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Peter Schattner and Mahiko Suzuki

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THE RENORMALIZATION GROUP AND THE SMALL MOMENTUM LIMIT OF FIELD THEORY

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Abstract: Renormalization group techniques are developed which, in contrast to those previously discussed, do not involve expanding a massive field theory around its massless limit. The methods are useful in infrared problems. The validity of perturbative expansions of the renormalization group functions is examined in quantum electrodynamics. The technique is illustrated in quantum electrodynamics and in a σ -type model of the pion.

1. Introduction

The methods of the renormalization group have provided a technique for extracting information from field theories even in the presence of strong coupling. Generally, the applications of these

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methods have been restricted to kinematic regions where the scale of all or some of the momenta is much larger than any of the masses or dimensional couplings in the theory. If only some of the momenta are scaled to infinity, Wilson's operator product expansion (1) must be used to supplement the renormalization group. In the deep Euclidean region, particle masses are found to be ignorable, or capable of being treated by small correction terms to the solution for massless theories. As a result, one succeeds in reducing the problem to one where there is essentially only one mass scale. However, one is often faced with a problem where there are more than one intrinsically important mass scales, for example, in the infrared momentum region (2).[†] It is proposed in the present paper to solve a slightly different version of the renormalization group equation and indicate how the solution might be applicable to obtain information about the small momentum behavior of a field theory. There already exist formal solutions of renormalization group equations valid at arbitrary momenta (3,4,5),^{††} but they are useful in the deep Euclidean region where the effective masses can be expanded in the Taylor series around zero. Because of explicit inclusion of different scale parameters, the present solution is interesting in its own right as well as being more useful for application in the infrared region.

The paper is organized as follows: In section 2 the arguments leading to the renormalization group equation are briefly reviewed and it is explained how the equation we solve is related to the "new

[†] The effects of differing mass scales in the renormalization group are studied in ref. 2 from a different viewpoint.

^{††} The relation between ref. 3 and ref. 4 is explained in ref. 5.

renormalization group equation" (3,4,5) and the Callan-Symanzik equation (6,7). We then give the general solution to our renormalization equation. In the Appendix, the proof is elaborated for the solution given in section 2. In Section 3 we examine under what circumstances a perturbative expansion is justified for the various characteristic functions of the renormalization group. Quantum electrodynamics is carefully examined in this respect. In section 4, the application of our approach is illustrated first in the infrared limit of quantum electrodynamics with the electron mass kept nonvanishing and then in a σ -type model incorporating a partially conserved axial vector current (PCAC). The last section contains a short summary and discussion of the results.

2. Renormalization Group Equations and Their Solutions

There are at least two different approaches to the derivation of renormalization group equations in field theory. In the first approach, the starting point is the scaling Ward identities of the unrenormalized cut-off theory (6). The second method, which is useful only for renormalizable theories, proceeds by examining scale changes in renormalization points of the renormalized Green's functions. Working in the unrenormalized theory has the advantage of being relevant to nonrenormalizable theories. For renormalizable theories, however, the second method tends to be more transparent and to lead to equations which are easier to solve for arbitrary nondeeper-Euclidean momenta. We shall use this latter approach exclusively.

Following S. Weinberg (3), our starting point is the simple but fundamental equation expressing multiplicative renormalizability of Green's functions:

$$\Gamma(p_i, g, m, \mu) = Z_\Gamma(g, m/\Lambda, \mu/\Lambda) \Gamma_u(p_i, g_0, m_0, \Lambda), \quad (2.1)$$

where p_i are the external momenta; g and m (g_0 and m_0) are the renormalized (unrenormalized) coupling constant and mass; Λ is some cutoff or regulator mass used to define the unrenormalized amplitude; μ symbolizes any explicit dependence on the renormalization point; Γ and Γ_u are respectively renormalized and unrenormalized Green's functions; and Z_Γ is the wave function renormalization constant. Unless explicitly stated otherwise we will restrict ourselves generally in this section to theories with one mass and one dimensionless coupling constant. The rather straightforward generalizations to more complicated situations will be mentioned later.

Renormalization group equations are derived by varying (2.1) with respect to the scale of the renormalization point, μ , and noting that Γ_u is μ independent. Now different renormalization prescriptions will lead to different μ dependences of Z_Γ and Γ , and consequently to somewhat different renormalization group equations. The work of Weinberg (3) and 't Hooft (4) has shown that changing renormalization prescriptions can affect the ease of solution of the resulting equation and also its applicability.

The essence of the Weinberg-'t Hooft approach is to renormalize the theory in such a way that the renormalization constants are independent of the masses. For example, in Weinberg's method, one renormalizes all vertices and propagators at external momenta characterized by the scale " μ " and with all internal propagators massless. Mass renormalization is done multiplicatively as $m = m_0 Z_\theta^{-1}$, where Z_θ is the renormalization constant for a mass insertion on an internal line. As a result of this prescription,

eq. (2.1) becomes;

$$\Gamma(p, g, m, \mu) = Z_T(g, \mu/\Lambda) \Gamma_u(p, g_0, m_0, \Lambda) \quad (2.2)$$

The μ independence of Γ_u and naive dimension analysis of Γ together lead one to

$$\left\{ \kappa \frac{\partial}{\partial \kappa} - \beta(g) \frac{\partial}{\partial g} + (1 + \gamma_\theta(g)) m \frac{\partial}{\partial m} + \gamma_\Gamma(g) - d \right\} \Gamma(\kappa p, g, m, \mu) = 0, \quad (2.3)$$

where

$$\beta(g) \equiv \mu \frac{\partial}{\partial \mu} g \Big|_{g_0, \Lambda \text{ fixed}}, \quad (2.4)$$

$$\gamma_\Gamma(g) \equiv \mu \frac{\partial}{\partial \mu} \ln Z_\Gamma \Big|_{g_0, \Lambda \text{ fixed}}, \quad (2.5)$$

$$\gamma_\theta(g) \equiv \mu \frac{\partial}{\partial \mu} \ln Z_\theta \Big|_{g_0, \Lambda \text{ fixed}} = -\mu \frac{\partial}{\partial \mu} \ln m \Big|_{g_0, m_0, \Lambda \text{ fixed}}, \quad (2.6)$$

and d is the canonical dimension of Γ . The fact that eq. (2.6) is homogeneous and that the functions β , γ_Γ , and γ_θ depend on only one variable g make this equation easy to solve at arbitrary values of κ . Indeed, defining an "effective coupling constant and mass" by

$$\kappa \frac{\partial}{\partial \kappa} \tilde{g}(\kappa) = \beta(\tilde{g}(\kappa)); \quad \tilde{g}(1) = g, \quad (2.7)$$

$$\kappa \frac{\partial}{\partial \kappa} \tilde{m}(\kappa) = -\tilde{m}(1 + \gamma_\theta(\tilde{g}(\kappa))); \quad \tilde{m}(1) = m, \quad (2.8)$$

one observes that

$$\Gamma(\kappa p, g, m, \mu) \equiv \kappa^d \Gamma(p, \tilde{g}(\kappa), \tilde{m}(\kappa), \mu) \exp \left\{ - \int_0^{\ln \kappa} \gamma_\Gamma(\tilde{g}(\kappa')) d \ln \kappa' \right\} \quad (2.9)$$

satisfies eq. (2.3). Here the effective mass $\tilde{m}(\kappa)$ is related to the physical mass in a complicated manner. This is not a drawback if one wishes to analyze the solutions in the ultraviolet region where the effects of mass terms are small so that Γ may be expanded around $\tilde{m}(\kappa) = 0$. However, if we want to apply the renormalization group in the infrared region, where masses are important, field theory should not be expanded around the corresponding massless theory. We will develop below an alternative method of the renormalization group that is particularly suitable to studying the infrared limit of field theories involving nonvanishing masses.

