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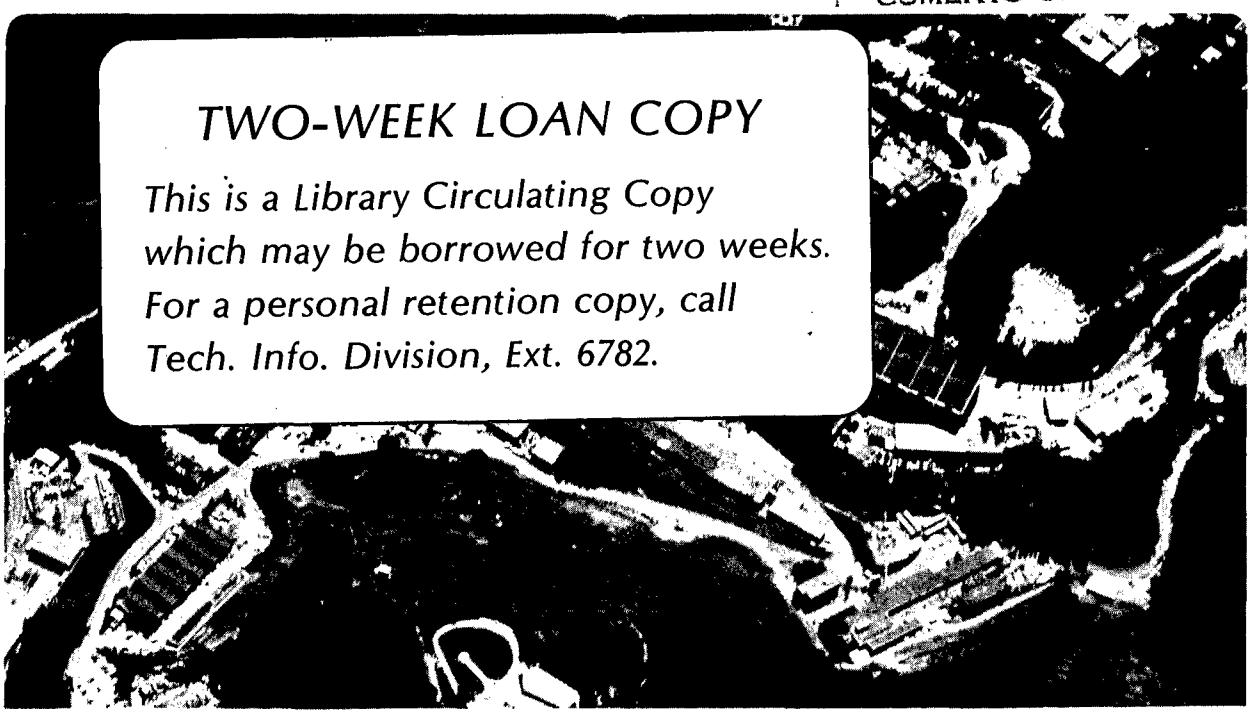

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AN ELEMENTARY DERIVATION OF THE MAGNETIC FLUX QUANTUM

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ABSTRACT

Starting with the well-known de Broglie relation $mv = h/\lambda$ that holds for a particle in zero magnetic field we give an elementary (freshman physics) derivation of the generalized de Broglie relation that holds for a charged particle in a circular orbit in a cylindrically symmetric magnetic field. We make no use of "div, grad, curl, and all that", and do not introduce canonical momentum, the vector potential, or the Schroedinger equation. This generalized de Broglie relation is then applied to two examples: (1) a single charged particle in a uniform external magnetic field and (2) a superconducting hollow cylinder. In both cases we find the result that the flux Φ enclosed by the orbit (in example 1) or trapped by the cylinder (example 2) obeys the relation $\Phi=nh/q$, but that the sameness of these two results is "accidental", since superconducting Cooper pairs have velocities about a million times too small for them to be in equilibrium "cyclotron orbits" in the magnetic field they experience. We also show that this de Broglie relation gives the correct value (i.e., the Schroedinger theory value) for the London penetration distance. In the appendices (junior physics course level) we show that this de Broglie relation also implies (correctly) the Meissner effect, and that it (correctly) insists on $\Phi=nh/q$ for the superconducting flux, ruling out, for example, $(n+\frac{1}{2})h/q$. Less correctly, it gives for the quantized cyclotron-orbit energy levels of example (1) the result $E=n\hbar\omega$ whereas the correct (Schroedinger theory) result is well known to be $(n+\frac{1}{2})\hbar\omega$.

1. INTRODUCTION

One of the most fascinating results of quantum mechanics is the famous prediction that the magnetic flux trapped in a superconducting ring is quantized: $\Phi=nh/q$, where Φ is the trapped flux (SI units), h is Planck's constant, $q=2e$ is the charge of the superconducting Cooper electron pair, and n is an integer. It would be gratifying if a derivation of this prediction could be presented to students at the freshman physics level. Unfortunately for the freshman, the beautiful treatment by Feynman[1] requires familiarity with the concept of the vector potential, and with the Schroedinger equation. It is therefore accessible only at the junior physics level.

It was therefore quite stimulating to read the interesting "plausibility derivation" by Higbie[2] of this famous result. In abbreviated form, Higbie's argument goes as follows: Consider a single Cooper pair circulating on the inner wall of a superconducting ring of radius r. Assume that the flux this single pair would generate is that of a uniform magnetic field B which would confine the pair to an equilibrium "cyclotron" orbit of radius r. That equilibrium requires $qvB=mv^2/r$, where m and v are the mass and velocity of the pair. Now demand that an integer number n of de Broglie waves of wavelength λ fit into the circumference: $2\pi r=n\lambda$. Then use the de Broglie relation, $mv=h/\lambda$. Combine these three equations to obtain $\Phi=\pi r^2 B = nh/2q$, which is half of the correct value given by the Schroedinger theory. The missing factor of two is then supplied by an additional argument involving a second Cooper pair.

The virtue of this derivation is that it is at the freshman physics level. Unfortunately, Higbie's assumptions about superconducting pairs are incorrect not by just a factor of two but by many orders of magnitude. The magnetic field experienced by a single Cooper pair circulating in a reasonable sized superconducting ring is of order 10^{18} times greater than that produced by the pair itself. That is because (in a reasonable geometry) there are of order 10^{18} pairs generating the field. Furthermore, the magnetic field experienced by a circulating Cooper pair is not even

approximately the field that would keep it in an equilibrium cyclotron orbit. Instead, pairs experience fields about 10^6 times stronger than that. Put differently, their velocities are about 10^{-6} times too small for them to be in cyclotron orbits in the field they experience. Cyclotron orbits have nothing to do with either the radial equilibrium of the pairs or with the size of the flux quantum, for a superconducting ring.

How is it possible to be off by a factor of 10^6 in velocity and still come within a factor of 2 of the correct formula? This is partly accomplished by using the de Broglie relation $h/\lambda = mv$. This relation is incomplete when there is a magnetic field: something else (s.e.) must be added to mv. This s.e. turns out to be about a million times larger than mv for a reasonable superconductor geometry. The correct de Broglie relation is $mv+s.e. = h/\lambda$. After throwing away the s.e. one can compensate by taking mv to be a million times larger than it actually is.

