly generous with his time and ideas. I consider myself extremely fortunate to have wandered into his course at UCLA in 2010.

I also wish to thank the other members of my committee: John Garnett, who taught me so much during my time in graduate school; Peter Petersen, who never turned me away when I showed up at his office with geometry questions; and Michael Gutperle, who graciously donated his time to a student outside his department.

Many other mathematicians contributed helpful discussions and comments regarding the work presented here. In addition to the members of my committee, I am indebted in particular to Raanan Schul, Matthew Badger, David Bate, Pekka Pankka, and Enrico Le Donne.

I would also like to acknowledge the general support of the UCLA mathematics department faculty and staff, and to thank the department for the Beckenbach Fellowship that partially funded this research. Parts of this dissertation were written while I attended the Winter 2014 trimester on Random Walks and Asymptotic Geometry of Groups at the Institut Henri Poincaré in Paris, and I thank the organizers of that conference for their support. This research was also partly funded by NSF grant DMS-1162471.

Finally, though it's a lot to send between four cities on two continents, all the remaining gratitude I possess goes to my family: Mom, Dad, Adee, Savta, and Cara.

Vita

2009	A.B. in Mathematics and Certificate in Applications of Computing, Princeton University
2009-2010	Graduate Student Teaching Assistant, Department of Mathematics, UCLA
2010	M.A. in Mathematics, UCLA
2010-2013	Graduate Student Teaching and Research Assistant, Department of Mathematics, UCLA
2012	C. Phil. in Mathematics, UCLA
2013	Beckenbach Fellowship, Department of Mathematics, UCLA
2014	Graduate Student Teaching and Research Assistant, Department of Mathematics, UCLA

Publications

G.C. David, *Bi-Lipschitz Pieces between Manifolds.* Preprint, 2013. arXiv:1312.3911

G.C. David, *Tangents and rectifiability of Ahlfors regular Lipschitz differentiability spaces.* Preprint, 2014. arXiv:1405.2461

CHAPTER 1

Introduction

The theory of Lipschitz mappings is ubiquitous in analysis, both for its intrinsic interest and for its applications to problems in differential equations, geometry, and even computer science. There are two main reasons for the importance of Lipschitz mappings. First, their definition is extremely simple and widely applicable, and so they can be found in abundance on any metric space without any assumptions of smoothness. Second, despite the the simplicity of their definition, they often possess many rigidity properties and therefore their study can yield surprising analytic and geometric conclusions.

The focus of this dissertation is on Lipschitz and bi-Lipschitz mappings on abstract metric measure spaces, and the interactions between analytic properties of these mappings and geometric properties of the spaces. We will mostly be interested in spaces that, far from being smooth, may even admit no nice embedding into any Euclidean space. In addition to their classical appearances as "pathological" subsets of Euclidean space, non-smooth or fractal metric spaces now arise naturally in a number of settings in mathematics. Just to give a brief taste, we mention that such spaces appear as limits of Riemannian manifolds with suitable curvature bounds (see, e.g., [14], [11]), as boundaries of hyperbolic groups or other negatively curved spaces ([9], [6], [7]), as so-called "sub-Riemannian" manifolds ([12], [44]), as proper settings for generalizations of (quasi)conformal geometry ([30]), and in embedding problems related to questions in computer science ([17]).

The dissertation has two main parts. In the first part, Chapter 2, we focus on a type of rigidity phenomenon for Lipschitz mappings known as "bi-Lipschitz pieces". This idea, first discovered by David[1] [18] in Euclidean space, says roughly that if the image of a Lipschitz mapping between two suitable metric measure spaces is "large", then the mapping must in fact be bi-Lipschitz on a set of large measure, in a quantitative way. We prove a theorem of this type for Lipschitz mappings between classes of abstract metric spaces satisfying some topological assumptions, and we obtain some corollaries about the geometry of such spaces. A more substantial introduction to this chapter of the dissertation can be found in Section 1.2.

In the second part of the dissertation, Chapter 3, we study abstract metric spaces that possess a type of "differentiable structure" for real-valued Lipschitz functions. Recall first that Rademacher's theorem says that any Lipschitz function on Euclidean space is differentiable almost everywhere; though classical, this is still a surprising theorem. In 1999, Cheeger [13] gave a striking generalization of Rademacher's theorem to a large class of metric measure spaces, providing these spaces with a way of "differentiating" real-valued Lipschitz functions defined on them. The relationship between this differentiable structure and the geometry of the space is rather mysterious in general, but in the second part of the dissertation we give some connections between this structure and classical geometric notions of tangents and rectifiability. We introduce these ideas in more detail in Section 1.3.

In the remainder of this introduction, we provide some definitions and background and then give precise statements of the main results.

1.1 Basic definitions

We denote metric spaces by pairs (X, d) and metric measure spaces by triples (X, d, μ), although when the metric and measure are clear we call such a space simply X. If X is a metric space, we may also denote the metric on X by d_X. If E and F are subsets of a metric

[1]The Guy David mentioned here and many other times in this dissertation, including references [18] through [22], is a professor at Université Paris-Sud and has no relation to the author of this dissertation. I wish to apologize for any confusion generated by this amusing coincidence. I have adopted the middle initial "C" and thus only references [23] and [24] in this dissertation are mine.

space X, then we write

$$\operatorname{dist}(E,F)=\operatorname{dist}_X(E,F)=\inf\{d_X(x,y):x\in E, y\in F\}.$$

If (X,d) is a metric space, we denote open and closed balls in X by $B(x,r)$ and $\overline{B}(x,r)$, respectively, i.e., we have

$$B(x,r)=\{y\in X : d(x,y)<r\}$$

and

$$\overline{B}(x,r)=\{y\in X : d(x,y)\le r\}.$$

If the space X is not clear from context, we will sometimes clarify by writing a ball in X as $B_X(x,r)$ or $\overline{B}_X(x,r)$. Note that in general $\overline{B}(x,r)$ need not be the closure of $B(x,r)$. If B is an open or closed ball of radius r in X and $\lambda>0$, then we will write λB for the (respectively, open or closed) ball of radius λr with the same center as B.

Definition 1.1.1. A mapping $f\colon X\to Y$ between two metric spaces (X,d) and (Y,ρ) is called *Lipschitz* if there is a constant $C\ge 0$ such that

$$\rho(f(x_1),f(x_2))\le Cd(x_1,x_2)$$

for every pair of points $x_1,x_2\in X$. We denote the smallest possible Lipschitz constant of f by

$$\operatorname{LIP}(f)=\inf\{C:\rho(f(x_1),f(x_2))\le Cd(x_1,x_2) \text{ for all } x_1,x_2\in X\}.$$

Lipschitz mappings expand distances by no more than a fixed constant factor. If f is Lipschitz with constant C and we wish to emphasize this particular constant, we will say that f is *C-Lipschitz*.

Definition 1.1.2. A mapping $f\colon X\to Y$ is called *bi-Lipschitz*, or *C-bi-Lipschitz* to emphasize the constant, if there is a constant $C\ge 1$ such that

$$C^{-1}d(x_1,x_2)\le\rho(f(x_1),f(x_2))\le Cd(x_1,x_2).$$

A 1-bi-Lipschitz mapping is called an *isometry*.

Bi-Lipschitz mappings are precisely those that preserve distances, up to a fixed constant factor. If there is a bi-Lipschitz mapping from a space X onto a space Y, then we say X and Y are *bi-Lipschitz equivalent*. Most properties of interest in analysis on metric spaces are preserved under bi-Lipschitz equivalence.

We can also define the upper and lower pointwise Lipschitz constants of a Lipschitz function $f\colon X \to \mathbb{R}$ as

$$\mathrm{Lip}_f(x) = \limsup_{r\to 0} \frac{1}{r} \sup_{y\in \overline{B}(x,r)} |f(x) - f(y)|, \tag{1.1.1}$$

$$\mathrm{lip}_f(x) = \liminf_{r\to 0} \frac{1}{r} \sup_{y\in \overline{B}(x,r)} |f(x) - f(y)|. \tag{1.1.2}$$

These functions will not play a significant role in this dissertation, but they will be mentioned in some of the background material below.

A non-trivial Borel regular measure μ on a metric space (X, d) is called a *doubling measure* if there is a constant C such that, for every $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

If a metric space (X, d) supports a doubling measure, then it is a *doubling metric space*, which means that there is a constant $N \geq 1$, depending only on the doubling constant C of μ, such that, for every $x \in X$ and $r > 0$, the ball $B(x, 2r)$ can be covered by at most N balls of radius r. A collection of metric spaces is called *uniformly doubling* if every space in the collection is doubling with a uniform upper bound on the constant N. The doubling condition is a type of finite-dimensionality condition for a metric space; for example, every Euclidean space $\mathbb{R}^n$ is doubling, but no infinite-dimensional Banach space is doubling. See [28] for more on doubling metric spaces and doubling measures.

We denote the standard s-dimensional Hausdorff measure in a a metric space (X, d) by $\mathcal{H}^s$ (see [28], Section 8.3). If the particular space X is not clear from context, we denote this by $\mathcal{H}^s_X$.

Definition 1.1.3. For $s > 0$, a metric space (X, d) is called *Ahlfors s-regular* if there is a constant $C_0 > 0$ such that for all $x \in X$ and $r \leq \operatorname{diam} X$, we have

$$C_0^{-1} r^s \leq \mathcal{H}^s(\overline{B}(x,r)) \leq C_0 r^s.$$

As above, we call a collection of metric spaces *uniformly Ahlfors s-regular* if every space in the collection is Ahlfors s-regular with the same constant C_0.

Ahlfors s-regularity is a strong, scale-invariant version of Hausdorff s-dimensionality. Note that if X is Ahlfors s-regular, then it is automatically a doubling metric space, and furthermore the measure $\mathcal{H}^s$ is a doubling measure on X.

1.1.1 Dyadic cubes

An important tool in classical analysis on Euclidean spaces is the dyadic cube decomposition of $\mathbb{R}^d$: for each $j \in \mathbb{Z}$ we can write $\mathbb{R}^d$ as a disjoint union of"half-open" cubes of the form

$$\left[n_1 2^j, (n_1+1)2^j\right) \times \left[n_2 2^j, (n_2+1)2^j\right) \times \cdots \times \left[n_d 2^j, (n_d+1)2^j\right),$$

where $n_i \in \mathbb{Z}$.

These cube decompositions have many nice properties. Informally, the different levels are nested nicely, and each individual cube in the jth level is approximately a metric ball of radius 2^j.

If X is a complete metric space that is Ahlfors s-regular with constant C_0, we can equip X with a type of "dyadic decomposition" analogous to that of Euclidean space. This was essentially first discovered by David in [18], but the formulation given by Semmes in [56], Section 2.3, is the easiest to apply here. It says that there exists $j_0 \in \mathbb{Z} \cup \{\infty\}$ (with $2^{j_0} \leq \operatorname{diam} X < 2^{j_0+1}$ if X is bounded) such that for each $j < j_0$, there exists a partition Δ_j of X into measurable subsets $Q \in \Delta_j$ such that

- $Q \cap Q' = \emptyset$ if $Q, Q' \in \Delta_j$ and $Q \neq Q'$.
- If $j \leq k < j_0$ and $Q \in \Delta_j, Q' \in \Delta_k$, then either $Q \subseteq Q'$ or $Q \cap Q' = \emptyset$.

- $C_0^{-1}2^j \le \operatorname{diam} Q \le C_0 2^j$ and $C_0^{-1}2^{sj} \le \mathcal{H}^s(Q) \le C_0 2^{sj}$.

- For every $j < j_0$, $Q \in \Delta_j$, and $\tau > 0$, we have

$$\mathcal{H}^s(\{x \in Q : \operatorname{dist}(x, X \setminus Q) \le \tau 2^j\}) \le C_0 \tau^{1/C_0} \mathcal{H}^s(Q)$$

  and

$$\mathcal{H}^s(\{x \in X \setminus Q : \operatorname{dist}(x, Q) \le \tau 2^j\}) \le C_0 \tau^{1/C_0} \mathcal{H}^s(Q).$$

Note that these dyadic cubes are not necessarily closed or open, but merely measurable. The first two properties indicate that the cubes are disjoint and properly nested. The third and fourth properties indicate that the cubes are "ball-like". Indeed, it follows from the third and fourth conditions that for every $j < j_0$ and $Q \in \Delta_j$, there exists $x \in Q$ such that

$$B(x, c_0 2^j) \subseteq Q \subseteq B(x, C_0 2^j).$$

All the constants in the cube decomposition depend only on s and the Ahlfors-regularity constant of the space, and so we have denoted the larger constant above also by C_0, the constant in the Ahlfors regularity of X.

1.1.2 Tangents and weak tangents

In this subsection we will provide a non-rigorous introduction to the concept of a tangent of a metric space. The rigorous definitions will have to wait until Sections 2.3.1 and 3.2. In this subsection we merely introduce some basic principles.

The notion of a tangent of a metric space will be important in both Chapters 2 and 3. To define it, we will need to introduce the theory of pointed Gromov-Hausdorff convergence. Perhaps unfortunately, we use two slightly different versions of this theory, one in Chapter 2 and one in Chapter 3. The theory of Gromov-Hausdorff convergence is a generalization to abstract metric spaces of the notion of Hausdorff convergence for subsets of Euclidean space. Pointed Gromov-Hausdorff convergence allows one to make sense of the notion of

$$\lim_{i \to \infty} (X_i, d_i, p_i),$$

where (X_i, d_i, p_i) are "pointed metric spaces", i.e., (X_i, d_i) are metric spaces and the points $p_i \in X_i$ are thought of as "base points". The main utility of this generalization is that here the spaces X_i are not *a priori* required to all be subsets of some ambient metric space.

Consider a metric space (X, d) and a point p in X. Fix a sequence λ_i of positive real numbers such that $\lambda_i \to 0$ as $i \to \infty$. A *tangent* of X at x is a complete pointed metric space $(X_\infty, d_\infty, p_\infty)$ such that

$$(X_\infty, d_\infty, p_\infty) = \lim_{i\to\infty} (X, \lambda_i^{-1} d, p). \tag{1.1.3}$$

Roughly speaking, a tangent of X at p describes the infinitesimal behavior of the space X near the point p, **along a specific sequence of scales** λ_i. It is obtained by rescaling the space around p by larger and larger factors and then passing to a limit.

There is no reason that this limit should exist in general, but if X is a doubling metric space then such a limit will always exist along a subsequence. Thus, if X is doubling then the collection $\mathrm{Tan}(X, p)$ of tangents of X at p is always non-empty. However, $\mathrm{Tan}(X, p)$ will in general contain many non-isometric metric spaces; most metric spaces do not have unique tangents.

If (X, d) is $\mathbb{R}^n$ with its standard metric $|\cdot|$, then at every point $p \in X$, every tangent of X at p is isometric to $\mathbb{R}^n$. This is simply because the rescaled spaces $(\mathbb{R}^n, \lambda_i^{-1}|\cdot|)$ are all isometric to $\mathbb{R}^n$, so from the perspective of Gromov-Hausdorff convergence the limit in (1.1.3) is simply the limit of a constant sequence. This uses the fact that $\mathbb{R}^n$ comes equipped with dilations around each point p, i.e., the mappings

$$x \mapsto \lambda^{-1}(x - p) + p.$$

These mappings fix p and multiply distances by λ^{-1}, and so are isometries between the pointed metric spaces.

$$(\mathbb{R}^n, |\cdot|, p) \text{ and } (\mathbb{R}^n, \lambda^{-1}|\cdot|, p).$$

Other spaces admitting such dilations, such as finite-dimensional Banach spaces or the Heisenberg group, also have tangents that are isometric to themselves (see [12], [43]).

It is also true, as one would expect, that at every point of a smooth Riemannian n-manifold, every tangent is isometric to $\mathbb{R}^n$ (see [11], Ex. 8.2.4).

We now describe a way of weakening the notion of a tangent. Consider a metric space (X, d). Let λ_i be a sequence of positive real numbers that is bounded above, but that does not necessarily converge to zero. Let p_i be a sequence of points in X. A *weak tangent* of X is a pointed Gromov-Hausdorff limit

$$(X_\infty, d_\infty, p_\infty) = \lim_{i\to\infty} (X, \lambda_i^{-1} d, p_i).$$

Weak tangents generalize the notion of tangents in two ways: the sequence of scales λ_i need not approach zero, and the base points p_i are allowed to move rather than being fixed at p. Of course, every tangent of X is also a weak tangent. On the other hand, the pointed metric space (X, d, p) is itself a weak tangent of X for every $p \in X$ (simply by taking $\lambda_i = 1$ and $p_i = p$ for all i), though it need not be a tangent of X.

Similar caveats apply to weak tangents as to tangents: the limit above need not exist, but if X is doubling then any choice of $\{\lambda_i\}$ and $\{p_i\}$ will yield a sub-sequential limit. For different choices of λ_i or p_i these (sub-sequential) limits may be completely different.

Suppose (X, d, μ) is a doubling metric space. There are two important general principles regarding the tangents and weak tangents of X.

- ([21] Lemma 9.12, [43] Proposition 3.1) If p is a point of μ-density of a subset $E \subset X$, then

  $$\mathrm{Tan}(E, p) = \mathrm{Tan}(X, p).$$

  In other words, the infinitesimal behavior of E at p is exactly that of X. Again we emphasize that this does not mean that there is a unique tangent of X or E at p.

- ([21] Lemma 9.5, [43] Theorem 1.1) "(Weak) tangents of (weak) tangents are (weak) tangents:" If Y is a weak tangent of X, and Z is a weak tangent of Y, then Z is a weak tangent of X. In addition, for μ-almost every point $p \in X$, if Y is a tangent of X at p and Z is a tangent of Y at $q \in Y$, then Z is also a tangent of X at p.

We will use these two principles at various points throughout this dissertation, and we will always be explicit about which particular version we use.

1.1.3 Rectifiability and unrectifiability

The notion of rectifiability was originally studied for subsets of Euclidean space, where many strong theorems about rectifiable sets were proven; standard references for these facts are Federer's book [25] or Mattila's book [47]. All the definitions we give here are valid for abstract metric spaces, though much less is known about rectifiability in this general setting.

For a non-negative integer n, a metric space X is called *n-rectifiable* if X can be written as

$$X = Z \cup \bigcup_{i=1}^{\infty} f_i(S_i),$$

where $\mathcal{H}^n(Z) = 0$, the sets S_i are subsets of $\mathbb{R}^n$, and the mappings $f_i \colon S_i \to X$ are Lipschitz. (The spaces we have called "n-rectifiable" are sometimes called by the more specific name "countably $\mathcal{H}^n$-rectifiable".)

Some of the first general results about rectifiable abstract metric spaces were obtained by Kirchheim [38]. Here we summarize some immediate consequences of his results, though much more is proven in [38].

Theorem 1.1.4 ([38] Lemma 4 and Theorem 9)**.** *Let (X, d) be an n-rectifiable metric space. Then the following statements hold:*

(i) *X can be written as*

$$X = Z \cup \bigcup_{i=1}^{\infty} g_i(E_i),$$

where $\mathcal{H}^n(Z) = 0$, the sets E_i are subsets of $\mathbb{R}^n$, and the mappings $g_i \colon E_i \to X$ are bi-Lipschitz.

(ii) *Suppose X supports a doubling measure μ that is absolutely continuous with respect to $\mathcal{H}^n$. Then for μ-a.e. $x \in X$, there exists a norm $\|\cdot\|_x$ on $\mathbb{R}^n$ such that every tangent of X at x is isometric to $(\mathbb{R}^n, \|\cdot\|_x)$.*

Part (i) of Theorem 1.1.4 is a consequence of Lemma 4 of [38], which says that Lipschitz maps from $\mathbb{R}^n$ into metric spaces admit decompositions into bi-Lipschitz pieces. We will discuss different quantitative versions of this type of decomposition extensively in Section 1.2 and Chapter 2, so we say no more about it here.

Part (ii) of Theorem 1.1.4 is a specialization to doubling metric measure spaces of Theorem 9 of [38]. In Kirchheim's paper, it is not stated in the language of tangents because a general non-doubling metric measure space need not admit tangents, but for our purposes in Chapter 3 (see Corollary 1.3.10) the statement above will suffice.

The following much stronger quantitative form of rectifiability was proposed by David and Semmes, and is the one we will use more frequently in the rest of this dissertation.

Definition 1.1.5. An Ahlfors d-regular space X is called *uniformly rectifiable* (in dimension d) if there exist constants $\alpha > 1$ and $0 < \beta \leq 1$ such that for every open ball B in X, there is a subset $E \subset B$ with $\mathcal{H}^d(E) \geq \beta \mathcal{H}^d(B)$ and an α-bi-Lipschitz map $f\colon E \to \mathbb{R}^d$.

We will call X *locally uniformly rectifiable* if for every $r > 0$, there exist constants α and β, depending on r, such that for every open ball B in X of radius less than r, there is a subset $E \subset B$ with $\mathcal{H}^d(E) \geq \beta \mathcal{H}^d(B)$ and an α-bi-Lipschitz map $f\colon E \to \mathbb{R}^d$.

We emphasize that the notions of uniform d-rectifiability and local uniform d-rectifiability apply only to Ahlfors d-regular spaces.

We now turn to the notion of unrectifiability; just as for rectifiability, we will introduce two variants, the second stronger than the first.

A metric space (X, d) is called *purely d-unrectifiable* if $H^d(f(E)) = 0$ whenever $E \subseteq \mathbb{R}^d$ and $f\colon E \to X$ is Lipschitz. This is the classical notion of unrectifiability used for subsets of Euclidean space; see [25] for examples of purely unrectifiable sets.

The notion of unrectifiability we will use in Chapter 3 of the dissertation is stronger than pure unrectifiability. This definition was introduced by Ambrosio and Kirchheim in [1].

Definition 1.1.6. For $s > 0$, a metric space X is said to be *strongly s-unrectifiable* if

$$\mathcal{H}^s(f(X)) = 0$$

for every $N \in \mathbb{N}$ and every Lipschitz map $f\colon X \to \mathbb{R}^N$,

Note that the parameter s in the definition of strong unrectifiability need not be an integer.

Remark 1.1.7. By Lemma 5.2 of [1] (or by Theorem 1.2.1 below) if $d \in \mathbb{N}$ then any strongly d-unrectifiable space is also purely d-unrectifiable. However, the converse is not true. Indeed, there are well-known non-trivial examples of purely unrectifiable subsets of Euclidean space, but clearly no subset of Euclidean space with positive $\mathcal{H}^s$-measure can be strongly s-unrectifiable.

In [1], Theorem 7.4, Ambrosio and Kirchheim construct, for each $s > 0$, an example of a strongly s-unrectifiable metric space with positive $\mathcal{H}^s$-measure. In Theorem 1.3.12 below, we will show that many interesting known metric spaces are in fact strongly unrectifiable.

1.2 Bi-Lipschitz pieces

In this section, we introduce the main topic and results of Chapter 2 of this dissertation, on which the author's preprint [23] is based. The main theorem is Theorem 1.2.2.

There are nowadays many different theorems of the following general form: Let (X, d, μ) and (Y, ρ, ν) be metric measure spaces (satisfying some assumptions), and let $f\colon X \to Y$ be a Lipschitz map whose image has positive ν-measure. Then f must be bi-Lipschitz on a subset of large measure, in a quantitative way.

This class of theorems is not true in general, and later on in Section 2.9 we will mention some interesting cases where it fails. Perhaps the most basic example is the "snowflaked" metric space

$$X = (\mathbb{R}, |\cdot|^{1/2}),$$

equipped with the measure $\mathcal{H}^2$. As we will see in Proposition 2.9.1 below, there is a surjective Lipschitz map f from X onto $[0,1]^2 \subset \mathbb{R}^2$, but no Lipschitz mapping from X to $\mathbb{R}^2$ can admit any set of positive measure on which it is bi-Lipschitz.

In spite of this example (and the others in Section 2.9), there are a number of situations in which results of the type "Lipschitz implies bi-Lipschitz pieces" can be proven.

As mentioned above, this idea started with [18], where David examined the case in which (X, d, μ) is Ahlfors d-regular and Y is $\mathbb{R}^d$ with the standard metric and Lebesgue measure. David showed that if, in addition to these assumptions, f satisfies a certain technical condition that we will discuss below, then it is quantitatively bi-Lipschitz on a set of large measure. By verifying his technical condition, David then applied this theorem to show that if an L-Lipschitz map f from the unit cube $[0,1]^d$ into $\mathbb{R}^d$ has an image of Lebesgue measure at least $\delta > 0$, then f is M-bi-Lipschitz on a set of Lebesgue measure θ in the cube, where θ and M depend only on L and δ.

Quite different methods were then invented by Jones [33] and David [19] to show the result in the case $X = [0,1]^d$ and $Y = \mathbb{R}^D$ equipped with d-dimensional Hausdorff measure, where $D \geq d$. In 2009, Schul [53] showed the result in the case where $X = [0,1]^d$ and Y is an arbitrary metric space, again equipped with d-dimensional Hausdorff measure. To be precise, let us state (a somewhat weakened version of) Schul's result, which generalizes those we have mentioned above.

Theorem 1.2.1 (Schul)**.** *Let (Y, ρ) be any metric space, and let $f\colon [0,1]^d \to Y$ be a 1-Lipschitz map. Let $0 < \alpha < 1$ be given. Then there are sets $F_1, \dots, F_l \subset [0,1]^d$ such that, for each $1 \leq i \leq l$ and every $x, y \in F_i$, we have*

$$\alpha|x-y| \leq \rho(f(x), f(y)) \leq |x-y|,$$

and in addition

$$\mathcal{H}^d\left(f([0,1]^d \setminus \cup_{i=1}^l F_i)\right) < c\alpha.$$

The constant l depends only on d and α, and the constant c depends only on d.

In addition to these results, we also mention that Meyerson [48] used techniques of Jones and David to show a "Lipschitz implies bi-Lipschitz pieces" result when X and Y are Carnot groups.

Although Theorem 1.2.1 is quite powerful, in that it allows arbitrary targets, it still (like the results of Jones and David) requires that the map have a Euclidean domain. (Meyerson's generalization to Carnot groups still relies heavily on the rigid algebraic structure of these objects.) As our snowflake example above shows, one must put some condition on the domain of the mapping for a suitable theorem to hold. Nonetheless, it is interesting to ask when bi-Lipschitz pieces results can be proven for mappings between general, non-Euclidean, non-Carnot metric spaces. Our main theorem, Theorem 1.2.2 below, gives a result in this vein.

In this result, we do not use the later methods of [33], [19], and [53], but rather the original method of David [18], which required verifying a certain technical condition on the Lipschitz map and the spaces in question. Originally, this applied only in the case $Y = \mathbb{R}^d$, but later Semmes [56] generalized David's theorem to the case of arbitrary target spaces Y. We apply this theorem of Semmes and adapt David's original argument to show the bi-Lipschitz pieces result for Lipschitz maps between certain types of abstract manifolds:

Theorem 1.2.2. *Let X and Y be Ahlfors s-regular, linearly locally contractible, complete, oriented, topological d-manifolds, for $s > 0$, $d \in \mathbb{N}$. Suppose in addition that Y has d-manifold weak tangents.*

Suppose I_0 is a dyadic 0-cube in X and $z\colon I_0 \to Y$ is a Lipschitz map. Then for every $\epsilon > 0$, there are measurable subsets $E_1, \dots, E_l \subset I_0$, such that $z|_{E_i}$ is M-bi-Lipschitz for each i, and

$$\left| z\left(I_0 \setminus \bigcup_{i=1}^{l} E_i \right) \right| < \epsilon \, |I_0| \, .$$

The constants l and M depend only on ϵ, the Lipschitz constant of z, the data of X, and the space Y.

Here $|\cdot|$ simply denotes s-dimensional Hausdorff measure $\mathcal{H}^s$. The definitions of the phrases "linearly locally contractible" and "d-manifold weak tangents" will be given below in Section 2.1; both are conditions that control the topology of a metric space in a quantitative way. The phrase "the data of X" refers to the collection of constants associated to X: the dimensions d and s, the constant C_0 appearing in the Ahlfors regularity of X, and the constants L and r_0 appearing in the linear local contractibility of X.

To our knowledge, Theorem 1.2.2 is the first "bi-Lipschitz pieces" result in which neither source nor target are required to have any Euclidean or Carnot group structure.