We first give our prescription of renormalization.

(1) The coupling constant is renormalized at $p_1^2 = -\mu^2$ with μ being an arbitrary value. All the results will remain valid even if the renormalization point is changed to, for instance, $p_1^2 = -(c_1 \mu)^2$ where c_1 's are just numbers.

(2) The mass is subtractively renormalized also at $p^2 = -\mu^2$.

Later we will study the case that the mass is renormalized on shell. In either case, the prescription that the mass not be renormalized at a light-like point makes our solution useful not only in the ultraviolet region, but also in the infrared region. Renormalization constants are computed with conventional massive internal propagators. Now the Z factors are mass dependent

$$Z = Z(g, m/\Lambda, \mu/\Lambda) \quad (2.10)$$

in contrast to that in Weinberg's prescription (3).

The renormalization group equation follows in the standard manner. The μ independence of Γ_u in (2.1) gives us

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta(g, m/\mu) \frac{\partial}{\partial g} + \gamma_m(g, m/\mu) m \frac{\partial}{\partial m} - \gamma_\Gamma(g, m/\mu) \right\} \Gamma(\kappa p, g, m, \mu) = 0, \quad (2.11)$$

where

$$\beta(g, m/\mu) \equiv \mu \frac{\partial}{\partial \mu} g \Big|_{g_0, m_0, \Lambda \text{ fixed}}, \quad (2.12)$$

$$\gamma_m(g, m/\mu) \equiv \mu \frac{\partial}{\partial \mu} \ln m \Big|_{g_0, m_0, \Lambda \text{ fixed}}, \quad (2.13)$$

$$\gamma_\Gamma(g, m/\mu) \equiv \mu \frac{\partial}{\partial \mu} \ln Z_\Gamma \Big|_{g_0, m_0, \Lambda \text{ fixed}}. \quad (2.14)$$

As usual β , γ_m , and γ_Γ are cutoff independent since they appear in equations for renormalized amplitudes. One can derive explicit formulas to express them in terms only of the renormalized $\Gamma^{(2)}$ and $\Gamma^{(3)}$ (or $\Gamma^{(4)}$) (6). Using standard dimension analysis of Γ transforms (2.11) into

$$\left\{ \kappa \frac{\partial}{\partial \kappa} - \beta(g, m/\mu) \frac{\partial}{\partial g} - (-1 + \gamma_m(g, m/\mu)) m \frac{\partial}{\partial m} + \gamma_\Gamma(g, m/\mu) - d \right\} \times \Gamma(\kappa p, g, m, \mu) = 0. \quad (2.15)$$

This equation resembles closely that of Weinberg (2.3), but the important difference is that β , γ_m , and γ_Γ are functions of two variables. This makes the equation more difficult to solve, and the usual procedure[†] is to go immediately into the ultraviolet region and assume that $\Gamma(\kappa p, g, m, \mu)$ is well approximated in that region by some function $\Gamma_{as}(\kappa p, g, \mu)$, where Γ_{as} satisfies (2.15) with $\beta(g, m/\mu)$ and $\gamma_\Gamma(g, m/\mu)$ replaced by $\beta(g, 0)$ and $\gamma_\Gamma(g, 0)$, respectively. However, we can show that with a certain minor technical assumption eq. (2.15) can also be solved at arbitrary momenta.

We define an "effective" mass $m(\kappa)$ and an "effective" coupling constant $g(\kappa)$ through the solutions to coupled differential equations

$$\kappa \frac{d}{d\kappa} g(\kappa) = \beta(g(\kappa), m(\kappa)/\mu), \quad (2.16)$$

$$\kappa \frac{d}{d\kappa} m(\kappa) = \left\{ -1 + \gamma_m(g(\kappa), m(\kappa)/\mu) \right\} m(\kappa), \quad (2.17)$$

with initial conditions

$$g(1) = g \quad \text{and} \quad m(1) = m. \quad (2.18)$$

The solution to eq. (2.15) is then given by

$$\Gamma(\kappa p, g, m, \mu) = \kappa^d \Gamma(p, g(\kappa), m(\kappa), \mu) \times \exp \left\{ - \int_0^{\ln \kappa} \gamma_\Gamma(g(\kappa'), m(\kappa')) d \ln \kappa' \right\}. \quad (2.19)$$

[†] For a clear exposition of the relation of different renormalization procedures to different renormalization equations, see section V of ref. 3.

If one regards $\{-1 + \gamma_m(g, m/\mu)\}m$ as the function β for a "dimensional coupling constant" m , eq. (2.19) looks very natural. Since to our knowledge there has been no detailed explicit proof given for such a problem, however, the proof is elaborated in the Appendix for the sake of completeness. In the case that the mass m is renormalized on shell and therefore μ independent, the renormalization group equation (2.15) is modified into

$$\left\{ \kappa \frac{\partial}{\partial \kappa} - \beta(g, m/\mu) \frac{\partial}{\partial g} + m \frac{\partial}{\partial m} + \gamma_\Gamma(g, m/\mu) - d \right\} \Gamma(\kappa p, g, m, \mu) = 0, \quad (2.15)$$

and the solution is given by (2.19) with $m(\kappa) = m/\kappa$. In this case eq. (2.16) is much easier to solve. We will study quantum electrodynamics in this method in section 4.