That still does not explain why the incorrect assumption of equilibrium cyclotron orbits for the Cooper pairs miraculously gives an answer within a factor of two of the correct one. That is because in spite of the formula $mv=h/\lambda$ being incorrect by a factor of a million for a superconductor, it is only wrong by a factor of two for a charged particle in a uniform external field. In that case the "something else" turns out to have half the magnitude of mv, and the opposite sign, so that the sum, mv+s.e., equals $\frac{1}{2}mv$. The correct de Broglie relation then gives $\frac{1}{2}mv=h/\lambda$ and also gives $\emptyset=nh/q$ for the flux enclosed by the orbit, whereas the incorrect relation $mv=h/\lambda$ gives the incorrect result $\emptyset=\frac{1}{2}nh/q$. For the superconductor the error in this approach is not, however, a factor of $\frac{1}{2}$ but a factor of a million, as stated above.

An elementary derivation follows. The quantum physics is at the level of the Bohr atom and makes no mention of vector potential or of canonical momentum. It is thus accessible to a student who has had only freshman physics. First we obtain the de Broglie relation for a particle in a circular orbit in a magnetic field having cylindrical symmetry. As our first application we consider a particle in an equilibrium cyclotron orbit in a uniform external magnetic field and find that the flux enclosed by the orbit is nh/q, which is the same formula as gives the famous flux quantum of superconductivity. That will lend credibility to the incorrect assumption that Cooper pairs are in equilibrium cyclotron orbits. Then we apply the de Broglie relation to a long superconducting hollow cylinder and again find that the flux is nh/q. But then we calculate the velocity of Cooper pairs and find that they are much too slow to be in equilibrium cyclotron orbits. Next we calculate the magnitude of the "London penetration distance" λ_o, of the magnetic field into the superconductor, and obtain the same result as in the Schroedinger theory.[2] Finally we answer the question "How can the circulating pairs be in radial equilibrium if they are not in equilibrium cyclotron orbits?"

The main text is kept elementary (no integrals, no differential equations) by taking the Meissner effect to be a given experimental fact that is independent of the de Broglie relation. In Appendices A and B (which are at the level of a junior physics course) we show that instead the Meissner effect is actually implied by the de Broglie relation, with no need to invoke the Schroedinger theory. In Appendix B we show that an ambiguity that arises in the choice of integer n versus half integer $n+\frac{1}{2}$ can be resolved in favor of n, without invoking the results of the Schroedinger theory.

2. DE BROGLIE RELATION

We want an elementary derivation of the de Broglie relation for a charged particle moving in a circular orbit in a magnetic field. Start with the more familiar case of a particle moving in a circle of radius r under the influence of a central Coulomb electric field, as in the Bohr atom. In that case we know we get correct results if we demand that an integer number n of de Broglie waves fit into one orbit:

$$2\pi r = n\lambda \tag{1}$$

and that the de Broglie relation be given by

$$mv = h/\lambda. \tag{2}$$

Together these give the quantum relation

$$mv = nh/2\pi r \tag{3}$$

from which follow the familiar results of the Bohr atom.

Now consider a particle in a quantized Bohr orbit having a particular value of n. Build, surreptitiously, a frictionless hollow rigid "doughnut" that encloses the orbit without disturbing it. Then slowly "turn down" the magnitude of the central positive charge that gives the Coulomb field. That is, slowly reduce to zero the radial electric field felt by the circulating particle. Since we do this very slowly, and n can only change by the discontinuous jump of an integer, we expect n to remain fixed: $\Delta n = 0$. Of course our frictionless rigid doughnut will have to take over more and more of the radial force that maintains radial equilibrium, as we turn down the Coulomb field. During all this, will mv change? No, because all the forces

are always radial: $\Delta(mv)=0$. Therefore our quantum relation (3) continues to hold even after the central charge is entirely gone and the particle is confined radially by the frictionless tube alone. Next we start slowly turning on a cylindrically symmetric magnetic field normal to the plane of the orbit. Because we do it very slowly n will remain unchanged: $\Delta n = 0$. However, mv no longer remains constant: $\Delta(mv) \neq 0$. The magnetic flux contained within the orbit at radius r, $\emptyset(< r)$, is changing with time. According to Faraday's Law, that will give an induced electric field tangential to the particle orbit, and the particle will be accelerated. Since mv does not remain constant we hope to find a new quantity that does stay constant, and that reduces to mv when there is no magnetic field. We can then hypothesize that we should replace the left sides of Eqs. (2) and (3) with that new quantity.

The induced emf ε at radius r is given by $\varepsilon=-d\emptyset(<r)/dt$. But ε also equals the induced azimuthal electric field, E, times the circumference $2\pi r$. Therefore

$$E = \frac{-d\emptyset(< r)/dt}{2\pi r} .$$

The change in particle mv during time interval Δt is given by

$$\Delta(mv) = F\Delta t = qE\Delta t = -q\Delta\emptyset(< r)/2\pi r,$$

i.e.,

$$\Delta[mv + q\emptyset(< r)/2\pi r] = 0 \qquad (4)$$

The left side of Equation (4) is the desired generalization of the result $\Delta(mv) = 0$ that we found while turning down the Coulomb field, and which maintained the de Broglie relation of Eq.(2)and the quantization relation given by Eq.(3). Therefore the "something" to be added to mv in Eq.(2)

is just $q\emptyset(<r)/2\pi r$. Combining this generalized de Broglie relation with the still-valid quantum condition of Eq.(1) gives us the generalization of Eq.(3):

$$mv(r) + q\emptyset(<r)/2\pi r = nh/2\pi r. \qquad (5)$$

The quantum relation (5) gives all of the results of this article. We will assume it holds more generally than its "derivation". We assume it gives all the allowed states for a single particle in a circular orbit with enclosed flux $\emptyset(<r)$.

It is important to realize that we still need our "rigid frictionless tube" in order to maintain radial equilibrium and keep r fixed while we change the flux $\emptyset(<r)$. Our "tube" will turn out to be a radial electric field, but it will not be due to a point charge in the case of a superconducting ring.

In our derivation we imagined that we started in a particular hydrogen-atom-like Bohr orbit, because for that case we know that Eqs.(1), (2), and (3) give the right answers (that is, answers that agree with experiment). However, that implies a constraining relationship between v and r (to give radial equilibrium, balancing the centrifugal force and the Coulomb attraction). Now that we have our "rigid frictionless tube" we can do away with that constraint. Therefore our allowed values of v, r, $\emptyset(<r)$, and the integer n are any values that satisfy Eq.(5). The velocity v(r) can be positive or negative (clockwise or counterclockwise circulation), the flux $\emptyset$ can be positive or negative, and the integer n can be either positive or negative. We even assume $n=0$ is allowed (infinite wavelength).

Of course, we must keep in mind the fact that we are using the concept of a classical orbit with well defined radius r, and that if we get the same result as the Schroedinger theory we must count ourselves as perhaps lucky.

3. PARTICLE IN A UNIFORM EXTERNAL MAGNETIC FIELD

As our first application of the quantum relation (5) we consider a single particle of charge q traveling with velocity v in a circle of radius r in a uniform magnetic field B normal to the plane of the circle. The frictionless tube confines the particle to radius r, independent of v. The circulating particle can be thought of as an electric current loop, so that it makes a contribution to the total magnetic field; but we assume this contribution is completely negligible compared with the external field. (Even if it were not, we would have to exclude the action of the single particle's self-produced field on itself.) The flux contained within the orbit is $\emptyset(<r) = \pi r^2 B$, so that Eq.(5) becomes

$$mv(r) + \tfrac{1}{2}qrB = nh/2\pi r \qquad (6)$$

The sign conventions in Eq.(6) are such that if the fingers of the right hand curl around in the direction of positive v then the thumb points along positive qB. Because of the frictionless tube any values of v and B that satisfy Eq.(6) are allowed.

