UC Santa Barbara

Departmental Working Papers

Title

Simple Economies with Multiple Equilibria

Permalink

<https://escholarship.org/uc/item/6qv909xs>

Authors

Bergstrom, Ted C
Shimomura, Ken-Ichi
Yamato, Takehiko

Publication Date

2008-07-09

Abstract

This paper studies a general class of pure exchange economies that have multiple equilibria, which extend an example presented by Shapley and Shubik [8]. For such economies, we find easily verified conditions that determine whether there are multiple equilibria. We also provide simple methods for constructing economies in which arbitrary pre-specified sets of prices are equilibria. These economies have simple comparative statics, since prices at interior competitive equilibrium depend on the parameters of utility but not on the endowment quantities.

Simple Economies with Multiple Equilibria

Theodore C. Bergstrom¹ Ken-Ichi Shimomura²,
Takehiko Yamato^{3 4}

July 9, 2008

¹Department of Economics, University of California Santa Barbara email: tedb@econ.ucsb.edu

²Research Institute for Economics and Business Administration, Kobe University, email: ken-ichi@rieb.kobe-u.ac.jp

³Graduate School of Decision Science and Technology, Tokyo Institute of Technology, email: yamato@valdes.titech.ac.jp

⁴This paper is a result of discussions among the authors that began during Bergstrom's visit to Kobe University in September 2006. We are grateful to RIEB at Kobe University for funding this visit and providing a stimulating working environment. Shimomura and Yamato thank the Division of the Humanities and Social Sciences at California Institute of Technology for their hospitality during the period when this draft was written.

1 Introduction

Lloyd Shapley and Martin Shubik [8] found a clever trick to construct a simple exchange economy that has three distinct competitive equilibria. There are two consumers and two goods. Both consumers have quasi-linear utility functions, but their utilities are linear in opposite goods, and each consumer's initial endowment includes only the good in which his own utility is linear. This trick works because strong income effects act in the opposite direction from substitution effects for both consumers, and consequently the aggregate excess demand function is increasing in price over part of its domain.

The Shapley-Shubik paper consists of a single specific example, in which the nonlinear parts of the utility functions are exponential. Although their example does not admit a closed form solution, Shapley and Shubik exhibit numerical solutions for three distinct competitive equilibria.

In this paper, we explore several families of two-person¹ economies of the Shapley-Shubik type, where the nonlinear portions of utility functions take alternative special forms. We find general conditions that are necessary and sufficient for the existence of multiple equilibria and we show how to solve for equilibrium prices, given specified utility functions.

In order to create examples to illustrate a theoretical point or to use in an experiment, it is often desirable to “work backwards” by choosing a set of distinct prices that are to be the equilibria and then finding parameters of utility functions that make these prices the equilibria of the resulting economy. This is often easier than finding the equilibrium prices corresponding to pre-specified parameters.

We are aware of two somewhat related articles in the literature. Alok Kumar and Martin Shubik [5] use computational methods to explore the geometry of the core as the number of replicas of the two participants in the Shapley-Shubik example is increased. Steven Gjerstad [4] constructs

¹This discussion applies equally well to “replica economies” with two types of consumers and equal numbers of each type.

examples of relatively simple economies with multiple equilibria in which consumers have differing homothetic constant-elasticity-of-substitution utility functions.

2 Shapley-Shubik Economies

We define the family of *Shapley-Shubik economies* as follows. There are two consumers and two goods. Consumer 1 is endowed with a positive amount $\bar{x}$ of good X and zero units of good Y . Consumer 2 is endowed with a positive amount $\bar{y}$ of Y and zero units of X . Consumers 1 and 2 have utility functions

$$\begin{aligned} U_1(x, y) &= x + f_1(y) \\ U_2(x, y) &= f_2(x) + y, \end{aligned} \tag{1}$$

where the $f_i(\cdot)$ are strictly concave, continuously differentiable functions from $[0, \infty)$ to the extended real line and where $\lim_{z \rightarrow \infty} f'_i(z) \leq 0$. For $i = 1, 2$, let $B_i = \lim_{z \rightarrow 0} f'_i(z)$.²

Excess demand depends only on relative prices. Therefore we can make good X the *numeraire* and set its price equal to one. Define $x_i(p)$ and $y_i(p)$ to be consumer i 's demands for goods X and Y when the price of X is 1 and the price of Y is p . In a two-commodity economy, Walras' law implies that if excess demand is zero in one market it will also be zero in the other. Therefore if we denote excess demand for good Y by $E(p)$, a price $\bar{p}$ will be an equilibrium price if and only if $E(\bar{p}) = 0$. An equilibrium price $\bar{p}$ will be stable or unstable under the usual competitive dynamics, depending on whether $E'(\bar{p}) < 0$ or $E'(\bar{p}) > 0$.

Since $f'_i(\cdot)$ is a strictly decreasing function, we can define the *inverse*

²We will consider Shapley-Shubik economies in which $B_i = f'_i(0)$ is finite and positive as well cases in which $f'_i(z)$ becomes arbitrarily large as z becomes small, in which case $B_i = \infty$.

marginal utility function $\phi_i(p) = f_i'^{-1}(p)$ on $(0, B_i]$.³ If B_i is finite, it must be that $\phi_i(B_i) = 0$ and for all $p \in (0, B_i)$, $\phi_i(p) > 0$ and $\phi_i'(p) < 0$.

Consumer 1's budget constraint is

$$x_1 + py_1 = \bar{x}. \quad (2)$$

If Consumer 1 demands a positive amount of both goods at price p , her marginal rate of substitution between y and x must be equal to p . This implies that $f_1'(y_1(p)) = p$ and hence

$$y_1(p) = f_1'^{-1}(p) = \phi_1(p). \quad (3)$$

Consumer 2's budget equation is

$$x_2 + py_2 = p\bar{y}. \quad (4)$$

At an interior solution for Consumer 2, it must be that $f_2'(x_2(p)) = 1/p$ and hence

$$x_2(p) = f_2'^{-1}\left(\frac{1}{p}\right) = \phi_2\left(\frac{1}{p}\right). \quad (5)$$

If $x_2(p)$ is given by Equation 5, then Consumer 2's budget equation implies that

$$y_2(p) - \bar{y} = -\frac{1}{p}x_2(p) = -\frac{1}{p}\phi_2(1/p) \quad (6)$$

From Equations 3 and 6, it follows that: if both consumers choose positive quantities of both goods at price p , then excess demand for Y at price p is

$$E(p) = \phi_1(p) - (1/p)\phi_2(1/p). \quad (7)$$

Definition 1. *Consumer i 's demands are interior at price p if i demands positive amounts of both goods at price p . A price p is an interior competitive*

³Conversely, if $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are positive valued strictly decreasing functions, we can recover utility functions $f_i(\cdot)$ such that $\phi_i(p) = f_i'^{-1}(p)$.

equilibrium price if demands of both consumers are interior at price p and if $E(p) = 0$.

From Equation 7 and the definition of interior competitive equilibrium, it is immediate that:

Remark 1. *The price p is an interior competitive equilibrium price if and only if demands of both consumers are interior at p and*

$$\phi_1(p) = \frac{1}{p}\phi_2(1/p). \quad (8)$$

A consumer's demand can fail to be interior in two different ways. One possibility is that he demands none of the good in which his utility is non-linear. If $B_1 = \infty$, then at every price p , Consumer 1 will buy some positive amount of Y . But if $B_1 = f'_1(0)$ is finite, then at any price $p \geq B_1$, Consumer 1 will buy none of Y . Consumer 2's utility is nonlinear in X and the price of X relative to Y is $1/p$. If $B_2 = \infty$, Consumer 2 will buy a positive amount of X at any price. If $B_2 = f'_2(0)$ is finite, then Consumer 2 will demand none of X if $B_2 \leq 1/p$. The lowest price at which a consumer demands none of a good is commonly known as its "choke price."

Consumer 1's demand is linear in X and he will demand a positive amount of X only if after purchasing $\phi(p)$ units of Y at price p , he has income left to spend on X . Since Consumer 1's income is $\bar{x}$, this is the case if $p\phi(p) < \bar{x}$. Consumer 2's demand is linear in Y and he consumes positive amounts of Good Y at price p only if after purchasing $\phi(1/p)$ units of X , he has income left to spend on Y . Since Consumer 2's income is $p\bar{y}$, this is the case if $\phi_2(1/p) < p\bar{y}$.