The generalization to more coupling constants and/or masses is straightforward. One has to solve a set of more complicated differential equations. It helps in simplifying them to use the on-shell mass renormalization whenever possible. Another simple and interesting generalization is to theories with two masses where external momenta and one of the particle masses m become very small as compared with the other mass M . With m renormalized on shell, the renormalization group equation becomes

$$\left\{ \kappa \frac{\partial}{\partial \kappa} - \beta(g, m/\mu, M/\mu) \frac{\partial}{\partial g} - (-1 + \gamma_M(g, m/\mu, M/\mu)) \right. \\ \left. \times M \frac{\partial}{\partial M} + \gamma_\Gamma(g, m/\mu, M/\mu) - d \right\} \times \Gamma(\kappa p, g, \kappa m, M, \mu) = 0, \quad (2.15'')$$

and its solution is given by

$$\Gamma(\kappa p, g, \kappa m, M, \mu) = \kappa^d \Gamma(p, g(\kappa), m, M(\kappa), \mu) \\ \times \exp \left\{ - \int_0^{\ln \kappa} \gamma_\Gamma(g(\kappa'), M(\kappa')) d \ln \kappa' \right\}, \quad (2.19')$$

where $g(\kappa)$ and $M(\kappa)$ are given by the solutions to

$$\kappa \frac{d}{d\kappa} g(\kappa) = \beta(g(\kappa), m/\mu, M(\kappa)/\mu), \quad (2.16')$$

$$\kappa \frac{d}{d\kappa} M(\kappa) = \left\{ -1 + \gamma_M(g(\kappa), m/\mu, M(\kappa)/\mu) \right\} M(\kappa), \quad (2.17')$$

with $g(1) = g$ and $M(1) = M$. Later in section 4, we will use this method to study soft pion amplitudes in the limit of the pion mass being zero.

3. Perturbation expansions of renormalization group functions

We will want to use our expressions for the Green's functions in the infrared region. We would like to do this for theories which are infrared stable. For any practical application we have to calculate β and γ 's in perturbation expansion in the coupling constant g . The perturbation expansion in g is justified in the conventional method of the renormalization group in which β and γ 's depend only on g , not on the mass m . If one has in mind applications to ultraviolet problems, one may calculate β and γ 's in the massless theory. In the present method, however, β and γ 's are functions of two variables g and m/μ . We must not expand β or

γ 's around $m/\mu = 0$ when we are trying to use our method in the infrared region. This difference is important and to be paid special attention to, when β and γ 's are calculated in perturbation expansion. In the infrared limit, $m(\kappa)$ increases indefinitely. We must examine in each theory whether or not the dependence on m/μ destroys the validity of the perturbation expansion in $g(\kappa)$, even if $g(\kappa) \rightarrow 0$ as $\kappa \rightarrow 0$. We choose here quantum electrodynamics as an example to show that the smallness of $g(\kappa)$ does justify the expansion even in the presence of $m(\kappa)/\mu$ ($\rightarrow \infty$ as $\kappa \rightarrow 0$).

In quantum electrodynamics, the Γ 's are the proper self-energy parts, the proper vertex part, or else are Greens' functions to be calculated from them through skeleton diagrams. The function β is, for instance, expanded as

$$\beta(e(\kappa), m(\kappa)/\mu) \equiv \mu \frac{\partial}{\partial \mu} \ln \left\{ 1 - \frac{\partial}{\partial q^2} \Pi(q^2) \right\} \bigg|_{q^2 = -\mu^2} \bigg|_{e_0, m_0, \Lambda \text{ fixed}}$$

$$= f_1(m(\kappa)/\mu) e^3(\kappa) + f_2(m(\kappa)/\mu) e^5(\kappa) + \dots \quad (3.1)$$

In order to retain only the first term in calculating $e(\kappa)$, one has to prove that $f_n(m(\kappa)/\mu)$ is not so singular at $\kappa \rightarrow 0$ as to compensate higher powers of $e(\kappa)$. The asymptotic behavior of $f_n(m(\kappa)/\mu)$ at $m(\kappa) \rightarrow \infty$ with μ fixed, or equivalently at $\mu \rightarrow 0$ with $m(\kappa)$ fixed is obtained from that of $\Pi(q^2)$, the proper self-energy part of photon. $\Pi(q^2)$, viewed as a function of q^2 , is singular on the positive real axis starting at $q^2 = 0$. The diagrams that contain

intermediate states consisting of only photons may give rise to a singularity at $q^2 = 0$, while intermediate states containing at least one e^+e^- pair contribute to singularities at or above $4m^2$. The part of $\Pi(q^2)$ due to the latter class of intermediate states is regular at $q^2 = 0$, and therefore, when it is substituted into (3.1), it contributes to $f_n(m(\kappa)/\mu)$ terms like

$$f_n(m(\kappa)/\mu) = c_1^{(n)} \left(\frac{\mu}{m(\kappa)} \right)^2 + c_2^{(n)} \left(\frac{\mu}{m(\kappa)} \right)^4 + \dots \quad (3.2)$$

The N photon intermediate states give rise to an absorptive part of the form of

$$\text{Im } \Pi(q^2) \underset{q^2 \rightarrow 0}{\sim} (q^2)^{N-2} \times |A(q^2)|^2 \theta(q^2) \quad (N \geq 3), \quad (3.3)$$

where $(q^2)^{N-2}$ comes from the N photon phase volume and $A(q^2)$ is the $(\gamma \rightarrow N\gamma)$ amplitude. (See fig. 1.) If $A(0) \neq 0$, this absorptive part for $N = 3$ would lead to a dispersive part for $\Pi(q^2)$ like

$$\Pi(q^2) \underset{q^2 \rightarrow 0}{\sim} q^2 \ln q^2 \times |A(0)|^2. \quad (3.4)$$

If this were the case, $f_n(m(\kappa)/\mu)$ could be more singular than $f_1(m(\kappa)/\mu)$, thus invalidating the simple perturbation expansion. However, gauge invariance demands that $A(q^2)$ must vanish with a certain power of q^2 . The dangerous multiphoton diagrams, therefore, are less singular than they might look at first sight so that the leading term of $f_n(m/\mu)$ at $m/\mu \rightarrow \infty$ is of the order of $(\mu/m)^2$ for all n . The same method of exploiting gauge invariance can be useful to show that the infrared limit (with electrons far off shell) of the proper vertex function is regular when the scale of the external momenta approaches zero. The proper self-energy of the electron is

regular at $p_\mu = 0$, since the singularity starts at $p^2 = m^2$. We have thus shown that the proper self-energy and vertex parts behave at $\kappa \rightarrow 0$ in the same way as their lowest orders. Validity of the perturbation expansion can be demonstrated for γ_m and γ_Γ in the same way as for β . It should be emphasized here that the applicability of the perturbative expansion is proven through gauge invariance. In the problem of chiral symmetry breaking to be discussed in the following section, the PCAC condition or the Adler zero plays the role of gauge invariance. In theories without such a condition, perturbative expansions must be treated with special caution.

4. Applications

We will illustrate our method of the renormalization group equation in two examples. The first example is the infrared limit of the quantum electrodynamics of a massless photon interacting with a massive fermion. Here the infrared limit means the limit in which all external four momenta approach zero, and therefore the fermion is off shell. We will see how the nonvanishing fermion mass affects the effective coupling constant $e(\kappa)$. The second example is the soft pion limit of pion amplitudes where the pion mass is scaled to zero as well as the external momenta. We will obtain some insight into the corrections due to chiral symmetry breaking.