We now apply Eq.(6) to the special case where, after the final value of B has been achieved, the particle is in an equilibrium "cyclotron orbit"; in that case we can remove the frictionless tube. (We needed it to maintain the radius constant during the build up of the field.) For such an orbit, Newton's 2nd law, "ma = F", gives $mv^2/r = qvB$ (in MKS units). That gives

$$mv = -qrB \qquad (7)$$

where the minus sign comes from using the same "right-hand rule" sign convention as for Eq.(6). Combining Eqs.(6) and (7) gives $-\tfrac{1}{2}qrB = nh/2\pi r$; or, dropping the minus sign, the flux contained within the orbit is

$$\emptyset(<r) = \pi r^2 B = nh/q \qquad (8)$$

The fact that Eq.(8) is identical with the formula for the trapped flux in a superconducting ring is more or less an accident, since, as we shall show later, the superconducting pairs are far from being in cyclotron orbits.

For a charged particle in a uniform magnetic field it is easier to think of ways to measure the particle's kinetic energy $\frac{1}{2}mv^2$ than to measure the flux enclosed by the orbit. We can easily show that Eq.(8) implies quantized kinetic energy: solve Eq.(7) for r and substitute into Eq.(8) to get

$$\frac{1}{2}mv^2 = n(h/2\pi)qB/m = n\hbar\omega \qquad (9)$$

where $\hbar$ equals $h/2\pi$ and $\omega = qB/m$ is the cyclotron angular frequency.

The quantized energy differences implied by Eq.(9) have actually been observed in the beautiful "(g-2)/2" experiment, using trapped single electrons and inducing quantum jumps between neighboring cyclotron orbits ($\Delta n=1$) by means of microwave quanta having energy $\hbar\omega$.[3]

At this point we should restrain our enthusiasm. The Schroedinger theory does not give Eq.(9); it gives[4]

$$\frac{1}{2}mv^2 = (n+\tfrac{1}{2})\hbar\omega \ . \qquad (10)$$

The additional term $\frac{1}{2}\hbar\omega$ is the "zero-point" energy. It is not detectable in the (g-2)/2 experiment, which measures only energy differences. If one asks why the simple de Broglie relation missed this term one might look at the assumption of Eq.(1): $2\pi r = n\lambda$. Is it that the Schroedinger theory wants to fit a half integer number of wavelengths into an orbit? No. It is rather that there is no well defined orbit, and r therefore has no well defined value.[4]

In the case of quantized trapped flux in a superconducting ring the Schroedinger theory[1] gives $\Phi = nh/q$, not $(n+\frac{1}{2})h/q$. That is also what we will find from the de Broglie relation (5). Perhaps we should attribute that agreement to good luck. But, remarkably, if we generalize our de Broglie relation so as to replace n by n+k, where n is the usual integer and k is an unknown constant, we will find (App.B) that for a solid superconducting cylinder we must set k=0; otherwise there is no solution.

4. THE DIFFERENCE BETWEEN ONE AND 10^{18} PARTICLES

In the theory of superconductivity it turns out that, at absolute zero, all the Cooper electron pairs are in the same "single-particle state".[1] If we assume (following Higbie) that in that state the Cooper pairs in a superconducting ring are in equilibrium cyclotron orbits under the influence of the magnetic field traversing the ring and experienced by the pairs, then Eq.(8) should apply, and we thus find a quantized trapped flux that agrees with the correct value.[1]

Why is this wrong? The quantity $\emptyset(<r)$ that appears in Eq.(5) represents the <u>total</u> flux inside radius r. For a single particle in an external field we can neglect the flux produced by the particle itself. That is how we obtained Eq.(8). But when we have a superconductor and vary an external field the change in $\emptyset(<r)$ is due not only to the change in the external field but also to changes in the flux produced by changes in the velocities of the huge number of Cooper pairs distributed through the superconductor. When we turn up the external field the flux change that the pairs produce is, by Lenz's Law, opposite in sign to the change in external flux, and, because we have a superconductor, equal in magnitude to the change in external flux. Thus the change in total flux, integrated over all radii r, is zero. However, the new total flux, even though equal in magnitude and sign to the old flux, is distributed differently in space, because a different fraction of it is due to the circulating Cooper pairs. Thus at each r there is a change in $\emptyset(<r)$, with equal amounts of positive and negative changes at different r so that the integrated change is zero. In order to use Eq.(5) we must learn how the magnetic field is distributed in space. Note also that the v(r) that appears in Eq.(5) is proportional to the current density at radius r, and this in turn contributes to the part of $\emptyset(<r)$ due to the superconducting currents. Thus we also need to know how v(r) is distributed in space. We must consider the superconductor in more detail.

5. SUPERCONDUCTING CURRENTS

We assume that Eq.(5), derived for a single particle, can be applied to a superconducting cylinder that (for our geometry) will have about 10^{18} contributing Cooper electron pairs. Each pair has charge q=-2e and a mass m which we take to be $2m_e$, where m_e is the mass of a free electron. (The effective mass of a Cooper pair is a difficult subject that we avoid.) We assume the superconductor is at absolute zero and that all pairs are in the same state, i.e., all have the same value of n in Eq.(5), which we assume holds for each pair. It is important to realize that $\emptyset(<r)$ in Eq.(5) is the total magnetic flux at radius less than r, i.e., it is the flux due to all of the Cooper pairs plus the flux due to any and all external sources of flux.

Consider a long hollow cylinder of length ℓ made of lead (which becomes superconducting at low enough temperature). This hollow cylinder has outer radius r_2 and inner radius r_1, with r_1 and r_2 both small compared with the length ℓ. Initially the lead cylinder is at room temperature. Inside the hollow region there is a very long cylindrical permanent magnet of radius less than r_1 that carries an unquantized external flux $\emptyset_o$ through the hollow region. See Fig.1. For simplicity we make the permanent magnet very long compared with the lead cylinder so that in our first discussion we can neglect leakage flux and "return" flux from the permanent magnet.

