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UNIVERSITY OF CALIFORNIA, SAN DIEGO

Stability of Gas-Fluidized Beds

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

Kevin Matthew Mandich

Committee in charge:

Professor Robert J. Cattolica, Chair
Professor Prabhakar R. Bandaru
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2013

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The dissertation of Kevin Matthew Mandich is approved,
and it is acceptable in quality and form for publication
on microfilm and electronically:

Chair

University of California, San Diego

2013

DEDICATION

To my parents, Gina and Steve, and to my brother and best friend, Michael.

EPIGRAPH

There is nothing new under the sun. It has all been done before.

“A Study in Scarlet”

Arthur Conan Doyle

“Excellent!” I cried. “Elementary,” said he.

“The Memoirs of Sherlock Holmes”

Arthur Conan Doyle

Education never ends, Watson. It is a series of lessons with the greatest for the last.

“His Last Bow”

Arthur Conan Doyle

TABLE OF CONTENTS

Signature Page	iii
Dedication	iv
Epigraph	iv
Table of Contents	vi
List of Figures	ix
List of Tables	xii
Acknowledgements	xiii
Vita	xv
Abstract of the Dissertation	xvii
Chapter 1 Introduction	1
1.1 Motivation for Studying Fluidized Beds	1
1.2 Hydrodynamic Stability Theory	2
1.3 Summary of Dissertation	4
References	5
Chapter 2 Hydrodynamic Theory	6
2.1 Fluidized Bed Behavior	6
2.1.1 Fluidization Regimes	6
2.1.2 Particle Characteristics	8
2.2 Governing Equations	9
2.3 The Extension of Kinetic Theory to Granular Flows	12
References	15
Chapter 3 Literature Review	17
References	20
Chapter 4 Stability of an Unbounded Fluidized Bed	22
4.1 Problem Description	22
4.2 Governing Equations	23
4.3 Base-State Solution: Homogeneous Fluidization	24
4.4 Linearization and Assumed Form of Perturbations	25
4.4.1 Numerical Method	27
4.5 Results	29
4.6 Separated Disturbance Modes	35
4.6.1 Longitudinal Disturbance	35
4.6.2 Transverse Disturbance	38

	4.7 Summary and Observations	43
	References	44
Chapter 5	Stability of a Vertical Fluidized Bed	46
	5.1 Problem Description	46
	5.2 Governing Equations	47
	5.3 Base-State Solutions	47
	5.4 Linear Stability Analysis	49
	5.4.1 Numerical Method	52
	5.5 Results	53
	References	63
Chapter 6	Stability of an Inclined Bed	65
	6.1 Problem Description	65
	6.2 Governing Equations	66
	6.3 Base-State Solution	67
	6.4 Linearization and Assumed Form of Perturbation	68
	6.5 Numerical Method	70
	6.6 Results	77
	6.6.1 Base-State Solutions	77
	6.6.2 Stability Analysis Results	78
	6.7 Discussion	85
	References	86
Chapter 7	Analytic Solutions for 2D and 3D Vertical Fluidized Beds	88
	7.1 Two-Dimensional Bed	88
	7.2 Three-Dimensional Bed	91
	7.3 Results	94
	7.4 Transverse Wave Solution	97
	References	99
Chapter 8	Experimental Determination of the Dominant Frequency	101
	8.1 Experimental Setup - Vertical Bed	102
	8.2 Experimental Setup - Inclined Bed	105
	8.3 Pressure Measurements	107
	8.4 Experimental Results	107
	8.4.1 Pressure Drop	107
	8.4.2 Pressure Time Signal	112
	8.4.3 Comparisons Between Experiment and Simulation	117
	References	122
Chapter 9	Conclusions	124
Appendix A	Discretized Equations - Inclined Bed	128
Appendix B	Discretized Equations - Cylindrical Bed	132

Appendix C	Analytic Solution of the Limiting Case	136
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LIST OF FIGURES

Figure 2.1:	Fluidization regimes [1]. Experiments and numerical analysis of the present work are concerned with the regimes represented by (a) through (d).	7
Figure 2.2:	Geldart chart of particle characterization [2]. Locations of the five materials used in the experiments are noted.	8
Figure 2.3:	Granular flow regimes [21]. The model developed in the present analysis deals only with the viscous regime.	12
Figure 4.1:	Growth constant traces corresponding to the longitudinal mode (solid lines, function of m) and transverse modes (dashed lines, function of k) for varying values of Re , as marked. (b) Growth constant traces for each mode with $Re = 0.780$ calculated by including and ignoring the fluid-phase convective, viscous, and body force terms, as marked. $R^{-1} = 2000$, $e_p = 0.90$	30
Figure 4.2:	Contour plots of the growth constants corresponding to the leading mode in the wavenumber plane for (a) $Re = 0.715$ (b) $Re = 0.850$. $R^{-1} = 2000$, $e_p = 0.90$	31
Figure 4.3:	Growth constants of the dominant mode as a function of the Reynolds number and solid-phase volume concentration. Dark-shaded regions correspond to a dominant transverse wave, while non-shaded regions denote longitudinal dominance. $R^{-1} = 3000$, $e_p = 0.90$	32
Figure 4.4:	Growth constants of the dominant mode as a function of the phase density ratio and solid-phase volume concentration. Dark-shaded regions correspond to a dominant transverse wave, while non-shaded regions denote longitudinal dominance. $Re = 0.780$, $e_p = 0.90$	33
Figure 4.5:	Growth constants of the dominant (a) transverse and (b) longitudinal modes as a function of the solid-phase volume concentration for various values of the restitution coefficient, as marked. $Re = 0.780$	34
Figure 4.6:	Traces corresponding to dimensionless values of the individual and summed terms of equation (4.46), as marked. $e_p = 0.90$	38
Figure 4.7:	Traces corresponding to dimensionless values of the individual and summed terms of equation (4.61): σ_2 , solid line; χ_1 , dashed line; χ_2 , dot-dashed line. The line at zero is included for reference. $e_p = 0.90$	41
Figure 5.1:	(a) Growth constants of the leading mode for the first several azimuthal modes $n = 0-5$, marked by arrow in increasing order, and (b) Growth constants (dots) of the first several leading modes, as well as the frequency (dashes) of the leading mode, for the axisymmetric disturbance, as a function of k . $H = 100$, $Re = 0.6326$, $Fr = 1.1472$, $R = 4.8 \cdot 10^{-4}$	54

Figure 5.2:	Eigenvectors of ϕ' for the (a,e) $n = 1$ (b,f) $n = 2$ (c,g) $n = 3$ (d,h) $n = 4$ azimuthal modes. The latter four plots depict the radial profiles of ϕ' for the first several leading eigenvalues of each angular mode, in order of decreasing growth constant: <i>i, ii, iii</i> . $L = 100$, $Re = 0.6326$, $Fr = 1.1472$, $R = 4.8 \cdot 10^{-4}$	56
Figure 5.3:	(a) Contours of the growth constant (thick line) and frequency (thin line) as a function of Re and k for the (a) $n = 0$ and (b) $n = 1$ modes. $L = 100$, $R = 4.8 \cdot 10^{-4}$	58
Figure 5.4:	(a) Growth constant contours of the dominant travelling mode (solid lines) and the stationary mode (dashed lines), and (b) frequency (thick lines) and wavenumber (thin lines) corresponding to the dominant travelling mode, as a function of L and $\bar{\phi}$. $R = 4.8 \cdot 10^{-4}$	60
Figure 5.5:	(a) Growth constant contours of the dominant travelling mode (solid lines) and the stationary mode (dashed lines), and (b) frequency (thick lines) and wavenumber (thin lines) corresponding to the dominant travelling mode, as a function of R^{-1} and $(Re^2/Fr)^{1/3}$. $L = 100$	61
Figure 5.6:	Growth constant (dots) and frequency (dashes) of the leading $n = 0$ mode at various values of e_p , as marked. Frequencies are marked by arrow in order of increasing e_p . $L = 100$, $Re = 0.6326$, $Fr = 1.1472$, $R = 4.8 \cdot 10^{-4}$	63
Figure 6.1:	Base-state solutions for the (a) dense and (b) dilute systems, in order of increasing inclination angle, as marked by arrow: $\theta = 0^\circ$, 15° , 30° , 45° . $L = 100$, $d\bar{P}_g/dx = -1.0 \cos(\theta)$, $C_1 = 1.0$	77
Figure 6.2:	Growth rates (dots) of the first several leading eigenvalues and frequency (dashes) of the leading eigenvalue as a function of k for (a) $\theta = 0^\circ$ (b) $\theta = 40^\circ$. σ_i values corresponding to the b modes have been reduced by a factor of 6 for convenience.	79
Figure 6.3:	Eigenvectors ϕ' corresponding to the four leading eigenmodes in figure 6.2 for (a) $\theta = 0^\circ$ (b) $\theta = 40^\circ$	80
Figure 6.4:	Contour plots of the (a) growth rate, $\sigma_{r,max}$, and (b) frequency, $\sigma_{i,max}$, of the leading eigenvalue as a function of bed inclination and longitudinal wavenumber. The b_1 mode comprises the upper-left-hand side of each plot. The a_1 mode is leading in the middle portion of the graph, and the a_2 mode is on the right side of each plot.	82
Figure 6.5:	Contour plots showing the evolution of the solid-phase volume fraction ϕ , beginning with the base-state solution and continuing in time with the superimposed dominant instability mode. Plots show the dominant instability modes from (a) figure 6.2(a) and (b) figure 6.2(b). Dark areas are dilute – small ϕ – and light areas are dense.	84
Figure 6.6:	Growth rates corresponding of eigenmode at $\theta = 40^\circ$ for several combinations of boundary condition parameters: (i) $e_w = 0.9$, $\phi_s = 1.0$ (ii) $e_w = 0.9$, $\phi_s = 0.3$ (iii) $e_w = 0.3$, $\phi_s = 0.3$ (iv) $e_w = 0.3$, $\phi_s = 1.0$ (iv) $e_w = 0.1$, $\phi_s = 0.1$	85

Figure 7.1:	Growth rates of first several leading eigenvalues for the (a) planar and (b) cylindrical beds. $\bar{\phi} = 0.60$, $L = 50$, $e_p = 0.95$	95
Figure 7.2:	Growth rate contours of first dominant transverse mode of the 3D system as a function of particle diameter and density. Dark lines denote barriers between the Geldart regimes.	96
Figure 8.1:	Schematic of the vertically-oriented fluidized bed experiment. Hatches on the tube are used for voidage measurements.	103
Figure 8.2:	Photograph of the vertically-oriented fluidized bed experiment. Tubes connected to the pressure taps converge on the signal conditioning and amplification circuit, which is not visible.	104
Figure 8.3:	Schematic of the inclined fluidized bed experiment. The tube is tilted such that the pressure tap P1 is on the side furthest to the ground.	108
Figure 8.4:	Photograph of the inclined fluidized bed experiment. The third set of pressure taps is behind the tube, facing the camera.	109
Figure 8.5:	Normalized pressure drop vs. mean gas velocity for each of the five bed materials at several angles of inclination, as shown.	111
Figure 8.6:	Pressure time signals of the vertical bed corresponding to the pressure taps numbered. Data was acquired for Material 3 with a normalized gas velocity of $U/u_{mf} = 1.3$	112
Figure 8.7:	Time signal of the difference between the pressure signals P3 and P4 of figure 8.6.	113
Figure 8.8:	Frequency spectra of several pressure time signals of the vertical bed, as marked on the vertical axis.	114
Figure 8.9:	Pressure time signals of three adjacent pressure taps of the 20° inclined bed, as marked on the vertical axis. $U/u_{mf} = 1.3$	115
Figure 8.10:	Frequency spectra of the time signals shown in figure 8.9, as marked on the vertical axis.	116
Figure 8.11:	Dimensionless dominant frequencies from experiment (circles) and numerics (dot-dashes), as a function of normalized fluidization velocity, for the five bed materials tested.	118
Figure 8.12:	(a) Dominant frequencies from experiment, determined from the frequency spectra of P15-P12 (dots), P12-P9 (circles), and P6-P3 (diamonds); (b) Frequency of the leading a mode; (c) Frequency of the leading b mode. The experimental frequency of the vertical bed (solid square) is included for comparison. The frequencies of the b mode are plotted on a different scale because of the large discrepancy in magnitude with the other two plots for each material.	121

LIST OF TABLES

Table 2.1:	Dimensionless functions used in the constitutive relations for the closure form employed.	11
Table 6.1:	Basic simulation parameters employed. Values are given in dimensional variables.	71
Table 8.1:	Properties of the bed materials used in each experiment. Diameters are mean values extrapolated from sieve size distributions provided by the manufacturer.	102

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Chapter 6, in part, will be submitted for publication in *International Journal of Multiphase Flow*, "Stability of an Inclined, Gas-Fluidized Bed," by K. Mandich and R. Cattolica. The thesis author is the primary investigator in this publication.

Chapter 7, in part, has been submitted for publication in *Physics of Fluids*, "Stability of a Vertical, Gas-Fluidized Bed," by K. Mandich and R. Cattolica. The thesis author is the primary investigator in this publication.

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ABSTRACT OF THE DISSERTATION

Stability of Gas-Fluidized Beds

by

Kevin Matthew Mandich

Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

University of California, San Diego, 2013

Professor Robert J. Cattolica, Chair

Applied mathematical techniques are employed to investigate the hydrodynamic stability of three gas-fluidized bed problems. The first that is considered is an unbounded bed subjected to a uniform fluid-phase pressure drop. The dispersion relation is solved numerically to determine the stability characteristics as a function of the bed parameters. Analytic solutions are derived for the cases of purely transverse and purely longitudinal disturbances. Long-wavelength analyses performed on each reveal the relevant stability mechanisms. Several of these are novel mechanisms stemming from the extension of kinetic gas theory to rapid granular flows used to close the equations of motion.

The linear stability analysis is then applied to two bounded problems: a cylindrically-bound vertical bed and a planar bed whose bounding walls are inclined from the vertical. The base states and linear stability analyses for both problems are solved

numerically to determine the complex frequency. In the 3D vertical bed, no particle movement is allowed in the base state, while this restriction is relaxed for the 2D inclined bed to allow for the non-uniform solid-phase pressure distribution.

It is found that the axisymmetric disturbance is dominant in the cylindrical bed. The dependence of its growth rate on particle diameter and density follows previously-published results for the likelihood of a gas-fluidized bed to exhibit bubbling at minimum fluidization as a function of these parameters. At low angles of inclination θ , the eigenmodes of the inclined bed and their characteristics exhibit similar behavior to those of the cylindrical bed. Analytic solutions derived for the limiting cases of zero shear at the walls for both beds explain these similarities. The dominant mode of the inclined bed at large θ exhibits an eigenmode whose time-evolution yields oscillating regions of very high and low voidage adjacent to the top wall, similar to the development and propagation of bubbles observed in experiment.

Pressure time signals were obtained during experiments performed on lab-scale vertical and inclined beds. A Fourier analysis yielded dominant experimental frequencies, which were compared to those from the numerical methods. The frequencies as a function of fluidization velocity and θ show qualitatively similar behavior between the stability analysis and experimental observations.

Chapter 1

Introduction

1.1 Motivation for Studying Fluidized Beds

The uses for fluidized beds in chemical and industrial engineering processes are widespread. The flow of one or more interstitial fluids through a granular medium with a large surface area-to-volume ratio provides an environment with large potential for heat and mass transfer, turbulent fluid and particle mixing, particle separation, and fluid residence times. With the numerous amount of parameters inherent in fluidized bed design, there are many options for optimizing the phenomenon to best suit a particular application. Processes include heat exchangers, temperature-sensitive reactions, hydrocarbon cracking, combustion and incineration, gasification and carbonization, calcination, particle coating, adsorption, reduction, biofluidization, and many others [1]. It is with this optimization in mind that the work on the stability of fluidized beds described in this dissertation was performed.

The fundamental fluid mechanics describing the multiphase nature of fluidized bed dynamics is well-established. The subject is challenging and comprehensive, and has received considerable attention in every decade over the past century. However, due to the inherent complexity of even the most basic two-phase systems, relatively little practical information can be extracted from the physics without a large computational effort. As such, much of the information used in fluidized bed design comes from empirical data. Richardson and Zaki [2] wrote a classic paper describing sedimentation and fluidization, presenting experimental results on fluid-solid drag and extending them to empirical pseudo-laws. Geldart [3] published experimental work identifying the fluidiza-

tion properties of particles as a function of their diameter and density, classifications which are still used frequently today. Ergun [4] correlated the frictional pressure drop through fixed beds of a fixed length and porosity, and combined the results with the theoretical pressure drop required to suspend the particles to derive an expression for the minimum fluidization velocity of a particle assembly.

The store of empirical data available is large. However, the potential range of applications and parameters is even larger, and it is of interest to develop a means of predicting the dynamics of a process in which the parameters may be treated as inputs. In particular, it is desirable to predict the onset of bubbling in many fluidized bed applications. Some processes may desire high levels of mixing via the propagation of regions of extreme voidage – otherwise known as bubbles – while minimizing the pressure drop over the bed and, therefore, the velocity of the interstitial fluid. Other designs may require the opposite, attempting to maximize the heat and mass transfer characteristics of the bed while minimizing the effects of large-scale, bubble-induced mixing. The Geldart [3] chart provides an excellent empirical roadmap for the tendency of a bed to bubble, although it is only valid for atmospheric air at STP and at one fluidization velocity – that of minimum fluidization.

The present work comprises a model used to predict the onset of bubbling for any combination of fluidized bed parameters: particle density and diameter; porosity; fluid-phase pressure drop; fluidization velocity; particle-particle restitution coefficient; particle-wall restitution coefficient; particle-wall friction; fluid type (gas or liquid); fluid density and viscosity; and, related to the latter two, thermodynamic temperature. The well-known work of Geldart [3], along with the experimental data obtained as part of the current investigation, serves as a benchmark with which to compare the model.

1.2 Hydrodynamic Stability Theory

Aside from empirical data, other forms of analysis are available from which to gather information on the creation of bubbles in fluidized beds. Since the equations governing the motion of fluidized beds are complex, analytic solutions exist only for the most basic flows. Generally, these equations yield steady-state solutions in simple geometries. Time-dependent flows, complex geometries, and attempts to derive information regarding the onset of bubbling lie beyond the scope of analytic derivations.

A recent tool used is a computational fluid dynamics (CFD) simulation, in which

the relevant equations of motion are discretized and linearized, and solved directly in space and time. Such models typically fall into either of two categories: discrete element methods (DEM), in which the fluid phase is modeled as a continuum and the particles as individual entities [5]; and Eulerian-Eulerian computations, in which both the fluid and solid phases are treated as interpenetrating continua [6]. The former method is generally much more computationally time-intensive than the latter, as the equations of motion must be computed for each particle (up to 10^{15} for large models) at each time step. The dual-continuum model not only saves computational resources but becomes more appropriate as the domain size increases, although it fails to capture the dynamics of individual particles.

Advantages of such full-scale CFD models include the ability to solve the system in complex geometries, the potential to add heat transfer and chemical kinetics models, the ability to resolve fine-scale structures (including bubbles and streamers), and the qualitative reproduction of natural fluidized bed phenomena. However, even the simplest models require large computational costs. Another disadvantage is the difficulty in validating these models through experiment as well as alternative computational models. Moreover, with regard to the current problem, such analyses yield no information concerning the underlying mechanisms of bubbling in fluidized beds.

A simpler analysis comes from the field of hydrodynamic stability theory. In this type of problem, a small perturbation is introduced into the steady-state solution of a simple flow [7]. The resulting equations of motion are linearized by neglecting perturbation terms of second-order and higher, as the amplitudes of these perturbations are assumed to be small. These equations are then solved by assuming an exponential-sinusoidal time dependence as well as a Fourier mode solution in one or more of the spatial dimensions, as the problem permits. At the simplest (e.g. problems with cyclic boundary conditions), problems reduce to solving a system of algebraic equations; the more complex require solving high-order systems of linear ODEs with non-constant coefficients in one or two spatial dimensions. The base-state solution of one of the current problems requires solving a highly-coupled system of nonlinear ODEs. Even so, the computational cost compared to solving a system of non-linear partial differential equations is minimal. Problems may be solved rapidly, and, as such, large-scale parametric analyses of problems may be performed with simplicity.

As will be demonstrated here, certain simplifications can lead to analytic expres-

sions regarding the (in)stability of fluidized systems and the mechanisms thereof. The field of linear stability has been successful in identifying and quantifying several well-known phenomena, including the Kelvin-Helmholtz shear layer instability, the Rayleigh-Taylor “finger” instability, and Rayleigh-Bénard convection cells [7, 8]. The primary disadvantage of this method lies in the breakdown of the linearized assumption after a period of time, when the exponentially-increasing amplitudes of unstable eigenmodes grow to a point where the nonlinear terms become appreciable. For example, a voidage perturbation in a fluidized bed will increase with time, eventually becoming large enough (e.g. a bubble) to a point where the small-amplitude assumption is invalidated. However, as the current problem is focused on the creation of bubbles as a function of the system parameters, the initial formation of such disturbances is of great interest.

1.3 Summary of Dissertation

An overview of the basic behavior of fluidized beds and an introduction to the hydrodynamic theory are presented in chapter 2. The governing equations for the fluid phase and the continuous particle phase are presented, along with the introduction of the pseudo-thermal granular temperature of the solid phase. In chapter 3 a review of the available literature regarding bubble formation in fluidized beds is presented, with specific attention to the applications of stability theory in this field.

Chapter 4 presents a linear stability analysis of an unbounded fluidized bed with disturbance wavenumber vector ranging from the purely horizontal to the purely vertical. Analytic criteria for the instability of transverse and longitudinal disturbances are derived, and several primary stability mechanisms are identified. This is followed by the analysis of two bounded fluidized bed problems. In chapter 5 the stability of a 3D bed bounded by a vertically-oriented, cylindrical tube is demonstrated. A stability analysis of a 2D bed bounded by walls tilted to an arbitrary angle from the vertical is given in chapter 6. Analytic solutions corresponding to the limiting cases of free-slip boundary conditions for the vertical 2D bed and the 3D bed with axisymmetric disturbances are introduced in chapter 7.

In chapter 8, experiments of a lab-scale vertical and inclined fluidized bed are described. The results of the experiment, along with comparisons between the dominant disturbance frequency obtained from experiment and the stability analysis, are also presented. A summary and conclusion of the work, along with an outlook for potential

future work, is presented in chapter 9.

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Chapter 2

Hydrodynamic Theory

2.1 Fluidized Bed Behavior

2.1.1 Fluidization Regimes

A fixed fluidized bed consists of a collection of particles bound by rigid walls over a distributor plate. The particle assembly is referred to as the *solid* or *particle phase*, and the interstitial fluid that lies in between the particles comprises the *fluid phase*. With the introduction of fluid flow – via the distributor plate at the bottom of the assembly – the particles are subject to a drag force from the fluid phase. The velocity at which the fluid travels through the bed is the *fluidization velocity*. The point at which the drag force induced by the relative velocity difference between the two phases equals the body force of the particles is known as the *minimum fluidization velocity*, denoted u_{mf} . This force balance may be expressed as:

$$\frac{\Delta P_g}{\bar{h}} = \phi(\rho_s - \rho_g)g \quad (2.1)$$

Here, ΔP_g is the pressure drop over the particle assembly of mean height $\bar{h}$, g is gravitational acceleration, ρ_s and ρ_g are the solid- and fluid-phase densities, and ϕ is the volume fraction of the solid phase. Assuming only two phases, the volume fraction of the fluid phase is defined as $1 - \phi$, a quantity also known as the *voidage*. Here, a subscript g is used for the fluid phase since the present work considers only gas. The subscripts s and g will be used throughout this work to differentiate between the phases.

As the fluidization velocity increases, the bed enters various flow regimes of increasing intensity. The possible behaviors are shown in figure 2.1. With increasing flu-

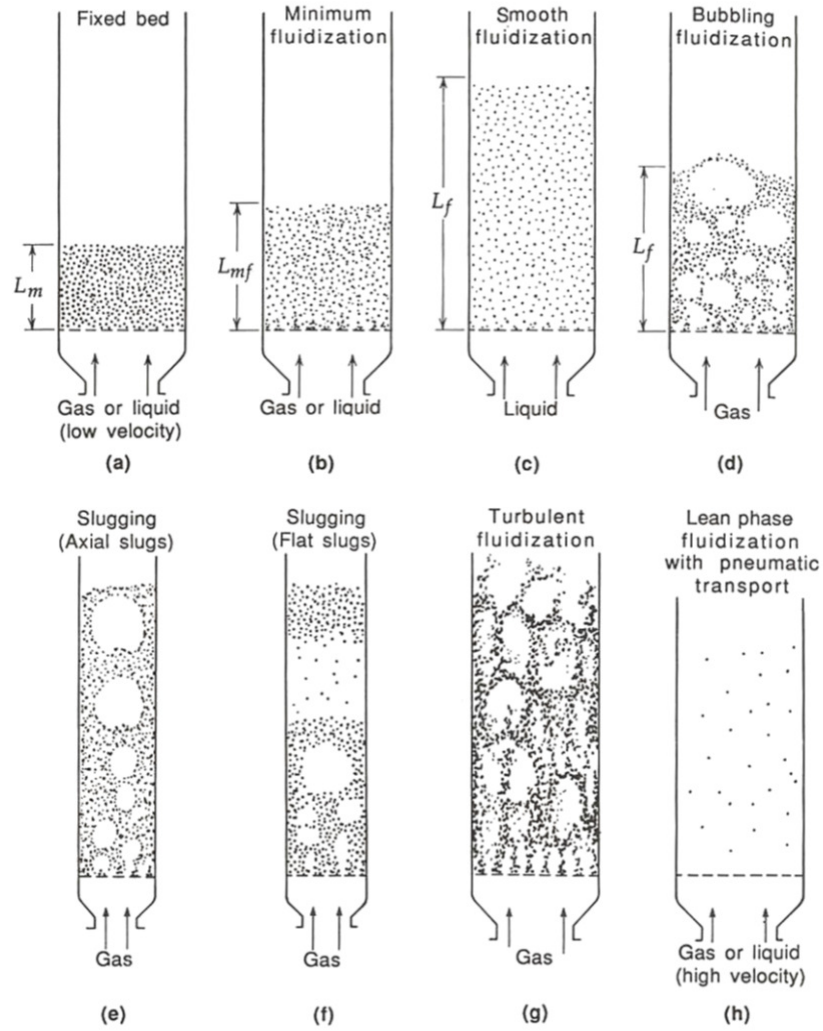


Figure 2.1: Fluidization regimes [1]. Experiments and numerical analysis of the present work are concerned with the regimes represented by (a) through (d).

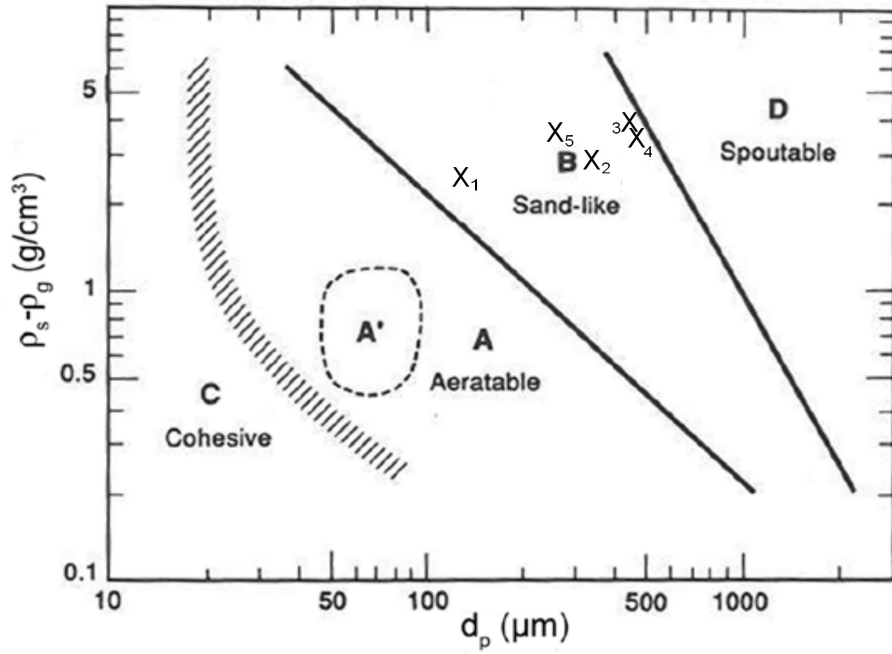


Figure 2.2: Geldart chart of particle characterization [2]. Locations of the five materials used in the experiments are noted.

idization velocity, a fixed bed (2.1a) increases in mean height $\bar{h}$ (2.1b-c). After a certain point, bubbling may occur (2.1d). For relatively narrow beds, the size of bubbles may approach the width of the bed, resulting in slugging (2.1e-f). With further increases in the pressure drop, the bed becomes fully turbulent (2.1g-h). The present problem focuses on determining the stability of the fixed and smooth fluidization regimes to perturbations which may evolve into bubbles. Since the presence of bubbles invalidates the small-amplitude assumption, the current analysis is limited to the regimes ranging from figures 2.1(a-c).

2.1.2 Particle Characteristics

The general behavior of a fluidized bed in the regimes between smooth fluidization and bubbling depends upon the particle density and diameter. Geldart [2] published an empirical road map of fluidized bed behavior as a function of these two parameters. This is shown in figure 2.2, along with the approximate locations of the five bed materials evaluated in the current analysis. Specific diameter and density measurements, along with other relevant material parameters, of the five materials are presented in chapter 8.

With regard to fluidization behavior, there are four primary particle classifications [2]. These are known as the Geldart A, B, C, and D particle groups. Geldart C particles are very small and light (e.g. powders) and generally don't fluidize because cohesive mechanisms between particles tend to clump particles together. Geldart A particles are slightly larger and denser than those of the C group. Such particles fluidize easily, although bubbling is generally deferred until fluidization velocities which are significantly above that of u_{mf} . The heaviest and most dense particles fall into the Geldart D category. Bubbling is delayed for these particles as well, although behavior of such beds is erratic, with channeling and spouting commonly seen. The Geldart B particles comprise the most commonly-used group in fluidization engineering. Beds with these particles fluidize very well, and exhibit bubbling right at the onset of minimum fluidization. The particles used in the current experiments all lie within the Geldart B domain. One of the materials lies near the Geldart A-B boundary, and two lie close to the B-D boundary.

2.2 Governing Equations

A common strategy for modeling the dynamics of fluidized beds consists of averaging the equations of motion for individual particles such that the point variables – velocity and pressure – may be replaced by local mean variables [4–9]. In this way both the fluid and particle phases are treated as interpenetrating continua. The region over which the averaging takes place must be large enough to contain many particles, yet small compared with the length scales of the macroscopic fluctuations of the system. Derivation of the averaged equations of motion have taken on several different forms. However, the end results are similar in form. The reader is referred to the relevant citations for details of the process. In the present work, the averaged equations studied by Agrawal [10] are employed:

$$\frac{\partial \phi}{\partial \hat{t}} + \hat{\nabla} \cdot (\phi \hat{\mathbf{u}}) = 0 \quad (2.2)$$

$$\frac{\partial (1 - \phi)}{\partial \hat{t}} + \hat{\nabla} \cdot (1 - \phi) \hat{\mathbf{v}} = 0 \quad (2.3)$$

$$\frac{\partial (\rho_s \phi \hat{\mathbf{u}})}{\partial \hat{t}} + \hat{\nabla} \cdot (\rho_s \phi \hat{\mathbf{u}} \hat{\mathbf{u}}) = -\hat{\nabla} \cdot \hat{\boldsymbol{\sigma}}^s - \phi \hat{\nabla} \cdot \hat{\boldsymbol{\sigma}}^g + \hat{\mathbf{F}} + \phi \rho_s g \mathbf{f} \quad (2.4)$$

$$\frac{\partial (\rho_g (1 - \phi) \hat{\mathbf{v}})}{\partial \hat{t}} + \hat{\nabla} \cdot (\rho_g (1 - \phi) \hat{\mathbf{v}} \hat{\mathbf{v}}) = -(1 - \phi) \hat{\nabla} \cdot \hat{\boldsymbol{\sigma}}^g - \hat{\mathbf{F}} + (1 - \phi) \rho_g g \mathbf{f} \quad (2.5)$$

The first two equations represent mass conservation for the solid and fluid phases, respectively, and the last two represent momentum conservation for each phase. Here, the solid- and fluid-phase velocities are given by $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$. Each of the three problems investigated in this thesis use different nomenclature for the phasic velocities, which are introduced at the beginning of their respective chapters. The stress tensors are represented by $\hat{\boldsymbol{\sigma}}$, $\hat{\mathbf{F}}$ is the inter-phase interaction term, and $\mathbf{f}$ is the unit body force vector.

The fluid-phase stress tensor takes on the common Newtonian representation:

$$\hat{\boldsymbol{\sigma}}^g = \hat{P}_g \mathbf{I} - \mu_g \left(\nabla \hat{\mathbf{v}} + \nabla \hat{\mathbf{v}}^T - \frac{2}{3} (\nabla \cdot \hat{\mathbf{v}}) \mathbf{I} \right) \quad (2.6)$$

$\hat{P}_g$ is the fluid-phase pressure and μ_g is the viscosity of atmospheric air at STP. The inter-phase interaction term is the same as that used in previous fluidized bed studies [12, 13], based upon the [14] correlation:

$$\hat{\mathbf{F}} = (1 - \phi) \frac{(\rho_s - \rho_g)g}{\hat{u}_t} f_0(\phi) (\hat{\mathbf{v}} - \hat{\mathbf{u}}) = (1 - \phi) \beta (\hat{\mathbf{v}} - \hat{\mathbf{u}}) \quad (2.7)$$

The function $f_0(\phi)$, of which N depends upon the Reynolds number, is displayed in Table 2.1. The fluidization velocity range considered in the current analysis corresponds to fluidized bed regimes ranging from smooth fluidization to the beginning of bubbling [1]. Considering the rather small corresponding range of Reynolds numbers investigated, a constant value of $N = 3$ is chosen for the Richardson-Zaki exponent [14], following the works of Anderson [12] and Liu [13]. The denominator $\hat{u}_t$ is the terminal velocity, defined empirically [1] for small particles:

$$\hat{u}_t = u_t^* \left(\frac{\rho_g^2}{\mu_g(\rho_s - \rho_g)g} \right)^{-1/3} \quad (2.8)$$

where

$$u_t^* = \left(\frac{18}{(d_p^*)^2} + \frac{0.591}{(d_p^*)^{0.5}} \right)^{-1}, \quad d_p^* = d_p \left(\frac{\rho_g(\rho_s - \rho_g)g}{\mu_g^2} \right)^{1/3} \quad (2.9)$$

and d_p is the mean particle diameter. For some of the data sets, it is useful to compute the Reynolds number such that the relative fluidization velocity – U/u_{mf} – remains the same, where U is the mean superficial gas velocity, and u_{mf} is the minimum fluidization velocity. The former quantity is defined in chapter 5. The latter quantity is the superficial gas velocity at which the interphase drag force equals the bulk body force of the bed. Since the particles are not guaranteed to be in contact with one another, u_{mf} may be solved for by combining equations 2.4 and 2.5, and solving for the fluid-phase velocity:

$$u_{mf} = \frac{\phi \hat{u}_t}{f_0(\phi)} \quad (2.10)$$

Table 2.1: Dimensionless functions used in the constitutive relations for the closure form employed.

$f_0 = \phi(1 - \phi)^{1-N}$
$f_1 = \phi(1 + 4\eta\phi g_o)$
$f_2 = \frac{2+\alpha}{3\sqrt{\pi}} \left[\frac{5\pi}{96\eta(2-\eta)} \left(1 + \frac{8}{5}\phi\eta g_o \right) (g_o^{-1} + \frac{8}{5}\eta(3\eta - 2)\phi) + \frac{8}{5}\eta\phi^2 g_o \right]$
$f_3 = \frac{8}{3\sqrt{\pi}}\eta\phi^2 g_o$
$f_4 = \frac{25\sqrt{\pi}}{16\eta(41-33\eta)} \left(1 + \frac{12}{5}\eta\phi g_o \right) (g_o^{-1} + \frac{12}{5}\eta^2(4\eta - 3)\phi) + \frac{4}{\sqrt{\pi}}\eta\phi^2 g_o$
$f_{4h} = \frac{25\sqrt{\pi}}{16\eta(41-33\eta)} \left((\eta g_o)^{-1} + \frac{12}{5}\eta \right) \frac{12}{5}\eta(2\eta - 1)(\eta - 1) \frac{\partial}{\partial \phi}(\phi^2 g_o)$
$f_5 = \frac{81\phi}{g_o\sqrt{\pi}}$
$f_6 = \frac{48}{\sqrt{\pi}}\phi^2\eta(1 - \eta)g_o$
$f_7 = \frac{\sqrt{3}\pi\phi\phi_s g_o}{6\phi_p}$
$\eta = 0.5(1 + e_p)$
$g_o = (1 - (\phi/\phi_{max})^{1/3})^{-1}$

An alternate form of the minimum fluidization velocity is presented by Kunii [1]. Ergun [3] proposed the following expression for the frictional pressure drop through a fixed bed of height $\bar{h}$:

$$\frac{\Delta P_g}{\bar{h}} = 150 \frac{\phi^2}{(1 - \phi)^3} \frac{\mu_g U}{(\phi_s d_p)^2} + 1.75 \frac{\phi}{(1 - \phi)^3} \frac{\rho_g U^2}{\phi_s d_p} \quad (2.11)$$

Combining this expression with the force balance at minimum fluidization 2.1 results in a quadratic relation for the minimum fluidization velocity:

$$\frac{1.75}{(1 - \phi)^3 \phi_s} Re_{p,mf}^2 + \frac{150\phi}{(1 - \phi)^3 \phi_s^2} Re_{p,mf} - Ar = 0 \quad (2.12)$$

Here, $Re_{p,mf} = \frac{d_p u_{mf} \rho_g}{\mu_g}$ is the particle Reynolds number at minimum fluidization, and $Ar = \frac{d_p^3 \rho_g (\rho_s - \rho_g) g}{\mu_g^2}$ is the Archimedes number, representing the ratio of gravitational forces to viscous forces. For small particle Reynolds numbers – $Re_{p,mf} < 20$ – the first term in 2.12 may be ignored [1]. All of the bed materials in the present study fall under this category, and so the simplified expression for the minimum fluidization velocity is expressed as the reduced version of equation 2.12:

$$u_{mf} = \frac{d_p^2 (\rho_s - \rho_g) g}{150 \mu_g} \frac{(1 - \phi)^3 \phi_s^2}{\phi} \quad (2.13)$$

The solid-phase stress term is closed through the well-established use of the kinetic theory of granular flows, which requires an extra mechanical energy conservation equation. This process is summarized in the following section.

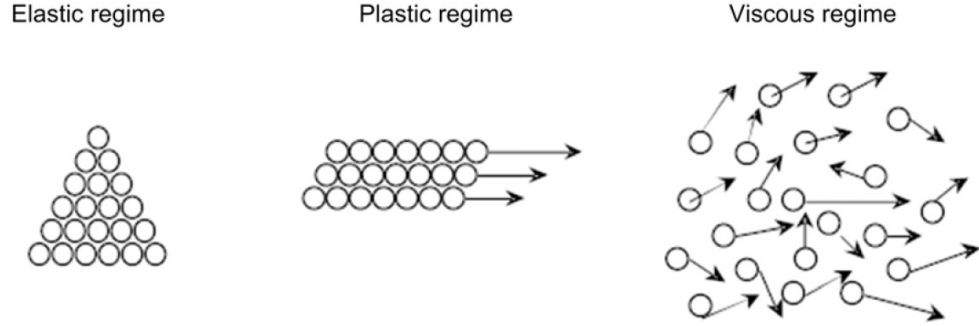


Figure 2.3: Granular flow regimes [21]. The model developed in the present analysis deals only with the viscous regime.

2.3 The Extension of Kinetic Theory to Granular Flows

Early attempts to close the equations of motion (2.2-2.5) included assuming the effective solid-phase pressure and viscosity to be functions of the solids volume fraction ϕ [15–18]. Although such ad-hoc approximations are generally able to capture the qualitative behavior of fluid-particle systems, more recent expressions for the solid-phase transport variables have been developed, based upon first principles. Specifically, the extension of the kinetic theory of gases to the field of granular flows provides such a formulation – see [19] and [20] for reviews. This formulation is appropriate for the “viscous” regime of granular flow, characterized by the chaotic particle motion of the solid phase. In the case of fluidized beds, this regime begins at the onset of minimum fluidization, when the particles are no longer guaranteed to be in contact with one another – that is, beyond the elastic and plastic regimes of granular flows in which friction and enduring contact between particles yield the dominant forces. In the viscous regime, particle collisions dominate. Figure 2.3 presents a diagram to visualize the different regimes of granular flow.

The instantaneous velocity vector of a particle c at a location $\mathbf{r}$ and time t is defined as $\mathbf{u}_p(\mathbf{r}, t)$. Taking the ensemble average yields the particle drift velocity $\mathbf{u}_s = \langle \mathbf{u}_p(\mathbf{r}, t) \rangle$. This quantity may be thought of as the bulk velocity of the material. $\mathbf{u}_s$ is manifested, for example, as the visible bulk upward and downward flow of particles corresponding to slugging in a fluidized bed. The fluctuation velocity – which, in the case of gas-fluidized beds, is generally not distinguishable by the human eye – is defined

as:

$$\mathbf{w}(\mathbf{r}, t) = \mathbf{u}_p(\mathbf{r}, t) - \mathbf{u}_s(\mathbf{r}, t) \quad (2.14)$$

The ensemble average of the fluctuation velocity is zero by definition. Because of this, it is necessary to describe the intensity of particle velocity fluctuations by the average of its square, or the variance. In a manner analogous to the kinetic energy of molecular motion in a gas, the granular temperature is defined as the ensemble average of the squared fluctuation velocity:

$$\frac{3}{2}T = \frac{1}{2} \left\langle (\mathbf{u}_p - \mathbf{u}_s)^2 \right\rangle \quad (2.15)$$

It is important to note the differences between granular temperature and the thermodynamic temperature of a gas. The granular temperature is not a thermodynamic property, as in a gas, but rather a function of the viscous and inertial forces imparted by the fluid phase. The kinetic theory of gases enforces equipartition of energy and the Maxwell-Boltzmann velocity distribution. Kinetic theory assumes elastic collisions between molecules; the same is not true for granular flows. Energy is dissipated by particle-particle and particle-wall collisions, as well as viscous dissipation, and compensation is required for the energy that is lost in these ways to maintain a granular temperature. This follows from the notion that the time-averaged granular temperature of this system remains constant. Also worth noting is the difference between mean-free paths of molecules in a gas and particles in a rapid granular flow. It is generally known that the mean-free path of a molecule in a gas under moderate conditions is several orders of magnitude larger than its diameter. However, dense granular flows have mean-free paths that are on the order of the particle diameters. Another difference pointed out by Campbell [28] is that gases need not have an external driving force to have a temperature. However, granular temperature depends upon this driving force and cannot exist without it. This is a result of particle inelasticity and viscous dissipation of the interstitial fluid phase.

The balance for the pseudo-thermal kinetic energy (PTE), as well as the constitutive relations for particle-phase pressure and viscosity, is derived by solving for relations of low-order moments of the particle velocity distribution function [22–25]. Closure is acquired by assuming a modified Maxwellian distribution function. The published derivations are lengthy and not reproduced here. The balance of the pseudo-thermal

kinetic energy of particle velocity fluctuations is represented by [10]

$$\frac{\partial}{\partial t} \left(\frac{3}{2} \rho_s \phi \hat{T} \right) + \hat{\nabla} \cdot \left(\frac{3}{2} \rho_s \phi \hat{T} \hat{\mathbf{u}} \right) = -\hat{\boldsymbol{\sigma}}^s : \hat{\nabla} \hat{\mathbf{u}} - \hat{\nabla} \cdot \hat{\mathbf{q}} + \hat{\Gamma}_s - \hat{J}_c - \hat{J}_v \quad (2.16)$$

The first term on the right-hand side represents the production of PTE due to shear, and the second term represents energy flux $\hat{\mathbf{q}}$. The third term is generation of PTE through interaction with the fluid phase, and the last two terms respectively represent energy dissipation due to inelastic collisions and viscous dissipation.

The present set of constitutive relations comes from the work of Lun [24] and Gidaspow [23]. In adopting these relations, we assume a particle phase consisting of hard, smooth particles governed by nearly elastic collisions. Although some granular flow studies account for frictional contributions to the solid-phase stress tensor [26], only collisional effects are considered in the present work on account of the suspended nature of particles in fluidized beds in the smooth fluidization regime. In addition, particle spin is neglected, and particles of large Stokes numbers are considered. As such, the velocity distribution is assumed to be governed by particle collisions rather than viscous interaction with the fluid phase. An isotropic velocity distribution is assumed, despite the expected anisotropy resulting from the unidirectional flow of the fluid. In doing so, it is asserted that collisions efficiently transfer kinetic energy from the vertical to the horizontal, and that the granular-temperature-generated instability mechanisms result from compaction of the particle assembly and the small amount of particle inelasticity [27]. The constitutive relations for the solid-phase stress, as well as the energy flux, generation, and dissipation of PTE are represented as

$$\hat{\boldsymbol{\sigma}}^s = \rho_s \left(f_1 \hat{T} - d_p f_3 \hat{T}^{1/2} (\hat{\nabla} \cdot \hat{\mathbf{u}}) \right) \mathbf{I} - 2\rho_s d_p f_2 \hat{T}^{1/2} \mathbf{S} \quad (2.17)$$

$$\hat{\mathbf{q}} = -\rho_s d_p \left(f_4 \hat{T}^{1/2} \hat{\nabla} \hat{T} + f_{4h} \hat{T}^{3/2} \hat{\nabla} \phi \right) \quad (2.18)$$

$$\hat{\Gamma}_s = \frac{\mu_g^2}{d_p^3 \rho_s} |\hat{\mathbf{v}} - \hat{\mathbf{u}}|^2 f_5 \hat{T}^{-1/2} \quad (2.19)$$

$$\hat{J}_c = \frac{\rho_s}{d_p} f_6 \hat{T}^{3/2} \quad (2.20)$$

$$\hat{J}_v = 3 \frac{(\rho_s - \rho_g)g}{\hat{u}_t} f_0 \hat{T} \quad (2.21)$$

where the stress tensor is defined as

$$\mathbf{S} = \frac{1}{2} (\hat{\nabla} \hat{\mathbf{u}} + \hat{\nabla} \hat{\mathbf{u}}^T) - \frac{1}{3} (\hat{\nabla} \cdot \hat{\mathbf{u}}) \mathbf{I} \quad (2.22)$$

The dimensionless functions f_1 through f_6 are listed in Table 2.1. The radial distribution function $g_o(\phi)$ behaves such that the solid-phase pressure approaches ∞ as $\phi \rightarrow \phi_p$, constraining the solids volume fraction to remain below the value at close packing, ϕ_p . The term e_p is the particle-particle restitution coefficient. The diffusive flux, as well as the particle-phase pressure (the first term in equation 2.17), is represented as the sum of kinetic and collisional contributions. The governing equations (2.2-2.5, 2.16) are non-dimensionalized using velocity and length scales which are problem-dependent. As such, the non-dimensionalization schemes are deferred to the respective chapters of each problem.

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Chapter 3

Literature Review

The pioneering paper of Jackson [1] was one of the first to apply hydrodynamic stability theory to a fluidized bed, an analysis rendered possible by the continuum assumption for the solid phase. He demonstrated that gas- and liquid-fluidized beds were unstable to small disturbances which traveled vertically through the bed. The mechanism responsible was linked to a volume-dependent drag force function. Voidage disturbances were found to induce a drag force fluctuation on the particles. Due to the larger particle-to-fluid inertia, particles would overshoot in response to this forcing. The higher degree of instability for gas beds as compared to liquid beds determined by the authors agrees with this mechanism. However, due to lack of a term representing resistance to shear in the solid phase, longitudinal disturbances were found to be unstable for all wavenumbers.

