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A METHOD TO SMOOTH EXPERIMENTALLY-DETERMINED
PARTIAL-WAVE PHASE SHIFT*

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August 9, 1972

ABSTRACT

A method to smooth an experimentally-measured phase shift is proposed. This method is based on circular application of a generalized N/D method in which CDD pole parameters are handled without ambiguity. The output solution is guaranteed to be free from causality-violating singularities, and the numerical calculation is simpler than in previous prescriptions with the same objective.

1. INTRODUCTION

Experimentally-determined phase shifts for elastic scattering processes usually contain ambiguities. For example, πN partial-wave phase shifts from various groups¹⁻⁴ are bumpy in the energy region where the inelasticity builds up. The bumpy behavior of the phase shifts suggests the existence of nearby singularities, e.g., resonance poles, but there is naturally fear that some of this bumpy behavior may be caused by experimental error and not correspond to genuine singularities of the analytic S matrix. One category of spurious singularities are those with complex positions on the first sheet (usually called "causality violating singularities"). Smoothing procedures typically fit the data to analytic functions that lack such inadmissible singularities, being singular only in regions permitted by general S-matrix principles. The purpose of this article is to propose a method of smoothing, based on circular application of the N/D method. This method leads to a simpler smoothing procedure than do previous N/D methods. We also demonstrate that CDD poles can be handled without ambiguity.

We need to review briefly previous attempts to use N/D integral equations in order to make clear the differences between this and the previous works. The standard way to perform an N/D calculation⁵ is first to estimate the nearby "left-hand" singularities of partial-wave amplitude from experimental data on cross channel reactions. The distant "left-hand" singularities, the input inelastic function and CDD pole parameters are considered to be adjustable parameters. Inserting these inputs into the N/D integral equations, one tries by adjustment of the input parameters to obtain an output phase shift which is a good fit to experiment. There have also been investigations

different from this standard method but parallel to it. We mention three: One method^{6,7} avoids usage of the N/D integral equations. Instead, the partial-wave dispersion relation is solved directly by some variational technique implementing unitarity, though the input potential is parametrized as in the standard N/D approach. The second approach uses Blankenbecler-Sugar equation⁸⁻¹¹ to include multi-particle channels. The third approach we want to mention is a multi-channel N/D framework,¹²⁻¹⁴ instead of the single-channel formalism of the above described standard approach. The parametrization of the left-hand singularities of all the coupled reactions replaces both the input inelastic function and the CDD pole parameters. (In practice, of course, it is impossible to include all coupled channels.)

The adjustment of input parameters in the above-mentioned methods is coupled to the procedure for solving the integral equations, that is, all the input parameters must be adjusted simultaneously and the integral equation must be solved once after each adjustment. This situation makes the practical calculation very complicated if we require the output phase shift to become so accurate that we can determine whether or not the bumpy behavior of the experimental phase shift is caused by causality-violating singularities. Our proposal is based on the observation that the N/D method is circular and that the standard approach corresponds only to half of the circle.

Our method starts from a tentative phase shift based on experiment instead of on the parametrization of the input potential. This experimental phase shift is used to calculate an "experimental potential," which is defined to be the amplitude minus its physical (right-hand) cut and which will contain causality violating singularities if the tentative phase shift contains these singularities. In

the next step the experimental potential is fitted by a function containing the well known nearby unphysical region singularities,^{15,16} plus some properly-parametrized distant singularities whose positions are restricted by general principles. The fitted potential is, of course, free from causality violating singularities. If used as the input potential to the N/D integral equation, the output solution is generated to be free from causality-violating singularities so long as some circularity constraints are satisfied by adjusting the input inelastic quantities. In this procedure all the parameters of the input potential are fixed before we solve the N/D integral equations. Only the inelastic function and CDD pole parameters need to be adjusted from their experimental value during the procedure to solve the integral equations, and thus we can obtain a high accuracy output solution with relatively less effort in the numerical analysis.

We note that there is no CDD pole ambiguity in this method. In the R-function method¹⁷ CDD poles correspond to zeros of the partial-wave amplitude in the physical region, and in the Frye-Warnock method¹⁸ they are the zeros of the partial-wave S matrix in the physical region. Thus all CDD parameters are fixed by giving the experimental phase shift (see the next section for a review of this subject). We also note that our method requires a tentative experimental phase shift for the entire range of physical energies up to infinity. The high energy part is obtained by partial-wave projection of some high energy extrapolation of the elastic scattering amplitude (e.g., we may use a Regge pole fit). Thus our smoothing procedure also removes any possible causality-violating singularities (if any) caused by non-smooth transition from high energy to low energy.

In this article we formulate the smoothing procedure in detail for the πN phase shift and restrict consideration to the R-function method. The kinematics and the definition of various amplitudes for πN elastic scattering are summarized in the Appendix. In Sec. II, the reformulation of the N/D method and the handling of CDD poles are reviewed, while the circular character of the N/D method is discussed in Sec. III. The procedure of smoothing is outlined in detail in Sec. IV, and a rough but illustrative example is given by performing the first part of the smoothing of a πN P_{33} -wave in Sec. V. Some overall assessment of the proposed procedure is given in the last section.