These considerations motivate the following *interiority conditions*:

Definition 2. *For a Shapley-Shubik economy, the interiority conditions are satisfied at $p > 0$ if $1/B_2 < p < B_1$, $p\phi_1(p) < \bar{x}$, and $\phi_2(1/p) < p\bar{y}$.*

and the following remark:

Remark 2. *In a Shapley-Shubik economy, demands of both consumers are interior at price p if and only if the interiority conditions of Definition 2 are satisfied at p .*

We have the following result:

Lemma 1. *In a Shapley-Shubik economy where $B_i = \lim_{z \rightarrow 0} f'(z)$ and where $B_1 > 1/B_2$, excess demand for Y is negative for all prices $p \geq B_1$, and positive at all prices $p \leq 1/B_2$.*

Proof. Let $E_i(p)$ denote excess demand for Y by Consumer i . For all prices $p \geq B_1$, $E_1(p) = 0$. Consumer 2's budget is $x_2 + py_2 = p\bar{y}$. Rearranging the budget constraint we see that $E_2(p) = -x_2(p)/p$, where $x_2(p)$ is the amount of good 2 demanded by Consumer 2. Therefore $E_2(p)$ will be negative if $x_2(p)$ is positive. Consumer 2 will demand a positive amount of X so long as $1/p < B_2$. But $p \geq B_1 > 1/B_2$ and hence $1/p < B_2$. It follows that $x_2(p) > 0$ and hence $E_2(p) < 0$. Therefore $E(p) = E_1(p) + E_2(p) < 0$ for all $p \geq B_1$. A similar argument shows that if $p \leq 1/B_2$, then $E_2(p) = 0$ and $E_1(p) > 0$. Therefore $E(p) > 0$ for all $p \leq 1/B_2$. □

Applying Lemma 1, we can show that if the choke prices B_1 and B_2 are finite and if initial endowments are sufficiently large, then there exists an interior competitive equilibrium.

Theorem 1. *In a Shapley-Shubik economy where $B_1 = f'_1(0)$ and $B_2 = f'_2(0)$ are finite and $B_1 > 1/B_2$, there exists at least one competitive equilibrium $\bar{p} \in (1/B_2, B_1)$. The price $\bar{p}$ is an interior competitive equilibrium if and only if $\bar{p}\phi_1(\bar{p}) < \bar{x}$, $\phi_2(1/\bar{p}) < \bar{p}\bar{y}$, and $\phi_1(\bar{p}) = (1/\bar{p})\phi_2(1/\bar{p})$.*

Proof. In a Shapley-Shubik economy, utility functions are strictly quasi-concave and continuous, and incomes are positive whenever prices are positive. Therefore, by a well-known result on the continuity of demand functions, the excess demand function $E(p)$ is continuous over $(1/B_2, B_1)$. Ac-

cording to Lemma 1, $E(p) > 0$ for all $p \leq 1/B_2$ and $E(p) < 0$ for all $p \geq B_1$. Continuity implies that $E(p) = 0$ for some $\bar{p} \in (1/B_2, B_1)$. □

Shapley-Shubik economies yield a strong comparative statics result that is a counterpart to Paul Samuelson's Non-substitution Theorem [7] for an economy where commodities are produced from other commodities with constant returns to scale and where there is only one non-produced input. In Samuelson's economy, competitive equilibrium prices are determined by the "supply-side" alone, and are not affected by changes in the demand functions. For a Shapley-Shubik economy, interior competitive equilibrium prices are determined by the "demand-side" alone and are not affected by changes in the supply of Goods X and Y so long as endowments remain sufficiently large to satisfy the interiority conditions. The following "neutrality theorem" follows from Remark 1 and Theorem 1:

Theorem 2. *If $\bar{p}$ is an interior competitive equilibrium price for a Shapley-Shubik economy with initial endowments $\bar{x}$ and $\bar{y}$, then $\bar{p}$ will also be an interior competitive equilibrium price for endowments x and y , so long as $\bar{p}\phi_1(\bar{p}) < x$ and $\phi_2(1/\bar{p}) < \bar{p}y$.*

2.1 Mirror-symmetric Shapley-Shubik Economies

We define a *mirror-symmetric* Shapley-Shubik economy as one in which $U_1(x, y) = U_2(y, x)$ for all relevant (x, y) and where $\bar{x} = \bar{y}$. This requires that $f_1(x) = f_2(x) = f(x)$ for some function f . The consumers' utility functions are:

$$\begin{aligned} U_1(x, y) &= x + f(y) \\ U_2(x, y) &= f(x) + y. \end{aligned} \tag{9}$$

Let $B = \lim_{z \rightarrow 0} f'(0)$ and let $\phi_1(p) = \phi_2(p) = \phi(p)$ for all $p \in (0, B]$.

Remark 3. *In a mirror-symmetric Shapley-Shubik economy where $B > 1$ and $\phi(1) < \min\{\bar{x}, \bar{y}\}$, there is an interior competitive equilibrium at price $p = 1$.*

Proof. When $p = 1$, $E(p) = \phi(p) - (1/p)\phi(1/p) = \phi(1) - \phi(1) = 0$. From Theorem 1, it follows that $p = 1$ will be an interior competitive equilibrium price if the interiority conditions are satisfied. We see that $1/B < 1 < B$. The remaining interiority conditions are satisfied if $\phi(1) < \min\{\bar{x}, \bar{y}\}$. $\square$

Notice also that in a mirror-symmetric Shapley-Shubik economy, $E(1/p) = \phi(1/p) - p\phi(p) = -pE(p)$. Therefore $E(p) = 0$ if and only if $E(1/p) = 0$, and so:

Remark 4. *In a mirror-symmetric Shapley-Shubik economy, if p is an interior competitive equilibrium and if the interiority conditions are satisfied at $1/p$, then $1/p$ is also an interior competitive equilibrium.*

A simple mathematical result allows us to identify mirror-symmetric economies that have multiple equilibria.

Lemma 2. *Let F be a continuous real-valued function on the interval $[x_1, x_2]$, and assume that $F(x_1) > 0$ and $F(x_2) < 0$. If for some $p^* \in (x_1, x_2)$, $F(p^*) = 0$ and $F'(p^*) > 0$, there exist at least three distinct solutions of the equation $F(p) = 0$ in the interval (x_1, x_2) .*

Proof. Since $F(p^*) = 0$ and $F'(p^*) > 0$, there must be some $\hat{p} \in (x_1, p^*)$ such that $F(\hat{p}) < 0$. Since $\lim_{p \rightarrow x_1} F(p) > 0$, continuity of F implies that there is some $p' \in (x_1, \hat{p})$ such that $F(p') = 0$. A similar argument establishes the existence of $p'' > p^*$ such that $F(p'') = 0$. $\square$

Theorem 3. *In a mirror-symmetric Shapley-Shubik economy, if $\phi(1) < \min\{\bar{x}, \bar{y}\}$ and $\phi(1) + 2\phi'(1) > 0$, there must be at least three competitive equilibria; one at price $p = 1$, one at a price $\bar{p} > 1$ and one at price $1/\bar{p} < 1$.*

Proof. From Remark 3 it follows that $E(1) = 0$. Direct calculation shows that

$$E'(p) = \phi'(p) + \frac{1}{p^2}\phi(1/p) + \frac{1}{p^3}\phi'(1/p) \quad (10)$$

and therefore $E'(1) = \phi(1) + 2\phi'(1)$. Applying Lemma 2 with $p^* = 1$ we know that there are at least three equilibria. Remark 4 informs us that if $\bar{p}$ is an equilibrium, so is $1/\bar{p}$. $\square$

It is noteworthy that in a mirror-symmetric Shapley-Shubik economy with multiple equilibria, the “fair” equilibrium outcome in which $p = 1$ and where the two consumption bundles are mirror images is unstable. The only stable competitive equilibria divide the gains from trade unequally.

Remark 5. *In a mirror-symmetric Shapley-Shubik economy, the symmetric outcome with $p = 1$ is a competitive equilibrium, but it is unstable. The two stable equilibria are at a reciprocal pair of prices $\bar{p} > 1$ which favors Consumer 2 and $1/\bar{p} < 1$ which favors Consumer 1.*

2.2 Rescaling Commodities

It is often convenient to reduce a Shapley-Shubik economy to a simpler canonical form by linear transformations of the variables x and y . This is equivalent to changing the units in which commodities X and Y are measured.

Let preferences be represented by the utility functions in Equations 1. Where $k_x > 0$ and $k_y > 0$, we can restate these utilities in terms of variables $x' = x/k_x$ and $y' = y/k_y$ as follows:

$$\begin{aligned} U_1(x', y') &= k_x x' + f_1(k_y y') \\ U_2(x', y') &= f_2(k_x x') + k_y y'. \end{aligned} \quad (11)$$

Since preferences are invariant to monotone transformations of the utility function, the preferences represented by the utilities in 11 can also be repre-

sented by the utility functions:

$$\begin{aligned}\tilde{U}_1(x', y') &= x' + \frac{1}{k_x} f_1(k_y y') \\ \tilde{U}_2(x', y') &= \frac{1}{k_y} f_2(k_x x') + y',\end{aligned}\tag{12}$$

Thus a Shapley-Shubik economy with the functions $f_1(y)$ and $f_2(x)$ can be transformed into an equivalent Shapley-Shubik economy with the functions $\tilde{f}_1(y') = f_1(k_y y')/k_x$ and $\tilde{f}_2(x') = f_2(k_x x')/k_y$. Equilibrium price p' for commodity Y relative to commodity X in the rescaled economy corresponds to a price $p = (k_x/k_y)p'$ where goods are scaled as in the original economy.

3 Quadratic Shapley-Shubik Economies

3.1 Mirror-symmetric Quadratic Utilities

There is an easily calculated closed-form solution for the equilibria of quadratic mirror-symmetric Shapley-Shubik economies. The nonlinear portions of the utility functions in such economies can be expressed in the canonical form, $f(z) = az - z^2/2$.⁴ Then the utility functions are:

$$\begin{aligned}U_1(x, y) &= x + ay - \frac{1}{2}y^2 \\ U_2(x, y) &= ax - \frac{1}{2}x^2 + y.\end{aligned}\tag{13}$$

⁴This discussion can be extended to allow for free disposability by making the function f non-decreasing, by letting $f(z) = a^2/2$ for all $z \geq a$. None of the results of this section are altered by this change, since in the economies considered, both consumers have strictly positive marginal utilities at all equilibrium prices.