Note that Theorem 1.2.2 implies in particular the type of result mentioned at the beginning of this section: if the image of z has positive measure in Y, then z is bi-Lipschitz on a set of definite size in I_0. This stronger conclusion, in which the domain of the mapping admits a decomposition into pieces on which the mapping is bi-Lipschitz and a "garbage" piece of small image, is typical and appears in the works [33], [19], [56], [53], and [48] mentioned above.

The main engine in the proof of Theorem 1.2.2 is a theorem of Semmes, Theorem 2.2.2 below, which is a generalization of a result of David. This theorem reduces the problem of finding bi-Lipschitz pieces to the problem of verifying a certain technical condition of David, see Definition 2.2.1 below.

Even under the assumptions $s = d$ and $Y = \mathbb{R}^d$, Theorem 1.2.2 appears to be new if X is not a subset of some Euclidean space. This observation has a consequence for the geometry of the space X. Namely, we can apply Theorem 1.2.2 and another theorem of Semmes [57] to show that some abstract manifolds are uniformly rectifiable. Note that in this case we require that the Ahlfors regularity dimension and the topological dimension of X coincide. Snowflaked metric spaces such as $(\mathbb{R}^n, |\cdot|^{1/2})$ provide counterexamples in the absence of this assumption.

Theorem 1.2.3. *An Ahlfors d-regular, linearly locally contractible, complete, oriented topological d-manifold is locally uniformly rectifiable. The local uniform rectifiability constants α and β depend on the scale r and otherwise only on the data of the space.*

In particular, a compact, Ahlfors d-regular, linearly locally contractible, oriented topological d-manifold is uniformly rectifiable, with constants depending only on the data of the space.

If X admits a bi-Lipschitz embedding into some Euclidean space, then Theorem 1.2.3 follows from work of David and Semmes in [22]. (Indeed the techniques in [22] work under much weaker topological assumptions on X.) However, examples of Semmes [54] and Laakso [42] show that such an embedding need not always exist.

Other corollaries of Theorems 1.2.2 and 1.2.3 will be given in Section 2.8 below.

1.3 Lipschitz differentiability spaces

We now describe Lipschitz differentiability spaces, the main objects of study in Chapter 3 of this dissertation (on which the author's preprint [24] is based). We first recall what is perhaps the most famous classical fact about Lipschitz functions on Euclidean spaces: the following theorem of Rademacher.

Theorem 1.3.1 (Rademacher)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Then f is differentiable almost everywhere with respect to Lebesgue measure. In other words, for almost every $x \in \mathbb{R}^n$, there is a linear map $df_x\colon \mathbb{R}^n \to \mathbb{R}$ such that*

$$\lim_{y\to x} \frac{|f(y) - f(x) - df_x(y-x)|}{|y-x|} = 0.$$

In 1989, Pansu ([49], Theorem 2) gave a generalization of Rademacher's theorem to a certain class of spaces known as Carnot groups. A Carnot group is a type of topological group equipped with a left-invariant metric and a notion of dilation. This structure allows one to define difference quotients analogous to those in Theorem 1.3.1. Pansu showed that Lipschitz maps between Carnot groups are differentiable, in the sense that at almost every x these difference quotients converge to group homomorphisms.

In 1999, Cheeger [13] generalized Rademacher's theorem in a different direction. To do so, he had to define a notion of differentiation in abstract metric spaces. This is formulated in terms of measurable charts covering the space.

Definition 1.3.2. A *Lipschitz differentiability space* is a metric measure space (X, d, μ) satisfying the following condition: There are positive measure sets ("charts") U_i partitioning X, positive integers n_i (the "dimensions of the charts"), and Lipschitz maps $\phi_i \colon U_i \to \mathbb{R}^{n_i}$ with respect to which any Lipschitz function is differentiable almost everywhere, in the sense that for almost every $x \in U_i$, there exists a unique $df_x \in \mathbb{R}^{n_i}$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - df_x \cdot (\phi_i(y) - \phi_i(x))|}{d(x, y)} = 0.$$

If x is in a chart U and f is differentiable at x, we will sometimes write the derivative df_x of f at x with respect to U as $d^U f_x$.

Remark 1.3.3. The term "Lipschitz differentiability space" was not used by Cheeger but rather was coined later by Bate [3]. In [35], Keith calls such a space a "metric measure space supporting a strong measurable differentiable structure".

Note that if $(U, \phi \colon U \to \mathbb{R}^n)$ is a chart in a Lipschitz differentiability space X, then we may without loss of generality assume that ϕ is a Lipschitz function defined on all of X. This follows from the well-known McShane extension theorem for Lipschitz maps (see [28], Theorem 6.2).

The particular choice of chart structure $\{(U_i, \phi_i \colon U_i \to \mathbb{R}^{n_i})\}$ on X is not unique. However, the following simple lemma shows that the dimension of a chart is invariant under changes of the chart structure in the following sense.

Lemma 1.3.4. *If $\{(U_i, \phi_i \colon U_i \to \mathbb{R}^{n_i})\}$ and $\{(V_j, \psi_j \colon V_j \to \mathbb{R}^{m_j})\}$ are two sets of charts for a Lipschitz differentiability space (X, d, μ), then $n_i = m_j$ whenever $\mu(U_i \cap V_j) > 0$.*

Proof. Suppose to the contrary that $\mu(U_i \cap V_j) > 0$ for some $i, j \in \mathbb{N}$ but $n = n_i > m_j = m$. Writing $\phi_i = (\phi_i^1, \ldots, \phi_i^n)$, we find that for almost every point $x \in U_i \cap V_j$, the derivatives

$$(d^{V_j} \phi_i^1)_x, \ldots, (d^{V_j} \phi_i^n)_x$$

are n vectors in $\mathbb{R}^m$, and so linearly dependent. Thus, there are real numbers $a_1, \dots, a_n$, not all zero, such that

$$\left(d^{V_j} \sum_{k=1}^{n} a_k \phi_i^k\right)_x = \sum_{k=1}^{n} a_k (d^{V_j} \phi_i^k)_x = 0,$$

and therefore

$$\lim_{y \to x} \frac{|(a_1, \dots, a_n) \cdot (\phi_i(y) - \phi_i(x))|}{d(x, y)} = 0.$$

It immediately follows that the two different vectors $(0, 0, \dots, 0)$ and $(a_1, \dots, a_n)$ both serve as $d^{U_i} f_x$ for the constant function $f \equiv 0$. This violates the uniqueness required in Definition 1.3.2. $\square$

It is immediate from Rademacher's theorem that $\mathbb{R}^n$, equipped with the usual metric and Lebesgue measure, is a Lipschitz differentiability space consisting of the single n-dimensional chart

$$(U = \mathbb{R}^n, \phi(x) = x).$$

As noted above, this choice of chart structure is in no way unique. For example, we could have also replaced $\phi(x) = x$ by any Lipschitz function whose (standard Euclidean) derivative has full rank n almost everywhere. We could also have partitioned U into multiple charts in an arbitrary measurable way. By Lemma 1.3.4 however, all these charts would necessarily have dimension n.

In a similar way, Pansu's theorem can be used to show that any Carnot group, equipped with its Haar measure and left-invariant Carnot-Carathéodory metric, is a Lipschitz differentiability space (see [13], Remark 4.66.) In the case of Carnot groups, the dimension of the differentiable structure can be strictly less than the Hausdorff dimension of the space. For example, the Heisenberg group is topologically 3-dimensional and Ahlfors 4-regular (making it of Hausdorff dimension 4), but its differentiable structure consists of 2-dimensional charts.

In addition, if a complete n-rectifiable metric space (X, d) admits a doubling measure μ such that μ is absolutely continuous with respect to $\mathcal{H}^n$, then $(X, d, \mathcal{H}^n)$ is a Lipschitz differentiability space, with charts of dimension n. This follows from part (i) of Kirchheim's

Theorem 1.1.4 above. More general statements can be given if one weakens the doubling assumption, as in [35], Remark 2.1.5.

The previous examples are very rigid, in the sense that they admit nice group structures (in the case of Euclidean space and Carnot groups) or nice parametrizations by spaces that do (in the case of rectifiable metric spaces). In [13], Cheeger showed that a large class of metric measure spaces, many without any such group structure or good parametrizations, are Lipschitz differentiability spaces. Although this class in particular does not play a significant role in the remainder of this dissertation, we feel it is necessary to introduce it here for background purposes. Recall the definition of Lip_f from (1.1.1).

Definition 1.3.5. A metric measure space (X, d, μ) is a *PI space* if μ is a doubling measure on X and (X, d, μ) satisfies a "$(1, p)$-Poincaré inequality" for some $1 \leq p < \infty$: There is a constant $C > 0$ such that, for every compactly supported Lipschitz function $f: X \to \mathbb{R}$ and every ball B in X,

$$\fint_B |f - f_B| d\mu \leq C(\operatorname{diam} B) \left(\fint_{CB} (\mathrm{Lip}_f)^p d\mu \right)^{1/p}.$$

(Here the notations $\fint_E g d\mu$ and g_E both denote the average value of the function g on the set E, i.e., $\frac{1}{\mu(E)} \int_E g d\mu$.)

This type of abstract Poincaré inequality on metric spaces was introduced by Heinonen and Koskela in [30] in connection with the analysis of quasiconformal mappings. The version stated above is not quite the same as that of Heinonen-Koskela, but it is equivalent in our setting (see [34]) and easier to state. The role of the Poincaré inequality is to control the average fluctuation of f on a given ball simply by integrating its pointwise Lipschitz constant. If a complete metric measure space is a PI space, then many other properties follow, including connectedness and "quasiconvexity": any two points can be joined by a rectifiable curve whose length is bounded by a fixed constant factor times the distance between the points (see [13], Theorem 17.1). The main result of [13] is that PI spaces are Lipschitz differentiability spaces:

Theorem 1.3.6 ([13], Theorem 4.38). *Every PI space X is a Lipschitz differentiability space, and the dimensions n_i of the charts U_i are bounded by a uniform constant depending only on the constants associated to the doubling property and Poincaré inequality of X.*

All Euclidean spaces and Carnot groups are PI spaces, and so in some sense Cheeger's theorem generalizes the theorems of Pansu and Rademacher mentioned above. However, many even stranger examples of PI spaces exist, as in [9], [41], [16]. Unlike Carnot groups, these examples have no group structure or manifold structure of any kind.

After Cheeger's work, it became a subject of interest both to find conditions weaker than the Poincaré inequality that imply Lipschitz differentiability, and also to explore the consequences of assuming that a space is a Lipschitz differentiability space: see, for example, [35], [36], [26], [4], [3].

The Poincaré inequality is certainly not necessary for Lipschitz differentiability. Indeed, Bate and Speight ([4], Theorem 2.4) have shown that any subset of positive measure in a Lipschitz differentiability space is also a Lipschitz differentiability space. (For doubling spaces, this is somewhat easier; see [2].) Such a subset may of course be totally disconnected and thus fail to admit a Poincaré inequality.

In [35], Theorem 2.3.1, Keith gave a condition weaker than the Poincaré inequality that implies Lipschitz differentiability. We state Keith's theorem only for doubling spaces, although he proved a stronger result. Recall the definitions of Lip_f and lip_f from (1.1.1) and (1.1.2).

Theorem 1.3.7 ([35], Theorem 2.3.1). *Let X be a doubling metric measure space that satisfies the following "*Lip-lip *condition": There is a constant $K > 0$ such that, for every Lipschitz function $f\colon X \to \mathbb{R}$, the inequality*

$$\mathrm{Lip}_f(x) \leq K \mathrm{lip}_f(x)$$

holds for μ-almost every $x \in X$. Then X is a Lipschitz differentiability space. The dimensions n_i of the charts U_i are bounded by a uniform constant depending only on K and the doubling constant of the measure μ.

For conditions that follow from, rather than imply, Lipschitz differentiability, we will mention only one result of Bate [3], which will play a role later on. This result is a consequence of Bate's study of certain measure decompositions known as "Alberti representations" in metric measure spaces. The following notation, taken from [3], is useful: Let $\Gamma(X)$ be the collection of all bi-Lipschitz functions of the form

$$\gamma : D_\gamma \to X$$

where $D_\gamma \subset \mathbb{R}$ is a non-empty compact set containing 0. We think of elements of $\Gamma(X)$ as "broken curves" in X. Bate's result gives a local description of Lipschitz differentiability spaces in terms of these broken curves.

Theorem 1.3.8 ([3], Corollary 6.7)**.** *Let (U, ϕ) be an n-dimensional chart in a complete Lipschitz differentiability space (X, d, μ). Then for almost every $x \in U$, there exist $\gamma_1^x, \dots, \gamma_n^x \in \Gamma(X)$ such that each $(\gamma_i^x)(0) = x$, 0 is a density point of $(\gamma_i^x)^{-1}(U)$, and the derivatives $(\phi \circ \gamma_i^x)'(0)$ exist and are linearly independent.*

Theorem 1.3.8 is the starting point for the results in Chapter 3 of this dissertation, in which we investigate the consequences of the Lipschitz differentiability property for the tangents and rectifiability of Ahlfors regular spaces. We now state these results.

Theorem 1.3.9. *Let X be a complete, Ahlfors n-regular Lipschitz differentiability space containing a chart U of dimension n. Then for $\mathcal{H}^n$-almost every point $x \in U$, every tangent of X at x is uniformly rectifiable. In particular, at almost every point of U, there is a tangent of X that is bi-Lipschitz equivalent to $\mathbb{R}^n$.*

The constants in the uniform rectifiability depend on the point x, but not on the particular sequence of scales defining the tangent.

If one applies Kirchheim's theorem, Theorem 1.1.4 above, to this fact, one immediately obtains the following corollary:

Corollary 1.3.10. *Let X be a complete, Ahlfors n-regular Lipschitz differentiability space containing a chart U of dimension n. Then at $\mathcal{H}^n$-almost every point $x \in U$, there is a tangent of X that is isometric to $\mathbb{R}^n$ equipped with a metric induced by a norm.*

Remark 1.3.11. If X is an Ahlfors n-regular Lipschitz differentiability space, then the dimension k of any chart $(U, \phi\colon U \to \mathbb{R}^k)$ satisfies $k \leq n$ (see Corollary 3.8.5), although this inequality may be strict. Thus, Theorem 1.3.9 and Corollary 1.3.10 are about the case in which the dimension is extremal.

In contrast to Theorem 1.3.9, one may ask whether a differentiable structure of dimension strictly less than the Ahlfors regularity dimension implies a type of unrectifiability of the space. This is in fact the case. Recall the definition of strong unrectifiability given in Definition 1.1.6.

Theorem 1.3.12. *Suppose that $s > 0$ and that X is an Ahlfors s-regular Lipschitz differentiability space containing a chart U of dimension k, with $k < s$. Then U is strongly s-unrectifiable in the sense of Ambrosio-Kirchheim.*

Remark 1.3.13. Theorem 1.3.12 shows that Ahlfors s-regular Lipschitz differentiability spaces with charts of dimension less than s provide additional examples of strongly unrectifiable spaces. (Note that, by Remark 1.3.11, any Ahlfors s-regular Lipschitz differentiability space with non-integer s satisfies the condition automatically.) In addition to all non-abelian Carnot groups, there are now numerous other interesting constructions of such spaces, including those of Bourdon-Pajot [9], Laakso [41], and Cheeger-Kleiner [16].

1.4 Outline of the dissertation

We now give a detailed outline of the dissertation.

Chapter 2 is about the "bi-Lipschitz pieces" phenomenon described in Section 1.2 above; its main result is Theorem 1.2.2. The chapter begins with some additional background and describes the underlying method of David and Semmes that is the foundation for the proof.

In Section 2.3, we define a form of pointed Gromov-Hausdorff convergence and prove some results about the Gromov-Hausdorff limits of manifolds needed in the proof of Theorem 1.2.2. Section 2.4 is devoted to Proposition 2.4.2, which is weaker than than Theorem 1.2.2 but whose proof uses similar ideas in a simpler context and thus serves as a "warm-up" for the proof of the main theorem. Sections 2.5 and 2.6 contain the proof of the main Theorem 1.2.2, and Section 2.7 gives the proof of Theorem 1.2.3 on uniform rectifiability. We conclude the chapter with Section 2.8, which discusses two embedding results which follow from Theorem 1.2.2 and results of Semmes, and Section 2.9, which summarizes various examples in which "bi-Lipschitz pieces" results fail; these are due to various authors.

Chapter 3 is about Lipschitz differentiability spaces. Most of the chapter is devoted to the proof of Theorem 1.3.9. This requires another type of pointed Gromov-Hausdorff convergence (defined in Section 3.2) and a generalization of a result about tangent spaces due to Le Donne [43] (Section 3.3). Sections 3.4 and 3.5 contain some general results about tangents of doubling Lipschitz differentiability spaces. Sections 3.6 and 3.7 contain the proofs of Theorems 1.3.9 and 1.3.12, respectively. Finally, in Section 3.8 we present some further corollaries. These include generalizations to Lipschitz differentiability spaces of some results of Cheeger [13] related to non-embedding for PI spaces — Corollaries 3.8.1, 3.8.2, 3.8.4, and 3.8.5 — as well as a rigidity result, Corollary 3.8.8, for Lipschitz differentiability spaces admitting quasi-Möbius symmetries, in the spirit of Bonk-Kleiner [6].

Chapters 2 and 3 are almost entirely independent of each other, although Chapter 3 uses a result of David and Semmes introduced in Section 2.2 of Chapter 2.

CHAPTER 2

Bi-Lipschitz pieces between manifolds

In this chapter we will prove a "bi-Lipschitz pieces" type result, Theorem 1.2.2, of the general form described in Section 1.2. A number of consequences will be deduced, namely Theorem 1.2.3 and Corollaries 2.8.1 and 2.8.4. The author's preprint [23] is based on the material in this chapter.

2.1 Additional definitions

In this chapter, whenever we speak of a measure we will mean s-dimensional Hausdorff measure in an Ahlfors s-regular space. To simplify notation, we therefore always write $|A|$ or, to avoid confusion, $|A|_X$ for the s-dimensional Hausdorff measure of a set A in a space X. We now give some more specialized definitions that will be needed in this chapter.

The following condition gives a quantitative bound on the local topology of a metric space.

Definition 2.1.1. A metric space (X, d) is called *linearly locally contractible* if there are constants $L, r_0 > 0$ such that every open ball $B \subset X$ of radius $r < r_0$ is contractible inside a ball with the same center of radius Lr. We may abbreviate the condition as LLC or (L, r_0)-LLC to emphasize the constants.

Remark 2.1.2. In some contexts, the abbreviation LLC refers to the weaker condition of "linear local connectivity". We do not use this condition in this dissertation.

The class of source and target spaces we consider in this chapter are complete, oriented topological d-manifolds that are Ahlfors s-regular and LLC. If X is such a space, the phrase "the data of X" refers to the collection of constants associated to X: the dimensions d and s, the constant C_0 appearing in the Ahlfors regularity of X, and the constants L and r_0 appearing in the LLC property of X.

There is also an additional constraint on the class of target spaces for which our theorem applies. This requires the notion of convergence of a sequence of pointed metric spaces, which we introduce in Definition 2.3.3 below.

Definition 2.1.3. We say a complete metric space (Y, ρ) has *d-manifold weak tangents* if the following holds: Whenever r_i is a sequence of positive real numbers that is bounded above, p_i are points in Y, and $(Y, \frac{1}{r_i}\rho, p_i)$ converges (as in Definition 2.3.3) to a space $(Y_\infty, \rho_\infty, p_\infty)$, then Y_∞ is a topological d-manifold.

Remark 2.1.4. Note that Definition 2.1.3 includes the assumption that Y itself is a topological d-manifold, by taking $r_i = 1$ and $p_i = p$ for all i.

Remark 2.1.5. While Definition 2.1.1 is rather standard, Definition 2.1.3 is more unusual, and somewhat restrictive. Here are some examples of spaces that satisfy it:

- The simplest example is $\mathbb{R}^d$ for $d \geq 1$. Indeed, if $Y = \mathbb{R}^d$, then all the pointed metric spaces $(Y, \frac{1}{r_i}\rho, p_i)$ are isometric to $(\mathbb{R}^d, |\cdot|, 0)$ by rescaling and translating. Therefore, the limiting space of this sequence is also $\mathbb{R}^d$, which is a topological d-manifold.

- For the same reasons, every Carnot group G, equipped with its Carnot-Carathéodory metric, has d-manifold weak tangents, where d is the topological dimension of G. For the definition of Carnot groups, see [12], Chapter 2.

- If X is a compact, doubling metric space with d-manifold weak tangents, and Y is quasisymmetric to X, then Y has d-manifold weak tangents. This follows, e.g., from [37], Lemmas 2.4.3 and 2.4.7. For the definition and properties of quasisymmetric mappings, see [28].

- Similarly, if G is a topologically d-dimensional Carnot group, and Y is quasisymmetric to G, then Y has d-manifold weak tangents (even if Y has larger Hausdorff dimension than G). This includes all "snowflaked" Carnot groups, i.e., metric spaces of the form (G, ρ^α), where $0 < \alpha \leq 1$ and (G, ρ) is a Carnot group.

- The Cartesian product of two spaces (X, d_X) and (Y, d_Y) with n- and m-manifold weak tangents, respectively, (equipped, e.g., with the metric $d((x,y),(x',y')) = d_X(x,x') + d_Y(y,y')$) has $(n+m)$-manifold weak tangents.

- Any complete, doubling, linearly locally contractible topological 2-manifold has 2-manifold weak tangents. Indeed, by Proposition 2.3.19 below, every weak tangent of such a space is a homology 2-manifold (see Definition 2.3.18), and the only homology 2-manifolds are topological 2-manifolds (see [10], Theorem V.16.32).

- Suppose a compact metric space Z has the property that every triple of points can be blown up to a uniformly separated triple by a uniformly quasi-Möbius map. (This condition was studied by Bonk and Kleiner in [6] and is satisfied by boundaries of hyperbolic groups equipped with their visual metrics.) Then Z has d-manifold weak tangents if and only if Z is itself a topological d-manifold. This follows from [6], Lemma 5.3. (Note that the definition of a weak tangent given in [6] is different than ours, in that it requires the sequence of scales $1/r_i$ tend to infinity. However, the proof of Lemma 5.3 in [6] works the same way without this restriction.)

2.2 Background

In [18], condition (9), David introduced the following condition for a Lipschitz map defined on a dyadic cube in an Ahlfors-regular space. Though David gave the condition for maps into $\mathbb{R}^d$, in [56], Condition 9.1, Semmes re-formulated David's condition for arbitrary target spaces. This is the formulation we give here. Recall that $|\cdot|$ denotes s-dimensional Hausdorff measure.

Definition 2.2.1. Let (X, d) be an Ahlfors s-regular metric space with a system of dyadic cubes as in Subsection 1.1.1. Let (Y, ρ) be a metric space. Let I_0 be a 0-cube in X, and

$z\colon I_0 \to Y$ be a Lipschitz map. We will say that z satisfies *David's condition* on I_0 if the following holds:

For every $\lambda, \gamma > 0$, there exist $\Lambda, \eta > 0$ such that, for every $x \in I_0$ and $j < j_0$, if T is the union of all j-cubes intersecting $B(x, \Lambda 2^j)$, and if $T \subseteq I_0$ and $|z(T)| \geq \gamma |T|$, then either:

(i) $z(T) \supseteq B(z(x), \lambda 2^j)$, or

(ii) there is a j-cube $R \subset T$ such that

$$|z(R)|/|R| \geq (1 + 2\eta)|z(T)|/|T|.$$

As in Theorem 1.2.2, it is convenient to phrase David's condition for 0-cubes because that is how we will use it, although it makes sense for cubes of all sizes. Note that, given the definition of our cubes in Subsection 1.1.1, a space may contain no 0-cubes at all, but one can always create some by rescaling the space and relabeling the levels of the cubes.

In essence, David's condition says the following: At every location and scale within I_0, if the map z does not collapse the measure of a ball too much, then one of two things must happen: either (i) the image of this ball contains a ball of comparable size (centered at the image of its center), or (ii) some sub-cube of this ball is expanded by a larger factor than the ball itself. The upshot of (i) is that the map z does not "fold" at this location and scale.

To take a concrete example, suppose $I_0 = [0, 1]^2 \subset \mathbb{R}^2$ and z is the map

$$z(x, y) = \left(\left| x - \frac{1}{2} \right|, y \right) : I_0 \to \mathbb{R}^2,$$

which folds the square in half along its central vertical axis. If T is well away from the folding line $\{x = \frac{1}{2}\}$, then z essentially acts isometrically on T and so condition (i) of David's condition holds. If T is centered on the folding line, then $|z(T)|/|T| = 1/2$ and (i) fails, but some sub-square R of T to the left or right of the folding line satisfies $|z(R)|/|R| = 1$, so (ii) holds. (Of course, this was just a heuristic explanation; to truly verify David's condition we would have to keep careful track of the constants involved.)

Theorem 10.1 of [56], which is a generalization of Theorem 1 of [18], says the following.

Theorem 2.2.2 ([56], Theorem 10.1)**.** *Let (X, d) be an Ahlfors s-regular metric space with a system of dyadic cubes as above. Let (Y, ρ) be an arbitrary metric space. Let I_0 be a 0-cube in X, and $z\colon I_0 \to Y$ be a Lipschitz map. Suppose that z satisfies David's condition on I_0.*

Then for every $\epsilon > 0$, there are measurable subsets $E_1, \dots, E_l \subset I_0$, such that $z|_{E_i}$ is M-bi-Lipschitz, and

$$\left| z\left(I_0 \setminus \bigcup_{i=1}^{l} E_i \right) \right| < \epsilon \left| I_0 \right| .$$

The constants l and M depend only on ϵ, the constants associated to the Ahlfors-regularity of the space X, the Lipschitz constant of z, and the numbers Λ and η from David's condition (for $\lambda = 1$ and γ depending only on ϵ and the Lipschitz constant of z.)

We will apply Theorem 2.2.2 and a modification of the proof of Theorem 2 of [18] to prove Theorem 1.2.2. It is worth noting that, in Theorem 1.2.2, the condition that Y is Ahlfors s-regular can be relaxed to the condition that Y is doubling and satisfies the upper mass bound

$$\mathcal{H}^s(\overline{B}_Y(x, r)) \leq C_0 r^s.$$

It is only this half of the Ahlfors regularity of Y that is used in the proof.

On the other hand, the fact that X and Y are have the same topological dimension d is crucial in the setting of Theorem 1.2.2. In Proposition 2.9.1 below, we will give a counterexample to Theorem 1.2.2 in which X and Y satisfy all the assumptions of the theorem, except that they are manifolds of different topological dimensions.

A few further remarks on the statement of Theorem 1.2.2 are in order.

Remark 2.2.3. That Theorem 1.2.2 gives dependence of constants on the space Y (and not just its data) is a consequence of our compactness style of proof. However, the proof of Theorem 1.2.2 can be modified slightly to reduce the dependence on Y in the following manner. Let Y be a complete, oriented d-manifold that is LLC, Ahlfors s-regular, and has d-manifold weak tangents. Suppose that Y' is LLC, Ahlfors s-regular, and is η-quasisymmetric to Y, by a quasisymmetry that maps balls in Y' of radius 1 to sets of uniformly bounded diameter. Then Theorem 1.2.2 holds for maps $z\colon X \to Y'$ with constants depending only on the space Y, the data of Y', and the quasisymmetry function η (as well as the data of z and X).

In particular, if $\xi \geq \xi_0 > 0$, then the theorem holds for target space $Y' = (Y, \xi\rho)$ with l, M depending only on Y and ξ_0 (as well as on ϵ and the data of X and z), and not on ξ itself. That is because this rescaling is quasisymmetric (with $\eta(t) = t$) and does not alter the data of Y, other than changing the contractibility radius r_0 to r_0/ξ_0.

Remark 2.2.4. We have phrased Theorem 1.2.2 for 0-cubes to parallel Theorem 2 of [18]. However, it is easy to see that the following statement also holds:

Suppose $j_1 < j_0$, Q_0 is a dyadic j-cube in X, $j \leq j_1$, and $z\colon Q_0 \to Y$ is Lipschitz. Then the conclusion of Theorem 1.2.2 holds for z on Q_0, i.e., for every $\epsilon > 0$, there are measurable subsets $E_1, \ldots, E_l \subset Q_0$, such that $z|_{E_i}$ is M-bi-Lipschitz, and

$$\left| z\left(I_0 \setminus \bigcup_{i=1}^{l} E_i \right) \right| < \epsilon\, |I_0|$$

Here l and M depend only on ϵ, the Lipschitz constant of z, j_1, the space Y, and the data of X.