2. CDD POLE PARAMETERS AND THE GENERALIZED N/D METHOD

The N/D method has recently been reconstructed,¹⁹⁻²² eliminating the assumption of power-boundedness for partial-wave amplitudes. Here we follow the style of Ref. 21 and review the reconstruction briefly in order to demonstrate the way to determine CDD pole parameters from a given phase shift. We use 0^- , $\frac{1}{2}^+$ kinematics (e.g., πN), summarized in the Appendix, and restrict consideration to the case of one CDD pole, the generalization to any number of CDD poles being straightforward.

A phase function $\Theta_{\ell\pm}(W)$ is introduced from the parametrization

$$f_{\ell\pm}(W) = |f_{\ell\pm}(W)| e^{i\Theta_{\ell\pm}(W)}, \quad (0 \leq \Theta_{\ell\pm}(W) \leq \pi)$$

while a $\tilde{D}$ function is defined as

$$\tilde{D}(W) = \exp \left\{ -\frac{W}{\pi} \int_{W_0}^{\infty} dW' \frac{\Theta(W')}{W'(W' - W)} \right\}, \quad (2.1)$$

where

$$W_0 = m + \mu,$$

the subscript $\ell\pm$ being omitted as understood. The function $\tilde{D}(W)$ will have a pole at $W = W_c$, if

$$\lim_{\epsilon \rightarrow 0_+} \Theta(W_c - \epsilon) = \pi$$

and

$$\lim_{\epsilon \rightarrow 0_+} \Theta(W_c + \epsilon) = 0.$$

We assume that one such pole in fact appears in the partial-wave amplitude. A new function $D(W)$ is defined as

$$D(W) = (W_0 - W) \tilde{D}(W) \quad (2.2)$$

so as to be free from the pole at $W = W_0$.

For the convenience of later application in Sec. 5, and to simplify the present discussion we assume that the πN elastic scattering amplitude satisfies the specific Regge parametrization of Ref. 23; then the asymptotic behavior of a partial-wave amplitude, neglecting the nonessential logarithmic factors, becomes

$$f_{\ell\pm}(W) \xrightarrow{W \rightarrow +\infty} \frac{1}{W},$$

and

$$\tan \theta(W) = \frac{\text{Im } f_{\ell\pm}(W)}{\text{Re } f_{\ell\pm}(W)} \xrightarrow{W \rightarrow +\infty} C \quad (C < 0).$$

The assumption of this specific asymptotic behavior for the partial-wave amplitude is not essential to our discussion,²¹ but makes it simple. With such asymptotic behavior for the phase, the D function behaves asymptotically like

$$D(W) \xrightarrow{W \rightarrow +\infty} W^{1+a} \quad \left(\frac{1}{2} < a < 1\right). \quad (2.3)$$

An N function is defined to be

$$N(W) = D(W) f_{\ell\pm}(W) \quad (2.4)$$

while the potential is

$$V(W) = f_{\ell\pm}(W) - \frac{1}{\pi} \int_{W_0}^{\infty} dW' \frac{\text{Im } f_{\ell\pm}(W')}{W' - W} \quad (2.5)$$

Also needed is a C function:

$$C(W) = N(W) - D(W) V(W) = \frac{D(W)}{\pi} \int_{W_0}^{\infty} dW' \frac{\text{Im } f_{\ell\pm}(W')}{W' - W} \quad (2.6)$$

The partial-wave unitarity relation can be expressed as

$$\text{Im} \left\{ \frac{1}{f_{\ell\pm}(W)} \right\} = -q_s R(W)$$

or

$$R(W) = \frac{\text{Im } f_{\ell\pm}(W)}{q_s |f_{\ell\pm}(W)|^2} \quad (2.7)$$

where $R(W) = 1$ in the elastic region and $R(W) \geq 1$ in the inelastic region. From partial-wave unitarity and the definitions of $N(W)$ and $C(W)$, i.e., Eqs. (2.4) and (2.6), we have

$$\text{Im } D(W) = \begin{cases} -q_s R(W) N(W) & W \geq W_0 \\ 0 & W < W_0, \end{cases} \quad (2.8)$$

and

$$\text{Im } C(W) = \begin{cases} q_s R(W) \cdot V(W) \cdot N(W) & W \geq W_0 \\ 0 & W < W_0. \end{cases}$$