3.1.1 Solving for equilibria

With the utility functions of Equation 13, the inverse marginal utilities are given by $\phi(p) = f'^{-1}(p) = a - p$. If demands are interior at price p , excess demand is given by

$$E(p) = \phi(p) - \frac{1}{p}\phi(1/p) = a - p - a\frac{1}{p} + \frac{1}{p^2} \quad (14)$$

From Equation 14 it follows that $E(1) = 0$ and that

$$E'(1) = \phi(1) + 2\phi'(1) = a - 3. \quad (15)$$

Therefore $E'(1) > 0$ if and only if $a > 3$. The interiority conditions of Definition 2 are satisfied at $p = 1$ when $\bar{x} > \phi(1) = a - 1$, and $\bar{y} > \phi(1) = a - 1$. These facts together with Theorem 2 allow us to conclude that:

Remark 6. *In a quadratic mirror-symmetric Shapley-Shubik economy, where $f(z) = az - z^2/2$, where $a > 3$, and where $\bar{x} > a - 1$ and $\bar{y} > a - 1$, there is an unstable interior competitive equilibrium at $p = 1$, and there are two additional distinct competitive equilibrium prices, $\bar{p} > 1$ and $1/\bar{p} < 1$, both of which are stable.*

If p is an interior competitive equilibrium price, then $E(p) = 0$ and hence $pE(p) = 0$. Applying Equation 14 and factoring, we find that

$$pE(p) = pa - a - p^2 + \frac{1}{p} = (p - 1) \left(a - \left(1 + p + \frac{1}{p}\right) \right) \quad (16)$$

The interiority conditions are satisfied at p and

$$0 = (p - 1) \left(a - \left(1 + p + \frac{1}{p}\right) \right) \quad (17)$$

Equation 17 is satisfied at p if and only if $p = 1$ or $a = 1 + p + 1/p$. The

latter equation is equivalent to:

$$\bar{p}^2 + (1 - a)\bar{p} + 1 = 0 \quad (18)$$

Equation 18 has two distinct real roots if and only if $a > 3$. These roots, which can be found by applying the quadratic formula are reciprocals, $\bar{p}$ and $1/\bar{p}$.

3.1.2 Constructing economies with specified equilibrium prices

We can also work backwards, starting with specified prices and finding a parameter value a such that these prices are competitive equilibria in a mirror-symmetric quadratic economy with this parameter. To do this, choose any price $\bar{p} > 1$, and set $a = 1 + \bar{p} + 1/\bar{p}$. In the mirror-symmetric Shapley-Shubik economy with quadratic parameter a , the three solutions of Equation 17 will be $\bar{p}$, 1, and $1/\bar{p}$. It is readily verified that $a = 1 + \bar{p} + 1/\bar{p} > 3$ for all $\bar{p} > 1$. The interiority conditions are satisfied at $p = 1$, $p = \bar{p}$, and $p = 1/\bar{p}$ if $\bar{x} > 1 + \bar{p}$ and $\bar{y} > 1 + \bar{p}$. Therefore:

Remark 7. *For any $\bar{p} > 1$, there exists a mirror-symmetric Shapley-Shubik economy in which the non-linear portion of each consumer's utility function is $f(z) = az - z^2/2$ with $a = 1 + \bar{p} + 1/\bar{p}$. If the initial endowments are $\bar{x} > 1 + \bar{p}$ and $\bar{y} > 1 + \bar{p}$, then there are exactly three interior competitive equilibria, occurring at prices at $p = \bar{p}$, $p = 1$, and $p = 1/\bar{p}$.*

Example 1. To construct a mirror-symmetric quadratic economy in which the three equilibrium prices are $1/2$, 1, and 2, let $a = 1 + 2 + \frac{1}{2} = \frac{7}{2}$. Then $\phi(p) = 7/2 - p$, and the utility functions are

$$\begin{aligned} U_1(x, y) &= x + \frac{7}{2}y - \frac{1}{2}y^2 \\ U_2(x, y) &= \frac{7}{2}x - \frac{1}{2}x^2 + y \end{aligned} \quad (19)$$

Theorem 3 implies that prices $1/2$, 1 , and 2 are interior equilibria if $\bar{x} > 3$ and $\bar{y} > 3$.

Figure 1 graphs the excess demand functions of Consumers 1 and 2, as well as the aggregate excess demand function for Good Y . Consumer 1's demand curve for Y is given by the linear equation $y = 7/2 - p$ for $p \leq 7/2$ and by $y = 0$ for $p > 7/2$. Since Consumer 1 has no initial endowment of good Y , his excess demand for Y is equal to his demand. Consumer 2's demand for Y is given by the equation

$$\bar{y} - \frac{1}{p} \left(\frac{7}{2} - \frac{1}{p} \right) \quad (20)$$

for $p \geq 2/7$ and $y = \bar{y}$ for $p < 2/7$. Consumer 2's excess demand for Y is then

$$-\frac{1}{p} \left(\frac{7}{2} - \frac{1}{p} \right) \quad (21)$$

for $p \geq 2/7$ and 0 for $p < 2/7$. As Figure 1 shows, Consumer 2's excess demand for Y is an increasing function of its price over a range of prices. This is a result of an income effect. As p increases, the value of Consumer 2's endowment increases. Since Y is a "normal good" for Consumer 2 and since her demand for X does not depend on price, the positive income effect of the price increase outweighs the substitution effect and Consumer 2's demand for Y increases with its price. As the figure shows, the aggregate excess demand curve crosses the horizontal axis three times; at $p = 1/2$, $p = 1$, and $p = 2$.

Figure 2 shows the offer curves for this economy in an Edgeworth box with initial endowments $\bar{x} = \bar{y} = 4$. The offer curves intersect three times. At the equilibrium point A , consumption bundles of Consumers 1 and 2, are $(1, 1.5)$ and $(3, 2.5)$ respectively. At equilibrium B , Consumer 1 consumes the bundle $(1.5, 2.5)$ and Consumer 2 consumes $(2.5, 1.5)$. At Equilibrium C , Consumer 1 consumes the bundle $(2.5, 3)$ and Consumer 2 consumes $(1.5, 1)$.

Figure 1: Excess Demand as a Function of Price (Quadratic Utility)

3.2 General Quadratic Utilities

We next consider Shapley-Shubik economies in which the nonlinear portions of consumers' utility functions are quadratic but not identical. For each consumer i , let

$$f_i(z) = \frac{a_i}{b_i}z - \frac{1}{2b_i}z^2, \quad (22)$$

where $a_i > 0$ and $b_i > 0$. The corresponding demand functions are $\phi_i(p) = a_i - b_i p$ for p in the interval $(0, a_i/b_i]$.

3.2.1 Constructing economies with specified equilibrium prices

For any three distinct positive prices, p_1 , p_2 , and p_3 , if initial endowments are large enough, we can construct a quadratic Shapley-Shubik economy for

Figure 2: Offer Curves in an Edgeworth Box (Quadratic Utility)

which each of these prices is a competitive equilibrium and where the utility functions are of the form:

$$\begin{aligned} U_1(x, y) &= x + a_1y - \frac{1}{2}y^2 \\ U_2(x, y) &= y + a_2x - \frac{1}{2b}x^2. \end{aligned} \quad (23)$$

We prove this assertion by showing a simple way to choose parameters a_1 , a_2 , and b to construct such an economy.

With the utility functions in 23, if the interiority conditions are satisfied at price p , excess demand is

$$E(p) = \phi_1(p) - \frac{1}{p}\phi_2(1/p) = a_1 - p - a_2\frac{1}{p} + b\frac{1}{p^2}. \quad (24)$$

At an interior competitive equilibrium, $E(p) = 0$ if and only if

$$p^2E(p) = -p^3 + a_1p^2 - a_2p + b = 0. \quad (25)$$

The cubic equation 25 has three distinct positive roots, p_1 , p_2 , and p_3 if and only if

$$-p^3 + a_1p^2 - a_2p + b = (p_1 - p)(p_2 - p)(p_3 - p). \quad (26)$$

If we carry out the multiplication on the right side of Equation 26, we see that p_1 , p_2 , and p_3 are equilibrium prices if and only if the following conditions are satisfied.

$$p_1p_2p_3 = b \quad (27)$$

$$p_1 + p_2 + p_3 = a_1 \quad (28)$$

$$p_1p_2 + p_1p_3 + p_2p_3 = a_2 \quad (29)$$

From Equations 27 and 29 it follows that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a_2}{b}. \quad (30)$$

For any choice of prices $p_1 > 0$, $p_2 > 0$, and $p_3 > 0$, we choose parameters b , a_1 and a_2 to satisfy Equations 27-29.⁵ The quadratic Shapley-Shubik economy with these parameters will have competitive equilibria at p_1 , p_2 , and p_3 if the interiority conditions are satisfied at each of these prices. From Equation 28 it follows that $\phi_1(p_i) = a_1 - p_i > 0$ for $i = 1, 2, 3$, and from Equation 30 it follows that $\phi_2(p_i) = a_2 - bp_i > 0$ for $i = 1, 2, 3$. The remaining interiority conditions will be satisfied if $\bar{x} > p_i\phi_1(p_i)$ and $p_i\bar{y} > \phi_2(1/p_i)$ for $i = 1, 2, 3$. The equilibria at p_1 and p_3 will be stable, while the equilibrium at p_2 will be unstable.