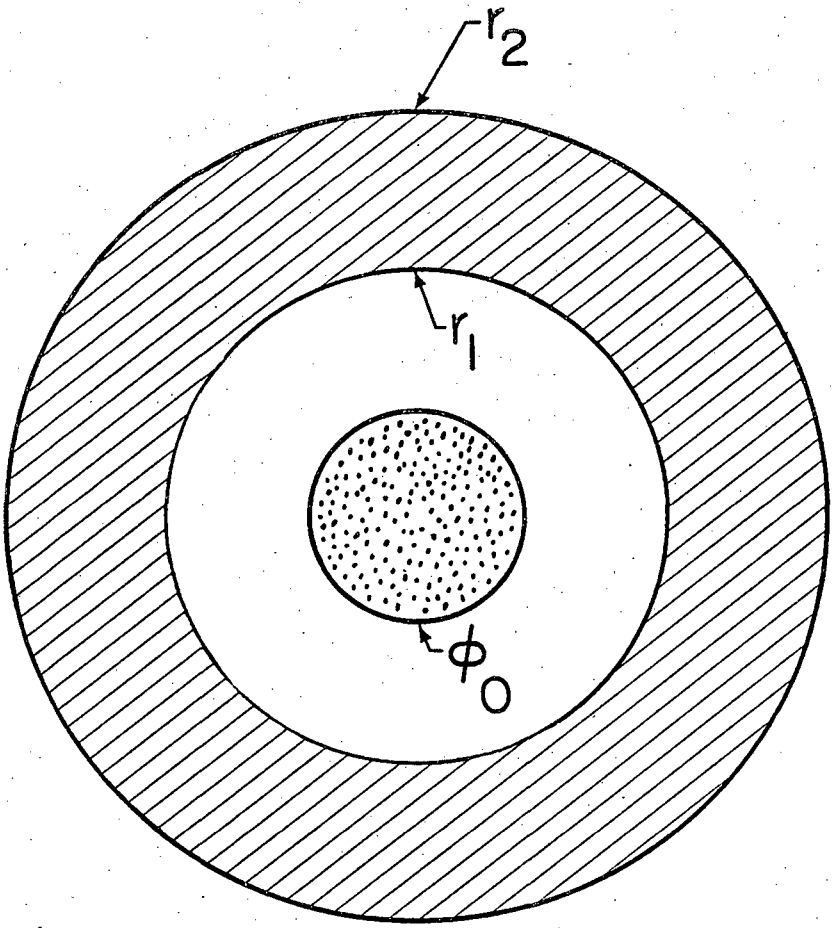

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Fig. 1. Room temperature. End view of long hollow lead cylinder having inner radius r_1 of order 1 cm and outer radius r_2 of order 2 cm. A very long permanent magnet carries unquantized external flux ϕ_o through the hollow region, the magnetic field lines being indicated as dots. Leakage flux and "return" flux in the lead are neglected. The warm lead is not superconducting.

Now cool the superconductor until it becomes superconducting; continue cooling all the way to absolute zero. Assume Eq.(5) applies to each of the Cooper pairs. We will find (Appendices A and B) that Eq.(5) implies the Meissner effect[1], according to which the total magnetic field (due both to external sources and to superconducting currents) must vanish in the interior of the superconductor: both the total magnetic field and the superconducting currents are confined to very thin surface layers (about 10^{-6} cm thick). If we had not assumed our permanent magnet to be very long compared with the hollow lead cylinder there would have been leakage and return flux from the permanent magnet in the material of the warm lead cylinder. When the cylinder became superconducting, the Meissner effect would demand the appearance of "spontaneously induced" surface currents on the cylinder surfaces, which are at $r=r_1$ and r_2. The "purpose" of these currents is to produce magnetic fields that will combine with the external fields to give a total field of zero in the main body of the superconductor (everywhere except very near the surfaces). Any leakage or return flux from the very long permanent magnet would give a small uniform external field throughout the region from r=0 to r_2. This would be cancelled by a small induced current I_2 on the outer surface of the cylinder, at $r=r_2$. That is because such a current is equivalent to the current in a long solenoid, and it is shown in every freshman physics course that such a current produces a uniform field at smaller radii than that of the current, and negligible field at larger radii. Since the surface current I_2 is not of present interest to us we assume there was no leakage or return flux from the permanent magnet and that therefore I_2 remains zero. (In Appendices A and B we consider the case where there is a uniform external field and I_2 is then not zero.) For the same reasons (Meissner effect and no leakage or return flux) there are no spontaneously induced currents or magnetic fields anywhere in the body of the cylinder, between r_2 and r_1 (but not including the surface at r_1).

6. FLUX QUANTIZATION

After the lead has become superconducting Eq.(5) applies. That implies the Meissner effect, according to which the current density $J(r)$ vanishes in the main body of the superconductor. But $J(r)=Nqv(r)$, where $v(r)$ is the Cooper pair velocity in Eq.(5), and N is the number of Cooper pairs per unit volume. (N is just $\frac{1}{2}N_e$, where N_e is the number of superconducting electrons per unit volume.) Consider a radius r that is sufficiently larger than r_1 (say 10^{-4} cm greater than r_1) that J(r) is zero, and hence v(r) is zero. Then according to Eq.(5) the total flux within that radius r is quantized and has the value $\Phi(<r) = nh/q$. That is the derivation!

7. INTEGER ROUND OFF FLUX

We seem to have a problem. The total flux for radius r a few times 10^{-6} cm greater than r_1 is quantized: $\Phi(<r)=nh/q$. But the permanent magnet's contribution Φ_o is unquantized. The solution is that in general there must be a small induced current I_1 on the inner surface at $r=r_1$. Otherwise we cannot satisfy Eq.(5). The sign and magnitude of I_1 depends on the exact value of Φ_o. The surface current I_1 provides a small unquantized flux which we shall call the "integer round off" flux. This integer-round-off flux will be some fraction (positive or negative) of a flux quantum h/q, such that the total flux is quantized. The unquantized flux due to I_1 might be expected to have magnitude equal or less than 1 quantum, i.e., "round off to the nearest integer". That turns out to be the case.[5] The situation is summarized in Fig. 2.

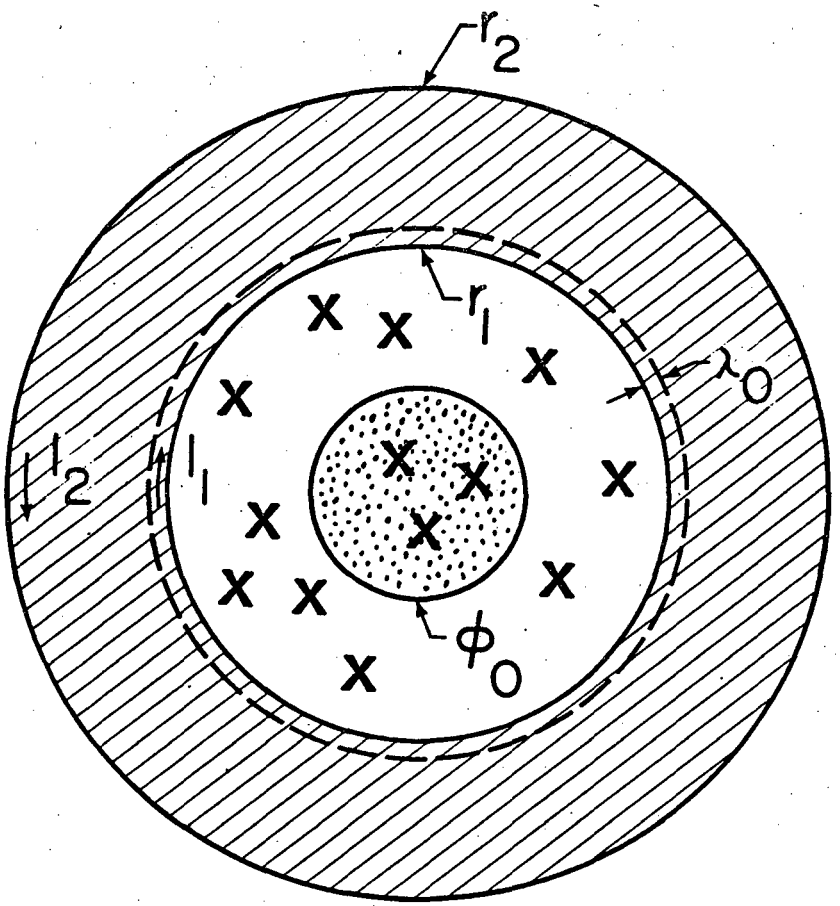

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Fig. 2. Superconducting. Because of the Meissner effect the only currents are the surface currents I_1 and I_2, but I_2 is zero if there was no magnetic field in the superconductor when it was warm. The small current I_1 is contained in a layer of thickness $\lambda_o \simeq 3 \times 10^{-6}$ cm, and provides an "integer-round-off" flux of less than one flux quantum, indicated by the crosses. The total flux, ϕ_o + round-off, is quantized. The round-off field is uniform for $r < r_1$ and falls off exponentially along with the current density, in the thin surface layer at radius r_1.