1. Quantum electrodynamics

We consider the infrared limit of the photon-electron system described by the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m_0)\psi - e_0 \bar{\psi}\gamma_\mu \psi A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (4.1)$$

Although this limit has little bearing on actual physical processes, it is a simple and illustrative exercise. For $m_0 = 0$ the theory is free asymptotically in the infrared limit and the effective coupling constant $e(\kappa)$ is given by

$$e^2(\kappa) \simeq \left[-\frac{1}{6\pi^2} \ln \kappa \right]^{-1} \quad (4.2)$$

But, here we will keep the electron mass nonvanishing to see how $e(\kappa)$ is modified near $\kappa = 0$.

The electron mass is renormalized on the physical mass shell, or it is defined through a zero on the positive real axis of the inverse electron propagator,

$$S_F^{-1}(\gamma p) \Big|_{\gamma p = m} = 0 \quad (4.3)$$

With this mass renormalization the differential equation for $m(\kappa)$ is greatly simplified, as we pointed out in section 2, into

$$\kappa \frac{dm(\kappa)}{d\kappa} = -m(\kappa) \quad (4.4)$$

and therefore

$$m(\kappa) = \frac{1}{\kappa} m \quad (4.5)$$

The multiplicative renormalization Z_1 , Z_2 , and Z_3 is done at an arbitrarily chosen space-like point $-\mu^2$,

$$q_\mu Z_1^{-1} \equiv \frac{1}{4} \text{Tr}(\gamma_\mu q^\nu \Gamma_\nu) \Big|_{(p+q)^2 = p^2 = -\mu^2} \quad (4.6)$$

$$Z_2^{-1} \equiv 1 - \Sigma_1(p^2) \Big|_{p^2 = -\mu^2}, \quad (4.7)$$

$$Z_3^{-1} \equiv 1 - \frac{\partial}{\partial q^2} \Pi(q^2) \Big|_{q^2 = -\mu^2}, \quad (4.8)$$

where the right-hand sides are related to the unrenormalized proper vertex and inverse propagators through

$$\Gamma_\mu(p, q) = \gamma_\mu \Gamma_1 + \sigma_{\mu\nu} q^\nu \Gamma_2 + p_\mu (\gamma q) \Gamma_3 + \dots, \quad (4.9)$$

$$S_F^{-1}(p) = (\gamma p)(1 - \Sigma_1) - m - \Sigma_2, \quad (4.10)$$

$$D_{F\mu\nu}(q) = (g_{\mu\nu} - q_\mu q_\nu / q^2)(q^2 - \Pi)^{-1}. \quad (4.11)$$

The vertex renormalization Z_1 is so chosen as to maintain gauge invariance. The Ward-Takahashi identity proves that $\text{Tr}(\gamma_\mu q^\nu \Gamma_\nu)$ is a function only of $(p+q)^2$ and p^2 , not of q^2 and further that Z_1 and Z_2 thus defined are related just as in the case of the on-shell renormalization,

$$Z_1 = Z_2. \quad (4.12)$$

A one-particle-irreducible Green's function satisfies the renormalization group equation given in (2.15'),

$$\left(\kappa \frac{\partial}{\partial \kappa} - \beta \frac{\partial}{\partial e} + m \frac{\partial}{\partial m} + \gamma_\Gamma - d \right) \Gamma(\kappa p, e, m, \mu) = 0. \quad (4.13)$$

The function β is calculated to the single loop order as

$$\beta = \frac{e^3}{12\pi^2} f_1(m/\mu), \quad (4.14)$$

where

$$f_1(m/\mu) \equiv 1 + \frac{\frac{12m^4}{\mu^4}}{1 + \frac{4m^2}{\mu^2}} + \frac{\frac{12m^4}{\mu^4} \left(1 + \frac{2m^2}{\mu^2}\right)}{\left[1 + \frac{4m^2}{\mu^2}\right]^{3/2}} \ln \left(\frac{\sqrt{1 + \frac{4m^2}{\mu^2}} - 1}{\sqrt{1 + \frac{4m^2}{\mu^2}} + 1} \right). \quad (4.15)$$

In contrast to the case of $m_0 = m = 0$, the effective coupling $e(\kappa)$ is not expected to shrink to zero as $\kappa \rightarrow 0$, but it approaches a nonvanishing finite value. The single loop approximation to β is justified not simply by going to a sufficiently small κp , but rather by demanding, as an additional requirement, that the initial value $e^2(1)$ can be chosen to be small enough to ignore higher-loop corrections in $e^2(1)$. This requirement is expected to be met in real quantum electrodynamics with $\alpha = 1/137$.

Solving the differential equation for $e(\kappa)$, we obtain

$$e^2(\kappa) \approx \left[\frac{1}{e^2(1)} - \frac{1}{6\pi^2} \int_0^{\ln \kappa} f_1(m/\kappa' \mu) d \ln \kappa' \right]^{-1}, \quad (4.16)$$

which reduces to (4.2) if $\frac{m}{\mu} \ll \kappa \ll 1$. As κ approaches zero ($\kappa \ll \frac{m}{\mu}$),

$$e^2(\kappa) \approx \left[\frac{1}{e^2(1)} + \frac{7}{192\pi^2} \cdot \frac{\mu^2}{m^2} (1 - \kappa^2) + O(\kappa^4) \right]^{-1}. \quad (4.17)$$

The κ dependence of $e^2(\kappa)$ is modified near $\kappa = 0$ due to $m \neq 0$, as is shown in fig. 2.

2. Soft pion amplitudes at $m_\pi \rightarrow 0$

We apply our method to the infrared asymptotic behavior of π - σ systems in the limit of both the pion mass and external momenta scaled to zero. We first consider a system of π and σ in which all renormalizable coupling constants are treated as independent, but PCAC holds up to small corrections of order ϵ . The asymptotic behavior is expanded in two independent small parameters, $(\ln \kappa)^{-1}$ and ϵ . The results are given in eqs. (4.42), (4.43), and (4.47). We next study the σ model in which the renormalizable couplings are related to each other through the pion mass, the σ mass, and one independent coupling constant. Corrections to PCAC are related to the scale parameter κ . The asymptotic behavior is now expanded in a single parameter κ . The results are given in the one loop approximation (see eqs. (4.33), (4.34), and (4.36)). They are compared with the previous results without the renormalization group. In both cases we search for solutions with specific asymptotic behavior in κ , as postulated in (4.39), which the infrared stableness suggests. We will not discuss on whether or not those solutions are really chosen by nature, nor whether or not there is another solution with a nonzero fixed point in any of the coupling constants.

