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On the mod p cohomology of pro-p Iwahori subgroups

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On the mod p cohomology of pro-p Iwahori subgroups

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosphy

in

Mathematics

by

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2022

The dissertation of Daniel Kongsgaard is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

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$\mathcal{B}$ / $\mathcal{B}^+$	($=\mathcal{T}\mathcal{N}$ / $=\mathcal{T}\mathcal{N}^+$) the Borel subgroups of $\mathcal{G}$ corresponding to Φ^- / Φ^+
Δ	a (fixed) basis of the root system Φ
$\mathcal{G}$	a (fixed) split and connected reductive algebraic $\mathbb{Z}_p$-group
$\mathfrak{g}$	$=\mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N$, the Lazard Lie algebra corresponding to N
G_ν	$:= \{g \in G : \omega(g) \geq \nu\}$
$G_{\nu+}$	$:= \{g \in G : \omega(g) > \nu\}$
$\operatorname{gr} G$	$:= \bigoplus_{\nu>0} \operatorname{gr}_\nu G$ (a graded Lie algebra over $\mathbb{F}_p[\pi]$)
$\operatorname{gr}_\nu G$	$:= G_\nu / G_{\nu+}$
h	the Coxeter number of $\mathcal{G}$
$H^\bullet(\mathfrak{g}, \,\cdot\,)$	the cohomology of the Lie algebra $\mathfrak{g}$
$H^\bullet_{\mathrm{cts}}(G, \,\cdot\,)$	the continuous group cohomology of a topological group G
$H^\bullet_{\mathrm{dsc}}(G, \,\cdot\,)$	the discrete group cohomology of a topological group H
$H^{s,t}$	$= \operatorname{gr}^s H^{s+t}$ for some cohomology H
$H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$	$= H^{s+t}\big(\operatorname{gr}^s \operatorname{Hom}_{\mathbb{F}_p}(\bigwedge^\bullet \mathfrak{g}, \mathbb{F}_p)\big)$
$\mathcal{N}$ / $\mathcal{N}^+$	the unipotent radical of $\mathcal{B}$ / $\mathcal{B}^+$
$\omega\colon G \setminus \{1\} \to (0, \infty)$	a p-valuation on G
p	a prime, $p \geq h-1$, where h is the Coxeter number of $\mathcal{G}$
Φ	$= \Phi(\mathcal{G}, \mathcal{T})$, the root system of $\mathcal{G}$ with respect to $\mathcal{T}$

Φ^+ / Φ^-	the positive/negative roots in Φ with respect to Δ
$\Phi^\vee$	the dual root system of Φ
$\pi\colon \operatorname{gr} G \to \operatorname{gr} G$	the direct sum of the maps $gG_{\nu+} \mapsto g^p G_{(\nu+1)+}$
$\operatorname{rank}(G,\omega)$	$:= \operatorname{rank}_{\mathbb{F}_p[P]} \operatorname{gr} G$ the rank of the pair (G,ω)
$\mathcal{T}$	a (fixed) split maximal torus of $\mathcal{G}$
$V_{\mathbb{F}_p}(\lambda)$	$= V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \mathbb{F}_p$
$V_{\mathbb{Z}}(\lambda)$	the Weyl module for $\mathcal{G}_{\mathbb{Z}}$ over $\mathbb{Z}$ with highest weight λ
W	the Weyl group corresponding to Φ and $\Phi^\vee$
X	$= X(\mathcal{T}) \cong X(\mathcal{T}_{\mathbb{Z}})$, the character group of $\mathcal{T}$
X^+	$= \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$

Cohomology of pro-p Iwahori subgroups

$\operatorname{diag}(a_1,\ldots,a_n)$	$(=(a_{ij}))$ the diagonal matrix with entries $a_{ii} = a_i$
$\operatorname{diag}_{i_1,\ldots,i_k}(a_1,\ldots,a_k)$	$(=(a_{ij}))$ the matrix with entries $a_{i_\ell i_\ell} = a_\ell$ for $\ell = 1,\ldots,k$, ones in the other diagonal entries and zeroes in all other entries
$\operatorname{diag}_{i,i+1}(u)$	$= \operatorname{diag}_{i,i+1}(u, u^{-1})$
$e_{i_1,\ldots,i_m}$	$= (\xi_{i_1} \wedge \cdots \wedge \xi_{i_m})^*$, the element of the dual basis of $\operatorname{Hom}_k\big(\bigwedge^m \mathfrak{g}, k\big)$ corresponding to $\xi_{i_1} \wedge \cdots \wedge \xi_{i_m}$ in the basis of $\bigwedge^m \mathfrak{g}$
E_{ij}	the matrix with 1 in the (i,j) entry and zeroes in all other entries
F	a finite extension of $\mathbb{Q}_p$
$\mathcal{G}$	a (fixed) split and connected reductive algebraic F-group
$\mathfrak{g}$	$= k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$, the Lazard Lie algebra corresponding to the pro-p Iwahori subgroup I

G	$= \mathcal{G}(F)$, a locally profinite group
g_{ij}	$= [g_i, g_j]$
h	the Coxeter number of $\mathcal{G}$
$H^\bullet(\mathfrak{g}, \,\cdot\,)$	the cohomology of the Lie algebra $\mathfrak{g}$
$H^\bullet(G, \,\cdot\,)$	the continuous group cohomology of the topological group G
$H^{s,t}$	$= \operatorname{gr}^s H^{s+t}$ for some cohomology H
$H^{s,t}$	$= H^{s,t}(\mathfrak{g}, k) = H^{s+t}\big(\operatorname{gr}^s \operatorname{Hom}_k(\bigwedge^\bullet \mathfrak{g}, k)\big)$
I_G	the pro-p Iwahori subgroup of G
k	a perfect field of characteristic p
ℓ	$= [F : \mathbb{Q}_p]$, the degree of the extension $F/\mathbb{Q}_p$
$\mathfrak{m}_F$	maximal ideal of the valuation ring $\mathcal{O}_F$
$\mathcal{O}_F$	the valuation ring of F
$O(p^r)$	for elements of $\mathcal{O}_F$ we write $x = y + O(p^r)$ if and only if $x - y \in p^r\mathcal{O}_F$
p	a prime, $p - 1 \geq eh$, where h is the Coxeter number of $\mathcal{G}$
Φ	$= \Phi(\mathcal{G}, \mathcal{T})$, the root system of $\mathcal{G}$ with respect to $\mathcal{T}$
ϖ_F	a uniformizer of F
$(\,\cdot\,)^\top$	the transpose matrix
$\mathcal{T}$	a (fixed) split maximal torus of $\mathcal{G}$
T	$= \mathcal{T}(F)$
$U_{\alpha,r}$	$= x_\alpha(\mathfrak{m}_F^r)$
$x_\alpha \colon F \xrightarrow{\cong} U_\alpha$	an isomorphism such that $t x_\alpha(x) t^{-1} = x_\alpha(\alpha(t)x)$ for $t \in T$ and $x \in F$

ξ_{ij}	$= [\xi_i, \xi_j]$
$(X^*(T), \Phi, X_*(T), \Phi^\vee)$	the root datum associated with $\Phi = \Phi((G), \mathcal{T})$

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Abstract of the Dissertation

On the mod p cohomology of pro-p Iwahori subgroups

by

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Doctor of Philosophy in Mathematics

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Professor Claus Sørensen, Chair

Let $\mathcal{G}$ be a split and connected reductive $\mathbb{Z}_p$-group and let $\mathcal{N}$ be the unipotent radical of a Borel subgroup. In the first chapter of this dissertation we study the cohomology with trivial $\mathbb{F}_p$-coefficients of the unipotent pro-p group $N = \mathcal{N}(\mathbb{Z}_p)$ and the Lie algebra $\mathfrak{n} = \mathrm{Lie}(\mathcal{N}_{\mathbb{F}_p})$. We proceed by arguing that N is a p-valued group using ideas of Schneider and Zábrádi, which by a result of Sørensen gives us a spectral sequence $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(N, \mathbb{F}_p)$, where $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, N$ is the graded $\mathbb{F}_p$-Lie algebra attached to N as in Lazards work. We then argue that $\mathfrak{g} \cong \mathfrak{n}$ by looking at the Chevalley constants, and, using results of Polo and Tilouine and ideas from Große-Klönne, we show that the dimensions of the $\mathbb{F}_p$-cohomology of $\mathfrak{n}$ and N agree, which allows us to conclude that the spectral sequence collapses on the first page.

In the second chapter we study the mod p cohomology of the pro-p Iwahori subgroups I of SL_n and GL_n over $\mathbb{Q}_p$ for $n = 2, 3, 4$ and over a quadratic extension $F/\mathbb{Q}_p$ for $n = 2$. Here we again use the spectral sequence $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(I, \mathbb{F}_p)$ due to Sørensen, but in this chapter

we do explicit calculations with an ordered basis of I, which gives us a basis of $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} I$ that we use to calculate $H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$. We note that the spectral sequence $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$ collapses on the first page by noticing that all maps on each page are necessarily trivial. Finally we note some connections to cohomology of quaternion algebras over $\mathbb{Q}_p$ and point out some future research directions.

Chapter 1

Introduction

The cohomology of Lie groups has a long history. In the late forties Chevalley and Eilenberg found that $H^*(G, \mathbb{R}) \cong H^*(\mathfrak{g}, \mathbb{R})$ for a connected compact Lie group G with Lie algebra $\mathfrak{g}$ (cf. [CE48]), and since then there has been much research into different types of Lie group cohomology. In particular, the mod p cohomology of a connected compact real Lie group has been well understood by Kac since the eighties (cf. [Kac85]), and the continuous mod p cohomology $H^*(G, \mathbb{F}_p)$ of an equi-p-valued compact p-adic Lie group G was already described by Lazard in the sixties (cf. [Laz65]). We note here that (except for Lazard's work) $H^*(G, \mathbb{R})$ and $H^*(G, \mathbb{F}_p)$ indicate the cohomology of G as a topological space, and not continuous group cohomology, which can be thought of as the cohomology of the classifying space BG.

This dissertation's main interest is the continuous mod p cohomology $H^*(G, \mathbb{F}_p)$ of compact p-adic Lie groups G for specific cases of G. Since p-adic Lie groups are totally disconnected, working with them requires very different methods than what Chevalley and Eilenberg or Kac used for real Lie groups, and we have to follow the ideas of Lazard (see [Laz65]) and Serre. In particular we need a p-valuation on G (and on the completed group algebras associated with G), and we work with the graded "Lazard" Lie algebra $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ attached to G. We will repeatedly use that Sørensen (in [Sør21]) showed that $H^*(\mathfrak{g}, \mathbb{F}_p)$ determines $H^*(G, \mathbb{F}_p)$ via a multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(G, \mathbb{F}_p).$$

When G is equi-p-valuable, we get that $\mathfrak{g}$ is concentrated in a single degree, and Lazard showed that $H^*(G, \mathbb{F}_p) \cong \bigwedge H^1(\mathfrak{g}, \mathbb{F}_p)$, while Sørensen showed that this also follows from the above spectral sequence. We are interested in cases where G is *not* equi-p-valuable, and we note that the spectral sequence of Sørensen allows us to work purely with G and $\mathfrak{g}$ without having to worry about the completed group algebras $\Lambda(G) = \mathbb{Z}_p[\![G]\!]$ and $\Omega(G) = \mathbb{F}_p[\![G]\!]$.

Before describing our particular results in the following paragraph, we emphasize the following remark of Sørensen from [Sør21]: It is known (due to Lazard) that any compact p-adic Lie group contains an open equi-p-valuable subgroup (see [Laz65, Chap. V 2.2.7.1]), which gives the impression that the distinction between p-valued and equi-p-valued groups is somewhat nuanced, which is true for some questions. But there are many examples of naturally occurring p-valuable groups G which are not equi-p-valuable, where detailed information about $H^*(G, \mathbb{F}_p)$ is important. For example unipotent groups (i.e., the $\mathbb{Z}_p$-points of the unipotent radical of a Borel in a split reductive group), Serre's standard groups with $e > 1$ as in [HKN11, Lem. 2.2.2], pro-p Iwahori subgroups for large enough p, and $1 + \mathfrak{m}_D$ where D is the quaternion division algebra over $\mathbb{Q}_p$ for $p > 3$ (or more generally a central division algebra over $\mathbb{Q}_p$). Sørensen explicitly calculates $H^*\big((1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\big)$ for $p > 3$ and uses it to describe $H^*(1 + \mathfrak{m}_D, \mathbb{F}_p)$, and he notes that $1 + \mathfrak{m}_D$ plays an important role both in number theory (in the Jacquet-Langlands correspondence for instance, see [JL70]) and algebraic topology, where $1 + \mathfrak{m}_D$ is known as the (strict) Morava stabilizer in stable homotopy theory, and $H^*(1 + \mathfrak{m}_D, \mathbb{F}_p)$ somehow controls certain localization functors with respect to Morava K-theory (see e.g. [Hen07]).

Our work in Chapter 2 will build on ideas of Lazard and Serre from their more general (but not yet finished) description of the case when G is not equi-p-valued, and especially the refinement of these ideas as described by Sørensen and Schneider in [Sør21] and [Sch11b]. We will focus on unipotent groups N originating from split and connected reductive $\mathbb{Z}_p$-groups, which is similar to recent work in the case of $\mathbb{Z}_p$ coefficients by Ronchetti (cf. [Ron20]). We note that this work can be considered a slight refinement of [Gro14] since we retain information about the cup product on $H^*(N, \mathbb{F}_p)$.

In Chapter 3 we focus on the case of pro-p Iwahori subgroups of SL_n and GL_n over $\mathbb{Q}_p$ for $n = 2, 3, 4$ or over quadratic extensions $F/\mathbb{Q}_p$ for $n = 2$. We explicitly calculate the algebra structure of $H^*(I, \mathbb{F}_p)$ for the pro-p Iwahori subgroups $I_{\mathrm{SL}_2(\mathbb{Q}_p)} \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ and $I_{\mathrm{GL}_2(\mathbb{Q}_p)} \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$, and we note that these are isomorphic as algebras to $H^*\big((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\big)$ and $H^*(1+\mathfrak{m}_D, \mathbb{F}_p)$ respectively. We finish the chapter by mentioning some future research directions and a conjecture on the connection between the mod p cohomology of $(1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}$ (resp. $1+\mathfrak{m}_D$) for central division algebras and $I_{\mathrm{SL}_n(\mathbb{Q}_p)}$ (resp. $I_{\mathrm{GL}_n(\mathbb{Q}_p)}$).

Finally, the appendix will end with a very brief description of other research (all joint) that I have participated in.

Chapter 2

Cohomology of Unipotent Groups

2.1 Introduction

In this chapter we show that the cohomology of certain unipotent groups can be found via a simpler cohomology calculation for related Lie algebras. This is done using a spectral sequence due to [Sør21].

2.1.1 Background and motivation

As mentioned in Chapter 1, we will focus on describing the continuous mod p cohomology of unipotent groups N originating from split and connected reductive $\mathbb{Z}_p$-groups in this chapter. To be precise, let $\mathcal{N}$ be the unipotent radical of a Borel in a split and connected reductive $\mathbb{Z}_p$-group, and let $N = \mathcal{N}(\mathbb{Z}_p)$ be the $\mathbb{Z}_p$-points of $\mathcal{N}$. In this chapter we will show that N is a p-valuable group (with a nice p-valuation), which will allow us to show that the Lazard Lie algebra $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N$ is isomorphic to $\mathfrak{n} = \operatorname{Lie}(\mathcal{N}_{\mathbb{F}_p})$. Using an idea of Große-Klönne (cf. [Gro14, Sect. 7]) we get that discrete mod p cohomology of $\mathcal{N}_{\mathbb{Z}}(\mathbb{Z})$ is isomorphic to the continuous mod p cohomology of $N = \mathcal{N}(\mathbb{Z}_p)$, and results of Polo and Tilouine (see [PT18]) allow us to compare the discrete cohomology of $\mathfrak{n}_{\mathbb{Z}} = \operatorname{Lie}(\mathcal{N}_{\mathbb{Z}})$ and $\mathcal{N}_{\mathbb{Z}}(\mathbb{Z})$, which by a short argument allows us to compare the dimensions of the continuous mod p cohomology of N and the mod p cohomology of $\mathfrak{g} \cong \mathfrak{n}$. This will let us conclude that the multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(N, \mathbb{F}_p)$$

due to Sørensen collapses at the first page, which gives us a description of $H^*(N, \mathbb{F}_p)$.

It is worth noting that this work started out as an attempt to better understand the proof of [Gro14, Theorem 7.1], in particular the part using the result of Grünenfelder (which by a remark of Polo and Tilouine might have a problem), but the work has since develop in a different direction, where the coefficients are more restricted, but we obtain a more precise (or indeed any) description of the cup product.

2.1.2 Notation and setup

Let p be an odd prime (that will be further restricted later).

Algebraic groups. We will work with schemes using the functorial approach and notation described in [Jan03]. In particular, given an integral domain R, we note that a *R-group functor* is a functor from the category of all R-algebras to the category of groups, a *R-group scheme* is a R-group functor that is an affine scheme over R when considered as a R-functor, and an *algebraic R-group* is a R-group scheme that is algebraic as an affine scheme. For more in depth introduction to these concepts, we refer to [Con14b] and [Jan03].

Base change. If R' is a R-algebra, then any R'-algebra A is in a natural way a R-algebra by combining the structural homomorphisms $R \to R'$ and $R' \to A$. We can therefore associate to each R-functor X a R'-functor $X_{R'}$ by $X_{R'}(A) = X(A)$ for any R'-algebra A. For any morphism $f \colon X \to X'$ of R-functors, we get a morphism $f_{R'} \colon X_{R'} \to X'_{R'}$ of k'-functors by $f_{R'}(A) = f(A)$ for any R'-algebra A. In this way we get a functor $X \mapsto X_{R'}$, $f \mapsto f_{R'}$ from the category of R-functors to the category of R'-functors, which we call the *base change* from R to R'.

Fixed $\mathbb{Z}_p$-groups and roots. We fix a split and connected reductive algebraic $\mathbb{Z}_p$-group $\mathcal{G}$ as well as a split maximal torus $\mathcal{T} \subseteq \mathcal{G}$. Let $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ be the *root system* of $\mathcal{G}$ with respect to $\mathcal{T}$. For any $\alpha \in \Phi$ we have the root subgroup $\mathcal{N}_\alpha \subseteq \mathcal{G}$ with Lie algebra $\operatorname{Lie} \mathcal{N}_\alpha = (\operatorname{Lie} \mathcal{G})_\alpha$. We fix a $\mathbb{Z}_p$-basis $(X_\alpha)_{\alpha \in \Phi}$ of $\operatorname{Lie} \mathcal{N}_\alpha$, and note that this choice gives rise to unique isomorphisms of group schemes $x_\alpha \colon \mathbb{G}_a \xrightarrow{\cong} \mathcal{N}_\alpha$ such that $(dx_\alpha)(1) = X_\alpha$. We furthermore fix a basis $\Delta \subseteq \Phi$ of the root system, so we get a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let $\mathcal{B} = \mathcal{T}\mathcal{N}$ and $\mathcal{B}^+ = \mathcal{T}\mathcal{N}^+$

denote the Borel subgroups of $\mathcal{G}$ corresponding to Φ^- and Φ^+, respectively, with unipotent radicals $\mathcal{N}$ and $\mathcal{N}^+$. Finally let $N = \mathcal{N}(\mathbb{Z}_p)$ and let $\mathfrak{n} = \mathrm{Lie}(\mathcal{N}_{\mathbb{F}_p})$ be the Lie algebra of $\mathcal{N}_{\mathbb{F}_p}$ over $\mathbb{F}_p$.

$\mathbb{Z}$-models. Let $\mathcal{G}_{\mathbb{Z}}$ be the Chevalley group over $\mathbb{Z}$ corresponding to $\mathcal{G}$ (cf. [Con14a, §1]), and consider the subgroups $\mathcal{T}_{\mathbb{Z}}, \mathcal{B}_{\mathbb{Z}}, \mathcal{N}_{\mathbb{Z}}$ corresponding to $\mathcal{T}, \mathcal{B}, \mathcal{N}$. Let furthermore $\mathfrak{n}_{\mathbb{Z}} = \mathrm{Lie}(\mathcal{N}_{\mathbb{Z}})$ be the Lie algebra of $\mathcal{N}_{\mathbb{Z}}$ over $\mathbb{Z}$, and note that $N = \mathcal{N}_{\mathbb{Z}}(\mathbb{Z}_p)$ and $\mathfrak{n} = \mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p$. (Note also that $(\mathcal{G}_{\mathbb{Z}})_{\mathbb{Z}_p} = \mathcal{G}$, so although we abuse notation a bit here, it wont be a problem.)