Indeed, if Q_0 is an j-cube for $j \leq j_1$, one need only apply Theorem 1.2.2 to the rescaled spaces $(X, 2^{-j}d)$ and $(Y, 2^{-j}\rho)$, and the same Lipschitz map z, relabeling the cubes so that Q_0 is a 0-cube. The rescaled spaces $(X, 2^{-j}d)$ and $(Y, 2^{-j}\rho)$ have the same data as X and Y, except that their LLC radii r_0 must be replaced by $2^{-j_1}r_0$. So we can apply Theorem 1.2.2 and Remark 2.2.3 to obtain this result.

David proved Theorem 1.2.2 in the case $X = Y = \mathbb{R}^d$ (see [18], Theorem 2). In doing so, he used a compactness argument to verify a modified version of what we have called David's condition. The general idea is the following: Consider a sequence of counterexample maps z_k, which in the case of $\mathbb{R}^d$ may all be defined on the unit cube, that fail both conditions of Definition 2.2.1 with increasingly worse constants as $k \to \infty$. Extract a sub-limit z, and by a careful argument show that z has constant Jacobian. Because z is in addition Lipschitz, it is a quasi-regular mapping, and a theorem of Reshetnyak implies that it is an open mapping. A degree argument then shows that, for k large, the image of the maps z_k must contain a fixed size ball around $z_k(0)$, with radius independent of k. For k large, this contradicts the assumption that the maps z_k fail the first condition of Definition 2.2.1.

In our setting, we follow a similar approach. The compactness argument of [18] is modified to be a Gromov-Hausdorff compactness argument; to make the degree theory work in this setting we require some results on the Gromov-Hausdorff limits of locally contractible manifolds: see Section 2.3 below. In addition, the theory of quasi-regular mappings and the result of Reshetnyak are not available to us. They are replaced by a topological theorem of Bonk and Kleiner (Theorem 2.3.29 below) on mappings of bounded multiplicity.

A completely different method for verifying David's condition in some situations is a type of detailed homotopy argument, as in [22], Chapter 9. This approach allows for much weaker topological assumptions on X, but it seems to rely on having $s = d$, $Y = \mathbb{R}^d$, and X embedded in some Euclidean space.

If X admits a bi-Lipschitz embedding into some Euclidean space, then Theorem 1.2.3 follows from work of David and Semmes in [22]. However, examples of Semmes [54] and Laakso [42] show that such an embedding need not always exist.

2.3 The main tools

In this section, we introduce the main concepts and results used in the proof of Theorem 1.2.2.

2.3.1 Convergence of metric spaces

We will use the notion of convergence of "mapping packages", a version of Gromov-Hausdorff convergence, that is described in Chapter 8 of [21]. All material in this subsection is from that source. A brief exposition of this material is also given in [35].

While the notation in this set-up is a bit more cumbersome than for other definitions of Gromov-Hausdorff convergence (for example the one we use in Section 3.2), the detailed results of [21] make it very flexible for discussing simultaneous convergence of metric spaces and mappings.

Definition 2.3.1. We say that a sequence $\{F_j\}$ of non-empty closed subsets of some Euclidean space $\mathbb{R}^N$ converges to a non-empty closed set $F \subseteq \mathbb{R}^N$ if

$$\lim_{j\to\infty} \sup_{x\in F_j\cap B(0,R)} \operatorname{dist}(x, F) = 0$$

and

$$\lim_{j\to\infty} \sup_{y\in F\cap B(0,R)} \operatorname{dist}(y, F_j) = 0$$

for all $R > 0$.

This convergence is stable under taking products, in the sense that if $\{F_j\}$ converges to F in $\mathbb{R}^N$ and $\{G_j\}$ converges to G in $\mathbb{R}^M$, then $\{F_j \times G_j\}$ converges to $F \times G$ in $\mathbb{R}^{N+M}$.

Definition 2.3.2. Suppose $\{F_j\}$ is a sequence of closed sets converging to a closed set F in $\mathbb{R}^N$ as in the previous definition. Let Y be a metric space and $\phi_j \colon F_j \to Y$, $\phi\colon F \to Y$ be mappings. We say that $\{\phi_j\}$ converges to ϕ if for each sequence $\{x_j\}$ in $\mathbb{R}^N$ such that $x_j \in F_j$ for all j and $x_j \to x \in F$, we have that

$$\lim_{j\to\infty} \phi_j(x_j) = \phi(x).$$

A *pointed metric space* is a triple (X, d, p), where (X, d) is a metric space and p is a point in X. All metric spaces that we consider are complete and doubling.

Definition 2.3.3. A sequence of pointed metric spaces $\{(X_j, d_j, p_j)\}$ converges to a pointed metric space (X, d, p) if the following conditions hold. There exists $\alpha \in (0, 1]$, $N \in \mathbb{N}$, and L-bi-Lipschitz embeddings $f_j \colon (X_j, d_j^\alpha) \to \mathbb{R}^N$, $f\colon (X, d^\alpha) \to \mathbb{R}^N$ with $f_j(p_j) = f(p) = 0$ for all j. Furthermore, we require that $f_j(X_j)$ converge to $f(X)$ in the sense of Definition 2.3.1, and that the real-valued functions $(x, y) \mapsto d_j(f_j^{-1}(x), f_j^{-1}(y))$ defined on $f_j(X_j) \times f_j(X_j)$ converge to the function $(x, y) \mapsto d(f^{-1}(x), f^{-1}(y))$ on $f(X)\times f(X)$, in the sense of Definition 2.3.2.

We only use Definition 2.3.3 when the metric spaces $\{(X_j, d_j)\}$ and (X, d) are uniformly doubling. In that case, embeddings f_j and f as in Definition 2.3.3 can always be found, by Assouad's embedding theorem (see [28], Theorem 12.2).

Definition 2.3.4. A *mapping package* consists of a pair of pointed metric spaces (M, d_M, p) and (N, d_N, q) as well as a mapping $g\colon M \to N$ such that $g(p) = q$.

Definition 2.3.5. A sequence of mapping packages $\{((X_j, d_j, p_j), (Y_j, \rho_j, q_j), h_j)\}$ is said to converge to another mapping package $((X, d, p), (Y, \rho, q), h)$ if the following conditions hold. The sequences $\{(X_j, d_j, p_j)\}$ and $\{(Y_j, \rho_j, q_j\}$ converge to (X, d, p) and (Y, ρ, q), respectively, in the sense of Definition 2.3.3. Furthermore, the maps $g_j \circ h_j \circ f_j^{-1}$ converge to $g \circ h \circ f^{-1}$ in the sense of Definition 2.3.2, where f_j, g_j, f, g are the embeddings of Definition 2.3.3.

The following proposition is a special case of Lemma 8.22 of [21].

Proposition 2.3.6. *Let $\{((X_j, d_j, p_j), (Y_j, \rho_j, q_j), h_j)\}$ be a sequence of mapping packages, in which all the metric spaces are complete and uniformly doubling, and in which the maps h_j are uniformly Lipschitz and satisfy $h_j(p_j) = q_j$. Then there exists a mapping package $((X, d, p), (Y, \rho, q), h)$ that is the limit of a subsequence of $\{((X_j, d_j, p_j), (Y_j, \rho_j, q_j), h_j)\}$.*

We will now describe some consequences of the convergence of a sequence of mapping packages, which are Lemmas 8.11 and 8.19 of [21].

Proposition 2.3.7. *Suppose a sequence of pointed metric spaces $\{(X_k, d_k, p_k)\}$ converges to the pointed metric space (X, d, p), in the sense of Definition 2.3.3.*

Then there exist (not necessarily continuous) mappings $\phi_k\colon X \to X_k$ and $\psi_k\colon X_k \to X$ such that:

- *For all k, $\phi_k(p) = p_k$ and $\psi_k(p_k) = p$.*
- *For all $R > 0$,*

$$\lim_{k\to\infty} \sup\{d_X(\psi_k(\phi_k(x), x)) : x \in B_X(p, R)\} = 0$$

and

$$\lim_{k\to\infty} \sup\{d_{X_k}(\phi_k(\psi_k(x), x)) : x \in B_{X_k}(p_k, R)\} = 0.$$

- *For all $R > 0$,*

$$\lim_{k\to\infty} \sup\{|d_{X_k}(\phi_k(x), \phi_k(y)) - d_X(x, y)| : x, y \in B_X(p, R)\} = 0$$

and

$$\lim_{k\to\infty} \sup\{|d_X(\psi_k(x), \psi_k(y)) - d_{X_k}(x, y)| : x, y \in B_{X_k}(p, R)\} = 0.$$

Proposition 2.3.8. *Suppose a sequence of mapping packages $\{((X_k, d_k, p_k), (Y_k, \rho_k, q_k), h_k)\}$ converges to a mapping package $((X, d, p), (Y, \rho, q), h)$, where the mappings h_k are uniformly Lipschitz and satisfy $h_k(p_k) = q_k$. Then there exist (not necessarily continuous) mappings $\phi_k \colon X \to X_k$ and $\psi_k \colon X_k \to X$ satisfying exactly the conditions of Proposition 2.3.7, and mappings $\sigma_k \colon Y \to Y_k$ and $\tau_k \colon Y_k \to Y$ satisfying the analogous properties of Proposition 2.3.7, such that in addition we have the following:*

For all $x \in X$,

$$\lim_{k\to\infty} \tau_k(h_k(\phi_k(x))) = h(x)$$

and this convergence is uniform on bounded subsets of X.

We will be interested in mapping packages in which the mappings h_k are defined only on subsets of the source spaces X_k. For this, we need the following fact, which is Lemma 8.32 of [21].

Lemma 2.3.9. *Suppose that $\{(X_k, d_k, p_k)\}$ is a sequence of pointed metric spaces that converges to the pointed metric space $\{(X, d, p)\}$ in the sense of Definition 2.3.3. Let $\{F_k\}$ be a sequence of nonempty closed sets with $F_k \subset X_k$ for each k. Suppose that*

$$\sup_k \operatorname{dist}_{X_k}(F_k, p_k) < \infty.$$

Then we can pass to a subsequence to get convergence to a nonempty closed subset F of X.

We make one final remark in this Subsection, which is Lemma 8.29 of [21].

Lemma 2.3.10. *Let the pointed metric spaces (X_j, d_j, p_j) converge to (X, d, p) in the sense of Definition 2.3.3. Suppose that (X_j, d_j) are Ahlfors s-regular, with Ahlfors regularity constant uniformly bounded (see Definition 1.1.3). Then (X, d) is Ahlfors s-regular, with constant controlled by the Ahlfors regularity constants of the spaces (X_j, d_j).*

2.3.2 Convergence of LLC spaces

Here we state some results that apply to the convergence of metric spaces (in the sense of the previous section) when those metric spaces also happen to be linearly locally contractible. The main goals are to show that a convergent sequence of uniformly LLC spaces has an LLC limit (essentially a result of Borsuk [8]), and to describe a result that improves Proposition 2.3.8 in this context.

The following basic fact about LLC spaces will be used a number of times.

Lemma 2.3.11. *Let X be a (L, r_0)-LLC space. Fix $x \in X$ and $r \in (0, r_0)$. Then there is a connected open set U satisfying*

$$B(x, r/(2L)) \subset U \subset B(x, r).$$

Proof. Consider a point $y \in X$ and a radius $r \in (0, r_0)$. Let $H\colon B(y, r/(2L)) \times [0,1] \to B(x, r/2)$ be a homotopy contracting $B(y, r/(2L))$ to a point. Define

$$E(y, r) = H(B(y, r/(2L)) \times [0,1]).$$

Then $E(y, r)$ is a connected subset of $B(y, r/2)$ containing $B(y, r/(2L))$.

Let $E_0 = E(x, r)$. For $i \in \mathbb{N}$, inductively define sets

$$E_i = \bigcup_{y \in E_{i-1}} E(y, 2^{-i} r).$$

By induction, each set E_i is connected. In addition, for each i we have the relation

$$E_i \subset \operatorname{int} E_{i+1}. \tag{2.3.1}$$

Now let

$$U = \bigcup_{i=1}^{\infty} E_i.$$

Then, as the union of connected sets that all contain the point x, the set U is connected. In addition, by (2.3.1) U is open: if $x \in U$, then, for some i,

$$x \in E_i \subset \operatorname{int} E_{i+1} \subset \operatorname{int} U.$$

Finally, if $y \in E_i \subset U$, then

$$d(x, y) \leq \left(2^{-(i+1)} + 2^{-i} + \cdots + 2^{-1}\right) r < r.$$

Thus, U is a connected open set, $U \subset B(x, r)$, and $U \supseteq E_0 \supseteq B(x, r/(2L))$. □

The following is our main lemma about convergence of uniformly LLC sets.

Lemma 2.3.12. *Let F_k be a sequence of closed sets in some Euclidean space $\mathbb{R}^N$ that are each (L, r_0)-LLC (as spaces equipped with the induced Euclidean metric). Suppose that $F_k \to F$ in the sense of Definition 2.3.1. Then F is LLC, with constants depending only on L and r_0.*

In the case of compact sets converging in the usual Hausdorff metric, Lemma 2.3.12 is due to Borsuk [8]. A similar localized version of the result was noted in [29]. Here we provide a proof, following the method of Borsuk.

The proof is somewhat technical, though the main idea is not difficult: For subsets of Euclidean space, the LLC property for a set E implies the existence of a retraction to E, from an open neighborhood of E of fixed size, that moves points by an amount proportional to their distance from E. We use the existence of these retractions on the limiting sets F_k to construct a retraction onto the limit F. This retraction can then be used to show that F is LLC. Because the convergence is local, there are some minor technical complications.

Proof of Lemma 2.3.12. For a set $E \subseteq \mathbb{R}^N$, let $U_\epsilon(E)$ denote the open ϵ-neighborhood of E. Let $B_R = B(0, R) \subset \mathbb{R}^N$.

We note first that the LLC property implies that there exist constants $0 < c < 1$ and $C = c^{-1} > 1$ such that each F_k admits a continuous retraction $r_k \colon U_c(F_k) \to F_k$ satisfying

$$|r_k(x) - x| \leq C \operatorname{dist}(x, F_k) \tag{2.3.2}$$

for $x \in U_c(F_k)$. The proof of this can be found in Section 13 of [8] (and does not require compactness of the sets).

Fix a ball $B = B(p, r) \cap F$ for $p \in F$ and $r < r_0/4$. Fix $R > \max\{4Lr, 12C\}$ large enough so that $\overline{B} \subset B_R$.

By passing to a subsequence, we may without loss of generality assume that, for all k,

$$\sup\{\operatorname{dist}(x, F_k) : x \in F \cap B_{10R}\} < c/4,$$

$$\sup\{\operatorname{dist}(x, F) : x \in F_k \cap B_{10R}\} < c/4.$$

It follows that

$$U := \bigcap_{k=1}^{\infty} U_c(F_k)$$

contains a $c/2$-neighborhood of $B_{9R} \cap F$ as well as of $\bigcup_{k=1}^{\infty}(B_{9R} \cap F_k)$.

For $k \in \mathbb{N}$, fix decreasing sequences

$$\eta_k = c^2 4^{-k}, \tag{2.3.3}$$

$$\eta_k' = c^2 4^{-k}/3. \tag{2.3.4}$$

We may now pass to a further subsequence of our sets on which we assume that

$$\sup\{\operatorname{dist}(x, F_k) : x \in F \cap B_{9R}\} < \eta_k'/8, \tag{2.3.5}$$

$$\sup\{\operatorname{dist}(x, F) : x \in F_k \cap B_{9R}\} < \eta_k'/8. \tag{2.3.6}$$

Let $U_k = U_{\eta_k}(F_k)$ and $V_k = U_{\eta_k'}(F_k)$. Then, if $x \in U_{k+1} \cap B_{7R}$, we have, by (2.3.3), (2.3.4), (2.3.5), and (2.3.6), that

$$\operatorname{dist}(x, F_k \cap B_{8R}) < \eta_k'.$$

Therefore, for every $0 < R' \leq 7R$,

$$(U_{k+1} \cap B_{R'}) \subset (V_k \cap B_{R'}) \subset (\overline{V}_k \cap B_{R'}) \subset (U_k \cap B_{R'}) \subset (U \cap B_{R'}). \tag{2.3.7}$$

We will now inductively construct a new sequence of retractions $s_k \colon U \cap B_{5R} \to F_k$ by modifying the maps r_k.

Let $s_1 = r_1$. Suppose that s_k has already been defined and in addition satisfies $s_k = r_k$ on $V_k \cap B_{5R}$. Let $f \colon U \to \mathbb{R}$ be a continuous function that is 0 on $U \setminus U_{k+1}$ and 1 on V_{k+1}. For $x \in U \cap B_{5R}$, define

$$s_{k+1}(x) = r_{k+1}((1 - f(x))s_k(x) + f(x)x).$$

We first check that s_{k+1} is well-defined, i.e., that for $x \in U \cap B_{5R}$, the point $(1 - f(x))s_k(x) + f(x)x$ is in U. If $x \in U \setminus U_{k+1}$, then $(1 - f(x))s_k(x) + f(x)x = s_k(x) \in F_k \subset U$, so s_{k+1} is well-defined. In the case $x \in U_{k+1}$, we have by (2.3.7) that $x \in V_k$. By our inductive assumption that $s_k = r_k$ on $V_k \cap B_{5r}$, we get

$$|x - s_k(x)| = |x - r_k(x)| \leq C\eta'_k < c.$$

Thus, every point on the line segment from x to $s_k(x)$ is in the c-neighborhood of F_k and so is in U.

By the definition of the function f, it is clear that $s_{k+1} = r_{k+1}$ on $V_{k+1} \cap B_{5R}$. Similarly, that s_{k+1} is the identity on points of $F_{k+1} \cap B_{5R}$ follows from the fact that, by definition, $s_{k+1} = r_{k+1}$ on F_{k+1}.

We now make the following claim: If $x \in U \cap B_{5R}$ and $s_k(x) \in B_{6R}$, then

$$|s_{k+1}(x) - s_k(x)| < 3C4^{-k}. \tag{2.3.8}$$

To prove this, we consider three cases.

(i) The case $x \in V_{k+1}$:

In this case, using (2.3.2) and the definitions of s_k and s_{k+1}, we get

$$|s_{k+1}(x) - s_k(x)| = |r_{k+1}(x) - r_k(x)| \leq |r_{k+1}(x) - x| + |x - r_k(x)| \leq C(\eta'_{k+1} + \eta'_k) < 3C4^{-k}.$$

(ii) The case $x \in U \setminus U_{k+1}$:

In this case, $s_{k+1}(x) = r_{k+1}(s_k(x))$. By assumption, $s_k(x) \in F_k \cap B_{6R}$ and therefore $\mathrm{dist}(s_k(x), F_{k+1}) < \eta'_k/4$ by (2.3.6). Therefore, by (2.3.2),

$$|s_{k+1}(x) - s_k(x)| = |r_{k+1}(s_k(x)) - s_k(x)| \leq C\eta'_k/4 < 3C4^{-k}.$$

(iii) The case $x \in U_{k+1} \setminus V_{k+1}$:

Note that $x \in U_{k+1} \cap B_{5R} \subset V_k \cap B_{5R}$, so $s_k(x) = r_k(x)$. Let

$$y = (1 - f(x))s_k(x) + f(x)x,$$

which is on the line segment L joining x to $s_k(x) = r_k(x)$. The diameter of L is therefore $|x - r_k(x)| \leq C\eta'_k$, by (2.3.2) and the fact that $x \in V_k$.

In addition, because $x \in U_{k+1}$, we have $\mathrm{dist}(x, F_{k+1}) < \eta_k$.

From these calculations, it follows that

$$\mathrm{dist}(y, F_{k+1}) \leq \mathrm{dist}(x, F_{k+1}) + \mathrm{diam}(L) \leq \eta_k + C\eta'_k,$$

and therefore, by (2.3.2), that

$$|s_{k+1}(x) - x| = |r_{k+1}(y) - x| \leq |r_{k+1}(y) - y| + |y - x| \leq C(\eta_k + C\eta'_k) + C\eta'_k \leq 2C4^{-k}.$$

From this, we see that

$$|s_{k+1}(x) - s_k(x)| \leq |s_{k+1}(x) - x| + |x - r_k(x)| < 2C4^{-k} + \eta_k < 3C4^{-k}.$$

This concludes the proof of the claim that $|s_{k+1}(x) - s_k(x)| < 3C4^{-k}$ if $x \in U \cap B_{5R}$ and $s_k(x) \in B_{6R}$.

Now note that

$$|s_1(x) - x| = |r_1(x) - x| \leq Cc = 1.$$

Therefore $s_1(x) \in B_{5.5R}$. Because $\sum_{k=0}^{\infty}(3C4^{-k}) \leq 6C < R/2$, it follows from the above claim that $s_k(x) \in B_{6R}$ for all k, and therefore that

$$|s_{k+1}(x) - s_k(x)| < 3C4^{-k}$$

for all $x \in U \cap B_{5R}$ and $k \in \mathbb{N}$.

It follows immediately from this and from (2.3.6) that the maps $s_k|_{U \cap B_{5R}}$ converge uniformly to a map

$$s \colon U \cap B_{5R} \to F \cap B_{6R}.$$

The map s is the identity on $F \cap B_{5R}$: if $x \in F \cap B_{5R}$, then by (2.3.5) and the definition of s_k we see that $s_k(x) = r_k(x)$. It follows that

$$|s(x) - x| = \lim_{k\to\infty} |s_k(x) - x| = \lim_{k\to\infty} |r_k(x) - x| \leq C \lim_{k\to\infty} \mathrm{dist}(x, F_k) = 0.$$

To finish the proof of the lemma, recall our fixed ball $B = B(p, r) \cap F$ in $F \cap B_R$. The map s, when restricted to $F \cap \overline{B}_{4R}$, is the identity. Therefore, for every positive number $\eta < r$ sufficiently small, there is a neighborhood $V \subset (U \cap B_{5R})$ of $F \cap \overline{B}_{4R}$ such that

$$x \in V \Rightarrow |s(x) - x| < \eta.$$

We may now choose k large so that $|s_k(x) - s(x)| < \eta$ for all $x \in U \cap B_{5R}$ (by uniform convergence) and in addition so that

$$F_k \cap B_{3R} \subset V.$$

Now we contract B in the following manner. First, consider the homotopy

$$h(x, t) = (1 - t)x + ts_k(x)$$

for $x \in B$ and $t \in [0, 1]$. Because $|s_k(x) - x| = |s_k(x) - s(x)| < \eta$, we have $h(B \times [0, 1]) \subset B_{3R}$. In addition, h deforms B onto a set $E \subset F_k \cap B_{3R}$ of diameter no more than $2r + 2\eta$. By our choices of r and η, $2r + 2\eta < 4r < r_0$, and therefore E is contractible inside a set $E' \subset F_k \cap B_{3R}$ of diameter $L(2r + 2\eta)$.

Let $g \colon B \times [0, 1] \to E' \subset (F_k \cap B_{3R})$ denote the homotopy of B onto a point that first deforms by h and then by the contraction in F_k. Then $s \circ g$ is a contraction of B to a point within the set $s(E') \subset F$, which has diameter no more than $L(2r + 2\eta) + 2\eta$.

In summary, if we recall that $\eta < r$, we have shown that the ball $B = B(p, r) \cap F$ is contractible within the ball $B' = B(p, (4L + 2)r) \cap F$ whenever $r < r_0/4$. This completes the proof. □

Lemma 2.3.13. *Suppose the pointed metric spaces (X_k, d_k, v_k) are (L, r_0)-LLC and converge to the pointed metric space (X, d, v) in the sense of Definition 2.3.3. Then (X, d) is LLC, with constants depending only on L and r_0.*

Proof. This follows immediately from Lemma 2.3.12 and Definition 2.3.3, as the "snowflake" transformations of Definition 2.3.3 distort the LLC constants in a quantitative way. □

To conclude this section, we give two lemmas which improve Propositions 2.3.7 and 2.3.8 in the setting of LLC spaces. They say that if a sequence of mapping packages converges, then the "almost-isometries" ϕ_k and ψ_k between the limiting spaces and the limit space can be taken to be continuous.

Definition 2.3.14. For $\eta > 0$, we say that continuous maps $f, g \colon M \to N$ between metric spaces are *η-homotopic* if they are homotopic by a homotopy $H \colon M \times [0, 1] \to N$ such that, for all $x \in M$ and $t \in [0, 1]$, we have

$$d_N(f(x), H(x, t)) < \eta.$$

Note in particular that if f and g are η-homotopic, then $d_N(f(x), g(x)) < \eta$ for all x.

Our next fact, Lemma 2.3.15 below, is an immediate consequence of Proposition 2.3.8 above, combined with Propositions 5.4 and 5.8 of [57]. (See also [50], Section 3, for a cleaner statement in the compact case.) Note that all our spaces are Ahlfors s-regular and thus have topological dimension bounded above by s, so those results apply.

Propositions 5.4 and 5.8 of [57], on which the proof of Lemma 2.3.15 is based, are important consequences of the linear local contractibility of the spaces X and $\{X_k\}$. Roughly speaking, they say that if a mapping into an LLC space is "roughly continuous" (as the maps ϕ_k and ψ_k from Proposition 2.3.7 are), then it is close to a continuous mapping, and if two continuous mappings into an LLC space are close, then they are η-homotopic for small η. The proofs of these facts use polyhedral approximations of the source space and an induction on the skeleta of the polyhedra. We encourage the reader to look at Semmes's paper [57] or Petersen's work [50] for the details.

Lemma 2.3.15. *Suppose the pointed metric spaces (X_k, d_k, v_k) are (L, r_0)-LLC, uniformly Ahlfors s-regular, and converge to the pointed metric space (X, d, v) in the sense of Definition 2.3.3.*

Fix a point $x \in X$ and a radius $R > 0$. Then there exist continuous mappings $f_k \colon \overline{B}_X(x, R) \to X_k$ and $g_k \colon \overline{B}_{X_k}(f_k(x), R) \to X$ satisfying the following conditions:

(i) *They almost preserve distances, in the sense that*

$$\lim_{k\to\infty} \sup\{|d_{X_k}(f_k(p), f_k(q)) - d_X(p, q)| : p, q \in B_X(x, R)\} = 0$$

and

$$\lim_{k\to\infty} \sup\{|d_X(g_k(p), g_k(q)) - d_{X_k}(p, q)| : p, q \in B_{X_k}(f_k(x), R)\} = 0.$$

(ii) *For every $0 < r < R$, we have*

$$\lim_{k\to\infty} \inf\{\eta : g_k \circ f_k|_{\overline{B}(x,r)} \text{ is } \eta\text{-homotopic to the inclusion map } \overline{B}(x, r) \to B(x, R)\} = 0$$

and

$$\lim_{k\to\infty} \inf\{\eta : f_k \circ g_k|_{\overline{B}(f_k(x),r)} \text{ is } \eta\text{-homotopic to the inclusion map } \overline{B}(f_k(x), r) \to B(f_k(x), R)\} = 0$$

(iii) *If x is the basepoint $v \in X$, then in addition we have*

$$\lim_{k\to\infty} d_k(f_k(v), v_k) = 0.$$

Proof. Take $\eta > 0$, which without loss of generality satisfies $r + 2\eta < R$ for the radii $r < R$ as in (ii). We will find, for all k sufficiently large, continuous mappings f_k and g_k as above that preserve distances up to additive error η and such that $f_k \circ g_k$ and $g_k \circ f_k$ are η-homotopic to the appropriate inclusion maps.

We will choose η'', η' small, with $0 < \eta'' < \eta' < \eta$, depending only on η and the (uniform) data of the space X and the sequence of spaces $\{X_k\}$. The precise way in which we choose η' and η'' will be described below.

By Proposition 2.3.7, there is an index $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, the maps $\phi_k \colon X \to X_k$ and $\psi_k \colon X_k \to X$ preserve distances up to additive error η'' on $\overline{B}(x, R)$ and $\overline{B}(f_k(x), R)$, respectively. By [57], Proposition 5.4, if η'' was chosen sufficiently small compared to η', then there exist continuous maps $f_k \colon \overline{B}(x, R) \to X_k$ and $g_k \colon \overline{B}(f_k(x), R) \to X$ such that

$$d_k(f_k(z), \phi_k(z)) < \eta' \text{ and } d(g_k(y), \psi_k(y)) < \eta' \tag{2.3.9}$$

for all points z, y in their respective domains. Part (i) of the lemma follows immediately from this by taking $\eta' < \eta/10$. Part (iii) also follows, because $\phi_k(v) = v_k$.