The first problem is a boson system consisting of a light neutral "pion" and a heavy scalar meson (hereafter called a σ -meson). The Lagrangian for this system is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_0^2 \phi^2 + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} M_0^2 \sigma^2 \\ & - \lambda_0 \phi^4 - 2\lambda'_0 \phi^2 \sigma^2 - \lambda''_0 \sigma^4 - g_0 \sigma \phi^2 - g'_0 \sigma^3, \end{aligned} \quad (4.18)$$

where ϕ and σ stand for the fields of π and σ , respectively. This Lagrangian would reduce to that of the simplest (truncated) σ -model if the coupling constants were subject to the constraints,

$$\lambda_0 = \lambda'_0 = \lambda''_0, \quad (4.19)$$

$$g_0 = g'_0, \quad (4.20)$$

$$g_0^2 / \lambda_0 = 2(M_0^2 - m_0^2). \quad (4.21)$$

But we will not impose these conditions here. The couplings are assumed to be independent and close to their values of the σ -model of the massless pion. With this small deviation of couplings from their σ -model values, PCAC theorems are valid up to corrections of order ϵ , a small parameter related to the deviation from the σ model. This fact will later be used to justify the single loop approximation to the function β of the renormalization group.

We are interested here in the limiting behavior of N-point pion amplitudes when the pion mass and external pion momenta both go to zero in the σ -like-model of (4.18) where PCAC holds approximately. The pion amplitudes are expanded in two parameters, the scale of the pion mass and momenta, κ , and the correction, ϵ , to PCAC. We renormalize the masses and coupling constants as follows:

(1) The pion mass m is renormalized on the physical mass shell through $\Delta_{\pi\pi}^{-1}(m^2) = 0$. The pion wave function renormalization $Z_{3,\pi}$ is done at an arbitrarily chosen space-time point $-u^2$,

$$Z_{3,\pi}^{-1} \equiv \left. \frac{\partial}{\partial q^2} \Delta_{\pi F}^{-1}(q^2) \right|_{q^2 = -\mu^2} \quad (4.22)$$

(2) The 4π vertex is renormalized at the space-like symmetric point $(q_i q_j) = -\frac{\mu^2}{3} (4\delta_{ij} - 1)$,

$$Z_{1,4\pi}^{-1} = \Gamma_{4\pi}(q_1^2 = q_2^2 = q_3^2 = q_4^2 = -\mu^2, \\ (q_1 + q_2)^2 = (q_1 + q_3)^2 = -\frac{4}{3}\mu^2) \quad (4.23)$$

(3) The σ -mass is renormalized at $-\frac{4}{3}\mu^2$ in line with (2) above,

$$M^2 = Z_{3,\sigma} \Delta_{\sigma F}^{-1}(-\frac{4}{3}\mu^2). \quad (4.24)$$

In the limit of $M \gg m$, the σ meson is highly unstable and the mass M as renormalized in (4.24) has little physical meaning. The σ wave function is renormalized at a light-like point,

$$Z_{3,\sigma}^{-1} = \left. \frac{\partial}{\partial p^2} \Delta_{\sigma F}^{-1}(p^2) \right|_{p^2=0} \quad (4.25)$$

(4) The $\sigma\pi\pi$ vertex is renormalized as

$$Z_{1,\sigma\pi\pi}^{-1} = \Gamma_{\sigma\pi\pi}(q_1^2 = q_2^2 = -\mu^2, (q_1 + q_2)^2 = -\frac{4}{3}\mu^2) \quad (4.26)$$

(5) The other vertices are renormalized at light-like points for simplicity,

$$Z_{1,\sigma\sigma\pi\pi}^{-1} = \Gamma_{\sigma\sigma\pi\pi}(p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0, \\ (p_1 + p_2)^2 = (p_1 + p_3)^2 = 0) \quad (4.27)$$

$$Z_{1,3\sigma}^{-1} = \Gamma_{3\sigma}(p_1^2 = p_2^2 = p_3^2 = 0) \quad (4.28)$$

$$Z_{1,4\sigma}^{-1} = \Gamma_{4\sigma}(p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0, \\ (p_1 + p_2)^2 = (p_1 + p_3)^2 = 0) \quad (4.29)$$

where p 's and q 's refer to the σ and π momenta, respectively. A one-particle-irreducible Green's function satisfies the renormalization group equation given in (2.15),

$$\left(\kappa \frac{\partial}{\partial \kappa} - \sum_i \beta_i \frac{\partial}{\partial f_i} - (-1 + \gamma_M) M \frac{\partial}{\partial M} + \frac{N_\pi}{2} \gamma_\pi + \frac{N_\sigma}{2} \gamma_\sigma - d \right) \\ \times \Gamma(\kappa q, f_i, \kappa m, M, \mu) = 0 \quad (4.30)$$

where f_i stands for dimensionless couplings $\lambda, \lambda', \lambda'', g^2/M^2$, and g'^2/M^2 , and N_π and N_σ are the numbers of external π and σ lines with γ_π and γ_σ being the anomalous dimensions of π and σ fields. Because of the off-shell σ -mass renormalization (4.24), we have the anomalous dimension term of the σ -mass,

$$\gamma_M \equiv \mu \frac{\partial}{\partial \mu} \ln M \Big|_{m_0, M_0, f_{0i}, \Lambda} \quad (4.31)$$

With the renormalization prescription given above, we obtain for the effective mass and couplings,

$$\kappa \frac{dM^2}{d\kappa} = - \left\{ 2 - \frac{1}{2\pi^2} \left(\frac{g^2}{2M^2} \right) \rho(m, \mu) \right\} M^2 \quad (4.32)$$

$$\kappa \frac{d\lambda}{d\kappa} = \frac{1}{2\pi^2} \left\{ 9\lambda^2 - 12\lambda \left(\frac{g^2}{2M^2} \right) + 4 \left(\frac{g^2}{2M^2} \right)^2 \right\} \rho(m, \mu) \quad (4.33)$$

$$\kappa \frac{d(g^2/2M^2)}{d\kappa} = \frac{1}{2\pi^2} \left\{ 6\lambda \left(\frac{g^2}{2M^2} \right) - 5 \left(\frac{g^2}{2M^2} \right)^2 \right\} \rho(m, \mu) \quad (4.34)$$

$$\kappa \frac{d\lambda'}{d\kappa} = 0 \quad (4.35)$$

$$\kappa \frac{d\lambda''}{d\kappa} = 0 \quad (4.36)$$

$$\kappa \frac{dg'}{d\kappa} = 0 \quad (4.37)$$

where $\rho(m, \mu)$ is a positive definite function given by

$$\rho(m, \mu) \equiv \int_0^1 \frac{\frac{4}{3} \mu^2 x(1-x)}{m^2 + \frac{4}{3} \mu^2 x(1-x)} dx \quad (4.38)$$

In writing these equations we keep in mind that we are looking for solutions with the following asymptotic behavior

$$M^2(\kappa) \sim \kappa^{-2} \quad \text{up to } \log \kappa$$

$$\lambda(\kappa) \text{ and } g^2/2M^2(\kappa) \sim (\log \kappa)^{-1},$$

$$\lambda'(\kappa), \lambda''(\kappa), \text{ and } g'^2(\kappa)/2M^2(\kappa) \sim \text{constant up to } \ln \kappa.$$

(4.39)