Total ordering of Φ^-. For any total ordering of Φ^- the multiplication induces an isomorphism of schemes $\prod_{\alpha \in \Phi^-} \mathcal{N}_\alpha \xrightarrow{\cong} \mathcal{N}$. For convenience we fix a total ordering which has the additional property that $\alpha_1 \geq \alpha_2$ if $\mathrm{ht}(\alpha_1) \leq \mathrm{ht}(\alpha_2)$. All products indexed by Φ^- are meant to be taken according to this ordering. Here we have the height function $\mathrm{ht}\colon \mathbb{Z}[\Delta] \to \mathbb{Z}$ given by $\sum_{\alpha \in \Delta} m_\alpha \alpha \mapsto \sum_{\alpha \in \Delta} m_\alpha$. In particular, since $\Phi \subseteq \mathbb{Z}[\Delta]$ the height $\mathrm{ht}(\beta)$ of any root $\beta \in \Phi$ is defined.

Coxeter number and p. Let h be the Coxeter number of $\mathcal{G}$ and assume from now on that $p \geq h - 1$.

Weyl group and module. Let $\Phi^\vee$ be the dual root system of Φ and let W be the corresponding Weyl group with length function ℓ on W. Let furthermore $X = X(\mathcal{T}) \cong X(\mathcal{T}_{\mathbb{Z}})$ be the character group of $\mathcal{T}$, and set

$$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

For any $\lambda \in X^+$, let $V_{\mathbb{Z}}(\lambda)$ be the Weyl module for $\mathcal{G}_{\mathbb{Z}}$ over $\mathbb{Z}$ with highest weight λ, and let $V_{\mathbb{F}_p}(\lambda) = V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \mathbb{F}_p$.

Lazard theory. We will introduce concepts from Lazard theory in next subsection, but we note now that we will let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, N$ be the Lazard Lie algebra corresponding to N.

Cohomology. For any ring R, we denote (using the Chevalley-Eilenberg complex) the Lie algebra cohomology of any R-Lie algebra $\mathfrak{g}$ by $H^\bullet(\mathfrak{g}, \cdot)$, while we write $H^\bullet_{\mathrm{dsc}}(G, \cdot)$ and $H^\bullet_{\mathrm{cts}}(H, \cdot)$ for the discrete (resp. continuous) group cohomology of a topological group G. Later we will introduce filtrations and then gradings on the cohomology, in which case we always use the notation $H^{s,t} = \operatorname{gr}^s H^{s+t}$ for any type of cohomology H.

Spectral sequences. Given a ring R, a cohomological spectral sequence is a choice of $r_0 \in \mathbb{N}$ and a collection of

- R-modules $E_r^{s,t}$ for each $s, t \in \mathbb{Z}$ and all integers $r \geq r_0$
- differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ such that $d_r^2 = 0$ and E_{r+1} is isomorphic to the homology of (E_r, d_r), i.e.,

$$E_{r+1}^{s,t} = \frac{\ker(d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r})}{\operatorname{im}(d_r^{s-r,t+r-1} \colon E_r^{s-r,t+r-1} \to E_r^{s,t})}.$$

For a given r, the collection $(E_r^{s,t}, d_r^{s,t})_{s,t \in \mathbb{Z}}$ is called the r-th page. A spectral sequence *converges* if d_r vanishes on $E_r^{s,t}$ for any s, t when $r \gg 0$. In this case $E_r^{s,t}$ is independent of r for sufficiently large r, we denote it by $E_\infty^{s,t}$ and write

$$E_r^{s,t} \Longrightarrow E_\infty^{s+t}.$$

Also, we say that the spectral sequence collapses at the r'-th page if $E_r = E_\infty$ for all $r \geq r'$, but not for $r < r'$. Finally, when we have terms E_∞^n with a natural filtration $F^\bullet E_\infty^n$ (but no natural double grading), we set $E_\infty^{s,t} = \operatorname{gr}^s E_\infty^{s+t} = F^s E_\infty^{s+t} / F^{s+1} E_\infty^{s+t}$.

2.1.3 Lazard theory

In this subsection we will briefly introduce elements of Lazard theory as presented in [Sch11a]. Let G be any abstract group and let the commutator be normalized as $[g, h] = ghg^{-1}h^{-1}$.

Definition 2.1. A *p-valuation* ω on G is a real valued function

$$\omega \colon G \setminus \{1\} \to (0, \infty)$$

which, with the convention that $\omega(1) = \infty$, satisfies

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$,

(c) $\omega([g,h]) \geq \omega(g) + \omega(h)$,

(d) $\omega(g^p) = \omega(g) + 1$

for any $g, h \in G$. ♠

For the rest of this subsection, let (G, ω) be a p-valued group, i.e., a group with a p-valuation.

For any real number $\nu > 0$ put

$$G_\nu := \{g \in G : \omega(g) \geq \nu\} \quad \text{and} \quad G_{\nu+} := \{g \in G : \omega(g) > \nu\},$$

and note that these are normal subgroups, cf. [Sch11a, Sect. 23].

The subgroups G_ν form a decreasing exhaustive and separated filtration of G with the additional properties

$$G_\nu = \bigcap_{\nu' < \nu} G_{\nu'} \quad \text{and} \quad [G_\nu, G_{\nu'}] \subseteq G_{\nu+\nu'}.$$

There is a unique (Hausdorff) topological group structure on G for which the G_ν form a fundamental system of open neighborhoods of the identity element. It will be called the *topology defined by* ω. We will assume that G is profinte in the topology defined by ω. Hence $G = \varprojlim_{\nu>0} G/G_\nu$ as topological groups, and thus G must be a pro-p-group since $\omega(g^p) = \omega(g) + 1$ implies that G/G_ν is a p-group (finite since G_ν is open).

We now form, for each $\nu > 0$, the subquotient group

$$\operatorname{gr}_\nu G := G_\nu / G_{\nu+}.$$

It is commutative by (c) and therefore will be denoted additively. We now consider the graded abelian group

$$\operatorname{gr} G := \bigoplus_{\nu > 0} \operatorname{gr}_\nu G.$$

An element $\xi \in \operatorname{gr} G$ is called, as usual, homogeneous (of degree ν) if it lies in $\operatorname{gr}_\nu G$. Furthermore, in this case any $g \in G_\nu$ such that $\xi = gG_{\nu+}$ is called a representative of ξ.

Note that $p\xi = 0$ for any homogeneous element $\xi \in \operatorname{gr} G$ since $\omega(g^p) = \omega(g) + 1$. Hence $\operatorname{gr} G$ in fact is an $\mathbb{F}_p$-vector space. Furthermore, by bilinear extension of the map

$$\operatorname{gr}_\nu G \times \operatorname{gr}_{\nu'} G \to \operatorname{gr}_{\nu+\nu'} G$$

$$(\xi, \eta) \mapsto [\xi, \eta] := [g, h]G_{(\nu+\nu')+},$$

for $\nu, \nu' > 0$, we obtain a graded $\mathbb{F}_p$-bilinear map

$$[\,\cdot\,,\,\cdot\,]\colon \operatorname{gr} G \times \operatorname{gr} G \to \operatorname{gr} G$$

which satisfies

$$[\xi, \xi] = 0 \qquad \text{for any } \xi \in \operatorname{gr} G.$$

One can check that $[\,\cdot\,,\,\cdot\,]$ satisfies the Jacobi identity, and thus $\operatorname{gr} G$ is a graded Lie algebra over $\mathbb{F}_p$, cf. [Sch11a, Sect. 23].

Now, noticing that the map

$$\operatorname{gr}_\nu G \to \operatorname{gr}_{\nu+1} G$$
$$gG_{\nu+} \mapsto g^p G_{(\nu+1)+}$$

is well defined and $\mathbb{F}_p$-linear, by considering for varying ν the direct sum of these maps, we can introduce an $\mathbb{F}_p$-linear map of degree one

$$\pi\colon \operatorname{gr} G \to \operatorname{gr} G.$$

We can and will therefore view $\operatorname{gr} G$ as a graded module over the polynomial ring $\mathbb{F}_p[\pi]$ in one variable over $\mathbb{F}_p$. Furthermore the Lie bracket on $\operatorname{gr} G$ is bilinear for the $\mathbb{F}_p[\pi]$-module structure, i.e., $\operatorname{gr} G$ is a Lie algebra over the ring $\mathbb{F}_p[\pi]$. For more details, we refer to [Sch11a, Sect. 25].

Definition 2.2. The pair (G, ω) is called of finite rank if $\operatorname{gr} G$ is finitely generated as an $\mathbb{F}_p[\pi]$-module. ♠

Note that G being of finite rank does not depend on the choice of the p-valuation, and assume from now on that (G, ω) is of finite rank. Note that $\operatorname{gr} G$ is finitely generated and torsionfree over the principal ideal domain $\mathbb{F}_p[\pi]$, and thus by the elementary divisor theorem $\operatorname{gr} G$ is free. We call

$$\operatorname{rank}(G, \omega) := \operatorname{rank}_{\mathbb{F}_p[\pi]} \operatorname{gr} G$$

the *rank* of the pair (G, ω).

For any $g \in G$ note that we then have a group homomorphism

$$c\colon \mathbb{Z} \to G$$
$$m \mapsto g^m.$$

Since G/N, for any $N \triangleleft G$, is a p-group, we obtain $c^{-1}(N) = p^{a_N}\mathbb{Z}$ for some $a_N \geq 0$. It follows that c extends uniquely to a continuous group homomorphism

$$\tilde{c}\colon \mathbb{Z}_p \to \varprojlim_{N \triangleleft G} \mathbb{Z}/p^{a_N}\mathbb{Z} \xrightarrow{c} \varprojlim_{N} G/N = G$$

which we always will write as $g^x := \tilde{c}(x)$. More generally, for any finitely many elements $g_1, \ldots, g_r \in G$, we have the continuous map

$$\begin{aligned} \mathbb{Z}_p^r &\to G \\ (x_1, \ldots, x_r) &\mapsto g_1^{x_1} \cdots g_r^{x_r} \end{aligned} \tag{2.1}$$

which depends on the order of the g_i and therefore is not a group homomorphism. However we introduce the following notation, where v_p denotes the usual p-adic valuation on $\mathbb{Q}_p$.

Definition 2.3. The sequence of elements $(g_1, \ldots, g_r)$ in G is called an *ordered basis* of (G, ω) if the map (2.1) is a bijection (and hence, by compactness, a homeomorphism) and

$$\omega(g_1^{x_1} \cdots g_r^{x_r}) = \min_{1 \leq i \leq r} (\omega(g_i) + v_p(x_i)) \qquad \text{for any } x_1, \ldots, x_r \in \mathbb{Z}_p.$$ ♠

Definition 2.4. For any $g \in G \setminus \{1\}$, we put $\sigma(g) := gG_{\omega(g)+} \in \operatorname{gr} G$. ♠

By [Sch11a, Remark 26.3], we note that for $g \in G \setminus \{1\}$ and $x \in \mathbb{Z}_p \setminus \{0\}$

$$\omega(g^x) = \omega(g) + v_p(x) \quad \text{and} \quad \sigma(g^x) = \bar{x}\pi^{v_p(x)} . \sigma(g), \tag{2.2}$$

where $\bar{x}$ is the image of $p^{-v_p(x)}x$ in $\mathbb{F}_p^\times$ (i.e., the first non-zero coefficient of $x = \sum_{k=0}^{\infty} a_k p^k$). We note that an ordered basis $(g_1, \ldots, g_d)$ of (G, ω) corresponds to an ordered $\mathbb{F}_p[\pi]$-basis $\big(\sigma(g_1), \ldots, \sigma(g_d)\big)$ of $\operatorname{gr} G$, cf. [Sch11a, Prop. 26.5].

Finally we let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G = \mathbb{F}_p \otimes_{\mathbb{F}_p} \operatorname{gr} G/\pi \operatorname{gr} G$, and note that this is a Lie algebra over $\mathbb{F}_p$ with an $\mathbb{F}_p$-basis of vectors $\xi_i = 1 \otimes \sigma(g_i)$.

2.1.4 Cohomology theories and the spectral sequence

One of the main results we use in this chapter is the spectral sequence introduced in [Sør21, §6.1], so in this subsection we aim to introduce the concepts needed to use this spectral sequence. We also mention a translation between continuous and discrete group cohomology for the groups we work with.

Let R be a ring and let $\mathfrak{g}$ be a R-Lie algebra with R a trivial (left) $\mathfrak{g}$-module. Then we use the cochain complex $C^\bullet(\mathfrak{g}, R) = \operatorname{Hom}_R(\bigwedge^\bullet \mathfrak{g}, R)$ to define Lie algebra cohomology, i.e., the cochain complex

$$0 \longrightarrow R \xrightarrow{\partial_1} \operatorname{Hom}_R(\mathfrak{g}, R) \xrightarrow{\partial_2} \operatorname{Hom}_R\Big(\bigwedge^2 \mathfrak{g}, R\Big) \xrightarrow{\partial_3} \cdots,$$

where the coboundary map ∂_n is given by

$$\partial_n(f)(x_1, \dots, x_n) = \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_n),$$

where $\widehat{x}_i$ means excluding x_i. For more details we refer to [CE56, Thm. 7.1] or [Fuk86, Chap. 1 §3], and note that we are considering the trivial action on R, which simplifies the formula slightly (cf. [Fuk86, Chap. 1 §3.2]).

Now consider $R = \mathbb{F}_p$ in the following and suppose that $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \cdots$ is a graded Lie algebra. Then $\bigwedge^n \mathfrak{g}$ is also graded by letting

$$\operatorname{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1 + \cdots + j_n = j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}.$$

Letting $\mathbb{F}_p$ be a $\mathbb{Z}$-graded (concentrated in degree 0) $\mathfrak{g}$-module, we get a grading

$$\operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big) = \bigoplus_{s \in \mathbb{Z}} \operatorname{Hom}^s_{\mathbb{F}_p}\Big(\bigwedge^n \mathfrak{g}, \mathbb{F}_p\Big)$$

where $\operatorname{Hom}^s_{\mathbb{F}_p}$ denotes the homogeneous $\mathbb{F}_p$-linear maps of degree s, cf. [FF74, Lem. 4.2]. One can check that this passes to bigrading of Lie algebra cohomology

$$H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = H^{s+t}\Big(\operatorname{gr}^s \operatorname{Hom}_{\mathbb{F}_p}\big(\bigwedge^\bullet \mathfrak{g}, \mathbb{F}_p\big)\Big).$$

See [Fuk86, Chap. 1 §3] for more details.

In the spectral sequence described in [Sør21, §6.1], we take $r_0 = 1$ (i.e., the spectral sequence starts from the first page) and $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$, where $\mathfrak{g} = \mathbb{F}_p \otimes \operatorname{gr} G$ indeed is (positively) $\mathbb{Z}$-graded.

Let now G be a topological group and $\mathbb{F}_p$ a G-module. Then we will define two types of group cohommology: continuous and discrete.

Continuous group cohomology $H^n_{\text{cts}}(G, \mathbb{F}_p)$ is the cohomology of the complex $C^\bullet(G, \mathbb{F}_p) = \mathcal{C}(G^\bullet, \mathbb{F}_p)$ of continuous maps $G \times G \times \cdots \times G \to \mathbb{F}_p$, i.e.,

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\partial_1} \mathcal{C}(G, \mathbb{F}_p) \xrightarrow{\partial_2} \mathcal{C}(G^2, \mathbb{F}_p) \xrightarrow{\partial_3} \mathcal{C}(G^3, \mathbb{F}_p) \xrightarrow{\partial_4} \cdots,$$

where the coboundary map ∂_n is given by

$$\partial_n(f)(g_1, \ldots, g_n) = f(g_2, \ldots, g_n) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n), \tag{2.3}$$

where n-th term is interpreted as $(-1)^n f(g_1, \ldots, g_{n-1})$, cf. [Sør21, §3] or [Ser02, §2]. Note again that our formula is slightly simpler since we only consider the trivial action on $\mathbb{F}_p$.

Discrete group cohomology $H^n_{\text{dsc}}(G, \mathbb{F}_p)$ is the cohomology of the complex $C^\bullet(G, \mathbb{F}_p) = \text{Hom}_G(\mathbb{Z}[G^\bullet], \mathbb{F}_p)$, i.e.,

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\partial_1} \text{Hom}_G(\mathbb{Z}[G], \mathbb{F}_p) \xrightarrow{\partial_2} \text{Hom}_G(\mathbb{Z}[G^2], \mathbb{F}_p) \xrightarrow{\partial_3} \cdots,$$

where the coboundary map δ_n is given by (2.3), see e.g. [Ser79, Chap. VII]. Note that this discrete cohomology can be viewed as continuous cohomology if we equip G with the discrete topology.

Note that [Sør21] gets the spectral sequence we are interested in by using an isomorphism to translate $H^\bullet_{\text{cts}}(G, \mathbb{F}_p)$ to $HH^\bullet(\Omega(G), \mathbb{F}_p)$ (essentially what is known as Mac Lane isomorphism) and introducing a $\mathbb{Z}$-filtration and grading on $HH^\bullet(\Omega(G), \mathbb{F}_p)$, which is used in the spectral sequence. Here $\Omega(G) = \mathbb{F}_p[\![G]\!]$ is the completed group algebra. We will skip the full details of this translation and just note that we get a $\mathbb{Z}$-filtration and grading on $H^\bullet(G, \mathbb{F}_p)$, which with $k = \mathbb{F}_p$ gives us the following, cf. [Sør21, Thm. 5.5–§6.1].

Theorem 2.5. *Let (G, ω) be a p-valuable group and $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ its Lazard Lie algebra. Then there is a convergent multiplicative spectral sequence collapsing at a finite stage,*

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}_{\text{cts}}(G, \mathbb{F}_p).$$

This means that each sheet E_r has a multiplication $E_r \otimes E_r \to E_r$ compatible with the (s,t)-bigrading and satisfying Leibniz formula. Furthermore $H^*(E_r) \cong E_{r+1}$ as algebras. I.e., the *multiplication on E_∞ is compatible with the cup product on $H^*(G, \mathbb{F}_p)$ in the sense that the following diagram commutes.*

$$\begin{array}{ccc} E_\infty^{s,n-s} \otimes E_\infty^{s',n'-s'} & \longrightarrow & E_\infty^{s+s',n+n'-s-s'} \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathrm{gr}^s H^n_{\mathrm{cts}}(G,\mathbb{F}_p) \otimes \mathrm{gr}^{s'} H^{n'}_{\mathrm{cts}}(G,\mathbb{F}_p) & \longrightarrow & \mathrm{gr}^{s+s'} H^{n+n'}_{\mathrm{cts}}(G,\mathbb{F}_p) \end{array}$$

♣

Remark 2.6. We note that [Fer+07, Thm. 2.10] implies that $H^n_{\mathrm{cts}}(N, \mathbb{F}_p) \cong H^n_{\mathrm{dsc}}(N, \mathbb{F}_p)$ for all n (with $N = \mathcal{N}(\mathbb{Z}_p)$ as above), if we can show that N is a pro-p group which is poly-$\mathbb{Z}_p$ by finite.

Definition 2.7. A group G is poly-$\mathbb{Z}_p$ if it has a normal series

$$G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n = 1$$

such that each factor group G_i/G_{i+1} is isomorphic to $\mathbb{Z}_p$.