Example 2. Let us construct a quadratic Shapley-Shubik economy with equilibrium prices $p_1 = 1/2$, $p_2 = 4$, and $p_3 = 5$. Using Equations 27-29, we

⁵This method of choosing parameters for a quadratic equation to generate desired solutions was applied in 1840 by Euler [3] (pages 253-254) to the study of Diophantine equations.

find $b = 10$, $a_1 = 9.5$, and $a_2 = 24.5$.

Simple calculations show that $p\phi_1(p) < \bar{x}$ for each of these prices if $\bar{x} > 22.5$ and that $\phi_2(1/p) < p\bar{y}$ if $\bar{y} > 9$. Therefore the interiority conditions will be satisfied at all three prices $p_1 = 1/2$, $p_2 = 4$, and $p_3 = 5$ if $\bar{x} > 22.5$ and $\bar{y} > 9$. Therefore there are three competitive equilibria prices, $p_1 = 1/2$, $p_2 = 4$, and $p_3 = 5$ in a quadratic Shapley-Shubik economy with utility functions

$$U_1(x, y) = x + 9.5y - \frac{1}{2}y^2 \quad (31)$$

$$U_2(x, y) = 24.5x - \frac{1}{20}x^2 + y. \quad (32)$$

3.2.2 Solving for equilibrium prices

Consider a Shapley-Shubik economy in which the functions f_i take the general quadratic form

$$f_i(z) = a_i z - \frac{1}{2b_i} z^2. \quad (33)$$

The task of finding competitive prices can be simplified by rescaling units of measurement of X and Y so that in the rescaled economy $b_1 = 1$ and $b_2 = 1$. To do so, make the linear transformation of variables $x' = x/k_x$ and $y' = y/k_y$ where

$$k_y = b_1^{2/3} b_2^{1/3} \text{ and } k_x = b_1^{1/3} b_2^{2/3}. \quad (34)$$

In the transformed economy,

$$f_i(z) = A_i z - \frac{1}{2} z^2 \quad (35)$$

for $i = 1, 2$, where $A_1 = a_1 k_y / b_1 k_x$ and $A_2 = a_2 k_x / b_2 k_y$.

The inverse marginal utility functions expressed in the transformed vari-

ables are $\phi_i(p) = A_i - p$. In competitive equilibrium,

$$E(p) = A_1 - p + \frac{1}{p} \left(A_2 - \frac{1}{p} \right) = 0. \quad (36)$$

Equation 36 is satisfied at p if and only if p is a root of the cubic equation

$$p^3 - A_1 p^2 + A_2 p - 1 = 0 \quad (37)$$

Closed form solutions for a general cubic function were first published in 1545 by G. Cardan and are known as Cardan's formulas. (A good discussion of solution methods for cubic equations is found in Dickson [2].) Cardan's formulas can be applied to find the solutions to Equation 37. For general values of A_1 and A_2 , these expressions are cumbersome and do not readily yield useful insight. On the other hand, calculating numerical solutions to Equation 37 for specific values of A_1 and A_2 is straightforward.⁶

General necessary and sufficient conditions for the existence of three positive real solutions to Equation 37 are available. As Dickson [2] explains, a cubic equation will have three distinct real roots if and only if its *discriminant* is strictly positive. The discriminant of Equation 37 turns out to be⁷

$$\Delta = 18A_1A_2 - 4A_1^3 + A_1^2A_2^2 - 4A_2^3 - 27. \quad (38)$$

Equation 37 had three real roots if and only if $\Delta > 0$. These roots will all be positive valued. To see this, note that if $p \leq 0$, then $p^3 - A_1 p^2 + A_2 p - 1 < 0$ and so p can not be a root of Equation 37.⁸ Where $\Delta > 0$, there will be competitive equilibria at each of the three positive roots of Equation 37

⁶"Cubic equation solvers" which output the solutions for a cubic with any specified real parameters are available on the internet. See for example www.1728.com/cubic.htm

⁷The discriminant of any equation is the product of the squares of the differences of its roots. Dickson [2] shows that the discriminant of a cubic equation of the form $x^3 + bx^2 + cx + d$ is $\Delta = 18bcd - 4b^3d + b^2c^2 - 4c^3 - 27d^2$.

⁸Euler [3], pages 256-257 provides a similar demonstration that real roots of a cubic with coefficients of alternating signs must be positive.

so long as the initial endowments $\bar{x}$ and $\bar{y}$ are large enough to satisfy the interiority.

4 Exponential Utilities

The example presented by Shapley-Shubik has utility functions:

$$\begin{aligned} U_1(x, y) &= x + 100(1 - e^{-y/10}) \\ U_2(x, y) &= y + 110(1 - e^{-x/10}) \end{aligned} \tag{39}$$

This example belongs to a general family of Shapley-Shubik economies that includes all utilities of the form:

$$\begin{aligned} U_1(x, y) &= x - A_1 e^{-b_1 y} \\ U_2(x, y) &= y - A_2 e^{-b_2 x}. \end{aligned} \tag{40}$$

By rescaling units of measurement, as in Section 2.2, we can convert a “four-parameter” economy with utility functions 40 to an equivalent “two-parameter” economy with utilities:

$$\begin{aligned} U_1(x, y) &= x - e^{a_1 - y} \\ U_2(x, y) &= y - e^{a_2 - x}. \end{aligned} \tag{41}$$

where $a_1 = \ln A_1 b_2$ and $a_2 = \ln A_2 b_1$. The inverse marginal utility functions are then $\phi_1(p) = a_1 - \ln p$ and $\phi_2(p) = a_2 - \ln p$.

4.1 General Exponential Utilities

We have found a closed-form expression, the sign of which determines the number of competitive equilibria for any Shapley-Shubik economy of exponential form. In Figure 3, we show the parameter regions of utilities for there are one, two, or three competitive equilibria, when utility is written in the canonical form of Equations 41. The point S in Figure 3 marks the parameter values a_1 and a_2 that correspond to the example presented by Shapley and Shubik. In Theorem 4, we characterize the parameter ranges over which there are one, two, or three equilibria.

Figure 3: Parameter ranges for multiple equilibria

Theorem 4. *In a Shapley-Shubik economy where $f_i(z) = -e^{a_i - z}$ for $i = 1, 2$ and where $\bar{x} > e^{a_1 - 1}$ and $\bar{y} > e^{a_2 - 1}$,*

(i) there are no more than three competitive equilibria.

(ii) there is a closed form function $G(a_1, a_2)$ such that there are one, two,

or three distinct competitive equilibria, depending on whether $G(a_1, a_2)$ is positive, zero, or negative.

(iii) there is exactly one competitive equilibrium if $\min\{a_1, a_2\} < 2$.

To prove Theorem 4, it is useful to define the function

$$F(p) = \frac{a_1 p - a_2}{1 + p} - \ln p. \quad (42)$$

for all $p \in (0, \infty)$. Our strategy of proof is to show by a series of lemmas that for all $p \in (0, \infty)$, the sign of $F(p)$ is the same as that of aggregate excess demand $E(p)$. Therefore a price p is a competitive equilibrium if and only if $F(p) = 0$. We then study the shape of the graph of $F(p)$ and display a closed form function $G(a_1, a_2)$ such that the sign of $G(a_1, a_2)$ determines the number of solutions to the equation $F(p) = 0$.

Lemma 3. *In a Shapley-Shubik economy with exponential utilities where $a_1 + a_2 > 0$, if $p \in (0, e^{-a_2}]$, then $E(p) > 0$ and $F(p) > 0$. If $p \in [e^{a_1}, \infty)$, then $E(p) < 0$ and $F(p) < 0$.*

Proof. Since $f_i(z) = -e^{a_i - z}$, we have $B_1 = f'_1(0) = e^{a_1}$ and $B_2 = f'_2(0) = e^{a_2}$. From Lemma 1 it follows that $E(p) > 0$ for $p \leq e^{-a_2}$ and that $E(p) < 0$ for $p \geq e^{a_1}$.