The D function satisfies a dispersion relation with two subtractions, according to its definition Eqs. (2.1) and (2.2) and its asymptotic behavior Eq. (2.3), while the C function satisfies a dispersion relation with one subtraction. Therefore we need three subtraction constants. We choose these to be as $D(0)$, $D'(0)$, and $C(0)$. The constant $D(0)$ is a normalization constant for both $D(W)$ and $N(W)$, and will not appear in their ratio. The constants $D'(0)$ and $C(0)$ come into the problem because of the physical region zero of the partial-wave amplitude $f_{\ell\pm}(W)$ at $W = W_c$. If $f_{\ell\pm}(W)$ does not have any zero in the physical region, only the constant $D(0)$ appears. Since $D'(0)$ and $C(0)$ constitute two parameters which must be supplied in addition to the potential $V(W)$ and the R function in the N/D integral equations (see below), we refer to them as a pair of CDD pole parameters. This definition of CDD pole is reasonable since in the $\tilde{D}$ function, there is a pole at $W = W_c$ due to the zero of $f_{\ell\pm}(W)$ at $W = W_c$. The values of $D'(0)$ and $C(0)$ can be written down from Eqs. (2.1), (2.2), and (2.6) to be

$$\gamma \equiv D'(0) = - \left\{ \frac{1}{W_c} + \frac{1}{\pi} \int_{W_0}^{\infty} dW' \frac{\Theta(W')}{W'^2} \right\} \quad (2.9)$$

and

$$\beta \equiv C(0) = \frac{1}{\pi} \int_{W_0}^{\infty} dW' \frac{\text{Im } f_{\ell\pm}(W')}{W'}$$

where the D function is normalized to be unity at $W = 0$. From Eq. (2.9) we see explicitly that the CDD pole parameters are determined uniquely if the partial-wave amplitude $f_{\ell\pm}(W)$ (and thus the

phase shift) is given. The dispersion relations for $D(W)$ and $C(W)$ can be written as

$$D(W) = 1 + \gamma W - \frac{W^2}{\pi} \int_{W_0}^{\infty} dW' \frac{q_s(W') R(W') N(W')}{W'^2 (W' - W)}$$

and

$$C(W) = N(W) - V(W) D(W) = \beta + \frac{W}{\pi} \int_{W_0}^{\infty} dW' \frac{q_s(W') R(W') V(W') N(W')}{W' (W' - W)}$$

Inserting the dispersion relation of $D(W)$ into that of $C(W)$, we obtain the N/D integral equation as

$$g(W) = v(W) + \int_{W_0}^{\infty} dW' K(W, W') g(W') \quad (2.10)$$

and

$$D(W) = 1 + \gamma W - \frac{W^2}{\pi} \int_{W_0}^{\infty} dW' \frac{q_s(W') R(W')}{W'} \cdot \frac{g(W')}{W' - W}, \quad (2.11)$$

where

$$g(W) = \frac{N(W)}{W},$$

$$v(W) = \frac{1}{W} \{ \beta + (1 + \gamma W) \cdot V(W) \}$$

and

$$K(W, W') = \frac{1}{\pi} \cdot \frac{q_s(W') R(W')}{W'} \cdot \frac{W' V(W') - W V(W)}{W' - W}.$$

3. CIRCULARITY OF THE N/D METHOD

The N/D method outlined in the previous section has a circular nature. Starting from a partial-wave amplitude $f_{\ell\pm}(W)$, we calculate the potential $V(W)$ from Eq. (2.5), the R function from Eq. (2.7), and the necessary CDD pole parameters from Eq. (2.9). Then inserting them into the N/D integral equations, an output partial-wave amplitude $f_{\ell\pm}(W)$ is obtained. Of course the output partial-wave amplitude should be identical to the input one. By a careful study of the construction of the N/D integral equations, we see that this circular nature is rather trivial²⁴ in the sense that any partial-wave amplitude will satisfy this circle, if it satisfies the unitarity relation in the elastic region, the unitarity bound in the inelastic region, the Hölder continuity,¹⁹ and if its potential makes the N/D integral equation Fredholm.

The above mentioned circularity of the N/D method guarantees the absence of ghost zeros from the output D function. But the absence of ghost zeros from the D function becomes uncertain if we start from giving an input potential, a R function and CDD pole parameters instead from an input partial-wave amplitude. Suppose such a set of input information is given, i.e., an input potential, a R function and CDD pole parameters from Eqs. (2.7), (2.5), and (2.9) respectively. We can reconstruct an output R function, an output potential and CDD pole parameters. The circularity between the input and the output R functions is trivial [it can be shown from Eqs. (2.4), (2.7), and (2.8)]. The circularity for the potential and CDD pole parameters is not obvious. Nevertheless if the latter circularity is also imposed, then the complete circularity of the N/D method guarantees the absence of ghost zeros from the D function.

We demonstrate the above argument by considering a simple example and investigate the nontrivial appearance of the circularity of the potential. We assume that there is no CDD pole, and the potential contains N poles only, so it can be written as

$$V(W) = \sum_{j=1}^N \frac{c_j}{W - W_j}$$

where

$$V^*(W) = V(W^*) .$$

The N/D integral equation becomes

$$N(W) = V(W) - \sum_{j=1}^N \frac{c_j W_j}{\pi} \cdot \frac{x_j}{W - W_j} , \quad (3.1)$$

where

$$x_j = \int_{W_0}^{\infty} dW' \frac{F(W') N(W')}{W' - W_j}$$

and

$$F(W') = \frac{q_s(W') R(W')}{W'}$$

Multiplying both sides by $F(W)/(W - W_i)$ and integrating over W from W_0 to ∞ , we obtain

$$x_i = t_i - \frac{1}{\pi} \sum_{j=1}^N \alpha_{ij} c_j x_j W_j \quad (i = 1, \dots, N) , \quad (3.2)$$