Pigford & Baron [2] and Anderson & Jackson [3] included this shear resistance by multiplying the shear stress of the solid phase by various numerical values of an effective particle-phase viscosity. Such a viscosity was found to decrease the growth rate of disturbances of all wavenumbers. The analysis also predicted a “least stable” disturbance, one with the highest growth rate at a particular wavenumber and predicted to be the dominant disturbance as per linear stability theory.

The effect of the solid-phase pressure was explored by Anderson & Jackson [3] and Batchelor [4]. Both found that the term representing a change in solid-phase pressure with volume fraction $\partial P_s / \partial \phi$ – also known as the bulk modulus of elasticity of the assembly – was found to reduce the growth constant at all wavenumbers and so exhibit a stabilizing effect by resisting compaction of the solid phase. Batchelor [4] took the analysis further and determined two contributions to this bulk elasticity term: the

transfer of particle momentum by fluctuations in ϕ and, thus, velocity fluctuations; and the repulsive force of particle collisions. The latter effect was found to be dominant for gas-fluidized beds. Medlin and Jackson [5] took into account the thickness of the distributor plate while solving the linearized equations of motion, and found that the stability of the bed decreases when the distributor pressure drop is spread over a finite thickness rather than one with zero thickness.

Didwania & Homsy [6] studied a novel Rayleigh-Taylor-type instability in which a particle assembly of large voidage was placed below that of an assembly with large ϕ . The large degree of instability predicted by the stability analysis did not match that seen in experiment, although they were able to use the stability results to postulate the existence of a yield stress in fluidized particle assemblies. The same authors [7] derived resonance conditions for a secondary transverse instability which gained energy from the primary longitudinal instability in a liquid-fluidized bed. By assuming a quadratic dependence of the primary nonlinear terms, postulating a small “sideband” transverse wavenumber, and substituting the experimentally-observed wavenumber into that of the linear longitudinal disturbance, they derived an estimate for this transverse wavenumber which matched that seen in experiment with reasonable accuracy.

Hernandez & Jimenez [8] introduced a generalized two-dimensional secondary instability into the primary longitudinal mode. The analysis predicted the existence of longitudinal streamers, which evolve temporally from bubble-like voidage perturbations. Goz [9] investigated both oblique travelling waves and standing travelling waves in fluidized beds. Goz took a different approach than that of Didwania & Homsy [7] by performing a stability analysis of the longitudinal travelling plane wave to transverse disturbances of finite wavenumbers. It was proposed that such bifurcations, when taken to higher orders, may lead to more complex wave patterns and ultimately to turbulence and bubbling. In a series of later papers [10–12] Goz confirmed the existence of such instabilities, and presented a bifurcation analysis of these vertical and oblique travelling waves.

Ganser and Drew [13] considered the nonlinear stability of a uniform, unbounded bed with rudimentary forms of the interphase drag and solid-phase viscosity terms. Despite the simple model, they found that gas-fluidized beds produced unstable disturbances whose amplitudes were much larger than those of the liquid-fluidized beds, which is consistent with the linear theory [3]. Anderson et al. [14] carried out a fully non-linear

integration of the equations of motion in two dimensions, assuming the base-state spatial periodicity in the vertical direction determined from linear stability theory. The temporal evolution of the disturbances revealed voidage structures which appear remarkably similar to bubbles seen in experiment, although the authors limited their analysis to just one gas- and liquid-fluidized bed parameter combination each.

The literature regarding the inclusion of granular temperature to the hydrodynamic stability of fluidized bed systems is sparse. It is of interest to include this complication since granular temperature theory provides a means of quantifying the contributions to particle-phase pressure due to interparticle collisions, as well as accounting for the slight inelasticity of particles. Koch & Sangani [15] investigated the stability of non-isothermal disturbances to the isothermal, homogeneous base state. This is the only non-isothermal analysis to date which attempts to identify the unstable mechanisms of such a system, although the authors limited their attention to longitudinal disturbances only. The authors found that the growth rate at large wavelengths has three contributions: the classically-recognized stabilizing and destabilizing mechanisms of the (1) bulk elasticity and (2) drag force fluctuation terms, respectively, and (3) a novel term representing the change in particle-phase pressure with fluctuations of the fluid velocity. Such a mechanism “tends to drive particles from dilute regions with fast-moving particles into dense regions with slow-moving particles [15].” It is also noted that the authors consider both isotropic and anisotropic granular temperature fields.

The work of Liu et al. [16] applied the problem of non-isothermal stability to a 2D, bounded, gas-fluidized bed. The governing equations and constitutive relations were the same as those used by Koch & Sangani [15]. The base state was acquired by directly solving the equations of motion. In doing so, the authors found that multiple solutions are possible. They focused their stability analysis on the solution representing the pneumatic flow regime of fluidization, characterized by very large fluid- and particle-phase velocities and very low solid-phase volume concentrations in the base state.

Didwania [20] derived stability limits for a class of purely transverse non-isothermal disturbances of an unbounded, homogeneous bed. However, it is shown in chapter 4 that such a system is stable to all horizontal disturbances due to the lack of a vertical momentum balance. Sergeev et al. [21] considered vertical disturbances in the same manner as Koch & Sangani [15] and determined stability conditions as a function of the level of dissipation due to interparticle collisions. The authors attempted to correlate the

wavelength of the dominant disturbance to the initial size of bubbles in a gas-fluidized bed.

It is apparent that the literature regarding application of hydrodynamic theory to these systems is extensive, and yet the reason for the onset of bubble formation in fluidized beds remains poorly defined. In reviewing the preceding works, a few trends are noticed. First, the majority of available literature relies upon ad-hoc approximations for the solid-phase transport variables to close the solid-phase momentum equation. Second, aside from a few pioneering works [1, 3, 15], there has been little attempt to understand the physical mechanisms by which voidage disturbances increase in amplitude, and these explore only one possible type of perturbation – the longitudinal disturbance. Many of the cited works take the stability analysis further and further through the linear and nonlinear regimes, oftentimes attaining solutions with temporal evolutions which appear to resemble experimentally-observed bubbles, without regard to the underlying causes of instability. In addition, most of the works consider unbounded assemblies which, while appropriate for ideal systems or beds in which the bounding walls are very far apart, provide little information on the stability of practical fluidized beds. There is also a lack of consideration of the transverse mode as the fundamental linear instability mode. And, finally, no work has yet attempted to provide an extensive parametric analysis of fluidized bed stability.

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Chapter 4

Stability of an Unbounded Fluidized Bed

4.1 Problem Description

The first problem under consideration is that of a unbounded, semi-infinite bed. Although this is an idealized system, the analysis yields theoretical and practical information regarding the instability mechanisms of the longitudinal and transverse disturbance classes. Furthermore, such an analysis may be appropriate for a bounded system whose distance between the walls is significantly larger than the wavelength of the dominant transverse disturbance.

The system under review consists of an unbounded assembly of spherical particles with constant particle diameter d_p , constant volume fraction ϕ , and constant pressure drop $\partial\bar{P}_g/\partial z$. The coordinate system is defined such that the positive z direction is aligned opposite to the direction of gravity. The base-state fluid velocity is unidirectional and constant in space and time, flowing in the positive z direction. In the absence of tilting or shear in the base state, along with the assumption of an isotropic granular temperature, it is possible to simplify the problem by considering only two spatial dimensions. This is accomplished by realizing that sinusoidal disturbances in the horizontal directions may be combined into one effective horizontal disturbance.

To visualize this, consider a general three-dimensional disturbance with wavenumber vector $\mathbf{k} = (k_1, k_2, m)^T$. With the homogeneous and isotropic assumptions, the disturbances $\mathbf{k} = (k_1, 0, m)^T$ and $\mathbf{k} = (0, k_2, m)^T$ are equivalent if $k_1 = k_2$. As such, an

effective horizontal wavenumber $k = (k_1^2 + k_2^2)^{1/2}$ may be considered and the momentum balances for each phase in one of the horizontal directions may be omitted. The velocities for the fluid and solid phases of the current problem are defined as $\mathbf{u}_g = (u_g, w_g)^T$ and $\mathbf{u}_s = (u_s, w_s)^T$, respectively.

4.2 Governing Equations

The equations of motion (2.2-2.5, 2.16) are non-dimensionalized using the particle diameter d_p and velocity U , where U is the constant base-state fluid velocity defined as the solution of 2.4-2.5. With this, the following dimensionless parameters are introduced:

$$\begin{aligned} \mathbf{x} &= \frac{\hat{\mathbf{x}}}{d_p}, \quad t = \frac{U}{d_p} \hat{t}, \quad (\mathbf{u}_g, \mathbf{u}_s) = \frac{1}{U}(\hat{\mathbf{u}}_g, \hat{\mathbf{u}}_s), \quad P_g = \frac{\hat{P}_g}{\rho_s U^2} \\ T &= \frac{\hat{T}}{U^2}, \quad Re = \frac{\rho_g U d_p}{\mu_g}, \quad R = \frac{\rho_g}{\rho_s}, \quad Fr = \frac{U^2}{g d_p} \end{aligned} \quad (4.1)$$

The resulting dimensionless equations of motion are:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}_s) = 0 \quad (4.2)$$

$$\frac{\partial(1-\phi)}{\partial t} + \nabla \cdot (1-\phi) \mathbf{u}_g = 0 \quad (4.3)$$

$$\frac{\partial(\phi \mathbf{u}_s)}{\partial t} + \nabla \cdot (\phi \mathbf{u}_s \mathbf{u}_s) = -\nabla \cdot \boldsymbol{\sigma}^s - \phi \nabla \cdot \boldsymbol{\sigma}^g + (1-\phi)\beta(\mathbf{u}_g - \mathbf{u}_s) + \frac{\phi}{Fr} \mathbf{f} \quad (4.4)$$

$$R \left(\frac{\partial((1-\phi) \mathbf{u}_g)}{\partial t} + \nabla \cdot ((1-\phi) \mathbf{u}_g \mathbf{u}_g) \right) = -\nabla \cdot \boldsymbol{\sigma}^s - (1-\phi)\beta(\mathbf{u}_g - \mathbf{u}_s) + \phi \frac{R}{Fr} \mathbf{f} \quad (4.5)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{3}{2} \phi T \right) + \nabla \cdot \left(\frac{3}{2} \phi T \mathbf{u}_s \right) &= -\boldsymbol{\sigma}^s : \nabla \mathbf{u}_s + \nabla \cdot (\lambda \nabla T + \lambda_h \nabla \phi) \\ &+ \frac{R^2}{Re^2} |\mathbf{u}_g - \mathbf{u}_s|^2 \Gamma_s - J_c - \frac{1-R}{Fr} J_v \end{aligned} \quad (4.6)$$

In these expressions, the phasic stress tensors are represented as:

$$\boldsymbol{\sigma}^g = -P_g \mathbf{I} + \frac{R}{Re} \left(\nabla \mathbf{u}_g + \nabla \mathbf{u}_g^T - \frac{2}{3} (\nabla \cdot \mathbf{u}_g) \mathbf{I} \right) \quad (4.7)$$

$$\boldsymbol{\sigma}^s = -P_s \mathbf{I} + 2(\mu_s + 3/5 \zeta_s) \mathbf{D}^s + \zeta_s (\nabla \cdot \mathbf{u}_s) \quad (4.8)$$

and the solid-phase deviatoric stress tensor is

$$\mathbf{D}^s = \frac{1}{2} (\nabla \mathbf{u}_s + \nabla \mathbf{u}_s^T) + \frac{1}{3} (\nabla \cdot \mathbf{u}_s) \quad (4.9)$$

The dimensionless variables are written as:

$$\begin{aligned}
\beta(\phi) &= u_t^{-1} f_0(\phi) & P_s(\phi, T) &= f_1(\phi) T \\
\mu_s(\phi, T) &= f_2(\phi) T^{1/2} & \zeta_s(\phi, T) &= \left(f_3(\phi) - \frac{2}{3} f_2(\phi) \right) T^{1/2} \\
\lambda(\phi, T) &= f_4(\phi) T^{1/2} & \lambda_h(\phi, T) &= f_{4h}(\phi) T^{3/2} \\
\Gamma_s(\phi, T) &= f_5(\phi) T^{-1/2} & J_c(\phi, T) &= f_6(\phi) T^{3/2} \\
J_v(\phi, T) &= 3u_t^{-1} f_0(\phi) T
\end{aligned} \tag{4.10}$$

For convenience, the terms representing production and dissipation of PTE due to viscous and collisional interactions have been combined into a single term:

$$\Gamma = \frac{R^2}{Re^2} |\mathbf{u}_g - \mathbf{u}_s|^2 \Gamma_s - J_c - \frac{(1-R)}{Fr} J_v \tag{4.11}$$

4.3 Base-State Solution: Homogeneous Fluidization

The well-known state of homogeneous fluidization [1] corresponds to the following set of dimensionless values.

$$\begin{aligned}
\mathbf{u}_g &= (0, 1)^T, \quad \mathbf{u}_s = (0, 0)^T, \quad \phi = \bar{\phi}, \quad P_g = \bar{P}_g(z) \\
T &= \bar{T}, \quad P_s = \bar{P}_s, \quad \mu_s = \bar{\mu}_s, \quad \lambda_s = \bar{\lambda}_s
\end{aligned} \tag{4.12}$$

With this, the conservations equations (6.2-6.6) reduce to:

$$\begin{aligned}
\frac{\partial \bar{P}_g}{\partial z} + \bar{\beta} + \frac{R}{Fr} &= 0 \\
\bar{\beta} + (R-1) \frac{\bar{\phi}}{Fr} &= 0 \\
\bar{\Gamma} &= 0
\end{aligned} \tag{4.13}$$

Here, and henceforth, an overbar on a variable refers to its base-state value. The granular temperature equation becomes, expanded:

$$\frac{R^2}{Re^2} |\mathbf{u}_g - \mathbf{u}_s|^2 f_5 \bar{T}^{-1/2} - f_6 \bar{T}^{3/2} - \frac{(1-R)}{Fr} \frac{3f_0}{u_t} \bar{T} = 0 \tag{4.14}$$

The value of the base-state granular temperature is determined numerically using a standard root-finding technique.

4.4 Linearization and Assumed Form of Perturbations

The equations of motion (6.2-6.6) are linearized by perturbing the variables about the base state:

$$\psi(\mathbf{x}, t) = \bar{\psi} + \psi'(\mathbf{x}, t) \quad (4.15)$$

where ψ is the seven-term solution vector defined as

$$\psi = (u_g \quad w_g \quad u_s \quad w_s \quad P \quad \phi \quad T)^T \quad (4.16)$$

We assume normal-mode perturbations of the form

$$\psi'(\mathbf{x}, t) = \hat{\psi} \exp(i(\mathbf{k} \cdot \mathbf{x} + \sigma t)) \quad (4.17)$$

where the wavenumber vector is $\mathbf{k} = (k, m)^T$. The wavenumbers k and m are real numbers, and the complex frequency is $\sigma = \sigma_r + i\sigma_i$. The real part of σ is the growth constant, and the imaginary part represents the frequency of the perturbations. A positive value of σ_r implies that the disturbance amplitude will increase with time, whereas a negative value leads to decay over time. A value of $\sigma_r = 0$ represents the state of marginal stability, with neither growth nor decay in time. The eigenvalue with the largest value of σ_r at a particular combination of k and m is referred to as the leading eigenvalue. The eigenvalue with the greatest real part is also referred to as the least-stable eigenvalue. The disturbance modes which have the highest value of σ_r over the separate k and m wavenumber spectra are the dominant transverse and dominant longitudinal modes, respectively. The disturbance having the largest σ_r over the entire $k - m$ space is the dominant mode.

The wavenumbers and complex frequency are non-dimensionalized using the inverse of the formulas for length and time, respectively, given in equation (4.1). The justification for this solution form lies in the assumption of a constant, mean fluid flow that does not depend upon position or time. This is an idealized flow arrangement but is a suitable approximation when considering a bed whose horizontal dimensions are much larger than the largest relevant wavelength. It also accurately represents the state of fluidized beds in the smooth fluidization regime. As noted by Sundaresan [2], this is seen “in the window of stable bed expansion, where the particles are essentially immobile until bubbling commences.” In other words, this analysis may be used to approximate the stability of a bed whose bounding walls have a separation length Δ much larger than the wavelength of the dominant transverse disturbance. This analysis is also expected

to be more suitable for beds whose wall separation length is much larger than the mean bubble size (a bed in which slugging may not occur), especially since the point of the current work is to identify disturbances which are assumed to lead into the formation of bubbles. This assumption is generally not suitable for lab-scale beds, whose dimensionless separation $L = \Delta/d_p$ is of $O(10^2)$ or less and whose bounding walls may have an appreciable effect on the base-state solutions and, thus, the flow stability. It is also noted that the presence of bubbles will significantly alter the base-state solution, although we ignore this defect because the current problem concerns only the instabilities leading to the formation of bubbles.

With the linearization form given by (4.15-4.17), the equations of motion (4.2-4.6) become:

$$\sigma\phi' + ik\bar{\phi}u'_s + im\bar{\phi}w'_s = 0 \quad (4.18)$$

$$(-\sigma - im)\phi' + ik(1 - \bar{\phi})u'_g + im(1 - \bar{\phi})w'_g = 0 \quad (4.19)$$

$$\begin{aligned} R(1 - \bar{\phi})(\sigma + im)u'_g = & -ik(1 - \bar{\phi})P'_g - \frac{R(1 - \bar{\phi})}{Re} \left(\frac{4}{3}k^2 + m^2 \right) u'_g \\ & - \frac{R(1 - \bar{\phi})}{Re} \left(\frac{1}{3}km \right) w'_g - (1 - \bar{\phi})\bar{\beta}(u'_g - u'_s) \end{aligned} \quad (4.20)$$

$$\begin{aligned} R(1 - \bar{\phi})(\sigma + im)w'_g = & -im(1 - \bar{\phi})P'_g - \frac{R(1 - \bar{\phi})}{Re} \left(k^2 + \frac{4}{3}m^2 \right) w'_g \\ & - \frac{R(1 - \bar{\phi})}{Re} \left(\frac{1}{3}km \right) u'_g - (1 - \bar{\phi})\bar{\beta}(w'_g - w'_s) \\ & - (1 - \bar{\phi})\bar{\beta}_\phi\phi' - \frac{R}{Fr}\phi' \end{aligned} \quad (4.21)$$

$$\begin{aligned} \bar{\phi}\sigma u'_s = & -ik(\bar{P}_{s\phi}\phi' + \bar{P}_{sT}T') - (\bar{\mu}_1(2k^2 + m^2) + \bar{\mu}_2k^2)u'_s - km(\bar{\mu}_1 + \bar{\mu}_2)w'_s \\ & - \frac{R\bar{\phi}}{Re} \left(\frac{4}{3}k^2 + m^2 \right) u'_g - \frac{R\bar{\phi}}{Re} \left(\frac{1}{3}km \right) w'_g + \bar{\beta}(u'_g - u'_s) \end{aligned} \quad (4.22)$$

$$\begin{aligned} \bar{\phi}\sigma w'_s = & -im(\bar{P}_{s\phi}\phi' + \bar{P}_{sT}T') - (\bar{\mu}_1(k^2 + 2m^2) + \bar{\mu}_2k^2)w'_s \\ & - km(\bar{\mu}_1 + \bar{\mu}_2)u'_s - \frac{R\bar{\phi}}{Re} \left(k^2 + \frac{4}{3}m^2 \right) w'_g - \frac{R\bar{\phi}}{Re} \left(\frac{1}{3}km \right) u'_g \\ & + \bar{\beta}(w'_g - w'_s) + \bar{\beta}_\phi\phi' - \frac{1}{Fr}\phi' \end{aligned} \quad (4.23)$$

$$\begin{aligned} \frac{3}{2}\sigma(\bar{\phi}T' + \bar{T}\phi') = & -i\bar{P}_s(ku'_s + mw'_s) - \bar{\lambda}(k^2 + m^2)T' + \bar{\lambda}_h(k^2 + m^2)\phi' \\ & + \bar{\Gamma}_{w_g}w'_g + \bar{\Gamma}_{w_s}w'_s + \bar{\Gamma}_{\phi}\phi' + \bar{\Gamma}_T T' \end{aligned} \quad (4.24)$$

Here, the solid-phase viscosities have been combined using the formulas $\bar{\mu}_1 = \bar{\mu}_s + \frac{3}{5}\bar{\zeta}_s$ and $\bar{\mu}_2 = \bar{\zeta}_s - \frac{2}{3}\bar{\mu}_s$. Terms that are dependent upon the perturbed variables ϕ' , T , etc. have been expanded through Taylor series in the form

$$P_s = \bar{P}_s + \phi' \left. \frac{\partial P_s}{\partial \phi} \right|_{\psi=\bar{\psi}} + T' \left. \frac{\partial P_s}{\partial T} \right|_{\psi=\bar{\psi}} + O(\psi'^2) \simeq \bar{P}_s + \bar{P}_{s\phi}\phi' + \bar{P}_{sT}T' \quad (4.25)$$

Here, and henceforth, a subscripted flow variable denotes partial differentiation with respect to that variable. Terms proportional to R may be ignored in the case of gas-fluidized beds. The interstitial gas is assumed to be atmospheric air at STP. The density ratios R^{-1} explored in the current problem range from 2.5×10^{-4} to 1.0×10^{-3} . Neglecting these values amounts to ignoring the convective, body-force, and viscous terms of the fluid phase, which is a common approximation with gas-fluidized beds [6]. As argued by Koch and Sangani [7], the fluid-phase viscous forces may be neglected in concentrated beds, assuming that the length scale over which the fluid velocity changes is larger than the particle diameter. These terms are kept in equations (4.18-4.24) for generality and are included in the solution. Figure 4.1(b) presents a pair of stability curves demonstrating the minor discrepancy resulting from keeping or discarding these terms. The dispersion relation obtained when keeping these terms is a hexic polynomial in σ . The preceding set of equations may be arranged in matrix form:

$$A\psi = \sigma B\psi \quad (4.26)$$

The eigenvalue problem $D = |A - \sigma B| = 0$ is solved to obtain the growth characteristics of the disturbance modes.

4.4.1 Numerical Method

The growth characteristics are determined by locating the least-stable eigenvalue σ and following its value as these quantities are changed. Results are presented as a function of dimensionless parameters R , Re , Fr , $\bar{\phi}$, and e_p , as well as the wavenumbers k and m . Standard atmospheric air with constant density and viscosity is assumed, and so varying the density ratio amounts to changing the particle density. The base-state fluid velocity was determined by solving the equations 4.13-4.14 given a specified

fluid-phase pressure drop and solids concentration. The effect of varying the fluid-phase pressure drop is explored by one data set (figure 4.3) in which the stability characteristics are plotted as a function of Re and $\bar{\phi}$.

A couple of procedures are combined to determine the eigenvalue σ . Following the technique of Mack [3], we compute the characteristic determinant D for a large number of values in the complex σ plane. From this we are able to trace the contour lines corresponding to $real(D) = 0$ and $imag(D) = 0$. The intersections of the contour lines corresponding to the zero real and imaginary parts of the determinant give the approximate location of eigenvalues. Although the eigenvalues of the current problem are generally well-separated, this method was complimented by the technique of Lessen et al. [4] to ensure that no eigenvalues were missed. Here, the determinant was computed for each point in a large contour of the complex σ plane at values close enough to approximate a closed contour. A result of the Cauchy integral theorem in complex analysis is that the number of times the determinant circles the origin in the complex D plane is equal to the number of zeros within the contour of the σ plane [4].

Once the approximate location of an eigenvalue is found, one may use an iterative procedure to converge on the eigenvalue, since $D(\sigma)$ is an analytic function of σ in the neighborhood of the actual value σ [5]. The following recursion relation is employed:

$$\sigma_{k+1} = \sigma_k - \left(\frac{\sigma_k - \sigma_{k-1}}{D_k - D_{k-1}} \right) D_k \quad (4.27)$$

To begin, we compute D_1 using σ_1 , which is the eigenvalue approximation found using the method of Mack [3]. A second guess $\sigma_2 = \sigma_1 + \epsilon$ is computed, along with the corresponding determinant D_2 , where ϵ is an arbitrary complex number with norm $\|\epsilon\| \ll 1$. From here, the iteration procedure continues until the following convergence criteria have been met [5]:

$$\|D_k - D_{k-1}\| \leq 10^{-10}, \quad \frac{\|\sigma_k - \sigma_{k-1}\|}{\|\sigma_k\|} \leq 10^{-8}, \quad \frac{\|D_k - D_{k-1}\|}{\|D_1\|} \leq 10^{-8} \quad (4.28)$$

The last condition was included to remove the remove any discrepancy of D_1 caused by changing the system parameters. Once the least-stable eigenvalue for a set of system parameters is found, a parameter of interest is incremented or decremented a small amount. The recursion relation is again used, using the previously-found value of σ as the initial guess. Continuation in this form leads to rapid convergence of the least-stable eigenvalue for a wide range of the flow variables. The determinant in each case was

determined by performing an LUP decomposition of the matrix A . When solving for the leading eigenvalue over the specified range of k , it was found that beginning at a non-zero value of k was much easier than preventing eigenvalue slippage at low values of the wavenumber. For example, the first run would begin at $k = 0.05$ and increase in k until the growth rate of the leading eigenvalue decreased below zero. Then, the procedure would begin again at $k = 0.05$ and decrease in very small increments until $k = 0$.

The eigenvalue search techniques of Mack [3] and Lessen et al. [4] were used periodically, and at various combinations of the system parameters, to ensure that the least-stable eigenvalue was the one being followed. This check was especially important in the regions of small wavenumbers, where slipping from the least-stable eigenvalue to a more stable one is common. Another check followed the more-stable eigenvalues to see whether their real part overtook σ_r as the system parameters were changed. As it turned out, in the range of investigated parameter values, the least-stable eigenvalue remained so. The numerical technique was checked through comparison with the analytical results of the simplified systems derived in section 4.6.

4.5 Results

The first computed data set compares so-called stability diagrams for either purely transverse – $\mathbf{k} = (k, 0)^T$ – or purely longitudinal – $\mathbf{k} = (0, m)^T$ – modes at several different Reynolds numbers. The Reynolds number is changed such that the superficial gas velocity is varied U/u_{mf} ranging from 1.05 to 1.25 while the other bed parameters remain constant. Figure 4.1 displays the growth constants of the leading eigenvalues for each mode as a function of their respective wavenumbers.

The growth constant traces for each mode are typical for most bed parameters. The value of the growth constant is zero at the origin and increases with respective wavenumber before reaching a maximum value and finally falling below zero. The region of the wavenumber space over which the growth constant is positive denotes the range of disturbances which will amplify in time. The drop in growth constant with increasing wavenumber is a result of viscous and conductive dampening, which is analogous to the same effect seen in single-fluid stability analyses. This results from the k^2 and m^2 dependence of the viscous and conductivity terms in (4.18-4.24), whose values dominate the stability of the flow at large wavelengths. Current parameter values were chosen so

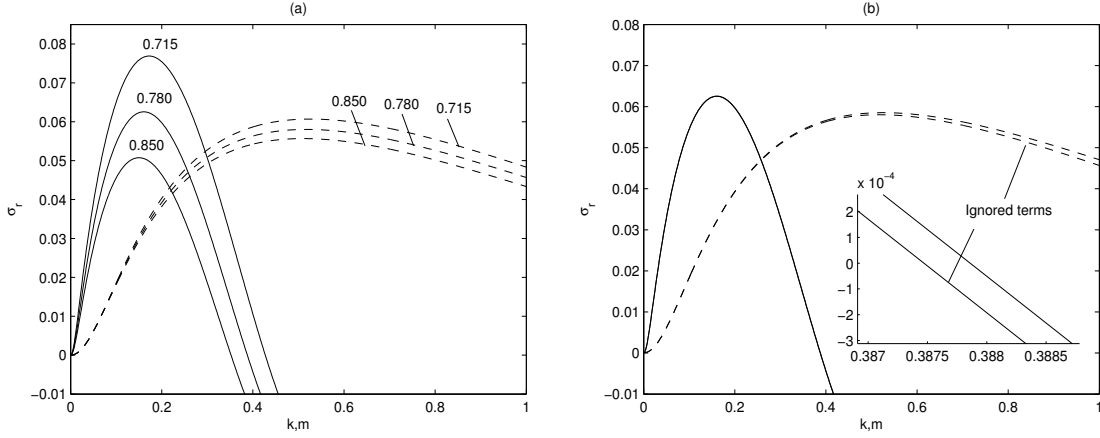


Figure 4.1: Growth constant traces corresponding to the longitudinal mode (solid lines, function of m) and transverse modes (dashed lines, function of k) for varying values of Re , as marked. (b) Growth constant traces for each mode with $Re = 0.780$ calculated by including and ignoring the fluid-phase convective, viscous, and body force terms, as marked. $R^{-1} = 2000$, $e_p = 0.90$.

that the dominant mode is longitudinal at a low Reynolds number (seen as the maximum of solid-line trace for $Re = 0.715$) but transverse at a higher Reynolds number (maximum of dashed line for $Re = 0.850$). Also notable is how the critical wavenumber the wavenumber after which all growth constants are negative is much larger in the case of transverse disturbances. The transverse disturbance is unstable over a much wider range of wavenumbers than the longitudinal. Figure 4.1(b) compares growth constant curves for each mode for two cases: one in which the fluid-phase convective, viscous, and body forces are included, and one in which they are ignored. The maximum growth constant for each mode changes by an almost indistinguishable amount, although the discrepancy is slightly higher for the transverse mode. The insert displays the point of marginal stability for the longitudinal mode for the purpose of showing that the longitudinal and transverse modes experience only a minor increase in the growth constant and wavenumber corresponding to marginal stability when the negligible fluid-phase terms are considered.

The growth constant traces of Figure 4.1 consider only disturbances with $k = 0$, $m \neq 0$ or $k \neq 0$, $m = 0$. In Figure 4.2 the growth constant corresponding to the leading mode is displayed as a function of horizontal and vertical wavenumbers. In this way disturbances of all wavenumber orientations are considered for the smallest

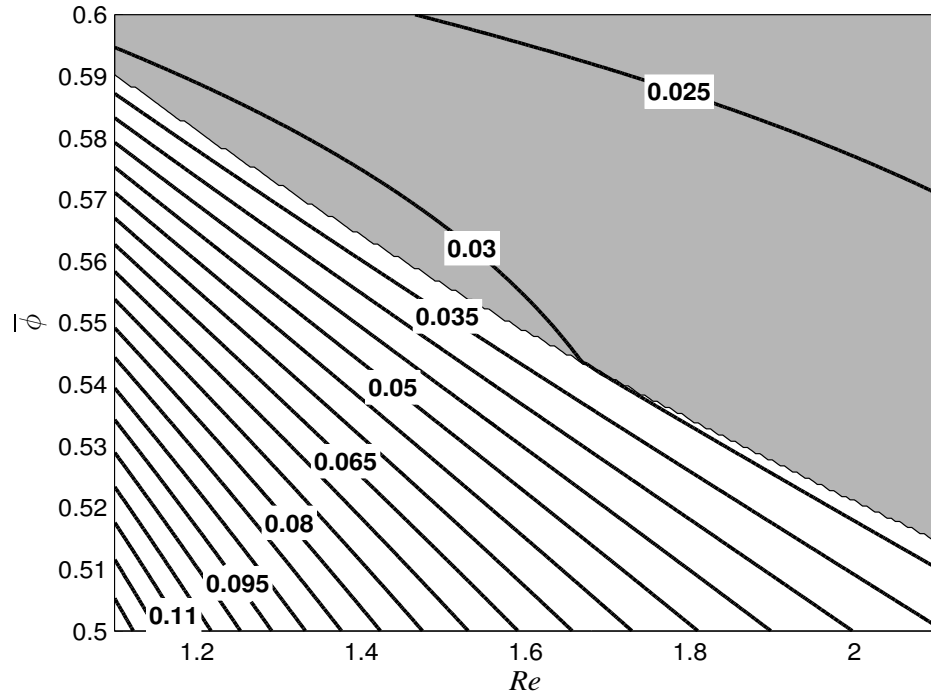


Figure 4.3: Growth constants of the dominant mode as a function of the Reynolds number and solid-phase volume concentration. Dark-shaded regions correspond to a dominant transverse wave, while non-shaded regions denote longitudinal dominance. $R^{-1} = 3000$, $e_p = 0.90$.

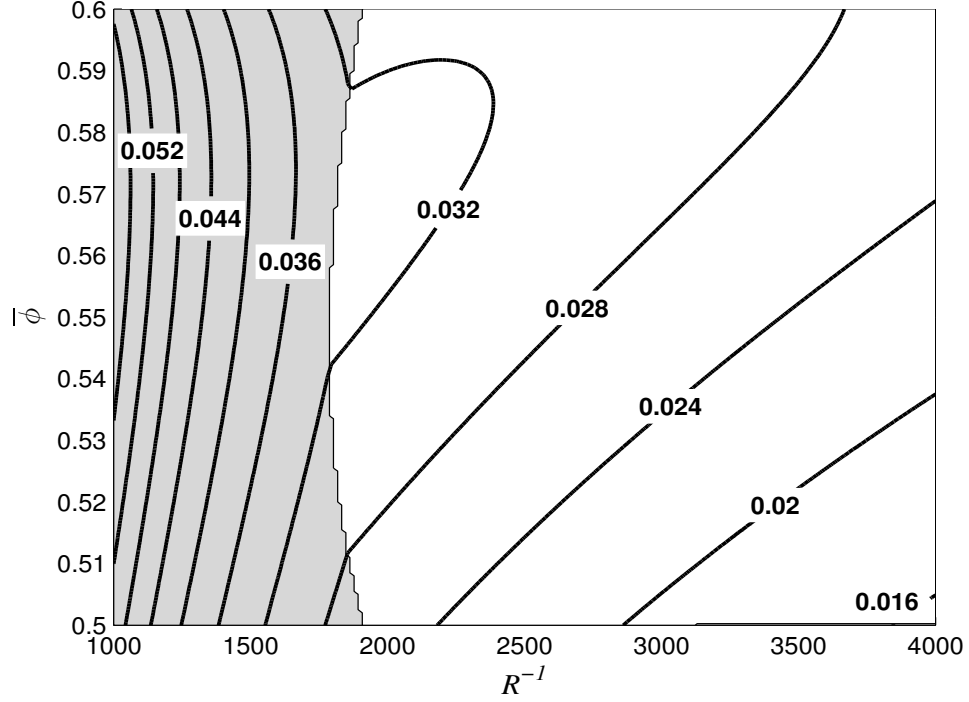


Figure 4.4: Growth constants of the dominant mode as a function of the phase density ratio and solid-phase volume concentration. Dark-shaded regions correspond to a dominant transverse wave, while non-shaded regions denote longitudinal dominance. $Re = 0.780$, $e_p = 0.90$.

unshaded. It is seen that the dominant mode of a system with a larger Reynolds number and solid-phase volume fraction is transverse. The growth constant of the dominant longitudinal mode changes rapidly with Re and $\bar{\phi}$, while the opposite is true for the transverse mode. Although the growth constants for each mode are not shown throughout the entire diagram, it is confirmed that the growth constants of each dominant disturbance mode remain within an order of magnitude of each other, and are positive over the given $Re - \bar{\phi}$ region. Likewise, the growth constant of both modes decrease with increasing Re and $\bar{\phi}$.

Growth constants for the dominant mode are plotted as a function of phase density ratio and solid-phase volume concentration in Figure 4.4. The nature of the disturbance mode is heavily dependent upon the particle density, with the lightest materials exhibiting a dominant transverse mode. The growth constants of each mode generally increase with the particle density, with the exception of the longitudinal mode at high

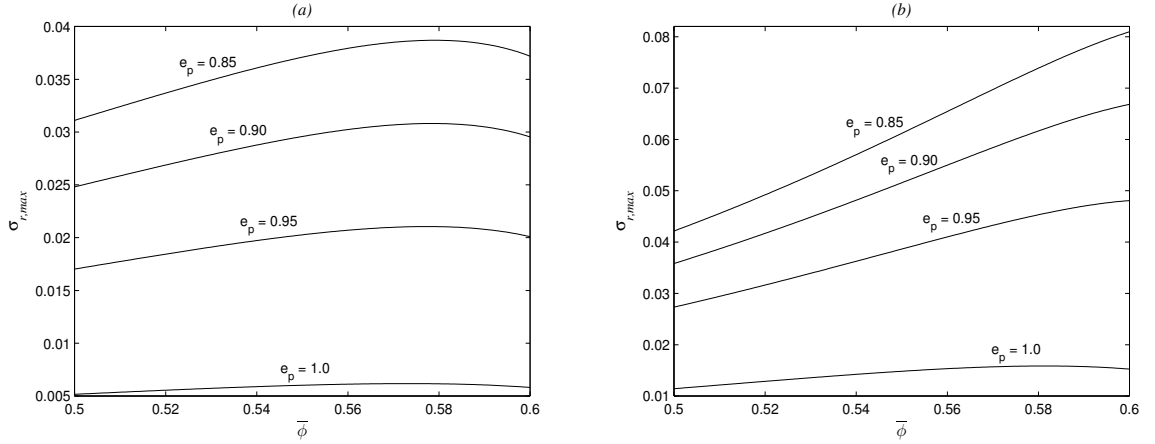


Figure 4.5: Growth constants of the dominant (a) transverse and (b) longitudinal modes as a function of the solid-phase volume concentration for various values of the restitution coefficient, as marked. $Re = 0.780$.

volume fractions. As opposed to the Reynolds number mode dependence, the growth constant of the transverse mode changes much more rapidly than the longitudinal disturbance, with respect to the density ratio. The growth constants of each mode again remain within an order of magnitude over the observed parameter space, indicating that both modes remain significant. The dependence of the modes and their growth constants with respect to $\bar{\phi}$ and Fr was found to be qualitatively similar to that seen in Figure 4.4, and so is not included here.

Figure 4.5 explores the effect of varying levels of particle restitution on the dominant longitudinal and transverse modes. The vertical mode exhibits a decrease in the growth constant as the particle becomes more elastic. This is a feature which has been recognized before in studies of longitudinal disturbances [7]. Interesting to note is the same behavior with the transverse mode, as depicted in Figure 5(a). The growth constants of both modes are very small for elastic particles and increase with the level of particle inelasticity. Although it is not represented on the figures, the wavenumber corresponding to the dominant disturbance also decreases with increasing e_p for each mode. The similar dependencies of the longitudinal and transverse disturbances on the restitution coefficient, as well as the other parameters, suggest similar stability mechanisms between the two modes. In the following section we explore these mechanisms and how they relate the two modes.

4.6 Separated Disturbance Modes

Since the dominant disturbance mode was found to be either purely longitudinal or purely transverse, we separate the modes into their respective simplified systems in an attempt to identify the instability mechanisms of each. To observe the leading contributions to the growth rate of the vertical mode, long wavelength analyses were also performed on the linearized equations of motion corresponding to each particular mode. Doing so amounts to solving for an estimate of the growth constant in the absence of short-wavelength dampening mechanisms, including viscosity and pseudo-thermal conductivity. As is evident from visual inspection of equations (4.18-4.24), these are the only terms proportional to squares of the wavenumbers. From this it is apparent that no fundamentally destabilizing short-wavelength terms will be missed by such an analysis.

4.6.1 Longitudinal Disturbance

The first system under consideration consists of purely longitudinal disturbances: $\mathbf{k} = (0, m)^T$. This is the classically-recognized disturbance of the isothermal system [8–10], and as such, it seems natural to extend such an analysis to a system in which non-isothermal disturbances are allowed. For simplicity, the convective, viscous, and body force terms of the fluid phase are neglected. As a further simplification, the terms corresponding to convection and conduction of PTE resulting from disturbances in the solid-phase volume fraction are ignored. These terms correspond to the second term on the left-hand side of equation (4.24), and the third term on the right-hand side of the same expression, respectively. It was found when solving for the full system (4.18-4.24) that these terms play a very minor role on the stability characteristics of the dominant mode, and may be safely ignored. With this in mind, the 1D, linearized equation of motion reduce to:

$$\sigma\phi' + im\bar{\phi}w'_s = 0 \quad (4.29)$$

$$-(\sigma + im)\phi' + im(1 - \bar{\phi})w'_g = 0 \quad (4.30)$$

$$0 = imP'_g + \bar{\beta}(w'_g - w'_s) + \bar{\beta}_\phi\phi' \quad (4.31)$$

$$\bar{\phi}\sigma w'_s = -im(\bar{P}_{s\phi}\phi' + \bar{P}_{sT}T') - m^2\bar{\xi}^s w'_s + \bar{\beta}(w'_g - w'_s) + \bar{\beta}_\phi\phi' - \frac{1}{Fr}\phi' \quad (4.32)$$

$$\frac{3}{2}\bar{\phi}\sigma T' = -im\bar{P}_s w'_s - m^2\bar{\lambda}T' + \bar{\Gamma}_{w_g}w'_g + \bar{\Gamma}_{w_s}w'_s + \bar{\Gamma}_\phi\phi' + \bar{\Gamma}_T T' \quad (4.33)$$

In the solid-phase momentum equation, $\bar{\xi}^s = 2\bar{\mu}_1^s + \bar{\mu}_2^s$. The determinant $D = |A - \sigma B| = 0$ is solved to obtain the dispersion relation, which is a cubic polynomial in σ :

$$\sigma^3 + \gamma_1 \sigma^2 + \gamma_2 \sigma + \gamma_3 = 0 \quad (4.34)$$

where the coefficients are written as

$$\gamma_1 = (m^2 \bar{\lambda} - \bar{\Gamma}_T) \left(\frac{3}{2} \bar{\phi} \right)^{-1} + A_1 \quad (4.35)$$

$$\gamma_2 = [(m^2 \bar{\lambda} - \bar{\Gamma}_T) A_1 + A_2] \left(\frac{3}{2} \bar{\phi} \right)^{-1} + A_3 \quad (4.36)$$

$$\gamma_3 = [(m^2 \bar{\lambda} - \bar{\Gamma}_T) A_3 + A_4] \left(\frac{3}{2} \bar{\phi} \right)^{-1} \quad (4.37)$$

and

$$A_1 = \frac{1}{\bar{\phi}} \left(\frac{\bar{\beta}}{1 - \bar{\phi}} + m^2 \bar{\xi}^s \right) \quad (4.38)$$

$$A_2 = -im \bar{P}_{sT} \left(\frac{\bar{\phi} \bar{\Gamma}_{w_g} + im(1 - \bar{\phi}) \bar{P}_s}{\bar{\phi}(1 - \bar{\phi})} \right) \quad (4.39)$$

$$A_3 = im \left(\frac{\bar{\beta}}{1 - \bar{\phi}} + \bar{\beta}_\phi - \frac{1}{Fr} - im \bar{P}_{s\phi} \right) \quad (4.40)$$

$$A_4 = m^2 \bar{P}_{sT} \left(\frac{\bar{\Gamma}_{w_g}}{1 - \bar{\phi}} + \bar{\Gamma}_\phi \right) \quad (4.41)$$

The dispersion relation was verified through numerical calculation of the determinant. Following the method of Koch & Sangani [7], the complex frequency is expanded into a Taylor series near $m = 0$:

$$\sigma = \sigma_r + i\sigma_i + m \frac{\partial}{\partial m} (\sigma_r + i\sigma_i) + \frac{m^2}{2} \frac{\partial^2}{\partial m^2} (\sigma_r + i\sigma_i) + O(m^3) \quad (4.42)$$

Several terms may be eliminated using the knowledge gained from the stability plots. For small m , it is observed that σ_r and σ_i are both very small, with values of zero at the origin ($m = 0$). Likewise, it is seen as $m \rightarrow 0$ that both $\partial \sigma_r / \partial m$ and $\partial^2 \sigma_i / \partial m^2$ approach zero. In addition, we may neglect terms of $O(m^3)$. The remaining terms yield

$$\sigma \simeq im\sigma_1 + m^2\sigma_2 \quad (4.43)$$

The first term represents the group velocity, of which $\sigma_1 = \partial \sigma_i / \partial m$, while

$$\sigma_2 = 0.5 (\partial^2 \sigma_r / \partial m^2) \quad (4.44)$$

is the group velocity dispersion. The sign of σ_2 determines the stability of the long-wavelength system, with a positive value being a sufficient but not necessary condition for instability. Substitution of this expression into the dispersion relation (4.34) and matching terms of $O(m)$ and $O(m^2)$ permits an expression for each term:

$$\sigma_1 = \frac{\bar{\phi}(1-\bar{\phi})}{\bar{\beta}} \left(\frac{1}{Fr} - \bar{\beta}_\phi - \frac{\bar{\beta}}{1-\bar{\phi}} \right) \quad (4.45)$$

$$\begin{aligned} \sigma_2 &= \frac{\bar{\phi}(1-\bar{\phi})}{\bar{\beta}\bar{\Gamma}_T} \left[\sigma_1\bar{\Gamma}_T - \left(\frac{\sigma_1 + \bar{\phi}}{\bar{\phi}(1-\bar{\phi})} \right) \bar{P}_{sT}\bar{\Gamma}_{w_g} + (\bar{P}_{sT}\bar{\Gamma}_\phi - \bar{P}_{s\phi}\bar{\Gamma}_T) \right. \\ &\quad \left. - \frac{3}{2}\bar{\phi}\sigma_1\bar{\beta}_\phi - \frac{3}{2} \left(\frac{1+\bar{\phi}}{1-\bar{\phi}} \right) \sigma_1\bar{\beta} + \frac{3}{2} \frac{\bar{\phi}}{Fr} \sigma_1 \right] \\ &= \frac{\bar{\phi}(1-\bar{\phi})}{\bar{\beta}\bar{\Gamma}_T} [\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 + \eta_6] \end{aligned} \quad (4.46)$$

The expression for the growth rate is similar to that obtained by Koch and Sangani [7], but with some differences. The authors cited also considered a long-vertical-wavelength analysis under the assumption that, at large wavelengths, granular conduction acts very slowly. In doing so, they assumed that the granular temperature is controlled by a local balance of the generation and dissipation terms. The current work considers the same circumstances, but allows for temporal variation of the perturbed granular temperature. This step results in a cubic dispersion relation, as opposed to the quadratic relation derived in Koch and Sangani [7], and yields expanded expressions for the granular temperature-generated mechanisms.

The first term in (4.46) represents the production and dissipation of energy resulting from fluctuations in the base-state granular temperature. The second term represents the sensitivity of the solid-phase pressure terms to fluctuations in the base-state granular temperature, which is itself a function of the base-state fluid velocity. The third term in (4.46) is a bulk elasticity term which, unlike the analogue found by Koch and Sangani [7], is negative for every value of the volume fraction and decreases rapidly as the bed approaches close packing. The quantity η_4 represents the aforementioned, classically-recognized destabilizing term [10]. The last two terms add to and detract from the long-wavelength growth constant, respectively, and may be lumped together because they share what is essentially the same mechanism. Figure 4.6 displays the value of σ_2 as a function of the volume fraction, as well as its term-by-term contributions. Each η term in Figure 4.6 from (4.46) is multiplied by the parenthesized quantity before the brackets.