Now fix $0 < r < R$. By Proposition 2.3.7 we may also assume that, for all $k \geq k_0$, we have

$$d(\phi_k(\psi_k(x)), x) < \eta' \text{ and } d(\psi_k(\phi_k(x)), x) < \eta',$$

in addition to the properties above.

As $r + 2\eta' < r + 2\eta < R$, we see that $f_k(\overline{B}(x, r)) \subset B(f_k(x), R)$, and so the composition $g_k \circ f_k$ is defined on $\overline{B}(x, r)$. Similarly, the composition $f_k \circ g_k$ is defined on $\overline{B}(f_k(x), r)$. By choosing $\eta'' < \eta'/10$ and using equation (2.3.9) and the properties of ϕ_k and ψ_k, we also see that

$$d(f_k(g_k(x)), x) < 2\eta' \text{ and } d(g_k(f_k(x)), x) < 2\eta'.$$

Therefore, if η' was chosen sufficiently small, depending on η and the data of the spaces X, $\{X_k\}$, Proposition 5.8 of [57] implies that

$$g_k \circ f_k|_{\overline{B}(x,r)} \text{ and } f_k \circ g_k|_{\overline{B}(f_k(x),r)}$$

are η-homotopic to the inclusions

$$\overline{B}(x, r) \to B(x, R) \text{ and } \overline{B}(f_k(x), r) \to B(f_k(x), R).$$

This proves part (ii) of the lemma. $\square$

The following additional fact is an immediate consequence of Lemma 2.3.8 and equation (2.3.9) above.

Lemma 2.3.16. *Suppose we have convergence of a sequence of mapping packages*

$$((X_k, d_k, p_k), (Y_k, \rho_k, q_k), h_k) \to ((X, d, p), (Y, \rho, q), h)$$

in the sense of Definition 2.3.5. Suppose that all the spaces involved are uniformly Ahlfors s-regular and uniformly LLC, and that the mappings $\{h_k\}$ and h are uniformly C-Lipschitz and satisfy $h_k(p_k) = q_k$ and $h(p) = q$. Then for all $R > 0$, there exist continuous mappings

$$f_k : \overline{B}_X(p, R) \to X_k \text{ and } g_k : \overline{B}_{X_k}(p_k, R) \to X$$

satisfying exactly the conditions of Lemma 2.3.15, and continuous mappings

$$\tilde{f}_k : \overline{B}_Y(q, R) \to Y_k \text{ and } \tilde{g}_k : \overline{B}_{Y_k}(q_k, R) \to Y$$

satisfying the analogous properties of Lemma 2.3.15, such that in addition we have that

$$\lim_{k\to\infty} \tilde{g}_k(h_k(f_k(x))) = h(x)$$

uniformly for $x \in \overline{B}_X(p, R/2C)$.

2.3.3 Convergence of manifolds

Here we state some facts on the convergence of metric spaces that are LLC topological manifolds. Our main goal is to give a proof of Proposition 2.3.19 below, which says that the limit of a sequence of uniformly Ahlfors regular, uniformly LLC, topological d-manifolds is a homology d-manifold (see Definition 2.3.18). This result essentially goes back to Begle [5] (see also [27]) and appears to be well-known, but we did not find a modern proof in the literature in the generality necessary here.

Below, H_* denotes singular homology with integer coefficients.

Lemma 2.3.17. *Let M be an (L, r_0)-LLC oriented topological d-manifold. Let $v \in M$ and let $K_1 \subset K_2$ be compact sets satisfying $v \in K_1 \subset B(v, r) \subset B(v, 2Lr) \subset K_2 \subset B(v, r_0)$. Then the following facts hold.*

(i) *The map* $j_*\colon H_p(M, M\setminus K_2) \to H_p(M, M\setminus K_1)$*, induced by inclusion, is trivial if* $p \neq d$*.*

(ii) *The map* $i_*\colon H_d(M, M\setminus K_2) \to H_d(M, M\setminus\{v\}) \cong \mathbb{Z}$*, induced by inclusion, is surjective.*

(iii) *With this notation, we also have* $\ker i_* \subseteq \ker j_*$ *in the top degree* $p = d$*.*

Proof. By use of the natural duality isomorphisms ([58], Theorem 6.2.17) we obtain the following commutative diagram. Here $\overline{H}$ denotes Čech cohomology, and all maps in the diagram are the natural maps induced by inclusion.

$$\begin{array}{ccccc} H_p(M, M\setminus K_2) & \xrightarrow{j_*} & H_p(M, M\setminus K_1) & \xrightarrow{k_*} & H_p(M, M\setminus\{v\}) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \overline{H}^{d-p}(K_2) & \xrightarrow{j^*} & \overline{H}^{d-p}(K_1) & \xrightarrow{k^*} & \overline{H}^{d-p}(\{v\}) \end{array} \tag{2.3.10}$$

If $p \neq d$, then j^* is trivial because K_1 is contractible in K_2, which proves (i).

Now let $p = d$. The map $i^* = k^*j^*\colon \overline{H}^0(K_2) \to \overline{H}^0(\{v\})$ is surjective, as $v \in K_2$, which proves (ii).

Finally, by Lemma 2.3.11, K_1 is entirely contained in a connected component E of K_2. Therefore, every connected component E' of K_2 that does not contain $\{v\}$ is in fact disjoint from K_1. It follows that if $i^*\phi = k^*j^*\phi$ is trivial in $\overline{H}^0(\{v\})$ for some $\phi \in \overline{H}^0(K_2)$, then $j^*\phi$ is already trivial in $\overline{H}^0(K_1)$. This proves claim (iii). $\square$

We now set up some definitions for the main result of this sub-section. A *Euclidean Neighborhood Retract* (ENR) is a locally compact space X which, for every $N \in \mathbb{N}$ and every topological embedding $e\colon X \to \mathbb{R}^N$, has the property that $e(X)$ is a retract of some open neighborhood of $e(X)$ in $\mathbb{R}^N$. Every locally compact LLC space with finite topological dimension is a Euclidean Neighborhood Retract (see [32], Theorem V.7.1).

Definition 2.3.18. A space M that is an ENR and that satisfies the condition

$$H_*(M, M\setminus\{x\}) = H_*(\mathbb{R}^d, \mathbb{R}^d\setminus\{0\}),$$

for all $x \in M$, is called a *homology d-manifold.*

Proposition 2.3.19. *Suppose $\{(X_k, d_k)\}$ is a sequence of uniformly Ahlfors s-regular, (L, r_0)-LLC oriented topological d-manifolds, $v_k \in X_k$, and the sequence of pointed metric spaces (X_k, d_k, v_k) converges to (X, d, v) in the sense of Definition 2.3.3. Then (X, d) is an LLC homology d-manifold.*

Proof. The fact that (X, d) is LLC is Lemma 2.3.13 above. As this statement is quantitative, we will denote the LLC constants of (X, d) also by (L, r_0).

The fact that X is a homology d-manifold can be proven by the methods of Begle [5], again as remarked in [29]. For convenience, we provide a proof using the tools introduced in this section.

We know that X is Ahlfors s-regular, and therefore it has finite Hausdorff dimension and thus finite topological dimension. Because X is also LLC, it is an ENR, as noted above. It now suffices to show that for every $x \in X$, the local integer (singular) homology groups $H_p(X, X \setminus \{x\})$ are isomorphic to $\mathbb{Z}$ if $p = d$ and trivial otherwise.

To set up the proof we need some notation.

Let $L' = 4L$. Fix an integer $p \geq 0$, a point $x \in X$, and a radius $R > 0$. In addition, for each $k \in \mathbb{N}$, fix continuous maps

$$f_k : B_X(x, R) \to X_k$$

$$g_k : B_{X_k}(f_k(x), R) \to X$$

as in Lemma 2.3.15. These maps have the property that, up to arbitrarily small additive error (decreasing to zero with k), they preserve distances and are inverses of each other.

For $n \in \mathbb{N}$, let

$$F_n = H_p(X, X \setminus \overline{B}(x, (L')^{-n} r_0))$$

and

$$G_n^k = H_p(X_k, X_k \setminus \overline{B}(f_k(x), (L')^{-n} r_0))$$

(Of course these groups depend on p, but we will make it clear from the context which value of p we take.)

Note that for $m \geq n$ there are natural maps $(i_{n,m})_*\colon F_n \to F_m$ and $(j^k_{n,m})_*\colon G^k_n \to G^k_m$ induced by inclusion.

Claim 2.3.20. *We have the direct limits*

$$F_\infty := \varinjlim F_n \cong H_p(X, X \setminus \{x\})$$

and

$$G^k_\infty := \varinjlim G^k_n \cong H_p(X_k, X_k \setminus \{f_k(x)\}) \cong \begin{cases} \mathbb{Z} & \text{if } p = d, \\ 0 & \text{if } p \neq d. \end{cases}$$

Proof of Claim 2.3.20. We will show the first direct limit; the proof of the second is identical. The proof follows from standard properties of direct limits and singular homology. There are natural maps $\phi_n\colon F_n \to H_p(X, X \setminus \{x\})$ induced by inclusion. To show that $F_\infty \cong H_p(X, X \setminus \{x\})$, we must show two statements (see, e.g., [46], Proposition A.4):

1. For every $a \in H_p(X, X \setminus \{x\})$, there exists $n \in \mathbb{N}$ and $b \in F_n$ such that $\phi_n(b) = a$.

2. If $b \in F_n$ and $\phi_n(b) = 0$, then $(i_{n,m})_*(b) = 0$ for some $m \geq n$.

To show (1), consider $a \in H_p(X, X\setminus\{x\})$. By excision and the fact that singular homology has compact support (see [58], 4.8.11), $a = j_*(c)$, where $c \in H_p(X, X \setminus U)$ for some open set U containing x, and

$$j_* : H_p(X, X \setminus U) \to H_p(X, X \setminus \{x\})$$

is the mapping induced by inclusion.

We now choose $n \in \mathbb{N}$ large enough so that $\overline{B}(x, (L')^{-n} r_0) \subset U$. There is a mapping

$$k_* : H_p(X, X \setminus U) \to H_p(X, X \setminus \overline{B}(x, (L')^{-n} r_0))$$

induced by inclusion.

Because all mappings are induced by inclusion, we have $\phi_n k_* = j_*$. Thus, if we let $b = k_*(c) \in H_p(X, X \setminus \overline{B}(x, (L')^{-n} r_0))$, we see that $\phi_n(b) = \phi_n k_*(c) = j_*(c) = a$. This proves part (1) of Claim 2.3.20.

To show part (2), suppose that $b \in F_n$ is such that $\phi_n(b) = 0 \in H_p(X, X \setminus \{x\})$. As before, using the fact that singular homology has compact support, we can write $b = l_*(c)$, where $c \in H_p(X, X \setminus U)$ for some open set U containing $\overline{B}(x, (L')^{-n})$ and

$$l_* : H_p(X, X \setminus U) \to F_n$$

is the mapping induced by inclusion.

By excision and [58], Theorem 4.8.13, we see that $i_*(c) = 0 \in H_p(X, X \setminus V)$, where $V \subset U$ is an open set containing x and

$$i_* : H_p(X, X \setminus U) \to H_p(X, X \setminus V)$$

is the mapping induced by inclusion.

We now choose $m \in \mathbb{N}$ large enough so that $\overline{B}(x, (L')^{-m} r_0) \subset V$. Let

$$h_* : H_p(X, X \setminus V) \to F_m$$

be induced by inclusion. Again because all mappings are compatible, we have

$$(i_{n,m})_*(b) = (i_{n,m})_* l_*(c) = h_* i_*(c) = h_*(0) = 0 \in F_m.$$

This completes the proof of Claim 2.3.20. □

Let $(i_n)_* \colon F_n \to F_\infty$ and $(j_n^k)_* \colon G_n^k \to G_\infty^k$ denote the natural inclusion maps.

The excision property of homology and the properties of f_k and g_k allow us to conclude the following: For all $n_0 \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that for all $n \leq n_0$ and $k \geq k_0$, there are group homomorphisms $a_n^k \colon F_n \to G_{n+1}^k$ and $b_n^k \colon G_n^k \to F_{n+1}$ that commute with the inclusion maps above, and that satisfy

$$b_{n+1}^k a_n^k = i_{n,n+2} \qquad \text{and} \qquad a_{n+1}^k b_n^k = j_{n,n+2}^k.$$

Indeed, a_n^k and b_n^k are simply the maps on homology induced by f_k and g_k, and so these properties follow from Lemma 2.3.15. The fact that a_n^k maps into G_{n+1}^k if $n \leq n_0$ and k

is sufficiently large follows from the fact that f_k preserves distances up to a small additive error, by Lemma 2.3.15.

In summary, for each n_0 there exists a k so that we have the following commutative diagram, in which the diagonal arrows do not exist past column n_0:

$$
\begin{tikzcd}
F_1 \arrow[r, "i_{1,2}"] \arrow[dr, "a_1^k"] & F_2 \arrow[r, "i_{2,3}"] \arrow[dr, "a_2^k"] & F_3 \arrow[r, "i_{3,4}"] \arrow[dr, "a_3^k"] & \dots \arrow[r, "i_{n_0-1,n_0}"] \arrow[dr, "a_{n_0-1}^k"] & F_{n_0} \arrow[r, "i_{n_0,n_0+1}"] & \dots F_\infty \\
G_1^k \arrow[r, "j_{1,2}^k"] \arrow[ur, "b_1^k"] & G_2^k \arrow[r, "j_{2,3}^k"] \arrow[ur, "b_2^k"] & G_3^k \arrow[r, "j_{3,4}^k"] \arrow[ur, "b_3^k"] & \dots \arrow[r, "j_{n_0-1,n_0}^k"] \arrow[ur, "b_{n_0-1}^k"] & G_{n_0}^k \arrow[r, "j_{n_0,n_0+1}^k"] & \dots G_\infty^k
\end{tikzcd}
\tag{2.3.11}
$$

Note that Lemma 2.3.17 translates to the following information in this setting:

(i) If $p \neq d$, then for all k and for all $m > n$, the map $j_{n,m}^k \colon G_n^k \to G_m^k$ is trivial.

(ii) If $p = d$, then for all k and n the map $j_n^k \colon G_n^k \to G_\infty^k$ is surjective.

(iii) If $p = d$, then for all k and n, we have $\ker j_n^k \subseteq \ker j_{n,n+1}^k$.

We wish to show that F_∞ is isomorphic to $\mathbb{Z}$ if $p = d$ and is trivial if $p \neq d$, just as each of the spaces G_∞^k are.

Consider first the case $p \neq d$. By (i), we have that for all k and for all $m > n$, the maps $j_{n,m}^k$ are trivial. It follows by the diagram that the maps $i_{n,n+3}$ are all trivial (as they factor through $j_{n,n+1}^k$ for some k) and therefore that F_∞ is trivial when $p \neq d$.

Now we consider the case $p = d$.

Claim 2.3.21. *In degree $p = d$, $i_2 \colon F_2 \to F_\infty$ is surjective.*

Proof of Claim 2.3.21. This is just diagram-chasing. We will freely use the three properties of the diagram (2.3.11) described above, and we encourage the reader to simply trace the proof in that diagram.

Fix $x \in F_\infty$. Then $x = i_m(x_m)$ for some $m \in \mathbb{N}$, by the definition of the direct limit. Fix k large so that diagram (2.3.11) has diagonal arrows $a_l := a_l^k$ and $b_l := b_l^k$ up to $l = m+3$. (We will suppress all superscripts k in the proof of this claim.)

Let $y_{m+1} = a_m(x_m) \in G_{m+1}$. Then some $y_1 \in G_1$ satisfies $j_1(y_1) = j_{m+1}(y_{m+1})$, by (ii), and so

$$j_{m+1}(y_{m+1}) = j_{m+1}j_{1,m+1}(y_1).$$

It follows, by (iii), that

$$j_{m+1,m+2}(y_{m+1}) = j_{1,m+2}(y_1).$$

Denote this element by $y_{m+2} \in G_{m+2}$.

Let $x_{m+3} = b_{m+2}(y_{m+2}) \in F_{m+3}$. We have

$$i_{2,m+3}b_1(y_1) = b_{m+2}j_{1,m+2}(y_1) = b_{m+2}(y_{m+2}) = x_{m+3}.$$

In addition,

$$\begin{aligned} x_{m+3} &= b_{m+2}(y_{m+2}) \\ &= b_{m+2}j_{m+1,m+2}(y_{m+1}) \\ &= b_{m+2}j_{m+1,m+2}a_m(x_m) \\ &= i_{m,m+3}(x_m). \end{aligned}$$

It follows that $i_{m+3}(x_{m+3}) = i_m(x_m) = x$, and so

$$i_2b_1(y_1) = i_{m+3}i_{2,m+3}b_1(y_1) = i_{m+3}(x_{m+3}) = x.$$

Thus, i_2 is surjective. $\square$

The following claim is also proven by a similar diagram chase.

Claim 2.3.22. *In dimension $p = d$,* $\ker i_n \subset \ker i_{n,n+3}$.

Proof of Claim 2.3.22. Suppose that $i_n(x_n) = 0$ for some $x_n \in F_n$. Then for some $m \geq n$, $i_{n,m}(x_n) = 0$. As in the previous claim, we now fix k large so that diagram (2.3.11) has diagonal arrows $a_l := a_l^k$ and $b_l := b_l^k$ up to column $l = m$. We then see that

$$j_{n+1,m+1}a_n(x_n) = a_m i_{n,m}(x_n) = a_m(0) = 0.$$

By (iii) above, it follows that $j_{n+1,n+2}a_n(x_n) = 0$. Thus,

$$i_{n,n+3}(x_n) = b_{n+2}j_{n+1,n+2}a_n(x_n) = b_{n+2}(0) = 0.$$

This completes the proof of Claim 2.3.22. □

Now fix k so that the diagonal arrows in diagram (2.3.11) exist up to $n = 10$. Let $G_\infty = G_\infty^k \cong \mathbb{Z}$. (Now that k is fixed we will again suppress the superscripts k.) We now define homomorphisms $\psi_n \colon F_n \to G_\infty \cong \mathbb{Z}$ by

$$\psi_n(x_n) = j_3 a_2 i_2^{-1} i_n(x_n).$$

Note that i_2 is surjective but not necessarily injective; nonetheless we have the following fact:

Claim 2.3.23. *The maps ψ_n are well-defined homomorphisms (i.e., independent of the choice of $i_2^{-1}i_n(x_n)$) and are compatible, in the sense that $\psi_m i_{n,m}(x_n) = \psi_n(x_n)$ for $m \geq n$.*

Proof of Claim 2.3.23. Suppose first that $i_2(x_2) = i_2(x_2')$ for some $x_2, x_2' \in F_2$. To show that ψ_n is well-defined we must show that

$$j_3 a_2(x_2) = j_3 a_2(x_2').$$

By Claim 2.3.22, $i_{2,3}(x_2) = i_{2,3}(x_2')$. Thus,

$$j_3 a_2(x_2) = j_4 j_{3,4}(x_2) = j_4 a_3 i_{2,3}(x_2) = j_4 a_3 i_{2,3}(x_2') = j_3 a_2(x_2').$$

This shows that ψ_n is well-defined. That ψ_n is a homomorphism is clear.

To see that $\psi_m(i_{n,m}(x_n)) = \psi_n(x_n)$, we note that if $i_2(x_2) = i_n(x_n)$, then $i_2(x_2) = i_m i_{n,m}(x_m)$. Thus,

$$\psi_m(i_{n,m}(x_n)) = j_3 a_2(x_2) = \psi_n(x_n).$$

$\square$

It follows that the maps ψ_n induce a homomorphism $h\colon F_\infty \to G_\infty \cong \mathbb{Z}$ satisfying $h \circ i_n = \psi_n$ for all n. We will show that h is injective and surjective.

Suppose $h(x) = 0$ for $x \in F_\infty$. By Claim 2.3.21, we can write $x = i_2 x_2$, for $x_2 \in F_2$. Therefore,

$$0 = h i_2(x_2) = \psi_2(x_2) = j_3 a_2(x_2).$$

Because $\ker j_3 \subseteq \ker j_{3,4}$, we have

$$j_{3,4} a_2(x_2) = 0 \in G_4.$$

It follows that

$$i_{2,5}(x_2) = b_4 j_{3,4} a_2(x_2) = 0 \in F_5$$

and therefore that $x = i_5 i_{2,5}(x_2) = 0 \in F_\infty$. This shows that h is injective.

To show that h is surjective, it suffices to show that $\psi_2 = j_3 a_2$ is surjective. Consider $y \in G_\infty$. Because j_1 is surjective, $y = j_1(y_1)$ for some $y_1 \in G_1$. Letting $x_2 = b_1(y_1)$, we see that

$$\psi_2(x_2) = j_3 a_2(x_2) = j_3 a_2 b_1(y_1) = j_1(y_1) = y.$$

This shows that h is surjective and is therefore an isomorphism $F_\infty \to G_\infty$.

We have therefore shown that $F_\infty \cong H_p(X, X \setminus \{x\})$ is isomorphic to $G_\infty \cong H_p(X_k, X_k \setminus \{x_k\})$, which completes the proof of Proposition 2.3.19. $\square$

2.3.4 Some basic degree theory

In this section, we give a degree-type lemma for close mapping packages. The idea here is quite simple, though the notation is cumbersome: If the limit of a suitable sequence of mappings is a homeomorphism, then sufficiently close limiting mappings should have images which contain a ball of fixed radius.

We now fix a set-up and some notation.

Let $\{(X_k, p_k)\}$ and $\{(Y_k, q_k)\}$ be two sequences of pointed metric spaces converging to (X, p) and (Y, q), respectively. Suppose that all the spaces are uniformly Ahlfors s-regular, (L, r_0)-LLC, homology d-manifolds, and furthermore that $\{Y_k\}$ are oriented topological d-manifolds and Y is a topological d-manifold.

For some fixed $R > 0$, let $F_k = \overline{B}(p_k, R)$ and assume also that the sequence of pointed metric spaces $\{(F_k, p_k)\}$ converges to the pointed metric space (F, p), where $F \subset X$ and $F \supset B(p, R)$ in X.

Finally, assume that the maps $w_k \colon F_k \to Y_k$ are uniformly C-Lipschitz and that we have convergence of the sequence of mapping packages:

$$\{(F_k, p_k), (Y_k, q_k), w_k\} \to \{(F, p), (Y, q), w\}$$

By Lemma 2.3.15, there are continuous mappings

$$\begin{aligned}
&f_k : F \to X_k, \\
&g_k : F_k \to X, \\
&\tilde{f}_k : \overline{B}_Y(q, 3CR) \to Y_k, \\
&\tilde{g}_k : \overline{B}_{Y_k}(q_k, 3CR) \to Y,
\end{aligned}$$

that almost preserve distances and are almost inverses, up to additive error that decreases to zero with k.

Fix an open set $A \subseteq F$ such that the set $w(A)$ lies within a single chart of Y homeomorphic to an open subset of $\mathbb{R}^d$.

Lemma 2.3.24. *Suppose that, for some $r, r' \in (0, r_0)$, the ball $B_X(z, 4Lr)$ is contained in A, the map $w|_A$ is a homeomorphism, and $w(B_X(z, r)) \supseteq B_Y(w(z), r')$. Then for all k sufficiently large, $w_k(B_{X_k}(f_k(z), 2r)) \supseteq B_{Y_k}(w_k(f_k(z)), r'/2)$.*

Proof. First of all, it is clear that in the proof we may assume without loss of generality that Y is an orientable d-manifold, because all arguments can be carried out in the orientable chart of Y containing $w(A)$. This will allow us to apply Lemma 2.3.17 to subsets of $w(A)$.

Now let $0 < \eta < r'/(100L)$. For all k sufficiently large, the maps f_k, g_k, $\tilde{f}_k$, $\tilde{g}_k$ preserve distances up to additive error η. In addition, again by Lemma 2.3.15, we may assume that, for all k large and for all $r < 2CR$, the map

$$\tilde{g}_k \circ \tilde{f}_k|_{B(p,r)}$$

is η-homotopic to the inclusion map of $B(p,r)$ into $B(p,R)$.

Fix $k \in \mathbb{N}$ large enough for this to hold; from now on, this k will be fixed, so we drop the subscript and denote the above maps by f, g, $\tilde{f}$, $\tilde{g}$. By Lemma 2.3.16 and [57], Proposition 5.8, we can also arrange that the maps $\tilde{f} \circ w$ and $w_k \circ f$, when restricted to A, are η-homotopic on F.

Let $B = \overline{B}_X(z,r)$. Fix $y \in B_{Y_k}(w_k(f_k(z)), r'/2)$.

First, because w is a homeomorphism on A, the induced mapping on relative homology,

$$w_* : H_d(A, A \setminus B) \to H_d(Y, Y \setminus w(B))$$

is an isomorphism. (Here we use excision for singular homology, see [58], Corollary 4.6.5.) Note that $H_d(Y, Y \setminus w(B))$ is non-trivial, by duality (e.g. [58], Theorem 6.2.17).

Let $V = B_Y(q, 2CR)$. The map $\tilde{f}$ induces a non-trivial map

$$\tilde{f}_* : H_d(V, V \setminus w(B)) \to H_d(Y_k, Y_k \setminus B'')$$

where $B'' = \overline{B}(y, r'')$, $r'' = r'/(10L)$. Indeed, if this map were trivial, then the map

$$\tilde{g}_* \tilde{f}_* : H_d(V, V \setminus w(B)) \to H_d(Y, Y \setminus \{a\}),$$

factoring as it does through the previous map, would be trivial for some $a \in w(B)$. But this map on homology is the same as that induced by inclusion, so this cannot be the case by the duality argument of Lemma 2.3.17 (ii).

It follows that the map

$$(\tilde{f} \circ w)_* = \tilde{f}_* w_* : H_d(A, A \setminus B) \to H_d(Y_k, Y_k \setminus B'')$$

is non-trivial.

Because $\tilde{f} \circ w$ and $w_k \circ f$ are η-homotopic, the map

$$(w_k \circ f)_* : H_d(A, A \setminus B) \to H_d(Y_k, Y_k \setminus B'')$$

is non-trivial.

This implies that

$$(w_k \circ f)_* : H_d(A, A \setminus B) \to H_d(Y_k, Y_k \setminus \{y\})$$

is non-trivial. Indeed, if not, then by Lemma 2.3.17 (iii),

$$(w_k \circ f)_* : H_d(A, A \setminus B) \to H_d(Y_k, Y_k \setminus B'')$$

would be trivial, but we just showed that it is not.

So we have shown that

$$(w_k \circ f)_* : H_d(A, A \setminus B) \to H_d(Y_k, Y_k \setminus \{y\})$$

is non-trivial. It follows from this that $y \in (w_k \circ f)(B)$, otherwise this map would factor through the trivial $H_d(Y_k \setminus \{y\}, Y_k \setminus \{y\})$.

Because $f(B) \subset B(f_k(z), 2r)$, we get that

$$y \in w_k(f(B)) \subset w_k(B(f_k(z), 2r)).$$

□

Later on, it will be convenient to work with a cohomological notion of local degree, which we introduce now. The following material is taken from [31]. For proofs, see [51], Chapter II.2.

Let H_c^* denote the Alexander-Spanier cohomology groups with compact supports and coefficients in $\mathbb{Z}$. (For the definition and properties of Alexander-Spanier cohomology, see [46].) The following definition is taken from [31], I.1.

Definition 2.3.25. A locally compact, connected, and locally connected Hausdorff space M is called a *generalized d-manifold* if:

- $H_c^p(U) = 0$ whenever $U \subseteq M$ is open and $p \geq d+1$.
- For every $x \in M$ and every open neighborhood U of x, there is another open neighborhood V of x contained in U such that

  $$H_c^p(V) = \begin{cases} \mathbb{Z} & \text{if } p = d, \\ 0 & \text{if } p = d-1, \end{cases}$$

  and the standard homomorphism $H_c^n(W) \to H_c^n(V)$ is surjective whenever W is an open neighborhood of x contained in V.
- X has finite topological dimension.

Remark 2.3.26. Any homology d-manifold is a generalized d-manifold, as noted in [31], Example 1.4 (c).

A generalized d-manifold X is said to be *oriented* if $H_c^d(X) = \mathbb{Z}$. In this case we can simultaneously orient all connected open subsets U of X via the isomorphism between $H_c^d(U)$ and $H_c^d(X)$.