where

$$\alpha_{ij} = \int_{W_0}^{\infty} dW' \frac{F(W')}{(W' - W_i)(W' - W_j)},$$

and

$$t_i = \sum_{j=1}^N \alpha_{ij} c_j.$$

Substituting Eq. (3.1) into the dispersion relation for $D(W)$,

$$D(W) = 1 - \frac{W}{\pi} \int_{W_0}^{\infty} dW' \frac{F(W')}{W' - W},$$

we have

$$D(W) = 1 - \frac{W}{\pi} \sum_{j=1}^N c_j \alpha_j(W) \cdot \left(1 - \frac{W_j x_j}{\pi}\right), \quad (3.3)$$

where

$$\alpha_j(W) = \int_{W_0}^{\infty} dW' \frac{F(W')}{(W' - W)(W' - W_j)}.$$

Taking the ratio N/D of Eqs. (3.1) and (3.3), and continue to

$W = W_i$, we have

$$\lim_{W \rightarrow W_i} (W - W_i) \frac{N(W)}{D(W)} = \frac{c_i \left(1 - \frac{W_i x_i}{\pi}\right)}{1 - \frac{W_i}{\pi} \sum_{j=1}^N c_j \alpha_{ij} \left(1 - \frac{W_j x_j}{\pi}\right)}.$$

Substituting Eq. (3.2) for x_i in the numerator, we have

$$\lim_{W \rightarrow W_i} (W - W_i) \frac{N(W)}{D(W)} = c_i \quad (i = 1, \dots, N).$$

This result indicates that the input potential is really a part of the unphysical region singularities of the output partial-wave amplitude.

In order to show the absence of the other unphysical region singularities (poles) from the output partial-wave amplitude, i.e., the absence of ghost zeros from the D function, the circularity of the potential must be imposed.

The circularity of the potential can be written, from Eq. (2.5),

as

$$\sum_{j=1}^N \frac{c_j}{W - W_j} = \frac{N(W)}{D(W)} - \frac{1}{\pi} \int_{W_0}^{\infty} dW' \frac{N(W')}{W' - W} \cdot \text{Im} \left\{ \frac{1}{D(W')} \right\}, \quad (3.4)$$

where $N(W)$ and $D(W)$ are given by Eqs. (3.1) and (3.3) respectively.

The parameters c_j 's and W_j 's may be considered as given, then the circularity equation of Eq. (3.4) becomes a nontrivial constraint for the R function, in other words, the R function must satisfy this constraint in order to guarantee the absence of ghost zeros from the D function.

4. THE PROCEDURE FOR SMOOTHING A PHASE SHIFT

We now outline our procedure for smoothing a phase shift based on the circularity of the N/D method. We discuss explicitly the πN phase shift as an example, the application to other kinds of phase shifts being straightforward.

Starting from an experimental low- to medium-energy phase shift for a certain partial wave, we combine this with the high-energy phase shift from the partial-wave projection of a phenomenological parametrization of the high-energy πN elastic amplitude, e.g., a Regge pole fit etc. We call this combined phase shift the experimental phase shift. The experimental input potential, R function, and CDD pole parameters (if their existence is indicated by the experimental phase shift) are calculated from their definitions by inserting the experimental phase shift into Eqs. (2.5), (2.7), and (2.9) respectively. In order to eliminate causality-violating singularities we next seek a "fitted" potential that approximates as well as possible the "input" potential but that contains only allowed unphysical region singularities.²⁵ The structure of the near-by unphysical singularities are known,^{15,16} but the distant unphysical-region singularities, whose positions are known, but whose magnitude is uncertain, must be parametrized and these parameters determined by a best fit to the experimental potential. This fitted potential as well as the experimental CDD pole parameters and the experimental R function are then used as the input to the N/D integral equation. Since the fitted potential is free from causality-violating singularities, the structure of the N/D integral equation guarantees a corresponding property for the N function so long as the input R function is continuous.

The R function and the CDD pole parameters then must be adjusted from their experimental input to make the circularity for the potential and CDD pole parameters be satisfied. The output partial-wave amplitude, after the complete circularity is imposed, is guaranteed to be free from causality violating singularities.

We note that in this smoothing procedure the determination of the parameters of the fitted potential is carried out before the integral equations are solved. This represents a considerable simplification of the effort in numerical analysis compared to the standard N/D approaches discussed in Sec. I, where the adjustment of the potential parameters is coupled to the procedure for solving the integral equations. Further the absence of causality violating singularities is not guaranteed in those standard N/D approaches due to the ignorance of the circularity of the potential and CDD pole parameters. Another characteristic of our smoothing procedure, which in principle also exists in the standard approach but usually is neglected, is the necessity to use an asymptotic high-energy phase shift. Thus our smoothing procedure automatically removes possible causality-violating singularities associated with nonsmooth transition from low energies to high energies.