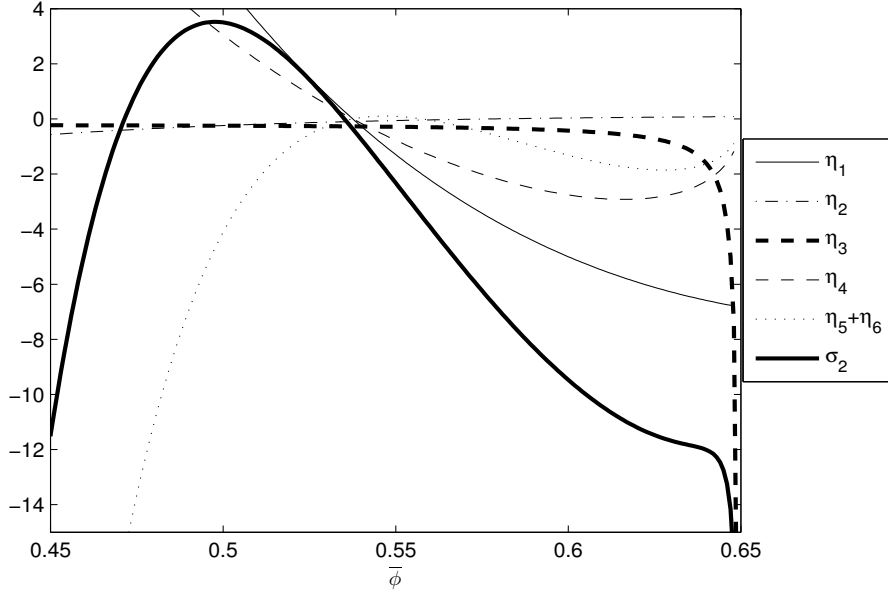


Figure 4.6: Traces corresponding to dimensionless values of the individual and summed terms of equation (4.46), as marked. $e_p = 0.90$.

Between solids volume fractions of 0.471 and 0.537, σ_2 assumes a positive value. For dilute flows, there is a balance primarily between the stabilizing action of the body force term, and the destabilizing effects of $\bar{\beta}_\phi$ and $\bar{\Gamma}_T$. The latter term, representing net energy generation due to collisional and viscous effects, contributes to instability until the point at which σ_1 switches signs. This is not shown in Figure 4.6, but occurs at a volume fraction of 0.537. As the bed becomes more densely-packed, energy losses due to collisional and viscous dissipation overcome the energy gained from the interstitial fluid. At higher volume fractions, contribution of $\bar{\beta}_\phi$ to the growth constant also becomes negative. After this point, the pressure term η_2 becomes positive, although its contribution to the growth constant is trumped by the other terms. The bulk elasticity term η_3 remains negative for all values of the volume fraction, although its stabilizing effect is most dramatic near close-packing.

4.6.2 Transverse Disturbance

To investigate the seemingly counterintuitive unstable transverse mode, we perform a similar analysis on the equations of motion in which disturbances are allowed only in the horizontal direction. Didwania [11] derived a dispersion relation for beds

which considered only horizontal velocities and disturbances. The equations of motion for this system are those (4.29-4.33) of the current work, although with w being replaced by u , m replaced with k , and without the following vertical-direction-specific terms: $\bar{\beta}_\phi$, Fr^{-1} , $\bar{\Gamma}_{w_f}$, and $\bar{\Gamma}_{w_s}$. The dispersion relation is a cubic polynomial of σ and is not reproduced here. We take the analysis a step further and, expanding the complex frequency using the transverse-wavenumber version of (4.43), obtain an analytic expression for the growth constant at large wavelengths:

$$\sigma_2 = \frac{\bar{\phi}(1 - \bar{\phi})}{\bar{\beta}\bar{\Gamma}_T} (\bar{P}_{sT}\bar{\Gamma}_\phi - \bar{P}_{s\phi}\bar{\Gamma}_T) \quad (4.47)$$

This is exactly the same as the third term in (4.46), which is negative for all values of the volume fraction and whose effect on the growth constant was found to be always stabilizing. If the inequality $|\frac{\bar{P}_{sT}\bar{\Gamma}_\phi}{\bar{P}_{s\phi}}| > |\bar{\Gamma}_T|$ holds, then (4.47) is positive and instability is possible. However, it is found that the inequality does not hold over the extensive parameter combinations tested, and the that long-wavelength growth constant estimate given by (4.47) is always negative. The set of 1D horizontal equations was also tested numerically, checking the results by use of the provided analytic dispersion relation, with the result that the leading eigenvalue of this system exhibits negative growth constants over all $k > 0$.

The unstable mechanism for the transverse disturbance must be identified using a different approach. Presently, we search for purely transverse modes but also allow for disturbances of the vertical velocities. This amounts to simply setting $m = 0$ in the linearized equations of motion (4.18-4.24). This simple allowance is essential in deriving stability conditions because it includes a term directly dependent upon the base-state fluid velocity. For added simplicity, the convective, viscous, and body force terms of the fluid phase are again neglected, resulting in the simplified relations:

$$\sigma\phi' + ik\bar{\phi}u'_s = 0 \quad (4.48)$$

$$-\sigma\phi' + ik(1 - \bar{\phi})u'_g = 0 \quad (4.49)$$

$$0 = ikP'_g + \bar{\beta}(u'_g - u'_s) \quad (4.50)$$

$$0 = \bar{\beta}(w'_g - w'_s) + \bar{\beta}_\phi\phi' \quad (4.51)$$

$$\bar{\phi}\sigma u'_s = -ik(\bar{P}_{s\phi}\phi' + \bar{P}_{sT}T') - k^2\bar{\xi}^s u'_s + \bar{\beta}(u'_g - u'_s) \quad (4.52)$$

$$\bar{\phi}\sigma w'_s = -k^2\bar{\mu}_1^s w'_s + \bar{\beta}(w'_g - w'_s) + \bar{\beta}_\phi\phi' - \frac{1}{Fr}\phi' \quad (4.53)$$

$$\frac{3}{2}\bar{\phi}\sigma T' = -ik\bar{P}_s u'_s - k^2\bar{\lambda}T' + \bar{\Gamma}_{w_g}w'_g + \bar{\Gamma}_{w_s}w'_s + \bar{\Gamma}_\phi\phi' + \bar{\Gamma}_T T' \quad (4.54)$$

Solving equation (7.45) for ϕ' , and making use of the fact that $\bar{\Gamma}_{w_g} = -\bar{\Gamma}_{w_s}$, the granular energy balance (4.54) may be re-written as:

$$\frac{3}{2}\bar{\phi}\sigma T' = -ik\bar{P}_s u'_s - k^2\bar{\lambda}T' - \bar{\Gamma}_{w_g}\frac{\bar{\beta}_\phi}{\bar{\beta}}\phi' + \bar{\Gamma}_\phi\phi' + \bar{\Gamma}_T T' \quad (4.55)$$

With this, the terms representing production of granular temperature due to fluctuations in the base-state fluid velocity are combined, representing production of granular temperature due to drag force fluctuations which result from perturbations in the base-state solid-phase volume concentration. Without the explicit dependence on the vertical velocity perturbations in the granular temperature equation, the relevant equations are reduced to (7.42), (7.43), (4.52), and (4.55). The resulting dispersion relation is a cubic polynomial in σ :

$$\sigma^3 + \zeta_1\sigma^2 + \zeta_2\sigma + \zeta_3 = 0 \quad (4.56)$$

The coefficients are written as

$$\zeta_1 = \frac{2}{3\bar{\phi}} \left[k^2 \left(\frac{3}{2}\bar{\xi}^s + \bar{\lambda} \right) + \frac{3\bar{\phi}}{2(1-\bar{\phi})}\bar{\beta} - \bar{\Gamma}_T \right] \quad (4.57)$$

$$\zeta_2 = \frac{2}{3\bar{\phi}} \left[k^4 \left(\bar{\lambda}\bar{\xi}^s \right) + k^2 \left(\frac{\bar{\beta}\bar{\lambda}}{1-\bar{\phi}} - \bar{\xi}^s\bar{\Gamma}_T + \bar{P}_s\bar{P}_{sT} + \frac{3}{2}\bar{\phi}^2\bar{P}_{s\phi} \right) - \frac{\bar{\beta}\bar{\Gamma}_T}{1-\bar{\phi}} \right] \quad (4.58)$$

$$\zeta_3 = \frac{2}{3\bar{\phi}} \left[k^4 \left(\bar{\lambda}\bar{P}_{s\phi} \right) + k^2 \left(\bar{P}_{sT}\bar{\Gamma}_\phi - \bar{P}_{s\phi}\bar{\Gamma}_T - \frac{\bar{\beta}_\phi}{\bar{\beta}}\bar{P}_{sT}\bar{\Gamma}_{w_f} \right) \right] \quad (4.59)$$

The dispersion relation was verified by numerically calculating the determinant. This is similar to the relation derived by Didwania [11], but with one critical difference. The coefficient given by (4.59) contains a term dependent upon the instability mechanism, $\bar{\beta}_\phi$, as well as the granular-temperature-generated mechanism $\bar{P}_{sT}\bar{\Gamma}_{w_f}$, neither of which are present when considering a transverse disturbance in a 1D system. The coefficients of (4.56) are all real, and so the real parts of all the roots are negative if and only if ζ_1 , ζ_2 , ζ_3 , and $\zeta_1\zeta_1 - \zeta_3$ are positive. The only relevant condition which is changed by the current analysis is the third, from which we obtain the following condition for instability:

$$k^2 < \frac{1}{\bar{\lambda}} \left(\bar{\Gamma}_T - \frac{\bar{P}_{sT}\bar{\Gamma}_\phi}{\bar{P}_{s\phi}} + \frac{\bar{P}_{sT}\bar{\Gamma}_{w_f}}{\bar{P}_{s\phi}} \frac{\bar{\beta}_\phi}{\bar{\beta}} \right) \quad (4.60)$$

Unlike the analogous expression (4.47) which does not account for vertical velocity perturbations, the right-hand-side of (4.60) becomes positive at some values of

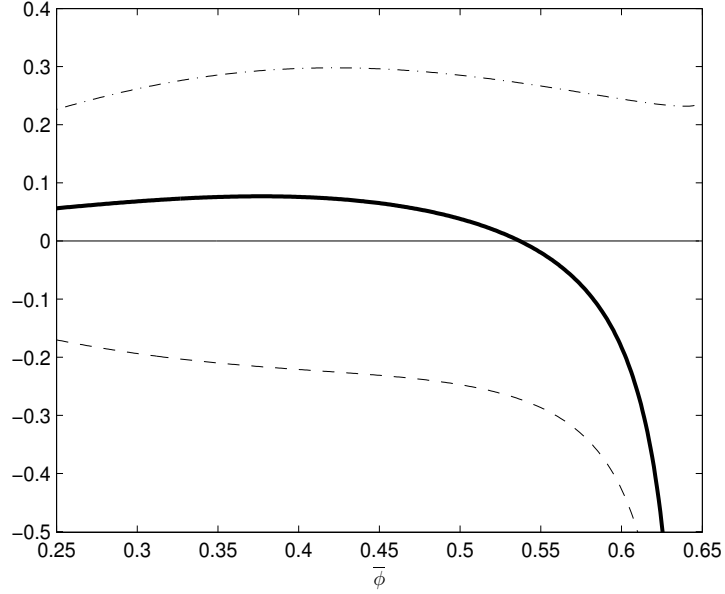


Figure 4.7: Traces corresponding to dimensionless values of the individual and summed terms of equation (4.61): σ_2 , solid line; χ_1 , dashed line; χ_2 , dot-dashed line. The line at zero is included for reference. $e_p = 0.90$.

the solid-phase volume concentration. Again expanding the complex frequency into a Taylor series near $k = 0$, and inserting into (4.56), we obtain $\sigma_1 = 0$, corresponding to a non-oscillatory mode. As it turns out – verified numerically – the leading mode for disturbances with wavenumber vector $\mathbf{k} = (k, 0)^T$ for all k is non-oscillatory. The corresponding expression for the growth constant of this mode at large transverse wavelengths is

$$\begin{aligned} \sigma_2 &= \frac{\bar{\phi}(1-\bar{\phi})}{\bar{\beta}} \left[\left(\frac{\bar{P}_{sT}\bar{\Gamma}_\phi}{\bar{\Gamma}_T} - \bar{P}_{s\phi} \right) - \frac{\bar{P}_{sT}\bar{\Gamma}_{wf}}{\bar{\Gamma}_T} \frac{\bar{\beta}_\phi}{\bar{\beta}} \right] \\ &= \frac{\bar{\phi}(1-\bar{\phi})}{\bar{\beta}} [\chi_1 + \chi_2] \end{aligned} \quad (4.61)$$

The first term in (4.61) is the same stabilizing mechanism found in (4.46) and (4.47). The second term is similar in form as the second term in (4.46). However, since $\bar{\Gamma}_T$ is always negative, this term is positive for all volume fractions and, as it turns out, has a higher absolute value than that of the first parenthesized term in (4.61) for certain bed parameters. The large-horizontal-wavelength growth constant, along with its two components, are plotted as a function of the volume fraction in Figure 4.7. As with

the large-longitudinal-wavelength growth constant, the effective bulk elasticity term decreases monotonically with the volume fraction and decreases very rapidly as the bed approaches close-packing. Thus we find that the transverse and longitudinal modes share the same dramatic stabilizing mechanism at large volume fractions.

The positive contribution to (4.61) remains so for all values of the solids volume fraction. This term is responsible for the flow instability up to a volume fraction of 0.537. This is the same value at which the wave speed of the large-longitudinal-wavelength disturbance, given by (4.45), changes signs. By analyzing the terms we may attempt to realize the physical process through which this unstable transverse mode amplifies. The term $\bar{\beta}_\phi$ is partially responsible for the instability of longitudinal waves through the drag force fluctuations induced by perturbations in the base-state, solid-phase volume fraction. The reactionary delay caused by the large settling time of particles to this disturbance, which occurs as a result of the large particle-to-fluid inertia, may add up over the period of the normal-mode disturbance for certain longitudinal wavenumbers. Since the dominant transverse wave is non-oscillatory ($\sigma_i = 0$), this mechanism acts to exacerbate local perturbations which are periodic in the transverse direction. Instead of adding over the period of a disturbance, the effect of this mechanism increases with the time-dependent increase in the amplitude of the voidage perturbation. If the time required to amplify the magnitude of these voidage perturbations - dictated by the growth constant of the disturbance - is smaller than the settling time of a particle, then this mode may amplify in time.

The term $\bar{P}_{sT}\bar{\Gamma}_{w_g}$ represents the change in the solid-phase pressure resulting from perturbations of the base-state granular temperature and, thus, the base-state fluid velocity. A perturbation in the base-state volume fraction yields a discrepancy between the phasic vertical velocities, as evidenced by equation (7.45). This increases the energy source resulting from the normal and shear stress interaction between the phases, which increases the granular temperature and the particle-phase pressure. The resulting mechanism tends to drive rapidly-moving particles from dilute regions to more compacted areas of hampered particle movement. As with the previously-mentioned term, this effect does not add over the cycle of a disturbance, but rather exacerbates horizontally-periodic perturbations of the volume fraction over time.

As mentioned, the leading transverse mode is non-oscillatory ($\sigma_i = 0$) for all wavenumbers k . With this information, it is possible to derive an expression relating

the growth rate of the dominant mode σ_{rD} and the corresponding wavenumber k_D . To begin, the dispersion relation (4.56) is differentiated with respect to k :

$$3\sigma^2 \frac{\partial \sigma}{\partial k} + 2\sigma \frac{\partial \sigma}{\partial k} \zeta_1 + \sigma^2 \frac{\partial \zeta_1}{\partial k} + \frac{\partial \sigma}{\partial k} \zeta_2 + \sigma \frac{\partial \zeta_2}{\partial k} + \frac{\partial \zeta_3}{\partial k} = 0 \quad (4.62)$$

Separating this expression into its imaginary and real parts yields two expressions:

$$\begin{aligned} & 6\sigma_i \sigma_r \frac{\partial \sigma_r}{\partial k} + 3(\sigma_i^2 - \sigma_r^2) \frac{\partial \sigma_i}{\partial k} + 2\sigma_i \frac{\partial \sigma_r}{\partial k} \zeta_1 \\ & + 2\sigma_r \frac{\partial \sigma_i}{\partial k} \zeta_1 + 2\sigma_i \sigma_r \frac{\partial \zeta_1}{\partial k} + \frac{\partial \sigma_i}{\partial k} \zeta_2 + \sigma_i \frac{\partial \zeta_2}{\partial k} = 0 \end{aligned} \quad (4.63)$$

$$\begin{aligned} & -6\sigma_i \sigma_r \frac{\partial \sigma_i}{\partial k} + 3(\sigma_i^2 - \sigma_r^2) \frac{\partial \sigma_r}{\partial k} - 2\sigma_i \frac{\partial \sigma_i}{\partial k} \zeta_1 \\ & + 2\sigma_r \frac{\partial \sigma_r}{\partial k} \zeta_1 + \frac{\partial \sigma_r}{\partial k} \zeta_2 + \sigma_r \frac{\partial \zeta_2}{\partial k} + \frac{\partial \zeta_3}{\partial k} = 0 \end{aligned} \quad (4.64)$$

The first expression, with $\sigma_{iD} = 0$, yields the trivial result of $\partial \sigma_i / \partial k = 0$ for all k . The dominant transverse mode satisfies the expression $\partial \sigma_r / \partial k|_{\sigma_r=\sigma_{rD}, k=k_D} = 0$. Inserting this into (4.64) results in a relation for the growth constant and wavenumber of the dominant mode:

$$\sigma_{rD}^2 + \sigma_{rD} (\varphi_1 k_D^2 + \varphi_2) + \varphi_3 k_D^2 + \varphi_4 = 0 \quad (4.65)$$

where

$$\varphi_1 = \frac{4\bar{\lambda}\bar{\xi}^s}{\bar{\phi}(3\bar{\xi}^s + 2\bar{\lambda})} \quad (4.66)$$

$$\varphi_2 = \frac{2}{\bar{\phi}(3\bar{\xi}^s + 2\bar{\lambda})} \left[\frac{\bar{\beta}\bar{\lambda}}{1 - \bar{\phi}} - \bar{\xi}^s \bar{\Gamma}_T + \bar{P}_s \bar{P}_{sT} + \frac{3}{2} \bar{\phi}^2 \bar{P}_{s\phi} \right] \quad (4.67)$$

$$\varphi_3 = \frac{4\bar{\lambda}\bar{P}_{s\phi}}{3\bar{\xi}^s + 2\bar{\lambda}} \quad (4.68)$$

$$\varphi_4 = \frac{2}{3\bar{\xi}^s + 2\bar{\lambda}} \left[\bar{P}_{sT} \bar{\Gamma}_\phi - \frac{\bar{\beta}_\phi}{\bar{\beta}} \bar{P}_{sT} \bar{\Gamma}_{w_g} \right] \quad (4.69)$$

4.7 Summary and Observations

A normal-mode stability analysis was performed on the two-phase equations of motion describing a gas-fluidized bed. The system was found to support unstable modes with wavenumber vectors ranging from the horizontal to the vertical. However, the dominant mode over the entire wavenumber spectrum was determined to be either purely longitudinal or purely transverse in nature. A parametric study of these separate modes

showed that the preferred disturbance depends upon the dimensionless parameters of the system. Furthermore, the dependencies of both modes upon the system parameters are similar, and the growth constants of each mode are within an order of magnitude of each other over the investigated parameter set, highlighting the importance of considering both disturbance modes in a stability analysis.

Investigations of each separated disturbance revealed that the transverse and longitudinal modes share similar stability mechanisms. The stabilizing term of the transverse mode is the same as that of the longitudinal mode, which is most effective at solid-phase volume fraction values near close-packing. The destabilizing mechanisms of the transverse mode are also the same as those of the vertical mode, although the long-wavelength growth constant of the latter is heavily influenced by the term representing granular temperature perturbations due to collisions at low volume fractions.

The current work considers particles with large Stokes numbers, the velocity distribution of which is assumed to be governed by particle collisions rather than viscous interactions with the fluid phase. A question arises from the resulting isotropic granular temperature assumption regarding the relevance of transverse modes in anisotropic systems in which the fluctuating energy source is stronger in the vertical direction. The transverse mode identified in the current work is expected to be as important as the longitudinal mode for two reasons. First, the destabilizing term in equation (4.61) is proportional to $\frac{\bar{\beta}_\phi}{\bar{\beta}}$, whose destabilizing effect is well-documented and is independent of the granular temperature. This quantity is at least an order of magnitude higher than the granular-temperature-generated portion of the term, $\frac{\bar{P}_{sT}\bar{\Gamma}_{wg}}{\bar{\Gamma}_T}$, for every set of parameters tested in the current work. The second reason is attributed to the work of Koch and Sangani [7], who considered both isotropic and anisotropic velocity variances. The authors noted that the anisotropic theory predicts that the larger source of fluctuating energy in the vertical direction stabilizes longitudinal disturbances. In addition, even in systems where fluid-solid interactions play a non-negligible role on particle velocity distributions, particle collisions still transfer energy from the vertical to the horizontal modes.

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Chapter 5

Stability of a Vertical Fluidized Bed

5.1 Problem Description

The present problems consider a homogeneous assembly of spherical particles with mean diameter d_p bounded by either a 3D cylindrical tube of radius $\hat{L}$. The base-state fluid velocity flows in the direction opposite to gravity. It is convenient to switch to the cylindrical-polar coordinate system: $\mathbf{x} = (r, \theta, z)^T$. The axial coordinate is defined such that the gravitational acceleration acts in the negative z direction. The solid- and fluid-phase velocities are defined as $\mathbf{u} = (u_r, u_\theta, u_z)^T$ and $\mathbf{v} = (v_r, v_\theta, v_z)^T$.

A homogeneous base-state solids volume concentration $\bar{\phi}$ is imposed, corresponding to the situation observed in experiment for mean flow velocities on the order of the minimum fluidization velocity. Along with the stipulation that no bulk velocity of the particle phase is allowed in the base-state – a phenomenon also observed in experiment near minimum fluidization – the base state solutions are greatly simplified. These simplifications come at the cost of disregarding any shear of the base-state fluid velocity, as well as restricting attention to an adiabatic boundary condition at the wall for the PTE. However, for the reasons mentioned in section 5.3, this minor allowance is not expected to have a significant effect of the stability characteristics of the flow.

5.2 Governing Equations

The equations of motion (2.2-2.5, 2.16) are non-dimensionalized using length scale $\hat{L}$ and velocity U , where U is the mean base-state fluid velocity defined as the solution of (2.5). With this, the following dimensionless parameters are introduced:

$$\begin{aligned} (r, z) &= \frac{1}{\hat{L}}(\hat{r}, \hat{z}), \quad t = \frac{U}{\hat{L}}\hat{t}, \quad (\mathbf{v}, \mathbf{u}) = \frac{1}{U}(\hat{\mathbf{v}}, \hat{\mathbf{u}}), \quad P_g = \frac{\hat{P}_g}{\rho_s U^2} \\ Re &= \frac{\rho_g U d_p}{\hat{\mu}_g}, \quad R = \frac{\rho_g}{\rho_s}, \quad Fr = \frac{U^2}{g d_p}, \quad L = \frac{\hat{L}}{d_p}, \quad \mu = \frac{\hat{\mu}_{eg}}{\hat{\mu}_g} \end{aligned} \quad (5.1)$$

The last term is included for convenience. This will be seen during the stability analysis when accounting for disturbances of the effective fluid-phase viscosity resulting from volume fraction perturbations. The resulting dimensionless equations of motion are:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) = 0 \quad (5.2)$$

$$\frac{\partial(1-\phi)}{\partial t} + \nabla \cdot (1-\phi) \mathbf{v} = 0 \quad (5.3)$$

$$\begin{aligned} \frac{\partial(\phi \mathbf{u})}{\partial t} + \nabla \cdot (\phi \mathbf{u} \mathbf{u}) &= -\nabla \cdot \left(P_s \mathbf{I} - \frac{\mu_s}{L} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{\zeta_s}{L} (\nabla \cdot \mathbf{u}) \right) \\ &\quad - \phi \nabla \cdot \left(P_g \mathbf{I} - \frac{R\mu}{HRe} (\nabla \mathbf{v} + \nabla \mathbf{v}^T) - \frac{2}{3} \frac{R\mu}{LRe} (\nabla \cdot \mathbf{v}) \right) \\ &\quad + \frac{L}{Fr} (1-R)(1-\phi) \beta (\mathbf{v} - \mathbf{u}) + \phi \frac{L}{Fr} \mathbf{f} \end{aligned} \quad (5.4)$$

$$\begin{aligned} \frac{\partial((1-\phi) \mathbf{v})}{\partial t} + \nabla \cdot ((1-\phi) \mathbf{v} \mathbf{v}) &= -(1-\phi) \nabla \cdot \left(R^{-1} P_g \mathbf{I} - \frac{\mu}{LRe} (\nabla \mathbf{v} + \nabla \mathbf{v}^T) - \frac{2}{3} \frac{\mu}{LRe} \nabla \cdot \mathbf{v} \right) \\ &\quad - \frac{L}{Fr} (R^{-1} - 1)(1-\phi) \beta (\mathbf{v} - \mathbf{u}) + (1-\phi) \frac{L}{Fr} \mathbf{f} \end{aligned} \quad (5.5)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{3}{2} \phi T \right) + \nabla \cdot \left(\frac{3}{2} \phi T \mathbf{u} \right) &= - \left(P_s \mathbf{I} - \frac{\mu_s}{L} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) - \frac{\zeta_s}{L} (\nabla \cdot \mathbf{u}) \right) : \nabla \mathbf{u} \\ &\quad + \nabla \cdot \left(\frac{\lambda}{L} \nabla T + \frac{\lambda_h}{L} \nabla \phi \right) \\ &\quad + \frac{LR^2 \mu^2}{Re^2} |\mathbf{v} - \mathbf{u}|^2 \Gamma_s - LJ_c - \frac{L}{Fr} (1-R) J_v \end{aligned} \quad (5.6)$$

5.3 Base-State Solutions

All base-state variables are assumed to be axisymmetric. Emulating the situation observed in experiment [1–3], we require no bulk movement of the solid phase

in the base state. Available boundary conditions for the fluid phase velocity and the granular temperature include that of Sinclair & Jackson [4] and Johnson & Jackson [5], respectively. The former is a force balance in the narrow region adjacent to the wall, and depends upon the parameter $\delta(\phi) = \delta_o \frac{\phi}{\phi_p}$, where δ_o represents the length over which the averaging of the equations of motion takes place. This condition is written as, in the present notation:

$$\frac{\partial \bar{v}_z}{\partial z} + \frac{L^2 Re}{\bar{\mu} Fr} (R^{-1} - 1) \frac{\delta \bar{\phi}}{\phi_p} \bar{\beta} \bar{v}_z + \frac{L Re}{\bar{\mu} R} \frac{\delta \bar{\phi}}{\phi_p} \frac{\partial \bar{P}_g}{\partial z} + \frac{2 \phi_p}{\delta \bar{\phi}} \bar{T} \bar{v}_z = 0 \quad (5.7)$$

The latter is an energy balance which depends upon the coefficient of restitution between the particles and the wall, e_w . This is written, with the current notation:

$$\mathbf{n} \cdot \mathbf{q} = L^3 (\mathbf{u} \cdot \mathbf{u}) \frac{\phi_s \sqrt{3} \pi \bar{\phi} \bar{T}^{1/2}}{6 \phi_p [1 - (\bar{\phi}/\phi_p)^{1/3}]} - L \frac{\sqrt{3} \pi \bar{\phi} \bar{T}^{3/2} (1 - e_w^2)}{4 \phi_p [1 - (\bar{\phi}/\phi_p)^{1/3}]} = 0 \quad (5.8)$$

The problem is simplified by taking the limit of the first condition as $\phi \rightarrow \phi_p$. The relevant boundary condition (5.7) simplifies to $\frac{L(1-R)}{Fr} \bar{\beta} \bar{v}_z + \frac{\partial \bar{P}_g}{\partial z} = 0$. The dimensional value of the base-state fluid velocity is then obtained by combining the axial fluid and solid momentum equations:

$$\hat{v}_z(r=0) = U = \frac{\rho_s g}{\bar{\beta}} \bar{\phi}, \quad (5.9)$$

with a wall boundary condition $\partial \bar{v}_z / \partial r = 0$. Hence, shear of the base-state fluid velocity is ignored. This is a reasonable approximation for fixed beds with fluidization velocities on the order of u_{mf} – implying [1] $\phi \sim \phi_p$ – which is the regime investigated in the present work. The second simplification consists of assuming $e_w = 1$ (i.e. adiabatic walls). In this case the relevant boundary condition (5.8) simplifies to $\partial \bar{T} / \partial r = 0$ at the wall. As a result, the present analysis is limited to beds in which no energy is lost at the wall due to particle collisions. With these two assumptions, the base state simplifies considerably. The base-state variables are defined as:

$$\begin{aligned} \mathbf{u} &= (0, 0, 0)^T, \quad \mathbf{v} = (0, 0, 1)^T, \quad T = \bar{T}, \quad \phi = \bar{\phi} \\ \beta &= \bar{\beta}, \quad P_g = \bar{P}_g(z), \quad P_s = \bar{P}_s, \quad \mu = \bar{\mu} \\ \mu_s &= \bar{\mu}_s, \quad \eta_s = \bar{\eta}_s, \quad \Gamma_s = \bar{\Gamma}_s, \quad J_c = \bar{J}_c, \quad J_v = \bar{J}_v \end{aligned} \quad (5.10)$$

The granular temperature equation (5.6) reduces to

$$\bar{f}_6 \bar{T}^{3/2} + \frac{3}{u_t Fr} (1 - R) \bar{f}_0 \bar{T} - \left(\frac{R \bar{\mu}}{Re} \right)^2 \bar{f}_5 \bar{T}^{-1/2} = 0 \quad (5.11)$$

The base-state granular temperature is acquired using a standard root-finding method.

5.4 Linear Stability Analysis

The governing equations (6.2-6.6) are linearized by perturbing the variables about the base state:

$$\psi(r, \theta, z, t) = \bar{\psi} + \tilde{\psi}(r, \theta, z, t) \quad (5.12)$$

where ψ is the nine-term solution vector defined as

$$\psi = (u_r \ u_\theta \ u_z \ v_r \ v_\theta \ v_z \ P_g \ \phi \ T)^T \quad (5.13)$$

As only initial perturbations of small magnitude are considered, perturbed terms of second order and higher are ignored. The dimensionless, linearized system is represented in operator form as:

$$\frac{\partial \tilde{\psi}}{\partial t} = \mathbf{D} \tilde{\psi} \quad (5.14)$$

$$\mathbf{B}_1 \tilde{\psi} = 0, \quad r = 1 \quad (5.15)$$

$$\mathbf{B}_0 \tilde{\psi} = 0, \quad r = 0 \quad (5.16)$$

We seek normal mode solutions of the form

$$\tilde{\psi}(r, \theta, z, t) = \psi'(r) \exp[i(kz + n\theta) + \sigma t] \quad (5.17)$$

Here, k is the longitudinal wavenumber and $\sigma = \sigma_r + i\sigma_i$ is the complex frequency, both of which are dimensionless. The wavenumber n measure periodicity in the azimuthal direction, with the value $n = 0$ corresponding to axisymmetric disturbances. The expanded form of $\mathbf{D}$ is presented:

$$\sigma \phi' = -\bar{\phi} \frac{\partial u_r'}{\partial r} - \frac{\bar{\phi}}{r} u_r' - \frac{in\bar{\phi}}{r} u_\theta' - ik\bar{\phi} u_z' \quad (5.18)$$

$$\sigma \phi' = (1 - \bar{\phi}) \frac{\partial v_r'}{\partial r} + \frac{(1 - \bar{\phi})}{r} v_r' + \frac{in(1 - \bar{\phi})}{r} v_\theta' + ik(1 - \bar{\phi}) v_z' - ik\phi' \quad (5.19)$$

$$\begin{aligned} \sigma v_r' = & -R^{-1} \frac{\partial P_g'}{\partial r} - \frac{L(R^{-1} - 1)}{Fr} \bar{\beta}(v_r' - u_r') - ikv_r' \\ & + \frac{\bar{\mu}}{LRe} \left(\nabla^2 v_r' + \frac{1}{3} \frac{\partial}{\partial r} (\nabla \cdot \mathbf{v}') - \frac{2in}{r^2} v_\theta' - \frac{v_r'}{r^2} \right) \end{aligned} \quad (5.20)$$

$$\begin{aligned} \sigma v_\theta' = & -\frac{inR^{-1}}{r} P_g' - \frac{L(R^{-1} - 1)}{Fr} \bar{\beta}(v_\theta' - u_\theta') - ikv_\theta' \\ & + \frac{\bar{\mu}}{LRe} \left(\nabla^2 v_\theta' + \frac{in}{3r} (\nabla \cdot \mathbf{v}') + \frac{2in}{r^2} v_r' - \frac{v_\theta'}{r^2} \right) \end{aligned} \quad (5.21)$$

$$\begin{aligned} \sigma v'_z - \frac{\sigma}{(1-\bar{\phi})} \phi' = & -ikR^{-1}P'_g - \frac{L(R^{-1}-1)}{Fr} (\bar{\beta}(v'_z - u'_z) + \bar{\beta}_\phi \phi') - ikv'_z \\ & + \frac{ik}{(1-\bar{\phi})} \phi' + \frac{\bar{\mu}}{LRe} \left(\nabla^2 v'_z + \frac{ik}{3} (\nabla \cdot \mathbf{v}') \right) \end{aligned} \quad (5.22)$$

$$\begin{aligned} \sigma \bar{\phi} u'_r = & -\bar{\phi} \frac{\partial P'_g}{\partial r} + \frac{\bar{\phi} \bar{\mu} R}{LRe} \left(\nabla^2 v'_r + \frac{1}{3} \frac{\partial}{\partial r} (\nabla \cdot \mathbf{v}') - \frac{2in}{r^2} v'_\theta - \frac{v'_r}{r^2} \right) \\ & + (1-\bar{\phi}) \frac{L(1-R)}{Fr} \bar{\beta}(v'_r - u'_r) - P_{s\phi} \frac{\partial \phi'}{\partial r} - P_{sT} \frac{\partial T'}{\partial r} \\ & + \frac{\bar{\mu}_s}{L} \nabla^2 u'_r + \frac{\bar{\mu}_s + \bar{\zeta}_s}{L} \frac{\partial}{\partial r} (\nabla \cdot \mathbf{u}') - \frac{2in\bar{\mu}_s}{Lr^2} u'_\theta - \frac{\bar{\mu}_s}{L} \frac{u'_r}{r^2} \end{aligned} \quad (5.23)$$

$$\begin{aligned} \sigma \bar{\phi} u'_\theta = & -\frac{in\bar{\phi}}{r} P'_g + \frac{\bar{\phi} \bar{\mu} R}{LRe} \left(\nabla^2 v'_\theta + \frac{in}{3r} (\nabla \cdot \mathbf{v}') + \frac{2in}{r^2} v'_r - \frac{v'_\theta}{r^2} \right) \\ & + (1-\bar{\phi}) \frac{L(1-R)}{Fr} \bar{\beta}(v'_\theta - u'_\theta) - \frac{in}{r} (\bar{P}_{s\phi} \phi' - \bar{P}_{sT} T') \\ & + \frac{\bar{\mu}_s}{L} \nabla^2 u'_\theta + \frac{\bar{\mu}_s + \bar{\zeta}_s}{L} \frac{in}{r} (\nabla \cdot \mathbf{u}') + \frac{2in\bar{\mu}_s}{Lr^2} u'_r - \frac{\bar{\mu}_s}{L} \frac{u'_\theta}{r^2} \end{aligned} \quad (5.24)$$

$$\begin{aligned} \sigma \bar{\phi} u'_z = & -ik\bar{\phi} P'_g + (1-\bar{\phi}) \frac{L(1-R)}{Fr} \left(\bar{\beta}(v'_z - u'_z) + \bar{\beta}_\phi \phi' - \frac{\bar{\beta}}{\bar{\phi}(1-\bar{\phi})} \phi' \right) + \frac{\bar{\mu}_s}{L} \nabla^2 u'_z \\ & + \frac{\bar{\phi} \bar{\mu} R}{LRe} \left(\nabla^2 v'_z + \frac{ik}{3} (\nabla \cdot \mathbf{v}') \right) - ik(\bar{P}_{s\phi} \phi' + \bar{P}_{sT} T') + \frac{\bar{\mu}_s + \bar{\zeta}_s}{L} ik(\nabla \cdot \mathbf{u}') \end{aligned} \quad (5.25)$$

$$\begin{aligned} \frac{3}{2} \sigma (\bar{\phi} T' + \bar{T} \phi') = & - \left(\bar{P}_s + \frac{3}{2} \bar{\phi} \bar{T} \right) (\nabla \cdot \mathbf{u}') + \frac{\bar{\lambda}_h}{L} \nabla^2 \phi' + \frac{\bar{\lambda}}{L} \nabla^2 T' \\ & + L (\bar{\Gamma}_{u_z} u'_z + \bar{\Gamma}_{v_z} v'_z + \bar{\Gamma}_T T' + \bar{\Gamma}_\phi \phi') \end{aligned} \quad (5.26)$$

Zero-shear conditions are imposed for the longitudinal and azimuthal fluid-phase velocity components at the wall, as in the base state. Likewise, an adiabatic condition is imposed for the perturbed granular temperature. The transverse components of velocity for each phase are set to zero, representing impermeability of the wall. Partial slip of the perturbed solids velocity at the wall is allowed, as in the previous problem. This condition is derived from the force balance near the wall presented by Johnson & Jackson [5]. Thus we have, at $r = 1$:

$$u'_r = v'_r = 0 \quad (5.27)$$

$$\left[\bar{\mu}_s \left(-\frac{1}{r} + \frac{\partial}{\partial r} \right) + L \bar{f}_7 \bar{T}^{1/2} \right] u'_\theta = 0 \quad (5.28)$$

$$\left[\bar{\mu}_s \frac{\partial}{\partial r} + L \bar{f}_7 \bar{T}^{1/2} \right] u'_z = 0 \quad (5.29)$$

$$\left(-\frac{1}{r} + \frac{\partial}{\partial r} \right) v'_\theta = 0 \quad (5.30)$$

$$\frac{\partial v'_z}{\partial r} = 0 \quad (5.31)$$

$$\frac{\partial T'}{\partial r} = 0 \quad (5.32)$$

At the centerline, statements of boundedness and smoothness of the perturbed variables are represented by the following expressions:

$$\lim_{r \rightarrow 0} \frac{\partial \mathbf{u}}{\partial \theta} = \lim_{r \rightarrow 0} \frac{\partial \mathbf{v}}{\partial \theta} = \lim_{r \rightarrow 0} \frac{\partial T}{\partial \theta} = \lim_{r \rightarrow 0} \frac{\partial P_g}{\partial \theta} = \lim_{r \rightarrow 0} \frac{\partial \phi}{\partial \theta} = 0 \quad (5.33)$$

The first expression may be expanded such that

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{\partial \mathbf{u}}{\partial \theta} = \lim_{r \rightarrow 0} \left(\frac{\partial}{\partial \theta} (\phi u_r) \mathbf{e}_r + \phi u_r \frac{\partial \mathbf{e}_r}{\partial \theta} + \frac{\partial}{\partial \theta} (\phi u_\theta) \mathbf{e}_\theta \right. \\ \left. + \phi u_\theta \frac{\partial \mathbf{e}_\theta}{\partial \theta} + \frac{\partial}{\partial \theta} (\phi u_z) \mathbf{e}_z + \phi u_z \frac{\partial \mathbf{e}_z}{\partial \theta} \right) \end{aligned} \quad (5.34)$$

Making use of the identities $\partial \mathbf{e}_r / \partial \theta = \mathbf{e}_\theta$, $\partial \mathbf{e}_\theta / \partial \theta = -\mathbf{e}_r$, and $\partial \mathbf{e}_z / \partial \theta = 0$, the preevious expression may be written as:

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{\partial \mathbf{u}}{\partial \theta} = \mathbf{e}_r \left(\frac{\partial}{\partial \theta} (\phi u_r) - \phi u_\theta \right) + \mathbf{e}_\theta \left(\frac{\partial}{\partial \theta} (\phi u_\theta) + \phi u_r \right) \\ + \mathbf{e}_z \left(\frac{\partial}{\partial \theta} (\phi u_z) \right) \end{aligned} \quad (5.35)$$

Linearizing about the base state, this becomes:

$$\lim_{r \rightarrow 0} \frac{\partial \mathbf{u}}{\partial \theta} = \mathbf{e}_r (in \bar{\phi} u'_r - \bar{\phi} u'_\theta) + \mathbf{e}_\theta (in \bar{\phi} u'_\theta + \bar{\phi} u'_r) + \mathbf{e}_z (in \bar{\phi} u'_z) \quad (5.36)$$

Applying the same analysis, the linearized expression for the fluid-phase velocity may be written:

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{\partial \mathbf{v}}{\partial \theta} = \mathbf{e}_r (in(1 - \bar{\phi}) u'_r - (1 - \bar{\phi}) u'_\theta) + \mathbf{e}_\theta (in(1 - \bar{\phi}) u'_\theta + (1 - \bar{\phi}) u'_r) \\ + \mathbf{e}_z (in(1 - \bar{\phi}) u'_z) \end{aligned} \quad (5.37)$$

For the equalities to hold, each component of the vectors in equations 5.36 and 5.37 must equal zero. With this, the previous expressions, along with the linearized version of the last three terms in equation 5.33, yield the following boundary conditions based upon the azimuthal wavenumber n :

$$in u'_r - u'_\theta = 0$$

$$\begin{aligned}
inu'_\theta + u'_r &= 0 \\
nu'_z &= 0 \\
inv'_r - v'_\theta &= 0 \\
inv'_\theta + v'_r &= 0 \\
(1 - \bar{\phi})nv'_z - n\bar{v}_z\phi' &= 0 \\
nT' = nP'_g = n\phi' &= 0
\end{aligned} \tag{5.38}$$

The conditions depend upon the value of the azimuthal wavenumber n . For the axisymmetric disturbance, $n = 0$, the boundary conditions are:

$$u'_r = u'_\theta = v'_r = v'_\theta = \frac{\partial u'_z}{\partial r} = \frac{\partial v'_z}{\partial r} = \frac{\partial T'}{\partial r} = 0 \tag{5.39}$$

For $|n| = 1$, two of the expressions become linearly dependent for each of the phase velocities. Following the work of Khorrami et al. [6], we take the limit of the linearized phasic continuity equations as $r \rightarrow 0$:

$$\lim_{r \rightarrow 0} \left[\sigma\phi' + \bar{\phi}\frac{\partial u'_r}{\partial r} + \frac{\bar{\phi}}{r}u'_r + \frac{in\bar{\phi}}{r}u'_\theta + ik\bar{\phi}u'_z \right] = 0 \tag{5.40}$$

$$\lim_{r \rightarrow 0} \left[-\sigma\phi' + (1 - \bar{\phi}) \left(\frac{\partial v'_r}{\partial r} + \frac{v'_r}{r} + \frac{in}{r}v'_\theta + ikv'_z \right) - ik\phi' \right] = 0 \tag{5.41}$$

With this, the centerline conditions for $|n| = 1$ become:

$$\begin{aligned}
iu'_r - u'_\theta &= iv'_r - v'_\theta = 0 \\
2\frac{\partial u'_r}{\partial r} + i\frac{\partial u'_\theta}{\partial r} &= 2\frac{\partial v'_r}{\partial r} + i\frac{\partial v'_\theta}{\partial r} = 0 \\
u'_z = v'_z = T' &= 0
\end{aligned} \tag{5.42}$$

And, for $|n| > 1$:

$$u'_r = u'_\theta = u'_z = v'_r = v'_\theta = v'_z = T' = 0 \tag{5.43}$$

5.4.1 Numerical Method

The numerical method used to solve the problem is much the same as the Chebyshev method described 6.5, with some minor modifications to account for the cylindrical-polar coordinate system. The specifics are deferred until that section, although the solution procedure is the same. The transformation from the domain $r : [0, 1]$ to $\alpha : [1, -1]$ is achieved using the relation

$$\alpha = 1 - 2r \tag{5.44}$$

When dealing with polar coordinates, the radial ordinate appears explicitly in the linearized equations. The transformed ordinate may be implemented by replacing r_j with $\frac{1-\alpha_j}{2}$. A staggered grid scheme was again employed for the ϕ' and P'_g variables. The solid-phase continuity (5.18) and fluid-phase θ -momentum (5.21) equations were resolved on the maxima of the N -degree polynomial, while the rest were resolved on the peaks of the polynomial of degree $N + 1$. A set of matrices is used to interpolate between the two grids [7]. The discretized equations are represented as

$$\sigma \mathbf{L}\psi = \mathbf{R}\psi \quad (5.45)$$

where $\mathbf{L}$ and $\mathbf{R}$ are square matrices of order $9N+7$. The individual components of the system are presented in Appendix B. The system (5.45) was solved using a complex QR solver available in MATLAB. The singularity in $\mathbf{R}$ is removed through row and column operations, reducing the rank of the matrices to $9N-7$. The leading and dominant eigenvalues are defined as before.

Convergence was determined by increasing the degree of the polynomial until the first several leading eigenvalues became independent of N . Convergence was generally achieved with $N = 60$ or less points. For the limiting case of axisymmetric disturbances and zero shear and conduction at the boundaries, only 20 points were required. The numerical scheme was validated by accurately reproducing the uniform shear case of the plane Couette flow of a granular medium studied by Alam & Nott [8], along with the pipe and narrow gap annulus Poiseuille flow stability results of Khorrami [7]. Accuracy was also checked through favorable comparison with the analytic solution of the limiting case presented in section 7.2.

Results are presented in terms of the dimensionless variables k , n , σ , Re , Fr , R^{-1} , L , and $\bar{\phi}$. The specular coefficient is assigned a value of 0.6 – unless otherwise noted – representing a moderate level of particle slip at the wall. The particle-particle restitution coefficient is assigned a nearly-elastic value of 0.95 in all of the computations except for one data set.

5.5 Results

Growth constants of the leading eigenvalue for the first several azimuthal modes are shown in figure 5.1(a). The leading eigenvalue at lower values of k correspond to different modes than the dominant disturbances for each of the angular modes n . This

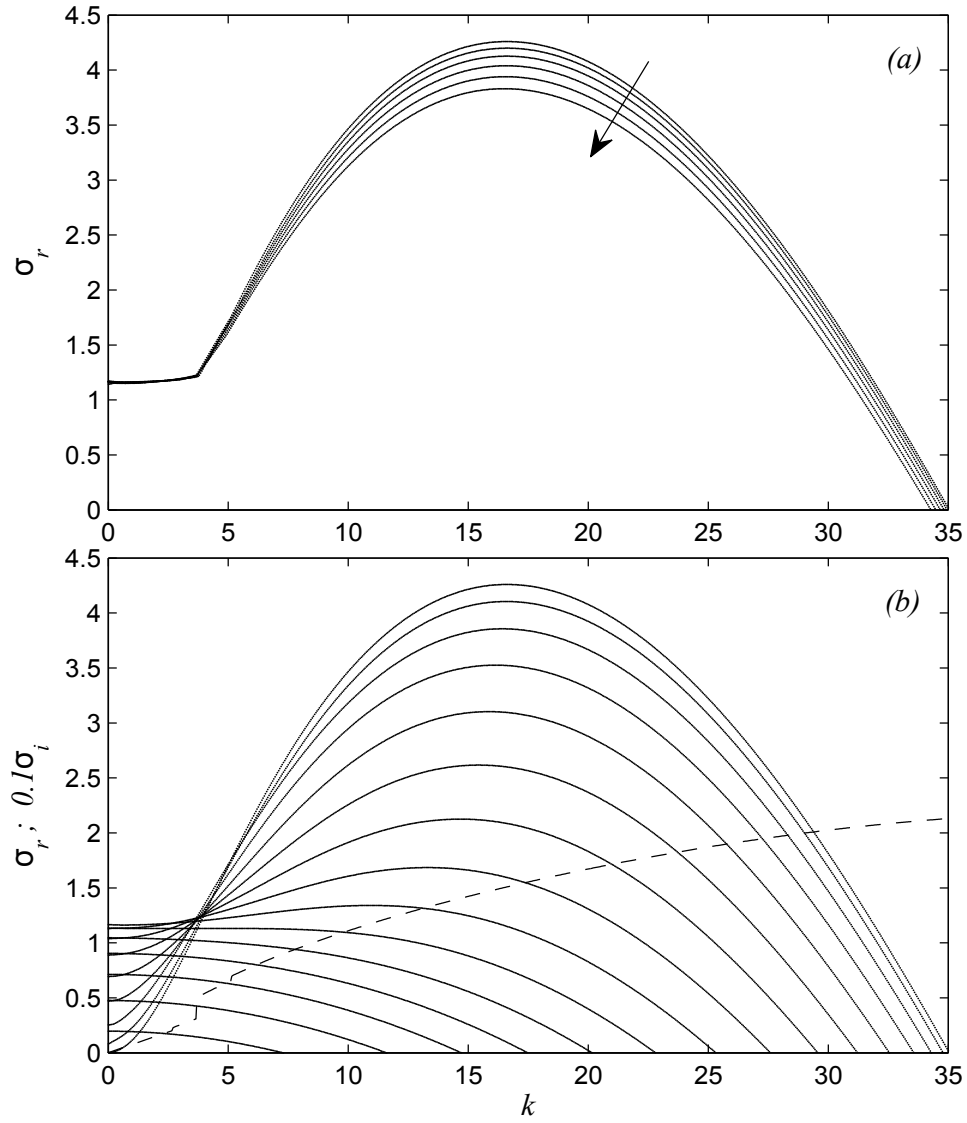


Figure 5.1: (a) Growth constants of the leading mode for the first several azimuthal modes $n = 0-5$, marked by arrow in increasing order, and (b) Growth constants (dots) of the first several leading modes, as well as the frequency (dashes) of the leading mode, for the axisymmetric disturbance, as a function of k . $H = 100$, $Re = 0.6326$, $Fr = 1.1472$, $R = 4.8 \cdot 10^{-4}$.

behavior is highlighted in figure 5.1(b), which presents growth constant traces of the first several leading eigenvalues for the $n = 0$ mode. The trace containing the dominant mode has a value of $\sigma_r = 0$ at $k = 0$, increases to the maximum value σ_{rD} , and decreases monotonically thereafter. At lower values of the longitudinal wavenumber k , there is a cascade of modes from one to the next. The leading mode in this region is one out of several which have the highest growth constant at a particular value of k . This is reflected in the dashed trace corresponding to the disturbance frequency of the leading mode. This frequency increases monotonically over the relevant wavenumber space. At small values of k , discontinuous jumps in the frequency denote the wavenumber values at which the leading mode switches from one to the next.