To complete the proof we show that $F(p) > 0$ for $p \leq e^{-a_2}$ and $F(p) < 0$ for $p \geq e^{a_1}$. If $p \leq e^{-a_2}$, then $-\ln p \geq a_2$, and therefore

$$F(p) \geq \frac{a_1 p - a_2}{1 + p} + a_2 = \frac{p}{1 + p}(a_1 + a_2) > 0. \quad (43)$$

If $p \geq e^{a_1}$, then $-\ln p \leq -a_1$ and therefore

$$F(p) \leq \frac{a_1 p - a_2}{1 + p} - a_1 = -\frac{a_1 + a_2}{1 + p} < 0. \quad (44)$$

□

Lemma 4. *In a Shapley-Shubik economy with exponential utilities, if $\bar{x} > e^{a_1-1}$ and $\bar{y} > e^{a_2-1}$, then the interiority conditions of Definition 2 apply for all $p \in (e^{-a_2}, e^{a_1})$.*

Proof. The interiority conditions are satisfied at p if and only if $p \in (e^{-a_2}, e^{a_1})$, $p\phi_1(p) < \bar{x}$, and $\phi_2(1/p) < p\bar{y}$. To show that $p\phi_1(p) < \bar{x}$, we differentiate $p\phi_1(p) = a_1p - p \ln p$ to find that $p\phi(p)$ is maximized when $a_1 - 1 = \ln p$ and hence when $p = e^{a_1-1}$. Therefore for all $p > 0$, $p\phi(p) \leq e^{a_1-1}\phi(e^{a_1-1}) = e^{a_1-1}(a_1 - (a_1 - 1)) = e^{a_1-1}$. We have assumed that $\bar{x} > e^{a_1-1}$. Therefore $p\phi_1(p) < \bar{x}$ for all $p > 0$. To show that $\phi_2(1/p) < p\bar{y}$ for all $p > 0$, we differentiate $(1/p)\phi_2(1/p)$ and find that $(1/p)\phi_2(1/p)$ is maximized when $p = e^{1-a_2}$. Therefore, for all $p > 0$, $(1/p)\phi_2(p) \leq e^{a_2-1}\phi_2(e^{a_2-1}) = e^{a_2-1}$. Since we assume that $\bar{y} > e^{a_2-1}$, it follows that $1/p\phi_2(p) < \bar{y}$ and hence $\phi_2(p) < p\bar{y}$. $\square$

Lemma 5. *In a Shapley-Shubik economy with exponential utilities, if $\bar{x} > e^{a_1-1}$ and $\bar{y} > e^{a_2-1}$, then for all $p \in (0, \infty)$, the sign of $F(p)$ is the same as the sign of $E(p)$.*

Proof. According to Lemma 4, the interiority conditions are satisfied at all p in the interval (e^{-a_2}, e^{a_1}) . Therefore, for all $p \in (e^{-a_2}, e^{a_1})$, $E(p) = \phi(p) - (1/p)\phi(1/p) = a_1 - \ln p - (1/p)(a_2 + \ln p)$. Rearranging the terms of $F(p)$, we see that

$$F(p) = \frac{p}{1+p} \left(a_1 - \ln p - \frac{1}{p}(a_2 + \ln p) \right) = \frac{p}{1+p} E(p) \quad (45)$$

for all $p \in (e^{-a_2}, e^{a_1})$. Since $p/(1+p) > 0$, the sign of $F(p)$ must be the same as that of $E(p)$ for all $p \in (e^{-a_2}, e^{a_1})$.

Lemma 3 informs us that the sign of $E(p)$ is the same as that of $F(p)$ for all p in the intervals $(0, e^{-a_2}]$ and $[e_1^a, \infty)$. Therefore the sign of $F(p)$ is the same as that of $E(p)$ on the entire interval $(0, \infty)$. $\square$

Figure 4 which graphs the function $F(p)$ with parameters $a_1 = \ln 10 =$

2.303 and $a_2 = \ln 11 = 2.398$ that apply to the Shapley-Shubik example. This graph will be useful in the exposition of the proof of Theorem 4.

Figure 4: The graph of $F(p)$

Proof. Theorem 4: According to Lemma 5, $E(p) = 0$ if and only if $F(p) = 0$. Calculation shows that

$$F'(p) = \frac{a_1 + a_2}{(1+p)^2} - \frac{1}{p}. \quad (46)$$

Then

$$-p(1+p)^2 F'(p) = p^2 - (a_1 + a_2 - 2)p + 1 \quad (47)$$

It follows that the sign of $F'(p)$ is opposite from the sign of the quadratic expression on the right side of Equation 47. This quadratic expression has at most two distinct real roots. Hence the slope of the graph of $F(p)$ can change signs at most twice over the interval (e^{-a_2}, e^{a_1}) . It follows that there cannot be more than three solutions to the equation $F(p) = 0$ and hence no more

than three distinct competitive equilibrium prices. This proves Assertion (i) of the theorem.

We will prove Assertion (ii) by exhibiting the desired function $G(a_1, a_2)$. To motivate this construction, it is helpful to consider Figure 4. As p ranges from 0 to ∞ , the graph of $F(p)$ first slopes downward, then upward, and then downward again. We will show that for any exponential Shapley-Shubik economy with $a_1 + a_2 > 4$, the graph of $F(\cdot)$ changes its direction of slope exactly twice. In the figure, the graph of $F(\cdot)$ crosses the horizontal axis three times, once in the upward-sloping segment and once in each of the downward-sloping segments. In general, the upward-sloping portion of the graph may lie either entirely above or entirely below the horizontal axis, in which case there is only one competitive equilibrium. If the upward-sloping portion of the graph crosses the horizontal axis as in Figure 4, then there will be exactly three competitive equilibria, one stable and two unstable.

If the graph of $F(\cdot)$ has a local minimum and maximum, they must occur at real roots of the quadratic expression in Equation 47. This equation has real roots if and only if $a_1 + a_2 \geq 4$. These roots are distinct if $a_1 + a_2 > 4$ and in this case, both roots must be positive. Let $p_L(a_1, a_2)$ be the smaller of these roots and $p_H(a_1, a_2)$ be the larger. It is readily verified that $p_L(a_1, a_2)p_H(a_1, a_2) = 1$ and hence $p_L(a_1, a_2) < 1 < p_H(a_1, a_2)$. A straightforward calculus argument shows that $p_L(a_1, a_2)$ is a local minimum and $p_H(a_1, a_2)$ is a local maximum of $F(p)$. It is also straightforward to demonstrate that $F'(p) < 0$ if $p < p_L(a_1, a_2)$, that $F'(p) > 0$ if $p_L(a_1, a_2) < p < p_H(a_1, a_2)$ and that $F'(p) < 0$ if $p > p_H(a_1, a_2)$.

If $F(p_L(a_1, a_2))$ and $F(p_H(a_1, a_2))$ are both positive or both negative, the graph of $F(p)$ can cross the horizontal axis at most once. Hence there can be no more than one competitive equilibrium. If $F(p_L(a_1, a_2))$ and $F(p_H(a_1, a_2))$ are of opposite signs, then it must be that as in Figure 4, $F(p_L(a_1, a_2)) < 0$, $F(p_H(a_1, a_2)) > 0$, and there are exactly three distinct solutions to the equation $F(p) = 0$. These include a stable equilib-

rium in the interval $(e^{-a_2}, p_L(a_1, a_2))$, an unstable equilibrium in the interval $(p_L(a_1, a_2), p_H(a_1, a_2))$, and a stable equilibrium in the interval $(p_H(a_1, a_2), e^{a_1})$. Define

$$G(a_1, a_2) = F(p_L(a_1, a_2)) F(p_H(a_1, a_2)). \quad (48)$$

Then $G(a_1, a_2)$ will be positive or negative depending on whether $F(p_L(a_1, a_2))$ and $F(p_H(a_1, a_2))$ are of the same or opposite signs. Therefore there are three equilibria if $G(a_1, a_2)$ is negative, and one equilibrium if $G(a_1, a_2)$ is positive. If $G(a_1, a_2) = 0$, then either $F(p_L(a_1, a_2))$ or $F(p_H(a_1, a_2))$ is zero. In this case, the the graph of $F(p)$ is tangent to the horizontal axis either at the local minimum $p_L(a_1, a_2)$ or the local maximum $p_H(a_1, a_2)$, in which case there will be exactly two equilibria, including one at the point of tangency. This completes the proof of Assertion (ii).

From Lemma 1 we see that any competitive equilibrium price must lie in the interval (e^{-a_2}, e^{a_1}) and from Lemma 4 it follows that the interiority conditions are satisfied for all p in this interval. Therefore, a competitive equilibrium must be a point p in the interval (e^{-a_2}, e^{a_1}) such that

$$E(p) = \phi(p) - \frac{1}{p}\phi(1/p) = a_1 - \ln p - \frac{1}{p}(a_2 + \ln p) \quad (49)$$

Differentiating Equation 49, we obtain

$$E'(p) = \frac{1}{p^2}(a_2 - 1 + \ln p - p). \quad (50)$$

With simple calculus, we see that $\ln p - p$ is maximized at $p = 1$ and that $\ln p - p \leq -1$ for all $p > 0$. Therefore

$$p^2 E'(p) = a_2 - 1 + \ln p - p \leq a_2 - 2 \quad (51)$$

It follows that if $a_2 < 2$, then $E'(p) < 0$ for all $p > 0$. In this case, $E(p)$ is a strictly decreasing function over the entire interval (e^{-a_2}, e^{a_1}) . It follows that there can be only one equilibrium when $a_2 < 2$. To show that there is

only one equilibrium if $a_1 < 2$, we note that a competitive equilibrium price p must solve the equation

$$0 = pE(p) = p(a_1 - \ln p) - (a_2 + \ln p) \quad (52)$$

Differentiating, we find that

$$\frac{d}{dp}pE(p) = a_1 - 1 - \ln p - \frac{1}{p} \quad (53)$$

With simple calculus, we find that $\ln p + 1/p$ is minimized at $p = 1$ and that $\ln p + 1/p > 1$ for all $p > 0$. Therefore it must be that $\frac{d}{dp}pE(p) \leq a_1 - 2$ for all $p > 0$. Therefore if $a_1 < 2$, then $pE(p)$ is a decreasing function of p and hence there can be only one competitive equilibrium price. This completes the proof of Assertion (iii). □

4.2 Mirror-symmetric Exponential Utilities

For mirror-symmetric Shapley-Shubik economies, the nonlinear portion of utilities take the exponential form $f(z) = -e^{a-z}$. In this case, we have a very simple necessary and sufficient condition for the existence of multiple equilibria.