A group is poly-$\mathbb{Z}_p$ by finite (virtually poly-$\mathbb{Z}_p$) if it contains a poly-$\mathbb{Z}_p$ subgroup of finite index. ♠

Note that [Con14b, Prop. 5.1.16(2) and Cor. 5.2.5] (as seen in the proof of [Con14b, Cor. 5.2.13] or [Con14b, Thm. 5.4.3]) gives us a composition series of $\mathcal{N}$ such that the successive quotients are $\mathbb{G}_a$, which implies that $N = \mathcal{N}(\mathbb{Z}_p)$ is poly-$\mathbb{Z}_p$ by finite since $\mathbb{G}_a(\mathbb{Z}_p) = \mathbb{Z}_p$. Thus, assuming that $\mathcal{N}(\mathbb{Z}_p)$ is a pro-p group, we get that

$$H^n_{\mathrm{cts}}(N, \mathbb{F}_p) \cong H^n_{\mathrm{dsc}}(N, \mathbb{F}_p) \qquad \text{for all } n. \tag{2.4}$$

△

2.1.5 Main result

We show first that N is p-valuable, which implies by [Sør21, §6.1] that we get a convergent multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}_{\mathrm{cts}}(N, \mathbb{F}_p). \tag{2.5}$$

We note that $\mathfrak{g} \cong \mathfrak{n}$ and then use ideas of [Gro14, §7] to transfer results from [PT18] about (the dimension of) $H^n(\mathfrak{n}_\mathbb{Z}, \mathbb{F}_p)$ and $H^n_{\mathrm{dsc}}(\mathcal{N}_\mathbb{Z}(\mathbb{Z}), \mathbb{F}_p)$ to $H^n(\mathfrak{n}, \mathbb{F}_p)$ and $H^n_{\mathrm{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p)$, giving us that

$$\sum_{s+t=n} \dim_{\mathbb{F}_p} H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\mathrm{cts}}(N, \mathbb{F}_p).$$

This implies that (2.5) collapses on the first page, and thus $H^{s,n-s}(\mathfrak{n},\mathbb{F}_p) \cong \operatorname{gr}^s H^n_{\mathrm{cts}}(N,\mathbb{F}_p)$. Noting that $E_\infty^{s,t} = E_1^{s,t}$, we get that the cup product on $E_1^{s,t} = H^{s,t}(\mathfrak{n},\mathbb{F}_p)$ (from $H^*(\mathfrak{n},\mathbb{F}_p)$) is compatible with the cup product on $H^*_{\mathrm{cts}}(N,\mathbb{F}_p)$ in the sense that the following diagram commutes.

$$\begin{array}{ccc} H^{s,n-s}(\mathfrak{n},\mathbb{F}_p)\otimes H^{s',n'-s'}(\mathfrak{n},\mathbb{F}_p) & \longrightarrow & H^{s+s',n+n'-s-s'}(\mathfrak{n},\mathbb{F}_p) \\ \cong\downarrow & & \downarrow\cong \\ \operatorname{gr}^s H^n_{\mathrm{cts}}(N,\mathbb{F}_p)\otimes \operatorname{gr}^{s'} H^{n'}(N,\mathbb{F}_p) & \longrightarrow & \operatorname{gr}^{s+s'} H^{n+n'}_{\mathrm{cts}}(N,\mathbb{F}_p) \end{array}$$

2.2 The p-valuation

In this section we will prove that N is p-valuable group, which we will need in multiple arguments later. It should be noted that this section, with the exception of Proposition 2.10, is a slightly rewritten version of [Sch11b] which expands on some of the arguments. Also, the proof of Proposition 2.10 is based on [Zab10, Lem. 1].

Note that as a set N is the direct product $N = \prod_{\alpha\in\Phi^-} x_\alpha(\mathbb{Z}_p)$, which allows us to introduce the function

$$\begin{aligned} \omega\colon N\setminus\{1\} &\to \mathbb{N} \\ \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha) &\mapsto \min_{\alpha\in\Phi^-}\big(v_p(a_\alpha)-\operatorname{ht}(\alpha)\big), \end{aligned} \tag{2.6}$$

where v_p denotes the usual p-adic valuation on $\mathbb{Z}_p$. Here it is important to note that we write any $g\in N$ uniquely as product

$$g = \prod_{\alpha\in\Phi^-} x_\alpha(a_\alpha)$$

by taking the product following the total ordering $\geq$ of Φ^- defined above. Now, with the convention that $\omega(1) \coloneqq \infty$, we define the descending sequence of subsets

$$N_m \coloneqq \{g\in N \mid \omega(g)\geq m\}$$

in N for $m\geq 0$, following the notation used for p-valuable groups. The goal of this section is to show that this ω is a p-valuation by a careful analysis of the sequence of subsets given by N_m.

Remark 2.8. If we are willing to restrict from $p+1\geq h$ to $p-1>h$, then we can restrict the p-valuation of the pro-p Iwahori subgroup of $\mathcal{G}$ introduced in Section 3.1 to a p-valuation on N. We prefer the above p-valuation because it will introduce a grading on $\mathfrak{g}$ that will directly correspond to the grading (by height) on $\mathfrak{n}$, whereas the restricted p-valuation is a scalar multiple of this p-valuation on a basis. $\triangle$

We first note that clearly $N_1 = N$, $\bigcap_m N_m = \{1\}$, and

$$\begin{aligned} N_m &= \prod_{\alpha\in\Phi^-} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \\ &= \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-1}} x_\alpha(p^{m-1}\mathbb{Z}_p)\cdots \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-(m-1)}} x_\alpha(p\mathbb{Z}_p) \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -m}} x_\alpha(\mathbb{Z}_p). \end{aligned} \tag{2.7}$$

In our analysis of this sequence it will be helpful to introduce the following two other filtrations of N. Firstly we will consider the filtration by congruence subgroups

$$N(m) := \ker\big(\mathcal{N}(\mathbb{Z}_p) \to \mathcal{N}(\mathbb{Z}/p^m\mathbb{Z})\big) = \prod_{\alpha\in\Phi^-} x_\alpha(p^m\mathbb{Z}_p) \tag{2.8}$$

for $m \geq 0$. Secondly, using the descending central series of the group $\mathcal{G}(\mathbb{Q}_p)$ defined by $C^1\mathcal{G}(\mathbb{Q}_p) := \mathcal{G}(\mathbb{Q}_p)$ and $C^{m+1}\mathcal{G}(\mathbb{Q}_p) := [C^m\mathcal{G}(\mathbb{Q}_p), \mathcal{G}(\mathbb{Q}_p)]$, we consider the filtration given by

$$N_{(m)} := N \cap C^m\mathcal{G}(\mathbb{Q}_p)$$

for $m \geq 1$. By [BT73, Prop. 4.7(iii)] we have that

$$N_{(m)} = \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -m}} x_\alpha(\mathbb{Z}_p), \tag{2.9}$$

and we note that the natural map

$$\prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)=-m}} x_\alpha(\mathbb{Z}_p) \to N_{(m)}/N_{(m+1)}$$

is an isomorphism of abelian groups, and that all the subgroups $N(m)$ and $N_{(m)}$ are normal in N.

We are now ready to prove the following lemma, which will help us when showing that ω is a p-valuation.

Lemma 2.9.

(i) $N_m = \prod_{1\leq i\leq m} N(m-i)\cap N_{(i)}$*, for any* $m \geq 1$*, is a normal subgroup of* N *which is independent of the choices made.*

(ii) $[N_\ell, N_m] \subseteq N_{\ell+m}$ *for any* $\ell, m \geq 1$.

(iii) N_m/N_{m+1}, *for any* $m \geq 1$, *is an* $\mathbb{F}_p$*-vector space of dimension equal to* $|\{\alpha \in \Phi^- | \mathrm{ht}(\alpha) \geq -m\}|$.

(iv) Let $g \in N_m$ *for some* $m \geq 1$. *If* $g^p \in N_{m+2}$, *then* $g \in N_{m+1}$. ♣

Proof. (i) Using (2.8) and (2.9) we note that

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-i}\mathbb{Z}_p) \subseteq N(m-i) \cap N_{(i)} \quad \text{and} \quad \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\leq -m}} x_\alpha(\mathbb{Z}_p) = N(0) \cap N_{(m)}$$

for $1 \leq i < m$, so by (2.7) it is clear that $N_m \subseteq \prod_{1\leq i\leq m} N(m-i) \cap N_{(i)}$. We also note, by (2.8) and (2.9), that

$$\begin{aligned}&\big(N(m-i) \cap N_{(i)}\big)\big(N(m-i-1) \cap N_{(i+1)}\big) \\ &\subseteq \Big(\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-i}} x_\alpha(p^{m-i}\mathbb{Z}_p)\Big)\big(N(m-i-1) \cap N_{(i+1)}\big)\end{aligned}$$

for any $1 \leq i < m$, so

$$\begin{aligned}&\prod_{1\leq i\leq m} N(m-i) \cap N_{(i)} \\ &\subseteq \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-1}} x_\alpha(p^{m-1}\mathbb{Z}_p) \cdots \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-(m-1)}} x_\alpha(p\mathbb{Z}_p)\big(N(0) \cap N_{(m)}\big) \\ &= N_m\end{aligned}$$

by induction, (2.7) and (2.9). This shows the equality and that N_m is normal clearly follows.

(ii) We first recall the following formulas for commutators

$$[gh, k] = g[h, k]g^{-1}[g, k] \quad \text{and} \quad [g, hk] = [g, h]h[g, k]h^{-1}. \tag{2.10}$$

Now, using (2.10), (i) and the fact that all the involved subgroups are normal, it is enough to show that

$$[N(\ell) \cap N_{(i)}, N(m) \cap N_{(j)}] \subseteq N(\ell + m) \cap N_{(i+j)}.$$

This further reduces to showing that

$$[N(\ell), N(m)] \subseteq N(\ell + m) \quad \text{and} \quad [N_{(i)}, N_{(j)}] \subseteq N_{(i+j)}.$$

The right inclusion is a well known property of the descending central series, so it follows from our definition of $N_{(m)}$. For the left inclusion it suffices, by (2.8) and (2.10), to show that

$$[x_\alpha(p^\ell \mathbb{Z}_p), x_\beta(p^m \mathbb{Z}_p)] \subseteq N(\ell+m)$$

for any $\alpha, \beta \in \Phi^-$. To show this inclusion we recall Chevalley's commutator formula, cf. [Con14b, Prop. 5.1.14],

$$[x_\alpha(a), x_\beta(b)] \in x_{\alpha+\beta}(c_{\alpha,\beta,1,1} ab\mathbb{Z}_p) \prod_{\substack{i,j\geq 1\\ i+j>2}} x_{i\alpha+j\beta}(c_{\alpha,\beta,i,j} a^i b^j \mathbb{Z}_p),$$

where $c_{\alpha,\beta,i,j} \in \mathbb{Z}_p$ and on the right hand side we use the convention is that $x_{i\alpha+j\beta} \equiv 1$ if $i\alpha + j\beta \notin \Phi$. From (2.8) and Chevalley's commutator formula the inclusion follows.

(iii) We note that

$$N(m-i) \cap N_{(i)} = \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -i}} x_\alpha(p^{m-i}\mathbb{Z}_p)$$

for $1 \leq i \leq m$, so the statement follows from (i) and (ii).

(iv) For any $1 \leq \ell \leq m$ we consider the chain of normal subgroups

$$N_{m+2}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell+1)}) \subseteq N_{m+1}(N_m \cap N_{(\ell)})$$

between N_{m+2} and N_m. By (2.10) and an argument like in (ii), we get that

$$[N_{m+1}(N_m \cap N_{(\ell)}), N_{m+1}(N_m \cap N_{(\ell)})] \subseteq N_{m+2}(N_m \cap N_{(\ell+1)}),$$

so the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian. Now looking carefully at the groups as sets, we see that

$$N_m \cap N_{(\ell)} = \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)\leq -\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p)$$

and thus (using Chevalley's commutator formula and the fact that $\mathrm{ht}(i\alpha + j\beta) \leq \mathrm{ht}(\alpha+\beta) < \mathrm{ht}(\alpha), \mathrm{ht}(\beta)$ to move the products for the $\mathrm{ht}(\alpha) = -\ell$ term)

$$N_{m+1}(N_m \cap N_{(\ell)}) = \prod_{\substack{\alpha\in\Phi^-\\ \mathrm{ht}(\alpha)>-\ell}} x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)$$

$$\cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} x_\alpha(p^{m-\ell}\mathbb{Z}_p)$$

$$\cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)<-\ell}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p).$$

Similarly

$$\begin{aligned} N_{m+2}(N_m \cap N_{(\ell+1)}) = & \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p) \\ & \cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} x_\alpha(p^{m+2-\ell}\mathbb{Z}_p) \\ & \cdot \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)\le-(\ell+1)}} x_\alpha(p^{\max(0,m+\mathrm{ht}(\alpha))}\mathbb{Z}_p), \end{aligned}$$

and since the quotient group

$$N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

is abelian, we see that it is isomorphic to

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} \frac{x_\alpha(p^{m-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

Here the subgroup

$$N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$$

corresponds to

$$\prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)>-\ell}} \frac{x_\alpha(p^{\max(0,m+1+\mathrm{ht}(\alpha))}\mathbb{Z}_p)}{x_\alpha(p^{\max(0,m+2+\mathrm{ht}(\alpha))}\mathbb{Z}_p)} \times \prod_{\substack{\alpha\in\Phi^- \\ \mathrm{ht}(\alpha)=-\ell}} \frac{x_\alpha(p^{m+1-\ell}\mathbb{Z}_p)}{x_\alpha(p^{m+2-\ell}\mathbb{Z}_p)}.$$

It follows that $N_{m+1}(N_m \cap N_{(\ell+1)})/N_{m+2}(N_m \cap N_{(\ell+1)})$ is the p-torsion subgroup of $N_{m+1}(N_m \cap N_{(\ell)})/N_{m+2}(N_m \cap N_{(\ell+1)})$.

Now let $g \in N_m$ for some $m \ge 1$. For $\ell = 1$ we have $g \in N_m = N_{m+1}(N_m \cap N_{(1)})$, since $N_{(1)} = N$, and clearly $g^p \in N_{m+2}(N_m \cap N_{(2)})$ because $g^p \in N_{(2)}$ by Chevalley's commutator formula and (2.9). Since $N_{m+1}(N_m \cap N_{(2)})/N_{m+2}(N_m \cap N_{(2)})$ is the p-torsion subgroup of $N_{m+1}(N_m \cap N_{(1)})/N_{m+2}(N_m \cap N_{(2)})$, it follows that $g \in N_{m+1}(N_m \cap N_{(2)})$ and thus $g^p \in N_{m+2}(N_m \cap N_{(3)})$ by Chevalley's commutator formula and (2.9). By induction on ℓ, we thus get that $g \in N_{m+1}(N_m \cap N_{(m+1)}) = N_{m+1}$. Here the last equality follows from the fact that $N_{(m+1)} \subseteq N_{m+1}$ by (2.7) and (2.9). $\square$

With this lemma, we are now ready to prove that ω is a p-valuation on N.

Proposition 2.10. *The function ω from* (2.6) *is a p-valuation on N, i.e., it satisfies for any $g, h \in N$:*

(a) $\omega(g) > \frac{1}{p-1}$,

(b) $\omega(g^{-1}h) \geq \min(\omega(g), \omega(h))$,

(c) $\omega([g,h]) \geq \omega(g) + \omega(h)$,

(d) $\omega(g^p) = \omega(g) + 1$. ♣

Proof. We note that (a) is obvious by our definition of ω, (c) follows from Lemma 2.9 (ii) and (d) follows from Lemma 2.9 (iv).

It only remains to show (b), which we will do by following the proof idea of [Zab10, Lem. 1], i.e., we are going to use triple induction. Here we note that all products $\prod_{\alpha \in \Phi^-} x_\alpha(a_\alpha)$ are in ascending order in Φ^- (so descending in height). For ease of notation, we prove equivalently that $\omega(gh^{-1}) \geq \min(\omega(g), \omega(h))$ for $g, h \in N$.

At first by induction on the number of non-zero coordinates among $(a_\beta)_{\beta \in \Phi^-}$ in $\prod_{\beta \in \Phi^-} x_\beta(a_\beta)$ we are reduced to the case where h is of the form $h = x_\beta(a_\beta)$ for some $\beta \in \Phi^-$ and $a_\beta \in \mathbb{Z}_p$. To see this let $h \in N \setminus \{1\}$ and write $h = \prod_{\beta \in \Phi^-} x_\beta(a_\beta)$ in our unique way (according to the ordering of Φ^-), and let α be the smallest element of Φ^- for which $a_\alpha \neq 0$ so that $h = x_\alpha(a_\alpha) \cdot h'$. Then $gh^{-1} = g(h')^{-1} \cdot x_\alpha(a_\alpha)^{-1}$ and thus strong induction will imply that

$$\begin{aligned} \omega(gh^{-1}) &\geq \min\bigl(\omega(g(h')^{-1}), v(a_\alpha) - \mathrm{ht}(\alpha)\bigr) \\ &\geq \min\bigl(\omega(g), \omega(h'), v(a_\alpha) - \mathrm{ht}(\alpha)\bigr) = \min\bigl(\omega(g), \omega(h)\bigr). \end{aligned}$$

Fix $h = x_\beta(a_\beta)$ and let now g be of the form $g = \prod_{k=1}^{r} x_{\alpha_k}(a_{\alpha_k})$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ in Φ^-. If $\beta > \alpha_r$, then $gh^{-1} = \prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_{\alpha_r}(a_{\alpha_r}) x_\beta(-a_\beta)$, so (b) is clearly true if $\beta > \alpha_1$

(by the definition of ω), and if $\beta = \alpha_r$, then $x_{\alpha_r}(a_{\alpha_r})x_\beta(-a_\beta) = x_\beta(a_{\alpha_r} - a_\beta)$ and (b) follows from $v_p(a-b) \geq \min(v_p(a), v_p(b))$ for $a, b \in \mathbb{Z}_p$.

On the other hand, if $\beta < \alpha_r$, then we write

$$\begin{aligned} gh^{-1} &= \prod_{k=1}^{r} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta) \\ &= \prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta) \cdot x_{\alpha_r}(a_{\alpha_r}) \cdot [x_{\alpha_r}(-a_{\alpha_r}), x_\beta(a_\beta)]. \end{aligned}$$

Now we use descending induction on β in the chosen ordering of Φ^- and suppose that the statement (b) is true for any g and any h' of the form $h' = x_{\beta'}(a_{\beta'})$ with $\beta' > \beta$. Note that the base case is trivial and recall that Φ^- is finite and totally ordered. Note furthermore that Chevalley's commutator formula gives us

$$[x_{\alpha'}(a_{\alpha'}), x_{\beta'}(a_{\beta'})] = \prod_{\substack{i\alpha'+j\beta' \in \Phi^- \\ i,j>0}} x_{i\alpha'+j\beta'}(c_{\alpha',\beta',i,j} a_{\alpha'}^i a_{\beta'}^j) \tag{2.11}$$

for any $\alpha', \beta' \in \Phi^-$, where $c_{\alpha',\beta',i,j} \in \mathbb{Z}_p$. Also, we have $\mathrm{ht}(i\alpha' + j\beta') \leq \mathrm{ht}(\alpha' + \beta') < \mathrm{ht}(\alpha'), \mathrm{ht}(\beta')$, so we can apply the induction hypothesis for $x_{\alpha_r}(a_{\alpha_r})$ and each $x_{i\alpha_r+j\beta}(c_{\alpha_r,\beta,i,j}(-a_{\alpha_r})^i a_\beta^j)$ in $[x_{\alpha_r}(-a_{\alpha_r}, x_\beta(a_\beta))]$, since $\alpha_r > \beta$ and all terms on the right side of (2.11) are larger than β (and α_r) in the ordering of Φ^-. We thus obtain

$$\begin{aligned} \omega(gh^{-1}) \geq \min\Big(&\min_{\substack{i\alpha_r+j\beta \in \Phi^- \\ i,j>0}} \omega(x_{i\alpha_r+j\beta}(c_{\alpha_r,\beta,i,j}(-a_{\alpha_r})^i a_\beta^j)), \\ &\omega(x_{\alpha_r}(a_{\alpha_r})), \omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\Big)\Big). \end{aligned} \tag{2.12}$$

Now, for $i, j > 0$ with $i\alpha' + j\beta' \in \Phi^-$,

$$\begin{aligned} \omega(x_{i\alpha'+j\beta'}(c_{\alpha',\beta',i,j} a_{\alpha'}^i a_{\beta'}^j)) &= v_p(c_{\alpha',\beta',i,j} a_{\alpha'}^i a_{\beta'}^j) - \mathrm{ht}(i\alpha' + j\beta') \\ &\geq v_p(c_{\alpha',\beta',i,j}) + v_p(a_{\alpha'}^i) + v_p(a_{\beta'}^j) - \mathrm{ht}(\alpha' + \beta') \\ &\geq v_p(a_{\alpha'}) - \mathrm{ht}(\alpha') + v_p(a_{\beta'}) - \mathrm{ht}(\beta') \\ &= \omega(x_{\alpha'}(a_{\alpha'})) + \omega(x_{\beta'}(a_{\beta'})) \\ &\geq \min\big(\omega(x_{\alpha'}(a_{\alpha'})), \omega(x_{\beta'}(a_{\beta'}))\big). \end{aligned} \tag{2.13}$$

So taking $\alpha' = \alpha_r$ and $\beta' = \beta$ and using (2.13) in (2.12), we get that

$$\omega(gh^{-1}) \geq \min\Big(\omega(x_{\alpha_r}(a_{\alpha_r})), \omega(x_\beta(a_\beta)), \omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\Big)\Big). \tag{2.14}$$

Finally induction on r will imply that

$$\omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k}) \cdot x_\beta(-a_\beta)\Big) \geq \min\Big(\omega\Big(\prod_{k=1}^{r-1} x_{\alpha_k}(a_{\alpha_k})\Big), \omega(x_\beta(a_\beta))\Big)$$
$$= \min\big(\min_{1\leq k\leq r-1} \omega(x_{\alpha_k}(a_{\alpha_k})), \omega(x_\beta(a_\beta))\big),$$

which by (2.14) implies that

$$\omega(gh^{-1}) \geq \min\big(\min_{1\leq k\leq r} \omega(x_{\alpha_k}(a_{\alpha_k})), \omega(x_\beta(a_\beta))\big)$$
$$= \min\big(\omega(g), \omega(h)\big),$$

thus finishing the proof. □

We have now shown that $N = \mathcal{N}(\mathbb{Z}_p)$ is a p-valuable group with the p-valuation ω introduced in (2.6), which is the main result of this section. Before continuing, we will clarify what this means based on Lazard theory as described in Section 2.1.