Some observations may be gathered from the growth constant traces in figure 5.1. First, the system supports disturbances that encompass a wide range of variability in the azimuthal and radial directions. This is different behavior, for example, than that seen in the stability of single-fluid Pipe Poiseuille flow [7]. In such a system, the $|n| = 1$ mode is expected to be the least stable, in the sense that its dominant travelling mode has a larger growth constant than that of any other value of n . The first non-axisymmetric mode allows two of the perturbed velocity components to be non-zero at the centerline, as opposed to the one allowed by axisymmetric disturbances. The lack of constraint for $|n| = 1$, combined with the larger amount of viscous dampening experienced for modes of $|n| > 1$, contributes to this mode hosting the dominant disturbance.

The same does not hold true for the current system. Aside from having different instability mechanisms, and in spite of the same velocity component relationship holding for both phasic velocities, the current system has two more perturbed flow variables that are zero at the centerline for $|n| = 1$: ϕ' and T' . Furthermore, it appears from the growth constant traces of figure 5.1(a) that the primary effect of changing the azimuthal wavenumber is an increased level of viscous and pseudo-thermal conductive dampening. This is opposed to the longitudinal wavenumber variation seen in figure 5.1(b), in which the largest growth constant occurs at some non-zero value of k .

Eigenvectors of the first several azimuthal modes are shown in figure 5.2. The dominant mode of the axisymmetric disturbance exhibits a large voidage perturbation concentrated at the centerline. Eigenfunctions of ϕ' for the $n = 1$ and $n = 3$ modes exhibit periodic regions of voidage and dense plugs between the centerline and the wall, while the $n = 2$ mode contains these in addition to thin regions of dense and light

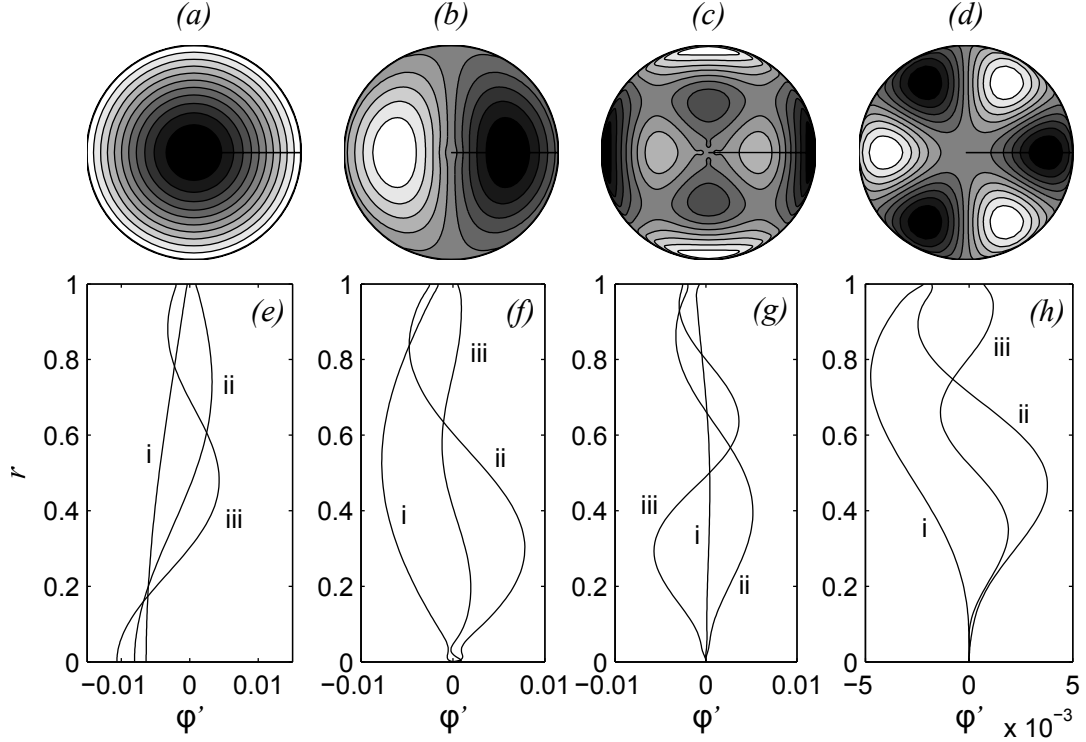


Figure 5.2: Eigenvectors of ϕ' for the (a,e) $n = 1$ (b,f) $n = 2$ (c,g) $n = 3$ (d,h) $n = 4$ azimuthal modes. The latter four plots depict the radial profiles of ϕ' for the first several leading eigenvalues of each angular mode, in order of decreasing growth constant: i , ii , iii . $L = 100$, $Re = 0.6326$, $Fr = 1.1472$, $R = 4.8 \cdot 10^{-4}$.

concentrations near the wall. The radial profiles of ϕ' for the first three leading modes of the axisymmetric disturbance seen in figure 5.2(e) resemble the first three modes of the limiting case of section 7.2, of which i is a physically-relevant modification of the previous uniform disturbance ($\Theta = 0$) resulting from the partial-slip boundary conditions of the solid-phase velocity.

Interesting to note is that the eigenvectors i , ii , and iii of figure 5.2(e) correspond to the growth constant traces with the three largest values of σ_R seen in figure 5.1(b), in order of decreasing σ_{rL} . Although only the eigenvectors of the first three leading modes are shown, this pattern continues down the line, with an increased level of variation in the radial direction associated with a lower growth constant for the large- k travelling modes. Thus we see that the modes with large growth constants at small values of k exhibit large levels of fluctuation in the radial direction. Also notable is how the leading modes (i) of the first four azimuthal disturbances shown in figures 5.2(e-h) exhibit similar amounts of variation in ϕ' with the radial direction. The same approximate longitudinal wavenumber k at which these modes support the largest growth constant – see figure 5.1(a) – suggests that periodicity in the angular direction serves only to dampen perturbations via solid-phase viscous stress and conduction of PTE.

Contours of the growth constant and frequency of the first two azimuthal modes are presented in the $Re - k$ plane in figure 5.3. Again the Reynolds number was changed by varying the relative fluidization velocity. As expected from the similar growth constant behavior seen in figure 5.1(a), the behavior and magnitude of each quantity is similar for $n = 0$ and $n = 1$. Although it is not immediately apparent from the plots, the growth constant of the axisymmetric mode is greater, by a narrow margin, than that for $n = 1$ over the entire parameter plane. The stationary disturbance is dominant at the lowest values of Re for each mode. As the Reynolds number increases, however, the travelling mode becomes dominant. Also visible is the rapid cascade of leading modes for disturbances of small k as the Reynolds number is increased. The plots of figure 5.3 give a clearer picture of the leading-mode dependence on Re and k first observed in figure 5.1(b). After a certain value of the longitudinal wavenumber, as Re increases the leading mode switches from the modes exhibiting dramatic fluctuations in the radial direction to the least-variant, dominant travelling mode. These travelling modes, which are leading over the majority of the plots in figure 5.1, correspond to the i profiles in figures 5.2(e-f). When the travelling mode is dominant, the disturbance frequency decreases with Re .

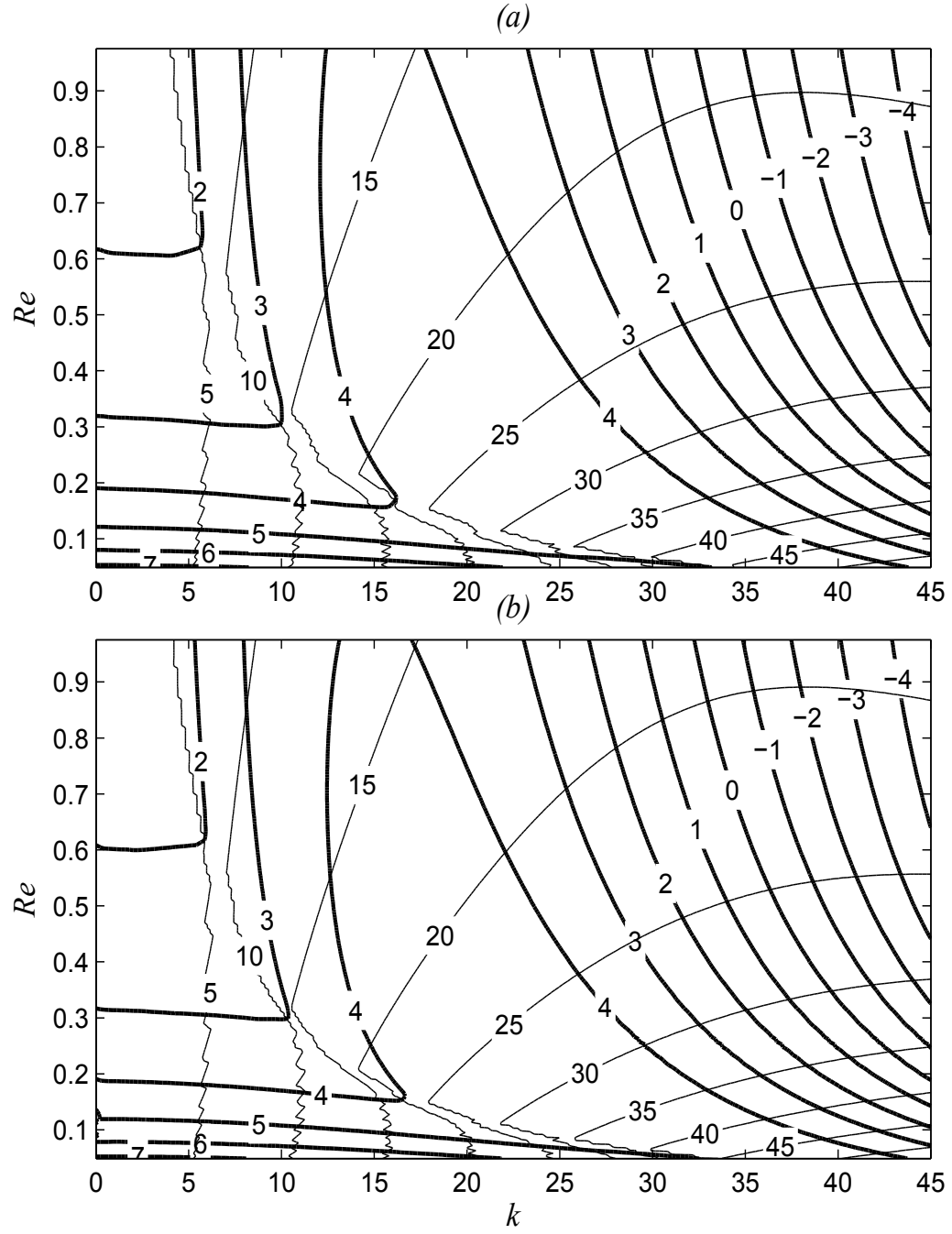


Figure 5.3: (a) Contours of the growth constant (thick line) and frequency (thin line) as a function of Re and k for the (a) $n = 0$ and (b) $n = 1$ modes. $L = 100$, $R = 4.8 \cdot 10^{-4}$.

The growth constant of the stationary mode decreases monotonically with the Reynolds number. Also noteworthy is how the range of k over which disturbances are unstable decreases with Re .

As it turns out, the axisymmetric ($n = 0$) mode is dominant over all parameter ranges considered. As such, further results will be presented in terms of this azimuthal mode only. The longitudinally-travelling disturbance ($k \neq 0$) is dominant throughout most parameter combinations. However, at certain parameter extremes, the non-oscillatory stationary mode $k = 0$ becomes dominant. For this reason growth constant and frequency results at a particular parameter combination will be presented in terms of both travelling and stationary modes.

Figure 5.4 displays contour plots of the travelling-mode growth constant and its corresponding frequency and wavenumber, along with the growth constant of the stationary mode. Here, L is varied by changing $\hat{L}$ while holding the particle diameter constant. Each point on the graph represents the leading travelling or stationary mode at that parameter combination. In this plot, the axisymmetric disturbance exhibited the largest growth constant. The growth constants of the travelling and stationary modes both increase with L and $\bar{\phi}$, although the former disturbance becomes stifled as the base-state bed approaches its closely-packed limit. However, considering the non-dimensionalization of the complex frequency using length scale $\hat{L}$, it is apparent that the dimensional growth rate of the travelling mode is virtually independent of the dimensional gap width $\hat{L}$. The travelling mode is the dominant disturbance over the entire $L - \bar{\phi}$ range considered, except for the very upper-right portion of the graph. It appears that, for closely-packed beds with a very large dimensionless gap width L , the stationary mode will be dominant. The frequency of the dominant travelling mode, σ_{iD} , as well as the corresponding wavenumber k_D , exhibits similar behavior as the growth constant.

Contours of the stationary and travelling mode growth constants, as well as the frequency and wavenumber of the latter, are presented in figure 5.5 as a function of the phase density ratio and the quantity

$$(Re^2/Fr)^{1/3} = \left(\frac{g\rho_g^2}{\hat{\mu}_g^2} \right)^{1/3} d_p \quad (5.46)$$

which is essentially the cube root of the Galileo number. Use of this quantity allows the results to be presented as a function of varying particle diameter, while holding

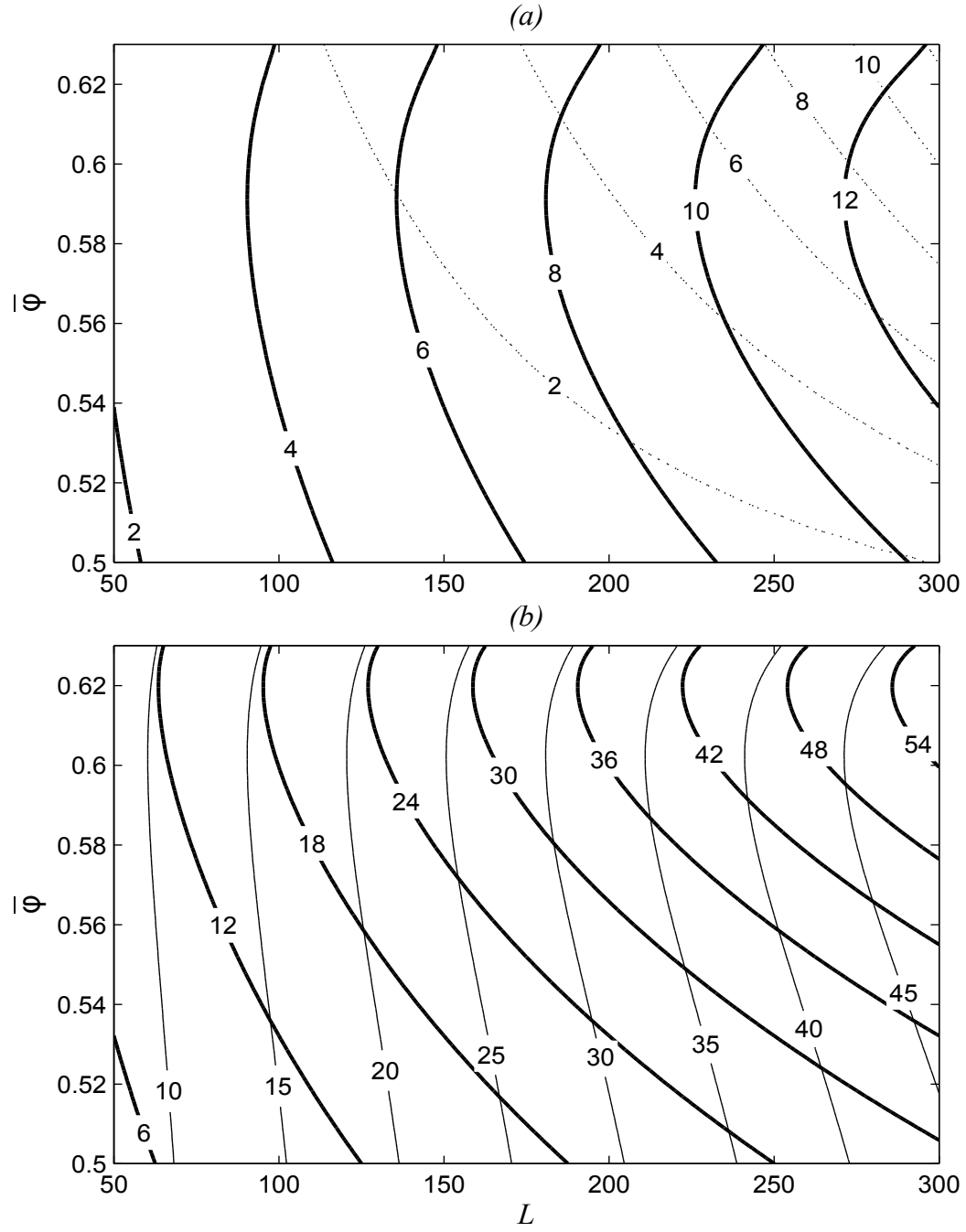


Figure 5.4: (a) Growth constant contours of the dominant travelling mode (solid lines) and the stationary mode (dashed lines), and (b) frequency (thick lines) and wavenumber (thin lines) corresponding to the dominant travelling mode, as a function of L and $\bar{\phi}$. $R = 4.8 \cdot 10^{-4}$.

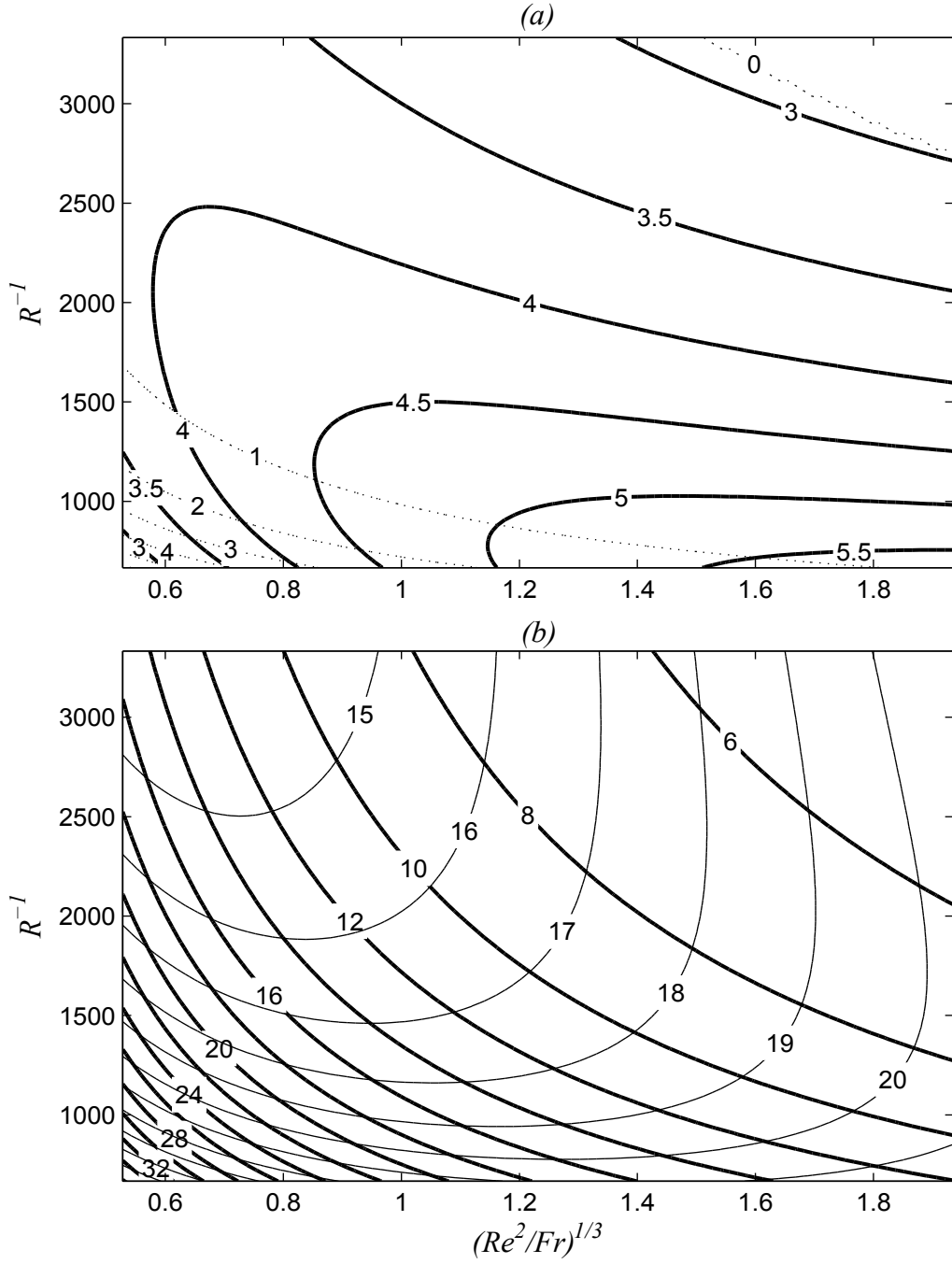


Figure 5.5: (a) Growth constant contours of the dominant travelling mode (solid lines) and the stationary mode (dashed lines), and (b) frequency (thick lines) and wavenumber (thin lines) corresponding to the dominant travelling mode, as a function of R^{-1} and $(Re^2/Fr)^{1/3}$. $L = 100$.

the relative fluidization velocity U/u_{mf} constant. As such, figure 5.5 plots the stability characteristics of the dominant modes as a function of varying particle diameter and density. In this way the current results may be compared to the empirical data of Geldart [9], who mapped the tendency of a particle assembly to bubble at minimum fluidization as a function of these two parameters. An interesting parallel exists between the two data sets. The lower-left-hand corner of figure 5.5 corresponds to the Geldart A classification, the particles of which are lightweight and very small. Fluidized beds containing this particle type exhibit a significant amount of bed expansion before the appearance of bubbles, which begin to form at fluidization velocities significantly past that of u_{mf} . The upper-right-hand portion of figure 5.5 represents Geldart D particle size and density combinations. Particles so characterized are large and heavy and generally do not bubble at lower fluidization velocities, instead giving way to spouting at significantly higher values of U/u_{mf} . The center portion of the plots in figure 5.5 correspond to the Geldart B regime. In contrast to the other regimes, particles so characterized exhibit bubbling right at minimum fluidization. A qualitative similarity exists between the experimentally-observed tendency for bubbling at minimum fluidization and the growth constant of the dominant travelling mode as a function of particle diameter and density. The stationary mode, however, does become dominant at the lowest values of particle density and diameter tested.

Figure 5.6 displays growth constant and frequency traces of the axisymmetric leading mode for various values of e_p . The strong destabilizing effect of particle inelasticity is apparent, with the perfectly elastic particle assembly exhibiting the lowest dominant travelling growth rate. The decrease in growth constant becomes increasingly more dramatic as e_p rises. The growth constant of the stationary ($k = 0$) mode, however, increases with increasing particle elasticity, although the travelling mode remains dominant even at $e_p = 1.0$. Also notable is the decrease in the wavenumber and frequency of the dominant disturbance as the particle elasticity increases. Likewise, the wavenumber range over which disturbances are unstable shrinks with increasing e_p . The value of the wavenumber at which the travelling short-wavelength travelling mode becomes leading over the small- k disturbance increases slightly with the restitution coefficient.

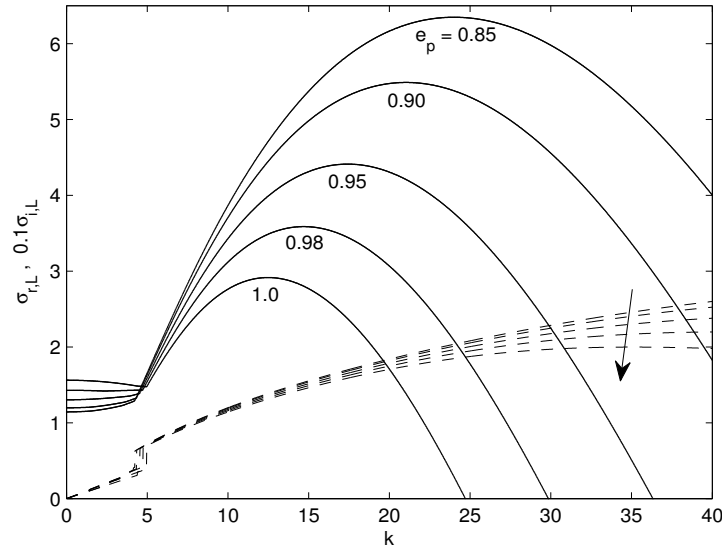


Figure 5.6: Growth constant (dots) and frequency (dashes) of the leading $n = 0$ mode at various values of e_p , as marked. Frequencies are marked by arrow in order of increasing e_p . $L = 100$, $Re = 0.6326$, $Fr = 1.1472$, $R = 4.8 \cdot 10^{-4}$.

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Chapter 5, in part, has been submitted for publication in *Physics of Fluids*, "Stability of a Vertical, Gas-Fluidized Bed," by K. Mandich and R. Cattolica. The thesis author is the primary investigator in this publication.

Chapter 6

Stability of an Inclined Bed

6.1 Problem Description

The first bounded system under consideration consists of an assembly of particles with mean diameter d_p and bound by walls which are inclined at an angle θ to the direction of gravity. The direction perpendicular to the base-state fluid velocity – the lateral direction – is represented by the ordinate y . The domain is defined as $-\hat{L} \leq y \leq \hat{L}$, where $y = \hat{L}$ denotes the top of the domain. Based upon the results of the analytic solutions presented in chapter 7, a 2D planar bed is considered as it is expected to be sufficient in describing the stability of the inclined system.

Inclination of the fluidized bed results in a considerably more complex problem. This is manifested through the non-zero body force term present in the lateral momentum equations. Because of this, the simplifying assumptions ($e_w = 1.0$, $\hat{\mu}_g \rightarrow 0$) made for the vertical bed formulations may not be used. When solving the steady-state equations, one must allow for shear in the base-state fluid velocity, as well as a non-uniform granular temperature.

For vertical and inclined fluidized beds with mean fluid velocity at or near minimum fluidization, it is observed in the experiment that there exists no bulk particle velocity – a non-zero $\mathbf{u}_s$ exists only when a perturbation develops into a bubble and propagates through the medium. Therefore, the base-state solution developed in the current chapter is valid for an ideal case in which the particles are suspended in a flow with constant fluid-phase pressure drop in the axial direction, which extends to infinity without interruption. In experiment, there does exist some lateral fluid-phase pressure

distribution resulting from the presence of the distributor plate. However, because of the difficulties in determining this distribution, the assumption of a constant pressure drop is employed. This will be seen when comparing the stability results of the limiting case of the present formulation when $\theta = 0^\circ$ to that of the vertical planar fluidized bed results. The similarity will be shown to stem from the difference between the base-state velocities $(\bar{u}_g - \bar{u}_s)$ and the instability mechanism $\bar{\beta}_\phi$.

6.2 Governing Equations

The dimensional governing equations are the same as those employed in the previous chapter, and are not reproduced here. However, due to the nature of the problem, a different set of dimensionless variables is used. Due to the asymmetry inherent with a tilted bed, the base-state fluid velocity and volume fraction may not be specified. Instead, the constant fluid-phase pressure drop must be specified, and the base-state variables obtained from solving the steady-state equations of motion.

As such, the only relevant velocity scale with which to non-dimensionalize the equations of motion is the particle terminal velocity $\hat{u}_t$. In addition, the fluid-phase pressure is now non-dimensionalized using the weight of the solid bed per unit height. Doing so allows one to maintain a constant specified pressure drop while varying the dimensionless bed width L . The dimensionless variables are then:

$$\begin{aligned} (x, y) &= \frac{1}{\hat{L}}(\hat{x}, \hat{y}), \quad t = \frac{\hat{u}_t}{\hat{L}}\hat{t}, \quad (\mathbf{u}_g, \mathbf{u}_s) = \frac{1}{\hat{u}_t}(\hat{\mathbf{u}}_g, \hat{\mathbf{u}}_s), \quad P_g = \frac{\hat{P}_g}{\rho_s g \hat{L}} \\ Re &= \frac{\rho_s \hat{u}_t d_p}{\mu_g}, \quad R = \frac{\rho_g}{\rho_s}, \quad Fr = \frac{\hat{u}_t^2}{g d_p}, \quad L = \frac{\hat{L}}{d_p} \end{aligned} \quad (6.1)$$

With this, the domain of the system becomes $-0.5 \leq y \leq 0.5$, and $y = 0.5$ corresponds to the top of the domain when the bed is tilted. The resulting dimensionless equations of motion are:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}_s) = 0 \quad (6.2)$$

$$\frac{\partial(1-\phi)}{\partial t} + \nabla \cdot (1-\phi) \mathbf{u}_g = 0 \quad (6.3)$$

$$\begin{aligned} \frac{\partial(\phi \mathbf{u}_s)}{\partial t} + \nabla \cdot (\phi \mathbf{u}_s \mathbf{u}_s) &= -\nabla \cdot \left(P_s \mathbf{I} - \frac{\mu_s}{L} (\nabla \mathbf{u}_s + \nabla \mathbf{u}_s^T) - \frac{\zeta_s}{L} (\nabla \cdot \mathbf{u}_s) \right) \\ &\quad - \phi \nabla \cdot \left(\frac{L}{Fr} P_g \mathbf{I} - \frac{1}{L Re} (\nabla \mathbf{u}_g + \nabla \mathbf{u}_g^T) - \frac{2}{3 L Re} (\nabla \cdot \mathbf{u}_g) \right) \\ &\quad + \frac{L}{Fr} (1-R)(1-\phi) \beta (\mathbf{u}_g - \mathbf{u}_s) + \phi \frac{L}{Fr} \mathbf{f} \end{aligned} \quad (6.4)$$

$$\begin{aligned}
\frac{\partial((1-\phi)\mathbf{u}_g)}{\partial t} + \nabla \cdot ((1-\phi)\mathbf{u}_g\mathbf{u}_g) = & - (1-\phi)\nabla \cdot \left(\frac{LR^{-1}}{Fr} P_g \mathbf{I} - \frac{1}{LRe} (\nabla \mathbf{u}_g + \nabla \mathbf{u}_g^T) \right. \\
& \left. - \frac{2}{3LRe} (\nabla \cdot \mathbf{u}_g) \right) + (1-\phi) \frac{L}{Fr} \mathbf{f} \\
& - \frac{L}{Fr} (R^{-1} - 1)(1-\phi)\beta(\mathbf{u}_g - \mathbf{u}_s)
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\frac{3}{2} \phi T \right) + \nabla \cdot \left(\frac{3}{2} \phi T \mathbf{u}_s \right) = & - \left(P_s \mathbf{I} - \frac{\mu_s}{L} (\nabla \mathbf{u}_s + \nabla \mathbf{u}_s^T) - \frac{\zeta_s}{L} (\nabla \cdot \mathbf{u}_s) \right) : \nabla \mathbf{u}_s \\
& + \frac{L}{Re^2} |\mathbf{u}_g - \mathbf{u}_s|^2 \Gamma_s - L J_c - \frac{L(1-R)}{Fr} J_v \\
& + \nabla \cdot \left(\frac{\lambda}{L} \nabla T + \frac{\lambda_h}{L} \nabla \phi \right)
\end{aligned} \tag{6.6}$$

where the dimensionless variables are written:

$$\begin{aligned}
\beta(\phi) &= f_0(\phi) & P_s(\phi, T) &= f_1(\phi)T & \mu_s(\phi, T) &= f_2(\phi)T^{1/2} \\
\zeta_s(\phi, T) &= \left(f_3(\phi) - \frac{2}{3} f_2(\phi) \right) T^{1/2} & \lambda(\phi, T) &= f_4(\phi)T^{1/2} & \lambda_h(\phi, T) &= f_{4h}(\phi)T^{3/2} \\
\Gamma_s(\phi, T) &= f_5(\phi)T^{-1/2} & J_c(\phi, T) &= f_6(\phi)T^{3/2} & J_v(\phi, T) &= 3f_0(\phi)T
\end{aligned} \tag{6.7}$$

6.3 Base-State Solution

The base-state variables are defined as:

$$\begin{aligned}
\mathbf{u}_s &= (\bar{u}_s(y), 0)^T, \mathbf{u}_g = (\bar{u}_g(y), 0)^T, T = \bar{T}(y), \phi = \bar{\phi}(y) \\
\beta &= \bar{\beta}(y), P_g = \bar{P}_g(x), P_s = \bar{P}_s(y), \mu_s = \bar{\mu}_s(y) \\
\zeta_s &= \bar{\zeta}_s(y), \Gamma_s = \bar{\Gamma}_s(y), J_c = \bar{J}_c(y), J_v = \bar{J}_v(y)
\end{aligned} \tag{6.8}$$

An overbar denotes the base-state value of a variable. Applying these values to 6.4-6.6, we obtain:

$$\frac{L(1-R)}{Fr} \bar{\beta} (\bar{u}_g - \bar{u}_s) + \frac{1}{L} \frac{\partial}{\partial y} \left(\bar{f}_2 \bar{T}^{1/2} \frac{\partial \bar{u}_s}{\partial y} \right) - \frac{L(1-R) \cos(\theta)}{Fr} \bar{\phi} = 0 \tag{6.9}$$

$$- \frac{L}{Fr} \frac{d\bar{P}_g}{dx} - \frac{L(1-R)}{Fr} \bar{\beta} (\bar{u}_g - \bar{u}_s) + \frac{1}{LRe} \frac{\partial^2 \bar{u}_g}{\partial y^2} - \frac{LR \cos(\theta)}{Fr} \bar{\phi} = 0 \tag{6.10}$$

$$\frac{\partial}{\partial y} (\bar{f}_1 \bar{T}) + \frac{L \sin(\theta)}{Fr} \bar{\phi} = 0 \tag{6.11}$$

$$\begin{aligned}
& \frac{1}{L} \bar{f}_2 \bar{T}^{1/2} \left(\frac{\partial \bar{u}_s}{\partial y} \right)^2 + \frac{1}{L} \frac{\partial}{\partial y} \left(\bar{f}_4 \bar{T}^{1/2} \frac{\partial \bar{T}}{\partial y} + \bar{f}_{4h} \bar{T}^{3/2} \frac{\partial \bar{\phi}}{\partial y} \right) \\
& - L \bar{f}_6 \bar{T}^{3/2} - \frac{3L(1-R)}{Fr} \bar{f}_o \bar{T} + \frac{L}{Re^2} |\bar{u}_g - \bar{u}_s|^2 \bar{f}_5 \bar{T}^{-1/2} = 0
\end{aligned} \tag{6.12}$$

This is subject to the boundary conditions

$$\pm \frac{\partial \bar{u}_g}{\partial y} + \frac{L^2 Re}{Fr} \frac{\delta}{\phi_p} \bar{\beta} \phi (\bar{u}_g - \bar{u}_s) + \frac{\phi_p}{\delta \phi} \bar{T} \bar{u}_g + \frac{L^2 Re}{Fr} \frac{\delta}{\phi_p} \frac{d\bar{P}_g}{dx} \bar{\phi} = 0, \quad y = \pm 0.5 \quad (6.13)$$

$$\pm \left(\bar{f}_4 \frac{\partial \bar{T}}{\partial y} + \bar{f}_{4h} \bar{T} \frac{\partial \bar{\phi}}{\partial y} \right) = L \bar{f}_8 \bar{u}_s^2 - L \bar{f}_7 \bar{T}, \quad y = \pm 0.5 \quad (6.14)$$

$$\pm \bar{f}_2 \frac{\partial \bar{u}_s}{\partial y} = L \bar{f}_8 \bar{u}_s, \quad y = \pm 0.5 \quad (6.15)$$

Derived from the partial-slip condition of Sinclair & Jackson [1], 6.13 is a force balance in the region of thickness $\delta(\phi/\phi_p)$ next to the wall. The latter two conditions, developed by Johnson & Jackson [2], equate the dissipation of energy at the boundary to the energy flux normal to the wall (6.14), and tangential solid-phase stress at the wall to the tangential momentum flux due to particle-wall collisions (6.15).

As with previous studies which considered non-inclined granular systems [1–4], the base-state system 6.9-6.15 must be solved by specifying either the mean volume fraction in the domain or the constant solid-phase pressure of the non-inclined system, $\bar{f}_1 \bar{T}$. The second choice provides a simpler means of solving the system and is employed. The lateral solid-phase momentum equation 6.11 is integrated to obtain an algebraic relation between $\bar{T}$ and $\bar{\phi}$:

$$\bar{f}_1 \bar{T} + \frac{L \sin(\theta)}{Fr} \int_Y^{0.5} \bar{\phi} dY = C_1 \quad (6.16)$$

Here, C is the specified dimensionless solid-phase pressure at the top of the domain ($y = 0.5$). With this, the number of equations and unknowns is reduced to three. In agreement with previous studies of gas-fluidized beds [5] it was found that ignoring the viscous and body force terms of the fluid phase has a negligible effect on the solution. For this reason, these terms are ignored in both the base-state solution and the stability analysis.

6.4 Linearization and Assumed Form of Perturbation

The governing equations 6.2-6.6 are again linearized by perturbing the variables about the base state:

$$\psi(x, y, t) = \bar{\psi}(y) + \tilde{\psi}(x, y, t) \quad (6.17)$$

where $\tilde{\psi}$ is the seven-term perturbation vector defined as

$$\tilde{\psi} = (\tilde{u}_s \tilde{u}_g \tilde{v}_s \tilde{v}_g \tilde{P}_g \tilde{\phi} \tilde{T})^T \quad (6.18)$$

Normal-mode solutions are sought:

$$\tilde{\psi}(x, y, t) = \psi'(y) \exp[ikx + \sigma t] \quad (6.19)$$

Inserting this into the governing equations 6.2-6.6, one obtains the linearized equations

$$\sigma \phi' = -ik\bar{\phi}u'_s - ik\bar{u}_s\phi' - \left(\bar{\phi}_y + \bar{\phi} \frac{\partial}{\partial y}\right) v'_s \quad (6.20)$$

$$-\sigma \phi' = ik(1 - \bar{\phi})u'_g - ik\bar{u}_g\phi' + \left(-\bar{\phi}_y + (1 - \bar{\phi}) \frac{\partial}{\partial y}\right) v'_g \quad (6.21)$$

$$\begin{aligned} \sigma \left(u'_g - \frac{\bar{u}_g}{(1 - \bar{\phi})} \phi'\right) = & -ik\bar{u}_g u'_g - \frac{ik\bar{u}_g^2}{1 - \bar{\phi}} \phi' + \bar{u}_{gy} v'_g - \frac{\bar{u}_g \bar{\phi}_y}{1 - \bar{\phi}} v'_g - ik \frac{LR^{-1}}{Fr} P'_g \\ & + \frac{L(R^{-1} - 1)}{Fr} [\bar{\beta}(u'_s - u'_g) + \bar{\beta}_\phi(\bar{u}_s - \bar{u}_g)\phi'] \end{aligned} \quad (6.22)$$

$$\sigma v'_g = -ik\bar{u}_g v'_g - ik \frac{LR^{-1}}{Fr} \frac{\partial P'_g}{\partial y} + \frac{L(R^{-1} - 1)}{Fr} \bar{\beta}(v'_s - v'_g) \quad (6.23)$$

$$\begin{aligned} \sigma(\bar{\phi}u'_s + \bar{u}_s\phi') = & -ik\bar{u}_s\bar{\phi}u'_s - ik\bar{u}_s^2\phi' - \bar{\phi}\bar{u}_{sy}v'_s - \bar{u}_s\bar{\phi}_y v'_s - ik\bar{\phi} \frac{L}{Fr} P'_g - \frac{L \cos \theta}{Fr} \phi' \\ & - ik(\bar{P}_{s\phi}\phi' + \bar{P}_{sT}T') + \frac{L(1 - R)}{Fr} (1 - \bar{\phi}) [\bar{\beta}(u'_g - u'_s) + \bar{\beta}_\phi(\bar{u}_g - \bar{u}_s)\phi'] \\ & + \left(-k^2 \frac{2\bar{\mu}_s + \bar{\zeta}_s}{L} + \frac{\bar{\mu}_s}{L} \frac{\partial^2}{\partial y^2}\right) u'_s + \left(ik \frac{\bar{\mu}_s + \bar{\zeta}_s}{L} \frac{\partial}{\partial y} + \frac{ik}{L} \frac{\partial \bar{\mu}_s}{\partial y}\right) v'_s \end{aligned} \quad (6.24)$$

$$\begin{aligned} \sigma \bar{\phi} v'_s = & -ik\bar{\phi}\bar{u}_s v'_s - \bar{P}_{s\phi y} \phi' - \bar{P}_{s\phi} \frac{\partial \phi'}{\partial y} - \bar{P}_{sTy} T' - \bar{P}_{sT} \frac{\partial T'}{\partial y} - \bar{\phi} \frac{L}{Fr} \frac{\partial P'_g}{\partial y} - \frac{L \sin \theta}{Fr} \phi' \\ & + \left(-k^2 \frac{\bar{\mu}_s}{L} + \frac{2\bar{\mu}_s + \bar{\zeta}_s}{L} \frac{\partial^2}{\partial y^2} + \frac{1}{L} \frac{\partial}{\partial y} (2\bar{\mu}_s + \bar{\zeta}_s) \frac{\partial}{\partial y}\right) v'_s + \frac{1}{L} \frac{\partial \bar{u}_s}{\partial y} (\bar{\mu}_{s\phi} \phi' + \bar{\mu}_{sT} T') \\ & + \frac{L(1 - R)}{Fr} (1 - \bar{\phi}) \bar{\beta}(v'_g - v'_s) + \left(ik \frac{\bar{\mu}_s + \bar{\zeta}_s}{L} \frac{\partial}{\partial y} + \frac{ik}{L} \frac{\partial \bar{\zeta}_s}{\partial y}\right) u'_s \end{aligned} \quad (6.25)$$

$$\begin{aligned} \frac{3}{2} \sigma (\bar{\phi} T' + \bar{T} \phi') = & -\frac{3}{2} ik (\bar{\phi} \bar{T} u'_s + \bar{\phi} \bar{u}_s T' + \bar{u}_s \bar{T} \phi') + \frac{3}{2} \left(\bar{\phi} \bar{T} \frac{\partial v'_s}{\partial y} + \bar{\phi} \bar{T}_y v'_s + \bar{\phi}_y \bar{T} v'_s\right) \\ & - \bar{P}_s \left(iku'_s + \frac{\partial v'_s}{\partial y}\right) + \frac{2\bar{\mu}_s}{L} \frac{\partial \bar{u}_s}{\partial y} \frac{\partial u'_s}{\partial y} + \frac{1}{L} \left(\frac{\partial \bar{u}_s}{\partial y}\right)^2 (\bar{\mu}_{s\phi} \phi' + \bar{\mu}_{sT} T') \\ & + \frac{\bar{\lambda}}{L} \left(-k^2 + \frac{\partial^2}{\partial y^2}\right) T' + \frac{\bar{\lambda}_h}{L} \left(-k^2 + \frac{\partial^2}{\partial y^2}\right) \phi' \\ & + \frac{1}{L} \frac{\partial}{\partial y} \left(\frac{\partial \bar{T}}{\partial y} (\bar{\lambda}_\phi \phi' + \bar{\lambda}_T T')\right) + \frac{1}{L} \frac{\partial}{\partial y} \left(\frac{\partial \bar{\phi}}{\partial y} (\bar{\lambda}_{h\phi} \phi' + \bar{\lambda}_{hT} T')\right) \\ & + L (\Gamma_\phi \phi' + \Gamma_T T' + \Gamma_{u_g} u'_g + \Gamma_{u_s} u'_s) \end{aligned} \quad (6.26)$$

The linearized boundary conditions are, at $y = \pm 0.5$,

$$v'_g = v'_s = 0 \quad (6.27)$$

$$\begin{aligned} \pm \left(\bar{\phi} \frac{\partial u'_g}{\partial y} + \bar{u}_{gy} \phi' \right) + \frac{2L^2 Re(1-R)}{Fr} \frac{\delta}{\phi_p} \bar{\phi} \beta (\bar{u}_g - \bar{u}_s) \\ + \frac{\phi_p}{\delta} (\bar{T} u'_g + \bar{u}_g T') + \frac{2L^2 Re}{Fr} \frac{\delta}{\phi_p} \bar{\phi} \frac{d\bar{P}_g}{dx} = 0 \end{aligned} \quad (6.28)$$

$$\pm \left(\bar{f}_2 \frac{\partial u'_s}{\partial y} + \bar{f}_{2\phi} \frac{\partial \bar{u}_s}{\partial y} \phi' \right) - L (\bar{f}_8 u'_s + \bar{f}_{8\phi} \bar{u}_s \phi') = 0 \quad (6.29)$$

$$\begin{aligned} \pm \left(\bar{f}_4 \frac{\partial T'}{\partial y} + \bar{f}_{4\phi} \frac{\partial \bar{T}}{\partial y} \phi' + \bar{f}_{4h} \bar{T} \frac{\partial \phi'}{\partial y} + \bar{f}_{4h} \frac{\partial \bar{\phi}}{\partial y} T' + \bar{f}_{4h\phi} \bar{T} \frac{\partial \bar{\phi}}{\partial y} \phi' \right) \\ - L (2\bar{f}_8 \bar{u}_s u'_s + \bar{f}_{8\phi} \bar{u}_s^2 \phi') + L (\bar{f}_7 T' + \bar{f}_{7\phi} \bar{T} \phi') = 0 \end{aligned} \quad (6.30)$$

Here, for convenience, the terms representing dissipation and generation of PTE are lumped together:

$$\Gamma = \frac{L}{Re} |\bar{u}_g - \bar{u}_s|^2 \bar{f}_5 \bar{T}^{-1/2} - L \bar{f}_6 \bar{T}^{3/2} - \frac{3L(1-R)}{Fr} \bar{f}_o \bar{T} \quad (6.31)$$

To focus on the effects the tilting angle has on the stability of the flow, solutions of the current problem are limited to one particular parameter combination, with the exception of one data set in which the effects of the boundary condition parameters are explored. Attention is primarily given to the base-state solutions, the resulting unstable eigenfunctions, and the time-dependent growth of these unstable modes as a function of the tilting angle. The basic simulation parameters used are given in table 6.1, presented in dimensional form.