Remark 8. *In a mirror-symmetric Shapley-Shubik economy with $f(z) = -e^{a-z}$, if $2 < a < \bar{x} = \bar{y}$ there will exactly three equilibrium prices; an unstable equilibrium with $\bar{p} = 1$ and two stable equilibria, $\bar{p} > 1$ and $1/\bar{p} < 1$.*

Proof. Note that $\phi(1) = a$ and $\phi(1) + 2\phi'(1) = a - 2$. Therefore the assumption that $2 < a < \bar{x}$ implies that $\phi(1) = a < \bar{x} = \bar{y}$ and $\phi(1) + 2\phi'(1) = a - 2 > 0$. From Theorem 3 it follows that there are at least three equilibria. From Theorem 4 it follows that there are no more than three equilibria. □

For an arbitrarily chosen parameter a , there is not a closed form solution for stable equilibrium prices. Solutions must be found by numerical methods.

Working backwards is easier. Starting with an arbitrarily chosen price $\bar{p} > 1$, there is a simple closed form solution for a parameter a such that a mirror-symmetric Shapley-Shubik economy in which $f(z) = -e^{a-z}$ has competitive equilibria at prices, $\bar{p}$, 1, and $1/\bar{p}$.

Remark 9. For any $\bar{p} > 1$, let

$$a = \frac{\bar{p} + 1}{\bar{p} - 1} \ln \bar{p}. \quad (54)$$

In a mirror-symmetric Shapley-Shubik economy where $f(z) = -e^{a-z}$ and where $\bar{x} = \bar{y} > 2\bar{p} \ln \bar{p}/(\bar{p} - 1)$, there are interior competitive equilibria at prices at $p = \bar{p}$, $p = 1$, and $p = 1/\bar{p}$.

Proof. There is an interior competitive equilibrium at $\bar{p} > 1$ if and only if the interiority conditions are satisfied at $\bar{p}$ and

$$E(\bar{p}) = a - \ln \bar{p} - \frac{1}{\bar{p}}(a + \ln \bar{p}) = 0 \quad (55)$$

Rearrangement of Equation 55 shows that Equation 55 is satisfied if and only if

$$a = \frac{\bar{p} + 1}{\bar{p} - 1} \ln \bar{p}. \quad (56)$$

The interiority conditions require that $1/f'(0) < \bar{p} < f'(0)$, $\bar{p}\phi(\bar{p}) < \bar{x}$, and $\phi(1/\bar{p}) < p\bar{x}$. We have

$$f'(0) = e^a = \bar{p}^{\frac{\bar{p}+1}{\bar{p}-1}}. \quad (57)$$

Then for $\bar{p} > 1$, $1/f'(0) = e^{-a} < \bar{p} < f'(0) = e^a$. Calculations show that $\bar{p}\phi(\bar{p}) = \bar{p}(a - \ln \bar{p}) = 2\bar{p} \ln \bar{p}/(1 - \bar{p})$ and $\phi(1/\bar{p}) = 2\bar{p} \ln \bar{p}/(1 - \bar{p})$. Therefore the interiority conditions $\bar{p}\phi(\bar{p}) < \bar{x}$ and $\phi(1/\bar{p}) < \bar{y}$ will be satisfied if $\bar{x} = \bar{y} > 2\bar{p} \ln \bar{p}/(\bar{p} - 1)$. $\square$

Example 3. To construct an economy with exponential utilities and equilibrium prices 2, 1 and $1/2$, we set $f(z) = -e^{a-z}$ where $a = ((2+1)/(2-1)) \ln 2$. Then $f(x) = -e^{3 \ln 2 - x} = -8e^{-x}$ and $\phi(p) = \ln 8 - \ln p$. There will be interior competitive equilibria at $\bar{p} = 2$ and $\bar{p} = 1/2$ if initial endowments are $\bar{x} = \bar{y} > 2\bar{p} \ln(\bar{p}/(\bar{p}-1)) = 4 \ln 2$.

5 Unbounded Marginal Utilities

5.1 Existence of Equilibrium

Theorem 1 proved the existence of competitive equilibrium for a Shapley-Shubik economy in which the nonlinear functions f_i have bounded marginal utilities. Some familiar examples of quasi-linear utility functions have unbounded marginal utilities with $\lim_{z \rightarrow 0} f'(z) = \infty$. These include the case where $f(z) = (1/a)z^a$ with $a < 1$ and that where $f(z) = a \ln z$ with $a > 0$.

In this case, the proof used in Theorem 1 does not apply. We construct a counterexample—a Shapley-Shubik economy in which the functions f_i are continuous and strictly quasi-concave but where there is no competitive equilibrium.⁹

Example 4. Consider a Shapley-Shubik economy in which $f_1(y) = y - y^2/2$ and $f_2(x) = k \ln x$ where $k > 1$. Assume also that $\bar{x} > 1$ and $\bar{y} > k$. Excess demand function for Y by Consumer 1 is $E_1(p) = 1 - p$ if $0 \leq p \leq 1$ and $E_1(p) = 0$ if $p > 1$. For Consumer 2, the inverse marginal utility function is $\phi_2(p) = k/p$. Consumer 2's demand for X is $x_2(p) = \phi_2(1/p) = pk$ for all $p > 0$. Since Consumer 2's budget constraint is $x_2(p) + py_2(p) = p\bar{y}$, we have $E_2(p) = y_2(p) - \bar{y} = -x_2(p)/p = -k$ for all $p > 0$. Therefore aggregate excess demand is $E(p) = E_1(p) + E_2(p) = 1 - k - p$ if $0 < p \leq 1$ and $E(p) = -k$ if $p > 1$. Since $k > 1$ it follows that $E(p) < 0$ for all $p > 0$. Therefore there can

⁹This example is an instance of a familiar problem in general equilibrium theory. If the initial allocation is not interior, then without additional assumptions, excess demand may be discontinuous at the boundary of the price simplex.

be no competitive equilibrium at a price $p > 0$. There is also no competitive equilibrium at $p = 0$, since if $p = 0$, Consumer 2's demand for Y is infinite.

Theorem 5 employs an assumption that ensures existence of at least one competitive equilibrium for a Shapley-Shubik economy, even with unbounded marginal utilities. This assumption implies that neither consumer is sated in either good over the range of feasible consumptions.

Theorem 5. *In a Shapley-Shubik economy there exists at least one competitive equilibrium price if $f'_1(\bar{y}) > 0$, $f'_2(\bar{x}) > 0$, and $f'_1(\bar{y}) < 1/f'_2(\bar{x})$.*

Proof. We begin the proof by showing that $E(p) > 0$ for all $p < f'_1(\bar{y})$ and that $E(p) < 0$ for all $p > 1/f'_2(\bar{x})$.

Since f'_1 is continuous and decreasing in y , it must be that for all $p < f'_1(\bar{y})$, $y_1(p) > \bar{y}$. Since $y_2(p) \geq 0$, it follows that for all $p < f'_1(\bar{y})$, $E(p) = y_1(p) + y_2(p) - \bar{y} > 0$.

For any price $p > 1/f'_2(\bar{x})$, it must be that $1/p < f'_2(\bar{x})$ and therefore $x_2(p) > \bar{x}$. Then $pE_2(p) = p(y_2(p) - \bar{y}) = -x_2(p) < -\bar{x}$. From Consumer 1's budget constraint, it follows that $pE_1(p) = py_1(p) \leq \bar{x}$. Therefore for all $p > 1/f'_2(\bar{x})$, $pE(p) = pE_1(p) + pE_2(p) < \bar{x} - \bar{x} = 0$.

Since the utility functions are continuous and strictly quasi-concave, excess demand is continuous on the interval $(0, \infty)$. We have shown that $E(p) > 0$ for $p < f'_1(\bar{y})$ and $E(p) < 0$ for $p > 1/f'_2(\bar{x})$. Continuity of f implies that there must be a price $\bar{p}$ in the interval $(f'_1(\bar{y}), 1/f'_2(\bar{x}))$ such that $E(\bar{p}) = 0$.

□

5.2 Power Function Utilities

Consider a Shapley-Shubik economy in which the nonlinear portions of utilities are power functions, of the form:

$$f_i(z) = \frac{1}{b_i} z^{b_i} \quad (58)$$

where $b_i < 1$. Then for all $z > 0$, $f'_i(z) = z^{b_i-1}$ and the inverse marginal utility function is of the constant elasticity form

$$\phi_i(p) = p^{1/(b_i-1)}. \quad (59)$$

We note that for all $b_1 < 1$, $\lim_{z \rightarrow 0} = \infty$.¹⁰

Let us consider Shapley-Shubik utility functions with power function utilities that are also mirror-symmetric. It will be convenient to express these in the form:

$$f(x) = \frac{\epsilon}{1 + \epsilon} x^{\frac{\epsilon+1}{\epsilon}} \quad (60)$$

where $\epsilon < 0$. The range of $f'(\cdot)$ is the entire interval $(0, \infty)$. The inverse marginal utility function is $\phi(p) = f'^{-1}(p) = p^\epsilon$.