We will not use any sophisticated facts about cohomology below, but only the following object and its basic properties: Let X and Y be oriented generalized d-manifolds, and let $f\colon X \to Y$ be continuous. For any relatively compact domain D in X, and for every $y \in Y \setminus f(\partial D)$, we can associate an integer called the local degree $\mu(y, D, f)$. In the following lemma, we collect the only properties of μ we will need.

Lemma 2.3.27. *For continuous maps f and g between oriented generalized d-manifolds X and Y, and a relatively compact domain $D \subseteq X$, the local degree μ has the following properties:*

- *The function $y \to \mu(y, D, f)$ is constant on each connected component of $Y \setminus f(\partial D)$.*
- *If $y \notin f(\overline{D})$, then $\mu(y, D, f) = 0$.*
- *If $f\colon D \to f(D)$ is a homeomorphism, then $\mu(y, D, f) = \pm 1$ for each $y \in f(D)$.*
- *If $y \in Y \setminus f(\partial D)$ and if $f^{-1}(y) \subset D'$, where D' is a domain contained in D such that $y \in Y \setminus f(\partial D')$, then*
$$\mu(y, D, f) = \mu(y, D', f)$$

Proof. These facts can all be found in [31], 2.3 or [51], II.2. $\square$

2.3.5 The Bonk-Kleiner theorem on mappings of bounded multiplicity

This material is taken from [6].

Definition 2.3.28. A map f between spaces X and Y is of *bounded multiplicity* if there is a constant $N \in \mathbb{N}$ such that $\# f^{-1}(y) \leq N$ for all $y \in Y$.

The following result of Bonk and Kleiner provides a partial substitute, in our setting, for Reshetnyak's theorem on quasi-regular mappings. (See the discussion of David's proof in Subsection 2.2.)

Theorem 2.3.29 ([6], Theorem 3.4)**.** *Suppose X is a compact metric space, every non-empty open subset of X has topological dimension at least d, and $f\colon X \to \mathbb{R}^d$ is a continuous map of bounded multiplicity. Then there is an open subset $V \subseteq f(X)$ with $\overline{V} = f(X)$ such that $U = f^{-1}(V)$ is dense in X and $f|_U\colon U \to V$ is a covering map.*

2.4 Warm-up: Getting bi-Lipschitz weak tangents

In this section, we prove a result that is much weaker than Theorem 1.2.2, but whose proof illustrates some of the techniques used in the proof of Theorem 1.2.2. Nothing in this section is needed in the proof of Theorem 1.2.2, so a reader who is solely interested in that proof can skip this section without missing anything needed later in the chapter.

Let (X, d) and (Y, ρ) be metric spaces and $f\colon X \to Y$ be Lipschitz. Define a *weak tangent of f* to be a mapping package

$$((X_\infty, d_\infty, x_\infty), (Y_\infty, \rho_\infty, y_\infty), f_\infty)$$

for which there is a sequence of positive real numbers λ_n, bounded above, and a sequence $x_n \in X$ such that, in the sense of Definition 2.3.5, we have

$$((X, \lambda_n^{-1}d, x_n), (Y, \lambda_n^{-1}\rho, f(x_n)), f) \to ((X_\infty, d_\infty, x_\infty), (Y_\infty, \rho_\infty, y_\infty), f_\infty)$$

as $n \to \infty$.

Note that the spaces X_∞ and Y_∞ here are weak tangents of the spaces X and Y, as in Definition 2.1.3 or Subsection 1.1.2.

We will say that f has a bi-Lipschitz weak tangent at x if, for one of its weak tangent mapping packages, the mapping f_∞ which arises is bi-Lipschitz.

Suppose that f is Lipschitz and that X and Y are doubling metric spaces. Suppose also that X is equipped with a doubling measure, and that x is a point of density of a set $E \subset X$ such that $f|_E$ is bi-Lipschitz. Consider any sequence $\lambda_n \to 0$. Then every weak tangent of f along the sequence of scales $\{\lambda_n\}$ and the sequence of points $\{x_n = x\}$ yields a mapping f_∞ that is bi-Lipschitz. This is a standard fact, and its proof is very similar to that given in Proposition 2.9.1 below.

Thus, a mapping having a positive-measure set on which it is bi-Lipschitz is a much stronger condition than a map merely having a bi-Lipschitz weak tangent.

In the setting of Theorem 1.2.2, one can give a simpler argument which shows that the mapping has a bi-Lipschitz weak tangent. This argument is really contained in [6], though our context is slightly different.

In the proof, we will need one definition that we have not yet introduced, coming from Chapter 12 of [21]. (This will not be used in the proof of the main Theorem 1.2.2.)

Definition 2.4.1. A Lipschitz mapping $f : M \to N$ between two metric spaces is said to be *David-Semmes regular* if there is a constant $C > 0$ such that, for every ball $B \subseteq N$ of radius r, the set $f^{-1}(B)$ can be covered by at most C balls of radius Cr.

In particular, David-Semmes regular maps always have bounded multiplicity.

Proposition 2.4.2. *Let X and Y be Ahlfors s-regular, linearly locally contractible, complete, oriented, topological d-manifolds, for $s, d \geq 1$. Suppose in addition that Y has d-manifold weak tangents.*

Suppose that $f\colon X \to Y$ is Lipschitz and has $|f(X)| > 0$. Then f has a bi-Lipschitz weak tangent.

Proof. The first step is to examine the spaces X_∞ and Y_∞. By Proposition 2.3.19 and the assumption that Y has d-manifold weak tangents, we see that X_∞ is a homology d-manifold and Y_∞ is a topological d-manifold. We also have, by Proposition 2.3.13, that Y_∞ is (L, r_0)-LLC, for some constants L and r_0.

The next step is to apply Proposition 12.8 of [21]. This says that we can find a weak tangent

$$((X_\infty, d_\infty, x_\infty), (Y_\infty, \rho_\infty, y_\infty), f_\infty)$$

of f such that f_∞ is a David-Semmes regular map. In particular, this means that f_∞ is a mapping of bounded multiplicity, in the sense of Definition 2.3.28.

We would now like to apply Theorem 2.3.29 to f_∞. Fix a small open ball $B \subset X_\infty$. We can choose B so small that $f_\infty(\overline{B})$ lies in a set $V \subset Y_\infty$ which is homeomorphic to an open set in $\mathbb{R}^d$, and which has diameter less than the contractibility radius r_0 of Y_∞.

Let $K = \overline{B}$. Then every open subset of K contains an open subset of the homology d-manifold X_∞ and thus has topological dimension at least d. Because we also know that f_∞ has bounded multiplicity on K, we can apply Theorem 2.3.29.

In particular, we obtain an open set $U \subset K \subset X$ such that f_∞, when restricted to U, is a homeomorphism. Let $V' \subset f_\infty(U)$ be a small open set such that

$$\operatorname{dist}(V', Y_\infty \setminus f_\infty(U)) > L \operatorname{diam} V'$$

and let $U' = f_\infty^{-1}(V') \cap U$.

We claim that f_∞ is in fact bi-Lipschitz on U'. We already know it to be Lipschitz, so it suffices to establish the other bound. Fix $x, y \in U'$ and consider $f(x), f(y) \in V' \subset Y_\infty$. Let $r = \rho_\infty(f(x), f(y))$; note that $r < r_0$ by our assumptions.

First of all, there is a compact connected set $S \subset \overline{B}(f_\infty(x), Lr) \subset f_\infty(U)$ containing $f_\infty(x)$ and $f_\infty(y)$. Indeed, by our assumptions, the compact set $\overline{B}(f(x), r)$ is contractible within $\overline{B}(f_\infty(x), Lr)$. If H is the homotopy realizing this contractibility, then

$$S = H(\overline{B}(f_\infty(x), r) \times [0,1])$$

contains $f_\infty(x)$ and $f_\infty(y)$ and is compact, connected, and contained in $\overline{B}(f_\infty(x), Lr) \subset f_\infty(U)$.

Now consider $E = f_\infty^{-1}(S) \cap U$. Because f_∞ is a homeomorphism on U, we have that E is a compact, connected set in U that contains x and y. Because f_∞ is David-Semmes regular with constant $C > 0$, E is contained in the union of at most C balls of radius CLr in X_∞. It follows that $\operatorname{diam} E \leq 2C^2 Lr$.

Thus,

$$d_\infty(x, y) \leq \operatorname{diam} E \leq 2C^2 Lr = 2C^2 L \rho_\infty(f_\infty(x), f_\infty(y)),$$

and so f_∞ is bi-Lipschitz on U'.

To complete the proof of the Proposition, we take another weak tangent of f_∞ along a sequence of scales $\{\lambda_n\}$ tending to zero and a fixed base-point sequence $\{x_n = x \in U'\}$. This yields a weak tangent of f_∞ which is globally bi-Lipschitz. That this is also a weak tangent of f itself is a standard fact (see [21], Lemma 9.22). □

2.5 Setting up the proof of Theorem 1.2.2

Recall that if a complete metric space X is Ahlfors s-regular with constant C_0, then we equip X with a type of dyadic cube decomposition Δ_n for each $n < j_0$, where j_0 is a fixed top scale in $\mathbb{Z} \cup \{\infty\}$. This was introduced in Subsection 1.1.1. Associated to these cubes are another constant c_0, which depends only on s and C_0. We will heavily use the four properties of this cube decomposition presented in Subsection 1.1.1.

We now introduce the following notation:

$$\tilde{B}_n(x, r) = \bigcup \{Q \in \Delta_n : Q \cap B(x, r) \neq \emptyset\}$$

By Theorem 2.2.2, to prove Theorem 1.2.2 it suffices to show the following proposition, which is just a restatement of David's condition, formulated in Definition 2.2.1.

Proposition 2.5.1. *Let $d \in \mathbb{N}$ and $s > 0$. Suppose (Y, ρ) is LLC, Ahlfors s-regular, and has d-manifold weak tangents. For all C_0, L, r_0, M and for all $\lambda, \gamma \geq 0$, there exist $\Lambda, \eta > 0$ such that the following holds:*

Let X be a complete, oriented, topological d-manifold that is Ahlfors s-regular with constant C_0 and (L, r_0)-LLC. Let I_0 be a 0-cube and $z \colon I_0 \to Y$ an M-Lipschitz map. If $x \in X$, $n \in \mathbb{Z}$, and $T = \tilde{B}_n(x, \Lambda 2^n) \subseteq I_0$ satisfies $|z(T)|/|T| \geq \gamma$, then one of the following holds:

(i) *$z(T) \supseteq B(z(x), \lambda 2^n)$, or*

(ii) *there is an n-cube $R \subset T$ such that*

$$|z(R)|/|R| \geq (1 + 2\eta)|z(T)|/|T|$$

We emphasize that in Proposition 2.5.1 the constants Λ and η depend only on the "input" constants λ and γ, as well as the "data" d, s, C_0, L, r_0, M, and the space Y.

We will actually prove the following similar statement, which implies Proposition 2.5.1. (This is analogous to Lemma 4 of [18].)

For $r > 0$, define n_r to be the largest integer n such that

$$10C_0 2^n \leq r. \tag{2.5.1}$$

Proposition 2.5.2. *Let $d \in \mathbb{N}$ and $s > 0$. Suppose (Y, ρ) is LLC, Ahlfors s-regular, and has d-manifold weak tangents. For all C_0, L, r_0 and for all $\gamma > 0$, there exist $\tau, \sigma > 0$ such that the following holds:*

Let X be a complete, oriented, topological d-manifold that is Ahlfors s-regular with constant C_0 and (L, r_0)-LLC. If $v \in X$, $0 < r \leq C_0$, $T = \tilde{B}_{n_r}(v, r)$, and $z \colon T \to Y$ is 1-Lipschitz satisfying $|z(T)|/|T| \geq \gamma$, then one of the following holds:

(i) *$z(T) \supseteq B(z(v), \tau r)$, or*

(ii) *there is a dyadic cube $R \subset T$ of diameter at least τr such that*

$$|z(R)|/|R| \geq (1 + \sigma)|z(T)|/|T|$$

As before, the constants τ and σ in Proposition 2.5.2 depend only on d, s, C_0, L, r_0, and γ, as well as the space Y.

Lemma 2.5.3. *Proposition 2.5.2 implies Proposition 2.5.1.*

Proof. Suppose that Proposition 2.5.2 is true but that Proposition 2.5.1 fails. The failure of Proposition 2.5.1, first of all, implies the existence of dimensions $d \in \mathbb{N}$, $s > 0$, and a space (Y, ρ). It also implies that for some data C_0, L, r_0, M, some constants $\lambda, \gamma > 0$ and every $\Lambda, \eta > 0$, there exists an Ahlfors s-regular, LLC, complete oriented topological d-manifold X (with data given by C_0, L, r_0), a 0-cube $I_0 \subset X$, and

$$T = \tilde{B}_n(x, \Lambda 2^n) \subset I_0,$$

as well as an M-Lipschitz $z \colon T \to Y$ with $|z(T)|/|T| \geq \gamma$ such that

- $z(T) \not\supset B(z(x), \lambda 2^n)$, and

- for every n-cube $R \subset T$,

$$|z(R)|/|R| \leq (1+2\eta)|z(T)|/|T|.$$

In the proof, our goal is to choose Λ large enough and η small enough to reach a contradiction.

We now reduce to the 1-Lipschitz case by letting $\tilde{z}\colon T \to (Y, \frac{1}{M}\rho)$. Then $\tilde{z}\colon T \to (Y, \frac{1}{M}\rho)$ satisfies

- $|\tilde{z}(T)|/|T| \geq \tilde{\gamma} = \gamma/M^s$
- $\tilde{z}(T) \not\supset B_{\frac{1}{M}Y}(z(x), \lambda 2^n/M)$, and
- for every n-cube $R \subset T$,

$$|\tilde{z}(R)|/|R| \leq (1+2\eta)|\tilde{z}(T)|/|T|.$$

Let $T' = \tilde{B}_{n_r}(x,r)$ for $r = \Lambda 2^n/10$. Note that, as $T \subset I_0$, we have $\operatorname{diam} T \leq \operatorname{diam} I_0$ and so $r \leq C_0$.

Note also that $T' \subseteq T$, by a simple triangle inequality argument. On the other hand, as we may choose $\Lambda > C_0$, we have

$$B(x, \Lambda 2^n/10) \subset T' \subset T \subset B(x, (\Lambda + C_0)2^n) \subset B(x, 2\Lambda 2^n),$$

and so the relative measure $|T'|/|T|$ is bounded below by a constant depending only on s and C_0.

If $\Lambda > 200C_0$, then T', and therefore also $T \setminus T'$, is a disjoint union of n-cubes. Indeed, in this case $n_r \geq n$, and T is a disjoint union of n_r-cubes, each of which is a disjoint union of n-cubes.

It follows from the second property of $\tilde{z}$ above that

$$\frac{|\tilde{z}(T')|}{|T'|} \leq (1+2\eta)\frac{|\tilde{z}(T)|}{|T|}$$

and

$$\frac{|\tilde{z}(T \setminus T')|}{|T \setminus T'|} \leq (1+2\eta)\frac{|\tilde{z}(T)|}{|T|}.$$

Therefore

$$\begin{aligned}
|\tilde{z}(T')| &\geq |\tilde{z}(T)| - |\tilde{z}(T \setminus T')| \\
&\geq |\tilde{z}(T)| - (1+2\eta)\frac{|\tilde{z}(T)|}{|T|}|T \setminus T'| \\
&= (|T| - (1+2\eta)|T \setminus T'|)\,\frac{|\tilde{z}(T)|}{|T|} \\
&= ((1+2\eta)|T'| - 2\eta|T|)\,\frac{|\tilde{z}(T)|}{|T|} \\
&\geq ((1+2\eta)|T'| - 2\eta C|T'|)\,\frac{|\tilde{z}(T)|}{|T|} \\
&\geq (1 - C'\eta)|T'|\frac{|\tilde{z}(T)|}{|T|} \\
&\geq \frac{\tilde{\gamma}}{3}|T'|
\end{aligned}$$

if η is small depending on γ. (Here C and C' depend only on the Ahlfors regularity constants s and C_0.)

Now, apply Proposition 2.5.2 to $\tilde{z}\colon T' \to (Y, \frac{1}{M}\rho)$ with γ as $\tilde{\gamma}/3$. We obtain τ and σ. Note that τ and σ depend only on the data d, s, C_0, L, r_0, M, the space Y, and the constant γ.

If $\Lambda > \max\{\frac{10\lambda}{M\tau}, \frac{10C_0}{\tau}\}$ and η is sufficiently small relative to σ, we get that either

- $\tilde{z}(T) \supseteq \tilde{z}(T') \supseteq B_{\frac{1}{M}Y}(\tilde{z}(x), \tau\Lambda 2^n/10) \supset B_{\frac{1}{M}Y}(\tilde{z}(x), \lambda 2^n/M)$, or
- there is a dyadic cube $R \subset T'$ of diameter at least $\tau\Lambda 2^n$ such that

$$|\tilde{z}(R)|/|R| \geq (1+\sigma)|\tilde{z}(T')|/|T'| \geq (1+\sigma)(1-C'\eta)|\tilde{z}(T)|/|T| \geq (1+3\eta)|\tilde{z}(T)|/|T|.$$

In the first case we contradict the assumption that the first conclusion in Proposition 2.5.1 fails. In the second case, note that R is a cube at scale larger than n (because $\tau\Lambda 2^n > C_0 2^n$) and therefore a disjoint union of n-cubes. At least one of those n-cubes R' must then also satisfy

$$|\tilde{z}(R')|/|R'| \geq (1+3\eta)|\tilde{z}(T)|/|T|,$$

which contradicts the assumption that the second conclusion of Proposition 2.5.1 fails. $\square$

2.6 Proof of Proposition 2.5.2

We will use the notation of the previous section; recall especially the definition of n_r from (2.5.1).

Suppose now that Proposition 2.5.2 is false. Then there exists constants $d, s, C_0, L, r_0, \gamma$, and a space (Y, ρ) that is LLC, Ahlfors s-regular and has d-manifold weak tangents, such that the following holds:

For each $k \in \mathbb{N}$, there is a space Z_k that is Ahlfors s-regular with constant C_0 and that is a (L, r_0)-LLC, complete oriented topological d-manifold. In addition, for each k, there is a radius $0 < r_k \leq C_0$, a subset $T_k = \tilde{B}_{n_{r_k}}(v_k, r_k) \subset Z_k$ and a 1-Lipschitz map $z_k \colon Z_k \to Y$ satisfying $|z_k(T_k)|_Y \geq \gamma |T_k|_{Z_k}$ and such that:

(i) $z_k(T_k) \not\supseteq B(z_k(v_k), \frac{1}{k} r_k)$, and

(ii) for every dyadic cube $R \subseteq T_k$ of diameter at least r_k/k, we have

$$\frac{|z_k(R)|}{|R|} \leq \left(1 + \frac{1}{k}\right) \frac{|z_k(T_k)|}{|T_k|}.$$

Let X_k be the metric space

$$\left(Z_k, \frac{1}{r_k} d_{Z_k}\right)$$

Let $S_k \subset X_k$ denote the corresponding rescaled version of T_k. (Of course S_k and T_k are the same set, but we consider them as subsets of different spaces.) Then

$$B(v_k, 1) \subseteq S_k \subseteq B(v_k, 2).$$

and

$$C_0 \leq |S_k|_{X_k} \leq 2^s C_0$$

Note that S_k has a dyadic cube decomposition given by the rescaled versions of cubes in T_k. The following additional technical fact about this decomposition of S_k is obvious but useful.

Lemma 2.6.1. *For every $0 < r < 1/20$ and every $k \in \mathbb{N}$, the set S_k can be written as a disjoint union of measurable sets R_j satisfying*

- $(2C_0^2)^{-1}r \leq \operatorname{diam} R_j \leq r$, *and*
- $(2C_0^2)^{-s}r^s \leq |R_j|_{X_k} \leq r^s$

Proof. Choose n such that

$$C_0 2^n \leq r r_k \leq 2C_0 2^n$$

If $r < 1/20$, then

$$2C_0 2^n \leq 2r r_k < r_k/10 \leq 2C_0 2^{n_{r_k}}$$

and so $n \leq n_{r_k}$.

Therefore, we can write T_k as a disjoint union of dyadic cubes in Δ_n. The rescaled versions of these cubes in S_k are now immediately seen to satisfy the required properties. □

For each k, we also consider the rescaled target spaces

$$Y_k = (Y, \rho_k) = \left(Y, \left(\frac{\gamma |T_k|}{|z_k(T_k)|} \right)^{1/s} \frac{1}{r_k} \rho \right).$$

Let $w_k \colon S_k \to Y_k$ be the map z_k (making the natural identification between points of Z_k and points of X_k). Then each w_k is Lipschitz with constant

$$\left(\frac{\gamma |T_k|}{|z_k(T_k)|} \right)^{1/s} \leq 1.$$

In addition, the maps w_k satisfy

$$|w_k(S_k)| = \gamma |S_k|$$

for all k. (The extra rescaling factor $\left(\frac{\gamma |T_k|}{|z_k(T_k)|} \right)^{1/s}$ in the target Y is to ensure this last convenient fact.)

Finally, the two important properties of z_k pass to w_k in the following way:

$$w_k(S_k) \not\supseteq B(w_k(v_k), \frac{1}{k}) \tag{2.6.1}$$

and for every dyadic cube $R \subseteq S_k$ of diameter at least $1/k$, we have

$$\frac{|w_k(R)|}{|R|} \leq \left(1 + \frac{1}{k}\right) \frac{|w_k(S_k)|}{|S_k|} = \left(1 + \frac{1}{k}\right) \gamma \tag{2.6.2}$$

Let $F_k = \overline{B(v_k, 1/2)} \subset S_k \subset X_k$. We may now consider the following sequence of mapping packages (see Definition 2.3.4):

$$\{((F_k, d_{X_k}, v_k), (Y_k, \rho_k, w_k(v_k)), w_k)\}.$$

Note that all the spaces in the above mapping packages are complete and uniformly doubling, and the mappings w_k are uniformly 1-Lipschitz. By applying Proposition 2.3.6, we obtain a subsequence of this mapping package that converges to a limit

$$\{(F, d, v), (M, d', q)), w)\}.$$

In addition, by Lemma 2.3.9 we may assume that along this subsequence we also have the convergence of the sequence of ambient source spaces (X_k, d_{X_k}, v_k) to a space (X, d, v) that contains F as a subset. (We continue to index this sequence by the original parameter k.)

The following diagram may be useful for keeping track of this convergence. The dotted arrows represent convergence of spaces in the sense of Definition 2.3.3.

$$\begin{tikzcd}
X_k \arrow[d, dotted] & \supset & S_k & \supset & F_k \arrow[r, "w_k"] \arrow[d, dotted] & Y_k \arrow[d, dotted] \\
X & & & \supset & F \arrow[r, "w"] & M
\end{tikzcd} \tag{2.6.3}$$

We now know, by Proposition 2.3.19, that the space X is an LLC, Ahlfors s-regular, homology d-manifold. In addition, by Lemmas 2.3.10 and 2.3.13 and the assumption that Y has d-manifold weak tangents, the space M is an Ahlfors s-regular, LLC, topological d-manifold. Finally, it is clear that the set $F \subset X$ contains the open ball $B(v, 1/2)$.

The space X is a generalized d-manifold (see Definition 2.3.25), so we may now fix an open subset of $B(v, 1/2) \subset F$ which has H_c^d isomorphic to $\mathbb{Z}$, i.e., is itself an oriented generalized d-manifold. We will only work in this oriented subset of X from now on.

Let A be a small open ball in X (of diameter smaller than half the contractibility radius of X) centered at v and compactly contained in this oriented open subset. Because M is a manifold and w is Lipschitz, by making A small enough, we may assume that $w(\overline{A})$ lies in a single chart of M. Let $K = \overline{A}$, which is compact.

We now investigate the limit map w.

Lemma 2.6.2. *The map $w|_K$ is of bounded multiplicity on K. In other words, there exists $N \in \mathbb{N}$ such that for every $x \in M$, there are at most N points in $w^{-1}(x) \cap K$.*

Proof. We will show that there exists N such that for all $r < 1/20$ and every $y \in M$, $w^{-1}(B(y,r)) \cap K$ is contained in the union of N balls of radius r in X. This clearly suffices to prove the lemma. (This essentially shows the stronger statement that w is a David-Semmes regular mapping, as in Definition 2.4.1, but we do not need this here.)

Recall from Propositions 2.3.7 and 2.3.8 that there are "almost-isometries" $\phi_k\colon F \to F_k \subset X_k$ and $\sigma_k\colon Y \to Y_k$, which, on some fixed ball, preserve distances up to an additive error that tends to zero as k approaches infinity. In addition, it follows immediately from those propositions that

$$\lim_{k\to\infty} \rho_k\left(w_k(\phi_k(x)), \sigma_k(w(x))\right) = 0$$

locally uniformly on $F \subset X$.

Fix a ball $B(y,r)$ in M. Let $E = w^{-1}(B(y,r)) \cap K$. Let $E_k = \phi_k(E) \subseteq X_k$. Note that if k is sufficiently large, we have both that $E_k \subset S_k$ and $w_k(E_k) \subset B(\sigma_k(y), 2r)$. By Lemma 2.6.1 we may write S_k as a disjoint union of cubes Q, each satisfying

$$(2C_0^2)^{-1} r \leq \operatorname{diam} Q \leq r$$

and

$$(2C_0^2)^{-d} r^s \leq |Q| \leq r^s.$$

We will call these cubes "r-sized".

Let $\mathcal{Q}$ denote the collection of r-sized cubes in S_k that intersect E_k, and let $N_k = \#\{Q \in \mathcal{Q}\}$. Because w_k is 1-Lipschitz on S_k,

$$w_k(Q) \subset B(\sigma_k(y), (2+2C_0)r) \subset Y_k$$

for all $Q \in \mathcal{Q}$.

Therefore, dividing S_k into those r-sized cubes that are in $\mathcal{Q}$ and those that are not (and taking all Hausdorff measures with respect to X_k and Y_k) we see that

$$\begin{aligned}
\gamma|S_k| = |w_k(S_k)| &\leq \left|\bigcup_{Q\in\mathcal{Q}} w_k(Q)\right| + \left|\bigcup_{Q\notin\mathcal{Q}} w_k(Q)\right| \\
&\leq |B(\sigma_k(y), (2+2C_0)r)| + \gamma(1+1/k) \sum_{Q\subset S_k, Q\in\Delta_{n_r}\setminus\mathcal{Q}} |Q| \\
&= |B(\sigma_k(y), (2+2C_0)r)| + \gamma(1+1/k)\left(|S_k| - \sum_{Q\in\mathcal{Q}} |Q|\right) \\
&\leq C_1 r^s + \gamma(1+1/k)\left(|S_k| - N_k C_2 r^s\right)
\end{aligned}$$

where C_1 depends only on C_0 and the Ahlfors-regularity constant of M, and $C_2 = (2C_0^2)^{-s}$.

Rearranging this inequality yields

$$N_k \leq \frac{C_1 r^s + \frac{1}{k}|S_k|}{\gamma(1+\frac{1}{k})C_2 r^s}.$$

Because the measures $|S_k|$ are uniformly bounded, we see that for all k sufficiently large (depending on r, but that is fine), we have

$$N_k \leq \frac{2C_1}{C_2\gamma}.$$

Since each cube in $\mathcal{Q}$ is contained in a ball of radius $2C_0 r$ in X_k, and each X_k is doubling with constant depending only on C_0 and d, we get that E_k is contained in a union of N balls of radius r, where N depends only on s, C_0 and γ. (This holds for all k sufficiently large.)

It immediately follows that the same holds for E (with a possibly larger N) by using the distance-preserving properties of ψ_k and ϕ_k for k large, and the fact that X is doubling. This proves the lemma. $\square$

Remark 2.6.3. In the proof of Lemma 2.6.2, we used the fact that w is a limit of mappings w_k, each of which does not multiply the measure of cubes of size at least $1/k$ by much more than the factor γ. The proof would be somewhat simpler if we knew that w itself does not expand the measure of any cube by more than a factor γ, because then the computations above could all be carried out in the limit $w\colon X \to M$, rather than in the limiting objects $w_k\colon X_k \to Y_k$. Unfortunately, it is not clear that this "non-expanding" property of the maps w_k passes directly to the limit map w. The same issue arises in Lemma 2.6.4 below.