We note that

$$\operatorname{gr} N \coloneqq \bigoplus_{m\geq 1} N_m/N_{m+1}$$

is a graded $\mathbb{F}_p$-vector space, and recall the following well known result, cf. [Laz65] or [Sch11a, Sect. 25].

Proposition 2.11. $\operatorname{gr} N$ *is a Lie algebra over the polynomial ring* $\mathbb{F}_p[\pi]$ *in one variable* π *where*

$$[gN_{\ell+1}, hN_{m+1}] \coloneqq [g,h]N_{\ell+m+1} \quad \textit{and} \quad \pi(gN_{m+1}) \coloneqq g^p N_{m+2},$$

and as an $\mathbb{F}_p[\pi]$*-module* $\operatorname{gr} N$ *is free of rank* $|\Phi^-|$. ♣

2.3 Spectral sequence and cohomology

Recall that $N = \mathcal{N}(\mathbb{Z}_p)$, $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N$ and $\mathfrak{n} = \operatorname{Lie}(\mathcal{N}_{\mathbb{F}_p})$. In this section we will first look at the spectral sequence from [Sør21] (cf. Theorem 2.5), i.e.,

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}_{\text{cts}}(N, \mathbb{F}_p),$$

and note that we can work with the left side using that $H^{s,t}(\mathfrak{g},\mathbb{F}_p) \cong H^{s,t}(\mathfrak{n},\mathbb{F}_p)$. Afterwards, we will use results from [PT18] to argue that the spectral sequence collapses on the first page.

We will start by showing that $\mathfrak{g} \cong \mathfrak{n}$, for which we will need the following lemma, which is again from [Sch11b].

Lemma 2.12. $\operatorname{gr} N \cong \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n}$ *as graded Lie algebras (where* π *has degree* 1*).* ♣

Proof. We first note that the elements X_α, where X_α is our fixed $\mathbb{Z}_p$-basis of $\operatorname{Lie}\mathcal{N}_\alpha$, reduce modulo p to an $\mathbb{F}_p$-basis $\{\overline{X}_\alpha\}_{\alpha\in\Phi^-}$ of $\mathfrak{n}$. On the other hand all

$$\sigma\big(x_\alpha(1)\big) \in \operatorname{gr}_{-\operatorname{ht}(\alpha)} N,$$

with $x_\alpha(1) \in N_{-\operatorname{ht}(\alpha)}$, form an $\mathbb{F}_p[\pi]$-basis of $\operatorname{gr} N$, cf. [Sch11a] Proposition 26.5. Hence the map

$$\begin{aligned} \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n} &\to \operatorname{gr} N \\ f \otimes \overline{X}_\alpha &\mapsto f \,.\, \sigma\big(x_\alpha(1)\big) \end{aligned}$$

is an isomorphism of graded modules. Chevalley's commutator formula (cf. [Con14b, Prop. 5.1.14]) says that there are p-adic integers $c_{\alpha,\beta} = c_{\alpha,\beta,1,1}$ such that $[X_\alpha, X_\beta] = c_{\alpha,\beta} X_{\alpha+\beta}$ and

$$[x_\alpha(1), x_\beta(1)] \in x_{\alpha+\beta}(c_{\alpha,\beta}) N_{-\operatorname{ht}(\alpha)-\operatorname{ht}(\beta)+1} = x_{\alpha+\beta}(1)^{c_{\alpha,\beta}} N_{-\operatorname{ht}(\alpha)-\operatorname{ht}(\beta)+1},$$

where $X_{\alpha+\beta} = 0$ and $x_{\alpha+\beta} \equiv 1$ if $\alpha+\beta \notin \Phi$. This implies that the image of the above map is a Lie subalgebra, and thus that the map is an isomorphism of Lie algebras. We note that $\mathfrak{n}$ is graded by the height function, which corresponds to the grading on $\operatorname{gr} N$ by the definition of ω in (2.6). □

Now $\operatorname{gr} N \cong \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n}$ implies that $\mathfrak{g} \cong \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \mathbb{F}_p[\pi] \otimes_{\mathbb{F}_p} \mathfrak{n} \cong \mathfrak{n}$, where both $\mathfrak{g}$ and $\mathfrak{n}$ is graded by the height function. From this it clearly follows that $H^{s,t}(\mathfrak{g},\mathbb{F}_p) \cong H^{s,t}(\mathfrak{n},\mathbb{F}_p)$. (Note that this can also be seen directly by looking at the Chevalley constants.)

By [PT18, §2.10] (using that $p \geq h-1$) and the Universal Coefficient Theorem (as used in [PT18, §3.8]), we get an $\mathbb{F}_p$-vector space isomorphism

$$H^n(\mathfrak{n}_{\mathbb{Z}},\mathbb{F}_p) = H^n(\mathfrak{n}_{\mathbb{Z}}, V_{\mathbb{F}_p}(0)) \cong \bigoplus_{\substack{w\in W\\ \ell(w)=n}} V_{\mathbb{F}_p}(w\cdot 0),$$

where $V_{\mathbb{F}_p}(0) = \mathbb{F}_p$ with the trivial action (concentrated in degree 0). Similarly, by the corollary in [PT18, §3.8], we have an $\mathbb{F}_p$-vector space isomorphism

$$\operatorname{gr} H^n_{\mathrm{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = \operatorname{gr} H^n_{\mathrm{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), V_{\mathbb{F}_p}(0)) \cong \bigoplus_{\substack{w \in W \\ \ell(w)=n}} V_{\mathbb{F}_p}(w \cdot 0).$$

Here the grading on cohomology will not be important, since we just need that

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\mathrm{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p). \tag{2.15}$$

We now equip $\mathcal{N}_{\mathbb{Z}}(\mathbb{Z})$ with the discrete topology and claim that

$$H^n_{\mathrm{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = H^n_{\mathrm{cts}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) \cong H^n_{\mathrm{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Here the first equality is clear since $\mathcal{N}_{\mathbb{Z}}(\mathbb{Z})$ is equipped with the discrete topology. To see the isomorphism, first note that $\mathbb{Z}$ is a discrete group, $\mathbb{Z}_p$ is a profinite group, and the homomorphism $\mathbb{Z} \to \mathbb{Z}_p$ has dense image in $\mathbb{Z}_p$. So we have homomorphisms

$$H^n_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{F}_p) \to H^n_{\mathrm{cts}}(\mathbb{Z}, \mathbb{F}_p)$$

for all $n \geq 0$ from [Ser02, Sect. I §2.6]. Now both $H^0_{\mathrm{cts}}(\mathbb{Z}, \,\cdot\,)$ and $H^0_{\mathrm{cts}}(\mathbb{Z}_p, \,\cdot\,)$ are the functor of taking invariant, both $H^1_{\mathrm{cts}}(\mathbb{Z}, \,\cdot\,)$ and $H^1_{\mathrm{cts}}(\mathbb{Z}_p, \,\cdot\,)$ are what [Gro14] calls the functor of taking "coinvariants" (giving the group of continuous crossed-homomorphisms of G into $\,\cdot\,$, cf. [Ser02, I. §2]), and all $H^n_{\mathrm{cts}}(\mathbb{Z}, \,\cdot\,)$ and $H^n_{\mathrm{cts}}(\mathbb{Z}_p, \,\cdot\,)$ vanish for $n \geq 2$, so $\mathbb{Z}$ is "good" in the sense of [Ser02, Section I §2.6 Exercise 2]. Thus [Ser02, Section I §2.6 Exercise 2(d)] implies that the homomorphisms

$$H^n_{\mathrm{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p) \to H^n_{\mathrm{cts}}(\mathcal{N}(\mathbb{Z}), \mathbb{F}_p) \qquad n \geq 0,$$

induced by the homomorphism $\mathcal{N}(\mathbb{Z}) \to \mathcal{N}(\mathbb{Z}_p)$, are all isomorphisms. To see this one can consider a filtration of $\mathcal{N}(\mathbb{Z})$ with subquotients isomorphic with $\mathbb{Z}$, and its parallel filtration of $\mathcal{N}(\mathbb{Z}_p)$ with subquotients isomorphic with $\mathbb{Z}_p$ as in [Gro14, Sect. 7], which will make it follow directly from [Ser02, Section I §2.6 Exercise 2(d)].

Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\mathrm{dsc}}(\mathcal{N}_{\mathbb{Z}}(\mathbb{Z}), \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n_{\mathrm{cts}}(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Now $\mathfrak{n} = \mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p$, and $H^n(\mathfrak{g}, \mathbb{F}_p) \cong H^n(\mathfrak{n}, \mathbb{F}_p)$ (since $\mathfrak{g} \cong \mathfrak{n}$) is the cohomology of the complex

$$C^\bullet(\mathfrak{n}, \mathbb{F}_p) = \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^{\bullet} \mathfrak{n}, \mathbb{F}_p\Big)$$

while $H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$ is the homology of the complex

$$C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^{\bullet} \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big).$$

Here $\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module and $(\bigwedge^\bullet \mathfrak{n}_{\mathbb{Z}}) \otimes \mathbb{F}_p \cong \bigwedge^\bullet(\mathfrak{n}_{\mathbb{Z}} \otimes \mathbb{F}_p) \cong \bigwedge^\bullet \mathfrak{n}$, so we have natural isomorphisms

$$\operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^{\bullet} \mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p\Big) \cong \operatorname{Hom}_{\mathbb{F}_p}\Big(\Big(\bigwedge^{\bullet} \mathfrak{n}_{\mathbb{Z}}\Big) \otimes \mathbb{F}_p, \mathbb{F}_p\Big) \cong \operatorname{Hom}_{\mathbb{F}_p}\Big(\bigwedge^{\bullet} \mathfrak{n}, \mathbb{F}_p\Big).$$

These isomorphisms are clearly compatible with the differentials, so $C^\bullet(\mathfrak{n}, \mathbb{F}_p) \cong C^\bullet(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$, and thus $H^n(\mathfrak{n}, \mathbb{F}_p) \cong H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p)$. Hence

$$\dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{n}_{\mathbb{Z}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p).$$

Now $\dim_{\mathbb{F}_p} H^n(\mathfrak{n}, \mathbb{F}_p) = \dim^n_{\mathbb{F}_p}(\mathfrak{g}, \mathbb{F}_p)$ and $N = \mathcal{N}(\mathbb{Z}_p)$ implies that

$$\sum_{s+t=n} \dim_{\mathbb{F}_p} H^{s,t}(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(\mathfrak{g}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(N, \mathbb{F}_p),$$

so the multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(N, \mathbb{F}_p)$$

collapses on the first page, since the dimension of $E_r^{s,t}$ is non-increasing as r increases. Since the spectral sequence collapses on the first page, we get that $E_1^{s,t} = E_\infty^{s,t}$, so

$$\operatorname{gr}^s H^n(N, \mathbb{F}_p) \cong H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \cong H^{s,t}(\mathfrak{n}, \mathbb{F}_p),$$

giving us a good description of $H^n(\mathcal{N}(\mathbb{Z}_p), \mathbb{F}_p)$. Furthermore, we can describe the cup product, by calculating it in $H^*(\mathfrak{g}, \mathbb{F}_p)$ or $H^*(\mathfrak{n}, \mathbb{F}_p)$, cf. Theorem 2.5 for the details. I.e., we have shown:

Theorem 2.13. *Let $N = \mathcal{N}(\mathbb{Z}_p)$ be the $\mathbb{Z}_p$-points of $\mathcal{N}$, where $\mathcal{N}$ is the unipotent radical of a Borel in a split and connected reductive $\mathbb{Z}_p$-group, and let and $\mathfrak{n} = \operatorname{Lie} \mathcal{N}_{\mathbb{F}_p}$. Then ω from (2.6) gives a p-valuation on N, and if we let $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N$ be the Lazard Lie algebra of N, then $\mathfrak{g} \cong \mathfrak{n}$.*

Furthermore, there is a convergent multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}_{\text{cts}}(N, \mathbb{F}_p)$$

collapsing at the first page, so $\operatorname{gr}^s H^n(N, \mathbb{F}_p) \cong H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \cong H^{s,t}(\mathfrak{n}, \mathbb{F}_p)$*, and the cup product on* $H^*(\mathfrak{g}, \mathbb{F}_p)$ *is compatible with the cup product on* $H^*(N, \mathbb{F}_p)$ *in the sense that the following diagram commutes.*

$$\begin{array}{ccc} H^{s,n-s}(\mathfrak{g}, \mathbb{F}_p) \otimes H^{s',n'-s'}(\mathfrak{g}, \mathbb{F}_p) & \longrightarrow & H^{s+s',n+n'-s-s'}(\mathfrak{g}, \mathbb{F}_p) \\ \cong\downarrow & & \downarrow\cong \\ \operatorname{gr}^s H^n_{\text{cts}}(G, \mathbb{F}_p) \otimes \operatorname{gr}^{s'} H^{n'}_{\text{cts}}(G, \mathbb{F}_p) & \longrightarrow & \operatorname{gr}^{s+s'} H^{n+n'}_{\text{cts}}(G, \mathbb{F}_p) \end{array}$$

♣

2.4 Example: $N \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$

In the case of $\mathcal{G} = \mathrm{SL}_3$ (in this case $h = 3$, so $p \geq 3$), we can take $\mathcal{T}$ to be the diagonal matrices in SL_3 ($\det = 1$), $\mathcal{B}$ upper triangular matrices in SL_3 and

$$\mathcal{N} = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{SL}_n .$$

Furthermore we can take $\Phi^- = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$ with

$$X_{\alpha_1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_1}(A)(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{\alpha_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_2}(A)(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{\alpha_3} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_3}(A)(a) = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for any $\mathbb{Z}_p$-algebra A and $a \in A$. Here $\operatorname{ht}(\alpha_1) = \operatorname{ht}(\alpha_2) = -1$ and $\operatorname{ht}(\alpha_3) = -2$, and explicit calculations show that, in $N = \mathcal{N}(\mathbb{Z}_p)$, $g_1 = x_{\alpha_1}(1), g_2 = x_{\alpha_2}(1), g_3 = x_{\alpha_3}(1)$ is an ordered basis of (N, ω). Thus (cf. [Sch11a, Prop. 26.5]) $\sigma(g_1), \sigma(g_2), \sigma(g_3)$ is a basis of the $\mathbb{F}_p[\pi]$-module $\operatorname{gr} N$, and ξ_1, ξ_2, ξ_3 is a basis of $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N$, where $\xi_i = 1 \otimes \sigma(g_i)$. Furthermore $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$, where $\mathfrak{g}^1 = \operatorname{span}(\xi_1, \xi_2)$ and $\mathfrak{g}^2 = \operatorname{span}(\xi_3)$.

The only non-trivial commutator among the g_i's is $[g_1, g_2] = x_{\alpha_3}(-1) = g_3^{-1}$, which implies (cf. [Sch11a, Rem. 26.3]) that $\sigma([g_1, g_2]) = -\sigma(g_3)$ and thus $[\xi_1, \xi_2] = -\xi_3$. In particular $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^2$.

Now $H^1(\mathfrak{g}, \mathbb{F}_p) = \mathrm{Hom}_k(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \mathbb{F}_p) = H^{-1,2}(\mathfrak{g}, \mathbb{F}_p)$, and, since $\bigwedge^3 \mathfrak{g} = \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2$ is degree 4, $H^3(\mathfrak{g}, \mathbb{F}_p) = H^{-4,7}(\mathfrak{g}, \mathbb{F}_p)$. A version of Poincaré duality (cf. [Fuk86, Chap. 1 §3.6–7]) gives us that $H^1 \times H^2 \to H^3$ with $H^{-1,2} \times H^{s,t} \to H^{-4,7}$ is only non-trivial for $(s, t) = (-3, 5)$, so $H^2(\mathfrak{g}, \mathbb{F}_p) = H^{-3,5}(\mathfrak{g}, \mathbb{F}_p)$. By considering the maps $d_r^{s,t}\colon E_r^{s,t} \to E_1^{s+r,t+1-r}$, we see that the spectral sequence collapses at the first page, so this gives us a description of $H^*(N, \mathbb{F}_p)$, and we note that the only non-trivial cup product is $H^1(N, \mathbb{F}_p) \times H^2(N, \mathbb{F}_p) \to H^3(N, \mathbb{F}_p)$.

We note that we skipped some of the details above since we will go through many examples of this kind of computation in Chapter 3, so we refer to there for more of the details in this type of argument.

Chapter 3

Cohomology of pro-p Iwahori Subgroups

3.1 Intoduction

In this chapter we will calculate the cohomology over perfect fields k (or just $k = \mathbb{F}_p$) of a collection of pro-p Iwahori subgroups of SL_n and GL_n over $\mathbb{Z}_p$ or $\mathcal{O}_F$ for quadratic extensions $F/\mathbb{Q}_p$.

3.1.1 Background and motivation

In this chapter we will focus on describing the continuous mod p cohomology of pro-p Iwahori subgroups of SL_n and GL_n over $\mathbb{Q}_p$ for $n = 2, 3, 4$ or over quadratic extensions $F/\mathbb{Q}_p$ for $n = 2$.

We start by introducing the techniques we use throughout this chapter, and then we explicitly calculate the algebra structure of $H^*(I, \mathbb{F}_p)$ for the pro-p Iwahori subgroups $I_{\mathrm{SL}_2(\mathbb{Q}_p)} \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ and $I_{\mathrm{GL}_2(\mathbb{Q}_p)} \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$, and we note that these are isomorphic as algebras to $H^*\big((1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\big)$ and $H^*(1 + \mathfrak{m}_D, \mathbb{F}_p)$ respectively. This part heavily relies on results of Sørensen and Fuks (see [Sør21] and [Fuk86]). Afterwards we fully describe the cohomological dimensions (but not the cup products, which are quite complicated) of the pro-p Iwahori subgroups of $\mathrm{SL}_3(\mathbb{Q}_p)$, $\mathrm{GL}_3(\mathbb{Q}_p)$, $\mathrm{SL}_2(F)$ and $\mathrm{GL}_2(F)$, where F is a quadratic extension of $\mathbb{Q}_p$. We also roughly describe the cohomological dimensions of the pro-p Iwahori subgroups of $\mathrm{SL}_4(\mathbb{Q}_p)$ and $\mathrm{GL}_4(\mathbb{Z}_p)$, but we note that we run into some problems here. In particular the multiplicative spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p) \Longrightarrow H^{s+t}(I, \mathbb{F}_p)$$

of Sørensen (see [Sør21]) collapses on the first page in all the previous examples, but for pro-p Iwahori subgroups of $\mathrm{SL}_4(\mathbb{Q}_p)$ or $\mathrm{GL}_4(\mathbb{Q}_p)$ we no longer (trivially) get this. Here $\mathfrak{g} = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$ is the graded Lazard Lie algebra associated with I.

This work can be seen as a continuation of the recent work on the mod p cohomology of pro-p Iwahori subgroups. E.g. the work by Schneider and Olivier (see [OS18; OS19; Sch15]) working with pro-p Iwahori-Hecke modules and the work by Koziol (see [Koz17]) computing $H^1(I, \pi)$ as a $\mathcal{H}$-algebra (where $\mathcal{H}$ is the pro-p Iwahori-Hecke algebra and π is a mod p principal series representation of $\mathrm{GL}_n(F)$ for some p-adic field F). Work by Cornut and Ray (cf. [CR16]) finding a minimal set of topological generators of the pro-p Iwahori subgroup of a split reductive group over $\mathbb{Z}_p$ is also relevant, since the number of generators can be used to find the cohomological dimension of $H^1(I, \mathbb{F}_p)$. Overall all of this work can be seen as part of the search for a mod p and p-adic local Langlands program.