8. TRAPPED QUANTIZED FLUX

We have exhibited quantized flux, $\Phi = nh/q$, but it is not yet in the form we desire--a trapped quantized flux entirely due to superconducting current. We need to get rid of the permanent magnet. Let us slowly (so as not to change the ingeter n) pull out the permanent magnet. As we do so we can no longer disregard the flux that emerges from one end of the permanent magnet and returns at the other. By the time we have removed the permanent magnet all of these flux lines will have passed through the superconductor. During the slow removal there will be new induced currents both at r_2 and r_1. When the magnet has been completely removed the current I_2 will have returned to zero (it was only needed to give zero net field in the body of the superconductor during the removal). After removal, the surface current I_1, which used to provide only the small "round off" flux, now provides the entire flux. This flux still has the same quantized value, nh/q, as before the removal. This flux is "trapped". We can pick up the hollow superconducting cylinder and carry it around with its trapped flux (provided we keep it cold). The trapped quantized flux nh/q equals the original unquantized flux Φ_o to within plus or minus about a half of a flux quantum, the round-off flux. The situation is summarized in Fig. 3.

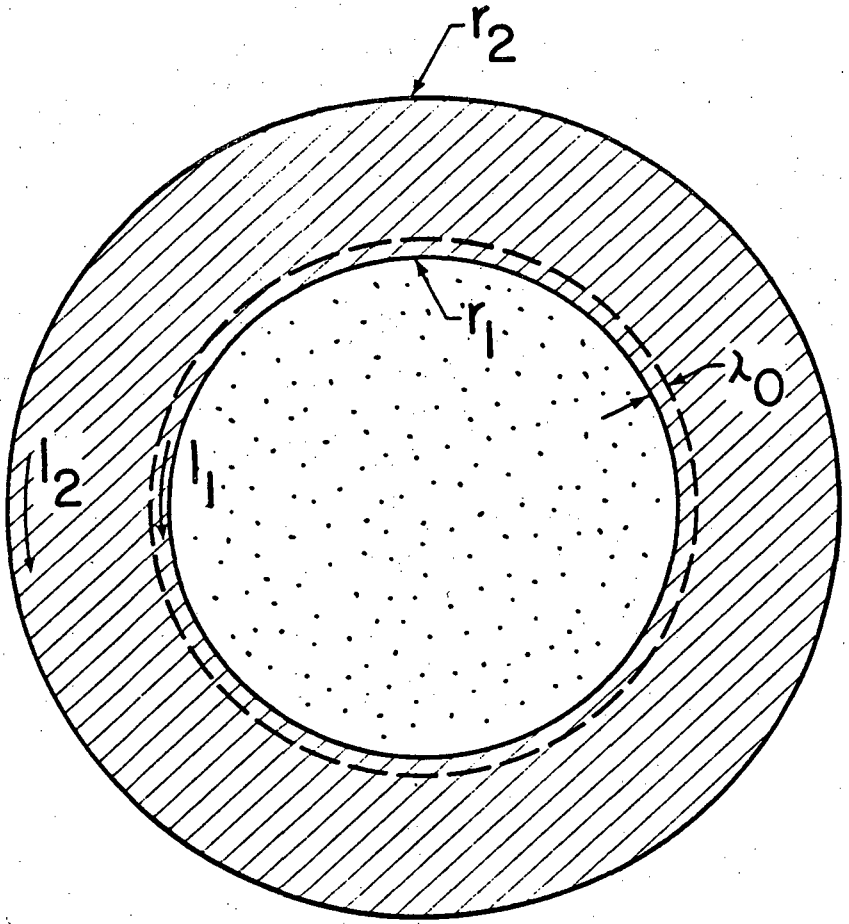

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Fig. 3. Trapped quantized flux. The permanent magnet has been slowly removed. The flux has the same quantized value as in Fig.2, but is now entirely provided by the large induced current I_1. A uniform field B_1 (indicated by dots) fills the hollow region and falls exponentially to zero, as does I_1, in the thin surface layer at radius r_1. The surface current I_2 is still zero.

9. VARIATION OF VELOCITY AND FIELD IN THE SURFACE LAYER

Now let us look more closely at the thin surface layer at $r=r_1$ that carries the surface current I_1, which provides the integer round-off flux before removal of the permanent magnet, or the entire flux after the removal. We demand that Eq.(5) hold for all r, with the same value of n. Start at r several times 10^{-6} cm larger than r_1, so we are well into the interior. Then $v(r)=0$ and the flux $\emptyset(<r)$ includes the entire quantized flux nh/q. Now progress to smaller r, approaching the surface at $r=r_1$. We will eventually reach radii where v(r) starts to grow from zero. As we pass through an increment dr that includes non-zero current density J(r) we will pass inside a small current increment $dI_1 = J(r)\ell dr$. This current increment gives magnetic field only at smaller radii than its radius. Thus the flux from dI_1 no longer lies entirely within our presently attained radius. Some of that flux lies outside. Thus $\emptyset(<r)$ decreases slightly, as v(r) increases, while we approach r_1 from larger radii. But that is just what we need to maintain the left side of Eq.(5) constant. By pursuing this line of reasoning carefully we can derive the exact relation between v(r) and the magnetic field B(r) within the thin layer that is needed in order to maintain Eq.(5). We do that in App.A. Here we shall simply say that it will turn out that v and B are always (at every r) proportional to one another and fall off together expenentially with the common factor $\exp-(r-r_1)/\lambda_o$, where λ_o is called the London penetration distance.

10. INDUCED VELOCITY OF COOPER PAIRS

Let us find the Cooper-pair velocities after the permanent magnet has been removed, and compare them with velocities that would give equilibrium cyclotron orbits. It will be sufficient to find the velocity $v = v_1$ at $r=r_1$ and relate it to the magnetic field $B=B_1$ there, because at larger r, v(r) and B(r) are proportional, with the same proportionality constant as at $r=r_1$ (See App.A.) It is a good enough approximation for our present purpose to say that the total trapped flux is ϕ_o, and that it is essentially all contained between r=0 and $r=r_1$ in the form of a uniform magnetic field B_1. That gives $\phi_o = B_1 \pi r_1^2$.