Equations (4.32)-(4.34) have been calculated up to one loop, and the terms of $O(m^2, \mu^2)$ in (4.32) and of $O(m^2/M^2, \mu^2/M^2)$ in (4.33) and (4.34) are suppressed as well as the contribution from high loop diagrams. In general, higher loops give $O(M^2(\ln \kappa)^{-2})$ and $O(M^2 \epsilon (\ln \kappa)^{-1})$ to (4.32), and $O((\ln \kappa)^{-3})$ and $O(\epsilon (\ln \kappa)^{-2})$ to (4.33) and (4.34). It is noted here that the higher loops do not accumulate a power of $\ln M^2$ in the right-hand sides of (4.32)-(4.34) if PCAC is exact, and that if PCAC is approximate, a power of $\ln M^2$ enters there with a coefficient of $O(\epsilon)$. These statements can be proved through arguments exactly parallel to those used in the case of quantum electrodynamics (see section 3). The "dangerous" multipion intermediate states are softened by the PCAC condition in the soft pion region just as the multiphoton intermediate states are softened by gauge invariance in the infrared region. Equations (4.35)-(4.37) are true to all orders according to our prescription of renormalization (5). They are solved trivially to be independent of κ .

Equations (4.33) and (4.34) can be combined into

$$\kappa \frac{d}{d\kappa} \left(\lambda(\kappa) - \frac{g^2(\kappa)}{2M^2(\kappa)} \right) = \frac{9}{2\pi^2} \rho(m, \mu) \left(\lambda(\kappa) - \frac{g^2(\kappa)}{2M^2(\kappa)} \right)^2 \quad (4.40)$$

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We must be careful about the higher loop corrections suppressed in (4.33) and (4.34) since $\lambda - \frac{g^2}{2M^2}$ is of the order ϵ of the PCAC corrections. The right-hand side of (4.40) may have the corrections of the order of $(\ln \kappa)^{-3}$. The solution to (4.40) is given by

$$\lambda(\kappa) - \frac{g^2(\kappa)}{2M^2(\kappa)} = \left[\frac{1}{\epsilon} - \frac{9}{2\pi^2} \rho(m, \mu) \ln \kappa \right]^{-1}, \quad (4.41)$$

where $\epsilon \equiv [\lambda(1) - g^2(1)/2M^2(1)]$. Our real interest is in the region where ϵ is much smaller than $(\ln \kappa)^{-1}$, since the PCAC corrections are expected to be typically of the order of the squared pion mass. We can now solve for $g^2(\kappa)/2M^2(\kappa)$ from (4.34) and (4.41) in the region of $\epsilon \ll (\ln \kappa)^{-1}$, and then $\lambda(\kappa)$ through (4.41). The solutions are given by

$$\lambda(\kappa) = \left[-\frac{\rho}{2\pi^2} \ln \kappa \right]^{-1} - \frac{17}{4} \epsilon, \quad (4.42)$$

$$g^2(\kappa)/2M^2(\kappa) = \left[-\frac{\rho}{2\pi^2} \ln \kappa \right]^{-1} - \frac{21}{4} \epsilon, \quad (4.43)$$

where the terms of $O((\ln \kappa)^{-2})$ and $O(\epsilon(\ln \kappa)^{-1})$ are suppressed in the right-hand sides. From (4.32) the solution to $M^2(\kappa)$ is obtained as

$$M^2(\kappa) = M^2(1)(\kappa^2 \ln \kappa)^{-1}. \quad (4.44)$$

From (4.37) and (4.44), we find that $g'^2(\kappa)/2M^2(\kappa) \sim \ln \kappa$. These asymptotic behaviors are consistent with our assumption (4.39).

We now look into the asymptotic behavior of N-point pion amplitudes. One finds from our renormalization prescription (4.22) and (4.25) that the anomalous dimensions of the pion and σ fields are given by

$$\gamma_\pi = \frac{g^2}{2M^2} \times O\left(\frac{m^2}{M^2}, \frac{\mu^2}{M^2}\right), \quad (4.45)$$

$$\gamma_\sigma = 0, \quad (4.46)$$

in the one-loop approximation. The one-loop approximation is justified by the same argument as for the β 's. We can then calculate the one-particle-irreducible Green's functions through our general formula (2.19'). It can be shown by the study of perturbation diagrams that in the limit of m^2 , μ^2 , and, $(q_1, q_j) \ll M^2$ of N-pion amplitudes, in general, the couplings λ' , λ'' , and $g'^2/2M^2$ enter with an extra suppressing power of M^{-2} as compared with λ and $g^2/2M^2$; each time that the $\sigma\pi\pi$, σ^4 , or σ^3 coupling enters, it brings more σ propagators in than the π^4 or $\sigma\pi\pi$ coupling does. Therefore, the leading asymptotic behaviors of the N-pion amplitudes expanded in the two parameters ϵ and $(\ln \kappa)^{-1}$ are obtained from the corresponding tree diagrams that contain only λ and $g^2/2M$. One can calculate in this way that the 4π amplitude is

$$T_{4\pi}(\kappa q_1, \kappa m, M) = \lambda(\kappa) - \frac{g^2(\kappa)}{M^2(\kappa)} + O\left(\frac{\lambda(\kappa)(q_1 q_j)}{M^2}\right),$$

$$= \left[\frac{1}{\epsilon} - \frac{9}{2\pi^2} \rho \ln \kappa \right]^{-1} + O((\ln \kappa)^{-2}), \quad (4.47)$$

which reduces to $\sim \epsilon + O((\ln \kappa)^{-2})$ in the region of $\epsilon \ll (\ln \kappa)^{-1}$. In the present approach to PCAC corrections, where ϵ is taken as a parameter independent of κ , the big disadvantage is that the higher loop contribution is, in general, of $O((\ln \kappa)^{-2})$ and therefore that the usefulness of our result (4.47) is limited to the region of $(\ln \kappa)^{-2} \ll \epsilon \ll 1$.