6.5 Numerical Method

The primary numerical methods used to solve the problem are based upon Chebyshev pseudo-spectral collocation. This class of numerical techniques was chosen for several reasons. First, Chebyshev methods exhibit rapid convergence rates as the number of grid points is increased, especially when compared to standard finite difference methods. The use of Chebyshev polynomials also distributes the error evenly over the domain and

Table 6.1: Basic simulation parameters employed. Values are given in dimensional variables.

d_p	Particle diameter	$400\mu m$
ρ_s	Particle density	$1000kgm^{-3}$
ρ_g	Gas density	$1.2kgm^{-3}$
μ_g	Gas viscosity	$0.000018kgm\ s^{-1}$
$\hat{L}$	Bed width	$4cm$
δ	Fluid velocity BC length scale	$5d_p$
e_p	Particle-particle restitution coefficient	0.99
e_w	Particle-wall restitution coefficient	0.9
ϕ_s	Specularity coefficient	0.6

clusters the grid points near the boundaries, which is generally preferred in most computational fluids applications. Implementation of the boundary conditions is simplified, and implementation of the method as a whole is fairly straightforward.

An iterative relaxation method is required to solve the nonlinear, steady-state equations of motion. Once this solution is known, the linear equations may be solved to determine the eigenvalues of the stability problem. The linear method also requires the use of a staggered grid scheme for two of the flow variables. The background of the Chebyshev collocation method is explained, followed by the explicit matrix equations used to solve the linear and nonlinear problems. The main points of the method which pertain to the current problem are only summarized here; a full derivation is available in [6], [7], and [8].

Chebyshev polynomials of the first kind are defined as a recursive function:

$$\begin{aligned}
 C_0(\alpha) &= 1 \\
 C_1(\alpha) &= \alpha \\
 C_{k+1}(\alpha) &= 2\alpha T_k(\alpha) - C_{k-1}(\alpha)
 \end{aligned} \tag{6.32}$$

Alternately, one may define Chebyshev polynomials on the interval $(-1, 1)$ by the relation

$$C_k(\alpha) = \cos(k \cos^{-1} \alpha) \tag{6.33}$$

Since the range of the current problem is $(0.5, -0.5)$, the following relation is employed to transform from the physical variable y to the Chebyshev variable α

$$\alpha = -2y \tag{6.34}$$

such that, at the top of the domain, $\alpha = -1$, and at the bottom, $\alpha = 1$. The collocation points, based upon Gauss-Lobatto quadrature, are defined as the maxima of the chebyshev polynomial $C_N(\alpha)$:

$$\alpha_j = \cos\left(\frac{\pi j}{N}\right), \quad j = 0, 1, \dots, N \quad (6.35)$$

Interpolant polynomials a are constructed to represent the flow variables at the collocation points. The granular temperature T is used as an example, although the same applies to the phasic velocity components. The variable may be written as

$$T_k(\alpha) = \sum_{k=0}^N a_k C_k(\alpha) \quad (6.36)$$

Derivatives of the variable may then be written as:

$$\left. \frac{\partial T}{\partial \alpha} \right|_j = \sum_{k=0}^N D_{jk} T_k(\alpha), \quad j = 0, 1, \dots, N \quad (6.37)$$

$$\left. \frac{\partial^2 T}{\partial \alpha^2} \right|_j = \sum_{k=0}^N D_{jk}^2 T_k(\alpha), \quad j = 0, 1, \dots, N \quad (6.38)$$

The elements of the derivative matrices are

$$\begin{aligned} D_{jk} &= S \frac{c_j}{c_k} \frac{(-1)^{j+k}}{\alpha_j - \alpha_k}, \quad j \neq k \\ D_{jj} &= S \frac{-\alpha_j}{2(1 - \alpha_j^2)} \\ D_{00} &= -D_{NN} = \frac{S(2N^2 + 1)}{6} \end{aligned} \quad (6.39)$$

and

$$D_{jk}^2 = D_{ji} D_{ik} \quad (6.40)$$

Here, $c_j = 2$ for $j = 0, N$ and $c_j = 1$ for $1 \leq j \leq N - 1$. In addition, S is the scaling factor defined as $\partial \alpha / \partial y = -2$. The steady-state equations of motion, 6.9-6.12, must also be linearized. This is accomplished by writing an iteration in p , which is linear in the unknown (e.g. $\bar{T}^{p+1}$), so that when the iteration converges, the converged solution solves the original nonlinear system of ODEs. For example, one of the nonlinear terms may be represented as:

$$\bar{T} \frac{\partial \bar{T}}{\partial y} = \frac{\bar{T}^p \left. \frac{\partial \bar{T}}{\partial y} \right|^{p+1} + \bar{T}^{p+1} \left. \frac{\partial \bar{T}}{\partial y} \right|^p}{2} \quad (6.41)$$

With this, the steady-state equations may be written as

$$\begin{aligned}
0 = & \frac{L(1-R)}{Fr} \bar{\beta}_j^p (\bar{u}_{gj}^{p+1} - \bar{u}_{sj}^{p+1}) - \frac{L(1-R) \cos \theta}{Fr} \bar{\phi}_j^{p+1} \\
& + \frac{1}{L} \sum_{k=0}^N D_{jk} \bar{f}_{2k} \left| \left(\bar{T}_j^{1/2} \right)^p \sum_{k=0}^N D_{jk} \bar{u}_{sk} \right|^{p+1} + \frac{1}{L} \bar{f}_{2j}^p \sum_{k=0}^N D_{jk} \bar{T}_k^{1/2} \left| \sum_{k=0}^N D_{jk} \bar{u}_{sk} \right|^{p+1} \\
& + \frac{1}{L} \bar{f}_{2j}^p \left(\bar{T}_j^{1/2} \right)^p \sum_{k=0}^N D_{jk}^2 \bar{u}_{sk} \left| \right|^{p+1}
\end{aligned} \tag{6.42}$$

$$\frac{L}{Fr} \frac{d\bar{P}_g}{dx} = - \frac{L(1-R)}{Fr} \bar{\beta}_j^p (\bar{u}_{gj}^{p+1} - \bar{u}_{sj}^{p+1}) + \frac{1}{LRe} \sum_{k=0}^N D_{jk}^2 \bar{u}_{gk} \left| \right|^{p+1} - \frac{LR \cos \theta}{Fr} \bar{\phi}_j^{p+1} \tag{6.43}$$

$$\begin{aligned}
0 = & \frac{L}{Re^2} \left| \bar{u}_{gj}^p - \bar{u}_{sj}^p \right|^2 \bar{f}_{5j}^p \left(\bar{T}_j^{-1/2} \right)^p + \frac{1}{L} \bar{f}_{2j}^p \left(\bar{T}_j^{1/2} \right)^p \sum_{k=0}^N D_{jk} \bar{u}_{sk} \left| \sum_{k=0}^N D_{jk} \bar{u}_{sk} \right|^{p+1} \\
& + \frac{1}{L} \sum_{k=0}^N D_{jk} \bar{f}_{4k} \left| \left(\bar{T}_j^{1/2} \right)^p \sum_{k=0}^N D_{jk} \bar{T}_k \right|^{p+1} + \frac{1}{L} \bar{f}_{4j}^p \left(\bar{T}_j^{1/2} \right)^p \sum_{k=0}^N D_{jk} \bar{T}_k \left| \sum_{k=0}^N D_{jk} \bar{T}_k \right|^{p+1} \\
& + \frac{1}{L} \bar{f}_{4j}^p \left(\bar{T}_j^{1/2} \right)^p \sum_{k=0}^N D_{jk}^2 \bar{T}_k \left| \right|^{p+1} + \frac{1}{L} \sum_{k=0}^N D_{jk} \bar{f}_{4hk} \left| \left(\bar{T}_j^{1/2} \right)^p \sum_{k=0}^N D_{jk} \bar{\phi}_k \right|^{p+1} \bar{T}_j^{p+1} \\
& + \frac{1}{L} \bar{f}_{4hj}^p \left(\bar{T}_j^{1/2} \right)^p \sum_{k=0}^N D_{jk} \bar{\phi}_k \left| \sum_{k=0}^N D_{jk} \bar{T}_k \right|^{p+1} + \frac{1}{L} \bar{f}_{4hj}^p \left(\bar{T}_j^{1/2} \right)^p \sum_{k=0}^N D_{jk}^2 \bar{\phi}_k \left| \right|^{p+1} \bar{T}_j^{p+1} \\
& - L \bar{f}_{6j}^p \left(\bar{T}_j^{1/2} \right)^p \bar{T}_j^{p+1} - \frac{3L(1-R)}{Fr} \bar{f}_{oj}^p \bar{T}_j^{p+1}
\end{aligned} \tag{6.44}$$

Here, $j = 0, 1, \dots, N$. The boundary conditions are:

$$\begin{aligned}
0 = & \pm D_{\gamma k} \bar{u}_{gk} |^{p+1} + \frac{L^2 Re}{Fr} \frac{\delta}{\phi_p} \bar{\beta}_\gamma^p \bar{\phi}_\gamma^p (\bar{u}_{g\gamma}^{p+1} - \bar{u}_{s\gamma}^{p+1}) + 0.5 \frac{\phi_p}{\delta \bar{\phi}_\gamma} \left[\bar{T}_\gamma^{p+1} \bar{u}_{g\gamma}^p + \bar{T}_\gamma^p \bar{u}_{g\gamma}^{p+1} \right] \\
& + \frac{L^2 Re}{Fr} \frac{\delta}{\phi_p} \frac{d\bar{P}_g}{dx} \bar{\phi}_\gamma^p
\end{aligned} \tag{6.45}$$

$$0 = \pm \bar{f}_{4\gamma}^p D_{\gamma k} \bar{T}_k |^{p+1} \pm \bar{f}_{4h\gamma}^p D_{\gamma k} \bar{\phi}_k |^p \bar{T}_\gamma^{p+1} - L \bar{f}_{8\gamma}^p \bar{u}_{s\gamma}^p \bar{u}_{s\gamma}^{p+1} + L \bar{f}_{7\gamma}^p \bar{T}_\gamma^{p+1} \tag{6.46}$$

$$0 = \pm \bar{f}_{2\gamma}^p D_{\gamma k} \bar{u}_{sk} |^{p+1} - L \bar{f}_{8\gamma}^p \bar{u}_{s\gamma}^{p+1} \tag{6.47}$$

Here, the subscript γ takes on the value 0 for $\alpha = -1$ and N when $\alpha = 1$. The iteration procedure to solve the system begins with an initial guess for the solution

vector $\bar{\Psi} = (\bar{u}_g, \bar{u}_s, \bar{T})^T$, as well as the volume fraction profile $\bar{\phi}$. The linearized system 6.42-6.47 is then solved to obtain a new solution $\bar{\Psi}$. The discretized version of 6.16 is then solved at each point in the domain using a standard root-finding method:

$$\bar{f}_{1j}^{p+1} \bar{T}_j^{p+1} + \frac{L \sin(\theta)}{2Fr} \int_{\alpha_j}^{-1} \bar{\phi}_j^{p+1} d\alpha = C_1 \quad (6.48)$$

The entire process is repeated until two conditions are met. The first is that the norm of the difference between successive solution vectors falls below a prescribed value:

$$\|\bar{\Psi}_{p+1} - \bar{\Psi}_p\| < 10^{-10} \quad (6.49)$$

The second condition requires the residuals of each of the three equations 6.42-6.44 to fall below 10^{-8} . It is possible to employ the solid-phase lateral momentum equation 6.11 to eliminate all instances of the terms $\partial \bar{\phi} / \partial y$ and $\partial^2 \bar{\phi} / \partial y^2$:

$$\frac{\partial \bar{\phi}}{\partial y} = - \left[\bar{f}_1 \frac{\partial \bar{T}}{\partial y} + \frac{L \sin \theta}{Fr} \bar{\phi} \right] \left(\frac{\partial \bar{f}_1}{\partial \phi} \bar{T} \right)^{-1} \quad (6.50)$$

$$\begin{aligned} \frac{\partial^2 \bar{\phi}}{\partial y^2} = & \left[\left(\frac{\partial \bar{f}_1}{\partial \phi} \frac{\partial \bar{T}}{\partial y} + \frac{L \sin \theta}{Fr} \right) \left(\bar{f}_1 \frac{\partial \bar{T}}{\partial y} + \frac{L \sin \theta}{Fr} \bar{\phi} \right) \left(\frac{\partial \bar{f}_1}{\partial \phi} \bar{T} \right)^{-1} - \bar{f}_1 \frac{\partial^2 \bar{T}}{\partial y^2} \right] \left(\frac{\partial \bar{f}_1}{\partial \phi} \bar{T} \right)^{-1} \\ & + 2 \left[\bar{f}_1 \frac{\partial \bar{T}}{\partial y} + \frac{L \sin \theta}{Fr} \bar{\phi} \right] \left(\frac{\partial \bar{f}_1}{\partial \phi} \bar{T} \right)^{-2} \left[\frac{\partial \bar{f}_1}{\partial \phi} \frac{\partial \bar{T}}{\partial y} \right. \\ & \left. - \frac{\partial^2 \bar{f}_1}{\partial \phi^2} \bar{T} \left(\bar{f}_1 \frac{\partial \bar{T}}{\partial y} + \frac{L \sin \theta}{Fr} \bar{\phi} \right) \left(\frac{\partial \bar{f}_1}{\partial \phi} \bar{T} \right)^{-1} \right]^{-1} \end{aligned} \quad (6.51)$$

However, it was found that this does not noticeably increase the rate of convergence. The preferred method was to apply the differentiation matrices to the previously-computed value of the solids volume fraction – e.g. $\sum_{k=0}^N D_{jk} \bar{\phi}_k^p$ – as shown in equations 6.42-6.47.

The solution of the linear stability problem requires a different approach. No boundary conditions are available for the perturbed fluid phase pressure P'_g or solids volume fraction ϕ' . It is possible to remove all instances of P'_g in the linearized equations 6.20-6.26 through cross-differentiation and substitution of the fluid-phase momentum equations. However, doing so was found to produce two spurious eigenvalues. These are numerical artifacts and do not converge with increasing N . It is also possible to eliminate all instances of ϕ' using either of the linearized continuity equations. However, for one of the primary eigenvalues, this produces a singularity when k is very small, and so does not allow for convergence over the entire wavenumber spectrum. The use of artificial

boundary conditions (e.g. applying one of the momentum equations at each boundary) was also attempted, although this again produces a spurious eigenvalue for each of the attempted variables.

An alternative is to employ a staggered grid scheme such that boundary conditions for P'_g and ϕ' are not necessary. These variables are evaluated at the staggered points $\alpha_{j+1/2}$, which are defined as the zeroes of the polynomial $C_N(\alpha)$:

$$\alpha_{j+1/2} = \cos\left(\frac{\pi(2j+1)}{2N}\right), \quad j = 0, 1, \dots, N-1 \quad (6.52)$$

The pressure and volume fraction variables are represented by polynomials of degree $N-1$. For example,

$$P_g(\alpha) = \sum_{k=0}^{N-1} b_k(\alpha) P_g(\alpha_{k+1/2}) \quad (6.53)$$

The interpolating polynomial is written

$$b_j(\alpha) = \frac{(-1)^j \sin(\alpha_{j+1/2}) C_N(\alpha)}{N(\alpha - \alpha_{j+1/2})}, \quad j = 0, 1, \dots, N-1 \quad (6.54)$$

Two sets of interpolating matrices are employed to interpolate from the standard grid points to the staggered, and vice versa, so that

$$T_{j+1/2} = \sum_{k=0}^N M_{jk}^s T_k, \quad j = 0, 1, \dots, N-1 \quad (6.55)$$

$$P_{gj} = \sum_{k=0}^{N-1} M_{jk}^r P_{g,k+1/2}, \quad j = 0, 1, \dots, N \quad (6.56)$$

The elements of the interpolating matrices are given by:

$$M_{jk}^s = \frac{(-1)^{j+k+1} (1 - \alpha_{j+1/2}^2)^{1/2}}{c_k N (\alpha_{j+1/2} - \alpha_k)}, \quad j = 0, 1, \dots, N-1, \quad k = 0, 1, \dots, N \quad (6.57)$$

$$M_{jk}^r = \frac{(-1)^{j+k} (1 - \alpha_{k+1/2}^2)^{1/2}}{N (\alpha_j - \alpha_{k+1/2})}, \quad j = 0, 1, \dots, N, \quad k = 0, 1, \dots, N-1 \quad (6.58)$$

It is noted that an extra row of zero elements must be added to the end of $\mathbf{M}^s$, as well as an extra column to the end of $\mathbf{M}^r$, to make each matrix square. In the current notation, this is written as $M_{Nk}^s = 0$ and $M_{jN}^r = 0$. Interpolation of the differentiation matrix is accomplished by taking the derivative of the corresponding polynomial:

$$\left. \frac{dT}{d\alpha} \right|_{j+1/2} = \sum_{k=0}^{N-1} \frac{db_k(\alpha_j)}{d\alpha} T_{k+1/2} = \sum_{k=0}^{N-1} E_{jk}^r T_{k+1/2}, \quad j = 0, 1, \dots, N \quad (6.59)$$

The elements of the derivative matrix are:

$$\begin{aligned}
E_{jk}^r &= \frac{(-1)^{j+k+1}(1 - \alpha_{k+1/2}^2)^{1/2}}{N(\alpha_j - \alpha_{k+1/2})^2}, \quad j = 1, 2, \dots, N-1 \\
E_{0k}^r &= \frac{(-1)^k(1 - \alpha_{k+1/2}^2)^{1/2}}{N} \left(\frac{N^2}{1 - \alpha_{k+1/2}} - \frac{1}{(1 - \alpha_{k+1/2})^2} \right) \\
E_{0k}^r &= \frac{(-1)^{k+N}(1 - \alpha_{k+1/2}^2)^{1/2}}{N} \left(\frac{N^2}{1 + \alpha_{k+1/2}} - \frac{1}{(1 + \alpha_{k+1/2})^2} \right)
\end{aligned} \tag{6.60}$$

In the above expressions, $k = 0, 1, \dots, N-1$. The derivative in the physical domain is obtained by simply multiplying the corresponding interpolating polynomial by the original derivative matrix:

$$E_{jk}^s = M_{jm}^s D_{mk} \tag{6.61}$$

With this information, it is now possible to solve the linearized equations of motion 6.20-6.26. The corresponding discretized equations are cumbersome and are presented in Appendix A. The discretized equations may be represented as

$$\sigma \mathbf{L} \psi = \mathbf{R} \psi \tag{6.62}$$

$\mathbf{L}$ and $\mathbf{R}$ are square matrices of dimensions $(7N+5)$. Because the boundary conditions do not contain the eigenvalue σ , the eigenvalue coefficient matrix $\mathbf{R}$ is singular. This singularity was removed through row and column operations, thereby reducing the ranks of each matrix to $(7N-5)$. The system 6.62 was solved using a complex QR solver available in MATLAB. The eigenvalue with the largest real part at a particular value of k is the leading eigenvalue, with wavenumber k_{max} , frequency $\sigma_{i,max}$, and growth rate $\sigma_{r,max}$. That with the maximum value over the entire wavenumber spectrum is referred to as the dominant eigenvalue with wavenumber k_D , frequency $\sigma_{i,D}$, and growth rate $\sigma_{r,D}$.

Convergence was achieved by increasing N until the first several leading eigenvalues became independent of the number of collocation points used. This was generally achieved with $N = 60$ or less points. For the limiting case of no shear and zero angle of inclination, only 20 points were required. The numerical model was validated by accurately reproducing the pipe and narrow gap annulus Poiseuille flow stability results of Khorrami [7], as well as the uniform shear case of the plane Couette flow of a granular medium studied by Alam & Nott [3]. Accuracy was also checked through comparison with the analytic solution of the zero-shear case, described in chapter 7.

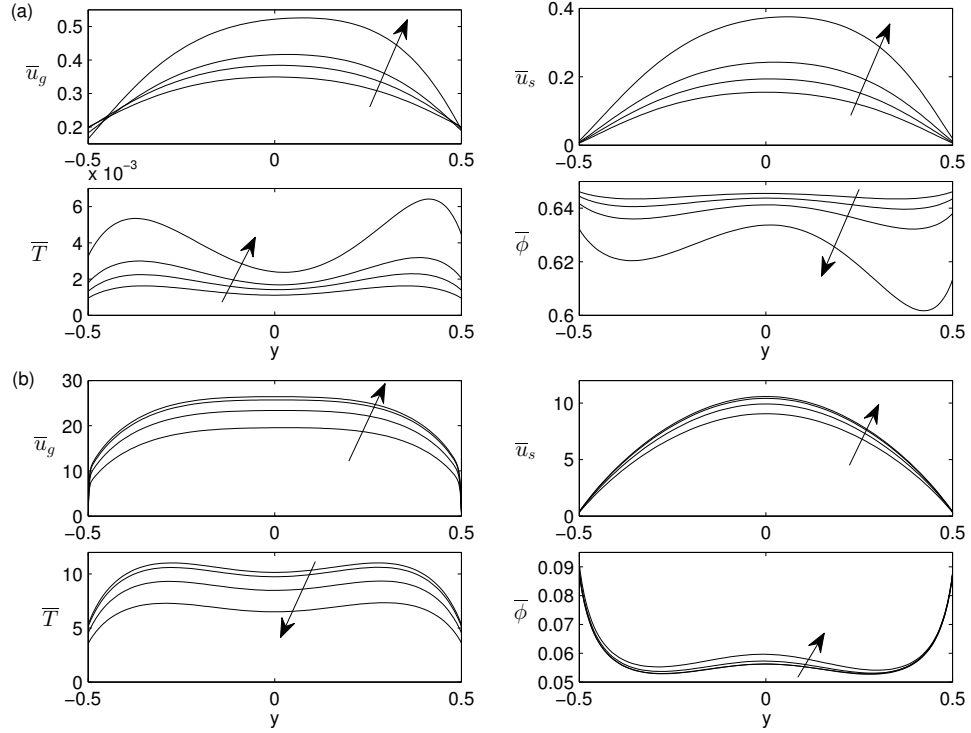


Figure 6.1: Base-state solutions for the (a) dense and (b) dilute systems, in order of increasing inclination angle, as marked by arrow: $\theta = 0^\circ, 15^\circ, 30^\circ, 45^\circ$. $L = 100$, $d\bar{P}_g/dx = -1.0 \cos(\theta)$, $C_1 = 1.0$

6.6 Results

6.6.1 Base-State Solutions

Figure 6.1 displays base-state solutions at various angles of inclination. As with the vertical bed results of Liu et al. [4], it was found that multiple solutions may exist for a particular combination of $d\bar{P}_g/dx$ and C_1 . The two sets of results in figure 6.1 represent the dense solution (a) and the dilute solution (b). The phasic velocities of the dilute solution are much higher than those of the dense system and represent fluidized beds in the pneumatic flow regime [9]. This agrees with the very low solid volume fraction values throughout the domain. The base-state profiles of the mean flow variables for the dilute case do not vary much with θ . For this reason, and because dense beds are of greater interest in experiment and stability theory, the rest of the results will focus on

the dense-bed solutions.

The dense solution shows an appreciable amount of variation in the mean-flow variables as the inclination angle changes. Specifically, the top region of the bed is characterized by a lower volume fraction, and vice versa. This is a result of the balance between the solid-phase pressure and the body force term (see equation 6.16). The solid-phase velocity is higher in the top region, although the relative gain is not as large as the fluid-phase velocity, as evidenced by the larger granular temperature near the top of the bed.

6.6.2 Stability Analysis Results

Growth rate traces of the first several leading eigenvalues are shown as a function of the wavenumber k in figure 6.2. The results are computed for two cases: with zero angle of inclination, and with $\theta = 40^\circ$. For both systems, several eigenvalues exist which have positive growth rates. Above a certain value of the wavenumber the system is stable to all disturbances, a feature commonly seen in linear stability analyses of fluid systems, and currently a result of the dampening mechanisms of the solid-phase viscosity and pseudo-thermal conductivity. The unstable modes are divided into two families. For $\theta = 0^\circ$, the a modes consist of a series of symmetric and anti-symmetric eigenmodes. The b modes are also symmetric and antisymmetric, although the shapes of their eigenmodes differ significantly. For $\theta > 0^\circ$, all eigenmodes lose their symmetry as a consequence of bed tilting. These eigenmode features are shown in figure 6.3 and are discussed shortly. An important result of the vertical bed is that the a_1 mode remains dominant for all values of the boundary conditions parameters e_w and ϕ_s between 0 and 1. This has implications regarding the use of the previous simplified vertical systems, and is discussed in the concluding section of this chapter.

At low angles of inclination, a_1 is dominant, while the b_1 mode exhibits a greater growth rate when $\theta \gtrsim 13^\circ$. Traces of the frequency corresponding to the leading eigenvalue at a particular value of k show that the leading mode is one of several, depending upon the value of the disturbance wavenumber. Notable is the significantly higher frequency of the b modes. In figure 6.2, the frequencies corresponding to the portions of the graph in which the b mode is leading are reduced by a factor of 6 for plotting convenience. With this, we see that the fluidized system tends to favor a highly-oscillatory mode as θ is increased. Figure 6.3 displays the eigenmodes ϕ' corresponding to the four

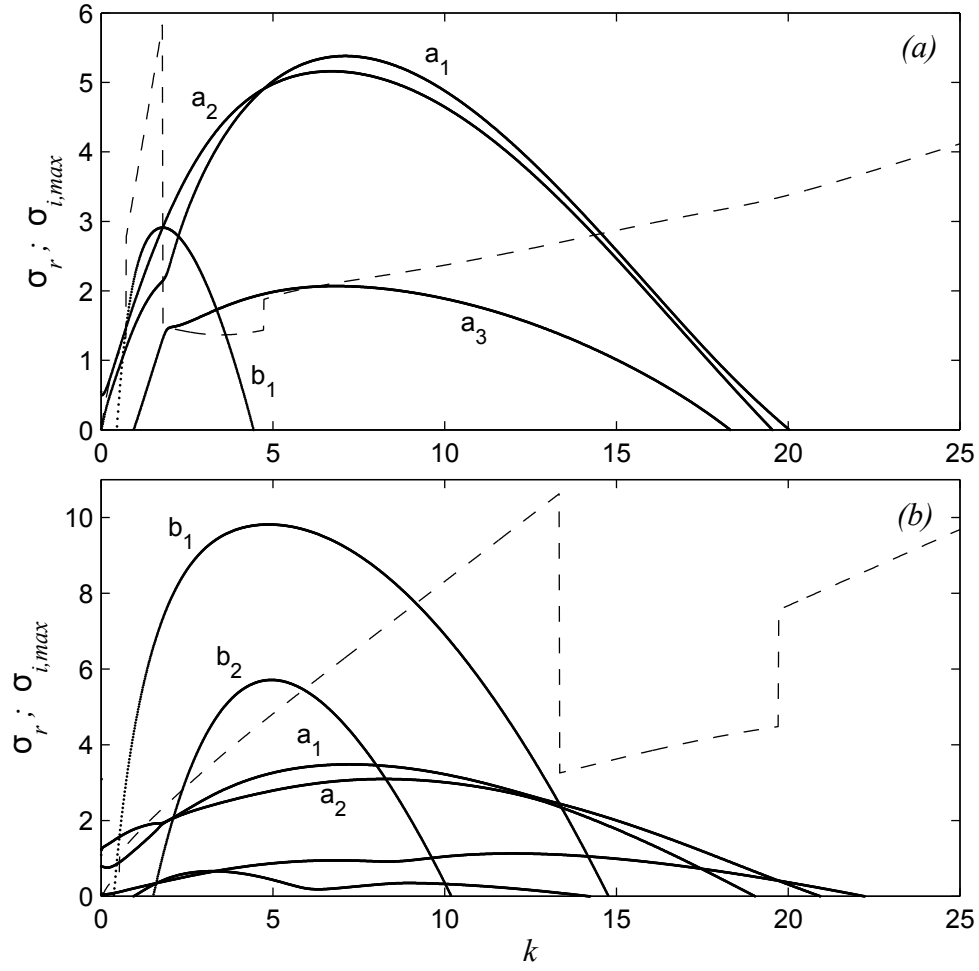


Figure 6.2: Growth rates (dots) of the first several leading eigenvalues and frequency (dashes) of the leading eigenvalue as a function of k for (a) $\theta = 0^\circ$ (b) $\theta = 40^\circ$. σ_i values corresponding to the b modes have been reduced by a factor of 6 for convenience.

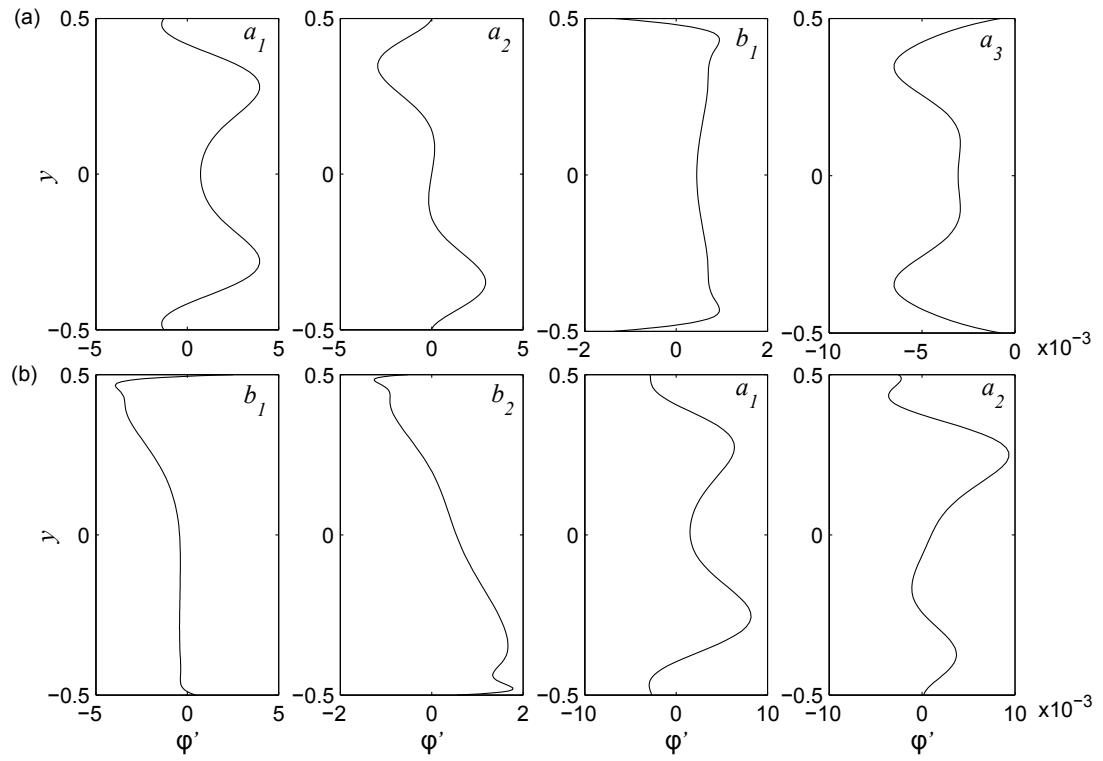


Figure 6.3: Eigenvectors ϕ' corresponding to the four leading eigenmodes in figure 6.2 for (a) $\theta = 0^\circ$ (b) $\theta = 40^\circ$.

leading modes seen in figure 6.2. The dominant mode a_1 of the vertical bed is symmetric about the centerline, while the next-highest leading eigenmode a_2 is antisymmetric. The former mode exhibits regions of growing voidage adjacent to the walls while the latter provides a voidage perturbation in one half of the domain. The a_3 mode continues the pattern of symmetry-antisymmetry which is found in the limiting case solution presented in the following section. The b_1 mode is also symmetric, although the profile for ϕ' is nearly constant throughout most of the domain with the exception of thin regions of large shear adjacent to the boundaries. This "boundary layer" mode becomes relevant when the bed is tilted, although it loses its symmetry when $\theta > 0^\circ$.

At $\theta = 40^\circ$, the ϕ' profile of the b_1 mode remains nearly constant in the lower half of the bed, while a region of voidage perturbation is present in the top half. The b_2 mode represents the second leading eigenvalue, antisymmetric but with a negative growth rate for all k when the bed is vertical. The a_1 and a_2 modes remain similar in shape as bed tilting increases, although with the first showing a region of dense growth in the lower part of the domain and the second in the upper half.

To visualize the evolution of the leading modes as the bed is tilted, the growth rate and frequency is plotted in the $k - \theta$ plane in figure 6.4. The three leading eigenmodes are separated by discontinuous jumps in $\sigma_{i,max}$ and $\partial\sigma_{r,max}/\partial k$. The former is more visible in figure 6.4(a), although the latter exist at the same locations in figure 6.4(b). The left region corresponds to the b_1 mode, the middle to the a_1 mode, and the right side of the plots to a_2 . The frequencies corresponding to b_1 were again reduced by a factor of 6. The a_1 mode is dominant from $0^\circ < \theta \lesssim 13^\circ$, while the b_1 mode is dominant at higher levels of inclination. Notable is the monotonic decrease in σ_r over all k for the a_1 mode as θ increases, and the monotonic increase in the same value for the b_1 mode. Although not shown on the contour plot, the b_2 mode exhibits the same behavior as its counterpart. The range of k over which the b_1 mode is leading also increases with θ . The wavenumber of the dominant mode increases as θ rises, with the exception of the discontinuous drop when dominance is transferred from a_1 to b_1 . Lastly, it is seen that the a_2 mode is leading at the largest values of k , although the range of k over which this mode is unstable is small.

The time evolution of the solid-phase volume concentration $\phi(x, y, t) = \bar{\phi}(y) + \tilde{\phi}(x, y, t)$ is displayed for the two cases of $\theta = 0^\circ$ and 40° , where

$$\tilde{\phi}(x, y, t) = \phi'(y)e^{ik_D x + (\sigma_{rD} + i\sigma_{iD})t} \quad (6.63)$$

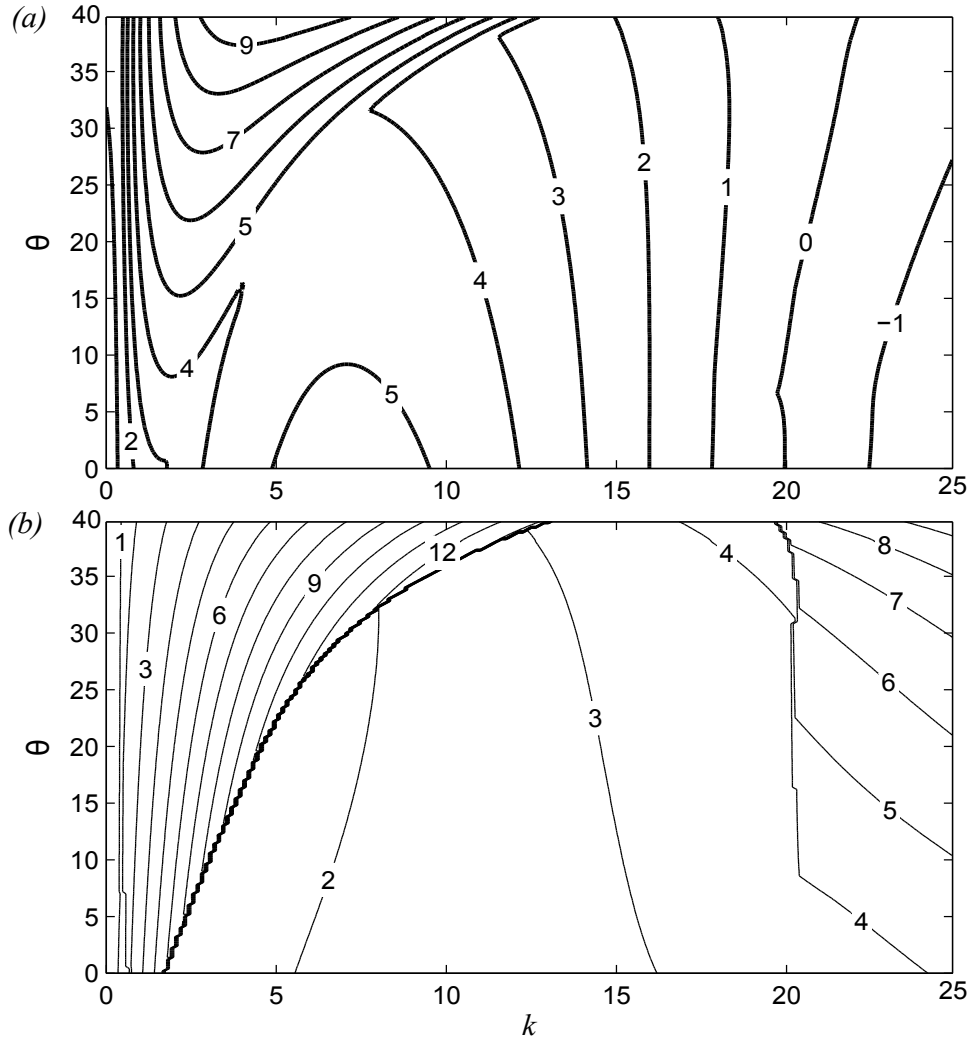


Figure 6.4: Contour plots of the (a) growth rate, $\sigma_{r,max}$, and (b) frequency, $\sigma_{i,max}$, of the leading eigenvalue as a function of bed inclination and longitudinal wavenumber. The b_1 mode comprises the upper-left-hand side of each plot. The a_1 mode is leading in the middle portion of the graph, and the a_2 mode is on the right side of each plot.

is the dominant linear mode with frequency σ_{iD} , growth rate σ_{rD} , and longitudinal wavelength k_D^{-1} . Figures 6.5(a) and 6.5(b) display the evolution of the dominant modes of the $\theta = 0^\circ$ and 40° beds, respectively. The volume fraction profile is displayed first in the base state, followed by the unstable mode for three different values of the dimensionless time. Only these two cases are shown because they represent the evolution of the a_1 and b_1 modes, which are the only dominant disturbances over the considered range of tilting angle. Dark areas correspond to dilute regions and vice versa. Each of the four plots shown for $\theta = 0^\circ$ and 40° are on the same gray scale within their respective systems.

A pattern exists for both cases. Dilute regions of the base-state solution ($\bar{\phi}$) yield time-amplified dilute regions of the perturbed solution (ϕ'). For the vertical bed, this is visible with the dilute bands in the upper and lower parts of the domain evolving into trains of very dilute regions alternating with heavily dense spots. The tilted bed shows the same pattern, with the initially dilute region near the top of the bed evolving into dramatically dilute pockets. This pattern is of particular interest for the tilted system because it agrees qualitatively with the evolution of bubbling observed in experiment [10, 11] in which disturbances originate in, and propagate through, the upper region of the tilted bed.

To conclude this section, the effects of the boundary condition parameters upon the boundary layer mode b_1 are investigated. Figure 6.6 displays growth rate traces of the leading eigenvalue for several combinations of the boundary condition parameters e_w and ϕ_s . It was found that the solutions of the base state and the subsequent stability analysis were largely insensitive to the fluid-phase parameter δ , and so the consequences of changing this are not shown. The primary results of modifying the other two parameters are evident from the graph, the most important of which being that the boundary layer mode b_1 remains dominant at $\theta = 40^\circ$. However, the growth rate and wavenumber of the dominant disturbance decrease when e_w is reduced – that is, when more pseudo-thermal energy is lost through the walls. In the same manner, traces (i) and (ii) reveal that decreasing the level of particle-wall friction yields a dominant mode with a larger growth rate and wavenumber. In agreement with intuition, the growth rate of the primary boundary layer mode is significantly increased when less energy and momentum are lost at the walls. It was also found that the angle at which the a_1 mode yielded dominance to the b_1 disturbance occurs at a larger tilting angle when e_w is decreased or ϕ_s is increased (the boundary layer mode does not become relevant until more severe θ ,

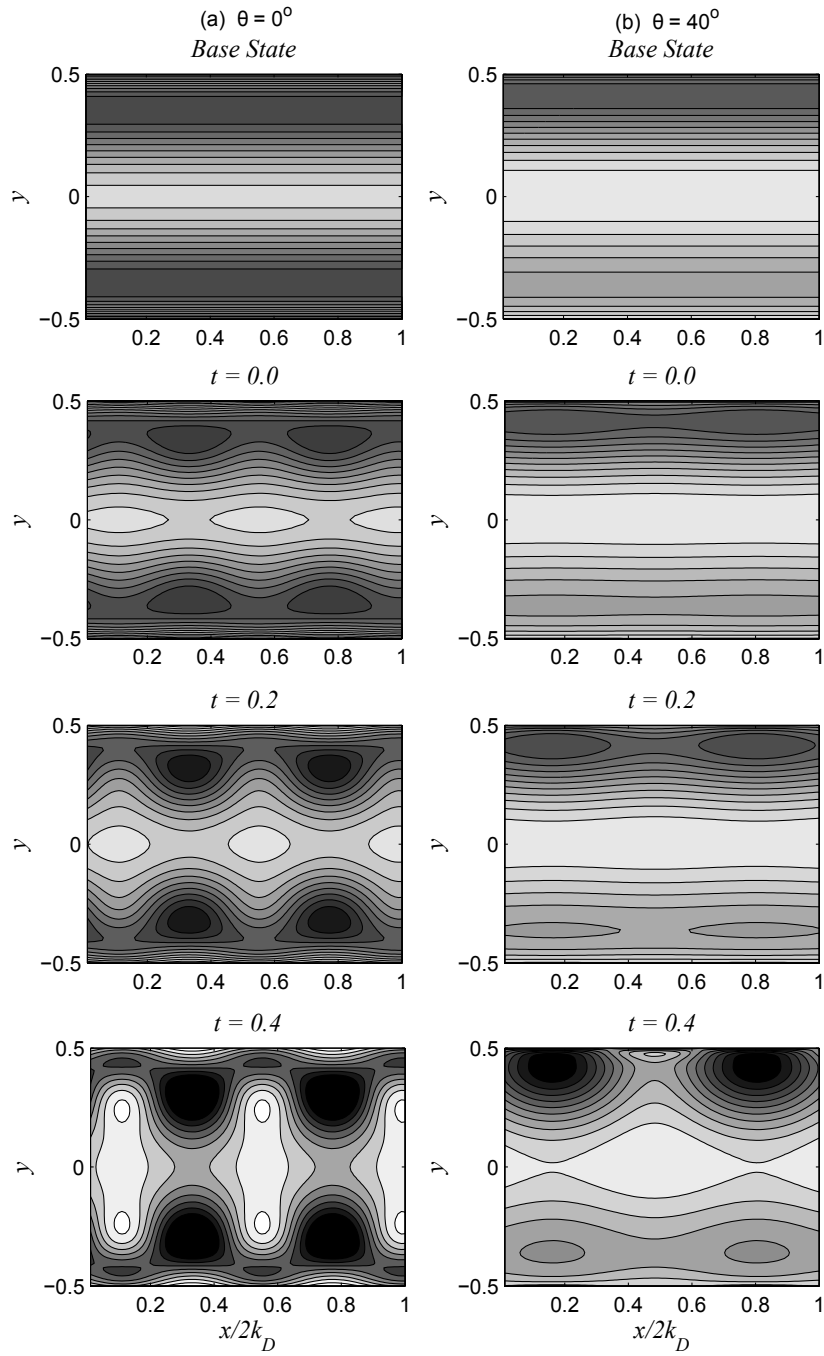


Figure 6.5: Contour plots showing the evolution of the solid-phase volume fraction ϕ , beginning with the base-state solution and continuing in time with the superimposed dominant instability mode. Plots show the dominant instability modes from (a) figure 6.2(a) and (b) figure 6.2(b). Dark areas are dilute – small ϕ – and light areas are dense.

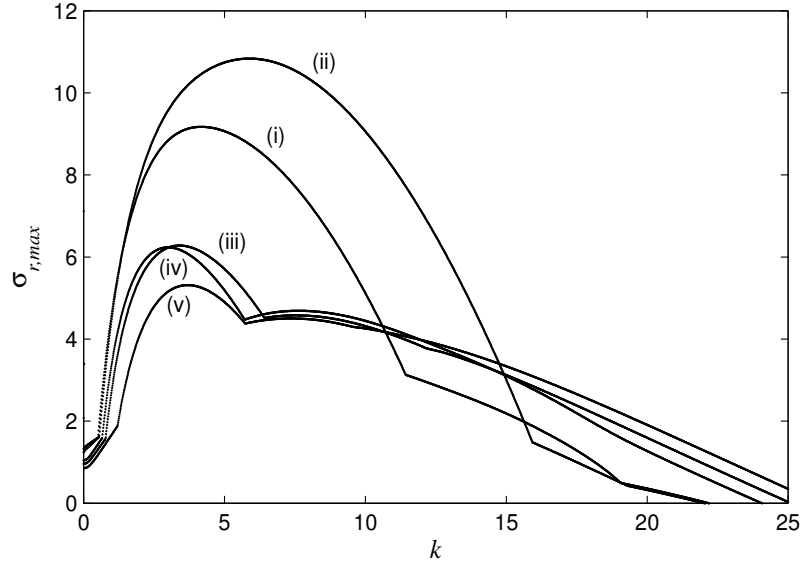


Figure 6.6: Growth rates corresponding of eigenmode at $\theta = 40^\circ$ for several combinations of boundary condition parameters: (i) $e_w = 0.9$, $\phi_s = 1.0$ (ii) $e_w = 0.9$, $\phi_s = 0.3$ (iii) $e_w = 0.3$, $\phi_s = 0.3$ (iv) $e_w = 0.3$, $\phi_s = 1.0$ (v) $e_w = 0.1$, $\phi_s = 0.1$

for lower e_w and higher ϕ_s).

6.7 Discussion

This chapter demonstrates that the dominant linear mode of the tilted system is one of two, depending on the value of θ . At low values of the tilting angle, the a_1 mode is dominant, while at larger θ the b_1 mode has the largest growth rate. For the most heavily inclined bed ($\theta = 0^\circ$), the boundary layer mode is found to be dominant over the entire range of the boundary condition parameters. The angle at which dominance transfers from a_1 to b_1 increases with the amount of momentum and energy dissipation at the boundaries.

An important result is found from the dominant and leading modes when $\theta = 0^\circ$. The a_1 mode is dominant for every combination of the parameters e_w and ϕ_s , meaning that the boundary layer mode is not relevant for the vertical bed. Combined with the alternating symmetric and antisymmetric modes a_1 , a_2 , a_3 , etc., in order of declining growth rate, a parallel is found between the results of the current chapter and those of the vertical planar bed in chapter 5. The leading eigenvalues of the simplified system

have eigenmodes which exhibit this same behavior, despite the obvious differences in shape due to the lack of shear in the base state. This is demonstrated with the analytic solution of the limiting vertical case presented in chapter 7. It is found that the simplified system is sufficient to describe the qualitative stability characteristics of the vertical bed.

The most significant similarity between the two systems deals with the classically-recognized drag force instability mechanism $\bar{\beta}_\phi$. The difference in base-state velocities $(\bar{u}_g - \bar{u}_s)$ is positive over the entire domain for both systems. Although the value of this difference varies over the domain for both systems and between both systems, the term to which this difference is proportional in the linearized equations of motion, $\bar{\beta}_\phi \phi'$, plays the second largest role in determining the stability of the bed (as demonstrated in chapter 4). The direct dependence of the drag force function on the particle diameter and density then largely explain the qualitative similarities between all three of the bounded systems considered.

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Chapter 6, in part, will be submitted for publication in *International Journal of Multiphase Flow*, "Stability of an Inclined, Gas-Fluidized Bed," by K. Mandich and R. Cattolica. The thesis author is the primary investigator in this publication.