Now look at Eq.(5) and think of how we can maintain the left side constant as we progress from r slightly greater than r_1, to $r=r_1$. We can neglect the tiny change in r as far as its effect in Eq.(5). The only variables are v(r) and $\phi(<r)$. For r slightly greater than r_1 we have v=0 and $\phi=\phi_o$. When we have reached $r=r_1$ we have $v=v_1$ and the flux $\phi(<r)$ must have been decreased by an amount $\Delta\phi$ such that the change in the left side of Eq.(5) is zero:

$$\Delta\, mv + q\, \Delta\phi/2\pi r_1 = 0$$

Or, since Δv equals v_1, we have, dropping a minus sign,

$$mv_1 = q\, \Delta\phi/2\pi r_1 \qquad (11)$$

The flux $\Delta\phi$ is the flux contained in the small annular ring of radius r_1 and effective thickness λ_o in which B falls exponentially from its maximum value of B_1 to zero. We can estimate $\Delta\phi$ as the maximum field B_1 times the area $2\pi r_1 \lambda_o$:

$$\Delta\phi = B_1\, 2\pi r_1 \lambda_o$$

Combining this with Eq.(11) gives

$$v_1 = qB_1\lambda_o/m \qquad (12)$$

Eq.(12) gives the correct ratio between v and B throughout the layer of surface current. [Eq.(12) is derived more rigorously in App.A.]

Let us compare the actual induced velocity v_1 with the velocity v_o that would give an equilibrium cyclotron orbit in the magnetic field B_1 experienced by Cooper pairs at $r=r_1$. According to Eq.(7) we have

$$v_o = qB_1 r_1/m \qquad (13)$$

Comparing (13) and (12) we see that the induced velocity v_1 is less than the cyclotron-orbit velocity v_o by a factor of λ_o/r_1. Since we shall find that λ_o is of order 10^{-6} cm, then in a geometry having $r_1 \simeq 1$ cm, we have $v_1/v_o \simeq 10^{-6}$. Thus the magnetic field tends to press the circulating pairs firmly against the inner wall of the hollow cylinder.This same factor of about 10^{-6} holds throughout the surface layer.

11. RADIAL EQUILIBRIUM

If the magnetic field is so strong (or the velocity so small), how do the Cooper pairs remain confined to a given radius? The simplest reply is that the inner wall of the hollow cylinder confines them. But then we must worry about whether the surface layer of thickness λ_o gets "squashed flat". It seems we might really need our "rigid frictionless tube" to confine the orbits and maintain radial equilibrium. The "frictionless tube" exists. It is provided by radial electric fields. The magnetic field produces a tiny inwards drift of the Cooper pairs. This induces a small negative charge on the inner surface and simultaneously uncovers the same amount of positive nuclear charge spread through the thin layer. This gives a "radial-Hall-effect" electric field that halts the radial drift and establishes radial equilibrium. The distance the pairs drift radially before equilibrium is established is tiny compared with λ_o. For a reasonable magnetic field $B_1 = 0.1$ T the radial drift is about $10^{-12}\ \lambda_o$. (See App. C) The "squashing" of the surface layer is thus negligible. The "tube" is very rigid.

12. LONDON PENETRATION DISTANCE

Let us estimate the penetration distance λ_o. We can do that because Eq.(12) relates the induced velocity (and hence the current density) to the trapped magnetic field and the penetration distance, and we know how to calculate the magnetic field of a long solenoid, given the current.

First we find the total current I_1. That equals the maximum current density $J_1 = Nqv_1$ times the effective cross sectional area $\ell\lambda_o$ seen by the current:

$$I_1 = Nqv_1\ell\lambda_o. \tag{14}$$

The magnetic field B_1 is equivalent to that inside a long solenoid and is given by Ampere's circuital law (MKS units):

$$B_1\ell = \mu_o I_1. \tag{15}$$

Combining Eqs. (15), (14), and (12) gives

$$\lambda_o^2 = m/Nq^2\mu_o. \tag{16}$$

(We derive Eq.(16) more rigorously in App. A.)

Setting $q = 2e$, $m = 2m_e$, and $N = \frac{1}{2}N_e$, where N_e is the number of conduction electrons per unit volume (twice the number of Cooper pairs per unit volume) we find

$$\lambda_o = \left(\frac{m_e}{N_e e^2 \mu_o}\right)^{\frac{1}{2}}. \tag{17}$$

This result is identical with that of the Schroedinger theory.[1]

For lead we take $N_e = 3\times10^{22}$ per cc (one conduction electron per lead nucleus). Express everything in MKS units. Then Eq.(17) gives $\lambda_o = 3\times10^{-6}$ cm.

13. NUMBER OF COOPER PAIRS

How many Cooper pairs contribute to I_1? The effective volume is the length ℓ times the perimeter $2\pi r_1$ times the effective thickness λ_o of the surface layer. Thus the number of pairs equals $N\, 2\pi r_1 \lambda_o$. Taking $N=1.5\times10^{22}$ per cc, $\ell = 10$ cm, and $r_1 = 1$ cm gives 3×10^{18} pairs.

DISCUSSION

It is remarkable that the quantum relation (5), based on the "old quantum theory", which retains the concept of a definite orbit, gives many of the results of the more sophisticated Schroedinger theory. It gives the Meissner effect (App.A), the correct value for the flux quantum, and the correct value for the London penetration distance. Surprisingly, we also find by considering the exact solution of Eq.(5) for a solid cylinder (App.B) that there is no freedom to replace n by, for example, $n+\frac{1}{2}$. That is surprising because it is often the case that the "old quantum theory" gives a result that agrees with the Schroedinger theory in the limit of large quantum numbers but disagrees for low quantum numbers. That is apparently the case when we try to apply Eq.(5) to finding the energy levels of a single particle in a uniform magnetic field. We don't get the "zero-point energy". Thus the result of App. B was a pleasant surprise.

APPENDIX A. THE MEISSNER EFFECT

Start with the generalized quantum relation, Eq.(5). Rather than induced velocity v(r) we work with the induced current density

$$J(r) = Nqv(r) \tag{18}$$

Multiply Eq.(5) by rNq/m to get the equivalent quantum relation

$$rJ(r) + (Nq^2/2\pi m)\phi(< r) = n\ Nqh/2\pi m \tag{19}$$

The flux $\phi(< r)$ is the sum of an external flux plus the induced flux due to the superconducting current:

$$\phi(< r) = \phi_{ext}(< r) + \phi_{in}(< r) \tag{20}$$

The induced flux is due to induced magnetic field B_{in}:

$$\phi_{in}(< r) = \int_o^r B_{in}(r')2\pi r'dr' \tag{21}$$

The induced magnetic field is obtained by applying Ampere's circuital law to a long solenoid of length ℓ:

$$B_{in}(r')\ell = \mu_o I(> r') \tag{22}$$

where μ_o comes from the MKS units, and $I(>r')$ is the induced current at larger radii than r':

$$I(> r') = \int_{r'}^{r_2} J(r'')\ \ell dr'' \tag{23}$$

Inserting (20) through (23) into (19) and defining

$$\lambda_o^2 = m/Nq^2\mu_o, \tag{24}$$

the quantum relation (19) becomes

$$rJ(r) + \phi_{ext}(< r)/(2\pi\mu_o\lambda_o^2) + (1/\lambda_o^2)\int_o^r r'dr' \int_{r'}^{r_2} J(r'')dr'' = nNqh/2\pi m \tag{25}$$

Eq.(25) relates current density and external flux in such a way that, as we shall see, the total magnetic field and current density go to zero in the body of the superconductor (provided r_2-r_1 is large compared with λ_o). That is the Meissner effect.