We finish the chapter by mentioning some observations on the nilpotency index of our calculated cohomology rings and discussing future research directions and a conjecture on the connection between the mod p cohomology of $(1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}$ (resp. $1+\mathfrak{m}_D$) for central division algebras and $I_{\mathrm{SL}_n(\mathbb{Q}_p)}$ (resp. $I_{\mathrm{GL}_n(\mathbb{Q}_p)}$). Finally we note that working with the Serre spectral sequence might allow us to generalize to all $I_{\mathrm{SL}_n(\mathbb{Q}_p)}$ for all n.

3.1.2 Setup and notation

Let p be an odd prime (further restricted later) and let k be a perfect field of characteristic p.

Field extension of $\mathbb{Q}_p$. We fix a finite extension of $F/\mathbb{Q}_p$ of degree ℓ with valuation ring $\mathcal{O}_F$ and maximal ideal $\mathfrak{m}_F = (\varpi_F) \subseteq \mathcal{O}_F$. Let $e = e(F/\mathbb{Q}_p)$ be the *ramification index* and $f = f(F/\mathbb{Q}_p)$ the *inertia degree* of the extension $F/\mathbb{Q}_p$. Let furthermore v be the valuation on F for which $v(p) = 1$, and thus $v(\varpi_F) = \frac{1}{e}$.

$\exp$ **and** $\log$**.** Given finite field extension $F/\mathbb{Q}_p$ with valuation ring $\mathcal{O}_F$ and maximal ideal $\mathfrak{m}_F$ with $p\mathcal{O}_F = \mathfrak{m}_F^e$, we get by [Neu99, Prop. (5.5)] (noting that we will ensure that $1 > \frac{e}{p-1}$ later) that the power series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{and} \qquad \log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots,$$

are two mutually inverse isomorphisms (and homeomorphisms)

$$\mathfrak{m}_F \underset{\log}{\overset{\exp}{\rightleftarrows}} U_F^{(1)}.$$

Note that this implies that a $\mathbb{Z}_p$-basis of $\mathfrak{m}_F$ translates to a $\mathbb{Z}_p$-basis of $U_F^{(1)} = 1 + \mathfrak{m}_F$ via exp.

Big-O notation. For elements of $\mathcal{O}_F$ we write $x = y + O(p^r)$ if and only if $x - y \in p^r \mathcal{O}_F$.

Matrices. Let E_{ij} denote the matrix with 1 in the (i, j) entry, and zeroes in all other entries, and write 1_n for the identity matrix in $M_n(F)$. Let $A = (a_{ij})$. We write $A = \mathrm{diag}(a_1, \ldots, a_n)$ for the diagonal matrix in $M_n(F)$ with entries $a_{ii} = a_i$ in the diagonal, and $A = \mathrm{diag}_{i_1,\ldots,i_k}(a_1, \ldots, a_k)$ for the diagonal matrix in $M_n(F)$ with entries $a_{i_\ell i_\ell} = a_\ell$ for $\ell = 1, \ldots, k$, ones in the other diagonal entries and zeroes in all other entries. Finally, we write $A^\top$ for the transpose matrix of A.

Dual basis. Let V be a k-vector space with basis $\mathcal{B} = (e_1, \ldots, e_d)$. Then we let $\mathcal{B}^* = (e_1^*, \ldots, e_d^*)$ be the dual basis of $\mathrm{Hom}_k(V, k)$ defined by $e_i^*(e_i) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. Now consider two vector spaces V and W with bases $\mathcal{B}_V$ and $\mathcal{B}_W$. Given a linear map $d \colon V \to W$ with matrix A when described in these bases, it is a well known fact from linear algebra that the dual map $d^* \colon \mathrm{Hom}_k(W, k) \to \mathrm{Hom}_k(V, k)$ has matrix $A^\top$ when described in the dual bases $\mathcal{B}_V^*$ and $\mathcal{B}_W^*$. We will often use this without mention and abuse notation writing d and $d^\top$ for these matrices.

Smith normal form. Let R be an integral domain and consider only non-zero matrices over R in this paragraph. Given an $n \times m$ matrix A, there exist invertible $m \times m$ and $n \times n$ matrices S and T such that

$$SAT = \begin{pmatrix} a_1 & 0 & \cdots & & & 0 \\ 0 & a_2 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & & \vdots \\ & & & a_r & & \vdots \\ & & & & 0 & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & & & 0 & 0 \end{pmatrix}$$

and the diagonal entries a_i satisfy $a_i \mid a_{i+1}$ for $i = 1, \ldots, r-1$. This matrix is called the Smith normal form of the matrix A. Given $n \times m$ matrices A, B, we write $A \overset{\mathsf{SNF}}{\sim} B$ if A and B have the same Smith normal form. This notation will mainly be used when B is already a matrix in Smith normal form. Finally we introduce the notation $A = \mathrm{SNF}_{n \times m}(a_1, \ldots, a_r, 0, \ldots, 0)$ for the $n \times m$ matrix with $a_{ii} = a_i$ for $i = 1, \ldots, r$ and zeroes in all other entries as above. In next subsection, we will note that the Smith normal form will be useful for our cohomology calculations.

Remark 3.1. In the case $R = \mathbb{Z}$, the Smith normal form of a matrix can be found using the following row and column operations, which are invertible over $\mathbb{Z}$.

(R1): swap rows R_i and R_j

(R2): multiply row R_i by -1

(R3): replace row R_i by $R_i + kR_j$ for some row $R_j \neq R_i$ and $k \in \mathbb{Z}$

(C1): swap columns C_i and C_j

(C2): multiply column C_i by -1

(C3): replace column C_i by $C_i + kC_j$ for some column $C_j \neq C_i$ and $k \in \mathbb{Z}$.

We will not do these calculations by hand in this chapter, and will instead utilize implementations in Sage and SymPy that can find the Smith normal form of a matrix over $\mathbb{Z}$. Here it is important to note that the SymPy implementation does not allow the use of the rules (R2) and (C2), so we get a small difference between the results of the calculations in SymPy and Sage, but it will only be a difference of sign on some entries in the diagonal. △

Lazard theory. For an introduction to Lazard theory see Section 2.1, or [Sch11a] for more details. In particular, note that the Lazard Lie algebra generalizes from $\mathbb{F}_p$ to general k of characteristic p. We will let $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$ be the Lazard Lie algebra corresponding to the pro-p Iwahori subgroup I. Furthermore, recall that a sequence of elements $(g_1, \ldots, g_r)$ in G is called an *ordered basis* of (G, ω) if the map $\mathbb{Z}_p^r \to G$ given by $(x_1, \ldots, x_r) \mapsto g_1^{x_1} \cdots g_r^{x_r}$ is a bijection (and hence, by compactness, a homeomorphism) and

$$\omega(g_1^{x_1} \cdots g_r^{x_r}) = \min_{1 \leq i \leq r} (\omega(g_i) + v(x_i)) \qquad \text{for any } x_1, \ldots, x_r \in \mathbb{Z}_p.$$

Algebraic groups. We will work with schemes using the functorial approach and notation described in [Jan03]. In particular, given an integral domain R, we note that a *R-group functor* is a functor from the category of all R-algebras to the category of groups, a *R-group scheme* is a R-group functor that is an affine scheme over R when considered as a R-functor, and an *algebraic R-group* is a R-group scheme that is algebraic as an affine scheme. For more in depth introduction to these concepts, we refer to [Con14b] and [Jan03].

Fixed groups and roots. We fix a split and connected reductive algebraic F-group $\mathcal{G}$, and consider the locally profinite group $G = \mathcal{G}(F)$. We then fix split maximal torus $\mathcal{T} \subseteq \mathcal{G}$ and let $T = \mathcal{T}(F)$. In T we have a maximal compact subgroup T^0 and its Sylow pro-p subgroup T^1.

Let $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ be the *root system* of $\mathcal{G}$ with respect to $\mathcal{T}$, and let $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ be the associated root datum. Fix a system of positive roots Φ^+ and let $\Delta \subseteq \Phi^+$ be the simple roots. For any $\alpha \in \Phi$ we have the root subgroup $\mathcal{U}_\alpha \subseteq \mathcal{G}$ with Lie algebra $\operatorname{Lie}\mathcal{U}_\alpha = (\operatorname{Lie}\mathcal{G})_\alpha$. We let $U_\alpha = \mathcal{U}_\alpha(F)$ and choose an isomorphism $x_\alpha \colon F \xrightarrow{\cong} U_\alpha$ such that $t x_\alpha(x) t^{-1} = x_\alpha(\alpha(t)x)$ for $t \in T$ and $x \in F$. For $r \in \mathbb{Z}_{\geq 0}$ we let $U_{\alpha,r} = x_\alpha(\mathfrak{m}_F^r)$.

Remark 3.2. In this chapter we write $\mathcal{U}$ instead of $\mathcal{N}$ since we try to stick to the notation of surrounding literature. △

Coxeter number and p. Let h be the Coxeter number of $\mathcal{G}$ and assume from now on that $p - 1 > eh$.

Pro-p Iwahori subgroups. We follow the definitions of [OS19] with $\mathcal{G}, \mathcal{T}$ and $\mathcal{U}_\alpha$ as above. Let I be the pro-p Iwahori subgroup of G (associated with a positive chamber as in [OS19], but we do not need the exact definition). We note by [OS19, Lem. 2.1(i)] and the proof of [OS19, Lem. 2.3] that I has the following factorization: Multiplication defines a homeomorphism

$$\prod_{\alpha \in \Phi^-} U_{\alpha,1} \times T^1 \times \prod_{\alpha \in \Phi^+} U_{\alpha,0} \xrightarrow{\cong} I, \tag{3.1}$$

where the products are ordered in an arbitrarily chosen way. For a more detailed introduction to these pro-p groups we refer to [OS19].

Pro-p Iwahori subgroups of $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$. In this chapter, we will only work with pro-p Iwahori subgroups of $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$, which simplifies the definitions. When $\mathcal{G} = \mathrm{GL}_n$ or $\mathcal{G} = \mathrm{SL}_n$, we can always take $\mathcal{T}$ the diagonal maximal torus, and we can take I to be the subgroup of $\mathcal{G}(\mathcal{O}_F)$ which is upper triangular and unipotent modulo ϖ_F. In this case we have that $U_{\alpha,1}$ for $\alpha \in \Phi^-$ correspond to entries below the diagonal and $U_{\alpha,0}$ for $\alpha \in \Phi^+$ corresponds to the entries above the diagonal.

p-valuation on I. By a recent preprint by Lahiri and Sørensen (cf. [LS22, Prop. 3.4]), we know (since $p - 1 > eh$) that I admits a p-valuation ω satisfying the properties:

(a) ω is compatible with Iwahori factorization (3.1) of I (cf. [LS22, Def. 3.3]).

(b) $\omega(x_\alpha(x)) = v(x) + \frac{\mathrm{ht}(\alpha)}{eh}$ where $\begin{cases} x \in \mathfrak{m}_F & \text{if } \alpha \in \Phi^-, \\ x \in \mathcal{O}_F & \text{if } \alpha \in \Phi^+. \end{cases}$

(c) $\omega(t) = \frac{1}{e} \cdot \sup\{n \in \mathbb{N} : t \in T^n\}$ for $t \in T^1$.

Ordered basis of I. Let $\{b_1, \ldots, b_\ell\}$ be a $\mathbb{Z}_p$-basis of $\mathcal{O}_F$, where $\ell = [F : \mathbb{Q}_p]$. Then $\big(x_\alpha(b_1), \ldots, x_\alpha(b_\ell)\big)$ is an ordered basis for $U_{\alpha,0}$ when $\alpha \in \Phi^+$, and $\big(x_\alpha(\varpi_F b_1), \ldots, x_\alpha(\varpi_F b_\ell)\big)$ is an ordered basis for $U_{\alpha,1}$ when $\alpha \in \Phi^-$. Furthermore, when G is semisimple and simply connected, we have that the simple coroots $\{\alpha^\vee : \alpha \in \Delta\}$ form a $\mathbb{Z}$-basis of $X_*(T)$, and thus $\big(\alpha^\vee(\exp(\varpi_F b_1)), \ldots, \alpha^\vee(\exp(\varpi_F b_\ell))\big)_{\alpha \in \Delta}$ form an ordered basis of T^1. By [LS22, Cor. 3.6], given orderings of Φ^+ and Φ^-, and assuming that G is semisimple and simply connected, we now get: the sequence of elements

- $\big(x_\alpha(\varpi_F b_1), \ldots, x_\alpha(\varpi_F b_\ell)\big)_{\alpha \in \Phi^-}$,
- $\big(\alpha^\vee(\exp(\varpi_F b_1)), \ldots, \alpha^\vee(\exp(\varpi_F b_\ell))\big)_{\alpha \in \Delta}$,
- $\big(x_\alpha(b_1), \ldots, x_\alpha(b_\ell)\big)_{\alpha \in \Phi^+}$

forms an ordered basis of (I, ω) (with ω from the previous paragraph) which is a saturated p-valued group. Here, [LS22] notes that the p-valuation from the previous paragraph on this basis is given by (cf. [LS22, Prop. 3.4])

$$\begin{cases} \omega\big(x_\alpha(\varpi_F b_\ell)\big) = \dfrac{1}{e} + \dfrac{\mathrm{ht}(\alpha)}{eh} & \alpha \in \Phi^- \\ \omega\big(\alpha^\vee(u_i)\big) = \dfrac{1}{e} & \alpha \in \Delta \\ \omega\big(x_\alpha(b_\ell)\big) = \dfrac{\mathrm{ht}(\alpha)}{eh} & \alpha \in \Phi^+. \end{cases} \tag{3.2}$$

We note that the above argument uses that $\exp\colon \mathfrak{m}_F = (\varpi_F) \to U_F^{(1)} = 1 + \mathfrak{m}_F$ takes a basis to a basis, and noting that $\{\varpi_F b_1, \dots, \varpi_F b_\ell\}$ is a $\mathbb{Z}_p$-basis of $\mathfrak{m}_F = \varpi_F \mathcal{O}_F$.

When $\mathcal{G} = \mathrm{SL}_n$, we have that $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n, i \neq j\}$ and can take

$$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n\},$$

where ε_i is the map that takes a diagonal matrix to its i-th diagonal entry. In this case $\alpha_i^\vee(u) = \mathrm{diag}(1, \dots, 1, u, u^{-1}, 1, \dots, 1) = \mathrm{diag}_{i,i+1}(u, u^{-1})$, where the non-trivial entries are the i-th and $(i+1)$-th entries. Since the second non-trivial entry of these matrices are always just the inverse of the first entry, we will abuse notation and write $\mathrm{diag}_{i,i+1}(u) = \mathrm{diag}_{i,i+1}(u, u^{-1})$. This together with the above gives us the following ordered basis (in the listed order and with a chosen ordering of $\{(i,j) : 1 \leq i, j \leq n\}$) in the case $\mathcal{G} = \mathrm{SL}_n$:

- $\big(1_n + \varpi_F b_1 E_{ij}, \dots, 1_n + \varpi_F b_\ell E_{ij}\big)_{1 \leq j < i \leq n}$,
- $\big(\mathrm{diag}_{i,i+1}(\exp(\varpi_F b_1)), \dots, \mathrm{diag}_{i,i+1}(\exp(\varpi_F b_\ell))\big)_{i=1,\dots,n-1}$,
- $\big(1_n + b_1 E_{ij}, \dots, 1_n + b_\ell E_{ij}\big)_{1 \leq i < j \leq n}$.

Here the p-valuation described in (3.2) is given by

$$\begin{cases} \omega\big(1_n + \varpi_F b_m E_{ij}\big) = \dfrac{1}{e} + \dfrac{j-i}{eh} & j < i, \\ \omega\big(\mathrm{diag}_{i,i+1}(\exp(\varpi_F b_m))\big) = \dfrac{1}{e} & i = 1, \dots, n-1, \\ \omega\big(1_n + b_m E_{ij}\big) = \dfrac{j-i}{eh} & i < j \end{cases} \tag{3.3}$$

on the above ordered basis.

Finally note that an ordered basis of GL_n can be obtained from an ordered basis of SL_n by adding non-trivial elements of the center, which in the above corresponds to adding $\big(\exp(\varpi_F b_1)1_n, \dots, \exp(\varpi_F b_\ell)1_n\big)$ to the middle item above (adding the root $\varepsilon_1 + \dots + \varepsilon_n$), and the p-valuation on these is still $\frac{1}{e}$.

Cohomology. We denote (using the Chevalley-Eilenberg complex) the Lie algebra cohomology of any k-Lie algebra $\mathfrak{g}$ by $H^\bullet(\mathfrak{g}, \,\cdot\,)$, while we write $H^\bullet(G, \,\cdot\,)$ for the continuous group cohomology of a topological group G. Here we let the entries distinguish between different types of cohomology without any ambiguity. As in Section 2.1, we introduce filtrations and then gradings on the cohomology and use the notation $H^{s,t} = \operatorname{gr}^s H^{s+t}$ for any type of cohomology H.

Spectral sequences. A cohomological spectral sequence is a choice of $r_0 \in \mathbb{N}$ and a collection of

- k-modules $E_r^{s,t}$ for each $s, t \in \mathbb{Z}$ and all integers $r \geq r_0$
- differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ such that $d_r^2 = 0$ and E_{r+1} is isomorphic to the homology of (E_r, d_r), i.e.,
$$E_{r+1}^{s,t} = \frac{\ker(d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r})}{\operatorname{im}(d_r^{s-r,t+r-1} \colon E_r^{s-r,t+r-1} \to E_r^{s,t})}.$$

For a given r, the collection $(E_r^{s,t}, d_r^{s,t})_{s,t\in\mathbb{Z}}$ is called the r-th page. A spectral sequence *converges* if d_r vanishes on $E_r^{s,t}$ for any s, t when $r \gg 0$. In this case $E_r^{s,t}$ is independent of r for sufficiently large r, we denote it by $E_\infty^{s,t}$ and write

$$E_r^{s,t} \Longrightarrow E_\infty^{s+t}.$$

Also, we say that the spectral sequence collapses at the r'-th page if $E_r = E_\infty$ for all $r \geq r'$, but not for $r < r'$. Finally, when we have terms E_∞^n with a natural filtration $F^\bullet E_\infty^n$ (but no natural double grading), we set $E_\infty^{s,t} = \operatorname{gr}^s E_\infty^{s+t} = F^s E_\infty^{s+t} / F^{s+1} E_\infty^{s+t}$.

3.1.3 Smith normal form and cohomology

It is well known that the Smith normal form of matrices are useful when calculating (co)homology over $\mathbb{Z}$ as follows.

Fact 3.3. *Given a complex*
$$\mathbb{Z}^n \xrightarrow{d_1} \mathbb{Z}^m \xrightarrow{d_2} \mathbb{Z}^\ell,$$
where d_1 and d_2 are $\mathbb{Z}$-linear maps with $d_2 \circ d_1 = 0$, the homology at the middle term is given by
$$\ker(d_2)/\operatorname{im}(d_1) \cong \bigoplus_{i=1}^{r} \mathbb{Z}/a_i\mathbb{Z} \oplus \mathbb{Z}^{m-r-s}.$$

Here $r = \text{rank}(d_1)$*,* $s = \text{rank}(d_2)$ *and* $a_1, \dots, a_r$ *are the non-zero diagonal elements of the Smith normal form of* d_1*.* ♣

We will not directly use this result, but instead we will follow the same ideas but reduce modulo p to get matrices over k (using the natural embedding $\mathbb{F}_p \hookrightarrow k$). Assuming that the non-zero diagonal entries a_i of the Smith normal form of a matrix d are in $\{1, 2, \dots, p-1\}$ (or more generally $\gcd(a_i, p) = 1$), we note that $a_i \pmod p \in k^\times$. So, given an $n \times m$ matrix d with integer entries such that

$$d \overset{\mathsf{SNF}}{\sim} \text{SNF}_{n\times m}(a_1, \dots, a_r, 0, \dots, 0),$$

where $a_1, \dots, a_r$ are non-zero and $\gcd(a_i, p) = 1$, we get by considering d as a matrix over k that

$$\begin{aligned} \dim_k \ker(d) &= m - r, \\ \dim_k \text{im}(d) &= r, \\ \dim_k \text{coker}(d) &= n - r. \end{aligned} \tag{3.4}$$

Remark 3.4. Note that finding the Smith normal form of all matrices used in our (co)homology calculations, will thus allow us to calculate (co)homological dimensions for p relatively prime to all non-zero diagonal entries of the Smith normal form matrices. This is what makes this method preferable to just calculating the rank of the matrices directly, since that would just allow us to find (co)homological dimensions for $p \gg 0$, but not give us the precise p it will work for. △

Remark 3.5. We assume here that $\mathfrak{g}$ can be lifted to a Lie algebra $\mathfrak{g}_\mathbb{Z}$ with the same Chevalley constants such that $\mathfrak{g} = \mathfrak{g}_\mathbb{Z} \otimes k$. In particular, we assume that these Chevalley constants are such that $\mathfrak{g}_\mathbb{Z}$ satisfy Jacobi's identity. This will not be a problem in the following sections, since we are working with Lie algebras that are well defined mod p for any large enough prime p with coefficients independent of p. In Section 3.11.2 we will see examples of Lie algebras where we need to work modulo a specific prime (we will do $p = 5$) and cannot lift easily to $\mathfrak{g}_\mathbb{Z}$. △

So, when calculating dimensions of homology over k of the middle term in a given complex

$$k^n \xrightarrow{d_1} k^m \xrightarrow{d_2} k^\ell,$$

where d_1 and d_2 can be described by matrices with integer entries, and $d_1 \overset{\mathsf{SNF}}{\sim} \text{SNF}_{m\times n}(a_1, \dots, a_r, 0, \dots, 0)$ and $d_2 \overset{\mathsf{SNF}}{\sim} \text{SNF}_{\ell\times m}(b_1, \dots, b_r, 0, \dots, 0)$, then we can do it as follows:

$n = 0$: The dimension of the homology of the middle term is $\dim_k \ker(d_2) = m - s$.