Note now that the set K is a compact set that is the closure of an open set in the homology d-manifold X. It follows that every relatively open subset of K contains an open subset of X and thus has topological dimension at least d (see [31], Remark 1.3(b)). Recall our assumption that $w(K)$ lies in a single chart of M. As w has bounded multiplicity on K, we can apply Theorem 2.3.29 to obtain a dense open subset V in $w(K)$ such that $U = w^{-1}(V) \cap K$ is dense in K and $w|_U$ is a covering map.

Lemma 2.6.4. *Every point in V has exactly one preimage in K under w.*

Proof. In other words, what we must show is that if $x \in U$ and $x' \in K$ with $x' \neq x$, then $w(x) \neq w(x')$. Suppose to the contrary that $w(x) = w(x') = y \in V$. As $x \in U$ and w is a covering map when restricted to U, we obtain a ball $B(x,r) \subset U$ such that $w|_{B(x,r)}$ is a homeomorphism and $w(B(x,r))$ contains a ball $B(y,r') \subset M$. Without loss of generality, we may take $r < d(x,x')/10C_0$ and $r < 1/20$.

Recall the the continuous "almost isometries" $f_k\colon K \to X$ from Lemma 2.3.15. By Lemma 2.3.24, for all k sufficiently large, we obtain $x_k = f_k(x) \in S_k$ such that $w_k(B(x_k, 2r))$ contains the ball $B(y_k, r'/2) \subset Y_k$, where $y_k = w_k(x_k)$. Also let $x'_k = f_k(x')$. For all k large, we have $\rho_k(w_k(x_k), w_k(x'_k)) < r/10$, because $w(x) = w(x')$.

Let $r_1 = \min\{r, r'\}$. By Lemma 2.6.1, we may write S_k as the disjoint union of sets Q such that

$$(2C_0^2)^{-1} r_1/10 \leq \operatorname{diam} Q \leq r_1/10$$

and

$$(2C_0^2)^{-d}(r_1/10)^s \leq |Q| \leq (r_1/10)^s.$$

One of these sets Q contains the point x_k'; let Q_0 denote that set. In addition, let T be the union of all these sets Q that intersect $B(x_k, 2r)$. Note that Q_0 is not in T by our choice of r and r_1. Then

$$w_k(Q_0) \subset B(y_k, r'/2) \subset w_k(T).$$

We now sum over all the sets Q in S_k as above that are not Q_0. Because $w_k(Q_0) \subseteq \bigcup_{Q \neq Q_0} w_k(Q)$, we have that

$$\begin{aligned} \gamma |S_k| = |w_k(S_k)| &\leq \sum_{Q \neq Q_0} |w_k(Q)| \\ &\leq \gamma(1 + 1/k) \sum_{Q \neq Q_0} |Q| \\ &\leq \gamma(1 + 1/k)(|S_k| - C_3 r_1^s) \end{aligned}$$

where $C_3 = (2C_0^2)^{-s}$.

Rearranging and recalling that $|S_k| \leq C_0 2^s = C_4$, we get

$$\gamma C_3 r_1^s \leq \frac{\gamma}{k}(C_4 - C_3 r_1^s),$$

which is a contradiction for k large. □

Lemma 2.6.5. *The map $w|_A \colon A \to M$ is an open mapping.*

Proof. We use the notion of local degree defined in Subsection 2.3.4, which we may apply to the oriented generalized d-manifold containing A.

Suppose w is not an open mapping on A. Then there is a point $x \in A$ and an open set $G \subseteq A$ containing x such that $y = w(x)$ is not an interior point of $w(G)$. Since w has bounded multiplicity, we can find a closed ball in G containing x and no other pre-images of y. Let B be a connected open subset of this ball containing x. Then $\overline{B} \cap w^{-1}(y) = \{x\}$.

We now claim that the local degree $\mu(y, B, w)$ is 0. Suppose to the contrary that $\mu(y, B, w) \neq 0$. Choose a small connected neighborhood N of y that does not intersect the compact set $w(\partial B)$. Then $\mu(y', B, w) \neq 0$ for all $y' \in N$. It follows (by Lemma 2.3.27) that $N \subseteq w(B)$, which contradicts our assumption that $y \notin \operatorname{int}(w(G))$. So $\mu(y, B, w) = 0$.

On the other hand, we can choose $x' \in B \cap U$ so that $y' = w(x') \in V$ is arbitrarily close to y. By Lemma 2.6.4, x' is the only pre-image of y'. As before, choose a small connected neighborhood $B' \subset B$ around x' so that $w|_{B'}$ is a homeomorphism and $\partial B'$ avoids the (finitely many) pre-images of y. Remember that B' contains the only pre-image x' of y' in B. It follows from Lemma 2.3.27 that

$$\mu(y', B, w) = \mu(y', B', w) = \pm 1.$$

Now, if y' is sufficiently close to y, then y' is in the same connected component of $M \setminus w(\partial B)$ as y. Because the local degree is locally constant (Lemma 2.3.27), we see that

$$\mu(y, B, w) = \mu(y', B, w).$$

But the left-hand side is 0 while the right-hand side is not. This completes the proof that w is an open mapping. $\square$

From the previous two lemmas it immediately follows that w is a homeomorphism on A. Indeed, we only need show it is injective. Suppose $w(x) = w(x')$. Choose small disjoint balls B and B' containing x and x', respectively. Then $w(B) \cap w(B')$ is an open set in $w(A)$ and therefore contains a point of V. This contradicts Lemma 2.6.4.

Because w is a homeomorphism, there are radii $r, r' > 0$ such that

$$w(B(v, r)) \supseteq B(w(v), r').$$

It follows by Lemma 2.3.24 and Lemma 2.3.15 that for all k sufficiently large,

$$w_k(S_k) \supseteq w_k(B(f_k(v), 2r)) \supseteq B(w_k(f_k(v)), r'/2) \supseteq B(w_k(v_k), r'/3).$$

This contradicts property (2.6.1) of w_k if k is large enough.

This completes the proof of Proposition 2.5.2 and thus of Theorem 1.2.2.

2.7 Proof of Theorem 1.2.3

Let X be an Ahlfors d-regular, LLC, oriented topological d-manifold. (We re-emphasize the fact that here the Ahlfors regularity dimension and the topological dimension of X must coincide.) We will apply Theorem 1.2.2 (in the case $Y = \mathbb{R}^d$) to a class of maps on X provided by a theorem of Semmes. These are given in the following result, which is a slightly weakened version of Theorem 1.29(a) of [57].

Theorem 2.7.1 ([57], Theorem 1.29(a))**.** *Let B be an open ball in X of radius $r > 0$. Then there is a surjective Lipschitz map f from X onto the standard d-dimensional unit sphere $\mathbb{S}^d$ with Lipschitz constant $\leq Cr^{-1}$ that is constant on $X \setminus B$. The constant C depends only on the data of X.*

Remark 2.7.2. In Theorem 2.7.1, it makes no difference whether one endows $\mathbb{S}^d$ with the standard Riemannian metric of diameter π or with the "chordal" metric arising from writing $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ and letting $d(x, y) = |x - y|$. These metrics are bi-Lipschitz equivalent. For convenience, we will use the latter.

Proof of Theorem 1.2.3. As above, write $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$. Consider the projection p from $\mathbb{S}^d$ onto the first d coordinates in $\mathbb{R}^{d+1}$. Then p is 1-Lipschitz and $|p(\mathbb{S}^d)| = \sigma_d$, the d-dimensional Hausdorff measure of the unit ball in $\mathbb{R}^d$.

Therefore, by post-composing the maps of Theorem 2.7.1 with p, we see that for every ball $B(x, r) \subseteq X$ there is a Cr^{-1}-Lipschitz map $g_B \colon B \to \mathbb{R}^d$ with $|g_B(B)| = \sigma_d$.

To show X is locally uniformly rectifiable, we must show that for all $R > 0$ there exists constants α, β such that for every ball B of radius at most R, there is a set $E \subseteq B$ and a map $f \colon E \to \mathbb{R}^d$ such that $|E| \geq \beta |B|$ and f is α-bi-Lipschitz.

Fix a ball $B = B(x, r)$, where $r < R$. Let n be such that $C_0 2^n < r \leq C_0 2^{n+1}$. Then B contains a dyadic cube $Q \in \Delta_n$.

As $c_0 2^n \geq \frac{c_0}{2C_0} r$, Q contains a ball B' of radius $\frac{c_0}{2C_0} r$. Let $g = g_{B'}$ be a map as above associated to B'. Then g is Lipschitz with Lipschitz constant bounded by $\frac{2CC_0}{c_0 r}$.

Therefore, the map $h = \frac{c_0 r}{2CC_0} g$ is 1-Lipschitz and $|h(B')| \geq c_5 r^d$, for $c_5 = \sigma_d(c_0/2CC_0)^d$.

Thus, $|h(Q)| \geq \delta|Q|$ for some constant δ depending only on the data of X. By choosing $\epsilon > 0$ sufficiently small in Theorem 1.2.2 (see Remark 2.2.4) we get that h is α-bi-Lipschitz on a set $E \subset Q \subset B$ of measure at least $\theta|Q| \geq \beta|B|$, where α and β depend only on R and the data of X. This proves Theorem 1.2.3. □

2.8 Consequences of Theorem 1.2.3

It is now possible to derive many corollaries which result immediately from applying deep theorems of David and Semmes on uniformly rectifiable sets to the conclusion of Theorem 1.2.3. We state two geometric examples below.

First of all, Theorem 1.2.3, in combination with a result of Semmes in [55], provides a quasisymmetric embedding result for suitable compact metric manifolds. For the definition and basic properties of quasisymmetric homeomorphisms, see [28].

Corollary 2.8.1. *Let X be an Ahlfors d-regular, LLC, compact, oriented topological d-manifold. Then X is quasisymmetrically equivalent to a space X' that is also an Ahlfors d-regular, LLC, compact, oriented topological d-manifold and that is a subset of some $\mathbb{R}^N$.*

Proof. By Theorem 1.2.3, the space X is uniformly rectifiable. Proposition 2.10 of [55], combined with equation (3.27) in that paper, shows that X can be quasisymmetrically deformed by a weight so that the resulting space admits a bi-Lipschitz embedding into some $\mathbb{R}^N$.

Both the deformation and the bi-Lipschitz embedding preserve the Ahlfors s-regularity of X. For the former, this is explained in the discussion following the proof of Lemma 4.4 in [55]; the latter is a general fact about bi-Lipschitz mappings.

Thus, if we let X' be the image of the deformed X under the bi-Lipschitz embedding, then X' is Ahlfors s-regular. Because it is quasisymmetrically homeomorphic to X, it is also a compact, LLC, oriented topological d-manifold. □

Remark 2.8.2. Every doubling metric space quasisymmetrically embeds in some Euclidean space by Assouad's theorem (see [28], Theorem 12.2), but in general this embedding first "snowflakes" the metric, increasing the Hausdorff dimension and destroying the rectifiability properties of the space. Corollary 2.8.1 is false if one replaces "quasisymmetrically" by "bi-Lipschitz", as examples of Semmes [54] and Laakso [42] show.

Once there is a nice embedding of the abstract metric space X as a uniformly rectifiable subset of Euclidean space, all the theory of these sets developed by David and Semmes can be applied. Here we merely mention one further example, which says that the image of the embedding in Corollary 2.8.1 can be taken to lie in a particularly nice subset of $\mathbb{R}^N$.

Recall the definition of David-Semmes regular maps, introduced in Definition 2.4.1. We define the following class of subsets of Euclidean space.

Definition 2.8.3. Let E be an Ahlfors d-regular subset of $\mathbb{R}^n$. We say that E is *quasisymmetrically d-regular* if $E = g(f(\mathbb{R}^d))$, where $f\colon \mathbb{R}^d \to Y$ is a quasisymmetric homeomorphism of $\mathbb{R}^d$ onto an Ahlfors d-regular space Y, and $g\colon Y \to \mathbb{R}^N$ is a David-Semmes regular mapping.

Quasisymmetrically d-regular sets admit bounded-multiplicity parametrizations by $\mathbb{R}^d$ in a controlled way.

The following corollary follows from a weakened version of the implication (C6)⇒(C7) in the main result of [20]. (The full version of the result should discuss deformations by A_1-weights, which we have not mentioned.)

Corollary 2.8.4. *Let X be an Ahlfors d-regular, LLC, compact, oriented topological d-manifold. Let X' be a quasisymmetrically equivalent subset of $\mathbb{R}^N$ provided by Corollary 2.8.1. Assume $N \geq 2d$. Then X' is a contained in a quasisymmetrically d-regular set $E \subset \mathbb{R}^N$.*

Proof. This follows from Corollary 2.8.1, Theorem 1.2.3, and the main result of [20] (specifically, the implication (C6)⇒(C7)). □

In general, it is not possible to find good (quasisymmetric or bi-Lipschitz) parametrizations of metric spaces such as those in Corollary 2.8.4 by standard spaces such as $\mathbb{S}^d$ or $\mathbb{R}^d$. Corollary 2.8.4 provides a weaker form of "parametrization", in that it yields a mapping onto but not into the space, and that is bounded-multiplicity rather than injective.

2.9 Counterexamples

To conclude, we wish to briefly describe some counterexamples regarding the class of "Lipschitz implies bi-Lipschitz" theorems discussed in Section 1.2 of this thesis. By this we mean the class of theorems that say that if $f\colon X \to Y$ is a Lipschitz mapping with positive-measure image, then f is bi-Lipschitz on a set of positive measure, quantitatively. None of these counterexamples are new, but they are scattered in a few different places in the literature and it may be convenient to collect them in one place. The first two can be found in Meyerson's paper [48], the third is due to David and Semmes [21], and the fourth is an example of Laakso [40].

The first counterexample shows that, in the setting of Theorem 1.2.2, the requirement that the two spaces have the same topological dimension is necessary. This proposition is proven by Meyerson in [48], Theorem 4.1. Here we give a slightly different argument.

Proposition 2.9.1. *There is an Ahlfors* 2*-regular, linearly locally contractible, complete oriented topological* 1*-manifold X and a Lipschitz map $f\colon X \to \mathbb{R}^2$ with positive measure image that is not bi-Lipschitz on any subset of positive measure.*

Proof. The metric space X will be the "snowflaked" space $(\mathbb{R}, |\cdot|^{1/2})$, equipped with two-dimensional Hausdorff measure (which is the same as one-dimensional Hausdorff measure on $(\mathbb{R}, |\cdot|)$). It is clear that X satisfies all the required properties.

It is well-known (see, e.g., [59], Theorem 7.3.1) that there is a space-filling curve $f\colon (\mathbb{R}, |\cdot|) \to \mathbb{R}^2$ that is Hölder continuous with exponent $1/2$ and whose image contains the unit square in $\mathbb{R}^2$. Therefore, when considered as a mappping $f\colon X \to \mathbb{R}^2$, f is Lipschitz, and it has positive-measure image.

However, no Lipschitz map from X to $\mathbb{R}^2$ can be bi-Lipschitz on a set of positive measure. Indeed, suppose that f is bi-Lipschitz on a set of positive measure E in X, with $f(0) = 0$. Let $E' = f(E) \subseteq \mathbb{R}^2$. Without loss of generality, we may assume that E is compact, that $0 \in \mathbb{R}$ is a point of density of E in X, and that $f(0) = 0 \in \mathbb{R}^2$ is a point of density of E' in $\mathbb{R}^2$. (We can always find such points.)

We now consider the sequences of mapping packages

$$\left\{\left((E, \frac{1}{n}d_X, 0), (E', \frac{1}{n}|\cdot|, 0), f)\right)\right\}. \tag{2.9.1}$$

Because $0 \in X$ is a point of density of E and $0 \in \mathbb{R}^2$ is a point of density of E', we have by [21], Lemmas 9.12 and 9.13, that, in the sense of pointed metric spaces,

$$\left(E, \frac{1}{n}d_X, 0\right) \to (X, d_X, 0)$$

and

$$\left(E', \frac{1}{n}|\cdot|, 0\right) \to \left(\mathbb{R}^2, |\cdot|, 0\right).$$

Therefore, some subsequence of the sequence of mapping packages in (2.9.1) converges to a mapping package

$$\left((X, d_X, 0), (\mathbb{R}^2, |\cdot|, 0), g\right).$$

The mapping g is bi-Lipschitz, because $f|_E$ is bi-Lipschitz. In addition, the map g is surjective. We may see this by passing to another subsequence along which the sequence of inverse mapping packages

$$\left\{\left((E', \frac{1}{n}|\cdot|, 0), (E, \frac{1}{n}d_X, 0), (f|_E)^{-1}\right)\right\}$$

converges to a mapping package

$$\left((\mathbb{R}^2, |\cdot|, 0), (X, d_X, 0), h\right).$$

It is then easy to see that $g(h(y)) = y$ for all $y \in \mathbb{R}^2$ and therefore that g is surjective.

So g is a bi-Lipschitz homeomorphism of X onto $\mathbb{R}^2$. But this is impossible, as X is homeomorphic to $\mathbb{R}$. $\square$

The two spaces in Proposition 2.9.1 satisfy all the conditions of Theorem 1.2.2, except that they are manifolds of different topological dimensions.

For the remaining three counterexamples that we mention here, we merely indicate the statements and refer the reader to the original sources for the proofs.

The second example is Theorem 4.2 of Meyerson's paper [48]. Let us first note that, as a consequence of Theorem 1.2.2, we know the following: Let X and Y be spaces as in Theorem 1.2.2. Let $U \subset X$ be an open set, and let $f\colon U \to Y$ be Lipschitz and satisfy $|f(U)| > 0$. Then there is a countable collection of measurable sets $E_i \subset U$ such that $f|_{E_i}$ is bi-Lipschitz for each i and $|f(U \setminus \cup E_i)| = 0$. (Here the sets E_i are not necessarily disjoint.) On the other hand, we have the following fact:

Proposition 2.9.2 ([48], Theorem 4.2)**.** *There is a doubling, LLC, complete, oriented topological* 2*-manifold* X *of Hausdorff dimension* 2*, an open set* $U \subset X$*, and a Lipschitz map* $f\colon U \to \mathbb{R}^2$ *that cannot be represented in the above manner. In other words, there is no countable collection of measurable sets* $E_i \subset U$ *such that* $f|_{E_i}$ *is bi-Lipschitz for each* i *and* $|f(U \setminus \cup E_i)| = 0$*.*

In particular, the conclusion of Theorem 1.2.2 does not hold for this choice of X and $Y = \mathbb{R}^2$. In this result, the space X can be chosen to be the sub-Riemannian manifold known as the Grushin plane. The source and target spaces in Proposition 2.9.2 satisfy all the conditions of Theorem 1.2.2, except that the source X is not Ahlfors 2-regular. The idea behind Proposition 2.9.2 is to reduce to Proposition 2.9.1, because the Grushin plane X contains a bi-Lipschitz equivalent copy of the snowflaked line $(\mathbb{R}, |\cdot|^{1/2})$ as a positive-measure subset.

If one completely relaxes the strong topological conditions imposed in Theorem 1.2.2, then one can find Lipschitz mappings between metric spaces with large images but no bi-Lipschitz pieces, even in the presence of very strong analytic assumptions on the spaces and mappings.

Proposition 2.9.3 ([21], Proposition 14.5)**.** *There is a compact, Ahlfors regular metric space*

X and a Lipschitz mapping $f\colon X \to X$ which is not bi-Lipschitz on any positive-measure subset. Furthermore, the mapping f can be taken to be a homeomorphism which is in addition David-Semmes regular and preserves measure, in the sense that $|f(K)| = |K|$ for all compact $K \subseteq X$.

The space X in Proposition 2.9.3 is a totally disconnected Cantor set. See Chapter 14 of [21] for the proof and some other related constructions.

In both the positive result Theorem 1.2.2 and the counterexample Propositions 2.9.1 and 2.9.3, the spaces in question may have no "good calculus", i.e., they may have no rectifiable curves and therefore no Poincaré inequality. (For the definition of Poincaré inequalities on metric measure spaces, see [28].) It is not known to what extent this type of calculus is helpful in proving "Lipschitz implies bi-Lipschitz" theorems, but in closing we wish to note the following theorem of Laakso [40], which shows that Ahlfors regular spaces with Poincaré inequalities may still fail to have such results.

Proposition 2.9.4 ([40]). *There exists an Ahlfors regular space X admitting a Poincaré inequality and a Lipschitz map $f\colon X \to X$ with positive-measure image such that there is no positive-measure subset of X on which f is bi-Lipschitz.*

In fact, in Laakso's example the mapping f does not even have any bi-Lipschitz weak tangents, in the sense of Section 2.4.

CHAPTER 3

Lipschitz differentiability spaces

As discussed in Section 1.3, the two main results in this chapter are Theorems 1.3.9 and 1.3.12. They present some consequences of the relationship between the Ahlfors regularity dimension of a Lipschitz differentiability space and the dimensions of the charts in its differentiable structure. If these coincide, the chart has uniformly rectifiable tangents, and if they differ, the chart is strongly unrectifiable.

Along the way, we will prove some general results about doubling Lipschitz differentiability spaces. The author's preprint [23] is based on the material in this chapter.

3.1 Outline of the proof of Theorem 1.3.9

Here we give a brief summary of the proof of Theorem 1.3.9. The starting point is a result of Bate, Theorem 1.3.8, that says that in a Lipschitz differentiability space, a generic point of a chart $(U, \phi\colon U \to \mathbb{R}^n)$ admits n distinct "broken curves" through it, along which ϕ is differentiable with derivatives pointing in n independent directions.

By modifying an idea of [43] — that a tangent remains a tangent after a change of base point — we upgrade this to a special property of the tangents (Y, y) of (X, x). Namely, such a tangent admits a Lipschitz map $G\colon Y \to \mathbb{R}^n$ (which comes from blowing up ϕ) such that *every* $z \in Y$ admits n bi-Lipschitz *lines* through it, pointing in "independent" directions, on which G is linear. (This is weaker than but similar to Cheeger's notion of a "generalized linear" function; see [13], Section 8.)

By a simple argument, such a map must be a Lipschitz quotient map, i.e., $G(B(x,r)) \supseteq$

$B(G(x), cr)$ for some constant $c > 0$ and every ball $B(x, r)$. We can then appeal to Semmes' Theorem 2.2.2 (or David's theorem from [18]), which implies that such a map from an Ahlfors n-regular metric space to $\mathbb{R}^n$ must be bi-Lipschitz on a large subset of every ball. This yields uniform rectifiability of the tangent.

To obtain a single tangent that is bi-Lipschitz equivalent to $\mathbb{R}^n$, we can take a further tangent at a point of density of such a subset. As tangents of tangents are tangents (see the principles introduced in 1.1.2), this yields a bi-Lipschitz map from a tangent of X onto $\mathbb{R}^n$.

Remark 3.1.1. We do not know, though it is a natural conjecture, whether Theorem 1.3.9 can be strengthened to show that an n-dimensional chart U in an Ahlfors n-regular Lipschitz differentiability space is itself n-rectifiable. It is possible to show that U is n-rectifiable if it admits a bi-Lipschitz embedding into some Euclidean space (see Corollary 3.8.1 below).

We now present the details. In Section 3.2 we define the version of Gromov-Hausdorff convergence used in this chapter, along with a variant which includes converging Lipschitz functions as well as spaces. In Section 3.3 we extend a result of Le Donne about tangent spaces to this setting. Sections 3.4 and 3.5 contain the proof that, in doubling Lipschitz differentiability spaces, blow-ups of the coordinate mappings are Lipschitz quotient maps. Sections 3.6 and 3.7 contain the proofs of Theorem 1.3.9 and Theorem 1.3.12, respectively. Finally, in Section 3.8 we present some further corollaries: non-embedding results analogous to those for PI spaces, a sharp dimension bound for differentiable structures, and a rigidity result for Lipschitz differentiability spaces admitting quasi-Möbius symmetries, in the spirit of Bonk-Kleiner [6].

3.2 Gromov-Hausdorff convergence of space-functions

We will denote metric spaces by pairs (X, d) and metric measure spaces by triples (X, d, μ). When the metric (and measure) are understood from context we will denote such a space simply by X. Our metric spaces are not necessarily assumed to be complete unless explicitly specified. Our measures μ will always be Borel regular measures, but they also are not necessarily assumed to be complete measures.

We will now define another type Gromov-Hausdorff convergence, first for sequences of metric spaces and then for pairs consisting of a metric space and a Lipschitz function. This version does not differ materially from that used Chapter 2 or in, for example, [11] or [39], but it is more convenient for our present purposes. The following preliminary definition will be useful.

Definition 3.2.1. A map $\phi\colon (X, d, x) \to (Y, d', y)$ between pointed metric spaces is called an *ϵ-isometry* if

(i) For all $a, b \in B_X(x, 1/\epsilon)$, we have $|d'(\phi(a), \phi(b)) - d(a, b)| < \epsilon$, and

(ii) for all $\epsilon \leq r \leq 1/\epsilon$, we have $N_\epsilon(\phi(B_X(x, r))) \supseteq B_Y(y, r - \epsilon)$.

Here $N_\epsilon(E)$ denotes the open ϵ-neighborhood of a subset E in a metric space Y. Note that we do not ask that $\phi(x) = y$, although it follows from the definition that $d'(\phi(x), y) \leq 2\epsilon$.

A sequence $\{(X_i, x_i)\}$, $i \in \mathbb{N}$, of pointed metric spaces converges to a metric space (X, x) in the pointed Gromov-Hausdorff sense if for all $\epsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that, for all $i > i_0$, there are ϵ-isometries

$$\phi_i : (X_i, x_i) \to (X, x) \text{ and } \psi_i : (X, x) \to (X_i, x_i).$$

If a sequence of pointed metric spaces is uniformly doubling, then it has a subsequence that converges in the pointed Gromov-Hausdorff sense (see, e.g., [11], Theorem 8.1.0). This notion of convergence can be associated to a distance function, as we indicate below.

Slightly modifying a definition of Keith [35], we will call a (X, x, f) a *space-function* if (X, x) is a pointed metric space and $f\colon X \to \mathbb{R}^n$ is a Lipschitz function, for some $n \in \mathbb{N}$ that will be clear from context. Note that, unlike in [35], the functions f in our space-functions are always Lipschitz, and they are allowed to map into $\mathbb{R}^n$ rather than $\mathbb{R}$. As an abuse of notation, we will call a space-function "doubling", "complete", etc. if the underlying space is doubling or complete, and we will call it L-Lipschitz if the function f is L-Lipschitz.

The notion of Gromov-Hausdorff convergence can be extended to space-functions as, for example, in [35] and [39]. We present a version of this here.

Definition 3.2.2. If $(X, x, f\colon X \to \mathbb{R}^n)$ and $(Y, y, g\colon Y \to \mathbb{R}^n)$ are space-functions, we define

$$\tilde{D}((X,d,x,f),(Y,d',y,g)) = \inf\Big\{\epsilon > 0 : \text{there exist } \phi : (X,d,x) \to (Y,d',y) \text{ and } \psi : (Y,d',y) \to (X,d,x) \text{ that are } \epsilon\text{-isometries, and such that } \sup_{B(x,1/\epsilon)} |f - g\circ\phi| < \epsilon \text{ and } \sup_{B(y,1/\epsilon)} |g - f\circ\psi| < \epsilon\Big\}$$

Lemma 3.2.3. *If we define $D = \min\{\tilde{D}, 1/2\}$, then $\tilde{D}$ is a "pseudo-quasi-metric", by which we mean the following:*

(i) *D is finite, non-negative, and symmetric.*

(ii) *The D-distance between two doubling space-functions (X, x, f) and (Y, y, g) is zero if and only if there is a surjective isometry $i\colon \overline{X} \to \overline{Y}$ such that $g \circ i = f$, where g and f are identified with their extensions to the completions $\overline{X}$ and $\overline{Y}$.*

(iii) *D satisfies the quasi-triangle inequality*

$$D\left((X,x,f),(Z,z,h)\right) \leq 2\left(D\left((X,x,f),(Y,y,g)\right) + D\left((Y,y,g),(Z,z,h)\right)\right).$$

Proof. It is clear from the definition that $\tilde{D}$, and therefore D, is finite, non-negative, and symmetric, and so (i) holds.

If $D((X,x,f),(Y,y,g)) = 0$ then there $1/i$-isometries $\phi_i\colon (X,x) \to (Y,y)$ such that

$$\sup_{B(y,i)} |g - f\circ\phi_i| < 1/i.$$

We can extend ϕ_i to a map from $\overline{X}$ to $\overline{Y}$ as a $2/i$-isometry. Because X and Y are doubling, $\overline{X}$ and $\overline{Y}$ are proper: closed balls are compact. Therefore the maps ϕ_i sub-converge uniformly on compact sets to an isometry from X to Y satisfying the conditions of the lemma.