In our example in the main text the external flux $\phi_{ext}(< r)$ is due to a permanent magnet carrying flux ϕ_o through the hollow space. In this Appendix we will instead consider the external flux to be due to a uniform external field B_o directed along the cylinder axis and filling all space. That gives

$$\phi_{ext}(< r) = B_o \pi r^2 \tag{26}$$

In order to solve Eq. (25) for J(r) we will differentiate it twice with respect to r and obtain a second order differential equation. In so doing we will lose the boundary conditions contained in (25), and will have to put them back when we have found a solution. First differentiate Eq. (25) once with respect to r. The first term gives $rdJ/dr + J$. The second term includes the factor $d\phi_{ext}(<r)/dr = 2\pi r B_o$. In differentiating the double integral with respect to r we simply erase the first integral sign and the dr', and replace r' by r wherever it appears. Thus (25) becomes

$$rdJ/dr + J + B_o r/(\mu_o \lambda_o^2) + (1/\lambda_o^2) r \int_r^{r_2} J(r'')dr'' = 0 \tag{27}$$

Notice that in obtaining (27) from (25) we have lost the quantum condition, that n be an integer. Any constant on the right side of Eq. (25) would have sufficed to give (27). Now divide (27) by r, and then differentiate with respect to r. The first two terms give $d^2J/dr^2 + (1/r)dJ/dr - J/r^2$; the next gives zero. Differentiating the integral gives a minus sign (because r is at the lower limit of the integral), erases the integral sign and the dr", and replaces r" by r. Then (27) becomes

$$d^2J/dr^2 + (1/r)dJ/dr - J/r^2 - (1/\lambda_o^2)J = 0 \tag{28}$$

In obtaining (28) from (27) we lost the numerical value of B_o and also lost the boundary condition that J is zero for $r > r_2$, and for $r < r_1$.

For convenience we introduce the dimensionless variable

$$x = r/\lambda_o. \tag{29}$$

Multiplying Eq.(28) by r^2 and using (29) we get

$$x^2(d^2J/dx^2) + x(dJ/dx) - (1+x^2)J = 0. \tag{30}$$

Equation (30) is an example of Bessel's modified differential equation[6]

$$x^2(d^2J/dx^2) + x(dJ/dx) - (s^2+x^2)J=0 \tag{31}$$

with, in our case, $s^2 = 1$. For the region where r is of order 1 cm, we have x of order 10^6. In that region we need the asymptotic solutions of (30) for large x. To obtain them consider (30). For large x neglect the "1" in the factor $(1 + x^2)$. (That also means we are considering s = 0 and s = 1 to be indistinguishable.) Then assume we can neglect $x(dJ/dx)$ compared with x^2J. After cancelling a common factor x^2 we then get $d^2J/dx^2 - J = 0$, with general solution

$$J(x) = C_1\exp(x) + C_2\exp(-x). \tag{32}$$

We easily verify that neglect of $x(dJ/dx)$ was justified for $x>>1$. The largest x for which there can be any non-zero J is $x = x_2 = r_2/\lambda_o$. Assume $J = J_2$ at $x = x_2$. As we go to smaller x, with r decreasing by as much as 1 cm and hency x decreasing by as much as 10^6, we must set $C_2 = 0$ in order to prevent "blow up" of J(x). That gives

$$J(x) = C_1\exp x = J_2\exp(x - x_2). \tag{33}$$

After r has decreased from r_2 to a few millimeters less than r_2, the current density (33) is negligible. However when we reach the inner surface at $r = r_1$ we have to consider again the general solution (32) and fit it to the boundary conditions at $r = r_1$.

Before doing that we shall verify that (33) provides exactly the right flux to cancel the external flux to zero inside the body of the superconductor, thus giving the Meissner effect. Go back to (27). Express it in terms of x. Divide by x. The result is

$$dJ/dx + J/x + B_o/(\lambda_o\mu_o) + \int_x^{x_2} J(x'')dx'' = 0 \qquad (34)$$

According to (33) we have $dJ/dx=J$. For our asymptotic solution we have neglected 1 compared with x, so we neglect J/x in (34). The integral, using (33), is just $J(x_2) - J(x)$. Then (34) becomes

$$J(x) + 0 + B_o/(\lambda_o\mu_o) + J(x_2) - J(x) = 0,$$

which gives

$$J(x_2) = J_2 = -B_o/(\lambda_o\mu_o). \qquad (35)$$

We can now find the induced magnetic field. Use (22) and (23) with (33) and (35) for J:

$$B_{in}(x) = \lambda_o\mu_o \int_x^{x_2} J(x'')dx'' = -B_o + B_o\exp(x - x_2). \qquad (36)$$

The total field, $B_o + B_{in}(x)$, equals B_o at $r = r_2$ and then falls exponentially to zero with decreasing radius. For r only 10^{-4} cm smaller than r_2 (and letting $\lambda_o = 3 \times 10^{-6}$cm) the magnetic field is down by a factor of about 10^{-15}, which we might as well call zero. Thus we have shown that Eq.(5) gives the Meissner effect. We may note that our Eq.(27), which gives the Meissner effect, would still hold if the "n" in Eq.(5) were replaced by n+½, or by any constant.

In all the results of this Appendix, so far, we have not used the fact that our cylinder is hollow, with inner radius r_1. If in fact the cylinder is solid, i.e., r_1 is zero, then the solution we have found can, to a first approximation, be extrapolated all the way to r = 0. That is only an approximation, because the solution given by (33) was to the asymptotic differential equation with x >> 1. In Appendix B we will use the exact equation so that

we can more carefully examine a solid cylinder all the way to the origin. But already we can see that, since the current I_2 we have found at $r = r_2$ is equivalent to the current in a long solenoid, it gives a uniform induced field for radii sufficiently smaller than r_2, and hence, in this example, cancels the external field B_o all the way to $r = 0$. For radii from slightly less than r_2 to $r = 0$, the state is indistinguishable from one for which there is no external field B_o and no surface current I_2 at $r = r_2$, just as we assumed in the main text.

Returning to our hollow cylinder with inner radius r_1, let us start at r_2 and proceed to smaller radii. The solution we have just found continues to hold as long as we have $x >> 1$. When we are at r slightly less than r_2 the induced surface current density has become completely negligible and the induced field has risen to its "cancellation" value of $-B_o$. As we approach the inner surface at $r = r_1$ the same differential equation, holds, namely $d^2J/dx^2 - J = 0$; but now, in order to satisfy the boundary conditions, we must superpose onto the existing solution (33) another term. This term will be due to surface current at $r = r_1$. Since we are still in the regime $x >> 1$, the new term must be picked from the general solution given by (32). In order to prevent "blow up" of this term for x increasing beyond $x = x_1$, we need the decreasing exponential. Thus we get

$$J(x) = J_1 \exp-(x - x_1). \qquad (37)$$

In the region between r_1 and r_2, J(x) is the superposition of (37) and (33). Of course, in most of that region they are both negligible, but each gives induced fields that extend to $r = 0$.

The magnetic field produced by I_1 is obtained by using (37) in Eqs. (22) and (23). The result is

$$B_{in}(x) = \mu_o \lambda_o J_1 \exp-(x-x_1). \qquad (38)$$

Eq.(38) gives the induced field for $r \geq r_1$. Of course for $r \leq r_1$ this field is constant and equal to $B_1 = \mu_o \lambda_o J_1$; this relation, with Eqs.(18) and (16) verifies our qualitatively derived Eq.(12).