$\ell = 0$: The dimension of the homology of the middle term is $\dim_k \operatorname{coker}(d_1) = m - r$.

$n, \ell \neq 0$: The dimension of the homology of the middle term is $\dim_k \frac{\ker(d_2)}{\operatorname{im}(d_1)} = m - s - r$, since $d_2 \circ d_1 = 0$.

Remark 3.6. Here the general formula is obviously just that the dimension of the homology of the middle term is $m - s - r$. Also, note that this is what we directly get from Fact 3.3 in the case $k = \mathbb{F}_p$, recalling that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(n, m)\mathbb{Z}$. △

Remark 3.7. When the dimensions of the vector spaces we work with get sufficiently large, the runtime of calculating the full Smith normal form of integer matrices becomes prohibitively high, so we can use an alternative solution. In this case, we can utilize that we know such a form exists, and that $\operatorname{rank}_{\mathbb{Z}}(A) = \operatorname{rank}_{\mathbb{Z}}(B)$ when $A \overset{\mathsf{SNF}}{\sim} B$. Considering the $n \times m$ matrix A with integer entries as a matrix over $\mathbb{R}$, we can then find the Singular value decomposition (SVD) of A, i.e., complex matrices U, Σ, V such that $A = U\Sigma V^*$. Here U is an $n \times n$ unitary matrix, Σ is a rectangular diagonal $n \times m$ matrix (a matrix like in the Smith normal form) with non-negative real numbers on the diagonal, and V is an $m \times m$ unitary matrix. Now $\operatorname{rank}_{\mathbb{R}} \Sigma = \operatorname{rank}_{\mathbb{Z}} A$ allows us to find dimensions of (co)homology as in the case where we know the Smith normal form, but we use information about which p exactly the calculations work for. Thus we will only be able to find the (co)homological dimensions for $p \gg 0$ in this case. △

3.2 Techniques

In this section we will describe how to calculate information about the cohomology of a p-valuable group by using its Lazard Lie algebra. Note that this section uses a lot of concepts and notation from Section 2.1.3.

Let (G, ω) be a p-valuable group and let k be a perfect field of characteristic p. In this section we will describe how the spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, k) \Longrightarrow H^{s+t}(G, k) \tag{3.5}$$

from [Sør21, §6.1] can be used to calculate information about the dimensions of $H^n(G,k)$ for varying n and information about the cup product on $H^*(G,k)$. After this, we will then briefly discuss how this applies to pro-p Iwahori subgroups I of GL_n or SL_n.

Recall that $\mathfrak{g}$ in the above spectral sequence is given by $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$, so to describe $\mathfrak{g}$, we first need a good description of the $\mathbb{F}_p[\pi]$-Lie algebra $\operatorname{gr} G$. To get this description, suppose that we have an ordered basis $(g_1, \dots, g_d)$ of G, so that $\omega(g) = \min_{i=1,\dots,d}\bigl(\omega(g_i) + v_p(x_i)\bigr)$ for $g = g_1^{x_1} \cdots g_d^{x_d}$, and recall that $\bigl(\sigma(g_1), \dots, \sigma(g_d)\bigr)$ is a basis of $\operatorname{gr} G$, where $\sigma(g) = gG_{\omega(g)+} \in \operatorname{gr} G$ for $g \neq 1$.

To understand the $\mathbb{F}_p[\pi]$-Lie algebra, we need to find $[\sigma(g_i), \sigma(g_j)] = \sigma\bigl([g_i, g_j]\bigr)$ for all $i, j = 1, \dots, d$. We recall from (2.2) that $\sigma(g^x) = \overline{x}\pi^{v_p(x)} . \sigma(g)$ for $g \in G \setminus \{1\}$ and $x \in \mathbb{Z}_p \setminus \{0\}$. Now, calculating $[g_i, g_j]$ for all $i, j = 1, \dots, d$, we can find $x_1, \dots, x_d \in \mathbb{Z}_p$ such that

$$[g_i, g_j] = g_1^{x_1} \cdots g_d^{x_d},$$

and thus

$$\bigl[\sigma(g_i), \sigma(g_j)\bigr] = \sigma\bigl([g_i, g_j]\bigr) = \sum_{\ell=1}^{d} \overline{x}_\ell \pi^{v_p(x_\ell)} . \sigma(g_\ell).$$

See the proofs of [Sch11a, Lem. 26.4 and Prop. 26.5] for more details.

Let $\{\ell_1, \dots, \ell_r\}$ be the subset of $\{1, \dots, d\}$ such that $v_p(x_{\ell_s}) = 0$ and $v_p(x_\ell) > 0$ for $\ell \notin \{\ell_1, \dots, \ell_r\}$, and recall that $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ has basis $1 \otimes \sigma(g_i)$. Since π acts trivially on k here, we see that

$$[\xi_i, \xi_j] = \bigl[1 \otimes \sigma(g_i), 1 \otimes \sigma(g_j)\bigr] = \sum_{s=1}^{r} \overline{x}_{\ell_s} \xi_{\ell_s}.$$

Now we have a basis $(\xi_1, \dots, \xi_d)$ of $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$, and we know all the structure constants.

Remark 3.8. Note that the structure constants are in $\mathbb{F}_p \subseteq k$ by the above, so we can lift them to structure constants in $\{0, 1, \dots, p-1\} \subseteq \mathbb{Z}$, which will be useful later. Also note that we will often (but not always) be able to lift $\mathfrak{g}$ to a $\mathbb{Z}$-Lie algebra $\mathfrak{g}_{\mathbb{Z}}$ with $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes k$. △

Assume from now on that the Lie algebra $\mathfrak{g}$ is unitary, i.e., that $[\xi_i, \xi_j] = \sum_{\ell=1}^{d} c_{ij\ell}\xi_\ell$ has $\sum_{j=1}^{d} c_{ijj} = 0$. This will be the case for all Lie algebras, we will work with in this chapter. Suppose furthermore that $\mathfrak{g}$ is a graded Lie algebra, graded by finitely many positive integers, $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^m$, which will also be the case for all Lie algebras we work with in this chapter.

Remark 3.9. Note that any p-valuable group G admits a p-valuation ω with values in $\frac{1}{m}\mathbb{Z}$ for some $m \in \mathbb{N}$, cf. [Sch11a, Cor. 33.3]. Thus we can reindex the filtration of G by letting $G^i = G_{\frac{i}{m}}$ for $i = 0, 1, \ldots$, and this translates to $\operatorname{gr}^i G = \operatorname{gr}_{\frac{i}{m}} G$ and $\mathfrak{g}^i = \mathfrak{g}_{\frac{i}{m}}$ in general. In the cases we care about there will be no zero graded part, which allows us to make the above assumption. △

Then $\bigwedge^n \mathfrak{g}$ is graded as well by letting

$$\operatorname{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}.$$

We note that, since $\mathfrak{g}$ is finite dimensional, there are only finitely many non-zero $\operatorname{gr}^j\big(\bigwedge^n \mathfrak{g}\big)$ we are interested in, and we can find a basis of each of these using our basis $(\xi_1, \ldots, \xi_d)$ of $\mathfrak{g}$.

Remark 3.10. When ordering the basis of $\operatorname{gr}^j\big(\bigwedge^n \mathfrak{g}\big) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$, we will do it as follows. First we order the $\mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$ by the lexicographical order on $(j_1, \ldots, j_n)$. Then we order the basis of each $\mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$ by the lexicographical order on equal j_ℓ's, i.e., if $\mathfrak{g}^1 = \operatorname{span}_k(\xi_1, \xi_3)$ and $\mathfrak{g}^2 = \operatorname{span}_k(\xi_2, \xi_4)$, then $\mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2$ has basis $\xi_1 \wedge \xi_3 \wedge \xi_2, \xi_1 \wedge \xi_3 \wedge \xi_4$. △

Assuming furthermore that k is $\mathbb{Z}$-graded (concentrated in degree 0), the space $\operatorname{Hom}_k\big(\bigwedge^n \mathfrak{g}, k\big)$ inherits the $\mathbb{Z}$-grading

$$\operatorname{Hom}_k\Big(\bigwedge^n \mathfrak{g}, k\Big) = \bigoplus_{s\in\mathbb{Z}} \operatorname{Hom}_k^s\Big(\bigwedge^n \mathfrak{g}, k\Big),$$

where Hom_k^s denotes the homogeneous k-linear maps of degree s, cf. [FF74, Lem. 4.2]. We note, by [Fuk86, Chap. 1 §3.7], that these gradings on the chain and cochain complexes transfer to gradings on the homology and cohomology. We write

$$H^{s,t} = H^{s,t}(\mathfrak{g}, k) = H^{s+t}\Big(\operatorname{gr}^s \operatorname{Hom}_k\big(\bigwedge^\bullet \mathfrak{g}, k\big)\Big).$$

Remark 3.11. We do not spend effort to describe the homology for a few reasons. First, we need the cohomology, not the homology, in our spectral sequence. Second, by [Fuk86, Chap. 1 §3.6], we have a version of Poincaré duality for Lie algebra cohomology, i.e., $H^n(\mathfrak{g}, k) \cong H_{n-d}(\mathfrak{g}, k)$, so we can easily describe the homology using the cohomology. Third, we will care about the cup product later, and we do not get a nice product in homology, cf. [Fuk86, Chap. 1 §3.2]. △

Now we have bases of all $\mathrm{gr}^j\big(\bigwedge^n \mathfrak{g}\big)$, and by [Fuk86, Chap. 1 §3.7] we get graded chain complexes that we can use to find the homology of $\mathfrak{g}$. Here the boundary maps of

$$\cdots \xrightarrow{d_4} \bigwedge^3 \mathfrak{g} \xrightarrow{d_3} \bigwedge^2 \mathfrak{g} \xrightarrow{d_2} \mathfrak{g} \xrightarrow{d_1} k \longrightarrow 0,$$

are given by

$$d_n(x_1 \wedge \cdots \wedge x_n) = \sum_{i<j} (-1)^{i+j}[x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_n,$$

and the coboundary maps

$$\cdots \xleftarrow{\partial_3} \mathrm{Hom}_k\Big(\bigwedge^2 \mathfrak{g}, k\Big) \xleftarrow{\partial_2} \mathrm{Hom}_k(\mathfrak{g}, k) \xleftarrow{\partial_1} \mathrm{Hom}_k(k, k) = k \longleftarrow 0,$$

are the dual maps of the boundary maps (see [Fuk86, Chap. 1 §3.1] for more details). Thus, if we use the dual basis of $\bigwedge^n \mathfrak{g}$ in $\mathrm{Hom}_k\big(\bigwedge^n \mathfrak{g}, k\big)$, we get that $\partial_n = d_n^\top$ as matrices, where $(\,\cdot\,)^\top$ is the transpose (cf. Section 3.1). Since we know bases and linear maps explicitly, and we know that the linear maps restrict to graded linear maps, we can now find matrices describing all graded linear maps

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) \to \mathrm{gr}^j\Big(\bigwedge^{n-1} \mathfrak{g}\Big),$$

and thus we can find matrices describing all graded linear maps

$$\mathrm{Hom}_k^s\Big(\bigwedge^{n-1} \mathfrak{g}, k\Big) \to \mathrm{Hom}_k^s\Big(\bigwedge^n \mathfrak{g}, k\Big).$$

Noting that all the structure constants can be lifted to $\mathbb{Z}$ (in the examples we work with) and looking at the formula for the boundary maps, it is clear that the above matrices describing the (co)boundary maps can be lifted to $\mathbb{Z}$. Finding the Smith normal form of these lifts, we can calculate the cohomology over k by Section 3.1.3 for p large enough. (In most examples, $p \geq 5$ will be enough.)

Suppose now that we have found the dimensions of $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$ for all s, t. To get information about the cohomology $H^n(G, k)$, we need to use the multiplicative spectral sequence (3.5), i.e.,

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, k) \Longrightarrow H^{s+t}(G, k)$$

and information about spectral sequences in general. We already know that this spectral sequence collapses at a finite page, and one can hope is that it will actually collapse at the first page. One way we can verify that the spectral sequence (in certain cases) collapses at the first page, is by considering the exact bidegrees of the differentials. We know that the differentials $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ have bidegree $(r, 1-r)$, so if the non-zero modules on the first pages is distributed in such a way that all differentials $d_r^{s,t}$ are trivial for all s, t and $r \geq 1$, then we can be sure that the spectral sequence collapses on the first page. This will become clearer when we look at examples in the next few sections.

Now note, by [Fuk86, Chap. 1 §3.7], that the cup product is compatible with the gradings on the Lie algebra cohomology, in particular

$$H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'}, \tag{3.6}$$

where $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$. Thus, since the spectral sequence is multiplicative, we can describe the cup product on $H^*(G, k)$ when the spectral sequence collapses on the first page. Some cup products will be trivially zero by (3.6), and for the rest of the cup product, we can calculate them with an explicit basis using the following.

For $f \in \mathrm{Hom}_k\big(\bigwedge^p \mathfrak{g}, k\big)$ and $g \in \mathrm{Hom}_k\big(\bigwedge^q \mathfrak{g}, k\big)$, we know from [CE56, Chap. XIII, Sect. 8], that the cup product in cohomology is induced by: $f \cup g \in \mathrm{Hom}_k\big(\bigwedge^{p+q} \mathfrak{g}, k\big)$ defined by

$$(f \cup g)(x_1 \wedge \cdots \wedge x_{p+q}) = \sum_{\substack{\sigma \in S_{p+q} \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(p+q)}} \mathrm{sign}(\sigma) f(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) g(x_{\sigma(p+1)} \wedge \cdots \wedge x_{\sigma(p+q)}). \tag{3.7}$$

Even when the spectral sequence does not (necessarily) collapse on the first page, we can still get some bounds on the dimensions of $H^n(G, k)$ that will allow us to draw some conclusions about the structure of $H^*(G, k)$.

In the rest of this chapter we will focus on using the techniques described in this section to get as much as possible information about the cohomology of $H^*(I, k)$, where I is the pro-p Iwahori subgroup of $\mathrm{SL}_n(\mathbb{Q}_p)$ or $\mathrm{GL}_n(\mathbb{Q}_p)$ $n = 2, 3, 4$ or the pro-p Iwahori subgroup of $\mathrm{SL}_2(F)$ or $\mathrm{GL}_2(F)$ for $F/\mathbb{Q}_p$ a quadratic extension.

3.3 $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$

In this section we will describe the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{SL}_2(\mathbb{Q}_p)$.

When I is the pro-p Iwahori subgroup in $\mathrm{SL}_2(\mathbb{Q}_p)$, we know by Section 3.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}^{\det=1} \subseteq \mathrm{SL}_2(\mathbb{Z}_p).$$

In this case, an obvious guess for an ordered basis (using that $(1+p)^{\mathbb{Z}_p} = 1+p\mathbb{Z}_p$) is

$$g_1' = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2' = \begin{pmatrix} 1+p & 0 \\ 0 & (1+p)^{-1} \end{pmatrix}, \qquad g_3' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Because we want to be able to describe the commutators using this ordered basis, we will at one point need to solve for x in equation of the form $(1+p)^x = y$. For this reason a better choice of ordered basis is (as described in Section 3.1)

$$g_1 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(-p) \end{pmatrix}, \qquad g_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.8}$$

In this case the above equations to solve translate to solving for x in $\exp(x) = y$, which we can easily do, as $x = \log(y)$, cf. Section 3.1.

3.3.1 Finding the commutators $[\xi_i, \xi_j]$

Now write

$$g_1^{x_1} g_2^{x_2} g_3^{x_3} = \begin{pmatrix} \exp(px_2) & x_3 \exp(px_2) \\ px_1 \exp(px_2) & px_1 x_3 \exp(px_2) + \exp(-px_2) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{3.9}$$

Furthermore, write $g_{ij} = [g_i, g_j]$ and $\xi_{ij} = [\xi_i, \xi_j]$. Then we are ready to find x_1, x_2, x_3 such that $g_{ij} = g_1^{x_1} g_2^{x_2} g_3^{x_3}$ for different $i < j$. (In the following we use that $\frac{1}{p-1} = 1 + p + p^2 + \cdots$ and $\log(1-p) = -p - \frac{p^2}{2} - \frac{p^3}{3} - \cdots$.)

We now list all non-identitiy commutators $g_{ij} = [g_i, g_j]$ and find $\xi_{ij} = [\xi_i, \xi_j]$ based on these. (For $g_{ij} = 1_2$ it is clear that $x_1 = x_2 = x_3 = 0$, and thus $\xi_{ij} = 0$.)

$g_{12} = \begin{pmatrix} 1 & 0 \\ p(1-\exp(-2p)) & 1 \end{pmatrix}$**:** Comparing g_{12} with (3.9), we see that $x_2 = x_3 = 0$. This leaves $a_{21} = px_1 = p(1-\exp(-2p)) = 2p^2 + O(p^3)$, which implies that $x_1 = 2p + O(p^2)$. Hence $\sigma(g_{12}) = 2\pi \,.\, \sigma(g_1)$, which implies that $\xi_{12} = 0$.

$g_{13} = \begin{pmatrix} 1-p & p \\ -p^2 & 1+p+p^2 \end{pmatrix}$**:** Comparing g_{13} with (3.9), we see that

$$a_{11} = \exp(px_2) = 1-p,$$

$$a_{12} = x_3 \exp(px_2) = x_3(1-p) = p,$$

$$a_{21} = px_1 \exp(px_2) = px_1(1-p) = -p^2,$$

and thus

$$x_2 = \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p),$$
$$x_3 = \frac{p}{1-p} = p + O(p^2),$$
$$x_1 = \frac{-p^2}{p(1-p)} = -p + O(p^2).$$

Hence $\sigma(g_{13}) = -\pi \,.\, \sigma(g_1) - \sigma(g_2) - \pi \,.\, \sigma(g_3)$, which implies that $\xi_{13} = -\xi_2$.

$g_{23} = \begin{pmatrix} 1 & \exp(2p)-1 \\ 0 & 1 \end{pmatrix}$**:** Comparing g_{23} with (3.9), we see that $x_1 = x_2 = 0$. This leaves $a_{12} = x_3 = \exp(2p) - 1 = 2p + O(p^2)$. Hence $\sigma(g_{23}) = 2\pi \,.\, \sigma(g_3)$, which implies that $\xi_{23} = 0$.

To clarify, we found that

$$\sigma(g_{12}) = 2\pi \,.\, \sigma(g_1),$$

$$\sigma(g_{13}) = -\pi \,.\, \sigma(g_1) - \sigma(g_2) - \pi \,.\, \sigma(g_3),$$

$$\sigma(g_{23}) = 2\pi \,.\, \sigma(g_3),$$

and recalling that $\xi_i = 1 \otimes \sigma(g_i)$ in $k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$, where π acts trivially on k, we get that

$$\xi_{12} = 0, \qquad \xi_{13} = -\xi_2, \qquad \xi_{23} = 0, \tag{3.10}$$

where $\xi_{ij} = [\xi_i, \xi_j]$.