Chapter 7

Analytic Solutions for 2D and 3D Vertical Fluidized Beds

7.1 Two-Dimensional Bed

We consider first the limiting case of the planar fluidized bed with full particle slip at the wall. Added simplifications include adiabatic walls ($e_w = 0$) as well as free slip for the solid ($\phi_s = 0$) and fluid phases. With this, the base-state boundary conditions simplify to:

$$\frac{\partial \bar{u}_g}{\partial y} = \frac{\partial \bar{u}_s}{\partial y} = \frac{\partial \bar{T}}{\partial y} = \frac{\partial \bar{\phi}}{\partial y} = 0, \quad y = \pm 0.5 \quad (7.1)$$

The solutions for the flow variables are then independent of y . One further simplification is made by eliminating the base-state solid-phase velocity profile $\bar{u}_s$. Since there is no shear present in this limiting case, the base-state fluid-phase velocity may replace the quantity $\bar{u}_g - \bar{u}_s$, essentially by shifting the frame of reference of the bed. With this, the linearized system of equations 6.20-6.26 reduces to:

$$\sigma \phi' = -ik \bar{\phi} u'_s - \bar{\phi} \frac{\partial}{\partial y} v'_s \quad (7.2)$$

$$- \sigma \phi' = ik (1 - \bar{\phi}) u'_g - ik \bar{u}_g \phi' + (1 - \bar{\phi}) \frac{\partial}{\partial y} v'_g \quad (7.3)$$

$$\begin{aligned} \sigma \left(u'_g - \frac{\bar{u}_g}{(1 - \bar{\phi})} \phi' \right) = & -ik \bar{u}_g u'_g - \frac{ik \bar{u}_g^2}{1 - \bar{\phi}} \phi' - ik \frac{LR^{-1}}{Fr} P'_g \\ & + \frac{L(R^{-1} - 1)}{Fr} [\bar{\beta}(u'_s - u'_g) - \bar{\beta}_\phi \bar{u}_g \phi'] \end{aligned} \quad (7.4)$$

$$\sigma v'_g = -ik\bar{u}_g v'_g - ik \frac{LR^{-1}}{Fr} \frac{\partial P'_g}{\partial y} + \frac{L(R^{-1}-1)}{Fr} \bar{\beta}(v'_s - v'_g) \quad (7.5)$$

$$\begin{aligned} \sigma \bar{\phi} u'_s = & -ik(\bar{P}_{s\phi} \phi' + \bar{P}_{sT} T') - ik\bar{\phi} \frac{L}{Fr} P'_g - \frac{L}{Fr} \phi' \\ & + \frac{L(1-R)}{Fr} (1-\bar{\phi}) [\bar{\beta}(u'_g - u'_s) + \bar{\beta}_\phi \bar{u}_g \phi'] \\ & + \left(-k^2 \frac{2\bar{\mu}_s + \bar{\zeta}_s}{L} + \frac{\bar{\mu}_s}{L} \frac{\partial^2}{\partial y^2} \right) u'_s + \left(ik \frac{\bar{\mu}_s + \bar{\zeta}_s}{L} \frac{\partial}{\partial y} \right) v'_s \end{aligned} \quad (7.6)$$

$$\begin{aligned} \sigma \bar{\phi} v'_s = & -\bar{P}_{s\phi} \frac{\partial \phi'}{\partial y} - \bar{P}_{sT} \frac{\partial T'}{\partial y} - \bar{\phi} \frac{L}{Fr} \frac{\partial P'_g}{\partial y} + \frac{L(1-R)}{Fr} (1-\bar{\phi}) \bar{\beta}(v'_g - v'_s) \\ & + \left(ik \frac{\bar{\mu}_s + \bar{\zeta}_s}{L} \frac{\partial}{\partial y} + \frac{ik}{L} \frac{\partial \bar{\zeta}_s}{\partial y} \right) u'_s + \left(-k^2 \frac{\bar{\mu}_s}{L} + \frac{2\bar{\mu}_s + \bar{\zeta}_s}{L} \frac{\partial^2}{\partial y^2} \right) v'_s \end{aligned} \quad (7.7)$$

$$\begin{aligned} \frac{3}{2} \sigma (\bar{\phi} T' + \bar{T} \phi') = & -\frac{3}{2} ik \left(\bar{\phi} \bar{T} u'_s + \bar{\phi} \bar{T} \frac{\partial v'_s}{\partial y} \right) - \bar{P}_s \left(ik u'_s + \frac{\partial v'_s}{\partial y} \right) \\ & + \frac{\bar{\lambda}}{L} \left(-k^2 + \frac{\partial^2}{\partial y^2} \right) T' + \frac{\bar{\lambda}_h}{L} \left(-k^2 + \frac{\partial^2}{\partial y^2} \right) \phi' \\ & L (\Gamma_\phi \phi' + \Gamma_T T' + \Gamma_{u_g} u'_g + \Gamma_{u_s} u'_s) \end{aligned} \quad (7.8)$$

The linearized boundary conditions reduce to:

$$v'_g = v'_s = \frac{\partial u'_g}{\partial y} = \frac{\partial u'_s}{\partial y} = \frac{\partial T'}{\partial y} = \frac{\partial \phi'}{\partial y} = \frac{\partial P'_g}{\partial y} = 0, \quad y = \pm 0.5 \quad (7.9)$$

The simplified system and boundary conditions allow the following symmetry relationships:

$$\begin{bmatrix} v'_g(y) \\ v'_s(y) \end{bmatrix} = \begin{bmatrix} \pm v'_g(-y) \\ \pm v'_s(-y) \end{bmatrix}, \quad \begin{bmatrix} u'_g(y) \\ u'_s(y) \\ T'(y) \\ \phi'(y) \\ P'_g(y) \end{bmatrix} = \begin{bmatrix} \mp u'_g(-y) \\ \mp u'_s(-y) \\ \mp T'(-y) \\ \mp \phi'(-y) \\ \mp P'_g(-y) \end{bmatrix} \quad (7.10)$$

A solution to this system is

$$\begin{bmatrix} v'_g(y) \\ v'_s(y) \end{bmatrix} = \begin{bmatrix} \check{v}_g \\ \check{v}_s \end{bmatrix} \sin \Theta(y \pm 0.5), \quad \begin{bmatrix} u'_g(y) \\ u'_s(y) \\ P'_g(y) \\ \phi'(y) \\ T'(y) \end{bmatrix} = \begin{bmatrix} \check{u}_g \\ \check{u}_s \\ \check{P}_g \\ \check{\phi} \\ \check{T} \end{bmatrix} \cos \Theta(y \pm 0.5) \quad (7.11)$$

Here, $\Theta = n\pi$, and n is a positive integer. Odd and even values of n correspond to the plus and minus values in front of v'_g and v'_s in 7.10, respectively. The resulting equations of motion are:

$$\sigma\check{\phi} + ik\bar{\phi}\check{u}_s + \bar{\phi}\Theta\check{v}_s = 0 \quad (7.12)$$

$$-\sigma\check{\phi} + ik(1 - \bar{\phi})\check{u}_g + (1 - \bar{\phi})\Theta\check{v}_g - ik\bar{u}_g\check{\phi} = 0 \quad (7.13)$$

$$\begin{aligned} \sigma \left(\check{u}_g - \frac{\bar{u}_g}{(1 - \bar{\phi})} \check{\phi} \right) = & -ik \frac{LR^{-1}}{Fr} \check{P}_g + \frac{L(R^{-1} - 1)}{Fr} (\bar{\beta}(\check{u}_s - \check{u}_g) - \bar{\beta}_\phi \bar{u}_g \check{\phi}) \\ & - \frac{ik\bar{u}_g^2}{(1 - \bar{\phi})} \check{\phi} - ik\bar{u}_g \check{u}_g \end{aligned} \quad (7.14)$$

$$\sigma\check{v}_g = \Theta \frac{LR^{-1}}{Fr} \check{P}_g + \frac{L(R^{-1} - 1)}{Fr} \bar{\beta}(\check{v}_s - \check{v}_g) - ik\bar{u}_g \check{v}_g \quad (7.15)$$

$$\begin{aligned} \sigma\bar{\phi}\check{u}_s = & -ik\bar{\phi} \frac{L}{Fr} \check{P}_g + (1 - \bar{\phi}) \frac{L(1 - R)}{Fr} (\bar{\beta}(\check{u}_g - \check{u}_s) + \bar{\beta}_\phi \bar{u}_g \check{\phi}) \\ & - k^2 \left(\frac{2\bar{\mu}_s + \bar{\zeta}_s}{L} \right) \check{u}_s - \Theta^2 \frac{\bar{\mu}_s}{L} \check{u}_s + ik\Theta \left(\frac{\bar{\mu}_s + \bar{\zeta}_s}{L} \right) \check{v}_s \\ & - ik(\bar{P}_{s\phi}\check{\phi} + \bar{P}_{sT}\check{T}) - \frac{L}{Fr} \phi' \end{aligned} \quad (7.16)$$

$$\begin{aligned} \sigma\bar{\phi}\check{v}_s = & \Theta\bar{\phi} \frac{L}{Fr} \check{P}_g + (1 - \bar{\phi}) \frac{L(1 - R)}{Fr} \bar{\beta}(\check{v}_g - \check{v}_s) + \Theta\bar{P}_{s\phi}\check{\phi} + \Theta\bar{P}_{sT}\check{T} \\ & - \Theta^2 \left(\frac{2\bar{\mu}_s + \bar{\zeta}_s}{L} \right) \check{v}_s - k^2 \frac{\bar{\mu}_s}{L} \check{v}_s - ik\Theta \left(\frac{\bar{\mu}_s + \bar{\zeta}_s}{L} \right) \check{u}_s \end{aligned} \quad (7.17)$$

$$\begin{aligned} \frac{3}{2}\sigma(\bar{\phi}\check{T} + \bar{T}\check{\phi}) = & -\bar{P}_s(ik\check{u}_s + \Theta\check{v}_s) + L(\bar{\Gamma}_{u_s}u'_s + \bar{\Gamma}_{u_g}u'_g + \bar{\Gamma}_T T' + \bar{\Gamma}_\phi \phi') \\ & + \frac{\bar{\lambda}_h}{L}(-k^2 - \Theta^2)\check{\phi} + \frac{\bar{\lambda}}{L}(-k^2 - \Theta^2)\check{T} - \frac{3}{2}\bar{\phi}\bar{T}(ik\check{u}_s + \Theta\check{v}_s) \end{aligned} \quad (7.18)$$

The dispersion relation may be obtained analytically in the form of a hexic polynomial in σ . However, due to the large number of terms and the complexity of the resulting solution, it is much more reasonable to solve the system through a simple algebraic procedure and still extract the desired information. The equations 7.12-7.18 may be represented in matrix form:

$$\mathbf{A}\check{\psi} = 0 \quad (7.19)$$

Here, $\mathbf{A}$ is a 6x6 matrix whose complex determinant yields the dispersion relation. We may then solve the problem $|A| = 0$ to locate the eigenvalues which satisfy this system

of equations. This solution is attained numerically; the procedures used are the same as described in section 4.4.1 for the case of an unbounded fluidized bed.

To obtain a reasonable analytic solution, the problem may be simplified further by neglecting the convective terms of the fluid phase. As shown in figure 4.1, omission of these terms has a very minor effect on the stability characteristics when considering gas-fluidized beds. Taking this step modifies only the fluid-phase momentum equations 7.14-7.15, which become:

$$0 = -ik\check{P}_g + (1 - R) (\bar{\beta}(\check{u}_s - \check{u}_g) - \bar{\beta}_\phi \bar{u}_g \check{\phi}) \quad (7.20)$$

$$0 = \Theta \check{P}_g + (1 - R) \bar{\beta}(\check{v}_s - \check{v}_g) \quad (7.21)$$

The resulting dispersion relation is a quartic polynomial:

$$\sigma^4 + \omega_1 \sigma^3 + \omega_2 \sigma^2 + \omega_3 \sigma + \omega_4 = 0 \quad (7.22)$$

The coefficients ω are presented in Appendix C.

7.2 Three-Dimensional Bed

The second bounded system which is amenable to an analytic solution is the cylindrical bed. The simplifications of no shear in the base state yield the following base-state boundary conditions:

$$\frac{\partial \bar{u}_g}{\partial r} = \frac{\partial \bar{T}}{\partial r} = \frac{\partial \bar{\phi}}{\partial r} = 0, \quad r = \pm 0, 1 \quad (7.23)$$

As an added simplification, only axisymmetric disturbances ($n = 0$) are considered. This is justified by the results of section 5.5, in which the axisymmetric disturbance was found to be the least stable over all azimuthal wavenumbers n . The relevant linearized equations are those (5.18-5.18) of the cylindrical system, but without the dependence on θ . With this, the azimuthal momentum equations for the fluid and solid phases are trivialized and need not be included. The resulting equations are:

$$\sigma \phi' = -\bar{\phi} \frac{\partial u'_r}{\partial r} - \frac{\bar{\phi}}{r} u'_r - ik \bar{\phi} u'_z \quad (7.24)$$

$$\sigma \phi' = (1 - \bar{\phi}) \frac{\partial v'_r}{\partial r} + \frac{(1 - \bar{\phi})}{r} v'_r + ik(1 - \bar{\phi}) v'_z - ik \bar{v}_z \phi' \quad (7.25)$$

$$\sigma v'_r = -ik \bar{v}_z v'_r - \frac{HR^{-1}}{Fr} \frac{\partial P'_g}{\partial r} - \frac{H(R^{-1} - 1)}{Fr} \bar{\beta}(v'_r - u'_r) \quad (7.26)$$

$$\begin{aligned} \sigma v'_z - \frac{\sigma \bar{v}_z}{(1-\bar{\phi})} \phi' &= -ik \bar{v}_z v'_z + \frac{ik \bar{v}_z^2}{(1-\bar{\phi})} \phi' - ik \frac{HR^{-1}}{Fr} P'_g \\ &\quad - \frac{H(R^{-1}-1)}{Fr} (\bar{\beta}(v'_z - u'_z) - \bar{\beta}_\phi \bar{v}_z \phi') \end{aligned} \quad (7.27)$$

$$\begin{aligned} \sigma \bar{\phi} u'_r &= -\bar{\phi} \frac{HR^{-1}}{Fr} \frac{\partial P'_g}{\partial r} + (1-\bar{\phi}) \frac{H(1-R)}{Fr} \bar{\beta}(v'_r - u'_r) - \bar{P}_{s\phi} \frac{\partial \phi'}{\partial r} \\ &\quad - \bar{P}_{sT} \frac{\partial T'}{\partial r} + \frac{\bar{\mu}_s}{H} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) u'_r \\ &\quad + \frac{\bar{\mu}_s + \bar{\zeta}_s}{H} \frac{\partial}{\partial r} \left(\frac{\partial u'_r}{\partial r} + \frac{u'_r}{r} + \frac{\partial u'_z}{\partial z} \right) - \frac{\bar{\mu}_s}{H} \frac{u'_r}{r^2} \end{aligned} \quad (7.28)$$

$$\begin{aligned} \sigma \bar{\phi} u'_z &= -ik \bar{\phi} \frac{H(R^{-1}-1)}{Fr} P'_g + (1-\bar{\phi}) \frac{H(1-R)}{Fr} (\bar{\beta}(v'_z - u'_z) + \bar{\beta}_\phi \bar{v}_z \phi') \\ &\quad - ik (\bar{P}_{s\phi} \phi' + \bar{P}_{sT} T') + \frac{\bar{\mu}_s}{H} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) u'_z \\ &\quad + \frac{\bar{\mu}_s + \bar{\zeta}_s}{H} ik \left(\frac{\partial u'_r}{\partial r} + \frac{u'_r}{r} + \frac{\partial u'_z}{\partial z} \right) - \frac{L}{Fr} \phi' \end{aligned} \quad (7.29)$$

$$\begin{aligned} \frac{3}{2} \sigma (\bar{\phi} T' + \bar{T} \phi') &= - \left(\bar{P}_s + \frac{3}{2} \bar{\phi} \bar{T} \right) \left(\frac{\partial u'_r}{\partial r} + \frac{u'_r}{r} + \frac{\partial u'_z}{\partial z} \right) \\ &\quad + \frac{\bar{\lambda}_h}{H} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \phi' + \frac{\bar{\lambda}}{H} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) T' \\ &\quad + H (\bar{\Gamma}_{u_z} u'_z + \bar{\Gamma}_{v_z} v'_z + \bar{\Gamma}_T T' + \bar{\Gamma}_\phi \phi') \end{aligned} \quad (7.30)$$

The base-state solutions are the same as those presented in section 5.3. The boundary conditions simplify to:

$$u'_r = v'_r = \frac{\partial u'_z}{\partial r} = \frac{\partial v'_z}{\partial r} = \frac{\partial T'}{\partial r} = 0, \quad r = 0, 1 \quad (7.31)$$

Equations (7.34-7.31) allow the following symmetry relationships:

$$\begin{bmatrix} u'_r(r) \\ v'_r(r) \end{bmatrix} = \begin{bmatrix} -u'_r(-r) \\ -v'_r(-r) \end{bmatrix}, \quad \begin{bmatrix} u'_z(r) \\ v'_z(r) \\ P'_g(r) \\ \phi'(r) \\ T'(r) \end{bmatrix} = \begin{bmatrix} u'_z(-r) \\ v'_z(-r) \\ P'_g(-r) \\ \phi'(-r) \\ T'(-r) \end{bmatrix} \quad (7.32)$$

A solution to (7.34-7.31) is

$$\begin{bmatrix} u'_r(r) \\ v'_r(r) \end{bmatrix} = \begin{bmatrix} \check{u}_r \\ \check{v}_r \end{bmatrix} J_1(\Omega r), \quad \begin{bmatrix} u'_z(r) \\ v'_z(r) \\ P'_g(r) \\ \phi'(r) \\ T'(r) \end{bmatrix} = \begin{bmatrix} \check{u}_z \\ \check{v}_z \\ \check{P}_g \\ \check{\phi} \\ \check{T} \end{bmatrix} J_0(\Omega r) \quad (7.33)$$

Here, J_0 and J_1 are zero- and first-order Bessel functions of the first kind, and Ω are the zeroes of J_1 (which are also the zeros of dJ_0/dr). The resulting equations of motion are:

$$\sigma\check{\phi} + ik\bar{\phi}\check{u}_z + \bar{\phi}\Omega\check{u}_r = 0 \quad (7.34)$$

$$-\sigma\check{\phi} + ik(1 - \bar{\phi})\check{v}_z + (1 - \bar{\phi})\Omega\check{v}_r - ik\bar{v}_z\check{\phi} = 0 \quad (7.35)$$

$$\begin{aligned} \sigma\check{v}_z - \frac{\sigma\bar{v}_z}{(1 - \bar{\phi})}\check{\phi} = & -ik\frac{LR^{-1}}{Fr}\check{P}_g + \frac{L(R^{-1} - 1)}{Fr}(\bar{\beta}(\check{u}_z - \check{v}_z) - \bar{\beta}_\phi\bar{v}_z\check{\phi}) \\ & - \frac{ik\bar{v}_z^2}{(1 - \bar{\phi})}\check{\phi} - ik\bar{v}_z\check{v}_z \end{aligned} \quad (7.36)$$

$$\sigma\check{v}_r = \Omega\frac{LR^{-1}}{Fr}\check{P}_g + \frac{L(R^{-1} - 1)}{Fr}\bar{\beta}(\check{u}_r - \check{v}_r) - ik\bar{v}_z\check{v}_r \quad (7.37)$$

$$\begin{aligned} \sigma\bar{\phi}\check{u}_z = & -ik\bar{\phi}\frac{L}{Fr}\check{P}_g + (1 - \bar{\phi})\frac{L(1 - R)}{Fr}(\bar{\beta}(\check{v}_z - \check{u}_z) + \bar{\beta}_\phi\bar{u}_g\check{\phi}) \\ & - k^2\left(\frac{2\bar{\mu}_s + \bar{\zeta}_s}{L}\right)\check{u}_z - \Omega^2\frac{\bar{\mu}_s}{L}\check{u}_z + ik\Omega\left(\frac{\bar{\mu}_s + \bar{\zeta}_s}{L}\right)\check{u}_r \\ & - ik(\bar{P}_{s\phi}\check{\phi} + \bar{P}_{sT}\check{T}) - \frac{L}{Fr}\phi' \end{aligned} \quad (7.38)$$

$$\begin{aligned} \sigma\bar{\phi}\check{u}_r = & \Omega\bar{\phi}\frac{L}{Fr}\check{P}_g + (1 - \bar{\phi})\frac{L(1 - R)}{Fr}\bar{\beta}(\check{v}_r - \check{u}_r) + \Omega\bar{P}_{s\phi}\check{\phi} + \Omega\bar{P}_{sT}\check{T} \\ & - \Omega^2\left(\frac{2\bar{\mu}_s + \bar{\zeta}_s}{L}\right)\check{u}_r - k^2\frac{\bar{\mu}_s}{L}\check{u}_r - ik\Omega\left(\frac{\bar{\mu}_s + \bar{\zeta}_s}{L}\right)\check{u}_z \end{aligned} \quad (7.39)$$

$$\begin{aligned} \frac{3}{2}\sigma(\bar{\phi}\check{T} + \bar{T}\check{\phi}) = & -\bar{P}_s(ik\check{u}_z + \Omega\check{u}_r) + L(\bar{\Gamma}_{u_z}u'_z + \bar{\Gamma}_{v_z}v'_z + \bar{\Gamma}_T T' + \bar{\Gamma}_\phi\phi') \\ & + \frac{\bar{\lambda}_h}{L}(-k^2 - \Omega^2)\check{\phi} + \frac{\bar{\lambda}}{L}(-k^2 - \Omega^2)\check{T} - \frac{3}{2}\bar{\phi}\bar{T}(ik\check{u}_z + \Omega\check{u}_r) \end{aligned} \quad (7.40)$$

As it turns out, after neglecting the fluid-phase convective terms and inserting these solutions into the equation of motion (7.34-7.40), this produces a system whose dispersion relation is nearly identical to that of the limiting case of the planar fluidized bed in the previous section. The only difference is that the discrete transverse wavenumber is represented as Θ in the planar system, and Ω in the cylindrical bed. The differences in the stability characteristics of each system arising from this discrepancy are explored in the following section.

7.3 Results

Figure 7.1 displays growth rate traces for the first several leading eigenvalues of the planar and cylindrical beds. Each curve represents a mode corresponding to a discrete transverse wavenumber. The curves correspond to $\Theta = n\pi$ and the first n zeroes of the Bessel function J_1 in figures 7.1 (a) and (b), respectively, where $n = 1, 2, \dots, 8$. The dominant mode corresponds to the first discrete transverse wavenumber for each case, corresponding to perturbed flow variables exhibiting the least amount of variability in the lateral direction. The dominant growth rate of the planar bed has a value of 3.446, while that of the cylindrical bed is 3.388. Likewise, the growth rate of the leading mode of the planar bed is higher than that of the cylindrical bed over all wavenumbers k .

As it turns out, the growth rate corresponding to the dominant mode of the 2D bed is slightly higher than that of the 3D bed over all bed parameters tested. As the analytic solution 7.22 is too complex to derive a condition for the dominant mode, defined for the highest value of σ_r at the point where $\partial\sigma_r/\partial k = 0$, this was tested by numerically solving the dispersion relation over a wide range of the dimensional fluidized bed parameters: d_p , ρ_s , $\bar{\phi}$, and e_p . This result is similar to that described by Squire's Theorem [1]. The cited author investigated the stability of flow between parallel walls, and determined that, at a particular value of the Reynolds number, the two-dimensional disturbance is always less stable than the corresponding three-dimensional disturbance. Although the results of Squire correspond to wavenumbers of longitudinal disturbances, the analogy is relevant due to the presence of unstable, purely transverse modes – $\sigma_r > 0$ for $(k = 0, \Theta \neq 0)$ and $(k = 0, \Omega \neq 0)$.

As a final application of the simplified analysis, the contour graph of the vertical bed (see section 5.5) is recreated using the simplifying boundary condition assumptions. However, in this case the results are presented as a function of the dimensional solid-phase

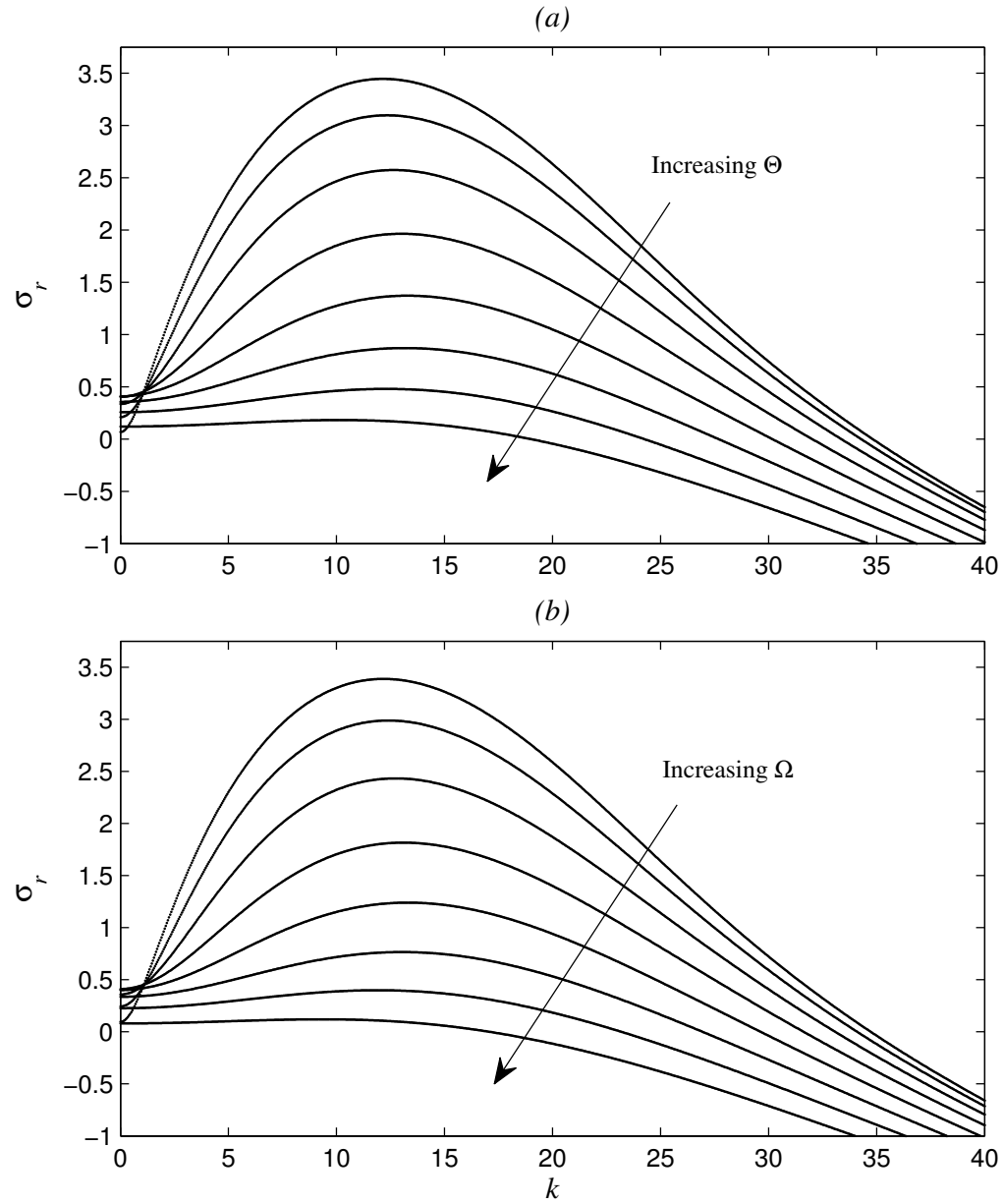


Figure 7.1: Growth rates of first several leading eigenvalues for the (a) planar and (b) cylindrical beds. $\bar{\phi} = 0.60$, $L = 50$, $e_p = 0.95$.

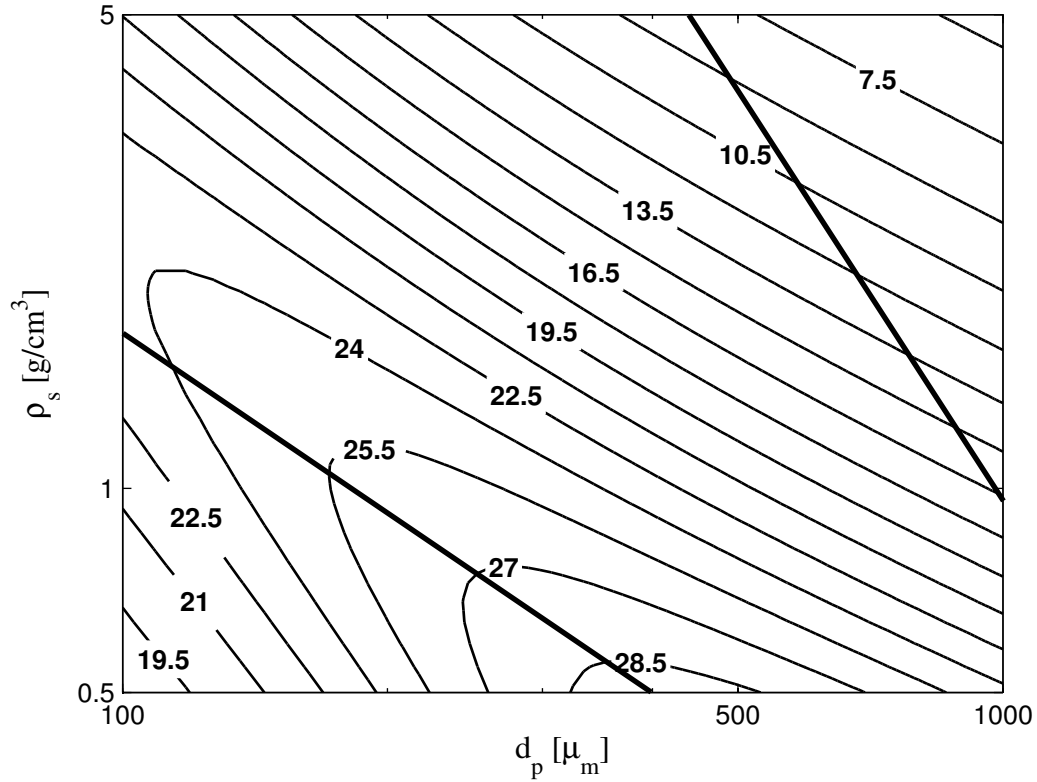


Figure 7.2: Growth rate contours of first dominant transverse mode of the 3D system as a function of particle diameter and density. Dark lines denote barriers between the Geldart regimes.

density and diameter, and the axes of the contour graph are shown in log scale. Doing so allows direct comparison with the Geldart chart (figure 2.2). In this case, the discrete transverse wavenumber Ω must be specified. The first physically relevant transverse mode corresponds to the first non-zero zero of the Bessel function J_1 , $\Omega_1 = 3.8316$. The dominant longitudinal mode then determined is the dominant mode of the entire system. The contour plot showing this value as a function of ρ_s and d_p is shown in figure 7.2.

Straight lines have been superimposed onto the graph to denote the barriers between the Geldart regimes. The lower-left region denotes Geldart A particles, the upper-right region is comprised of Geldart D particles, and Geldart B particles lie in the center. The growth rate exhibits similar dependence upon the bed material properties as that seen in figure 5.5. The largest growth rates are seen in the Geldart A and B

regions, with preference given to the latter. In the A regime, as the particle density and diameter decrease, so does the growth constant. In the D regime, an increase in ρ_s and d_p is associated with a decrease in the growth constant. The increased instability of the Geldart B regime is again represented, although presentation of the characteristics in log-scale shows that beds with particles along the Geldart A-B border exhibit the largest growth rates.

7.4 Transverse Wave Solution

This chapter is concluded by investigating the case of the non-oscillatory ($\sigma_i = 0$), layering ($k = 0$) modes present in both the 2D and 3D beds. Since the quartic dispersion relation is identical in form for each system, the following derivation uses the notation of the planar bed, although it is interchangeable with that of the cylindrical bed. To begin, the convective, viscous, and body force terms of the fluid phase are neglected, as before. As such, the derivation begins with the linearized equations 7.12-7.13, 7.16-7.18, and 7.20-7.21, although without the terms proportional to k . The fluid-phase pressure equation may be eliminated by cross-differentiation of the fluid-phase momentum equations, which results in the following expression:

$$u'_g - u'_s = -\frac{\bar{\beta}_\phi}{\bar{\beta}} \bar{u}_g \phi' \quad (7.41)$$

The dependence of u'_s and u'_g in the granular temperature equation 7.18 may be eliminated using 7.41, as well as the fact that $\Gamma_{u_g} = -\Gamma_{u_s}$. With this, the axial solid-phase momentum and combined fluid momentum equations are trivialized. The relevant equations are:

$$\sigma \check{\phi} + \bar{\phi} \Theta \check{v}_s = 0 \quad (7.42)$$

$$-\sigma \check{\phi} + (1 - \bar{\phi}) \Theta \check{v}_g = 0 \quad (7.43)$$

$$\sigma \bar{\phi} \check{v}_s = \frac{L(1-R)}{Fr} \bar{\beta} (\check{v}_g - \check{v}_s) + \Theta \bar{P}_{s\phi} \check{\phi} + \Theta \bar{P}_{sT} \check{T} - \Theta^2 \left(\frac{2\bar{\mu}_s + \bar{\zeta}_s}{L} \right) \check{v}_s \quad (7.44)$$

$$\begin{aligned} \frac{3}{2} \sigma (\bar{\phi} \check{T} + \bar{T} \check{\phi}) = & -\frac{3}{2} \bar{\phi} \bar{T} \Theta \check{v}_s - \Theta \bar{P}_s \check{v}_s - \frac{\Theta^2}{L} (\bar{\lambda}_h \check{\phi} + \bar{\lambda} \check{T}) \\ & + L \left(\bar{\Gamma}_\phi \phi' + \bar{\Gamma}_T T' - \bar{\Gamma}_{u_g} \frac{\bar{\beta}_\phi}{\bar{\beta}} \bar{u}_g \phi' \right) \end{aligned} \quad (7.45)$$

The resulting dispersion relation is a cubic polynomial:

$$\sigma^3 + \alpha_1 \sigma^2 + \alpha_2 \sigma + \alpha_3 = 0 \quad (7.46)$$

The coefficients are given by:

$$\alpha_1 = \left(\frac{3}{2}\bar{\phi}\right)^{-1} \left[\frac{3}{2} \frac{1}{1-\bar{\phi}} \frac{L}{Fr} \bar{\beta} - L\Gamma_T + \frac{\Theta^2}{L} \left(\frac{3}{2} (2\bar{\mu}_s + \bar{\zeta}_s) + \bar{\lambda}_s \right) \right] \quad (7.47)$$

$$\begin{aligned} \alpha_2 = \left(\frac{3}{2}\bar{\phi}\right)^{-1} & \left[\left(\frac{1}{1-\bar{\phi}} \frac{L}{Fr} \bar{\beta} + L\Theta^2 (2\bar{\mu}_s + \bar{\zeta}_s)^2 \right) \left(\Theta^2 \frac{\bar{\lambda}_s}{L} - L\bar{\Gamma}_T \right) \right. \\ & \left. + \frac{3}{2}\bar{\phi}\Theta^2 (\bar{\phi}\bar{P}_{s\phi} - \bar{T}\bar{P}_{sT}) + \Theta\bar{P}_{sT} \left(\frac{3}{2}\bar{\phi}\bar{T} + \Theta\bar{P}_s \right) \right] \end{aligned} \quad (7.48)$$

$$\begin{aligned} \alpha_3 = \left(\frac{3}{2}\bar{\phi}\right)^{-1} & \left[L\Theta^2 (\bar{\Gamma}_\phi \bar{P}_{sT} - \bar{\Gamma}_T \bar{P}_{s\phi}) + \frac{\Theta^4}{L} (\bar{\lambda}_s \bar{P}_{s\phi} - \bar{\lambda}_h \bar{P}_{sT}) \right. \\ & \left. - L\Theta^2 \bar{\Gamma}_{u_g} \bar{P}_{sT} \frac{\bar{\beta}_\phi}{\bar{\beta}} \bar{u}_g \right] \end{aligned} \quad (7.49)$$

The real parts of the roots are negative if and only if α_1 , α_2 , α_3 , and $\alpha_1\alpha_2 - \alpha_3$ are positive. The only relevant condition which breaks the sufficient condition for instability is the third, from which the upper bound on the transverse wavenumber for an unstable system is obtained:

$$\Theta^2 < L^2 \left[\frac{(\bar{\Gamma}_\phi \bar{P}_{sT} - \bar{\Gamma}_T \bar{P}_{s\phi}) - \bar{\Gamma}_{u_g} \bar{P}_{sT} \frac{\bar{\beta}_\phi}{\bar{\beta}} \bar{u}_g}{\bar{\lambda}_h \bar{P}_{sT} - \bar{\lambda}_s \bar{P}_{s\phi}} \right] \quad (7.50)$$

Notable is the L^2 dependence on the upper bound of Θ , demonstrating that beds with a larger dimensionless bed width allow disturbances with larger transverse variability of the perturbed functions. The complex frequency $\sigma(\Theta)$ may be expanded into a Taylor series near $\Theta = 0$:

$$\sigma \simeq i\Theta\sigma_1 + \Theta^2\sigma_2 + O(\Theta^3) \quad (7.51)$$

The first term represents the group velocity, of which $\sigma_1 = \partial\sigma_i/\partial k$, while

$$\sigma_2 = 0.5 (\partial^2\sigma_r/\partial k^2) \quad (7.52)$$

is the group velocity dispersion. The sign of σ_2 determines the stability of the long-wavelength system, with a positive value being a sufficient, but not necessary, condition for instability. Although the transverse wavenumber is discrete and the first non-zero

value $\Theta = \pi$ is not necessarily small, this method may still provide physical insight into the mechanisms through which these modes become unstable. The expansion 7.51 is inserted into the dispersion relation 7.46 and terms of $O(\Theta)$ and $O(\Theta^2)$ are matched. This yields $\sigma_1 = 0$, corresponding to a non-oscillatory mode, and the following expression for σ_2 :

$$\sigma_2 = \frac{Fr}{L} \frac{\bar{\phi}(1 - \bar{\phi})}{\bar{\beta}\bar{\Gamma}_T} \left[(\bar{\Gamma}_\phi \bar{P}_{sT} - \bar{\Gamma}_T \bar{P}_{s\phi}) - \bar{\Gamma}_{u_g} \bar{P}_{sT} \frac{\bar{\beta}_\phi}{\bar{\beta}} \bar{u}_g \right] \quad (7.53)$$

The long-transverse-wavelength instability mechanisms are similar to those found in the case of the unbounded bed in section 4.6.2. The first parenthesized term in the brackets represents the effective bulk elasticity term. This term again behaves such that the sum of its two components is found to be always negative, representing a stabilizing mechanism. As such, the term $(\bar{\Gamma}_\phi \bar{P}_{sT} - \bar{\Gamma}_T \bar{P}_{s\phi})$ is essentially the non-isothermal analogue of the isothermal bulk elasticity term $\bar{P}_{s\phi}$ [2, 3], which is found to stabilize the flow.

The second term represents the combination of two destabilizing mechanisms. The first is the drag force fluctuation term $\bar{\beta}_\phi$, the classically-recognized [2] inertial instability mechanism resulting from the large settling time of particles to disturbances in the volume fraction. The term $\bar{\Gamma}_{u_g} \bar{P}_{sT}$, similar in form to that identified by Koch & Sangani [4], represents the generation of solid-phase pressure caused by perturbations of the base-state granular temperature and, through association, the base-state fluid velocity. A volume fraction perturbation yields a discrepancy between the base-state phasic axial velocities, resulting in an increase of the granular temperature and the particle-phase pressure. The resulting mechanism drives rapidly-moving particles from regions of low ϕ to dense regions with lower particle movement. As with the drag force term, this effect exacerbates lateral perturbations of the volume fraction over time, rather than amplifying over the time period of a disturbance.

The terms inside the brackets of equation 7.53 are similar in form and behavior to those of the analogous expression for σ_2 of an unbounded bed, given by equation 4.61. The primary difference is a result of the impermeable walls: the former expression is multiplied by $L/Fr = \hat{L}g/U^2$.

References

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Chapter 8

Experimental Determination of the Dominant Frequency

Contour plots displaying the growth rate of the dominant linear disturbance as a function of particle diameter and density were found to qualitatively match the tendency of a bed to exhibit bubbling at minimum fluidization recorded by Geldart [1]. This result serves as a form of validation for the linear stability analysis and suggests that the dominant linear mode may remain relevant as disturbances increase in amplitude past the point at which the nonlinear terms become appreciable. A second form of comparison consists of determining the dominant frequency of fluidized bed perturbations existing in experiment. These disturbances do not necessarily indicate the presence of bubbles. In fact, it is found that the dominant disturbance identified in this manner has a frequency which is significantly higher than that of the visually-identified bubble frequency. Using a series of wall-mounted, non-intrusive pressure taps, the time signal of the fluid-phase pressure may be captured at several points along the bed. Conversion of the time signal to Fourier space allows one to identify the frequency component with the largest amplitude - that is, the dominant experimental frequency.

Two experiments were constructed and performed to accomplish this task: a vertically-oriented bed in a cylindrical tube, and a bed capable of being tilted to an arbitrary angle. The dominant frequencies of the experimental set-ups were compared to those of the corresponding stability analysis. Results are presented as a function of fluidization velocity and angle of inclination, as well as particle size and density. Five bed materials were employed during the experiments. Each of the materials consisted of

Table 8.1: Properties of the bed materials used in each experiment. Diameters are mean values extrapolated from sieve size distributions provided by the manufacturer.

Name	d_p [μm]	ρ_s [$\text{kg}\cdot\text{m}^{-3}$]	$\bar{\phi}_{rest}$	ϕ_s
Glass Spheres	134	1250	0.585	1.0
CarboAccucast ID50	255	3240	0.570	0.9
EconoProp 40/70	334	2700	0.578	0.9
CarboHSP 30/60	430	3560	0.562	0.9
CarboProp 30/60	443	3150	0.564	0.9

nearly-spherical, ceramic proppants provided by Carbo[®] Ceramics, along with the addition of soda-lime glass spheres. Table 8.1 lists the relevant properties of each material, numbered in order of increasing mean particle diameter. These materials were chosen because they represent a sizeable portion of the Geldart B regime of particle characterization. The glass spheres are assumed to have a sphericity of $\phi_s = 1.0$. Those of the remaining materials, as well as the density and the mean diameter, were obtained from the manufacturer. The mean solids volume fraction at rest, $\bar{\phi}_{rest}$, was determined using the method preceding equation 8.1 outlined in the following section.

8.1 Experimental Setup - Vertical Bed

The schematic of the vertically-oriented bed is shown in figure 8.1. The geometric constraints of the bed consist of the 8.89cm (3.5”) I.D. polycarbonate tube – with markings on the side for height measurements – and the distributor plate at the bottom of the tube. The top of the tube is open to atmospheric air. Air at STP flows in from a laboratory air supply, through a pair of Omega FL-3440C rotameters (0 - 66.3 liters/min at STP), and into a small reservoir which sits just below the distributor plate. A pressure gauge connected to the node joining the rotameter exits and the reservoir inlet provides a pressure compensation for the rotameter readings. Air flows from the reservoir, through the distributor plate, and into the bed before returning to the atmosphere. A photograph of the experimental setup is shown in figure 8.2.

Six pressure taps run along the length of the tube. These are marked as P1-P6 on the schematic, with the latter measuring the pressure inside of the reservoir. Holes were bored into the tube such that the pressure taps were flush with the inside of the tube. Wire mesh attached to the ends of each tap ensured that the taps were non-intrusive

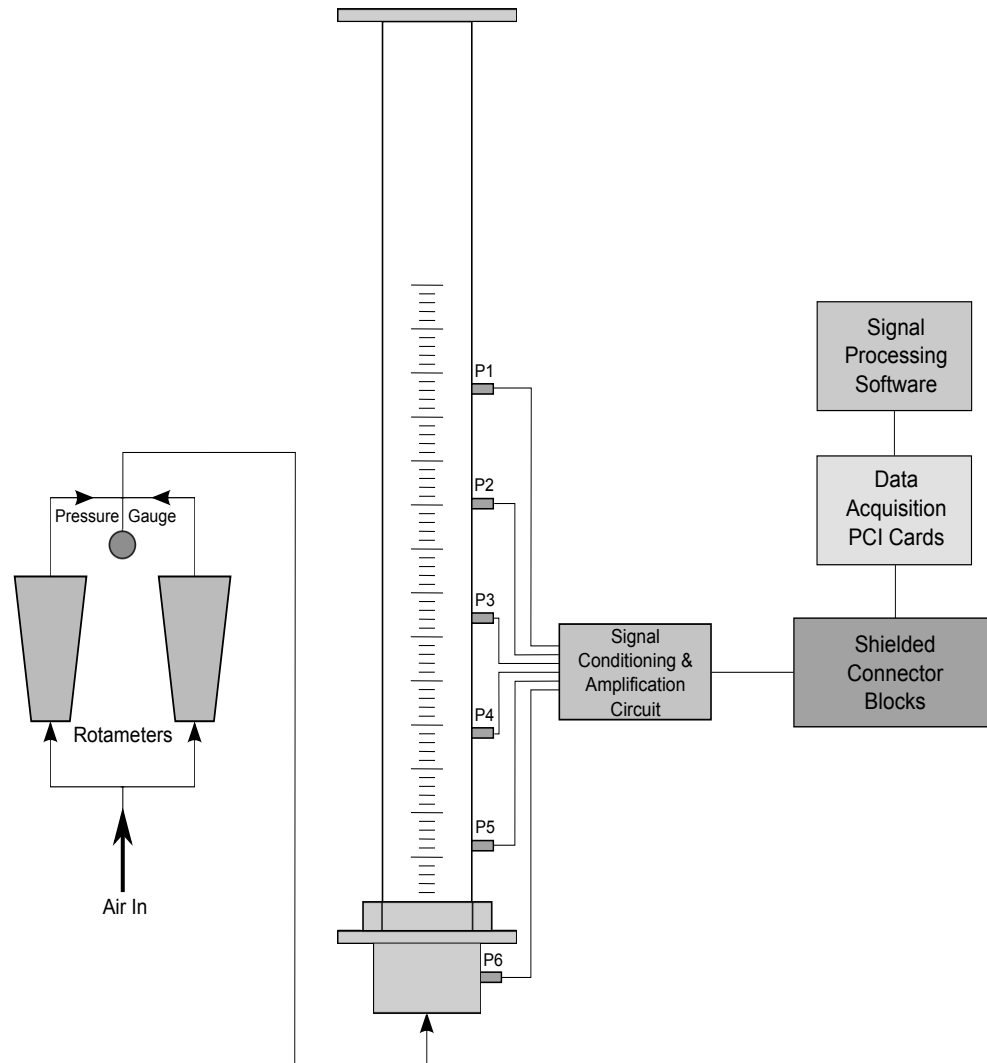


Figure 8.1: Schematic of the vertically-oriented fluidized bed experiment. Hatches on the tube are used for voidage measurements.



Figure 8.2: Photograph of the vertically-oriented fluidized bed experiment. Tubes connected to the pressure taps converge on the signal conditioning and amplification circuit, which is not visible.

to the bed, allowing flow of the fluid but not the particles into their respective tubes. Detailed information and drawings regarding the design and build of the pressure taps are presented in a previous work [2] and are not reproduced here.