In this economy, if $\bar{x} > 1$ and $\bar{y} > 1$, there will be an interior competitive equilibrium at $p = 1$. At this equilibrium, $E'(1) = \phi(1) + 2\phi'(1) = 1 + 2\epsilon > 0$ if $-1/2 < \epsilon < 0$. Therefore, according to Theorem 3, there must be at least three equilibria if $-1/2 < \epsilon < 0$.

However $p = 1$ is the only positive real solution to the equation

$$0 = \phi(p) - (1/p)\phi(1/p) = p^\epsilon - p^{-(1+\epsilon)}. \quad (61)$$

To find the other two equilibrium prices, we must look for “corner solutions.” There are two such equilibria. In one of them, Consumer 1 trades away all of his holding of good X and consumes only Y . At the other corner equilibrium, Consumer 2 trades away all of his initial Y and consumes only X .

At the corner equilibrium where Consumer 1 consumes only Y , Consumer 2 must consume $\bar{x}$ units of X and a positive amount of Y . Therefore it must be that $\bar{x} = \phi(1/p) = p^{-\epsilon}$ and hence $p = \bar{x}^{-1/\epsilon}$. At this price, Consumer 1 consumes $y_1 = \bar{x}/p = p^{-(1+\epsilon)}$ units of Y . It is straightforward to verify that

¹⁰In case $b_i < 0$, we must also define $f_i(0) = \lim_{z \rightarrow 0} f(0) = -\infty$.

with this quantity, Consumer 1's marginal rate of substitution between Y and X exceeds the price p , and hence Consumer 1 is at a corner solution. A similar line of reasoning shows that at the corner solution where Consumer 2 consumes only X , $\bar{y} = p^\epsilon$ and hence $p = \bar{y}^{1/\epsilon}$.

Example 5. Consider the mirror-symmetric Shapley-Shubik economy where $f(x) = -(1/2)x^{-2}$ and $\bar{x} = \bar{y} = 2$. In this economy, $\phi(p) = p^{-1/3}$. There is an interior competitive equilibrium at $p = 1$. There is a corner equilibrium where Consumer 1 trades away all of his initial endowment of X and consumes only Y . This occurs at a price $p = 8$. At this price, Consumer 2 will demand $\phi(1/p) = 2$ units of X and $1\frac{3}{4}$ units of Y . Consumer 1 trades all of his X for $1/4$ unit of Y . There is another corner equilibrium where Consumer 2 trades all of his initial endowment of Y and consumes only X . At this equilibrium, $p = 1/8$, Consumer 1 consumes 2 units of Y and $1\frac{3}{4}$ units of X , and Consumer 2 consumes no Y and $1/4$ unit of X .

Figure 5 shows an Edgeworth box and offer curves for this example, where Consumer 1 is endowed with $\bar{x} = 2$ units of x and no y and Consumer 2 is endowed with $\bar{y} = 2$ units of y and no x . The competitive equilibrium A is a corner solution for Consumer 1, who consumes no x . The equilibrium C is a corner solution for Consumer 2, who consumes no y . The equilibrium C is symmetric and interior for both consumers but is unstable.

6 Interior Endowments and Generalized Shapley-Shubik Economies

Our definition of a Shapley-Shubik economy requires the initial endowment to be at a corner of the Edgeworth box. Each consumer is endowed only with the commodity in which his utility is linear. Many of our results can be extended to *generalized Shapley-Shubik economies* where utility functions are as in a Shapley-Shubik economy, but where one or both consumers have

Figure 5: Offer Curves in an Edgeworth Box (Power Function Utility)

Figure 6: Endowments Leading to Multiple Equilibria (Quadratic Utility)

positive initial endowments of both goods.

In general, if a Shapley-Shubik economy has multiple equilibria, the generalized Shapley-Shubik economies constructed by reallocating the same total endowments will continue to have multiple equilibria so long as the initial endowments are “not too distant” from the initial endowments in the corresponding Shapley-Shubik economy. This is illustrated in the Edgeworth box drawn in Figure 6, which applies to the quadratic utility function used in Example 1, where the total initial endowments of X and Y are each 4 units. There are three competitive equilibria if and only if initial endowments are located in the shaded area.¹¹

Let us define a generalized Shapley-Shubik economy as follows. There are two consumers, with utility functions $U_1(x, y) = x + f_1(y)$ and $U_2(x, y) = f_2(x) + y$ where the functions f_i are concave and continuously differentiable, as in an ordinary Shapley-Shubik economy. Each consumer i is endowed with

¹¹Kumar and Shubik [5] displayed a similar figure showing the endowments that lead to multiple equilibria in the Shapley-Shubik example.

$\bar{x}_i$ units of X and $\bar{y}_i$ units of Y . Then $\bar{x}_1 + \bar{x}_2 = \bar{x}$ and $\bar{y}_1 + \bar{y}_2 = \bar{y}$ are the total supplies of X and Y .

As Example 4 demonstrates, in a Shapley-Shubik economy with unbounded utilities, existence of competitive equilibrium requires an additional assumption that is not needed for the case of bounded utility. Existence of equilibrium does not require this extra assumption in a generalized Shapley-Shubik economy where both consumers have endowments of both commodities.

Theorem 6. *In a generalized Shapley-Shubik economy, there exists at least one competitive equilibrium if both consumers have positive endowments of both commodities.*

Proof. Since the utility functions $f_i(\cdot)$ are strictly concave and continuously differentiable, it must be that $y_1(p) - \bar{y}_1 > 0$ if $p < f'_1(\bar{y}_1)$ and $y_1(p) - \bar{y}_1 < 0$ if $p > f'_1(\bar{y}_1)$. Similarly, it must be that $x_2(p) - \bar{x}_2 > 0$ if $1/p < f'_2(\bar{x}_2)$ and $x_2(p) - \bar{x}_2 < 0$ if $1/p > f'_2(\bar{x}_2)$. From the Consumer 2's budget constraint it follows that the sign of $y_2(p) - \bar{y}_2$ is opposite from the sign of $x_2(p) - \bar{x}_2$. Therefore $y_2(p) - \bar{y}_2 > 0$ if $p < 1/f'_2(\bar{x}_2)$ and $y_2(p) - \bar{y}_2 < 0$ if $p > 1/f'_2(\bar{x}_2)$. It follows that both consumers have positive excess demand for Y if $p < \min\{f'_1(\bar{y}_1), 1/f'_2(\bar{x}_2)\}$ and that both consumers have negative excess demand for Y if $p > \max\{f'_1(\bar{y}_1), 1/f'_2(\bar{x}_2)\}$. Therefore aggregate demand is positive for small enough $p > 0$ and negative for large enough p . Since demand is continuous, there must be at least one price at which aggregate demand is zero. $\square$

At an interior competitive equilibrium, Consumer 1's consumption of Y is $y_1(p) = \phi_1(p)$ and Consumer 2's consumption of X is $x_2(p) = \phi_2(1/p)$. From Consumer 2's budget constraint, it follows that

$$y_2(p) - \bar{y}_2 = \frac{1}{p} (\bar{x}_2 - \phi_2(1/p)) \quad (62)$$

At an interior equilibrium, excess demand for good Y is given by

$$E(p) = y_1(p) + y_2(p) - \bar{y}_2 - \bar{y}_1 = \phi_1(p) - \bar{y}_1 - \frac{1}{p}(\phi_2(1/p) - \bar{x}_2) \quad (63)$$

Let us define $\tilde{\phi}_1(p) = \phi_1(p) - \bar{y}_1$ and $\tilde{\phi}_2(p) = \phi_2(p) - \bar{x}_2$. Then from Equation 63 it follows that at an interior competitive equilibrium,

$$E(p) = \tilde{\phi}_1(p) - (1/p)\tilde{\phi}_2(1/p) = 0. \quad (64)$$

It follows from the two consumers' budget constraints that Consumer 1's consumption of X and Consumer 2's consumption of Y will be positive if and only if

$$p(\phi_1(p) - \bar{y}_1) < \bar{x}_1 \text{ and } \phi_2(1/p) - \bar{x}_2 < p\bar{y}_2. \quad (65)$$

The inequalities in 65 can be written as $p\tilde{\phi}_1(p) < \bar{x}_1$ and $\tilde{\phi}(1/p) < p\bar{y}_1$. Therefore we have the following generalization of Theorem 1.

Lemma 6. *In a generalized Shapley-Shubik economy, p is an interior competitive equilibrium price if and only if $p\tilde{\phi}_1(p) < \bar{x}_1$, $\tilde{\phi}_2(1/p) < p\bar{y}_2$, and $\tilde{\phi}_1(p) = (1/p)\tilde{\phi}_2(1/p)$.*

Since the equations for an interior equilibrium do not depend on initial endowments, the neutrality result of Theorem 2 extends to generalized Shapley-Shubik economies.