3.3.2 Describing the graded chain complex, $\operatorname{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$

Looking at (3.3) (with $e = 1$ and $h = 2$), we see that

$$\omega(g_1) = 1 - \frac{1}{2} = \frac{1}{2}, \qquad \omega(g_2) = 1, \qquad \omega(g_3) = \frac{1}{2}.$$

Hence $\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{2}} = \mathrm{span}_k(\xi_1, \xi_3)$ and $\mathfrak{g}^2 = \mathfrak{g}_1 = \mathrm{span}_k(\xi_2)$, cf. Remark 3.9.

Now we are ready to describe the graded chain complex

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$$

and its bases. We list the grading of $\bigwedge^n \mathfrak{g}$ for all n.

$n = 0:$

$$\mathrm{gr}^j(k) = \begin{cases} k & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$k: \quad 1.$$

$n = 1:$

$$\mathrm{gr}^j(\mathfrak{g}) = \begin{cases} \mathfrak{g}^2 & j = 2, \\ \mathfrak{g}^1 & j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1: \quad \xi_1, \xi_3,$$
$$\mathfrak{g}^2: \quad \xi_2.$$

$n = 2:$

$$\mathrm{gr}^j\Big(\bigwedge^2 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1 \wedge \mathfrak{g}^2: \quad \xi_1 \wedge \xi_2, \xi_3 \wedge \xi_2,$$
$$\mathfrak{g}^1 \wedge \mathfrak{g}^1: \quad \xi_1 \wedge \xi_3.$$

$n = 3:$

$$\mathrm{gr}^j\Big(\bigwedge^3 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2: \quad \xi_1 \wedge \xi_3 \wedge \xi_2.$$

$n \geq 4:$

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = 0 \text{ for all } j.$$

Table 3.1: Dimensions of $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$ for the $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ case.

$n \backslash j$	0	1	2	3	4
0	1				
1		2	1		
2			1	2	
3					1

We collect the above information about the dimensions of the chain complex of $\mathfrak{g}$ in Table 3.1, and note that we only need to consider non-zero (non-empty) entries of the table, when we calculate $H^{s,t} = H^{s,n-s}$ (where $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$). Also, recalling that

$$\mathrm{Hom}_k\Big(\bigwedge^n \mathfrak{g}, k\Big) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s\Big(\bigwedge^n \mathfrak{g}, k\Big),$$

we see that, with $j = -s$, we get the same table for dimensions of the graded hom-spaces in the cochain complex.

3.3.3 Finding the graded Lie algebra cohomology, $H^{s,t}(\mathfrak{g}, k)$

Remark 3.12. In this section we will calculate the cohomology directly instead of using the method described in Section 3.1.3, since the calculations are only with small matrices. To see how Section 3.1.3 is used, we refer to Section 3.5. △

We will now go through all different graded chain complexes one by one, using that gr^j in the chain complex corresponds to gr^s with $s = -j$ in the cochain complex. We note that the graded chain complex corresponds to vertical downwards arrows in Table 3.1, while the cochain complex corresponds to vertical upwards arrows. And finally, we reiterate that $H^n = H^n(\mathfrak{g}, k)$ and $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$ in the following.

In grade 0 we have the chain complex

$$0 \longrightarrow k \longrightarrow 0,$$

which gives us the grade 0 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^0(k, k) \longleftarrow 0.$$

So $H^0 = H^{0,0}$ with $\dim H^{0,0} = 1$.

In grade 1 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \longrightarrow 0,$$

which gives us the grade -1 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-1}(\mathfrak{g}, k) \longleftarrow 0.$$

So $\dim H^{-1,2} = 2$ by Table 3.1.

In grade 2 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{(1)} \mathfrak{g}^2 \longrightarrow 0,$$

since

$$\begin{gathered} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \to \mathfrak{g}^2 \\ \xi_1 \wedge \xi_3 \mapsto -[\xi_1, \xi_3] = \xi_2. \end{gathered}$$

This gives us the grade -2 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-2}\left(\bigwedge^2 \mathfrak{g}, k\right) \xleftarrow{(1)} \mathrm{Hom}_k^{-2}(\mathfrak{g}, k) \longleftarrow 0.$$

So with $d = (1)$, and comparing with Table 3.1,

$$\begin{gathered} \dim H^{-2,3} = \dim \ker(d) = 0, \\ \dim H^{-2,4} = \dim \operatorname{coker}(d) = 0. \end{gathered}$$

In grade 3 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -3 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-3}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \longleftarrow 0.$$

So $\dim H^{-3,5} = 2$ by Table 3.1.

In grade 4 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -4 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-4}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \longleftarrow 0.$$

So $\dim H^{-4,7} = 1$ by Table 3.1.

Table 3.2: Dimensions of $E_1^{s,t} = H^{s,t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ case.

t \ s	0	−1	−2	−3	−4
0	1				
1					
2		2			
3					
4					
5				2	
6					
7					1

Altogether, we see that

$$\begin{aligned} H^0 &= H^{0,0}, \\ H^1 &= H^{-1,2}, \\ H^2 &= H^{-3,5}, \\ H^3 &= H^{-4,7}, \end{aligned} \tag{3.11}$$

with dimension as described in Table 3.2.

3.3.4 Describing the group cohomology, $H^n(I, k)$

We note that all differentials $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ in Table 3.2 has bidegree $(r, 1-r)$, i.e., they are all below the $(r, -r)$ arrow going r to the left and r up in the table, where $r \geq 1$. Looking at Table 3.2, this clearly means that all differentials for $r \geq 1$ are trivial, and thus the spectral sequence collapses on the first page. Hence $H^{s,t}(\mathfrak{g}, k) = E_1^{s,t} \cong E_\infty^{s,t} = \operatorname{gr}^s H^{s+t}(I, k)$, and by (3.11) and Table 3.2 we get that

$$\dim H^n(I, k) = \begin{cases} 1 & n = 0, \\ 2 & n = 1, \\ 2 & n = 2, \\ 1 & n = 3. \end{cases} \tag{3.12}$$

Recalling that the spectral sequence is multiplicative, we also note, by Table 3.2, that $H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'}$ implies that the cup products

$$\operatorname{gr}^s H^n(I, k) \otimes \operatorname{gr}^{s'} H^{n'}(I, k) \to \operatorname{gr}^{s+s'} H^{n+n'}(I, k)$$

are trivial, except for the obvious ones with $H^0(I, k)$ and $H^1 \otimes H^2 \to H^3$. We now want to describe the cup product $H^1 \otimes H^2 \to H^3$.

Let $e_{i_1,\ldots,i_m} = (\xi_{i_1} \wedge \cdots \wedge \xi_{i_m})^*$ be the element of the dual basis of $\operatorname{Hom}_k\big(\bigwedge^m \mathfrak{g}, k\big)$ corresponding to $\xi_{i_1} \wedge \cdots \wedge \xi_{i_m}$ in the basis of $\bigwedge^m \mathfrak{g}$. Looking at the cochain complexes and descriptions of the maps above together with the known bases of the graded chain complexes, we get the following precise descriptions of the of the graded cohomology spaces $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$:

$$\begin{aligned} H^{-1,2} &= k[e_1, e_3], \\ H^{-3,5} &= k[e_{1,2}, e_{3,2}], \\ H^{-4,7} &= k[e_{1,3,2}]. \end{aligned} \tag{3.13}$$

Remark 3.13. Here we abuse notation and write $k[e_1, e_3]$ for the k-vector space with basis e_1 and e_3, and not the polynomial ring. We use the same notation from now on without notice, and note that this should not give rise to any confusion since it will only be used for basis elements of the $e_{i_1,\ldots,i_m}$. △

For $f \in \operatorname{Hom}_k(\bigwedge^p \mathfrak{g}, k)$ and $g \in \operatorname{Hom}_k(\bigwedge^q \mathfrak{g}, k)$, we recall from (3.7) that the cup product in cohomology is induced by: $f \cup g \in \operatorname{Hom}_k(\bigwedge^{p+q} \mathfrak{g}, k)$ defined by

$$(f \cup g)(x_1 \wedge \cdots \wedge x_{p+q}) = \sum_{\substack{\sigma \in S_{p+q} \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(p+q)}} \operatorname{sign}(\sigma) f(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) g(x_{\sigma(p+1)} \wedge \cdots \wedge x_{\sigma(p+q)}).$$

So, when finding

$$H^{-1,2} \otimes H^{-3,5} \xrightarrow{\cup} H^{-4,7},$$

we need to calculate $e_1 \cup e_{1,2}$, $e_1 \cup e_{3,2}$, $e_3 \cup e_{1,2}$ and $e_3 \cup e_{3,2}$ on the basis $\mathcal{B} = (\xi_1 \wedge \xi_3 \wedge \xi_2)$ of $\operatorname{gr}^4 \bigwedge^3 \mathfrak{g}$.

We first note that (3.7) simplifies to

$$(e_i \cup e_{j,k})(x_1 \wedge x_2 \wedge x_3) = \sum_{\substack{\sigma \in S_3 \\ \sigma(2) < \sigma(3)}} \operatorname{sign}(\sigma) e_i(x_{\sigma(1)}) e_{j,k}(x_{\sigma(2)} \wedge x_{\sigma(3)})$$

in these cases. Here the terms of the sum on the right is only non-zero if $x_{\sigma(1)} = \xi_i$ and $x_{\sigma(2)} \wedge x_{\sigma(3)} = \xi_j \wedge \xi_k$ (up to constants). In the case $e_1 \cup e_{1,2}$ (resp. $e_3 \cup e_{3,2}$), we can only have this if $x_1 \wedge x_2 \wedge x_3$ contains two copies of ξ_1 (resp. ξ_3), which implies that $x_1 \wedge x_2 \wedge x_3 = 0$. So $e_1 \cup e_{1,2} = 0$ and $e_3 \cup e_{3,2} = 0$. Alternatively, one can see this by plugging in $x_1 \wedge x_2 \wedge x_3 = \xi_1 \wedge \xi_3 \wedge \xi_2$ and simply calculating the right side.

In the case $e_1 \cup e_{3,2}$, (3.7) simplifies to

$$(e_1 \cup e_{3,2})(x_1 \wedge x_2 \wedge x_3) = \sum_{\substack{\sigma \in S_3 \\ \sigma(2) < \sigma(3)}} \operatorname{sign}(\sigma) e_1(x_{\sigma(1)}) e_{3,2}(x_{\sigma(2)} \wedge x_{\sigma(3)}).$$

When $x_1 \wedge x_2 \wedge x_3 = \xi_1 \wedge \xi_3 \wedge \xi_2$, we see that the terms on the right side are only non-zero if $x_{\sigma(1)} = \xi_1$, i.e., $\sigma(1) = 1$, and thus $\sigma = (1)$ since $\sigma(2) < \sigma(3)$. So $x_{\sigma(1)} = \xi_1$, $x_{\sigma(2)} \wedge x_{\sigma(3)} = \xi_3 \wedge \xi_2$ and $\operatorname{sign}(\sigma) = 1$, which gives us $(e_1 \cup e_{3,2})(\xi_1 \wedge \xi_3 \wedge \xi_2) = 1$. Hence $e_1 \cup e_{3,2} = e_{1,3,2}$.

In the case $e_3 \cup e_{1,2}$, (3.7) simplifies to

$$(e_3 \cup e_{1,2})(x_1 \wedge x_2 \wedge x_3) = \sum_{\substack{\sigma \in S_3 \\ \sigma(2) < \sigma(3)}} \operatorname{sign}(\sigma) e_3(x_{\sigma(1)}) e_{1,2}(x_{\sigma(2)} \wedge x_{\sigma(3)}).$$

When $x_1 \wedge x_2 \wedge x_3 = \xi_1 \wedge \xi_3 \wedge \xi_2$, we see that the terms on the right side are only non-zero if $x_{\sigma(1)} = \xi_3$, i.e., $\sigma(1) = 2$, and thus $\sigma = (1,2)$ since $\sigma(2) < \sigma(3)$. So $x_{\sigma(1)} = \xi_3$, $x_{\sigma(2)} \wedge x_{\sigma(3)} = \xi_1 \wedge \xi_2$ and $\operatorname{sign}(\sigma) = -1$, which gives us $(e_3 \cup e_{1,2})(\xi_1 \wedge \xi_3 \wedge \xi_2) = -1$. Hence $e_3 \cup e_{1,2} = -e_{1,3,2}$.

In conclusion, all the non-trivial and non-zero cup products (up to graded commutativity) are:

$$\begin{aligned} e_1 \cup e_{3,2} &= e_{1,3,2}, \\ e_3 \cup e_{1,2} &= -e_{1,3,2}. \end{aligned} \tag{3.14}$$

Now, since the spectral sequence collapses on the first page, all of the above work on the cup product of the Lie algebra cohomology transfers to the cup product on $H^*(I, k)$ as described above. In particular, since all $H^n(I, k)$ only have one graded component, this is a clear description of the cup product on $H^*(I, k)$, and not just a graded cup product.

Remark 3.14. Let D be the division quaternion algebra over $\mathbb{Q}_p$ for a prime $p > 3$ and let $G = (1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1}$, where $\mathrm{Nrd} = \mathrm{Nrd}_{D/\mathbb{Q}_p}$ is the norm form. From [Sør21, Sect. 6.3] (or from [Hen07, Prop. 7]) we know that there is an isomorphism

$$H^*(G, \mathbb{F}_p) \cong \mathbb{F}_p \oplus \mathbb{F}_D \oplus \mathbb{F}_D \oplus \mathbb{F}_p$$

of graded $\mathbb{F}_p$-algebras (where $\mathbb{F}_D \cong \mathbb{F}_{p^2}$ is viewed simply as a $\mathbb{F}_p$-vector space). I.e., $H^n(G, \mathbb{F}_p)$ has the same dimensions as described in (3.12) (with $k = \mathbb{F}_p$). [Sør21] also shows that the only non-trivial and non-zero cup product is $H^1(G, \mathbb{F}_p) \times H^2(G, \mathbb{F}_p) \to H^3(G, \mathbb{F}_p)$, which corresponds to the trace pairing $\mathbb{F}_D \times \mathbb{F}_D \to \mathbb{F}_p, (x, y) \mapsto \mathrm{Tr}(xy)$ (where $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{F}_D/\mathbb{F}_p}$ from [Neu99, Def. 2.5]). To be more explicit, let's assume that $p \equiv 3 \pmod 4$. Then $x^2 + 1$ is irreducible over $\mathbb{F}_p$, so we can write $\mathbb{F}_D = \mathbb{F}_p[\alpha]$ with $\alpha^2 = -1$, where $\mathbb{F}_p[\alpha]$ has $\mathbb{F}_p$-basis $1, \alpha$. Now, considering the maps $1\colon a + b\alpha \mapsto a + b\alpha$, $\alpha\colon a + b\alpha \mapsto -b + a\alpha$ and $\alpha^2 = -1\colon a + b\alpha \mapsto -a - b\alpha$, we see that the trace pairing is given by

$$\begin{aligned} \mathbb{F}_D \times \mathbb{F}_D &\to \mathbb{F}_p \\ (1, 1) &\mapsto \mathrm{Tr}(1) = 2, \\ (1, \alpha) &\mapsto \mathrm{Tr}(\alpha) = 0, \\ (\alpha, 1) &\mapsto \mathrm{Tr}(\alpha) = 0, \\ (\alpha, \alpha) &\mapsto \mathrm{Tr}(\alpha^2) = -2. \end{aligned}$$

This is (up to a multiple of 2) the same as the description of the cup product on $H^*(I, \mathbb{F}_p)$ above for $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$, so an interesting question is: Is there a nice relation between mod p representations of

$G = (1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}$ and I? We already have bijections between *certain* mod p representations of $D^\times$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ from the Jacquet-Langlands correspondence for GL_2 (cf. [JL70]), but by [Tok15, Rem. 4.5] irreducible representations of $D^\times$ are trivial on $1+\mathfrak{m}_D$, so we need something new if we want a correspondence between mod p representations of $G = (1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}$ and $I \subseteq \mathrm{SL}_2(\mathbb{Q}_p)$.

As a continuation to the question in the $p \equiv 3 \pmod 4$ case, we note that $\mathfrak{g}_D = \mathbb{F}_D \oplus \mathbb{F}_D^{\mathrm{Tr}=0}$ sitting in degree 1 and 2 has Lie bracket given by $[\overline{x}, \overline{y}] = \overline{xy^p} - \overline{yx^p}$ for any $\overline{x}, \overline{y} \in \mathbb{F}_D$ in degree 1 (and 0 otherwise) by [Sør21, (6.6)]. So, with $\mathbb{F}_D = \mathbb{F}_p[\alpha]$, we note that

$$[1, \alpha] = 1 \cdot \alpha^p - \alpha \cdot 1^p = -2\alpha,$$

since $p \equiv 3 \pmod 4$ and $\alpha^2 = -1$. Thus we have an obvious relation between $\mathfrak{g}$ and $\mathfrak{g}_D$ given by $\xi_1, \xi_3 \leftrightarrow 1, \alpha$ in degree 1 and $\xi_2 \leftrightarrow 2\alpha$ in degree 2. The question is whether we can lift this to a relation between (the representations of) I and G. Which we will explore in more detail in Section 3.11.1. △

It might even be the case, that we have a group isomorphism between $(1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}$ and I, which lifts the $\mathbb{F}_p$-Lie algebra isomorphism $\mathfrak{g} \cong \mathfrak{g}_D$ of Remark 3.14, but this seems less likely.

3.3.5 Lower p-series of I

One of the consequences of the cohomology calculations above is that I is not a uniformly powerful group. To see this, note by the proof of [HKN11, Thm. 3.3.3] that a uniformly powerful pro-p group is equi-p-valuable, and thus Lazard's famous isomorphism $H^*(G,k) \cong \bigwedge \mathrm{Hom}_k(\mathfrak{g}, k)$ for equi-p-valued groups G can be applied to uniformly powerful groups. We refer to [Sør21, Cor. 6.3] for a proof of the isomorphism with methods similar to what we use in this chapter. Now, since the dimensions from (3.12) do not match the dimensions of $\bigwedge \mathrm{Hom}_k(\mathfrak{g}, k)$ $(1,3,3,1$ since $\dim_k \mathfrak{g} = 3)$, we see that I cannot be uniformly powerful (or equi-p-valuable). This leads to the interesting question of, whether we can describe the lower p-series of I, and see that I is not uniformly powerful directly?

Before answering this question, we recall the following definitions, cf. [Dix+99, Def. 1.15, Cor. 1.20, Def. 3.1 and Def. 4.1].

Definition 3.15. A p-valued group G is *equi-p-valuable* if it admits a p-valuation ω and an ordered basis $(g_1, \dots, g_d)$ such that $\omega(g_i) = \omega(g_j)$ for all i, j. ♠

Definition 3.16. Let G be a finitely generated pro-p group. The *lower p-series* $\cdots \geq P_3(G) \geq P_2(G) \geq P_1(G)$ of G is given by $P_i(G)$, where $P_1(G) = G$ and

$$P_{i+1}(G) = P_i(G)^p \big[P_i(G), G\big]$$

for $i \geq 1$. ♠

Definition 3.17. Let p be an odd prime. A pro-p group G is *uniformly powerful* (often written as *uniform*) if

(i) G is finitely generated,

(ii) G is *powerful*, i.e., $G/\overline{G^p}$ is abelian, and

(iii) for all i, $[P_i(G) : P_{i+1}(G)] = [G : P_2(G)]$. ♠

To show directly that I is not uniformly powerful, we will calculate its lower p-series, for which we will introduce the notation $I_i = P_i(I)$ and work with generators of I. We already know that I is generated by g_1, g_2, g_3 (note that this is similar to [CR16, Thm. 2.4.1] but with $\exp(p)$ instead of $1 + p$ in the torus), and we now want to describe the generators of each $I_i = P_i(I)$. For this description we will use the following lemma.

Lemma 3.18. *Let J be the subgroup of I generated by $g_1^{p^{v_1}}, g_2^{p^{v_2}}, g_3^{p^{v_3}}$, and set $m_1 = \min(v_1, v_2)$, $m_2 = \min(v_1, v_3)$ and $m_3 = \min(v_2, v_3)$. Then*

(i) J^p is the subgroup generated by $g_1^{p^{v_1+1}}, g_2^{p^{v_2+1}}, g_3^{p^{v_3+1}}$, and

(ii) $[J, I]$ is the subgroup generated by $g_1^{p^{m_1+1}}, g_2^{p^{m_2}}, g_3^{p^{m_3+1}}$. ♣

Proof. By the definition of J, it is clear that J^p is the subgroup generated by $g_1^{p^{v_1+1}}, g_2^{p^{v_2+1}}, g_3^{p^{v_3+1}}$, so we only need to show (ii).