In the real σ -model the five coupling constants and the two masses are related to each other through the three independent parameters, the chiral symmetric coupling, the bare σ -mass, and the bare pion mass as a chiral symmetry breaking term. In such a model only one of the five couplings is to be chosen as independent, and the other four are to be calculated from it. Let us choose λ as the independent coupling together with the two independent mass parameters M^2 and m^2 . Calculating up to one loop, with the same renormalization prescription as before, we have found that $g^2/2M^2$ is related to λ through

$$g^2/2M^2 = \lambda + O\left(\frac{m^2}{M^2} \lambda, \frac{\mu^2}{M^2} \lambda^2\right). \quad (4.48)$$

where no power of $\ln M^2$ is multiplied with $(m^2/M^2)\lambda$ nor $(\mu^2/M^2)\lambda^2$. A factor of $\ln M^2$ could appear in the next higher order. For the other couplings, we have found to the same order

$$\lambda' = \lambda + \lambda^2 \times O(1), \quad (4.49)$$

$$\lambda'' = \lambda + \lambda^2 \times O(1), \quad (4.50)$$

$$g'^2/2M^2 = \lambda + \lambda \times O(1), \quad (4.51)$$

where $O(1)$ stands for the zeroth order in M^2 and $\ln M^2$. The next order correction is of the order of $(m^2/M^2)\ln M^2$ and $(\mu^2/M^2)\ln M^2$. It is plausible that (4.48)-(4.51) are valid to all orders in the perturbation expansion of the σ -model. Beyond a plausibility argument, we have been unable to prove this to all orders. We therefore restrict the rest of the argument to the one loop approximation improved by the renormalization group. The independent coupling λ satisfies

$$\kappa \frac{d\lambda}{d\kappa} = \frac{1}{2\pi^2} \rho(m, \mu) \lambda^2 + \lambda^2 \times O\left(\frac{m^2}{M^2}, \frac{\mu^2}{M^2}\right), \quad (4.52)$$

where the next higher order correction is $\lambda^2 \times O(M^{-4} \ln M^2)$. This leads us to the solution

$$\lambda(\kappa) = \left[-\frac{\rho}{2\pi^2} \ln \kappa + O(\kappa^2) \right]^{-1}, \quad (4.53)$$

and

$$M^2(\kappa) = M^2(1)(\kappa^2 \ln \kappa)^{-1}. \quad (4.54)$$

If we apply this result to the 4π amplitude, we find that

$$T_{4\pi}(\kappa q_1, \kappa m, M) = \lambda(\kappa) - \frac{g^2(\kappa)}{2M^2(\kappa)} + \lambda(\kappa) \times O\left(\frac{(q_1 q_j)}{M^2}\right) + \lambda^2(\kappa) f\left(\frac{m^2}{M^2}, \frac{\mu^2}{M^2}, \frac{(q_1 q_j)}{M^2}\right), \quad (4.55)$$

where $f\left(\frac{m^2}{M^2}, \frac{\mu^2}{M^2}, \frac{(q_1 q_j)}{M^2}\right)$ turns out to vanish linearly in the three variables with the higher order corrections of $\sim O(M^{-4} \ln M^2)$ as the variables approach zero. Combining (4.48) and (4.52) with (4.55), we find that as $\kappa \rightarrow 0$

$$T_{4\pi}(\kappa q_1, \kappa m, M) \simeq c \kappa^2 + O(\kappa^4 \ln \kappa), \quad (4.56)$$

where c is a κ -independent number. The leading correction term due to the chiral symmetry breaking is $\sim O(\kappa^2)$ without any $\ln \kappa$ factor. This is in agreement with the single loop calculation by Guralnik et al. (8) who investigated a more complicated version of σ -model, but our result contains more information because it is derived from the renormalization group, admittedly with the assumption that the renormalization group functions can be approximated by their one-loop expansions. Note that in our expression the logarithmic factors from $\lambda(\kappa)$ and $M^2(\kappa)$ are exactly cancelled to give $\sim O(\kappa^2)$ (the pion mass squared). However, we may have to expect logarithmic factors in higher order because the typical $\ln \kappa$ factors of the renormalization group may no longer be cancelled away. This logarithmic correction may be regarded as the type of PCAC corrections that Li and Pagels have been emphasizing (9).

5. Summary

The techniques of the renormalization group developed here are straightforward generalizations of those already known. In ultraviolet problems, they lead to the same results as the known techniques. Our method is particularly useful in infrared problems since the field theory need not be approximated by nor expanded around the corresponding

massless theory. However, the validity of the usual perturbation expansions is by no means automatic even if a theory is infrared stable in the ordinary sense. If $\log M$, with M being a mass scale much larger than external momenta, accumulates in power, one can not use perturbation expansions to calculate the relevant functions. This will happen in theories where the infrared behavior becomes more and more singular in higher order perturbation diagrams. This would be the case in a scalar meson theory in which the scalar meson mass is much lighter than other mass scales; there is no PCAC or the like in such a theory.

In general, the coupled differential equations for the effective coupling constants and masses are not easy to solve. The on-shell mass renormalization makes the differential equation for the effective mass decouple from the rest, so that the rest of the differential equations will become less formidable. In the σ -type model we have examined in this paper, we have been guided by PCAC to choose the renormalization prescription so as to decouple a certain linear combination of the coupling constants from the rest. Such symmetry consideration helps in simplifying the differential equations.

We have chosen the two applications to illustrate our method in relativistic quantum field theory. Considering its special usefulness in infrared problems, we expect that the present method of renormalization group will turn out to be relevant to those problems outside of relativistic quantum field theory that are formulated in different field theories (10) or even to those not formulated in any field theory (11).[†]

[†] Reference 11 reviews various applications of renormalization group techniques to problems in solid state physics. All of them are, in our language, infrared problems.

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Appendix

We prove here that Γ given in (2.19) of the text is really a solution to eq. (2.15). Our problem is in mathematical structure very similar to that of a massless theory involving two coupling constants, one dimensionless and the other dimensional, if we regard $(-1 + \gamma_m)m$ as β for m . Although our solution looks almost obvious from this viewpoint, the proof is by no means trivial. So we give it below.

Let us substitute our solution

$$\Gamma(\kappa p, g, m, \mu) = \kappa^d \Gamma(p, g(\kappa), m(\kappa), \mu) \times \exp \left\{ - \int_0^{\ln \kappa} \gamma_\Gamma(g(\kappa'), m(\kappa')) d \ln \kappa' \right\} \quad (A.1)$$

into the renormalization group equation

$$\left\{ \kappa \frac{\partial}{\partial \kappa} - \beta(g, m/\mu) \frac{\partial}{\partial g} + (1 - \gamma_m(g, m/\mu))m \frac{\partial}{\partial m} + \gamma_\Gamma(g, m/\mu) - d \right\} \Gamma(\kappa p, g, m, \mu) = 0 \quad (A.2)$$

We obtain

$$\left\{ F_g(\kappa) \frac{\partial}{\partial g(\kappa)} + F_m(\kappa) \frac{\partial}{\partial m(\kappa)} + F_d(\kappa) \right\} \Gamma(p, g(\kappa), m(\kappa), \mu) = 0, \quad (A.3)$$

with

$$F_g(\kappa) \equiv \beta(g(\kappa), m(\kappa)/\mu) - \beta(g, m/\mu) \frac{\partial g(\kappa)}{\partial g} \Big|_{\kappa, m \text{ fixed}} + (1 - \gamma_m(g, m/\mu)) m \frac{\partial g(\kappa)}{\partial m} \Big|_{\kappa, g \text{ fixed}}, \quad (A.4)$$

$$F_m(\kappa) \equiv (-1 + \gamma_m(g(\kappa), m(\kappa)/\mu)) m(\kappa) - \beta(g, m/\mu) \frac{\partial m(\kappa)}{\partial g} \Big|_{\kappa, m \text{ fixed}} + (1 - \gamma_m(g, m/\mu)) m \frac{\partial m(\kappa)}{\partial m} \Big|_{\kappa, g \text{ fixed}}, \quad (A.5)$$