There is a more effective method to check the absence of the ghost zeros (causality violating zeros or zeros below the elastic threshold) from the output D function than the explicit check of the circularity of the potential. Suppose the dispersion relation of D requires $N + 1$ subtractions, that is, we are considering the case with N CDD poles. A $\tilde{D}$ function can be constructed by Eq. (2.1) from the phase of the output D function. This phase should have N discontinuous jumps from π to 0 in the physical region in order to

generate N poles in $\tilde{D}$. A new D function is then obtained by multiplying the $\tilde{D}$ function with N zeros to cancel N poles of $\tilde{D}$. This new D function is apparently free from ghost zeros. The ratio between the output D and this new D function is regular on the whole complex energy plane. If this entire function approaches a constant asymptotically at any direction, the output D function is proportional to the new D function, and it is also free from ghost zeros.

5. AN ILLUSTRATIVE EXAMPLE (SMOOTHING OF A πN P_{33} -WAVE)

We smooth a πN P_{33} -wave phase shift in this section as an example illustrative of the procedure outlined in the previous section. The fitting technique used in this example is primitive and can be greatly improved by advanced numerical analysis. The last step of the smoothing procedure, i.e., the check of the circularity of the potential and CDD pole parameters has not been performed in this simple calculation. Instead the behavior of the output D function is investigated explicitly. It turns out that there is no obvious causality violating zeros (ghost zeros) in our solution.

The low-energy πN P_{33} -wave phase shift is taken from the CERN analysis,¹ while the high-energy phase shift is obtained from a partial-wave projection of Barger and Phillips' five Regge pole fit²³ to the πN elastic scattering amplitude. A portion of the combined experimental phase shift (from the πN threshold up to 3.3 GeV in center of mass energy) is shown in Fig. 1 by the dotted points. The transition from the CERN phase shift to the Regge phase shift occurs around 2.2 GeV. Substituting this combined experimental phase shift into Eq. (2.5), an experimental potential is obtained. To simplify the calculation we fit the experimental potential by a finite number of poles. In any serious attempt to implement this smoothing procedure, the known structure of nearby unphysical region singularities (they are usually logarithmic) should be used explicitly, instead of just poles. For our fitted potential in this example we use twenty-one poles; three in the interval between the origin of the W plane and the πN elastic threshold, one pair of complex conjugate poles at the position of the circle singularities²⁵ associated with the

t channel, and nine pairs of complex conjugate poles along the imaginary axis of the W plane. The positions of the poles are fixed, but their residues are considered as the adjustable parameters. Thus we have twelve adjustable parameters (the residues of a pair of complex conjugate poles are complex conjugate). These twelve adjustable parameters are then determined by a least square fit to the experimental potential. A portion of the fitted potential and the experimental one are shown in Fig. 2 up to about 3.5 GeV. A better fit can be obtained by an advanced numerical analysis, e.g., also take the positions of poles are adjustable parameters, and by considering some logarithmic singularities explicitly.

The experimental phase shift from which we started requires one CDD pole at infinity, whose presence reflects our definition of the partial-wave amplitude combined with the Regge behavior of Ref. 23, our partial-wave amplitude vanishing at infinity. This CDD pole could be removed by changing the definition of the partial-wave amplitude to let it approach a nonvanishing asymptote, but then a subtraction constant must be included in the definition of the potential. The physics would be unchanged. Using the notion of a CDD pole at infinity in this example, we obtain the CDD pole parameters (one subtraction constant for the C function and one for the D function). An experimental R function is also calculated from the combined experimental phase shift. The fitted potential, the experimental R function, and the experimental CDD pole parameters are used as the input to the N/D integral equations. The output solution of the integral equations has a Δ_{33} resonance with a mass shifted about 80 MeV below the experimental mass (1236 MeV).

The CDD pole parameters are varied about 10% from their experimental value to bring the output mass of the Δ_{33} resonance to its experimental value. The R function is adjusted by hand, starting from the experimental value to produce a resonant fit to the input phase shift. This adjusted R function has not been listed, instead the output solution translated to the output phase shifts are shown in Fig. 2 as the solid curve [we note that due to the trivial circularity of the R function, our input R function can be easily obtained from the output phase shift through Eq. (2.7)]. The circularity of the potential and CDD pole parameters is not checked in this simple example, but the values of the output D function at various energies are plotted in Fig. 3. The lack of clockwise circle in this figure indicates the absence of nearby ghost zeros from the D function. This can also be seen from the negativity of the imaginary part of the D function in the physical region, combining with the dispersion relation of D (the imaginary part of D at complex W will never vanish). We note that no systematic effort has been made in this example to obtain a best fit to the input phase shift.

6. DISCUSSION

The smoothing procedure outlined in the previous sections has been constructed on the ordinary partial-wave amplitude. We note that a similar smoothing procedure can be constructed on other partial-wave amplitudes,^{26,27} with which the partial-wave expansion converges faster than the ordinary partial-wave expansion.