Conversely, if such an isometry exists, then it is clear that $D((X,x,f),(Y,y,g)) = 0$. Therefore (ii) holds.

The quasi-triangle inequality (iii) for D follows from the fact that

$$(2(\epsilon+\delta))^{-1} \leq \min\{\epsilon^{-1} - 2\delta, \delta^{-1} - 2\epsilon\}$$

if $0 < \epsilon, \delta < 1/2$. Indeed, this is inequality exactly what is needed to show that the composition of an ϵ-isometry and a δ-isometry is a $2(\epsilon+\delta)$-isometry. $\square$

Although the function D is not a metric, the previous lemma says that it is similar enough for our application. We will therefore say that a sequence of space-functions (X_n, x_n, f_n) "converges in D" to a space-function (X, x, f) if

$$D((X_n, x_n, f_n), (X, x, f)) \to 0 \text{ as } n \to \infty.$$

The convergence in D of a sequence of space-functions implies that the pointed metric spaces converge in the pointed Gromov-Hausdorff sense (as defined above). Conversely, by a standard Arzéla-Ascoli type argument, if (X_n, x_n, f_n) are C-doubling, L-Lipschitz space functions with $\{f_n(x_n)\}$ bounded, and if $(X_n, d_n, x_n) \to (X, d, x)$ in the pointed Gromov-Hausdorff sense, then there is a subsequence $\{(X_{n_k}, x_{n_k}, f_{n_k})\}$ and a Lipschitz function $f \colon X \to \mathbb{R}$ such that

$$(X_{n_k}, x_{n_k}, f_{n_k}) \to (X, x, f)$$

in D.

If (X, x) is a pointed metric space, and $f \colon X \to \mathbb{R}^n$ is Lipschitz, then we denote by $\mathrm{Tan}(X, x, f)$ the collection of space functions (Y, y, g) such that Y is complete and

$$\left(\frac{1}{\lambda_i}X, x, \frac{1}{\lambda_i}(f - f(x))\right) \to (Y, y, g)$$

for some sequence of positive real numbers λ_i converging to zero. This is the collection of tangents of X at x. If X is doubling and f is Lipschitz, then $\mathrm{Tan}(X, x, f)$ is always non-empty, by the above standard facts about Gromov-Hausdorff convergence.

Lemma 3.2.4. *The following properties are preserved under Gromov-Hausdorff convergence of a sequence of space functions $\{(X_i, x_i, f_i)\} \to (X, x, f)$:*

- *If the functions f_i are all L-Lipschitz, then so is f.*
- *If the functions f_i are all L-bi-Lipschitz, then so is f.*
- *If the spaces X_i are uniformly doubling metric spaces, then X is doubling.*
- *If the spaces X_i are uniformly Ahlfors n-regular, then X is Ahlfors n-regular.*

Proof. The first three of these properties are easy to check, and the fourth can be found in, e.g., Lemma 8.29 of [21]. □

The following lemma about convergence of space-functions will be useful.

Lemma 3.2.5. *Suppose that (X, x, f) and (Y, y, g) are Lipschitz space-functions (mapping into Euclidean space of the same dimension). Suppose that $\phi\colon X \to Y$ is an ϵ-isometry such that $\phi(x) = y$ and*

$$\sup_{B(x,1/\epsilon)} |f - g \circ \phi| < \epsilon < 1.$$

Then

$$D\left((X, x, f), (Y, y, g)\right) < C\epsilon,$$

where C depends only on the Lipschitz constants of f and g.

Proof. For simplicity, we denote the metrics on X and Y both by d. Let $N \subset B(x, 1/\epsilon)$ be a maximal separated ϵ-net. In other words,

$$d(y, z) \geq \epsilon$$

if $y, z \in N$ and $y \neq z$, and

$$\mathrm{dist}(z, N) < \epsilon$$

for all $z \in B(x, 1/\epsilon)$. We can also arrange that $x \in N$.

The fact that ϕ is an ϵ-isometry implies that $\phi|_N$ is injective. Let $N' = \phi(N) \subset Y$. Because ϕ is an ϵ-isometry, we know that every point of $B(y, 1/2\epsilon)$ is within 3ϵ of a point in N'.

Let $\pi\colon Y \to N'$ denote any choice of closest-point projection, i.e., $\pi(Y) \subset N'$ and $d(y, \pi(y)) = \mathrm{dist}(y, N')$. Then π preserves distances up to an additive error of 6ϵ for points in $B(y, 1/2\epsilon)$. Let

$$\psi = (\phi|_N)^{-1} \circ \pi : Y \to X.$$

We first claim that ψ is a 7ϵ-isometry. Fix $y_1, y_2 \in B(y, 1/7\epsilon)$. We have

$$\begin{aligned}
|d(\psi(y_1), \psi(y_2)) - d(y_1, y_2)| &\leq |d(\phi^{-1}(\pi(y_1)), \phi^{-1}(\pi(y_2))) - d(\pi(y_1), \pi(y_2))| \\
&\quad + |d(\pi(y_1), \pi(y_2)) - d(y_1, y_2)| \\
&\leq \epsilon + 6\epsilon \\
&= 7\epsilon.
\end{aligned}$$

In addition, for $r \leq 1/(7\epsilon)$,

$$\psi(B(y, r)) \supseteq N \cap B(x, r - \epsilon)$$

and therefore

$$N_{7\epsilon}(\psi(B(y, r))) \supseteq B(x, r - 7\epsilon).$$

We now claim that

$$\sup_{B(y,1/7\epsilon)} |g - f \circ \psi| < C\epsilon,$$

where C depends only on the Lipschitz constant of g. For $z \in B(y, 1/7\epsilon)$, we have

$$\begin{aligned}
|g(z) - f(\psi(z))| &= |g(z) - f((\phi|_N)^{-1}(\pi(z)))| \\
&\leq |g(z) - g(\pi(z))| + |g(\pi(z)) - f((\phi|_N)^{-1}(\pi(z)))| \\
&\leq 6\epsilon \mathrm{LIP}(g) + \epsilon.
\end{aligned}$$

This completes the proof. $\square$

At this point, we remark that all spaces in this chapter are doubling and therefore separable, so they admit isometric embeddings into the Banach space $\ell^\infty(\mathbb{N})$. Thus "the set of all doubling metric spaces up to isometry" can be identified with a subset of the power set of $\ell^\infty(\mathbb{N})$, and so there are no set-theoretic difficulties with this object.

Though D is not a metric, we nonetheless let the *D-diameter* of a collection $\mathcal{C}$ of space functions be

$$\operatorname{diam}_D \mathcal{C} = \sup\{D((X,x,f),(Y,y,g)) : (X,x,f),(Y,y,g) \in \mathcal{C}\}.$$

Lemma 3.2.6. *Let $\mathcal{M}$ be a collection of doubling, L-Lipschitz space-functions (mapping into the same $\mathbb{R}^n$). Then for any $\eta > 0$, $\mathcal{M}$ is contained in a countable union of sets B_l, $l \in \mathbb{N}$, of D-diameter at most η.*

Proof. We consider the countable collection of all space-functions (X,x,f) such that

- X is finite and all distances between points of X are rational, and
- f takes values in $\mathbb{Q}^n \subset \mathbb{R}^n$.

Given $(Y,y,g) \in \mathcal{M}$, we will show that it is within D-distance $\eta/4$ of such a space-function. The quasi-triangle inequality for D (Lemma 3.2.3) then concludes the proof.

Let $\delta > 0$ be a small constant to be chosen later, depending only on η and L. Let N denote a finite maximal δ-net in $B(y,1/\eta) \subseteq Y$, which we assume contains y. The set N is finite because Y is doubling. We consider N as a metric space equipped with the restriction of the metric from Y.

By Kuratowski's theorem ([28], p. 99), N isometrically embeds into $(\mathbb{R}^m, ||\cdot||_{\ell^\infty})$ for some $m \in \mathbb{N}$. Here $(\mathbb{R}^m, ||\cdot||_{\ell^\infty})$ denotes $\mathbb{R}^m$ equipped with the metric induced by the norm

$$||x||_{\ell^\infty} = \max\{|x_i| : i = 1,\dots,m\}$$

for $x = (x_1,\dots,x_m) \in \mathbb{R}^m$.

We form a new metric space (N', d') in the following way: For each $a \in N \subset \mathbb{R}^m$, choose $a' \in \mathbb{Q}^m \subset \mathbb{R}^m$ within $\delta/8$ of a. Let (N', d') denote the metric space on the set of all these new points a' equipped with the restriction of the ℓ^∞ metric from $\mathbb{Q}^m$. Note that all distances in (N', d') are rational.

Let $\psi \colon N' \to N \subset Y$ be the obvious bijection between points of N' and points of N, and let $y' = \psi^{-1}(y)$. It is clear that ψ is an $(\eta/2)$-isometry if δ is sufficiently small depending on η. Let $f \colon N' \to \mathbb{R}$ be defined so that $f(x)$ is a rational number within $\eta/2$ of $g(\psi(x))$. Thus, $g \circ \psi$ is within $\eta/2$ of f by definition.

By Lemma 3.2.5,

$$D((Y, y, g), (N', y', f)) \leq C\delta \leq \eta/2,$$

where C depends only on L, and δ is chosen in addition to be less than $\eta/(2C)$. This proves the lemma. □

3.3 Moving the base points of tangents

This section is devoted to the proof of the following result, which is an extension of a result of Le Donne [43].

Proposition 3.3.1. *Suppose (X, d, μ) is a doubling metric measure space and $f \colon X \to \mathbb{R}^n$ is Lipschitz. Then, for μ-almost every $x \in X$, for all $(Y, y, g) \in \mathrm{Tan}(X, x, f)$, and for all $y' \in Y$, we have $(Y, y', g - g(y')) \in \mathrm{Tan}(X, x, f)$.*

As we have not assumed that the measure μ is complete, the exceptional set in Proposition 3.3.1 need not be measurable. We define the outer measure μ^* by

$$\mu^*(A) = \inf\{\mu(B) : B \text{ Borel}, \ B \supseteq A\}.$$

Proposition 3.3.1 says that the exceptional set on which the conclusion fails has outer measure zero. Such a set is contained in a Borel set of measure zero.

The point a is a *point of outer density* of a set A if $a \in A$ and

$$\lim_{r \to 0} \frac{\mu^*(A \cap B(x,r))}{\mu(B(x,r))} = 1.$$

Every subset of X with positive outer measure has a point of outer density. Indeed, for any such set $A \subseteq X$ there exists a Borel set $B \supset A$ with $\mu(B) = \mu^*(A) > 0$. We have that

$$\mu^*(A \cap E) = \mu(B \cap E)$$

for any Borel set $E \subseteq X$, from which it follows that any point of density of B is a point of outer density of A.

Lemma 3.3.2. *Let (X, d, μ) be a doubling metric measure space, $f \colon X \to \mathbb{R}^n$ be Lipschitz, and let $A \subset X$ be a subset with a point of outer density at $a \in A$. Then* $\mathrm{Tan}(A, a, f) = \mathrm{Tan}(X, a, f)$.

Proof. The proof of this is an easy modification of the proof of Proposition 3.1 in [43], which we omit. $\square$

Proof of Proposition 3.3.1. We closely follow the argument in [43]. Our goal is to show that the set

$$\big\{x \in X : \text{there exists } (Y, y, g) \in \mathrm{Tan}(X, x, f) \text{ and } y' \in Y \\ \text{such that } (Y, y', g - g(y')) \notin \mathrm{Tan}(X, x, f)\big\}$$

has outer measure zero.

Consider the collection $\mathcal{M}$ consisting of (X, x, f) and all its rescalings and tangents. Note that $\mathcal{M}$ is a collection of uniformly doubling, uniformly Lipschitz space-functions. Using Lemma 3.2.6, we see that for each $k \in \mathbb{N}$, there exist countably many collections B_l, of space-functions such that, for all $l \in \mathbb{N}$,

$$\mathrm{diam}_D(B_l) < 1/4k$$

and $\mathcal{M} \subseteq \cup B_l$.

It therefore suffices to show that, for all $k, l, m \in \mathbb{N}$, the set

$$\Big\{x \in X : \text{there exists } (Y, y, g) \in \mathrm{Tan}(X, x, f) \text{ and } y' \in Y \text{ such that}$$
$$(Y, y', g - g(y')) \in B_l \text{ and } D\left((Y, y', g - g(y')), \left(\frac{1}{t}X, x, \frac{1}{t}(f - f(x))\right)\right) > \frac{1}{k}$$
$$\text{for all } t \in (0, 1/m)\Big\}$$

has outer measure zero.

Suppose that, for some $k, l, m \in \mathbb{N}$, the set above has positive outer measure, and call it $A \subseteq X$. Let a be a point of outer density of A. Then there exists $(Y, y, g) \in \mathrm{Tan}(X, a, f)$ and $y' \in Y$ such that

$$(Y, y', g - g(y')) \in B_l$$

and

$$D\left((Y, y', g - g(y')), \left(\frac{1}{t}X, a, \frac{1}{t}(f - f(x))\right)\right) > \frac{1}{k},$$

for all $t \in (0, 1/m)$.

Because $(Y, y, g) \in \mathrm{Tan}(a, f) = \mathrm{Tan}(A, a, f)$, there are sequences $\lambda_n \to 0$ and $\epsilon_n \to 0$, as well as ϵ_n-isometries $\phi_n \colon (Y, y) \to (\frac{1}{\lambda_n}X, a)$ taking values in A and satisfying

$$\sup_{B(y, \epsilon_n^{-1})} \left| g - \frac{1}{\lambda_n}(f \circ \phi_n - f(x)) \right| \leq \epsilon_n.$$

Let $a_n = \phi_n(y') \in A \subseteq X$. Note that

$$d_X(a_n, a) = O(\lambda_n) \to 0 \tag{3.3.1}$$

as $n \to \infty$.

Consider the space-functions

$$\left(\frac{1}{\lambda_n}X, a_n, \frac{1}{\lambda_n}(f - f(a_n))\right).$$

We now make the following claim:

Claim 3.3.3. *In the distance D, we have the convergence*

$$\left(\frac{1}{\lambda_n}X, a_n, \frac{1}{\lambda_n}(f - f(a_n))\right) \to (Y, y', g - g(y'))$$

Proof of Claim 3.3.3. Consider the same mappings ϕ_n as before, now considered as mappings

$$\phi_n : (Y, y') \to \left(\frac{1}{\lambda_n}X, a_n\right).$$

We will first show that if n is sufficiently large, ϕ_n is a $2\epsilon_n$-isometry with these base points. By (3.3.1), if n is sufficiently large, then

$$B_Y(y', (2\epsilon_n)^{-1}) \subset B_Y(y, \epsilon_n^{-1})$$

and so ϕ_n satisfies property (i) of a $2\epsilon_n$-isometry.

In addition, if $r \leq (2\epsilon_n)^{-1}$ and n is sufficiently large, then

$$B_{\lambda_n^{-1}X}(a_n, r - 2\epsilon_n) \subset B_{\lambda_n^{-1}X}(x, 1/\epsilon_n - \epsilon_n).$$

Therefore, if $z \in B_{\lambda_n^{-1}X}(a_n, r - 2\epsilon_n)$ then z is within $\lambda_n^{-1}X$-distance ϵ_n of a point $\phi_n(w)$, where $w \in B_Y(y', 1/\epsilon_n)$. A simple application of the triangle inequality and the properties of ϕ_n shows that w must be in $B_Y(y', r)$. Therefore,

$$B_{\lambda_n^{-1}X}(a_n, r - 2\epsilon_n) \subset N_{2\epsilon_n}\left(B_Y(\phi_n(y'), r)\right)$$

which verifies property (ii) of a $2\epsilon_n$-isometry.

Thus, for n large, each mapping ϕ_n is a $2\epsilon_n$-isometry from (Y, y') to $(\frac{1}{\lambda_n}X, a_n)$. In addition, we have, for $z \in B(y', (2\epsilon_n)^{-1})$,

$$\begin{aligned}
\left|(g(z) - g(y')) - \frac{1}{\lambda_n}(f(\phi_n(z)) - f(a_n))\right| &\leq \left|g(z) - \frac{1}{\lambda_n}(f(\phi_n(z)) - f(a))\right| \\
&\quad + \left|g(y') - \frac{1}{\lambda_n}(f(a_n) - f(a))\right| \\
&= \left|g(z) - \frac{1}{\lambda_n}(f(\phi_n(z)) - f(a))\right| \\
&\quad + \left|g(y') - \frac{1}{\lambda_n}(f(\phi_n(y')) - f(a))\right| \\
&\leq \epsilon_n + \epsilon_n \\
&= 2\epsilon_n
\end{aligned}$$

Thus, the mappings $\phi_n \colon (Y, y') \to (\frac{1}{\lambda_n} X, a_n)$ each satisfy the conditions of Lemma 3.2.5, and so we see that, for some $C > 0$ independent of n,

$$D\left((Y, y', g - g(y')), \left(\frac{1}{\lambda_n} X, a_n, \frac{1}{\lambda_n}(f - f(a_n))\right)\right) \le C\epsilon_n \to 0.$$

$\square$

Therefore, for n sufficiently large, we have

$$D\left(\left(\frac{1}{\lambda_n} X, a_n, \frac{1}{\lambda_n}(f - f(a_n))\right), (Y, y', g - g(y'))\right) < \frac{1}{4k}. \tag{3.3.2}$$

Now, since $a_n \in A$, there are space-functions $(Y_n, y_n, g_n) \in \mathrm{Tan}(X, a_n, f)$ and points $y'_n \in Y_n$ such that

$$(Y_n, y'_n, g_n - g_n(y'_n)) \in B_l,$$

and

$$D\left(\left(\frac{1}{t} X, a_n, \frac{1}{t}(f - f(a_n))\right), (Y_n, y'_n, g_n - g_n(y'_n))\right) > 1/k,$$

for all $t \in (0, 1/m)$.

We then have, for n large,

$$\begin{aligned}
\frac{1}{k} &< D\left((Y_n, y'_n, g_n - g_n(y'_n)), \left(\frac{1}{\lambda_n} X, a_n, \frac{1}{\lambda_n}(f - f(a_n))\right)\right) \\
&\le 2\Bigg(D\left((Y_n, y'_n, g_n - g_n(y'_n)), (Y, y', g - g(y'))\right) \\
&\qquad + D\left((Y, y', g - g(y')), \left(\frac{1}{\lambda_n} X, a_n, \frac{1}{\lambda_n}(f - f(a_n))\right)\right)\Bigg) \\
&< 2\left(\frac{1}{4k} + \frac{1}{4k}\right),
\end{aligned}$$

where the first $\frac{1}{4k}$ term arises because both spaces are in B_l and the second comes from (3.3.2). This is a contradiction. $\square$

3.4 Relationship to Lipschitz differentiability

We now investigate Lipschitz differentiability spaces. From now on, all metric measure spaces are assumed to be doubling and complete (but not necessarily Ahlfors regular until the proof of Theorem 1.3.9).

Recall the following notation, taken from [3] and introduced earlier in Section 1.3. We write $\Gamma(X)$ be the collection of all bi-Lipschitz functions of the form

$$\gamma : D_\gamma \to X$$

where $D_\gamma \subset \mathbb{R}$ is a compact set containing 0.

We now bring in Theorem 1.3.8, the result of Bate [3] referenced in the introduction. This result provides, at almost every point x in a complete Lipschitz differentiability space X, elements of $\Gamma(X)$ passing through x in n "independent" directions (see Theorem 1.3.8 for the precise statement.)

The property given in the conclusion of Theorem 1.3.8 admits an improvement if one passes to tangents. An *L-bi-Lipschitz line* in a metric space X is an L-bi-Lipschitz map $l\colon \mathbb{R} \to X$.

Proposition 3.4.1. *Let (X, d, μ) be a complete doubling metric measure space and let $f\colon X \to \mathbb{R}^n$ be a Lipschitz function. Suppose that there is a set A of positive measure such that for every $x \in A$, there exists $\gamma^x \in \Gamma(X)$ with $\gamma^x(0) = x$, 0 a density point of D_{γ^x}, and such that $v_x = (f \circ \gamma^x)'(0)$ exists and is non-zero.*

Then for almost every $x \in A$, every $(Y, y, g) \in \mathrm{Tan}(X, x, f)$ has the following property:

There is $L \geq 1$ such that for every $z \in Y$, there exists an L-bi-Lipschitz line $l\colon \mathbb{R} \to Y$ with $l(0) = z$ that satisfies

$$g(l(t)) = g(z) + tv_x$$

for all $t \in \mathbb{R}$.

The constant L depends on the point x but not on the sequence of scales defining the tangent.

Proof. Because the conclusion is supposed to hold for almost every $x \in A$, we may assume that x is among the full-measure set of points for which the conclusion of Proposition 3.3.1 holds.

Consider any $(Y, y, g) \in \text{Tan}(X, x, f)$. There is a sequence $\{\lambda_n\}$ tending to zero such that

$$\left(\lambda_n^{-1} X, x, \lambda_n^{-1}(f - f(x))\right) \to (Y, y, g).$$

Fix ϵ_n-isometries $\phi_n \colon (Y, y) \to (\lambda_n^{-1} X, x)$ and $\psi_n \colon (\lambda_n^{-1} X, x) \to (Y, y)$ such that

$$\sup_{B(x,1/\epsilon_n)} |\lambda_n^{-1}(f - f(x)) - g \circ \phi| < \epsilon_n \text{ and } \sup_{B(y,1/\epsilon)} |g - \lambda_n^{-1}(f \circ \psi - f(x))| < \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

We first claim the following: there is an L-bi-Lipschitz line $l \colon \mathbb{R} \to Y$ such that $l(0) = y$ and $g(l(t)) = tv_x$ for $t \in \mathbb{R}$. In other words, we first claim that the conclusion of the proposition holds when z is actually the base point y of the tangent.

To find the line l, we blow up the curve γ^x at $t = 0 \in \mathbb{R}$, along the same sequence of scales $\{\lambda_n\}$. Although in this chapter we have not defined the Gromov-Hausdorff convergence of functions mapping into metric spaces other than $\mathbb{R}^n$, for this one can use the theory developed in [21], Chapter 8, which was introduced in Section 2.3.1 of this dissertation.

Passing to a subsequence and using again standard facts about blowups at points of density (see [21], Lemmas 9.12 and 9.13), this gives a bi-Lipschitz line l in Y through y.

By Proposition 2.3.8 (i.e., [21], Lemma 8.19) this line l has the following property: There are maps $\sigma_n \colon \mathbb{R} \to D_\gamma$ such that

$$\lim_{n \to \infty} \lambda_n^{-1} |\sigma_n(t) - \lambda_n t| = 0$$

and

$$l(t) = \lim_{n \to \infty} \psi_n(\gamma(\sigma_n(t)))$$

uniformly in t on bounded subsets of $\mathbb{R}$.

Recall also that g is given by the limit

$$g(z) = \lim_{n \to \infty} \frac{1}{\lambda_n} \left(f(\phi_n(z)) - f(x)\right)$$

uniformly on bounded subsets of Y.

Therefore, using the uniformity of the convergence and the Lipschitz property of f and γ^x, we have that

$$\begin{aligned}
g(l(t)) &= \lim_{n\to\infty} \frac{1}{\lambda_n} \left(f(\phi_n(l(t))) - f(x)\right) \\
&= \lim_{n\to\infty} \frac{1}{\lambda_n} \left(f(\phi_n(\psi_n(\gamma^x(\sigma_n(t))))) - f(x)\right) \\
&= \lim_{n\to\infty} \frac{1}{\lambda_n} \left(f(\gamma^x(\sigma_n(t))) - f(x)\right) \\
&= \lim_{n\to\infty} \frac{1}{\lambda_n} \left(f(\gamma^x(\lambda_n t)) - f(x)\right) \\
&= t(f \circ \gamma^x)'(0) \\
&= t v_x
\end{aligned}$$

for all $t \in \mathbb{R}$. Thus, we see that $(g \circ l)(t) = tv_x$.

This gives the conclusion of the proposition at the base point $y \in Y$. Now consider any point $z \in Y$. By Proposition 3.3.1, $(Y, z, g - g(z)) \in \mathrm{Tan}(X, x, f)$. Therefore, by the preceding argument, we get the conclusion of the proposition at the arbitrary point $z \in Y$. $\square$

Proposition 3.4.2. *Let $(U, \phi\colon U \to \mathbb{R}^n)$ be an n-dimensional chart in a complete doubling Lipschitz differentiability space. When they exist, let $\gamma_1^x, \dots, \gamma_n^x$ be the n "broken curves" through x provided by Theorem 1.3.8, and let $v_i^x = (\phi \circ \gamma_i^x)'(0)$, which are n linearly independent vectors in $\mathbb{R}^n$.*

Then for almost every $x \in U$, every $(Y, y, G) \in \mathrm{Tan}(X, x, \phi)$ has the following property: There exists $L \geq 1$ such that, for every $z \in Y$, there are n L-bi-Lipschitz lines $l_1, \dots, l_n$ with $l_i(0) = z$ that satisfy

$$G(l_i(t)) = G(z) + tv_i^x$$

for all $t \in \mathbb{R}$.

The constant L depends on the point x but not on the sequence of scales defining the tangent.

Proof. This follows immediately from the previous two results. $\square$

3.5 Obtaining Lipschitz quotient maps

Recall that a *Lipschitz quotient map* $f\colon X \to Y$ between metric spaces is a Lipschitz map for which there exists $c > 0$ such that

$$f(B(x,r)) \supseteq B(f(x), cr)$$

for any ball $B(x,r)$ in X. The constant c is called the *co-Lipschitz* constant of the map.

A simple reformulation of the Lipschitz quotient property is the following. A Lipschitz map $f\colon X \to Y$ is a Lipschitz quotient map with co-Lipschitz constant c if and only if

$$\mathrm{dist}_X(x, f^{-1}(y)) \leq c^{-1} d_Y(f(x), y)$$

for every $x \in X$ and $y \in Y$.

Corollary 3.5.1. *Let $(U, \phi\colon U \to \mathbb{R}^n)$ be an n-dimensional chart in a complete doubling Lipschitz differentiability space X. Then for almost every $x \in U$, every $(Y, y, F) \in \mathrm{Tan}(X, x, \phi)$ has the property that F is a Lipschitz quotient map onto $\mathbb{R}^n$.*

The Lipschitz and co-Lipschitz constants associated to the Lipschitz quotient map F depend on the point x, but not on the sequence of scales defining the tangent.

Proof. We may assume that x lies in the full measure set provided by Proposition 3.4.2. As in Proposition 3.4.2, we have n "broken curves" γ_i^x through x, from Theorem 1.3.8. Let $v_i = (\phi \circ \gamma_i^x)'(0)$, which are n linearly independent vectors in $\mathbb{R}^n$. To simplify the proof, we first fix a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ that sends each v_i to e_i, the ith standard basis vector of $\mathbb{R}^n$. Note that A is invertible, because $\{v_i\}$ is a linearly independent set.

Now let $\psi = A \circ \phi$. It is clear that (U, ψ) is still an n-dimensional chart, so we can apply Proposition 3.4.2 to obtain $L \geq 1$ and $(Y, y, G) \in \mathrm{Tan}(X, x, \psi)$ with the property that for every $z \in Y$, there are n L-bi-Lipschitz lines $l_1^z, \ldots, l_n^z$ with $l_i^z(0) = z$ that satisfy

$$G(l_i^z(t)) = G(z) + te_i$$

for all $t \in \mathbb{R}$.

We now show that G is a Lipschitz quotient map. As a tangent of a Lipschitz map, it is automatically Lipschitz. To establish the co-Lipschitz bound, it suffices (by the remark above) to show that there is a constant $C > 0$ such that, whenever $z \in Y$ and $p \in \mathbb{R}^n$,

$$\operatorname{dist}(z, G^{-1}(p)) \leq C|G(z) - p|. \tag{3.5.1}$$

Fix $z \in Y$ and $p \in \mathbb{R}^n$. Let $q = G(z) \in \mathbb{R}^n$. Write $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ and $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$.