Remark 10. *If $\bar{p}$ is an interior competitive equilibrium in a generalized Shapley-Shubik economy with initial endowments $(\bar{x}_1, \bar{x}_2)$ and $(\bar{y}_1, \bar{y}_2)$, then $\bar{p}$ will also be an interior competitive equilibrium if initial endowments are (x_1, x_2) and (y_1, y_2) so long as $\bar{p}\phi_1(\bar{p}) < x_1 + \bar{p}y_1$ and $\phi_2(1/\bar{p}) < \bar{p}y_2 + x_2$.*

6.1 Mirror-symmetric Interior Endowments

Within the class of generalized Shapley-Shubik economies, there is an interesting subclass of economies with interior endowments but with sufficient symmetry to allow clear, simple results. Consider a generalized Shapley-Shubik economy with mirror-symmetric utilities in which each person has a positive initial endowment of each good. We will say that there are mirror-symmetric endowments if the two consumers have equal initial endowments of the good in which they have nonlinear utility. When initial endowments are $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$, this means that $\bar{y}_1 = \bar{x}_2 = \bar{z}$ for some $\bar{z} > 0$. Where $\phi(p)$ is the inverse marginal utility function, let $\tilde{\phi}(p) = \phi(p) - \bar{z}$. For an interior competitive equilibrium, excess demand for good Y is zero if

$$E(p) = \tilde{\phi}(p) - \frac{1}{p}\tilde{\phi}(1/p) = 0 \quad (66)$$

and the interiority conditions will be satisfied at p if

$$p\tilde{\phi}(p) < \bar{x}_1 \text{ and } \tilde{\phi}(1/p) < p\bar{y}_2. \quad (67)$$

From Equation 66 it is immediate that if the interiority conditions are satisfied, there is a competitive equilibrium at price $p = 1$ and that if p is an equilibrium price, then $1/p$ is also an equilibrium price. Under the conditions of Theorem 2, there will be multiple equilibria if $\tilde{\phi}(1) + 2\tilde{\phi}'(1) > 0$. Since, by definition, $\tilde{\phi}(p) = \phi(p) - \bar{z}$, there will be multiple equilibria if $\phi(1) + 2\phi'(1) > \bar{z}$ and if the interiority conditions are satisfied.

Example 6. Consider an economy with the same quadratic utility functions as those in Example 1, but with initial endowments $(\bar{x}_1, \bar{y}_1) = (4, \bar{z})$ and $(\bar{x}_2, \bar{y}_2) = (\bar{z}, 4)$ where $\bar{z} > 0$. There will be three distinct competitive equilibria if $E'(1) > 0$ and if price $p = 1$ satisfies the interiority conditions. For this economy, $\tilde{\phi}(p) = 7/2 - \bar{z} - p$. Therefore $E'(1) = \tilde{\phi}(1) + 2\tilde{\phi}'(1) = 1/2 - \bar{z}$ and hence $E'(1) > 0$ if and only if $\bar{z} < 1/2$. The interiority conditions will

Figure 7: Offer Curves in an Edgeworth Box (The MWG Example)

be satisfied at $p = 1$ if $\tilde{\phi}(1) = 5/2 - \bar{z} < \min\{\bar{x}_1, \bar{y}_2\} = 4$, or equivalently if $z < 3/2$. It follows that there are three competitive equilibria if and only if $\bar{z} < 1/2$.

Example 7. Andreu Mas-Colell, Michael Winston, and Jerry Green present an example of an exchange economy with three competitive equilibria in their graduate economic theory textbook, *Microeconomic Theory* [6] (page 521). This example is a generalized Shapley-Shubik economy with mirror-symmetric power function utilities. In Section 5, we showed that if the initial endowment is at a corner of the Edgeworth box then, if there are multiple equilibria, two of these equilibria will be corner solutions. The Mas-Collell, Winston, Green example is constructed with symmetric interior endowments and this example has three interior competitive equilibria. Figure 7 shows offer curves and the Edgeworth box for the MWG example.

In this example, the utility functions of Consumers 1 and 2 are

$$U_1(x, y) = x - \frac{1}{8}y^{-8}$$

$$U_2(x, y) = -\frac{1}{8}x^{-8} + y \quad (68)$$

The initial endowments are $(\bar{x}_1, \bar{y}_1) = (2, \bar{z})$ and $(\bar{x}_2, \bar{y}_2) = (\bar{z}, 2)$. Then $\phi(p) = p^{-1/9}$ and $\tilde{\phi}(p) = p^{-1/9} - \bar{z}$. If p is an interior competitive equilibrium,

$$0 = \tilde{\phi}(p) - \frac{1}{p}\tilde{\phi}\left(\frac{1}{p}\right) = p^{-1/9} - p^{-8/9} - \bar{z}\left(1 - \frac{1}{p}\right). \quad (69)$$

Price p is a solution to Equation 69 if and only if

$$\bar{z} = \frac{p^{8/9} - p^{1/9}}{p - 1} \quad (70)$$

One can construct an example with any desired pair of equilibrium prices $\bar{p} > 1$ and $1/\bar{p}$ by choosing $\bar{z}$ to satisfy Equation 70. The MWG example is chosen to have equilibria at $p = 2$ and $p = 1/2$. This is the case when $\bar{z} = 2^{8/9} - 2^{1/9}$. To ensure that the solutions $p = 2$ and $p = 1/2$ are both interior solutions, we verify that $2\tilde{\phi}(2) < \bar{x}_1 = 2$ and $1/2\tilde{\phi}(1/2) < \bar{y}_2 = 2$.

7 Conclusion

The family of Shapley-Shubik economies is an interesting class of “special general equilibrium models” which are computationally manageable, yet permit complex outcomes such as multiple equilibria. One remarkable property of Shapley-Shubik economies with corner endowments is that if endowments are sufficiently large, equilibrium prices depend only on the demand functions and not on the quantity of goods available. For mirror-symmetric Shapley-Shubik economies (subject to simple interiority conditions) there will always be a competitive equilibrium at a price of 1. A simple calculation of the derivative of excess demand at price 1 determines whether there are multiple equilibria. In a mirror-symmetric economy, if any price is an equilibrium, then so is its reciprocal.

We show how to construct a quadratic Shapley-Shubik economy for which any three arbitrarily specified prices are competitive equilibria. We also find conditions on parameter values that determine whether the number of competitive equilibria is one, two or three. Finding the equilibria for pre-specified parameters is equivalent to solving a cubic equation.

Shapley and Shubik's paper presents a single example in which the non-linear portions of utility functions are exponential. They demonstrated by means of a numerical solution that this example has three distinct equilibria. We present a general formula for determining the number of competitive equilibria over this entire class of economies.

The initial endowments in a Shapley-Shubik economy are not interior to consumers' consumption sets, but "at a corner of the Edgeworth box". Each consumer is endowed only with the commodity in which his utility is linear. As Debreu [1] noted, this can result in pathological cases where competitive equilibrium does not exist. We present an example of a Shapley-Shubik economy that lacks a competitive equilibrium and then show that with the addition of relatively weak assumptions, existence of equilibrium can be guaranteed.

Our qualitative results for Shapley-Shubik economies are robust to perturbations in initial endowments that allow each consumer to have a positive endowment of each commodity. We show that such economies are isomorphic to standard Shapley-Shubik economies in which the utility functions are appropriately modified. In the modified economies, the necessary and sufficient conditions for multiple equilibria become more stringent as endowments move further from the corner. In generalized Shapley-Shubik economies, the neutrality result that interior equilibrium prices are not affected by small changes in endowments continues to hold.

In a Shapley-Shubik economy where the nonlinear portions of utility are power functions, the demand function over the range of interior solutions has constant elasticity. We show that in a mirror-symmetric Shapley-Shubik

economy with power functions and “inelastic demand”, there are three distinct competitive equilibria. Two of these are corner solutions, in which one consumer trades away his entire initial endowment to obtain some of the good possessed by the other consumer. The example of an economy with multiple equilibria found in Mas Collé, Winston and Green is a mirror-symmetric Shapley-Shubik economy with power function utilities, but where each consumer has a positive initial endowment of both goods. For this economy, there are three interior competitive equilibria.

References

- [1] Gerard Debreu. *Theory of Value*. Wiley, New York, 1959.
- [2] Leonard E. Dickson. *First Course in the Theory of Equations*. John Wiley & Sons, New York, 1922.
- [3] Leonard Euler. *Elements of Algebra*. Springer-Verlag, New York, 1972. First published in German in 1770. English translation by Rev. John Hewlett, 1840.
- [4] Steven Gjerstad. Multiple equilibria in exchange economies with homothetic, nearly identical preferences. Technical report, University of Minnesota, Department of Economics, University of Minnesota, Minneapolis, MN, 1996.
- [5] Alok Kumar and Martin Shubik. A computational analysis of core convergence in a multiple equilibrium economy. *Games and Economic Behavior*, 42:253–256, 2003.
- [6] Andreu Mas-Collé, Michael Winston, and Jerry Green. *Microeconomic Theory*. Oxford University Press, New York, 1995.

- [7] Paul A. Samuelson. Abstract of a theorem concerning substitutability in open Leontief models. In Tjalling C. Koopmans, editor, *Activity Analysis of Production and Allocation*. Wiley, 1951.
- [8] Lloyd Shapley and Martin Shubik. An Example of a Trading Economy with three competitive equilibria. *Journal of Political Economy*, 85(4):873–875, 1977.