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APPENDIX

We review the kinematics of πN scattering in this appendix. The notations follow that of Ref. 25 and Ref. 28. Let the four momenta of the incident and outgoing pions be q_1 and q_2 respectively, while those of the initial and final nucleons are p_1 and p_2 . The masses of pion and nucleon are denoted as μ and m . The Mandelstam variables s , t , and u are defined to be

$$s = (p_1 + q_1)^2$$

$$t = (q_2 - q_1)^2$$

$$u = (p_1 - q_2)^2.$$

The magnitude of the three momentum vector in the center of mass frame, q_s , and the cosine of the scattering angle, $\cos \theta$, are

$$q_s^2 = \frac{[s - (m + \mu)^2][s - (m - \mu)^2]}{4s}$$

and

$$\cos \theta = 1 + \frac{t}{2q_s^2}.$$

Two amplitudes f_1 and f_2 are defined²⁸ in terms of the invariant amplitudes $A(s, t, u)$ and $B(s, t, u)$ as

$$f_1 = \frac{E + m}{8\pi W} \{A + (W - m)B\}$$

and

$$f_2 = \frac{E - m}{8\pi W} \{-A + (W + m)B\},$$

where

$$W = (s)^{\frac{1}{2}}$$

and

$$E = (m + q_s)^{\frac{1}{2}}.$$

The partial-wave expansion of f_1 and f_2 ²⁸ are

$$f_1 = \sum_{\ell=0}^{\infty} \{f_{1+}(W) P'_{\ell+1}(\cos \theta) - f_{\ell-}(W) P'_{\ell-1}(\cos \theta)\}$$

and

$$f_2 = \sum_{\ell=1}^{\infty} \{f_{\ell-}(W) - f_{\ell+}(W)\} P'_{\ell}(\cos \theta).$$

By the help of the orthogonal relation

$$\int_{-1}^1 d(\cos \theta) P'_k(\cos \theta) \{P_{\ell+1}(\cos \theta) - P_{\ell-1}(\cos \theta)\} = \delta_{k\ell}$$

we obtain

$$f_{\ell\pm}(W) = \frac{1}{16\pi W^2} \{(W + m + \mu)(W + m - \mu) \cdot [A_{\ell}(s) + (W - m)B_{\ell}(s)] \\ + (W - m + \mu)(W - m - \mu) \cdot [-A_{\ell}(s) + (W + m)B_{\ell\pm 1}(s)]\}$$