Let $z_1 = l_1^z(p_1 - q_1)$. Then

$$G(z_1) = q + (p_1 - q_1)e_1 = (p_1, q_2, \ldots, q_n).$$

Let $z_2 = l_2^{z_1}(p_2 - q_2)$. Then

$$G(z_2) = G(z_1) + (p_2 - q_2)e_2 = (p_1, p_2, q_3, \ldots, q_n).$$

Repeating this n times, we obtain z_n such that $G(z_n) = p$. In addition,

$$\begin{aligned} d_Y(z_n, z) &\leq d(z, z_1) + d(z_1, z_2) + \cdots + d(z_{n-1}, z_n) \\ &\leq L|p_1 - q_1| + L|p_2 - q_2| + \cdots + L|p_n - q_n| \\ &\leq Ln^{1/2}|G(z) - p|. \end{aligned}$$

Because $z_n \in G^{-1}(p)$, this proves (3.5.1) and so concludes the proof that G is a Lipschitz quotient map with co-Lipschitz constant $c = (Ln^{1/2})^{-1}$. Now consider the space-function $(Y, y, F) \in \operatorname{Tan}(X, x, \phi)$ associated to the same sequence of scales as $(Y, y, G) \in \operatorname{Tan}(X, x, \psi)$. As $A \circ \phi = \psi$ and taking tangents is a linear operation on functions, we see that $A \circ F = G$. Therefore $F = A^{-1} \circ G$, and since A is bi-Lipschitz, F is also a Lipschitz quotient map.

The bi-Lipschitz constant of A depends only on the vectors $\{v_i\}$ and not on the sequence of scales defining the tangent. Therefore, the Lipschitz and co-Lipschitz constants of F also do not depend on the sequence of scales defining the tangent. $\square$

The following corollary summarizes two simple immediate consequences.

Corollary 3.5.2. *Let $(U, \phi\colon U \to \mathbb{R}^n)$ be an n-dimensional chart in a complete doubling Lipschitz differentiability space X. Then the following two facts hold:*

(i) *At almost every point of U, any tangent of ϕ maps onto $\mathbb{R}^n$.*

(ii) *For almost every point $x \in U$, there is a constant $c_0 > 0$ such that any tangent $(Y, y) \in \operatorname{Tan}(X, x)$ satisfies the lower mass bound*

$$\mathcal{H}^n(B(z, r)) \geq c_0 r^n$$

for all $z \in Y$ and $r > 0$.

Corollary 3.5.2 is an analog of Theorem 13.4 of [13] from the setting of PI spaces.

3.6 Uniformly rectifiable tangents

The proof of Theorem 1.3.9 will be an immediate application of Corollary 3.5.1 and Semmes' Theorem 2.2.2 stated in Chapter 2.

3.6.1 Bi-Lipschitz pieces for Lipschitz quotient maps

Recall David's condition 2.2.1 and the associated Theorem 2.2.2 of Semmes. (Actually, for this application we could equally well use the original result of David from [18].) Although in Chapter 2 we stated these only for 0-cubes, they apply to dyadic cubes of any scale by a simple rescaling (see also [56], Condition 9.1 and Theorem 10.1).

We now note that if (X, d) is Ahlfors n-regular and $z\colon X \to \mathbb{R}^n$ is a Lipschitz quotient map, then z trivially satisfies David's condition on any cube. Indeed, suppose z is a Lipschitz quotient map, so that $z(B(x, r)) \supseteq B(z(x), cr)$ for all $x \in X$ and $r > 0$. Fix a cube $I_0 \subseteq X$ and constants $\lambda, \gamma > 0$. Set $\Lambda = \lambda/c$ and η arbitrary. Let T be the union of all j-cubes in X touching $B = B(x, \Lambda 2^j)$, where $x \in I_0$. If $T \subseteq I_0$, then $B \subseteq T \subseteq I_0$. We therefore see that

$$z(T) \supseteq z(B) \supseteq B(z(x), c\Lambda 2^j) = B(z(x), \lambda 2^j).$$

Thus the first branch of David's condition is always satisfied.

The following is therefore an immediate consequence of Theorem 2.2.2, which we record so that we can reference it later.

Corollary 3.6.1. *Let (Y, d) be Ahlfors n-regular and let $f \colon Y \to \mathbb{R}^n$ be a Lipschitz quotient map. Then there exist constants $\alpha \geq 1$ and $\beta > 0$ such that, for every ball $B \subset Y$, f is α-bi-Lipschitz on some subset of B of measure at least $\beta \mathcal{H}^n(B)$. Here $\alpha, \beta > 0$ depend only on the Ahlfors regularity constant of the space Y and the Lipschitz and co-Lipschitz constants of f.*

In particular, Y is uniformly rectifiable, with constants α and β.

3.6.2 Proof of Theorem 1.3.9 and Corollary 1.3.10

We now apply Corollaries 3.5.1 and 3.6.1 to prove Theorem 1.3.9. Let X be an Ahlfors n-regular Lipschitz differentiability space containing a chart $(U, \phi \colon U \to \mathbb{R}^n)$ of dimension n. Note that, as mentioned above, any tangent Y of X is Ahlfors n-regular.

By Corollary 3.5.1, for almost every point x of U, there exists $(Y, y) \in \mathrm{Tan}(X, x)$ and a Lipschitz quotient map $G \colon Y \to \mathbb{R}^n$. It follows immediately from Corollary 3.6.1 that Y is uniformly rectifiable.

For the second part of the theorem, take a positive measure subset E of Y on which G is a bi-Lipschitz map. Fix a point of density y' of E such that $G(y')$ is a point of $\mathcal{H}^n$-density of $G(E) \subset \mathbb{R}^n$. Take a further tangent $(Z, z, H) \in \mathrm{Tan}(Y, y', G)$. Note that $(Z, z) \in \mathrm{Tan}(Y, y') \subset \mathrm{Tan}(X, x)$ (see [43], Theorem 1.1). In addition, it follows from Lemma 3.3.2 and a standard argument (which appears in the proofs of Proposition 2.9.1 and Corollary 3.8.1) that H is a bi-Lipschitz map from Z onto $\mathbb{R}^n$. This completes the proof of Theorem 1.3.9.

Corollary 1.3.10 is an immediate consequence of Theorem 1.3.9 and Theorem 9 of [38], again using the fact [43] that, at almost every point of X, tangents of tangents are tangents.

3.7 Proof of Theorem 1.3.12

We now consider an Ahlfors s-regular Lipschitz differentiability space X with a k-dimensional chart U, where $k < s$.

Fix any $N \in \mathbb{N}$ and any Lipschitz function $f\colon U \to \mathbb{R}^N$. We will show that $\mathcal{H}^s(f(U)) = 0$.

Without loss of generality, we may assume that f is 1-Lipschitz, $N \geq s$, and U is bounded. Write $f = (f_1, \dots, f_N)$, where $f_i\colon X \to \mathbb{R}$ for $1 \leq i \leq N$. We say that f is differentiable at $x \in U$ if each f_i is differentiable at x. In this case, we write Df_x for the $N \times k$ matrix whose ith row is $d(f_i)_x \in \mathbb{R}^k$.

Note that the subset of U on which f is non-differentiable has $\mathcal{H}^s$-measure zero, and thus so does its image under f. It therefore suffices to show that $\mathcal{H}^s(f(V)) = 0$, where $V \subseteq U$ is the subset on which f is differentiable. To do so, it suffices to show that the Hausdorff content $\mathcal{H}^s_\infty$ of $f(V)$ is zero (see [28], p. 61).

Fix $\delta > 0$. For each $x \in V$, choose $r_x \in (0,1)$ small so that

$$y \in B(x, 6r_x) \Rightarrow |f(y) - f(x) - Df_x \cdot (\phi(y) - \phi(x))| < \delta r_x$$

where $Df_x = (d(f_1)_x, d(f_2)_x, \dots, d(f_n)_x)$.

By a basic covering theorem, (see [28], Theorem 1.2), we may acquire a collection of balls $\{B_j = B(x_j, r_j)\}$, with $x_j \in V$ and $r_j = 5r_{x_j}$, covering V such that the collection $\{\frac{1}{5}B_j\}$ consists of pairwise disjoint sets.

Let P_j denote the k-dimensional affine space $f(x_j) + Df_{x_j}[\mathbb{R}^k] \subset \mathbb{R}^N$. Then

$$f(B_j) \subset N_{\delta r_j}(B(f(x_j), r_j) \cap P_j).$$

Thus, $f(B_j)$ can be covered by $C\delta^{-k}$ balls of radius $2\delta r_j$, where C depends only on k (cover the k-dimensional Euclidean ball $B(f(x_j), r_j) \cap P_j$ by balls of radius δr_j and then double the radii of these balls).

Note that because V is bounded and the balls $B(x_j, \frac{1}{5}r_j)$ are disjoint, we have that

$$\sum_j r_j^s = 5^s \sum_j (r_j/5)^s \leq C_0 5^s \mathcal{H}^s(N_1(V)) < \infty, \tag{3.7.1}$$

where the first inequality is because the collection $\{B(x_j, r_j/5)\}$ consists of disjoint subsets of $N_1(V)$, and the second inequality is because V is bounded and X is Ahlfors s-regular with constant C_0.

Thus,

$$\begin{aligned}
\mathcal{H}^s_\infty(f(V)) &\leq C \sum_j \delta^{-k}(2\delta r_j)^s \\
&\leq 2^s C \delta^{s-k} \sum_j r_j^s \\
&\leq 10^s C C_0 \mathcal{H}^s(N_1(V)) \delta^{s-k},
\end{aligned}$$

using (3.7.1).

Because $s - k > 0$ and $\mathcal{H}^s(N_1(V)) < \infty$, sending $\delta \to 0$ completes the proof of Theorem 1.3.12.

Remark 3.7.1. In [57], Semmes shows that a linearly locally contractible, Ahlfors n-regular n-manifold M admits a Poincaré inequality, and is therefore a Lipschitz differentiability space. Using Theorem 1.3.12 above, combined with the deep Theorem 1.29 of [57], one can give a straightforward proof that the differentiable structure of M consists of n-dimensional charts. This is done in the following way:

The fact that the charts in M have dimension at most n follows from Theorem 13.8 of [13], or alternatively from Corollary 3.8.5 below.

If a chart U in M had dimension $k < n$, then U would be strongly unrectifiable by Theorem 1.3.12. Fixing a point of density x of U, Theorem 1.29 of [57] provides, for all $j \in \mathbb{N}$, mappings $f_j \colon B(x, j^{-1}) \to \mathbb{S}^n$ that are Cj-Lipschitz, for some constant C, and whose images have full measure in the standard unit sphere $\mathbb{S}^n$. In other words, we have

$$\mathcal{H}^n(f_j(B(x, j^{-1}))) = \mathcal{H}^n(\mathbb{S}^n)$$

independent of $j \in \mathbb{N}$. On the other hand, by the strong unrectifiability of U we have

$$\mathcal{H}^n(f_j(B(x, j^{-1}) \cap U)) = 0.$$

Letting j tend to infinity, one easily obtains a contradiction to the assumption that x is a point of density of U.

Remark 3.7.2. In fact, as was shown in Theorem 1.2.3 of this dissertation (and in the publication [23]), the spaces described in Remark 3.7.1 are locally uniformly rectifiable in dimension n, which is much stronger than having an n-dimensional differentiable structure.

3.8 Additional corollaries

This section contains some further results that follow from Corollary 3.5.2 and Theorem 1.3.9.

3.8.1 Embedding and rectifiability

The fact that blowups of the coordinate functions are surjective (statement (i) in Corollary 3.5.2) appears to be new for Lipschitz differentiability spaces (as opposed to PI spaces, where it appears in Theorem 13.4 of [13]). In this section, we give some consequences of this fact. (While this dissertation was in preparation, we learned of Schioppa's paper [52], in Section 5 of which he also proves that the blowups of the coordinate functions are surjective in a Lipschitz differentiability space. The results in this subsection, which are all corollaries of that fact, can thus also be derived from Schioppa's work.)

Exactly as for PI spaces (see [13], Theorems 14.1 and 14.2), surjectivity of blowups gives the following consequences.

Corollary 3.8.1. *Let (X, d, μ) be a complete Lipschitz differentiability space with an n-dimensional chart (U, ϕ). Suppose that $F\colon X \to \mathbb{R}^N$ is a bi-Lipschitz embedding. Then for almost every $x \in U$, the set $F(X)$ has a unique tangent at $F(x)$ that is an n-dimensional linear subspace of $\mathbb{R}^N$.*

If in addition $\mathcal{H}^n(U) < \infty$ and $\mathcal{H}^n$ is absolutely continuous with respect to μ, then it follows that $F(U)$, and therefore U, is n-rectifiable.

Proof. The proof proceeds as for PI spaces. Consider any point of density x of U at which F is differentiable; we may also assume that $F(x)$ is a point of $F_*(\mu)$-density of $F(U)$. Note that the push-forward measure $F_*(\mu)$ is doubling on $F(X)$ as F is bi-Lipschitz.

Take a tangent $Z \subset \mathbb{R}^N$ of $F(U)$ at $F(x)$ along some sequence of scales. Note that this tangent Z can be realized as a subset of $\mathbb{R}^N$, as in [21], Lemma 8.2.

Simultaneously blow up X and the maps ϕ and F at x to obtain a tangent Y of X at x, a Lipschitz map $\tilde{\phi}\colon Y \to \mathbb{R}^n$, and a bi-Lipschitz map $\tilde{F}\colon Y \to Z \subset \mathbb{R}^N$. To summarize, we obtain, along some fixed sequence of scales,

$$(Z, z) \in \operatorname{Tan}(F(U), F(x)) = \operatorname{Tan}(F(X), F(x)),$$

$$(Y, y, \tilde{F}) \in \operatorname{Tan}(X, x, F), \text{ and}$$

$$(Y, y, \tilde{\phi}) \in \operatorname{Tan}(X, x, \phi),$$

where $\tilde{F}$ is a bi-Lipschitz map of Y into Z. (The fact noted above that $\operatorname{Tan}(F(U), F(x)) = \operatorname{Tan}(F(X), F(x))$ follows from Lemma 3.3.2 and the fact that $F_*(\mu)$ is doubling.)

In fact, by a similar argument as in Proposition 2.9.1, $\tilde{F}$ maps Y onto Z. To see this, we may use the type of convergence discussed in Subsection 2.3.1 to simultaneously blow up $F^{-1}\colon F(U) \to U$ at $F(x)$, yielding a bi-Lipschitz map $G\colon Z \to Y$ such that $\tilde{F}(G(w)) = w$ for all $w \in Z$. This shows that $\tilde{F}$ is surjective.

Now, because F is differentiable at x, its blowup $\tilde{F}$ can be written as

$$\tilde{F} = DF_x \circ \tilde{\phi},$$

where DF_x is a linear map $\mathbb{R}^n \to \mathbb{R}^N$ and $\tilde{\phi}$ is the blowup of ϕ. Note that DF_x must have full rank n, because $\tilde{F}$ is a bi-Lipschitz map.

Because $\tilde{\phi}$ is surjective (see Corollary 3.5.2), we have

$$Z = F(Y) = DF_x(\tilde{\phi}(Y)) = DF_x(\mathbb{R}^n),$$

which is a fixed n-dimensional linear subspace of $\mathbb{R}^N$ independent of the choice of scales defining the tangent Z. This shows that for μ-almost every $x \in U$, $F(U)$ has a unique tangent at $F(x)$ that is an n-dimensional linear subspace of $\mathbb{R}^N$.

The second part of the corollary, about the rectifiability of U, now follows from a well-known characterization of rectifiable sets in Euclidean space (see [47], Theorem 15.19). $\square$

Note that the proof of Corollary 3.8.1 shows that, if X is a Lipschitz differentiability space with an n-dimensional chart (U, ϕ), and if $F \colon X \to \mathbb{R}^N$ is bi-Lipschitz, then at almost every point $x \in U$, every tangent of X at x is bi-Lipschitz equivalent to $\mathbb{R}^n$. The following non-embedding result is therefore an immediate consequence.

Corollary 3.8.2. *Let X be a complete Lipschitz differentiability space with an n-dimensional chart (U, ϕ). Suppose there exists a set $A \subseteq U$ of positive measure such that for every $a \in A$, there exists $(Y, y) \in \mathrm{Tan}(X, a)$ that is not bi-Lipschitz equivalent to $\mathbb{R}^n$. Then X does not admit a bi-Lipschitz embedding into any Euclidean space.*

Remark 3.8.3. In the second part of Corollary 3.8.1, note that if $\mathcal{H}^n(U) = 0$ then the n-rectifiability of U holds for trivial reasons. However, if one appeals to a result recently announced by Csörnyei and Jones, it is possible to show that an n-dimensional chart U always has $\mathcal{H}^n(U) > 0$ (see [3], Remark 6.11).

The previous two results greatly restrict the subsets of Euclidean space that can admit differentiable structures. For example, we obtain the following non-existence result. Here $|\cdot|$ refers to the standard Euclidean metric.

Corollary 3.8.4. *Let E be a closed subset of some $\mathbb{R}^N$ that is Ahlfors s-regular, where $0 < s \leq N$. If s is not an integer, then $(E, |\cdot|, \mathcal{H}^s)$ is not a Lipschitz differentiability space.*

Proof. Suppose that s is not an integer but that $(E, |\cdot|, \mathcal{H}^s)$ is in fact a Lipschitz differentiability space. Because E is Ahlfors s-regular, so are all its tangents. On the other hand, by Corollary 3.8.1, some tangent of E must be a linear subspace of $\mathbb{R}^N$, and so must have integer Hausdorff dimension. This is a contradiction. $\square$

In particular, many self-similar fractals like the standard Sierpiński carpet and Sierpiński gasket cannot be Lipschitz differentiability spaces when equipped with their canonical measures. In general, Ahlfors regular spaces with non-integer Hausdorff dimension can be Lipschitz differentiability spaces and can even admit Poincaré inequalities (see [9], [41], [16]). Such spaces can never admit bi-Lipschitz embeddings into any Euclidean space. Indeed, in the case of PI spaces, stronger non-embedding results hold (see [15]).

Surjectivity of the blowups also implies a sharp bound on the dimension of a differentiable structure on a doubling space. This uses the notion of the Assouad dimension $\dim_A X$ of a metric space X; a definition can be found in [28], Definition 10.15.

Corollary 3.8.5. *Let X be a doubling Lipschitz differentiability space with an n-dimensional chart (U, ϕ). Then $n \leq \dim_A X$.*

Proof. This follows from two facts about Assouad dimension. First, the Assouad dimension of a space X is always at least the Hausdorff dimension $\dim_H X$ of X (see [45], Section 1.4.4). Second, the Assouad dimension of a tangent space is always at most the Assouad dimension of the original space ([45], Proposition 6.1.5).

Because in addition the blowups of the coordinates yield a Lipschitz map from a tangent Y of X onto $\mathbb{R}^n$, we have that

$$\dim_A X \geq \dim_A Y \geq \dim_H Y \geq n.$$

□

Note that, for example, the Assouad dimension of $\mathbb{R}^n$ is the same as the dimension of its differentiability charts, so Corollary 3.8.5 is sharp. Corollary 3.8.5 was first noted by Schioppa in [52], Section 5.

3.8.2 Spaces with quasi-Möbius symmetries

In [6], Bonk and Kleiner consider compact metric spaces that admit the following type of symmetries. For the definition of quasi-Möbius maps, see [6].

Definition 3.8.6. A compact metric space X *admits quasi-Möbius symmetries* if the following holds: every triple of points in X can be blown up to a uniformly separated triple by a uniformly quasi-Möbius map. In other words, there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ and a constant $\delta > 0$ such that for every triple of points $x, y, z \in X$, there is a η-quasi-Möbius map $g\colon X \to X$ such that the points $g(x)$, $g(y)$, $g(z)$ have mutual distance at least δ.

This condition is satisfied, for example, by the boundaries of hyperbolic groups equipped with their visual metrics.

Bonk and Kleiner show the following theorem.

Theorem 3.8.7 ([6], Theorem 6.1)**.** *If a compact Ahlfors n-regular metric space X admits quasi-Möbius symmetries and in addition has topological dimension n, then X is quasi-Möbius equivalent to the standard sphere $\mathbb{S}^n$.*

In other words, if a space admits quasi-Möbius symmetries and has extremal topological dimension, then it must be the standard sphere.

An immediate consequence of Theorem 1.3.9 and the methods of [6] is the following alternate version of Bonk and Kleiner's result, in which the assumption of extremal topological dimension is replaced by the assumption of extremal "differentiability dimension":

Corollary 3.8.8. *Let X be a compact Ahlfors n-regular Lipschitz differentiability space containing a chart U of dimension n. Suppose that X admits quasi-Möbius symmetries, as in Definition 3.8.6. Then X is quasi-Möbius equivalent to $\mathbb{S}^n$.*

Proof. By Theorem 1.3.9, X admits a tangent Y that is bi-Lipschitz equivalent to $\mathbb{R}^n$. It follows from Lemma 5.8 of [6] (see also the remark in the proof of Theorem 6.1 of that paper) that X is quasi-Möbius equivalent to $\mathbb{S}^n$. $\square$

The assumption that X is a Lipschitz differentiability space is strong, but it is somewhat natural in this context: In [7], Bonk and Kleiner show that if an Ahlfors regular space admits quasi-Möbius symmetries with no common fixed point and in addition is extremal for conformal dimension, then it supports a Poincaré inequality and is therefore a Lipschitz differentiability space.

References

[1] L. Ambrosio and B. Kirchheim. Rectifiable sets in metric and Banach spaces. *Math. Ann.*, 318(3):527–555, 2000.

[2] Z.M. Balogh, K. Rogovin, and T. Zürcher. The Stepanov differentiability theorem in metric measure spaces. *J. Geom. Anal.*, 14(3):405–422, 2004.

[3] D. Bate. Structure of measures in Lipschitz differentiability spaces. Preprint, 2013. arXiv:1208.1954.

[4] D. Bate and G. Speight. Differentiability, porosity and doubling in metric measure spaces. *Proc. Amer. Math. Soc.*, 141(3):971–985, 2013.

[5] E. Begle. Regular convergence. *Duke Math. J.*, 11(3):441–450, 1944.

[6] M. Bonk and B. Kleiner. Rigidity for quasi-Möbius group actions. *J. Diff. Geom.*, 61:81–106, 2002.

[7] M. Bonk and B. Kleiner. Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary. *Geom. Topol.*, 9:219–246, 2005.

[8] K. Borsuk. On some metrizations of the hyperspace of compact sets. *Fund. Math.*, 41(2):168–202, 1955.

[9] M. Bourdon and H. Pajot. Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings. *Proc. Amer. Math. Soc.*, 127(8):2315–2324, 1999.

[10] G. E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[11] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. Amer. Math. Soc., Providence, RI, 2001.

[12] L. Capogna, D. Danielli, S.D. Pauls, and J.T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[13] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9:428–517, 1999.

[14] J. Cheeger and T.H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46(3):406–480, 1997.

[15] J. Cheeger and B. Kleiner. Differentiability of Lipschitz functions from metric measure spaces to Banach spaces with the Radon-Nikodym property. *Geom. Funct. Anal.*, 19:127–183, 2009.

[16] J. Cheeger and B. Kleiner. Inverse limit spaces satisfying a Poincaré inequality. Preprint, 2013. arXiv:1312.5227.

[17] J. Cheeger, B. Kleiner, and A. Naor. Compression bounds for Lipschitz maps from the Heisenberg group to L_1. *Acta Math.*, 207(2):291–373, 2011.

[18] G. David. Morceaux de graphes Lipschitziens et intégrales singulières sur un surface. *Rev. Mat. Iberoamericana*, 4(1):73–114, 1988.

[19] G. David. *Wavelets and singular integrals on curves and surfaces*, volume 1465 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1991.

[20] G. David and S. Semmes. Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs. *Asterisque*, 193, 1991.

[21] G. David and S. Semmes. *Fractured Fractals and Broken Dreams.* Oxford University Press, New York, 1997.

[22] G. David and S. Semmes. Uniform rectifiability and quasiminimizing sets of arbitrary codimension. *Mem. Amer. Math. Soc.*, 144(687), 2000.

[23] G. C. David. Bi-Lipschitz pieces between manifolds. Preprint, 2013. arXiv:1312.3911.

[24] G. C. David. Tangents and rectifiability of Ahlfors regular Lipschitz differentiability spaces. Preprint, 2014. arXiv:1405.2461.

[25] H. Federer. *Geometric Measure Theory.* Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[26] J. Gong. Measurable differentiable structures on doubling metric spaces. Preprint, 2011. arXiv:1110.4279.

[27] K. Grove, P. Petersen V, and J.Y. Wu. Geometric finiteness theorems via controlled topology. *Invent. Math.*, 99(1):205–213, 1990.

[28] J. Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

[29] J. Heinonen and S. Keith. Flat forms, bi-Lipschitz parameterizations, and smoothability of manifolds. *Publ. Math. Inst. Hautes Études Sci.*, 113:1–37, 2011.

[30] J. Heinonen and P. Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, 181(1):1–61, 1998.

[31] J. Heinonen and S. Rickman. Geometric branched covers between generalized manifolds. *Duke Math. J.*, 113(3):465–529, 2002.

[32] S. Hu. *Theory of retracts.* Wayne State University Press, Detroit, 1965.

[33] P.W. Jones. Lipschitz and bi-Lipschitz functions. *Rev. Mat. Iberoamericana*, 4(1):115–121, 1988.

[34] S. Keith. Modulus and the Poincaré inequality on metric measure spaces. *Math. Z.*, 245(2):255–292, 2003.

[35] S. Keith. A differentiable structure for metric measure spaces. *Adv. Math.*, 183(2):271–315, 2004.

[36] S. Keith. Measurable differentiable structures and the Poincaré inequality. *Indiana Univ. Math. J.*, 53(4):1127–1150, 2004.

[37] S. Keith and T. Laakso. Conformal Assouad dimension and modulus. *Geom. Funct. Anal.*, 14(6):1278–1321, 2004.

[38] B. Kirchheim. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Amer. Math. Soc.*, 121(1):113–123, 1994.

[39] B. Kleiner and J. Mackay. Differentiable structures on metric measure spaces: A primer. Preprint, 2011. arXiv:1108.1324.

[40] T. Laakso. Lipschitz mappings with no bilipschitz tangents in spaces with decent calculus. *Reports of the Department of Mathematics, University of Helsinki*, Preprint 213, 1999.

[41] T. Laakso. Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. *Geom. Funct. Anal.*, 10(1):111–123, 2000.

[42] T. Laakso. Plane with A_∞-weighted metric not bi-Lipschitz embeddable to $\mathbb{R}^N$. *Bull. London Math. Soc.*, 34(6):667–676, 2002.

[43] E. Le Donne. Metric spaces with unique tangents. *Ann. Acad. Sci. Fenn. Math.*, 36(2):683–694, 2011.

[44] E. Le Donne. Lipschitz and path isometric embeddings of metric spaces. *Geom. Dedicata*, 166:47–66, 2013.

[45] J.M. Mackay and J.T. Tyson. *Conformal dimension*, volume 54 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2010.

[46] W. S. Massey. *Homology and cohomology theory*. Marcel Dekker Inc., New York, 1978. Monographs and Textbooks in Pure and Applied Mathematics, Vol. 46.

[47] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[48] W. Meyerson. Lipschitz and bilipschitz maps on Carnot groups. *Pacific J. Math.*, 263(1):143–170, 2013.

[49] P. Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. Math., (2)*, 129(1):1–60, 1989.

[50] P. Petersen, V. A finiteness theorem for metric spaces. *J. Differential Geom.*, 31(2):387–395, 1990.

[51] T. Rado and P. V. Reichelderfer. *Continuous transformations in analysis*. Springer-Verlag, Berlin, 1955.

[52] A. Schioppa. Derivations and Alberti representations. Preprint, 2013. arXiv:1311.2439.

[53] R. Schul. Bi-Lipschitz decomposition of Lipschitz functions into a metric space. *Rev. Mat. Iberoamericana*, 25(2):521–531, 2009.

[54] S. Semmes. On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights. *Rev. Mat. Iberoamericana*, 12(2):337–410, 1996.

[55] S. Semmes. Bilipschitz embeddings of metric spaces into Euclidean spaces. *Publ. Mat.*, 43(2):571–653, 1999.

[56] S. Semmes. Measure-preserving quality within mappings. *Rev. Mat. Iberoamericana*, 16(2):363–458, 2000.

[57] S. Semmes. Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincare inequalities. *Selecta Math. (N.S.)*, 2(2):155–295, 2009.

[58] E. H. Spanier. *Algebraic topology*. Springer-Verlag, New York, 1981.

[59] E. M. Stein and R. Shakarchi. *Real analysis*. Princeton Lectures in Analysis, III. Princeton University Press, Princeton, NJ, 2005.