Bed material was inserted into the top opening of the tube until the height of the compacted bed was approximately 5cm (1.97") above the P2 tap. Voidage – and, alternately, volume fraction ϕ – measurements were obtained by recording the mass of the particles m_b and the height h of the bed, along with the expression:

$$\epsilon = 1 - \phi = 1 - \frac{\rho_b}{\rho_s} = 1 - \frac{m_b}{\rho_s A_t h} \quad (8.1)$$

Here, A_t is the cross-sectional area of the tube and ρ_b is the bulk density of the bed. The mean gas velocity was determined by dividing the volumetric flow rate of the gas, Q – determined using the rotameters – by the area of the tube:

$$U = Q/A_t \quad (8.2)$$

Note that, for convenience, the same variable is used to denote the mean gas velocity in both the numerical and experimental methods. For the case of the inclined bed, the numerical profile is averaged using the expression

$$U = \frac{1}{\hat{L}} \int_{-\hat{L}/2}^{\hat{L}/2} \bar{u}_g dy \quad (8.3)$$

Each pressure tap was connected via 0.75cm (0.295") I.D. Tygon tubing to Omega PX26-005GV (5 psig-maximum) and PX-015GV (15 psig-maximum) gage pressure transducers. The reservoir probe was connected to the 15 psig-maximum transducer due to the higher pressure in the region below the porous plate. The transducers were connected to a signal conditioning circuit board which amplified the signals from the transducers to a 0-10V output. The resulting signals travelled to a National Instruments (NI) SCB-68 shielded I/O connector block and then sent to a NI PCI-MIO-16E-4 data acquisition (DAQ) card. From here, the captured 0-10V signals were recorded and processed using NI LabVIEW v6.1 software. A schematic of the circuit board, as well as the LabVIEW virtual instrument (VI) programs used to acquire, process, and store the data, are presented in Mandich [2].

8.2 Experimental Setup - Inclined Bed

The the tilted bed is similar to the vertical bed. The confining tube has an inner diameter of 6.35cm (2.5"). There are two additional sets of pressure taps along the wall.

The entire apparatus was mounted onto a frame with a hinge mechanism which allows for the bed to be tilted to an arbitrary angle. A photograph of the experiment is shown in figure 8.4. Calibration of the tilting angle was performed by measuring two sides of the triangle formed by the vertical post, the distance from this post to the reservoir, and the post acting as the hypotenuse. The usual trigonometric identity was employed to determine θ . The “bottom” of the bed is defined as that closest to the ground, and the top as the opposite.

The hardware and software for the inclined bed were the same as those used for the vertical bed, although the inclined bed employed 10 more pressure taps and transducers, two additional signal processing boards, a second shielded connector block, and another DAQ card. The corresponding LabVIEW VI program was also expanded to accommodate the 16 pressure signals.

The extra sets of pressure taps were employed due to the asymmetric nature of inclined granular flows. Inclined fluidized beds experience a slightly different flow evolution than their vertical counterparts as the mean gas velocity increases. Vertical beds normally traverse from the packed bed regime, through smooth fluidization, and on to bubbling. For beds inclined above approximately 10° , there is an extra regime of fluidization [3]. After the packed bed regime, the bed experiences separation between the top and bottom regions of the domain. Along the upper wall, a channel develops which is characterized by fluidization and bubble propagation. A distinct line reveals the borderline between the upper and lower regions. As U increases, this channel becomes larger, comprising more of the domain. After a certain fluidization velocity, the distinction between the two regions disappears, and the bed is fully fluidized. The initial channel development occurs at the usual minimum fluidization velocity, modified to account for the tilting angle:

$$u_{mf} = \frac{\phi \hat{u}_t}{f_0(\phi) \cos \theta} \quad (8.4)$$

Since the initial development of bubbles is of primary interest, the experimental work focuses on the narrow range of fluidization velocity corresponding to channeling in the upper region of the bed. Three sets of pressure taps were placed along the top, middle, and bottom of the bed to determine whether any discrepancy in the fluid-phase pressure exists among each region. As mentioned, beds at or below approximately 10° do not display this channeling behavior. It was found during experiment that beds above 35° exhibit channeling almost exclusively up to very high values of the mean gas velocity.

These angle values are approximate and vary depending upon the mean particle diameter and density. Because of this, the current experiment only considers beds in the range of $15^\circ \leq \theta \leq 30^\circ$.

8.3 Pressure Measurements

Pressure measurements were originally taken at a sampling rate of 200 Hz for a total of 30 seconds. However, it was found that the dominant frequency may be found using lower values for each parameter. For both the vertical and inclined beds, a sampling rate of 120 Hz over 10 seconds was found to be sufficient, for a total of 1,200 pressure measurements per pressure tap per run. The largest relevant frequency component was generally at or below 40 Hz. The sampling rate of 120 Hz, resolving frequencies up to 60 Hz, was then sufficient to capture disturbances at this frequency. Running the simulation for longer time periods yielded a frequency spectrum which allowed for greater resolution in determining the dominant experimental frequency. However, when reducing the run time from 30 to 10 seconds, there was no significant difference in the value of the dominant frequency.

As mentioned, pressure taps along the top, middle, and bottom of the inclined bed were employed to determine what variation, if any, the asymmetric flow had on the dominant frequency. As is demonstrated in the following section, the time signal of the uppermost tap varies significantly from the others, primarily due to the presence of the upper channel flow. However, the frequency spectrum and, most importantly, the frequency value of the dominant disturbance, are similar among all three pressure tap locations. Therefore, dominant frequency values of the inclined bed are produced by averaging the results of a set of adjacent pressure taps. The frequency values of the vertical bed are determined by averaging the results of three separate time runs for a single pressure tap.

8.4 Experimental Results

8.4.1 Pressure Drop

Pressure drop diagrams for each of the five bed materials are shown in figure 8.5. The pressure drop is normalized and rendered dimensionless by the weight of the bed, as well as the inclination angle. Each point represents the average of six separate runs,

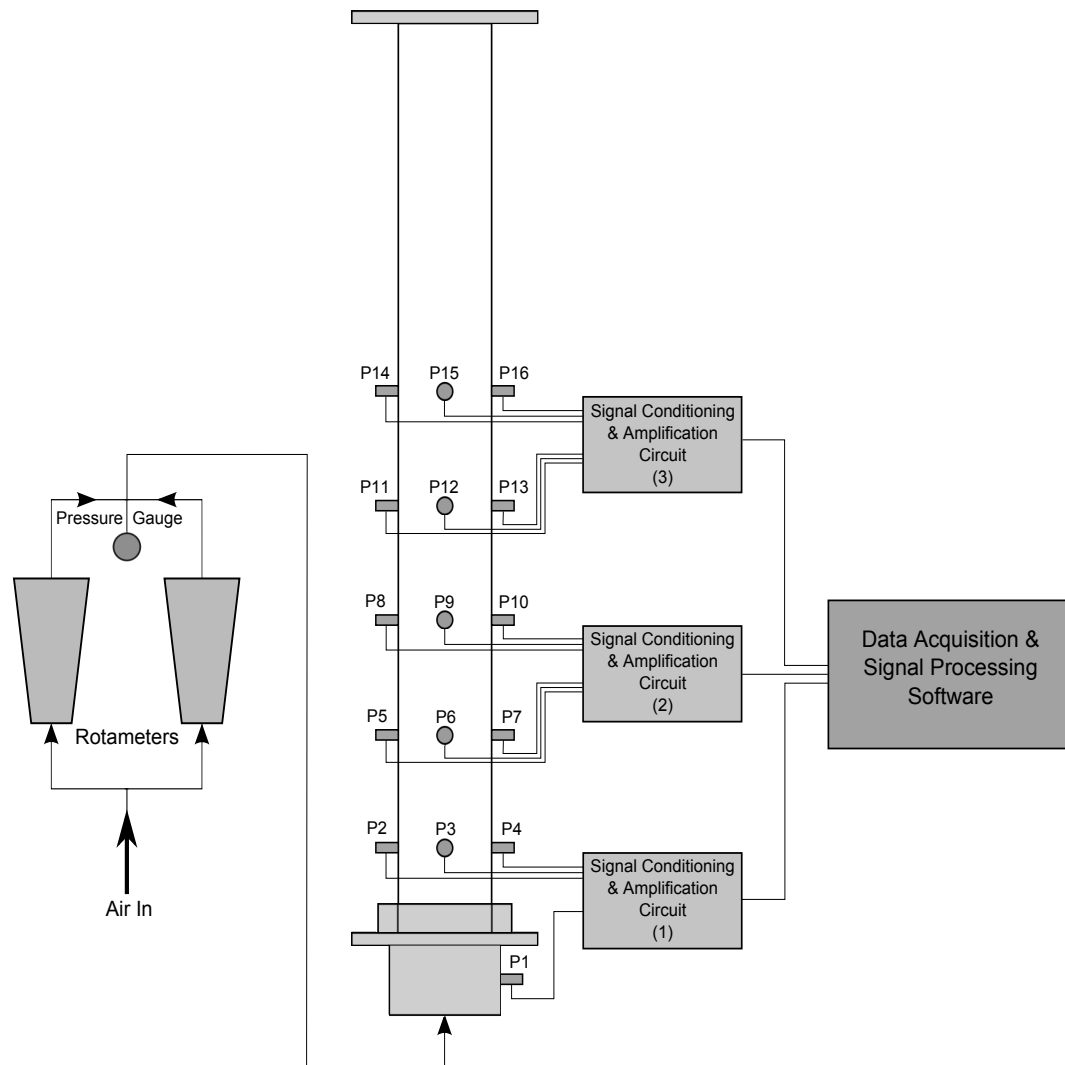


Figure 8.3: Schematic of the inclined fluidized bed experiment. The tube is tilted such that the pressure tap P1 is on the side furthest to the ground.

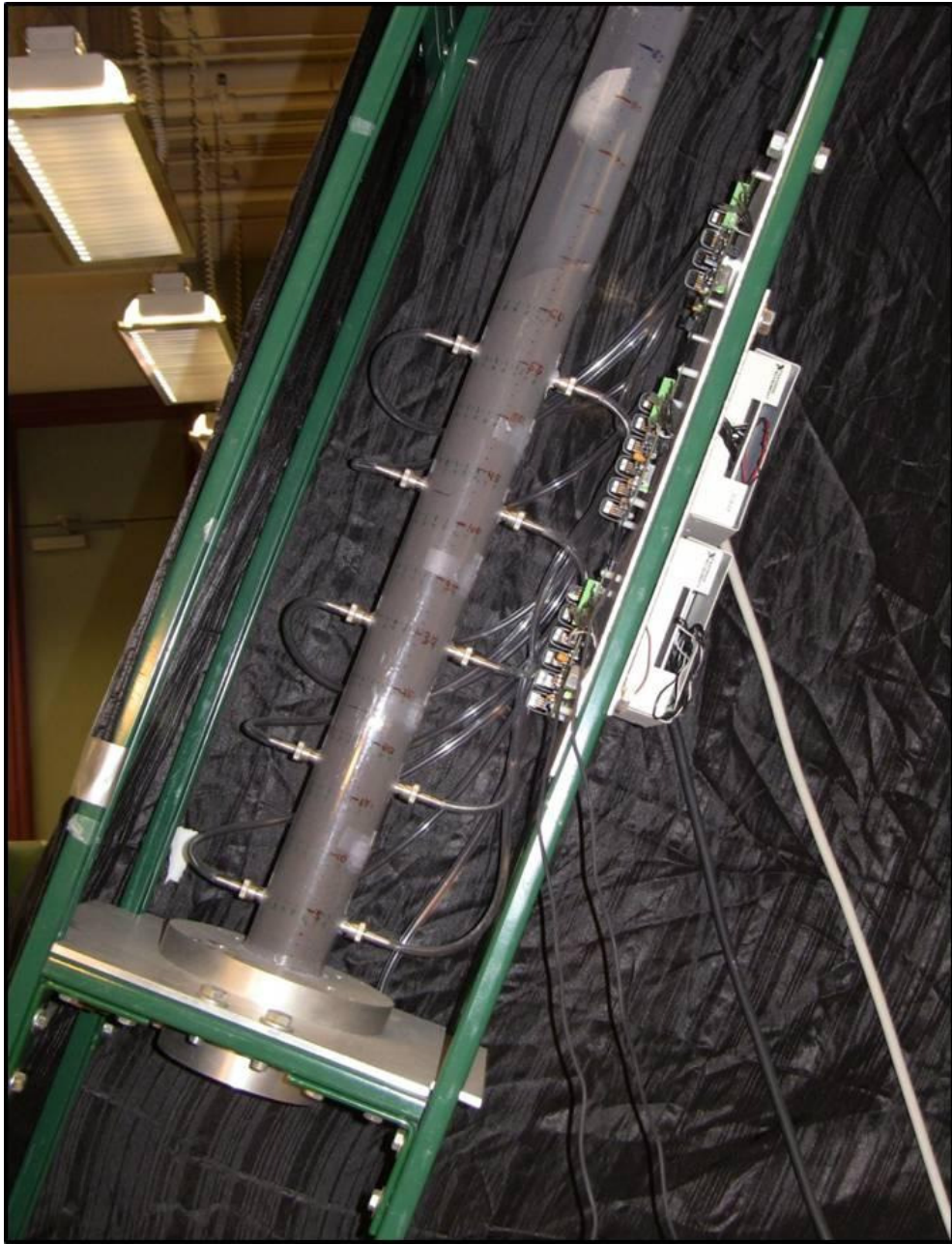


Figure 8.4: Photograph of the inclined fluidized bed experiment. The third set of pressure taps is behind the tube, facing the camera.

recorded for a total of 10 seconds each: three with the fluidization velocity beginning at zero and increasing, and the other three beginning at the maximum fluidization velocity and decreasing. This was done to dampen the effects of hysteresis which occur as a result of bed compaction in the unfluidized state [4]. The results portray only the inclined systems. The pressure drops of vertical systems are well known in the literature and similar in shape to those shown in figure 8.5. These are not shown, although the plots for materials 1, 3, and 5 may be found in Mandich [2].

A few trends are shared by all of the bed materials. The minimum fluidization velocity is defined as the point at which the pressure drop becomes invariant to any increase in the mean gas velocity. This separates the regime of bed expansion below u_{mf} to the state of fluidization beyond this gas velocity. In the current case of the inclined bed the onset of minimum fluidization is defined as the point at which channel flow begins. It is evident from the pressure drop plots that accurate definition of the point is difficult due to a transition region which ranges from the expanding bed to the fluidized state. This is especially true for bed materials whose particle diameter distribution is large – in the current experiment, materials 4 and 5 appear to exhibit the largest transition phases. However, some general conclusions may be inferred. The slope of the pressure drop in the region of the expanding bed is lower for beds with larger tilting angles. Likewise, the beginning of the transition from bed expansion to fluidization occurs at a higher gas velocity for more inclined beds. These trends hold for all five of the bed materials tested. However, all of the pressure drop traces collapse onto a value of $\frac{\Delta P \cos \theta}{\phi_o \rho_s g h} \simeq 1$ after the transition region.

The bed materials with the larger particles exhibit a second region of increase in the pressure drop as the gas velocity increases further. This represents the transition from channel flow to full fluidization of the bed. The increase in normalized pressure results from the beginnings of full bed collapse, which produces a decrease in the mean bed height $\bar{h}$, and thus $\bar{\phi}$. Notable at this point is the larger pressure drop increase for beds at the highest angles of inclination, opposite to the behavior seen in the initial bed expansion. The exception to these observations is Material 1, which does not appear to have a second region of pressure drop increase, and shows the least-inclined bed having the largest pressure drop at the highest values of U tested. The reasons for this are not clear, although it is expected to be a result of both the small amount of particle diameter variability and the perfect sphericity of this material. This is also expected to be the

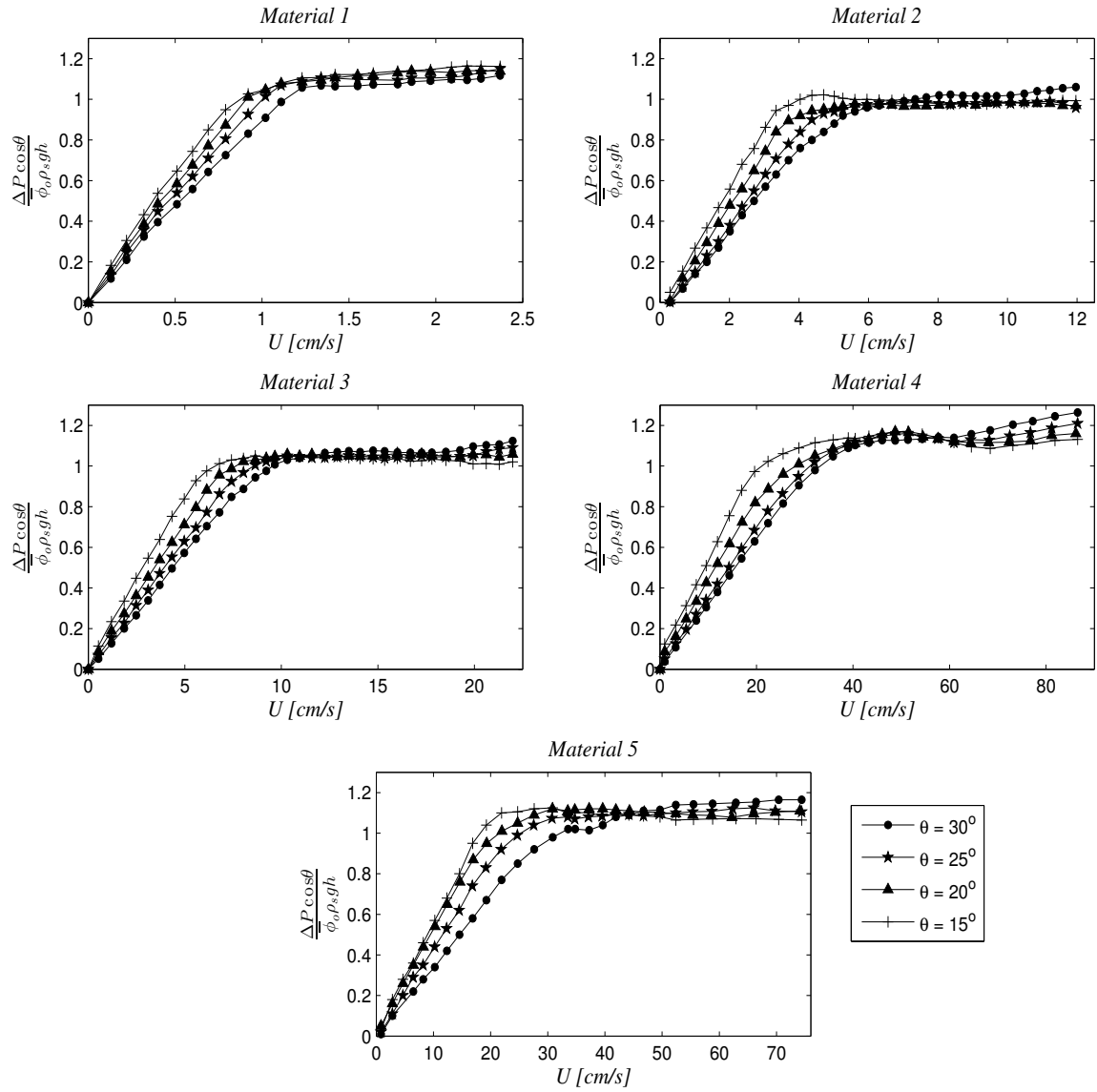


Figure 8.5: Normalized pressure drop vs. mean gas velocity for each of the five bed materials at several angles of inclination, as shown.

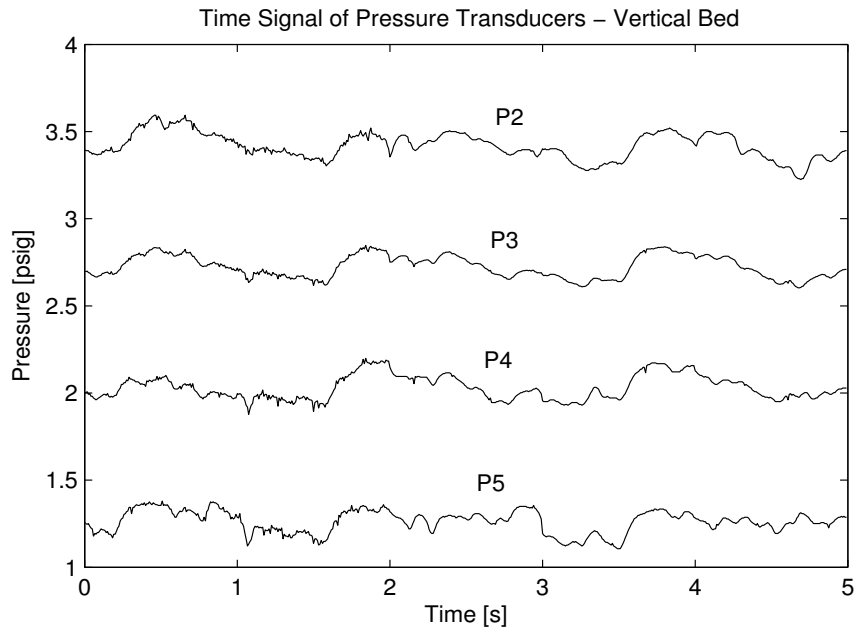


Figure 8.6: Pressure time signals of the vertical bed corresponding to the pressure taps numbered. Data was acquired for Material 3 with a normalized gas velocity of $U/u_{mf} = 1.3$.

reason behind the smooth and rapid transition from channel flow to full fluidization evident in the pressure drop plot for this material.

8.4.2 Pressure Time Signal

The pressure time signals for the vertical bed at various heights are shown in figure 8.6. The large-time-scale fluctuations are visible in all of the pressure signals. These are primarily a result of bubbling and slugging, the effect of which is to lift and lower the bed as they propagate through the medium. These changes in the hydrostatic pressure are manifested in the large-scale fluctuations visible in figure 8.6. As these fluctuations occur at the same time, the expected linear pressure drop throughout the bed is maintained. There are a few exceptions to this behavior. The dip in the pressure signal of P5 immediately following $t = 1$ s is visible in the other pressure traces, although the magnitude of the dip decreases with distance from the distributor plate. This behavior is seen in several other instances within the 5-second time period over which the pressure signals were captured, and is attributed to the uneven pressure distribution in the region immediately above the distributor plate.

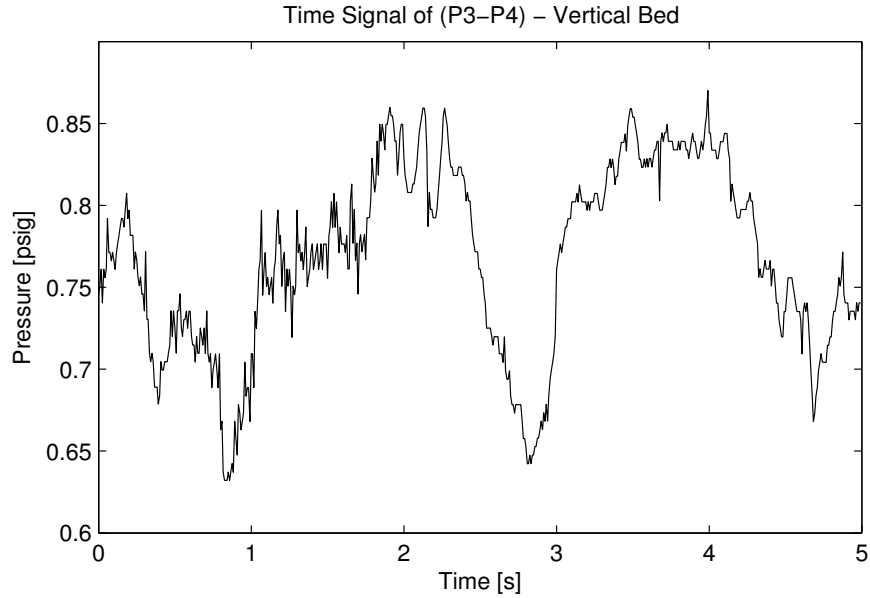


Figure 8.7: Time signal of the difference between the pressure signals P3 and P4 of figure 8.6.

In an attempt to eliminate the large-scale hydrostatic fluctuations from the pressure time signals, the traces corresponding to two adjacent pressure taps are subtracted. Figure 8.7 displays the difference between the pressure taps P3 and P4. In this plot a series of low- and high-frequency pressure perturbations is displayed. The largest disturbances – for example, the dip between $t = 2$ and $t = 3$ seconds – are likely products of the uneven pressure distribution of the bed immediately above the distributor plate. There exist numerous small-time-scale perturbations, although the dominant time scales are obscured in the time domain. This difficulty is overcome by applying a Fourier transform to the time signals.

Figure 8.8 displays the frequency spectra of the pressure signals from P3, P4, and P3-P4. As expected, the low-frequency components are the largest and increase as the frequency approaches the DC component. The location of the dominant relevant frequency is ambiguous when examining the frequency spectra of a single pressure signal. The frequency spectra $P3(f)$ and $P4(f)$ exhibit amplitude spikes above 20 Hz, although the largest amplitude is difficult to locate. The low-frequency components may be attenuated by taking the Fourier transform of the time signal $(P3 - P4)(t)$. As shown in figure 8.7, doing so removes a large portion of the hydrostatic pressure fluctuations. This is manifested in the frequency spectrum by significantly reduced amplitudes in the

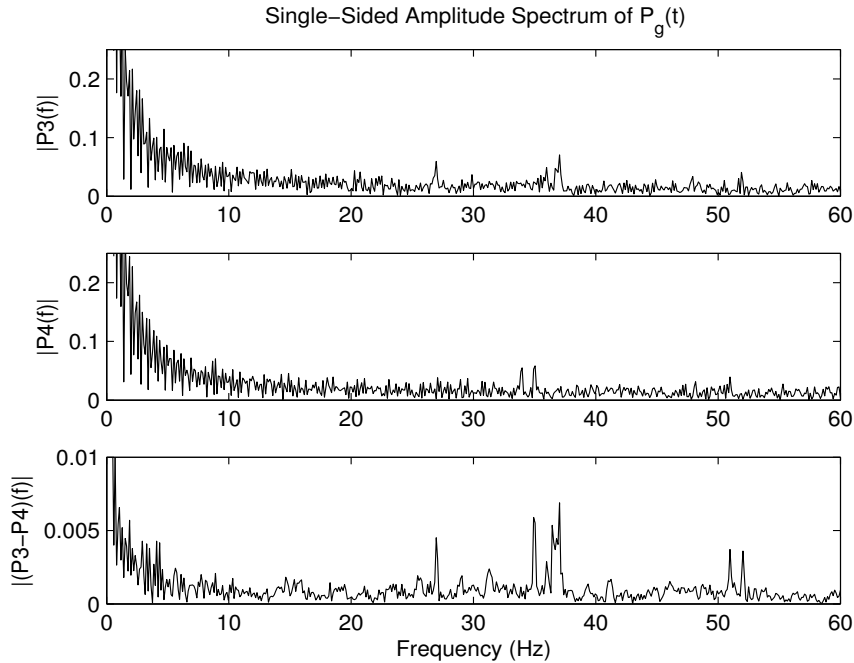


Figure 8.8: Frequency spectra of several pressure time signals of the vertical bed, as marked on the vertical axis.

region $f < 20$ Hz. A comparison of the vertical axis scales between the graphs portrays this. This carries the additional advantage of highlighting the higher-frequency components. Frequency components above 20 Hz are also attenuated, although not nearly to the extent of the low-frequency components. The dominant frequency for this run is identified as $f_D \simeq 37$ Hz. It was found during the course of the investigation that two frequency components which differed slightly in value had approximately the same amplitudes. For example, the two peaks at 37 Hz and 34 Hz seen in Figure 8.8 have similar values. If this were the case, the mean of the values was taken as the dominant frequency. The difference between two similar peaks generally differed by no more than 4 Hz.

Time signals of three adjacent pressure signals – P8, P9, and P10 – from the inclined system are displayed in figure 8.9. Results are shown for the bed inclined at 20° , although the pressure signals are similar in form over all tilting angles tested. As expected, the time signals P8 and P9 corresponding to the bottom and side of the bed, respectively, are nearly identical. The minor dip in the hydrostatic pressure seen between P8 and P9 corresponds to the slight vertical discrepancy between the two taps

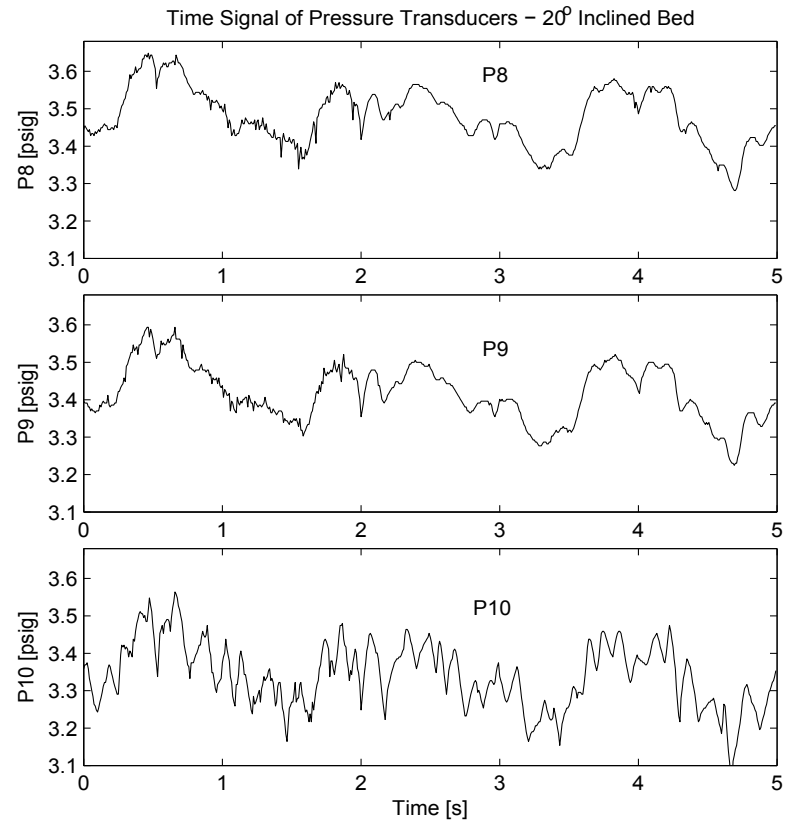


Figure 8.9: Pressure time signals of three adjacent pressure taps of the 20° inclined bed, as marked on the vertical axis. $U/u_{mf} = 1.3$

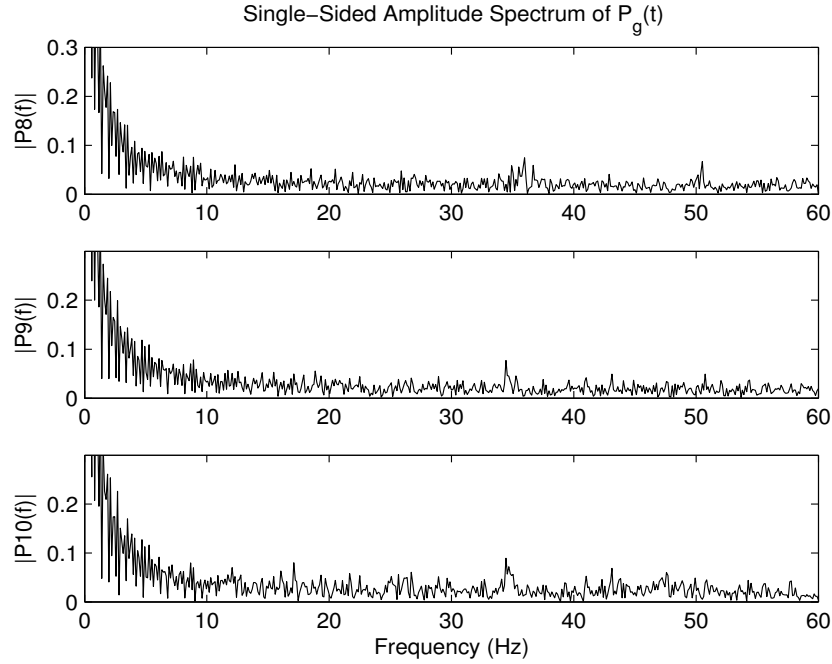


Figure 8.10: Frequency spectra of the time signals shown in figure 8.9, as marked on the vertical axis.

due to inclination of the bed. The pressure tap along the top side of the bed, P10, exhibits much of the same bulk variation in the pressure as the other signals, although there is a significant amount of variation in between the large troughs and valleys of the hydrostatic pressure fluctuations. This is expected to be a result of the presence of bubbles and other disturbances in the channel flow focused in the top region of the bed. In the vertical bed, bubbles tend to travel near the center of the domain. With the inclined bed, bubbles traverse the upper channel flow adjacent to the top of the domain. The regions of highest voidage – and, consequently, the largest fluctuations in volume fraction – are immediately adjacent to the upper wall. The peaks of the fluctuations of the P10 signal are slightly lower than that of the P9 signal, again an indication of the slight decrease in hydrostatic pressure due to inclination of the bed.

Frequency spectra of these three time signals are shown in figure 8.10. The significant differences between the time signals of P10 and both P8 and P9 are not manifested when viewing the frequency components of each signal. The fluctuations of P10 are obscured in the region $f < 10$ Hz and do not make a significant impact on the dominant mode, which is visible in all three signals. This result is of importance because

it reveals that the dominant perturbations are present throughout the domain. It also reveals that such disturbances exist even in the presence of bubbling, which suggests that a stability analysis may be relevant even after the onset of bubbling. The reason for this is not entirely clear, although one possible explanation deals with the nature of bubbling in fluidized beds. When bubbles appear, especially in the region of intermediate fluidization velocities $1.0 < U/u_{mf} \lesssim 1.5$, they are generally small, and the bed exhibits large time intervals between the presence of bubbles. This time interval is seen in the large-scale fluctuations of the pressure time signals of figures 8.6 and 8.9. In the regions of the bed through which bubbles are not currently traversing, the volume fraction remains nearly constant. The lateral fluid-phase pressure distribution also remains nearly constant, as seen when comparing the pressure signals of P8 and P9 in figure 8.10. The exception is seen with the fluctuating signal of P10, although the result is expected to be more valid for vertical beds due to symmetry. Therefore, a stability analysis may still retain validity in the regions of the bed between bubbles. However, due to the non-constant nature of the axial pressure drop, the assumption of a constant pressure drop dP_g/dx may not be suitable when bubbling is present in the system. This is discussed in the concluding chapter of the thesis.

8.4.3 Comparisons Between Experiment and Simulation

Validation of the numerical models developed in the present work is difficult. This stems from the fact that there are few flow variables which may be measured directly. The fluid-phase pressure is perhaps the most readily-measured quantity, although relevant disturbances in the pressure are often trumped by the larger hydrostatic fluctuations produced by bubbling, back-pressure waves caused by bed collapse, and other nonlinear disturbances. Qualitative similarities have been found between the current numerical work and experimental observations. The dependence of the growth constant on particle diameter and density bears qualitative resemblance to the tendency of fluidized beds to bubble at minimum fluidization as a function of the same particle characteristics mapped by Geldart [1]. The dominant instability mode of the inclined system yields a solid-phase volume fraction perturbation whose time evolution leads to the development of a dramatic voidage region adjacent to the upper wall. This is in agreement with the development and propagation of bubbles in the upper channel of the flow seen during the current inclined experiment.

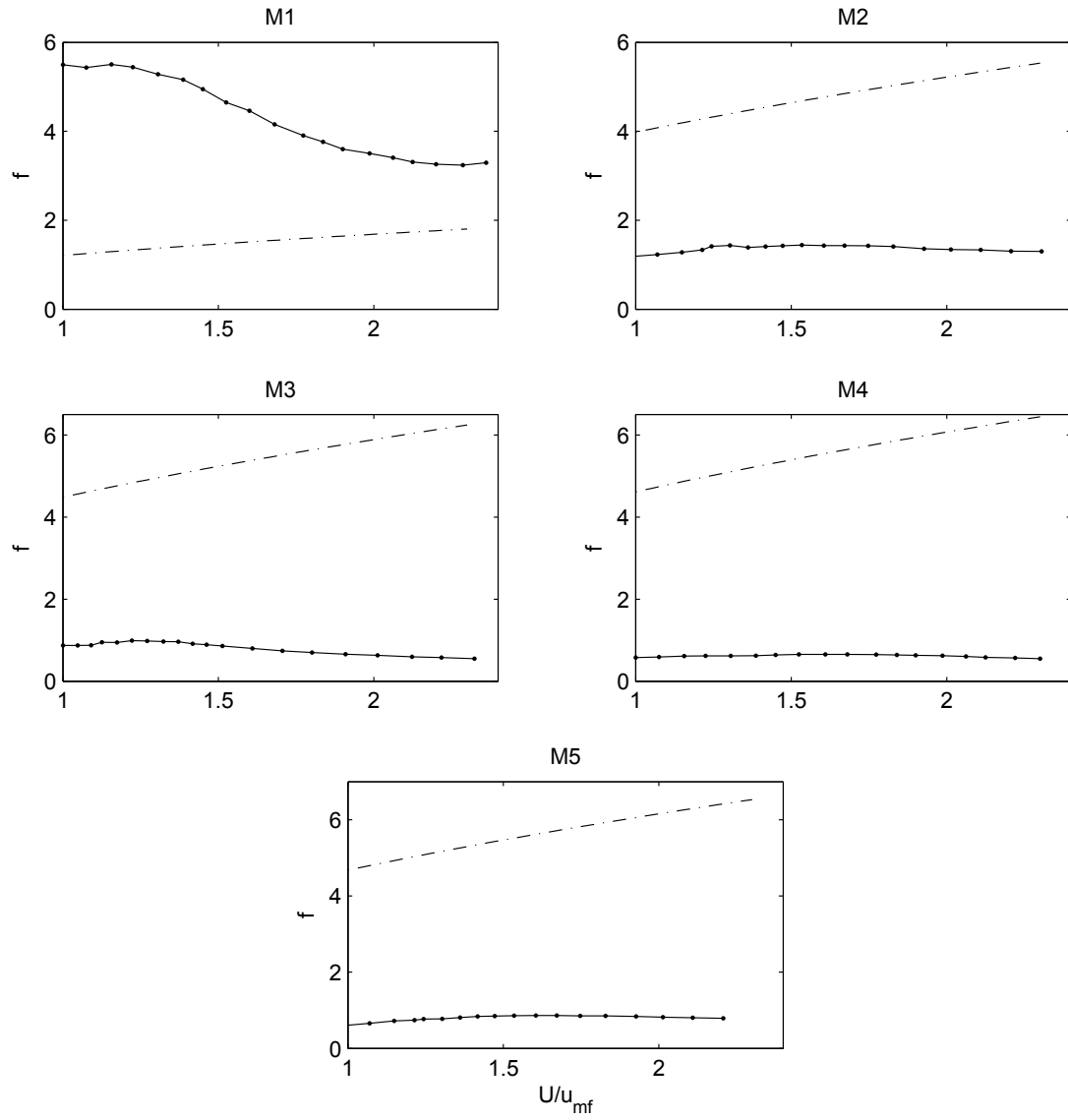


Figure 8.11: Dimensionless dominant frequencies from experiment (circles) and numerics (dot-dashes), as a function of normalized fluidization velocity, for the five bed materials tested.

In a more direct attempt at comparison, the dominant experimental disturbance frequencies are compared with those of the linear stability analysis. For the vertical bed, the frequency is presented as a function of fluidization velocity for the five materials, while for the inclined bed the tilting angle is treated as the independent variable. Since tilting angle has an effect upon the minimum fluidization velocity, the experiments were run such that the normalized fluidization velocity u/u_{mf} remained constant while varying θ .

Figure 8.11 compares the dominant experimental and numerical disturbance frequencies of the vertical bed as a function of U/u_{mf} . The frequencies of each method are rendered dimensionless by the bounding wall gap width, $\hat{L}$, as well as the terminal velocity of the bed material, $\hat{u}_t$. The numerical frequencies were determined using the model developed in chapter 5. The input parameters include: the particle density and diameter of the material; the mean solids volume fraction $\bar{\phi}_o$, determined by measuring the time-averaged bed height at the particular value of U/u_{mf} tested; and the fluid-phase pressure $d\bar{P}_g/dx$, which was determined by changing its value until the mean fluid velocity U matched that of the experiment. The disturbance frequency of the numerical method from chapter 5 was modified so that the dimensionless parameters matched those specified above: $f = \sigma_i(\hat{u}_t/\hat{U})$.

The numerical frequencies exhibit a monotonic increase as the fluidization velocity rises. The experimentally-determined frequencies exhibit an initial increase followed by a decrease. The exception to this is the lightest material, M1, for which the frequency decreases over the range of fluidization velocity above $U/u_{mf} \simeq 1.15$. A qualitative similarity exists between the frequency of the dominant disturbance and the fluidization velocity in the small range of U/u_{mf} after minimum fluidization. This is significant because the linear stability analyses of the present work are expected to be the most applicable in the regimes of smooth fluidization and the onset of bubbling. The reason for the decrease in experimental frequency after a certain value of U/u_{mf} is unknown. The dominant disturbance identified for each of the five materials was the "same" throughout the investigated range of fluidization velocity – that is, there was a smooth transition from increase to decrease in f with fluidization velocity, and no other disturbances were within the vicinity of the dominant disturbance. A possible explanation is also a caveat of the current experimental method – the dominant disturbance identified through experiment does not necessarily correspond to the dominant linear mode. However, the

behavior of the dominant experimental and numerical frequencies at small U/u_{mf} is a promising indicator, despite the differences in magnitude from the numerical method.

For a final comparison, the experimental and numerical frequencies are plotted as a function of the tilting angle. The numerical method used was that employed in chapter 6. The input parameters were the same as described above for the vertical bed, although the reader is again reminded that the numerical method for the inclined bed allows for a non-zero base-state solids velocity. The experiment was run such that the normalized fluid velocity was maintained at a constant value of $U/u_{mf} = 1.2$. This was performed with respect to the results shown in figure 8.11, along with the supposition that the linear stability analysis is most accurate at and around minimum fluidization. This value of U/u_{mf} was high enough to guarantee some visible degree of instability for all bed materials at the four angles of inclination tested. The numerical procedure involved locating the values of $d\bar{P}_g/dx$ and C which yield the correct value of the mean solid volume fraction and normalized fluidization velocity corresponding to each experimental run. As mentioned in chapter 6, one of two mode types is dominant, depending upon the tilting angle: the a modes at lower θ and the high-frequency boundary layer b modes at larger θ (see figure 6.2). The critical angle of transition from the a to the b modes, θ_c , depends upon the system parameters. For this reason both the leading a and b modes are included in this comparison.

Figure 8.12 displays the numerical and experimental frequencies as a function of the tilting angle θ for all five bed materials. The frequencies of the numerically-determined a -modes exhibit a monotonic decrease as the tilting angle is increased, with the exception of the parameter set corresponding to the M3 material. The discontinuous jump in frequency at $\theta = 34^\circ$ occurs as a result of the leading a mode switching from a_1 to a_2 . The frequencies of the b modes exhibit an initially slight increase followed by a monotonic decrease with the exception of the M4 material, whose leading b mode exhibits a decreasing frequency over the entire range of tilting angle.

The result for $\theta = 0^\circ$ is included for comparison between the vertical and inclined beds. Three separate data sets are presented in each of the experimental plots, corresponding to the frequency spectra of the pressure signals $P_{15}(t) - P_{12}(t)$, $P_{12}(t) - P_9(t)$, and $P_6(t) - P_3(t)$. Originally only the former spectrum was computed, but the latter two were computed to search for any pattern that may not exist in the upper region of the bed. An interesting pattern emerges when comparing the signals. The signals

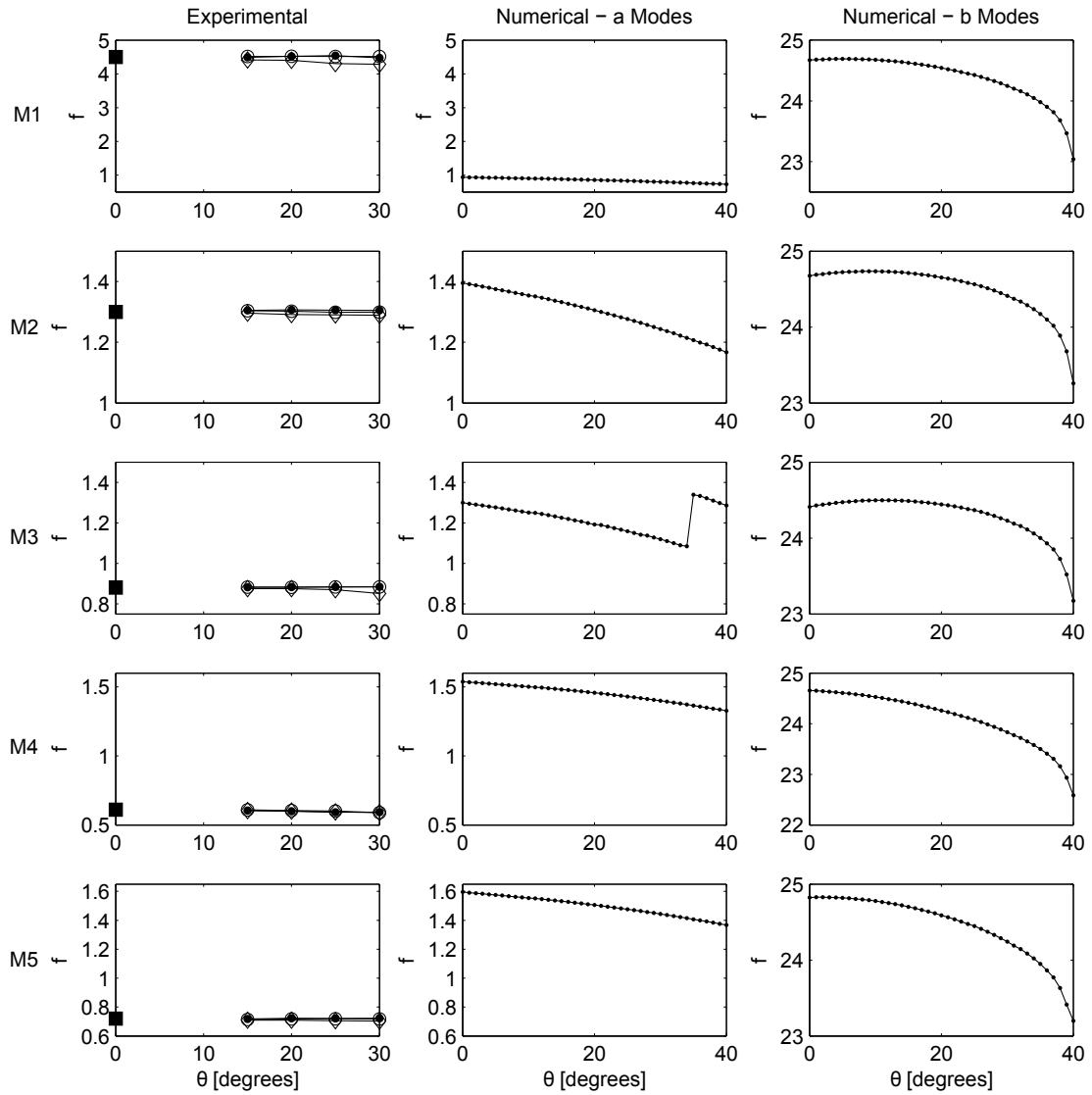


Figure 8.12: (a) Dominant frequencies from experiment, determined from the frequency spectra of P15-P12 (dots), P12-P9 (circles), and P6-P3 (diamonds); (b) Frequency of the leading a mode; (c) Frequency of the leading b mode. The experimental frequency of the vertical bed (solid square) is included for comparison. The frequencies of the b mode are plotted on a different scale because of the large discrepancy in magnitude with the other two plots for each material.

corresponding to the bottom region of the bed generally exhibit disturbance frequency components which are the lowest in magnitude, as well as decrease with an increased tilting angle. In addition, the dominant frequencies obtained from this signal all have a smaller magnitude than those of the vertical bed, which are denoted by solid squares in figure 8.12. The signals corresponding to the middle and top of the bed share qualitative and quantitative similarities with each other, although these exhibit mostly different patterns than that of the bottom signal (see plots corresponding to M1, M3, and M5 for examples of this).