where

$$A_{\ell}(s) = \int_{-1}^1 d(\cos \theta) A(s, t, u) P_{\ell}(\cos \theta)$$

and

$$B_{\ell}(s) = \int_{-1}^1 d(\cos \theta) B(s, t, u) P_{\ell}(\cos \theta) .$$

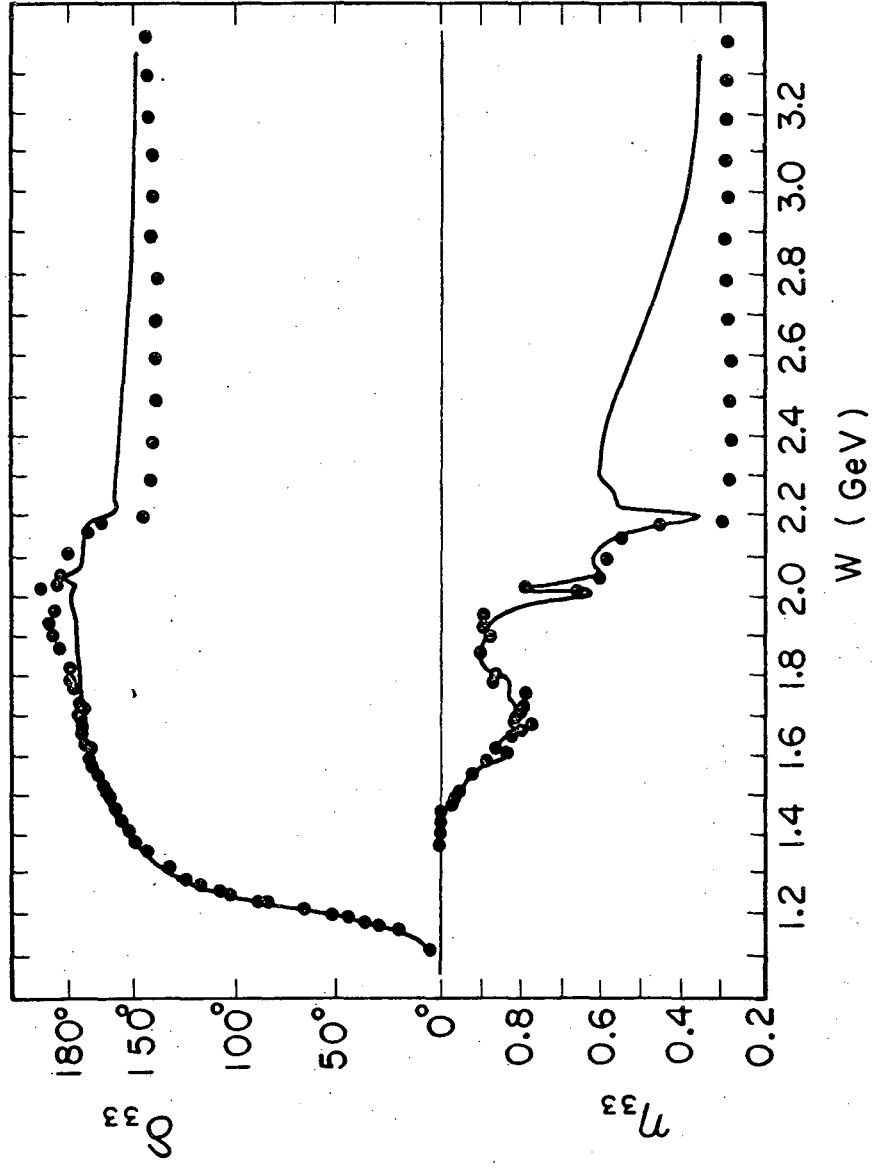
FOOTNOTES AND REFERENCES

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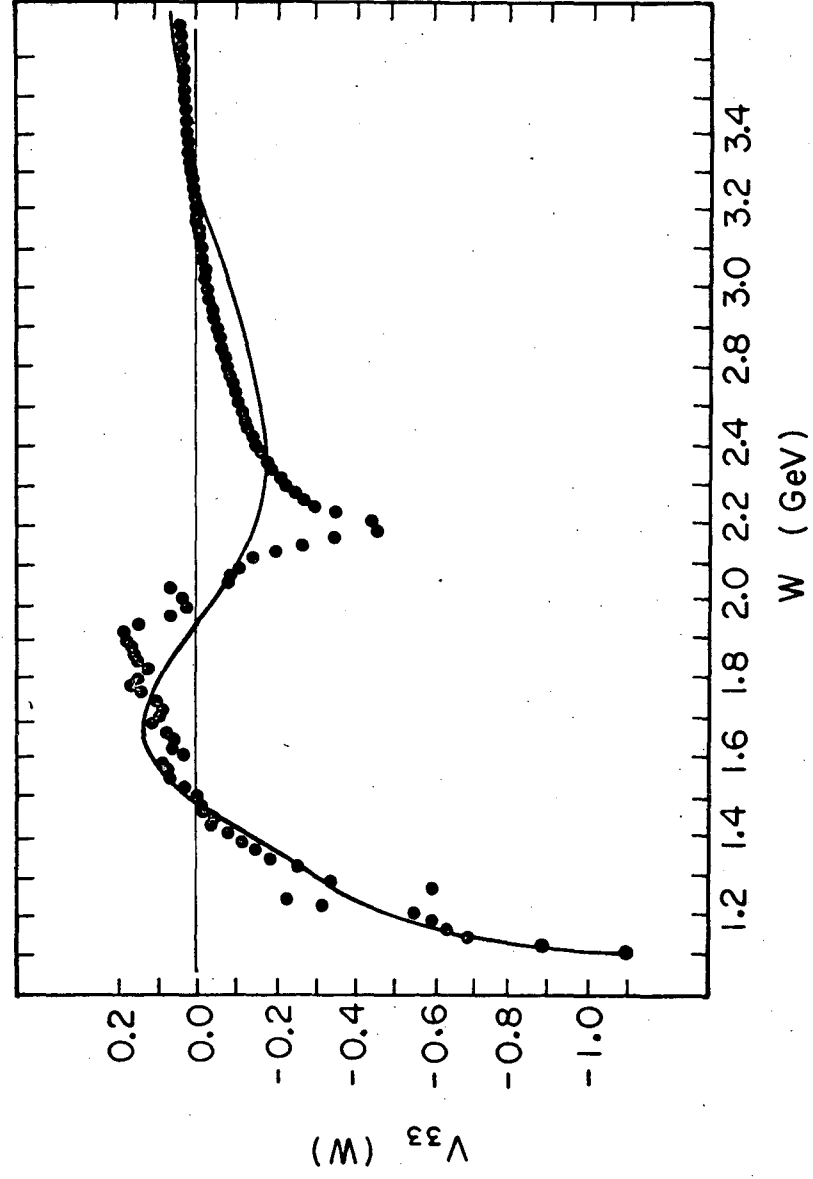
FIGURE CAPTIONS

- Fig. 1. The dotted points below 2.2 GeV are the CERN experimental phase shift for πN P_{33} -wave and those above 2.2 GeV are the Regge phase shift from the partial-wave projection of the Regge pole fit of Ref. 23. The solid curves are our output phase shift.
- Fig. 2. The dotted points represent the value of the experimental potential, and the solid curve is the fitted one.
- Fig. 3. The values of the output D function are plotted for various energies.