$$F_d(\kappa) \equiv \gamma_\Gamma(g, m/\mu) - \gamma_\Gamma(g(\kappa), m(\kappa)/\mu) + \int_0^{\ln \kappa} d \ln \kappa' \left\{ \beta(g, m/\mu) \frac{\partial g(\kappa')}{\partial g} \Big|_{\kappa', m} - (1 - \gamma_m(g, m/\mu)) m \frac{\partial g(\kappa')}{\partial m} \Big|_{\kappa', g} \right\} \frac{\partial \gamma_\Gamma(\kappa')}{\partial g(\kappa')} \Big|_{m(\kappa')} + \int_0^{\ln \kappa} d \ln \kappa' \left\{ \beta(g, m/\mu) \frac{\partial m(\kappa')}{\partial g} \Big|_{\kappa', m} - (1 - \gamma_m(g, m/\mu)) m \frac{\partial m(\kappa')}{\partial m} \Big|_{\kappa', g} \right\} \frac{\partial \gamma_\Gamma(\kappa')}{\partial m(\kappa')} \Big|_{g(\kappa')}. \quad (A.6)$$

We first show that $F_g(\kappa)$ and $F_m(\kappa)$ are identically equal to zero (at least in the region of κ analytically continued from the point $\kappa = 1$), by showing that $F_g(\kappa)$, $F_m(\kappa)$, and all of their derivatives vanish at $\kappa = 1$. To this end, we note the initial condition that

$$F_g(1) = F_m(1) = 0, \quad (A.7)$$

since $g(1) = g$ and $m(1) = m$ imply:

$$\frac{\partial g(\kappa)}{\partial g} \Big|_{m \text{ fixed}, \kappa=1} = 1, \quad (A.8)$$

$$\frac{\partial g(\kappa)}{\partial m} \Big|_{g \text{ fixed}, \kappa=1} = 0, \quad (A.9)$$

$$\frac{\partial m(\kappa)}{\partial g} \Big|_{m \text{ fixed}, \kappa=1} = 0, \quad (A.10)$$

$$\frac{\partial m(\kappa)}{\partial m} \Big|_{g \text{ fixed}, \kappa=1} = 1. \quad (A.11)$$

Next, taking the logarithmic derivative in κ of $F_g(\kappa)$ and $F_m(\kappa)$, we find after a little algebra,

$$\kappa \frac{\partial}{\partial \kappa} F_g(\kappa) = \frac{\partial \beta(g(\kappa), m(\kappa)/\mu)}{\partial g(\kappa)} \Big|_{m(\kappa)} F_g(\kappa) + \frac{\partial \beta(g(\kappa), m(\kappa)/\mu)}{\partial m(\kappa)} \Big|_{g(\kappa)} F_m(\kappa), \quad (A.12)$$

and

$$\kappa \frac{\partial}{\partial \kappa} F_m(\kappa) = m(\kappa) \frac{\partial \gamma_m(g(\kappa), m(\kappa)/\mu)}{\partial g(\kappa)} \bigg|_{m(\kappa)} F_g(\kappa) + \left\{ -1 + \gamma_m(g(\kappa), m(\kappa)/\mu) + m(\kappa) \frac{\partial \gamma_m(g(\kappa), m(\kappa)/\mu)}{\partial m(\kappa)} \bigg|_{g(\kappa)} \right\} F_m(\kappa) \quad (A.13)$$

Evaluating (A.12) and (A.13) at $\kappa = 1$ by use of the initial condition (A.7) and the assumption that the four coefficients of $F_g(\kappa)$ and $F_m(\kappa)$ on the right-hand sides of (A.12) and (A.13) exist and are finite at $\kappa = 1$, we conclude that

$$\frac{\partial F_g(\kappa)}{\partial \kappa} \bigg|_{\kappa=1} = \frac{\partial F_m(\kappa)}{\partial \kappa} \bigg|_{\kappa=1} = 0 \quad (A.14)$$

By taking further logarithmic derivatives of (A.12) and (A.13) and making the assumption that the n th derivatives of these four coefficients are finite for arbitrary n , we can conclude that $F_g(\kappa)$, $F_m(\kappa)$, and all of their derivatives vanish at $\kappa = 1$ and hence that at least in some neighborhood of $\kappa = 1$, both $F_g(\kappa)$ and $F_m(\kappa)$ vanish identically. The technical assumption made on the four coefficients above is satisfied if they are regular at $\kappa = 1$. This is the case in the applications that we will discuss later in section 4. If not, $g(\kappa)$ and $m(\kappa)$ themselves would be, in general, non-analytic in κ at $\kappa = 1$.

Having shown that $F_g(\kappa) = F_m(\kappa) \equiv 0$, it is now straightforward to show $F_d(\kappa) = 0$:

$$\begin{aligned} F_d(\kappa) &= \gamma_\Gamma(g, m/\mu) - \gamma_\Gamma(g(\kappa), m(\kappa)/\mu) \\ &+ \int_0^{\ln \kappa} d \ln \kappa' \beta(g(\kappa'), m(\kappa')/\mu) \frac{\partial \gamma_\Gamma(\kappa')}{\partial g(\kappa')} \bigg|_{m(\kappa')} \\ &+ \int_0^{\ln \kappa} d \ln \kappa' \left\{ -1 + \gamma_m(g(\kappa'), m(\kappa')/\mu) \right\} m(\kappa') \frac{\partial \gamma_\Gamma(\kappa')}{\partial m(\kappa')} \bigg|_{g(\kappa')} \\ &= \gamma_\Gamma(g, m/\mu) - \gamma_\Gamma(g(\kappa), m(\kappa)/\mu) \\ &+ \int_0^{\ln \kappa} d \ln \kappa' \frac{d}{d \kappa'} \gamma_\Gamma(g(\kappa'), m(\kappa')/\mu) \quad , \\ &= 0 \quad (A.15) \end{aligned}$$

We have used here eqs. (2.16) and (2.17) and the fact that $F_g(\kappa) = F_m(\kappa) = 0$. This completes the demonstration that at least in the region of κ where $g(\kappa)$ and $m(\kappa)$ are regular, eq. (A.1) is a solution of eq. (A.2). In the case that m is renormalized on shell, the proof remains valid with $\gamma_M \equiv 0$. The generalization to more renormalized coupling constants and masses is straightforward.

0 0 0 0 4 3 0 0 5 7 1

Figure Captions

Fig. 1: A dangerous diagram of $\Pi(q^2)$ in perturbation expansion.

Fig. 2: The effective coupling $e(\kappa)$ versus κ . The solid curve is the case of $m \neq 0$ as given in (4.16), while the broken curve is the case of $m = 0$.

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Fig. 1.

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Fig. 2.

0 0 0 4 3 0 0 3 7 3

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