The total flux, from r = 0 to a radius r that is sufficiently greater than r_1 (i.e., $r - r_1 >> \lambda_o$), is given by integrating (38) from $r = r_1$ to this sufficiently large r, and adding that flux to the flux $\pi r_1^2 B_1$ from $r<r_1$. The result is

$$\phi = (\pi r_1^2 + 2\pi r_1 \lambda_o) B_1 = nh/q \qquad (39)$$

where the setting of this flux equal to nh/q came from (19), with J(r) = 0 for $r - r_1 >> \lambda_o$. Recall that in this same region (from r=0 to r several times λ_o greater than r_1) the external magnetic field B_o is completely cancelled by the induced field due to the current I_2 on the outer surface.

If we now slowly turn down the external field to zero the surface current I_2 will also go to zero. The current I_1 will not change. We will be left with trapped flux given by Eq.(39). Note that not only the flux is quantized, but also the field value B_1 and, because of Eq.(12), the current density J_1 at $r=r_1$. That implies that the current I_1 is quantized.

APPENDIX B. THE QUANTUM NUMBER IS n, NOT n +½.

Consider a solid superconducting cylinder of radius r_2 in a uniform external field B_o directed along the axis. All of the discussion of Appendix A that led to the solution (33) still holds. But now we have no inner surface, so that we need not only the asymptotic solution given by (33) but also the solution all the way in to the origin. When we go all the way to the origin we can of course distinguish between s = 0 and s = 1 in (31). Even though our equation has s = 1 it turns out that the solution for s = 0 will be useful. By using a power series expansion at x = 0 we easily verify that the solutions of (31) for s = 0 and s = 1 that do not blow up at the origin are[6]

$$s = 0: \; I_o(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \tag{40}$$

$$s = 1: \; I_1(x) = \frac{x}{2} + \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \ldots \tag{41}$$

with

$$dI_o(x)/dx = I_1(x). \tag{42}$$

The function $I_o(x)$ resembles coshx; $I_1(x)$ resembles sinhx. At large x, I_o and I_1 are both proportional to exp(x), and are nearly indistinguishable as discussed after Eq.(31). We now take the exact solution of (30) corresponding to the asymptotic solution (33), to be

$$J(x) = A \, I_1(x) \tag{43}$$

with A as unknown coefficient. The quantum condition (25) expressed in terms of x becomes

$$xJ + \frac{B_o x^2}{2\lambda_o \mu_o} + \int_o^x x'dx' \int_{x'}^{x_2} J(x'')dx'' = \frac{nNqh}{2\pi m\lambda_o} . \tag{44}$$

Using (43) and (42) we find

$$\int_{x'}^{x_2} J(x'')dx'' = A[I_o(x_2) - I_o(x')]. \tag{45}$$

Then

$$\int_o^x x'dx' \int_{x'}^{x_2} J(x'')dx'' = \tfrac{1}{2} AI_o(x_2)x^2 - A \int_o^x x'I_o(x')dx'. \tag{46}$$

Multiplying (40) by x and integrating term by term, and then comparing with (41), we find

$$\int_o^x x'I_o(x')dx' = xI_1(x). \tag{47}$$

Inserting (47) into (46), and then (46) into (44) we find the happy result that the term $xI_1(x)$ cancels and we are left with a simple result that holds for all values of x:

$$\frac{1}{2} x^2[B_o/\lambda_o\mu_o + AI_o(x_2)] = \frac{nNqh}{2\pi m\lambda_o} . \tag{48}$$

The right side of (48) is independent of x. Therefore the left side must vanish. That gives

$$A = -B_o/\lambda_o\mu_o I_o(x_2) \tag{49}$$

and the exact result

$$n=0. \tag{50}$$

At this point we may recall our "enthusiasm tempering" discussion after Eq. (9) where we pointed out that in using the "old quantum theory" we could not be sure whether we should use n or n+½. In the present example of a solid cylinder in a uniform external field, Eq. (48) can only be satisfied for n=0. That is if we replace n by n+½ and then set n=0 there is no solution. We conclude that at least for this case we can resolve the ambiguity and firmly choose the quantum number to be n, not n+½, without resorting to the Schroedinger theory.

To complete the present example we find the magnetic field for all values of x. The field is obtained from Eqs.(22), (23), (45), and (49). The result is

$$B_{tot}(x) = B_{in}(x) + B_o$$
$$= B_o I_o(x)/I_o(x_2) \qquad (51)$$

This agrees with Eq.(33), except that (51) holds all the way in to x=0 whereas (33) holds only for $x \gg 1$. Remarkably, they agree at x=0 as well as at $x \gg 1$.

To summarize: for a solid cylinder the magnetic field is essentially excluded from the intire cylinder and flux-quantum integer n is exactly zero.

APPENDIX C. RADIAL EQUILIBRIUM: THE RADIAL HALL EFFECT

We have found that everywhere in the thin surface-current layer at $r = r_1$ the magnetic field is r_1/λ_o times larger than that which would give an equilibrium cyclotron orbit. Therefore the orbits of the Cooper pairs will drift to slightly smaller radii until a negative surface charge is induced on the inner surface, and a positive nuclear charge is slightly uncovered throughout the surface layer. The drift will continue until radial equilibrium is established between the induced radial electric field E_r and the magnetic field. Since we have found that the magnetic force is of order 10^6 times mv^2/r, the electric force must nearly cancel the magnetic force at all radii. At $r=r_1$ that gives $qE_r=qv_1B_1$ Since v and B each fall off exponentially with increasing r [see Eqs.(37) and (38)] that gives E_r as a function of r:

$$E_r=v_1B_1\exp\left[-2(r-r_1)/\lambda_o\right] \qquad (52)$$

Integrating Eq.(52) from $r=r_1$ to r_2, with $r_2-r_1 >> \lambda_o$ gives a radial "Hall effect" emf

$$V_r = \tfrac{1}{2}v_1B_1\,\lambda_o = \tfrac{1}{2}qB_1^2\lambda_o^2/m \qquad (53)$$

where in the last step we used Eq.(12) to eliminate v_1. For B_1=0.1T, $m=2m_e$, $q=2e$, and $\lambda_o = 3\times10^{-6}$ cm this gives V_r=0.8 microvolts.

How far do the Cooper pairs drift before they establish the equilibrium electric field? (It had better be small compared with λ_o or we are in deep trouble!) The largest drift needed will be where there is the largest B, at $r = r_1$. The radial field there is $E_r = v_1B_1 = q\lambda_oB_1^2/m$. The corresponding total induced charge Q is given by using Gauss' Law: $E_r 2\pi r_1 \ell = Q/\varepsilon_o$ Let the drift distance be d. Then Q is given by the pairs in the volume of the very thin layer of thickness d: $Q=Nq2\pi r_1 \ell d$. Putting all these together gives

$$d/\lambda_o = B_1^2 \varepsilon_o / Nm. \qquad (54)$$

For $B_o = 0.1T$ and $N = 1.5 \times 10^{22}$ per cc we get (putting everything first into MKS) $d/\lambda_o = 3 \times 10^{-12}$. Thus the "frictionless tube" provided by the radial electric field is extremely rigid. There is no tendency for the surface layer to be "squashed flat".

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