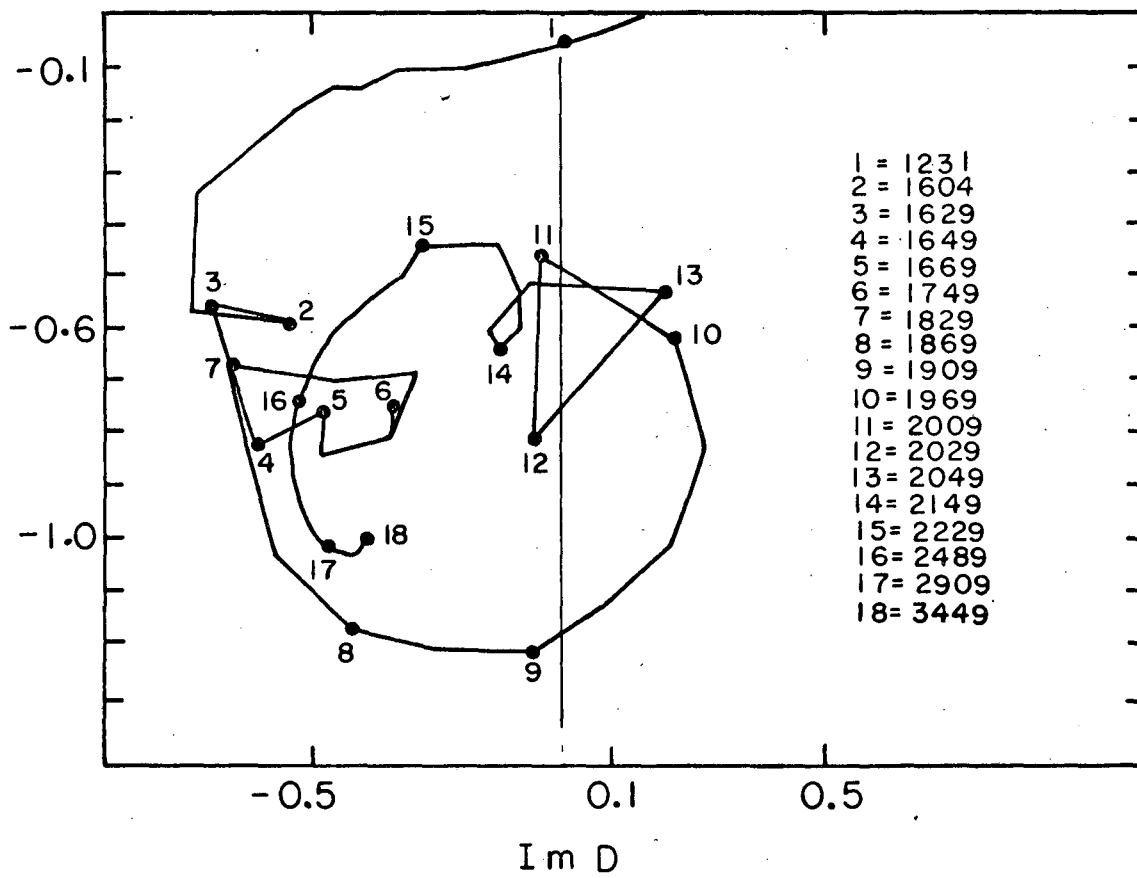
XBL728-3750

Fig. 1



XBL728 - 3749

Fig. 2



XBL728-3748

Fig. 3

UPDATE CONTROL CARD PARAMETERS

UPDATE ($p_1, p_2, \dots, p_n$)

- A Sequential-to-random copy
- B Random-to-sequential copy
- C Compile file output; COMPILE if C or omitted. No compile if C=0. Otherwise, output to file named.
- D Data width; 72 columns if no D. 80 columns if D.
- E Edit. No editing if no E.
- F Full update. If F omitted, corrected decks and those named on COMPILE cards are compiled. If F is specified, all decks are compiled.
- G Pullmod file. If omitted, pulled modifications are appended to source file. Otherwise, output is written on named file.
- I Input. If omitted, directives and text on INPUT. Otherwise on named file.
- K COMPILE card sequence. Takes precedence over C mode. If K, decks written on file COMPILE in COMPILE card sequence. If K=filename, decks written on named file.
- L List options. If omitted, options A, 1, 2, 3, and 4 selected. Options A, F, and 0-9 not separated by commas. Any use of 0 suppresses listing.
- M Merge input. Omitted, no merge. M, merge input on file MERGE. M=filename, input on named file.
- N New program library. Omitted, no new library. N, output on NEWPL. N=filename, output to named file.
- O List output file. Omitted or O, listings on OUTPUT. O=filename, output to named file.
- P Old program library. Omitted or P, library on OLDPL. P=filename, library on named file.
- Q Quick update. Takes precedence over F. Omitted, normal selective mode. If Q, only decks on COMPILE cards are processed.
- R Rewind. Omitted, files are automatically rewound. If R, no rewinds issued. If $R=c_1c_2\dots c_n$, then only files indicated by C, N, P, S are rewound.
- S Source output. Omitted and no T, no source output. S, output on SOURCE. S=filename, output on named file.
- T Source output excluding common decks (takes precedence over S). Omitted and no S, no source output. If T, output on SOURCE. T=filename, output on named file.
- U Debug mode. Omitted, fatal error ends execution. If F, fatal error does not end execution.
- Y Random NEWPL. If omitted, NEWPL is sequential.
- X Compressed compile file. Omitted, compile file not compressed.
- Z Compressed input file. Omitted, input file is not compressed.
- 8 80-column output on compile file. Omitted, 90-column card images. If 8, 80-column card images.
- * Master control character. Omitted, control character is *. If *=c, new character is c.
- / Comment control character. Omitted, comment character is /. If /=c, new comment character is c.
- 7 Truncate card images on COMPILE-No sequencing or trailing blanks.

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