There do exist some qualitative similarities between the dominant experimental and numerical frequencies. However, the same discrepancies appear as in the vertical bed – namely, the significant differences in magnitude between the experimental and numerical methods. Likewise, the decrease in frequency with the tilting angle seen in the experiment is very minimal, with values changing less than 3% for all materials, while those of the numerics show much more variance. The large differences in magnitude between the experimental frequencies and those of the numerical b modes also bring into question the relevance of these disturbances. However, the similar dependence of the frequency on θ , as well as the time evolution of the solid-phase volume fraction seen in figure 6.5, produce convincing arguments that the dominant mode determined from the model is important in describing the stability of inclined fluidized beds. A possible explanation regarding the discrepancy in magnitude between the experimental and numerical methods deals with the evolution of the linear mode into the nonlinear regime. If the dominant modes determined experimentally are a result of nonlinear interactions, the similar behavior of the frequency with increasing tilting angle suggests that either (1) the fundamental linear mode develops directly into the dominant disturbance seen in experiment, or (2) the nonlinear disturbances exhibit similar dependencies on the parameters of the system. In the case that either situation is true, the linear stability analysis would be a sufficient method to predict the evolution of disturbances all the way from the linear regime to the appearance of bubbles. However, proof of this hypothesis requires more work, and likely a new method of experimental verification, and remains beyond the scope of the present work.

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Chapter 9

Conclusions

Previous attempts to identify instability mechanisms leading to the formation of bubbles in fluidized beds have either focused only on the classic longitudinal mode or employed a closure set based upon ad-hoc approximations of the solid-phase transport variables. The current work completes the picture of linear stability in gas-fluidized beds by considering disturbance wavenumber vectors ranging from the axial to the transverse in an Eulerian-Eulerian system whose solid-phase transport variables are derived from the extension of kinetic gas theory to rapid granular flows.

A linear stability analysis was applied to three separate fluidized bed configurations. The first problem considered an unbounded bed subject to a unidirectional, constant, fluid-phase pressure drop. An analytic solution to the dispersion relation is possible, although due to the large number of terms is rendered impractical. The problem was instead solved numerically to identify and locate the eigenvalues. Analytical solutions corresponding to purely transverse and longitudinal disturbances were derived. It was found that the stability and instability mechanisms of each mode are similar, and augment the classically-recognized stabilizing and destabilizing mechanisms of the velocity- and voidage-based drag force fluctuations, respectively. These mechanisms are primarily a function of the ratio between the solid and fluid densities, and are independent of the solid-phase transport variable closure scheme used.

The primary stabilizing mechanism is the analogue of the isothermal bulk density, $\bar{P}_{s\phi} = \partial\bar{P}_s/\partial\phi$. When fluctuations in the granular temperature are considered, the expression responsible for the dampening of disturbances at small wavenumbers and large solid-phase volume fractions includes terms based upon the generation and

dissipation of PTE: $\bar{P}_{sT}\bar{\Gamma}_\phi - \bar{P}_{s\phi}\bar{\Gamma}_T$. At large wavenumbers, the dominant stabilizing mechanisms are the solid-phase viscous and pseudo-thermal conductive terms, $\bar{\mu}_s$ and $\bar{\lambda}_s$. The former is the same as in previous isothermal studies, although both contain explicit dependencies upon the granular temperature in the present study. Another stabilizing mechanism represents the sensitivity of the pressure to fluctuations in the base-state granular temperature and, thus, the base-state fluid velocity: $\bar{P}_{sT}\bar{\Gamma}_{wf}$. However, it was determined during the large-longitudinal-wavelength analysis that this mechanism actually contributes to instability at the highest values of $\bar{\phi}$. The value of the solid-phase volume fraction at which this mechanism becomes unstable is the value at which the phase velocity of the long-axial-wavelength disturbance equals zero. However, the relative positive and negative contributions of this mechanism to the disturbance growth rate were found to be minimal in comparison with the others.

A primary destabilizing mechanism corresponds to the term representing the production and dissipation of energy resulting from fluctuations in the base-state granular temperature, $\bar{\Gamma}_T$. This mechanism is the secondary destabilizing mechanism behind the drag force fluctuation term $\bar{\beta}_\phi$, and effectively represents the ability of a granular medium to propagate energy through particle collisions. As expected, this term, and thus the growth rate of both transverse and longitudinal disturbances, is highly dependent upon the particle-particle restitution coefficient. A larger value of e_w is associated with a greater level of energy transfer via particle collisions, a mechanism which is reflected in the decreasing growth rate with increasing e_w .

With the relevant mechanisms so identified, attention was turned to two bounded problems: a vertically-oriented bed bounded by a cylindrical tube, and an inclined planar bed. Separate solution philosophies were employed to solve each problem. Particle movement was allowed in the base state of the inclined bed, while none was allowed in the vertical bed. Allowing for particle movement equates to the ideal case of a time- and laterally-constant fluid-phase pressure drop, and corresponds to a bed which extends infinitely in the axial direction. A system with no particle movement more accurately portrays the scenario observed in experiment, in which there exists no bulk movement of the particle phase in the absence of bubbling. In reality, there must exist some minor temporal and spatial variation of the fluid-phase pressure drop resulting from turbulent mixing in the region just above the distributor plate. The first scenario is, then, a more accurate representation of a bed with an ideal pressure drop, while the second more

accurately portrays the fluidized bed behavior observed in experiment.

Despite this seemingly major difference in the base state, the planar bed yielded very similar stability results as that of the vertical bed when the tilting angle was set to 0° . The similarities were determined to stem from the similar profile of the base-state velocity difference, $\bar{\mathbf{u}} - \bar{\mathbf{v}}$, whose always positive value reflects the result that the fluid velocity is always greater than the solids velocity. The terms representing the two most influential instability mechanisms, $\bar{\beta}_\phi$ and $\bar{\Gamma}_T$, are directly proportional to this base-state velocity difference. This velocity difference profile was also found to be similar in shape and magnitude for the planar and cylindrical beds, all other parameters held constant, and is the root of the dominant mode similarities between each method.

The eigenmodes corresponding to the dominant frequencies were also similar for each case, and represented modified versions of the limiting-case solution functions of the planar and cylindrical beds. When full fluid and particle slip, as well as no conduction of PTE, were allowed at the wall, the linearized equations admit transverse-component sinusoidal and Bessel function solutions for the 2D and 3D beds, respectively. This solution was limited to the axisymmetric ($n = 0$) case of the cylindrical system, although this is no detriment to the dominant mode as the axisymmetric disturbance was found to yield the largest growth rate in the full numerical solution.

Validation of these models is difficult, although qualitative comparisons between numerical and experimental results – old and new – appear promising. The dependence of the vertical bed’s dominant mode growth constant on the particle diameter and density forms a parallel with the empirically-recorded tendency for bubbling near minimum fluidization published by Geldart in 1973. The time-evolution of the dominant inclined eigenmode ϕ' , superimposed onto the base-state solids volume fraction $\bar{\phi}$, yields a pattern of alternating dense and dilute regions adjacent to the top wall of the bed, similar to the development and propagation of bubbles observed in experiment.

A comparison between results of the experiments and the numerical methods consisted of locating the strongest frequency component of the fluid-phase pressure time signal obtained using non-intrusive probes in the wall of two lab-scale fluidized beds. These dominant frequencies were compared to those obtained numerically. There is some qualitative agreement between dominant frequency and fluidization velocity for the vertical bed in the range of fluidization immediately following minimum fluidization, $1 < U/u_{mf} \lesssim 1.3$. In the case of the tilted bed, there exists a qualitatively similar

decrease in dominant frequency with increasing angle of inclination.

The limitations of the methods used must be addressed. Most importantly, the linear stability analysis is valid under the premise of small perturbations. If these perturbations grow to such a size as to invalidate the small-amplitude assumption, the present analysis breaks down. However, the current work was undertaken with the proposition that the fundamental linear modes identified serve as the basis for the nonlinear interactions which evolve into bubbles. Therefore, there is additional work to be accomplished to test this hypothesis. A secondary linear stability analysis may provide further insight into the evolution of the fundamental linear mode. For example, a new base-state solution may be constructed such that: $\bar{\phi}_2(x, y, t) = \bar{\phi}(y) + \phi'(x, y, t)$. Here, $\bar{\phi}(y)$ is the base-state solution solved in the current 2D problem and $\phi'(x, y, t)$ is the perturbed volume fraction solved through the linear stability analysis. Therefore, the new base-state solution is also a function of x and t and, as such, numerous additional terms will appear in the stability analysis. The new solution for the volume fraction may be represented as $\phi_2(x, y, t) = \bar{\phi}_2(x, y, t) + \phi'_2(x, y, t)$, where $\phi'_2(x, y, t)$ is the solution acquired by introducing a perturbation around $\bar{\phi}_2$. The stability analysis would be carried out for various values of t corresponding to different amplitude magnitudes of the initial perturbation $\phi'(x, y, t)$.

Another more sophisticated method involves solving the fully non-linear perturbed system. The strategy for this would be similar to the secondary stability analysis: introduce a new solution $\phi_2(x, y, t) = \bar{\phi}_2(x, y, t) + \phi'_2(x, y, t)$, where $\bar{\phi}_2(x, y, t) = \bar{\phi}(y) + \phi'(x, y, t)$ is determined from the initial linear stability analysis. However, terms which are quadratic or higher in $\phi'_2(x, y, t)$ would not be ignored, resulting in a nonlinear system of equations. This complicates the analysis considerably, although not nearly as much as a full, time-dependent CFD solution of the equations of motion. The form of the perturbation would still take the form $\phi'_2(x, y, t) = \hat{\phi}'_2(y) \exp(ik_2x + \sigma t)$, thereby simplifying the x - and t -dependencies in the equations of motion. Here, k_2 is the axial wavenumber of the secondary disturbance. The procedure may take place in the same manner described in the preceding paragraph, although there is no practical limit on the size of the amplitudes. This method may also be more suitable to validation via the methods described in chapter 8, since the dominant modes identified via the fluid-phase pressure spectra do not necessarily represent the linear disturbances. These problems are left to future work.

Appendix A

Discretized Equations - Inclined Bed

Presented here are the discretized equations corresponding to the linearized equations and boundary conditions 6.20-6.26 of the inclined fluidized bed.

$$\begin{aligned} \sigma \phi'_{j+1/2} + ik \sum_{k=0}^N M_{jk}^s \bar{\phi}_k \sum_{k=0}^N M_{jk}^s u'_{sk} + ik \phi'_{j+1/2} \sum_{k=0}^N M_{jk}^s \bar{u}_{sk} \\ + \sum_{k=0}^N E_{jk}^s \bar{\phi}_k \sum_{k=0}^N M_{jk}^s v'_{sk} + \sum_{k=0}^N M_{jk}^s \bar{\phi}_k \sum_{k=0}^N E_{jk}^s v'_{sk} = 0 \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} -\sigma \sum_{k=0}^N M_{jk} \phi'_k + ik (1 - \bar{\phi}_j) u'_{gj} - ik \bar{u}_{gj} \sum_{k=0}^N M_{jk} \phi'_k \\ - v'_{sj} \sum_{k=0}^N D_{jk} \bar{\phi}_k + (1 - \bar{\phi}_j) \sum_{k=0}^N D_{jk} v'_{gk} = 0 \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \sigma \left(u'_{gj} - \frac{\bar{u}_{gj}}{(1 - \bar{\phi}_j)} \sum_{k=0}^N M_{jk} \phi'_k \right) - ik \bar{u}_{gj} u'_{gj} - \frac{ik \bar{u}_{gj}^2}{1 - \bar{\phi}_j} \sum_{k=0}^N M_{jk} \phi'_k + v'_{gj} \sum_{k=0}^N D_{jk} \bar{u}_{gk} \\ - \frac{\bar{u}_{gj}}{1 - \bar{\phi}_j} v'_{gj} \sum_{k=0}^N D_{jk} \bar{\phi}_k - ik \frac{LR^{-1}}{Fr} \sum_{k=0}^N M_{jk} P'_{gk} + \frac{L(R^{-1} - 1)}{Fr} \bar{\beta}_j (u'_{sj} - u'_{gj}) \\ + \frac{L(R^{-1} - 1)}{Fr} \bar{\beta}_{\phi j} (\bar{u}_{sj} - \bar{u}_{gj}) \sum_{k=0}^N M_{jk} \phi'_k = 0 \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned}
& \sigma \sum_{k=0}^N M_{jk}^s v'_{gk} - ik \sum_{k=0}^N M_{jk}^s \bar{u}_{gk} \sum_{k=0}^N M_{jk}^s v'_{gk} - ik \frac{LR^{-1}}{Fr} P'_{g,j+1/2} \\
& + \frac{L(R^{-1}-1)}{Fr} \sum_{k=0}^N M_{jk}^s \bar{\beta}_k \left(\sum_{k=0}^N M_{jk}^s v'_{sk} - \sum_{k=0}^N M_{jk}^s v'_{gk} \right) = 0 \quad (\text{A.4})
\end{aligned}$$

$$\begin{aligned}
& \sigma \left(\bar{\phi}_j u'_{sj} + \bar{u}_{sj} \sum_{k=0}^N M_{jk} \phi'_k \right) - ik \bar{u}_{sj} \bar{\phi}_j u'_{sj} - ik \bar{u}_{sj}^2 \sum_{k=0}^N M_{jk} \phi'_k - \bar{\phi}_j v'_{sj} \sum_{k=0}^N D_{jk} \bar{u}_{sk} \\
& - \bar{u}_{sj} v'_{sj} \sum_{k=0}^N D_{jk} \bar{\phi}_k - ik \bar{\phi}_j \frac{L}{Fr} \sum_{k=0}^N M_{jk} P'_{gk} - \frac{L \cos \theta}{Fr} \sum_{k=0}^N M_{jk} \phi'_k \\
& - ik \left(\bar{P}_{s\phi j} \sum_{k=0}^N M_{jk} \phi'_k + \bar{P}_{sTj} T'_j \right) + \frac{L(1-R)}{Fr} (1-\bar{\phi}_j) \bar{\beta}_j (u'_{gj} - u_{s'j}) \\
& + \frac{L(1-R)}{Fr} \bar{\beta}_{\phi j} (\bar{u}_{gj} - \bar{u}_{sj}) \sum_{k=0}^N M_{jk} \phi'_k - k^2 \frac{2\bar{\mu}_{sj} + \bar{\zeta}_{sj}}{L} u'_{sj} + \frac{\bar{\mu}_{sj}}{L} \sum_{k=0}^N D_{jk}^2 u'_{sk} \\
& + ik \frac{\bar{\mu}_{sj} + \bar{\zeta}_{sj}}{L} \sum_{k=0}^N D_{jk} v'_{sk} + \frac{ik}{L} v'_{sj} \sum_{k=0}^N D_{jk} \bar{\mu}_{sk} = 0 \quad (\text{A.5})
\end{aligned}$$

$$\begin{aligned}
& \sigma \bar{\phi}_j v'_{sj} - ik \bar{\phi}_j \bar{u}_{sj} v'_{sj} - \sum_{k=0}^N M_{jk} \phi'_k \sum_{k=0}^N D_{jk} \bar{P}_{s\phi k} - \bar{P}_{s\phi j} \sum_{k=0}^N E_{jk} \phi'_k - T'_j \sum_{k=0}^N D_{jk} \bar{P}_{sTk} \\
& - \bar{P}_{sTj} \sum_{k=0}^N D_{jk} T'_k - \frac{L}{Fr} \bar{\phi}_j \sum_{k=0}^N E_{jk} P'_{gk} - \frac{L \sin \theta}{Fr} \sum_{k=0}^N M_{jk} \phi'_k - k^2 \frac{\bar{\mu}_{sj}}{L} v'_{sj} \\
& + \frac{L(1-R)}{Fr} (1-\bar{\phi}_j) \bar{\beta}_j (v'_{gj} - v'_{sj}) + \frac{2\bar{\mu}_{sj} + \bar{\zeta}_{sj}}{L} \sum_{k=0}^N D_{jk}^2 v'_{sk} \\
& + \frac{1}{L} \sum_{k=0}^N D_{jk} (2\bar{\mu}_{sk} + \bar{\zeta}_{sk}) \sum_{k=0}^N D_{jk} v'_{sk} + \frac{1}{L} \left(\bar{\mu}_{s\phi j} \sum_{k=0}^N M_{jk} \phi'_k + \bar{\mu}_{sTj} T'_j \right) \sum_{k=0}^N D_{jk} \bar{\mu}_{sk} \\
& + ik \frac{\bar{\mu}_{sj} + \bar{\zeta}_{sj}}{L} \sum_{k=0}^N D_{jk} u'_{sk} + \frac{ik}{L} \sum_{k=0}^N D_{jk} \bar{\zeta}_{sk} u'_{sj} = 0 \quad (\text{A.6})
\end{aligned}$$

$$\begin{aligned}
& \frac{3}{2}\sigma \left(\bar{\phi}_j T'_j + \bar{T}_j \sum_{k=0}^N M_{jk} \phi'_k \right) - \frac{3}{2}ik \left(\bar{\phi}_j \bar{T}_j u'_{sj} + \bar{\phi}_j \bar{u}_{sj} T'_j + \bar{u}_{sj} \bar{T}_j \sum_{k=0}^N M_{jk} \phi'_k \right) \\
& + \frac{3}{2} \left(\bar{\phi}_j \bar{T}_j \sum_{k=0}^N D_{jk} v'_{sk} + \bar{\phi}_j v'_{sj} \sum_{k=0}^N D_{jk} \bar{T}_k + \bar{T}_j v'_{sj} \sum_{k=0}^N D_{jk} \bar{\phi}_k \right) - ik \bar{P}_{sj} u'_{sj} \\
& + \frac{2}{L} \bar{\mu}_{sj} \sum_{k=0}^N D_{jk} \bar{u}_{sk} \sum_{k=0}^N D_{jk} u'_{sk} + \frac{1}{L} \left(\bar{\mu}_{s\phi j} \sum_{k=0}^N M_{jk} \phi'_k + \bar{\mu}_{sTj} T'_j \right) \sum_{k=0}^N D_{jk} \bar{u}_{sk} \sum_{k=0}^N D_{jk} \bar{u}_{sk} \\
& - \bar{P}_{sj} \sum_{k=0}^N D_{jk} v'_{sk} - \frac{k^2}{L} \bar{\lambda}_j T'_j - \frac{1}{L} \bar{\lambda}_j \sum_{k=0}^N D_{jk}^2 T'_k - \frac{k^2}{L} \bar{\lambda}_{hj} \phi'_j - \frac{1}{L} \bar{\lambda}_{hj} \sum_{k=0}^N E_{jk}^2 \phi'_k \\
& + \frac{1}{L} \left(\bar{\lambda}_{\phi j} \sum_{k=0}^N M_{jk} \phi'_k + \bar{\lambda}_{Tj} T'_j \right) \sum_{k=0}^N D_{jk}^2 \bar{T}_k + \frac{1}{L} \left(\bar{\lambda}_{h\phi j} \sum_{k=0}^N M_{jk} \phi'_k + \bar{\lambda}_{hTj} T'_j \right) \sum_{k=0}^N D_{jk}^2 \bar{\phi}_k \\
& + \sum_{k=0}^N M_{jk} \phi'_k \sum_{k=0}^N D_{jk} \bar{T}_k \sum_{k=0}^N D_{jk} \bar{\lambda}_{\phi k} + \sum_{k=0}^N M_{jk} \phi'_k \sum_{k=0}^N D_{jk} \bar{\phi}_k \sum_{k=0}^N D_{jk} \bar{\lambda}_{hk} \\
& + T'_j \sum_{k=0}^N D_{jk} \bar{T}_k \sum_{k=0}^N D_{jk} \bar{\lambda}_{Tk} + T'_j \sum_{k=0}^N D_{jk} \bar{\phi}_k \sum_{k=0}^N D_{jk} \bar{\lambda}_{hTk} + \bar{\lambda}_{\phi j} \sum_{k=0}^N D_{jk} \bar{T}_k \sum_{k=0}^N E_{jk} \phi'_k \\
& + \bar{\lambda}_{h\phi j} \sum_{k=0}^N D_{jk} \bar{\phi}_k \sum_{k=0}^N E_{jk} \phi'_k + \bar{\lambda}_{Tj} \sum_{k=0}^N D_{jk} \bar{T}_k \sum_{k=0}^N D_{jk} T'_k + \bar{\lambda}_{hTj} \sum_{k=0}^N D_{jk} \bar{\phi}_k \sum_{k=0}^N D_{jk} T'_k \\
& L \left(\Gamma_{\phi j} \sum_{k=0}^N M_{jk} \phi'_k + \Gamma_{Tj} T'_j + \Gamma_{u_{gj}} u'_{gj} + \Gamma_{u_{sj}} u'_{sj} \right) = 0
\end{aligned} \tag{A.7}$$

Here, $j = 1, 2, \dots, N+1$. The linearized boundary conditions are, at $y = \pm 0.5$,

$$v'_g = v'_s = 0 \tag{A.8}$$

$$\begin{aligned}
& \pm \bar{\phi}_\gamma \sum_{k=0}^N D_{\gamma k} u'_{gk} \pm \sum_{k=0}^N M_{\gamma k} \phi'_\gamma \sum_{k=0}^N D_{\gamma k} \bar{u}_{gk} \\
& + \frac{2L^2 Re(1-R)}{Fr} \frac{\delta}{\phi_p} \bar{\phi}_\gamma \bar{\beta}_\gamma (\bar{u}_{g\gamma} - \bar{u}_{s\gamma}) \\
& + \frac{\phi_p}{\delta} (\bar{T}_\gamma u'_{g\gamma} + \bar{u}_{g\gamma} T'_\gamma) + \frac{2L^2 Re}{Fr} \frac{\delta}{\phi_p} \frac{d\bar{P}_g}{dx} \bar{\phi}_\gamma = 0
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
& \pm \bar{f}_{2\gamma} \sum_{k=0}^N D_{\gamma k} u'_{sk} \pm \bar{f}_{2\phi\gamma} \sum_{k=0}^N M_{\gamma k} \phi'_\gamma \sum_{k=0}^N D_{\gamma k} \bar{u}_{sk} - L \left(\bar{f}_{8\gamma} u'_{s\gamma} + \bar{f}_{8\phi\gamma} \bar{u}_{s\gamma} \sum_{k=0}^N M_{\gamma k} \phi'_\gamma \right) = 0
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
& \pm \bar{f}_{4\gamma} \sum_{k=0}^N D_{\gamma k} T'_k \pm \bar{f}_{4\phi\gamma} \sum_{k=0}^N M_{\gamma k} \phi'_\gamma \sum_{k=0}^N D_{\gamma k} \bar{T}_\gamma \pm \bar{f}_{4h\gamma} \bar{T}_\gamma \sum_{k=0}^N E_{\gamma k} \phi'_\gamma \\
& \pm \bar{f}_{4h\gamma} T'_\gamma \sum_{k=0}^N D_{\gamma k} \bar{\phi}_\gamma \pm \bar{f}_{4h\phi\gamma} \bar{T}_\gamma \sum_{k=0}^N M_{\gamma k} \phi'_\gamma \sum_{k=0}^N D_{\gamma k} \bar{\phi}_\gamma \\
& - L \left(2\bar{f}_{8\gamma} \bar{u}_{s\gamma} u'_{s\gamma} + \bar{f}_{8\phi\gamma} \bar{u}_{s\gamma}^2 \sum_{k=0}^N M_{\gamma k} \phi'_\gamma \right) \\
& + L \left(\bar{f}_{7\gamma} T'_\gamma + \bar{f}_{7\phi\gamma} \bar{T}_\gamma \sum_{k=0}^N M_{\gamma k} \phi'_\gamma \right) = 0 \tag{A.11}
\end{aligned}$$

The subscript γ takes on the value 0 when $\alpha = -1$ and N when $\alpha = 1$. Here, for convenience, the terms representing dissipation and generation of PTE are lumped together:

$$\Gamma = \frac{L}{Re} |\bar{u}_g - \bar{u}_s|^2 \bar{f}_5 \bar{T}^{-1/2} - L \bar{f}_6 \bar{T}^{3/2} - \frac{3L(1-R)}{Fr} \bar{f}_o \bar{T} \tag{A.12}$$

Appendix B

Discretized Equations - Cylindrical Bed

Presented here are the discretized equations corresponding to the linearized equations 5.18 - 5.26 of the cylindrical fluidized bed. The fluid-phase viscosity terms, which were found to be negligible during the course of the investigation, are not included in these equations for the sake of brevity.

$$\begin{aligned} \sigma \phi'_{j+1/2} + \sum_{k=0}^N M_{jk}^s \bar{\phi}_k \sum_{k=0}^N E_{jk}^s u'_{rk} + \frac{2}{1-\alpha_{j+1/2}} \sum_{k=0}^N M_{jk}^s \bar{\phi}_k \sum_{k=0}^N M_{jk}^s u'_{rk} \\ + \frac{2in}{1-\alpha_{j+1/2}} \sum_{k=0}^N M_{jk}^s \bar{\phi}_k \sum_{k=0}^N M_{jk}^s u'_{\theta k} + ik \sum_{k=0}^N M_{jk}^s \bar{\phi}_k \sum_{k=0}^N M_{jk}^s u'_{zk} = 0 \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} -\sigma \sum_{k=0}^N M_{jk} \phi'_k + ik (1 - \bar{\phi}_j) \sum_{k=0}^N D_{jk} v'_{rk} + \frac{2}{1-\alpha_j} (1 - \bar{\phi}_j) v'_{rj} \\ + \frac{2in}{1-\alpha_j} (1 - \bar{\phi}_j) v'_{\theta j} + ik (1 - \bar{\phi}_j) v'_z - ik \sum_{k=0}^N M_{jk} \phi'_k = 0 \end{aligned} \quad (\text{B.2})$$

$$-\sigma v'_{rj} - R^{-1} \sum_{k=0}^N E_{jk} P'_{gk} - \frac{L(R^{-1} - 1)}{Fr} \bar{\beta}_j (v'_{rj} - u'_{rj}) \quad (\text{B.3})$$

$$\begin{aligned}
& -\sigma \sum_{k=0}^N M_{jk}^s v'_{\theta k} - \frac{2inR^{-1}}{1-\alpha_{j+1/2}} P'_{g,j+1/2} - ik \sum_{k=0}^N M_{jk}^s v'_{\theta k} \\
& - \frac{L(R^{-1}-1)}{Fr} \left[\sum_{k=0}^N M_{jk}^s \bar{\beta}_k \left(\sum_{k=0}^N M_{jk}^s v'_{\theta k} - \sum_{k=0}^N M_{jk}^s u'_{\theta k} \right) \right] = 0 \quad (B.4)
\end{aligned}$$

$$\begin{aligned}
& - \left(\sigma v'_{rz} + \frac{1}{(1-\bar{\phi}_j)} \sum_{k=0}^N M_{jk} \phi'_k \right) - ikR^{-1} \sum_{k=0}^N M_{jk} P'_{gk} - ik v'_{zj} \\
& + \frac{ik}{(1-\bar{\phi}_j)} \sum_{k=0}^N M_{jk} \phi'_k - \frac{L(R^{-1}-1)}{Fr} \left(\bar{\beta}_j (v'_{zj} - u'_{zj}) + \bar{\beta}_{\phi j} \sum_{k=0}^N M_{jk} \phi'_k \right) = 0 \quad (B.5)
\end{aligned}$$

$$\begin{aligned}
& -\sigma \bar{\phi}_j u'_{rj} - \bar{\phi}_j \sum_{k=0}^N E_{jk} P'_{gk} + (1-\bar{\phi}_j) \frac{L(R^{-1}-1)}{Fr} \bar{\beta}_j (v'_{rj} - u'_{rj}) \\
& - \bar{P}_{s\phi j} \sum_{k=0}^N E_{jk} \phi'_k - \bar{P}_{sTj} \sum_{k=0}^N D_{jk} T'_k - \frac{8in\bar{\mu}_{sj}}{L(1-\alpha_j)^2} u'_{\theta j} - \frac{4\bar{\mu}_{sj}}{L(1-\alpha_j)^2} u'_{rj} \\
& + \frac{\bar{\mu}_{sj}}{L} \left(\sum_{k=0}^N D_{jk}^2 u'_{rk} + \frac{2}{1-\alpha_j} \sum_{k=0}^N D_{jk} u'_{rk} - \frac{4n^2}{(1-\alpha_j)^2} u'_{rj} - k^2 u'_{rj} \right) \\
& + \frac{\bar{\mu}_{sj} + \bar{\zeta}_{sj}}{L} \sum_{k=0}^N D_{jk} \left(\sum_{m=0}^N D_{km} u'_{rm} + \frac{2}{1-\alpha_k} u'_{rk} + \frac{2in}{1-\alpha_k} u'_{\theta k} + ik u'_{zk} \right) = 0 \quad (B.6)
\end{aligned}$$

$$\begin{aligned}
& -\sigma \bar{\phi}_j u'_{\theta j} - \frac{2in\bar{\phi}_j}{1-\alpha_j} \sum_{k=0}^N M_{jk} P'_{gk} + (1-\bar{\phi}_j) \frac{L(R^{-1}-1)}{Fr} \bar{\beta}_j (v'_{\theta j} - u'_{\theta j}) \\
& - \frac{2in}{1-\alpha_j} \left(\bar{P}_{s\phi j} \sum_{k=0}^N M_{jk} \phi'_k - \bar{P}_{sTj} T'_j \right) + \frac{8in\bar{\mu}_{sj}}{L(1-\alpha_j)^2} u'_{rj} - \frac{4\bar{\mu}_{sj}}{L(1-\alpha_j)^2} u'_{\theta j} \\
& + \frac{\bar{\mu}_{sj}}{L} \left(\sum_{k=0}^N D_{jk}^2 u'_{\theta k} + \frac{2}{1-\alpha_j} \sum_{k=0}^N D_{jk} u'_{\theta k} - \frac{4n^2}{(1-\alpha_j)^2} u'_{\theta j} - k^2 u'_{\theta j} \right) \\
& + \frac{\bar{\mu}_{sj} + \bar{\zeta}_{sj}}{L} \frac{2in}{1-\alpha_j} \left(\sum_{m=0}^N D_{jk} u'_{rk} + \frac{2}{1-\alpha_j} u'_{rj} + \frac{2in}{1-\alpha_j} u'_{\theta j} + ik u'_{zj} \right) = 0 \quad (B.7)
\end{aligned}$$

$$\begin{aligned}
& -\sigma \bar{\phi}_j u'_{zj} - ik \bar{\phi}_j \sum_{k=0}^N M_{jk} P'_{gk} - ik \left(\bar{P}_{s\phi j} \sum_{k=0}^N M_{jk} \phi'_k - \bar{P}_{sTj} T'_j \right) \\
& + (1 - \bar{\phi}_j) \frac{L(R^{-1} - 1)}{Fr} \left(\bar{\beta}_j (v'_{\theta j} - u'_{\theta j}) + \bar{\beta}_{\phi j} \sum_{k=0}^N M_{jk} \phi'_k - \frac{\bar{\beta}_j}{\bar{\phi}_j(1 - \bar{\phi}_j)} \sum_{k=0}^N M_{jk} \phi'_k \right) \\
& + \frac{\bar{\mu}_{sj}}{L} \left(\sum_{k=0}^N D_{jk}^2 u'_{zk} + \frac{2}{1 - \alpha_j} \sum_{k=0}^N D_{jk} u'_{zk} - \frac{4n^2}{(1 - \alpha_j)^2} u'_{zj} - k^2 u'_{zj} \right) \\
& + \frac{\bar{\mu}_{sj} + \bar{\zeta}_{sj}}{L} ik \left(\sum_{m=0}^N D_{jk} u'_{rk} + \frac{2}{1 - \alpha_j} u'_{rj} + \frac{2in}{1 - \alpha_j} u'_{\theta j} + ik u'_{zj} \right) = 0
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
& -\frac{3}{2} \sigma \left(\bar{\phi}_j T'_j + \bar{T}_j \sum_{k=0}^N M_{jk} \phi'_k \right) + L \left(\bar{\Gamma}_{u_z, j} u'_{jz} + \bar{\Gamma}_{v_z, j} v'_{jz} + \bar{\Gamma}_{Tj} T'_j + \bar{\Gamma}_{\phi j} \sum_{k=0}^N M_{jk} \phi'_k \right) \\
& - \left(\bar{P}_{sj} + \frac{3}{2} \bar{\phi}_j \bar{T}_j \right) \left(\sum_{m=0}^N D_{jk} u'_{rk} + \frac{2}{1 - \alpha_j} u'_{rj} + \frac{2in}{1 - \alpha_j} u'_{\theta j} + ik u'_{zj} \right) \\
& + \frac{\bar{\lambda}_{hj}}{L} \left(\sum_{k=0}^N E_{jk}^2 \phi'_k + \frac{2}{1 - \alpha_j} \sum_{k=0}^N E_{jk} \phi'_k - \frac{4n^2}{(1 - \alpha_j)^2} \sum_{k=0}^N M_{jk} \phi'_k - k^2 \sum_{k=0}^N M_{jk} \phi'_k \right) \\
& + \frac{\bar{\lambda}_j}{L} \left(\sum_{k=0}^N D_{jk}^2 T'_k + \frac{2}{1 - \alpha_j} \sum_{k=0}^N D_{jk} T'_k - \frac{4n^2}{(1 - \alpha_j)^2} T'_j - k^2 T'_j \right) = 0
\end{aligned} \tag{B.9}$$

The boundary conditions at the centerline depend upon the value of the azimuthal wavenumber. For $n = 0$:

$$\begin{aligned}
& u'_{r1} = u'_{\theta 1} = v'_{r1} = v'_{\theta 1} = 0 \\
& \sum_{k=0}^N D_{1k} u'_{zk} = \sum_{k=0}^N D_{1k} v'_{zk} = \sum_{k=0}^N D_{1k} T'_k = 0
\end{aligned} \tag{B.10}$$

For $|n| = 1$:

$$\begin{aligned}
& iu'_{r1} - u'_{\theta 1} = iv'_{r1} - v'_{\theta 1} = 0 \\
& 2 \sum_{k=0}^N D_{1k} u'_{rk} + i \sum_{k=0}^N D_{1k} u'_{\theta k} = 2 \sum_{k=0}^N D_{1k} v'_{rk} + i \sum_{k=0}^N D_{1k} v'_{\theta k} = 0 \\
& u'_{z1} = v'_{z1} = T'_1 = 0
\end{aligned} \tag{B.11}$$

For $|n| > 1$:

$$u'_{r1} = u'_{\theta 1} = u'_{z1} = v'_{r1} = v'_{\theta 1} = v'_{z1} = T'_1 = 0 \tag{B.12}$$

At the wall, the boundary conditions reduce to:

$$u'_{r,N+1} = v'_{r,N+1} = 0 \quad (\text{B.13})$$

$$\left[\bar{\mu}_{s,N+1} \left(\frac{-2}{1-\alpha_{N+1}} + \sum_{k=0}^N D_{N+1,k} \right) + L \bar{f}_{7,N+1} \bar{T}_{N+1}^{1/2} \right] u'_{\theta,N+1} = 0 \quad (\text{B.14})$$

$$\left[\bar{\mu}_{s,N+1} \sum_{k=0}^N D_{N+1,k} + L \bar{f}_{7,N+1} \bar{T}_{N+1}^{1/2} \right] u'_{z,N+1} = 0 \quad (\text{B.15})$$

$$\left(\frac{-2}{1-\alpha_{N+1}} + \sum_{k=0}^N D_{N+1,k} \right) v'_{\theta,N+1} = 0 \quad (\text{B.16})$$

$$\sum_{k=0}^N D_{N+1,k} v'_{z,N+1} = \sum_{k=0}^N D_{N+1,k} T'_{N+1} = 0 \quad (\text{B.17})$$

Appendix C

Analytic Solution of the Limiting Case

Presented here is the solution for the limiting case of a planar bed with full slip and zero pseudo-thermal energy loss at the walls (7.12-7.18), as well as the analogous case for the cylindrical bed (7.34-7.40). The solution is a quartic polynomial in σ :

$$\sigma^4 + \frac{\omega_1}{\omega^*}\sigma^3 + \frac{\omega_2}{\omega^*}\sigma^2 + \frac{\omega_3}{\omega^*}\sigma + \frac{\omega_4}{\omega^*} = 0 \quad (\text{C.1})$$

The coefficients are written:

$$\omega^* = \frac{A_1 C_5 G_5 + 2G_5 D_5 A_2}{A_1^2 C_5^2} \quad (\text{C.2})$$

$$\begin{aligned} \omega_1 = & \frac{G_5}{C_5 A_1} \left(-\frac{A_2 D_1}{A_1} + D_2 + \frac{D_3 B_1 A_2}{B_3 A_1} - \frac{B_2 D_3}{B_3} \right) - \frac{G_5}{C_5} \left(-\frac{C_1}{A_1} + \frac{B_1 C_3}{B_3 A_1} - \frac{C_3}{B_3} \right) \\ & + \frac{G_5 D_5}{A_1 C_5^2} \left(-\frac{C_1 A_2}{A_1} + C_2 + \frac{C_3 B_1 A_2}{B_3 A_1} - \frac{C_3 B_2}{A_3} \right) + \frac{G_6}{C_5 A_1} \\ & + \frac{A_2 D_5}{A_1 C_5} \left(-\frac{G_6}{A_1 C_5} - \frac{C_1 G_5}{A_1 C_5} + \frac{B_1 C_3 G_5}{A_1 B_3 C_5} - \frac{C_3 G_5}{B_3 C_5} \right) \\ & - \frac{D_5}{A_1 C_5} \left(\frac{G_1 G_6}{C_5} + \frac{A_2 B_1 G_2 G_6}{A_1 B_3 C_5} - \frac{B_2 G_2 G_6}{B_3 C_5} \right) \\ & + \left(G_1 + \frac{A_2 B_1 G_2}{A_1 B_3} - \frac{B_2 G_2}{B_3} \right) \left(-\frac{D_1 D_3 G_5}{A_1 C_5} + \frac{B_1 D_3 G_5}{A_1 B_3 C_5} - \frac{D_3 G_5}{B_3 C_5} \right) \\ & + \left(G_1 + \frac{A_2 B_1 G_2}{A_1 B_3} - \frac{B_2 G_2}{B_3} \right) \left(-\frac{C_1 D_5 G_5}{A_1 C_5^2} + \frac{B_1 C_3 D_5 G_5}{A_1 B_3 C_5^2} - \frac{C_3 D_5 G_5}{B_3 C_5^2} \right) \quad (\text{C.3}) \end{aligned}$$

$$\begin{aligned}
\omega_2 = & \left(-G_3 - \frac{B_1 G_2}{A_1 B_3} + \frac{G_2}{B_3} + \frac{C_1 G_6}{A_1 C_5} - \frac{B_1 C_3 G_6}{B_3 A_1 C_5} \right. \\
& \left. + \frac{C_3 G_6}{B_3 C_5} + \frac{C_3 B_4 G_5}{B_3 C_5} - \frac{C_4 G_5}{C_5} \right) \left(\frac{D_5}{A_1 C_5} - 1 \right) \\
& + \frac{A_2 B_4 D_3 G_5}{A_1 B_3 C_5} - \frac{A_2 D_4 G_5}{A_1 C_5} + \frac{A_2 B_4 C_3 D_5 G_5}{A_1 B_3 C_5^2} - \frac{A_2 C_4 D_5 G_5}{A_1 C_5^2} \\
& \frac{D_5}{C_5} \left(\frac{G_6}{A_1 C_5} + \frac{C_1 G_5}{A_1 C_5} - \frac{B_1 C_3 G_5}{B_3 A_1 C_5} + \frac{C_3 G_5}{B_3 C_5} \right) \left(C_2 - \frac{C_1 A_2}{A_1} + \frac{C_3 B_1 A_2}{B_3 A_1} - \frac{C_3 B_2}{A_3} \right) \\
& + \left(\frac{G_6}{A_1 C_5} + \frac{C_1 G_5}{A_1 C_5} - \frac{B_1 C_3 G_5}{B_3 A_1 C_5} + \frac{C_3 G_5}{B_3 C_5} \right) \left(D_2 - \frac{A_2 D_1}{A_1} + \frac{D_3 B_1 A_2}{B_3 A_1} - \frac{B_2 D_3}{B_3} \right) \\
& - \frac{D_5}{A_1 C_5} \left(-\frac{A_2 C_1 G_6}{A_1 C_5} + \frac{A_2 B_1 C_3 G_6}{A_1 B_3 C_5} - \frac{B_2 C_3 G_6}{A_3 C_5} + G_1 + \frac{A_2 B_1 G_2}{A_1 B_3} - \frac{B_2 G_2}{B_3} \right) \\
& \left(\frac{G_1 G_6}{C_5} + \frac{G_2 G_6 B_1 A_2}{A_1 B_3 C_5} - \frac{G_2 G_6 B_2}{B_3 C_5} - \frac{C_1 A_2 G_5}{A_1 C_5} + \frac{C_2 G_5}{C_5} + \frac{A_2 B_1 C_3 G_5}{A_1 B_3 C_5} \right. \\
& \left. - \frac{B_2 C_3 G_5}{A_3 C_5} \right) \left(-\frac{D_1 D_3}{A_1} + \frac{B_1 D_3}{A_1 B_3} - \frac{D_3}{B_3} - \frac{C_1 D_5}{A_1 C_5} + \frac{B_1 C_3 D_5}{A_1 B_3 C_5} - \frac{C_3 D_5}{B_3 C_5} \right) \quad (C.4)
\end{aligned}$$

$$\begin{aligned}
\omega_3 = & \left(-G_3 - \frac{B_1 G_2}{A_1 B_3} + \frac{G_2}{B_3} + \frac{C_1 G_6}{A_1 C_5} - \frac{B_1 C_3 G_6}{B_3 A_1 C_5} + \frac{C_3 G_6}{B_3 C_5} + \frac{C_3 B_4 G_5}{B_3 C_5} - \frac{C_4 G_5}{C_5} \right) \\
& \times \left(D_2 - \frac{A_2 D_1}{A_1} + \frac{A_2 B_1 D_3}{A_1 B_3} - \frac{B_2 D_3}{B_3} - \frac{A_2 C_1 D_5}{A_1 C_5} + \frac{C_2 D_5}{C_5} \right. \\
& \left. + \frac{A_2 B_1 C_3 D_5}{A_1 B_3 C_5} - \frac{B_2 C_3 D_5}{A_3 D_5} \right) \\
& + \left(G_4 - \frac{B_4 G_2}{B_3} - \frac{C_3 B_4 G_6}{B_3 C_5} + \frac{C_4 G_6}{C_5} \right) \left(\frac{A_2 D_5}{A_1 C_5} + 1 \right) \\
& \left(\frac{G_1 G_6}{C_5} + \frac{G_2 G_6 B_1 A_2}{A_1 B_3 C_5} - \frac{G_2 G_6 B_2}{B_3 C_5} + \frac{G_3 G_5}{C_5} + \frac{B_1 G_2 G_5}{A_1 B_3 C_5} - \frac{G_2 G_5}{B_3 C_5} \right) \\
& \times \left(D_4 - \frac{B_4 D_3}{B_3} + \frac{C_4 D_5}{C_5} - \frac{B_4 C_3 D_5}{B_3 C_5} \right) \\
& + \left(\frac{C_2 G_6}{C_5} - \frac{A_2 C_1 G_6}{A_1 C_5} + \frac{C_3 B_1 A_2 G_6}{A_1 B_3 C_5} - \frac{B_2 C_3 G_6}{A_3 C_5} \right) \left(\frac{B_1 D_3}{A_1 B_3} - \frac{D_3}{B_3} \right. \\
& \left. - \frac{D_1 D_3}{A_1} - \frac{C_1 D_5}{A_1 C_5} + \frac{B_1 C_3 D_5}{A_1 B_3 C_5} - \frac{C_3 D_5}{B_3 C_5} \right) \quad (C.5)
\end{aligned}$$

$$\begin{aligned}
\omega_4 = & \left(\frac{B_4 G_2}{B_3} - G_4 + \frac{C_3 B_4 G_6}{B_3 C_5} - \frac{C_4 G_6}{C_5} \right) \left(D_2 - \frac{A_2 D_1}{A_1} + \frac{D_3 B_1 A_2}{B_3 A_1} - \frac{B_2 D_3}{B_3} \right. \\
& \left. + \frac{C_2 D_5}{C_5} + \frac{C_3 B_1 A_2 D_5}{B_3 A_1 C_5} - \frac{C_3 B_2 D_5}{A_3 C_5} \right) \\
& \left(\frac{C_2 G_6}{C_5} - \frac{C_1 A_2 G_6}{A_1 C_5} + \frac{C_3 B_1 A_2 G_6}{B_3 A_1 C_5} - \frac{C_3 B_2 G_6}{A_3 C_5} \right) \left(D_4 - \frac{B_4 D_3}{B_3} \right. \\
& \left. + \frac{C_4 D_5}{C_5} - \frac{C_3 B_4 D_5}{B_3 C_5} \right) \quad (C.6)
\end{aligned}$$

Here, the variables A , B , C , D , and G are represented by:

$$\begin{aligned}
A_1 &= ik\bar{\phi} & D_1 &= -\frac{1}{\bar{\phi}} \left(\frac{\bar{\mu}_s + \bar{\zeta}_s}{L} ik\Theta \right) \\
A_2 &= \Theta\bar{\phi} & D_2 &= -\frac{1}{\bar{\phi}} \left(\frac{\bar{\mu}_s + \bar{\zeta}_s}{L} (-\Theta^2) + \frac{\bar{\mu}_s}{L} (-k^2) - \frac{L(1-R)}{Fr} \bar{\beta} \right) \\
B_1 &= ik(\bar{\phi} - 1) & D_3 &= -\frac{1}{\bar{\phi}} \left(\frac{L(1-R)}{Fr} \bar{\beta} \right) \\
B_2 &= \frac{k^2}{\Theta} (1 - \bar{\phi}) & D_4 &= -\frac{1}{\bar{\phi}} (\bar{P}_{s\phi}\Theta) \\
B_3 &= \left(\frac{k^2}{\Theta} + \Theta \right) (\bar{\phi} - 1) & D_5 &= -\frac{1}{\bar{\phi}} (\bar{P}_{sT}\Theta) \\
B_4 &= ik\bar{u}_g \left(1 + \frac{(1-\bar{\phi})\bar{\beta}_\phi}{\Theta\bar{\beta}} \right) & G_1 &= \frac{ik}{\Theta} L\bar{\Gamma}_{u_g} \\
C_1 &= -\frac{1}{\bar{\phi}} \left(\frac{\bar{\mu}_s + \bar{\zeta}_s}{L} (-k^2) + \frac{\bar{\mu}_s}{L} (-\Theta^2) \right) & G_2 &= -\frac{ik}{\Theta} L\bar{\Gamma}_{u_g} \\
C_2 &= -\frac{1}{\bar{\phi}} \left(\frac{\bar{\mu}_s + \bar{\zeta}_s}{L} (ik\Theta) + \frac{L(1-R)}{Fr} \frac{ik}{\Theta} \bar{\beta} \right) & G_3 &= \frac{\bar{P}_s}{\bar{\phi}} \\
C_3 &= -\frac{1}{\bar{\phi}} \left(\frac{L(1-R)}{Fr} \frac{ik}{\Theta} \bar{\beta} \right) & G_4 &= L\bar{\Gamma}_\phi - \frac{L\bar{\Gamma}_{u_g}\bar{\beta}_\phi}{\Theta\bar{\beta}} \bar{u}_g + \frac{\bar{\lambda}_h}{L} (-k^2 - \Theta^2) \\
C_4 &= -\frac{1}{\bar{\phi}} \left(-ik\bar{P}_{s\phi} \right. & G_5 &= -\frac{3}{2}\bar{\phi} \\
&\quad \left. + \frac{L(1-R)}{Fr} \bar{u}_g \left(\bar{\beta}_\phi - \frac{\bar{\beta}_\phi}{\Theta} - \frac{\bar{\beta}}{\bar{\phi}} \right) \right) & G_6 &= L\bar{\Gamma}_T + \frac{\bar{\lambda}}{L} (-k^2 - \Theta^2) \\
C_5 &= -\frac{1}{\bar{\phi}} (-ik\bar{P}_{sT}) & &
\end{aligned} \tag{C.7}$$