To find generators of $[J, I]$, it is enough to calculate the commutators of the generators of J and I, and we note that g and g^x generate the same subgroup for $g \in I$ if $x \in \mathbb{Z}_p^\times$. Now

$$[g_1^{p^{v_1}}, g_2] = \begin{pmatrix} 1 & 0 \\ p^{v_1+1}\big(1 - \exp(-2p)\big) & 0 \end{pmatrix} = g_1^{p^{v_1}(1-\exp(-2p))},$$

$$[g_1, g_2^{p^{v_2}}] = \begin{pmatrix} 1 & 0 \\ p\big(1-\exp(-2p^{v_2+1})\big) & \end{pmatrix} = g_1^{p^{v_2}(1-\exp(-2p^{v_2+1}))},$$

where $\frac{1}{p}\big(1-\exp(-2p)\big) \in \mathbb{Z}_p^\times$ and $\frac{1}{p}\big(1-\exp(-2p^{v_2+1})\big) \in \mathbb{Z}_p^\times$ (since they both have $v_p = 0$), so these commutators are correspond to generators $g_1^{p^{v_1+1}}$ and $g_1^{p^{v_2+1}}$. Since $g_1^{p^{v_1+1}}$ and $g_1^{p^{v_2+1}}$ clearly generate the same group as $g_1^{p^{m_1+1}}$, we have the subgroup generated by $g_1^{p^{m_1+1}}$ in $[J, I]$.

Similarly

$$[g_2^{p^{v_2}}, g_3] = \begin{pmatrix} 1 & \exp(2p^{v_2+1}) - 1 \\ 0 & 1 \end{pmatrix} = g_3^{\exp(2p^{v_2+1})-1},$$
$$[g_2, g_3^{p^{v_3}}] = \begin{pmatrix} 1 & p^{v_3}\big(\exp(2p) - 1\big) \\ 0 & 1 \end{pmatrix} = g_3^{p^{v_3}(\exp(2p)-1)},$$

where $\frac{1}{p^{v_2+1}}\big(\exp(2p^{v_2+1}) - 1\big) \in \mathbb{Z}_p^\times$ and $\frac{1}{p}\big(\exp(2p) - 1\big)$, so these commutators correspond to generators $g_3^{p^{v_2+1}}$ and $g_3^{p^{v_3+1}}$, which generate the same subgroup as $g_3^{p^{m_3+1}}$. Hence the subgroup generated by $g_1^{p^{m_1+1}}$ and $g_3^{p^{m_3+1}}$ is a subgroup of J.

Finally, comparing

$$[g_1^{p^{v_1}}, g_3] = \begin{pmatrix} 1 - p^{v_1+1} & p^{v_1+1} \\ -p^{2v_1+2} & p^{2v_1+2} + p^{v_1+1} + 1 \end{pmatrix} = g_1^{x_1} g_2^{x_2} g_3^{x_3}$$

with (3.9), we see that

$$a_{11} = \exp(px_2) = 1 - p^{v_1+1},$$
$$a_{12} = x_3 \exp(px_2) = x_3\big(1 - p^{v_1+1}\big) = p^{v_1+1},$$
$$a_{21} = px_1 \exp(px_2) = px_1\big(1 - p^{v_1+1}\big) = -p^{2v_1+2},$$

and thus

$$x_2 = \frac{1}{p}\log(1 - p^{v_1+1}) = -p^{v_1} + O(p^{v_1+1}),$$
$$x_3 = \frac{p^{v_1+1}}{1 - p^{v_1+1}} = p^{v_1+1} + O(p^{v_1+2}),$$
$$x_1 = \frac{-p^{2v_1+2}}{p(1 - p^{v_1+1})} = -p^{2v_1+1} + O(p^{2v_1+2}).$$

Using that $g_1^{m_1+1}$ and $g_3^{m_3+1}$ generate a subgroup of J and that $m_1 \le v_1$ and $m_3 \le v_3$, we can see that the above adds a generator $g_2^{\frac{1}{p}(\log(1-p^{v_1+1}))}$, and since

$$\frac{\frac{1}{p}\log(1 - p^{v_1+1})}{p^{v_1}} \in \mathbb{Z}_p^\times,$$

we see that this is equivalent to adding a generator $g_2^{p^{v_2}}$. Similarly, comparing

$$[g_1, g_3^{p^{v_3}}] = \begin{pmatrix} 1 - p^{v_3+1} & p^{2v_3+1} \\ -p^{v_3+2} & p^{2v_3+2} + p^{v_3+1} + 1 \end{pmatrix} = g_1^{x_1} g_2^{x_2} g_3^{x_3}$$

with (3.9), we see that

$$\begin{aligned} a_{11} &= \exp(px_2) = 1 - p^{v_3+1}, \\ a_{12} &= x_3 \exp(px_2) = x_3\bigl(1 - p^{v_3+1}\bigr) = p^{2v_3+1}, \\ a_{21} &= px_1 \exp(px_2) = px_1\bigl(1 - p^{v_3+1}\bigr) = -p^{v_3+2}, \end{aligned}$$

and thus

$$\begin{aligned} x_2 &= \frac{1}{p}\log(1 - p^{v_3+1}) = -p^{v_3} + O(p^{v_3+1}), \\ x_3 &= \frac{p^{2v_3+1}}{1 - p^{v_3+1}} = p^{2v_1+1} + O(p^{2v_1+2}), \\ x_1 &= \frac{-p^{v_3+2}}{p(1 - p^{v_3+1})} = -p^{v_3+1} + O(p^{v_3+2}). \end{aligned}$$

Using that $g_1^{m_1+1}$ and $g_3^{m_3+1}$ generate a subgroup of J and that $m_1 \leq v_1$ and $m_3 \leq v_3$, we can see that the above adds a generator $g_2^{\frac{1}{p}(\log(1-p^{v_3+1}))}$, and since

$$\frac{\frac{1}{p}\log(1 - p^{v_3+1})}{p^{v_3}} \in \mathbb{Z}_p^{\times},$$

we see that this is equivalent to adding a generator $g_2^{p^{v_3}}$. So we have added the generators $g_2^{p^{v_1}}$ and $g_2^{p^{v_3}}$, which is equivalent to adding the generator $g_2^{p^{m_2}}$.

Altogether, we see that $[J, I]$ is generated by $g_1^{p^{m_1+1}}$, $g_2^{p^{m_2}}$ and $g_3^{p^{m_3+1}}$. $\square$

Now $I = I_1$ is generated by g_1, g_2, g_3, so by Lemma 3.18 I^p is generated by g_1^p, g_2^p, g_3^p and $[I, I]$ is generated by g_1^p, g_2, g_3^p. Thus $I_2 = I^p[I, I]$ is generated by g_1^p, g_2, g_3^p, and we see that $I_2 = [I, I]$. Using Lemma 3.18 again, we see that $[I, I]^p$ is generated by $g_1^{p^2}, g_2^p, g_3^{p^2}$ and $\bigl[[I, I], I\bigr]$ is generated by g_1^p, g_2^p, g_3^p. So I_3 is generated by g_1^p, g_2^p, g_3^p, and we see that $I_3 = I^p$. We now claim that

$$I_i = \begin{cases} I^{p^n} & \text{if } i = 2n+1, \\ [I, I]^{p^{n-1}} & \text{if } i = 2n, \end{cases}$$

where I_{2n+1} is generated by $g_1^{p^n}, g_2^{p^n}, g_3^{p^n}$ and I_{2n} is generated by $g_1^{p^n}, g_2^{p^{n-1}}, g_3^{p^n}$. We will prove this by induction on i, where we already covered the base cases above. Assume first that $I_{2n} = [I, I]^{p^{n-1}}$

is generated by $g_1^{p^n}, g_2^{p^{n-1}}, g_3^{p^n}$. Then, by Lemma 3.18, I_{2n}^p is generated by $g_1^{p^{n+1}}, g_2^{p^n}, g_3^{p^{n+1}}$ and $[I_{2n}, I]$ is generated by $g_1^{p^n}, g_2^{p^n}, g_3^{p^n}$, so $I_{2n+1} = I_{2n}^p[I_{2n}, I]$ is generated by $g_1^{p^n}, g_2^{p^n}, g_3^{p^n}$ and thus $I_{2n+1} = I^{p^n}$. Assume now, on the other hand, that $I_{2n+1} = I^{p^n}$ is generated by $g_1^{p^n}, g_2^{p^n}, g_3^{p^n}$. Then, by Lemma 3.18, I_{2n+1}^p is generated by $g_1^{p^{n+1}}, g_2^{p^{n+1}}, g_3^{p^{n+1}}$ and $[I_{2n+1}, I]$ is generated by $g_1^{p^{n+1}}, g_2^{p^n}, g_3^{p^{n+1}}$, so $I_{2n+2} = I_{2n+1}^p[I_{2n+1}, I]$ is generated by $g_1^{p^{n+1}}, g_2^{p^n}, g_3^{p^{n+1}}$ and thus $I_{2n+2} = [I, I]^{p^n}$. Hence, by induction, we have proved:

Theorem 3.19. *Let I be the pro-p Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ and let g_1, g_2, g_3 be the ordered basis of I from* (3.8)*. Then the lower p-series is given by*

$$P_i(I) = \begin{cases} I^{p^n} & \text{if } i = 2n+1, \\ [I, I]^{p^{n-1}} & \text{if } i = 2n, \end{cases}$$

where $P_{2n+1}(I) = I^{p^n}$ is the subgroup generated by $g_1^{p^n}, g_2^{p^n}, g_3^{p^n}$ and $P_{2n}(I) = [I, I]^{p^{n-1}}$ is the subgroup generated by $g_1^{p^n}, g_2^{p^{n-1}}, g_3^{p^n}$.

Thus

$$[P_i(G) : P_{i+1}(G)] = \begin{cases} 1 & \text{if } i = 2n, \\ 2 & \text{if } i = 2n+1. \end{cases}$$

In particular, I is not uniformly powerful. ♣

3.4 $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$

In this section we will describe the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{GL}_2(\mathbb{Q}_p)$.

When I is the pro-p Iwahori subgroup in $\mathrm{GL}_2(\mathbb{Q}_p)$, we know by Section 3.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{Z}_p),$$

and, by Section 3.1, we have an ordered basis

$$\begin{aligned} g_1 &= \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(-p) \end{pmatrix}, \\ g_3 &= \begin{pmatrix} \exp(p) & 0 \\ 0 & \exp(p) \end{pmatrix}, \qquad g_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.15}$$

Since we just renamed some elements and added an element of the center of $\mathrm{GL}_2(\mathbb{Z}_p)$ when comparing to the ordered basis of $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ from Section 3.3, it is clear from Equation (3.10) that the only non-zero commutator in $\mathfrak{g} = k \otimes \operatorname{gr} G$ is

$$[\xi_1, \xi_4] = -\xi_2,$$

where $\xi_i = 1 \otimes \sigma(g_i)$ as usual.

3.4.1 Describing the graded chain complex, $\operatorname{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$

Looking at (3.3) (with $e = 1$ and $h = 2$) and the note about the GL_n case after (3.3), we see that

$$\omega(g_1) = \frac{1}{2}, \qquad \omega(g_2) = 1,$$
$$\omega(g_3) = 1 \qquad \omega(g_4) = \frac{1}{2}.$$

Hence $\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{2}} = \operatorname{span}_k(\xi_1, \xi_4)$ and $\mathfrak{g}^2 = \mathfrak{g}_1 = \operatorname{span}_k(\xi_2, \xi_3)$, cf. Remark 3.9.

Now we are ready to describe the graded chain complex

$$\operatorname{gr}^j\left(\bigwedge^n \mathfrak{g}\right) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n}$$

and its bases.

$n = 0:$

$$\operatorname{gr}^j(k) = \begin{cases} k & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$k: \quad 1.$$

$n = 1:$

$$\operatorname{gr}^j(\mathfrak{g}) = \begin{cases} \mathfrak{g}^2 & j = 2, \\ \mathfrak{g}^1 & j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1 : \quad \xi_1, \xi_4,$$
$$\mathfrak{g}^2 : \quad \xi_2, \xi_3.$$

$n = 2 :$

$$\mathrm{gr}^j\Big(\bigwedge^2 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 4, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^2 \wedge \mathfrak{g}^2 : \quad \xi_2 \wedge \xi_3,$$
$$\mathfrak{g}^1 \wedge \mathfrak{g}^2 : \quad \xi_1 \wedge \xi_2, \xi_1 \wedge \xi_3, \xi_4 \wedge \xi_2, \xi_4 \wedge \xi_3,$$
$$\mathfrak{g}^1 \wedge \mathfrak{g}^1 : \quad \xi_1 \wedge \xi_4.$$

$n = 3 :$

$$\mathrm{gr}^j\Big(\bigwedge^3 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 5, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 : \quad \xi_1 \wedge \xi_2 \wedge \xi_3, \xi_4 \wedge \xi_2 \wedge \xi_3,$$
$$\mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 : \quad \xi_1 \wedge \xi_4 \wedge \xi_2, \xi_1 \wedge \xi_4 \wedge \xi_3.$$

$n = 4 :$

$$\mathrm{gr}^j\Big(\bigwedge^4 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 6, \\ 0 & \text{otherwise.} \end{cases}$$

Bases:

$$\mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^4 : \quad \xi_1 \wedge \xi_4 \wedge \xi_2 \wedge \xi_3.$$

Table 3.3: Dimensions of $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$ for the $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ case.

$n \backslash j$	0	1	2	3	4	5	6
0	1						
1		2	2				
2			1	4	1		
3					2	2	
4							1

$n \geq 5$:

$$\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right) = 0 \text{ for all } j.$$

We collect the above information about the dimensions of the chain complex of $\mathfrak{g}$ in Table 3.3, and note that we only need to consider non-zero (non-empty) entries of the table, when we calculate $H^{s,t} = H^{s,n-s}$ (where $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$). Also, recalling that

$$\mathrm{Hom}_k\left(\bigwedge^n \mathfrak{g}, k\right) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s\left(\bigwedge^n \mathfrak{g}, k\right),$$

we see that, with $j = -s$, we get the same table for dimensions of the graded hom-spaces in the cochain complex.

3.4.2 Finding the graded Lie algebra cohomology, $H^{s,t}(\mathfrak{g}, k)$

We will now go through all different graded chain complexes one by one, using that gr^j in the chain complex corresponds to gr^s with $s = -j$ in the cochain complex. We note that the graded chain complex corresponds to vertical downwards arrows in Table 3.3, while the cochain complex corresponds to vertical upwards arrows. And finally, we reiterate that $H^n = H^n(\mathfrak{g}, k)$ and $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$ in the following.

In grade 0 we have the chain complex

$$0 \longrightarrow k \longrightarrow 0,$$

which gives us the grade 0 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^0(k, k) \longleftarrow 0.$$

So $H^0 = H^{0,0}$ with $\dim H^{0,0} = 1$.

In grade 1 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \longrightarrow 0,$$

which gives us the grade -1 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-1}(\mathfrak{g}, k) \longleftarrow 0.$$

So $\dim H^{-1,2} = 2$ by Table 3.1.

In grade 2 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \mathfrak{g}^2 \longrightarrow 0,$$

since

$$\begin{aligned} \mathfrak{g}^1 \wedge \mathfrak{g}^1 &\to \mathfrak{g}^2 \\ \xi_1 \wedge \xi_4 &\mapsto -[\xi_1, \xi_4] = \xi_2. \end{aligned}$$

This gives us the grade -2 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-2}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \xleftarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} \operatorname{Hom}_k^{-2}(\mathfrak{g}, k) \longleftarrow 0.$$

So with

$$d = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

and comparing with Table 3.3,

$$\dim H^{-2,3} = \dim \ker(d) = 1,$$

$$\dim H^{-2,4} = \dim \operatorname{coker}(d) = 0.$$

In grade 3 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -3 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-3}\left(\bigwedge^2 \mathfrak{g}, k\right) \longleftarrow 0.$$

So $\dim H^{-3,5} = 2$ by Table 3.1.

In grade 4 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} \mathfrak{g}^2 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

since

$$\begin{aligned}
&\mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \to \mathfrak{g}^2 \wedge \mathfrak{g}^2 \\
&\xi_1 \wedge \xi_4 \wedge \xi_2 \mapsto -[\xi_1, \xi_4] \wedge \xi_2 + [\xi_1, \xi_2] \wedge \xi_4 - [\xi_4, \xi_2] \wedge \xi_1 = \xi_2 \wedge \xi_2 = 0, \\
&\xi_1 \wedge \xi_4 \wedge \xi_3 \mapsto -[\xi_1, \xi_4] \wedge \xi_3 + [\xi_1, \xi_3] \wedge \xi_4 - [\xi_4, \xi_3] \wedge \xi_1 = \xi_2 \wedge \xi_3.
\end{aligned}$$

This gives us the grade -4 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-4}\left(\bigwedge^3 \mathfrak{g}, k\right) \xleftarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \operatorname{Hom}_k^{-4}\left(\bigwedge^2 \mathfrak{g}, k\right) \longleftarrow 0.$$

So with

$$d = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and comparing with Table 3.3,

$$\begin{aligned}
\dim H^{-4,6} &= \dim \ker(d) = 0, \\
\dim H^{-4,7} &= \dim \operatorname{coker}(d) = 1.
\end{aligned}$$

In grade 5 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -5 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-5}\left(\bigwedge^3 \mathfrak{g}, k\right) \longleftarrow 0.$$

So $\dim H^{-5,8} = 2$ by Table 3.1.

In grade 6 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \longrightarrow 0,$$

which gives us the grade -6 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-6}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \longleftarrow 0.$$

So $\dim H^{-6,10} = 1$ by Table 3.1.

Table 3.4: Dimensions of $E_1^{s,t} = H^{s,t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ case.

$t \backslash s$	0	-1	-2	-3	-4	-5	-6
0	1						
1							
2		2					
3			1				
4							
5				4			
6							
7					1		
8						2	
9							
10							1

Altogether, we see that

$$\begin{aligned} H^0 &= H^{0,0}, \\ H^1 &= H^{-1,2} \oplus H^{-2,3}, \\ H^2 &= H^{-3,5}, \\ H^3 &= H^{-4,7} \oplus H^{-5,8}, \\ H^4 &= H^{-6,10}, \end{aligned} \tag{3.16}$$

with dimension as described in Table 3.4.

3.4.3 Describing the group cohomology, $H^n(I,k)$

We note that all differentials $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ in Table 3.4 have bidegree $(r, 1-r)$, i.e., they are all below the $(r,-r)$ arrow going r to the left and r up in the table, where $r \geq 1$. Looking at Table 3.4, this clearly means that all differentials for $r \geq 1$ are trivial, and thus the spectral sequence collapses on the first page. Hence $H^{s,t}(\mathfrak{g},k) = E_1^{s,t} \cong E_\infty^{s,t} = \operatorname{gr}^s H^{s+t}(I,k)$, and by (3.16) and Table 3.4 we get that

$$\dim H^n(I,k) = \begin{cases} 1 & n = 0, \\ 3 & n = 1, \\ 4 & n = 2, \\ 3 & n = 3, \\ 1 & n = 4. \end{cases} \tag{3.17}$$

Recalling that the spectral sequence is multiplicative, we also note, by Table 3.4, that $H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'}$ implies that the cup products

$$\operatorname{gr}^s H^n(I,k) \otimes \operatorname{gr}^{s'} H^{n'}(I,k) \to \operatorname{gr}^{s+s'} H^{n+n'}(I,k)$$

are trivial except for the cases with H^0 and

$$\begin{aligned} &H^{-1,2} \cup H^{-2,3} \subseteq H^{-3,5}, \\ &H^{-1,2} \cup H^{-3,5} \subseteq H^{-4,7}, \\ &H^{-1,2} \cup H^{-5,8} \subseteq H^{-6,10}, \\ &H^{-2,3} \cup H^{-3,5} \subseteq H^{-5,8}, \\ &H^{-2,3} \cup H^{-4,7} \subseteq H^{-6,10}, \\ &H^{-3,5} \cup H^{-3,5} \subseteq H^{-6,10}, \end{aligned} \tag{3.18}$$

and the reverse of the above (which we can find using graded commutativity).

Next we want to describe these cup products.

Let $e_{i_1,\dots,i_m} = (\xi_{i_1} \wedge \cdots \wedge \xi_{i_m})^*$ be the element of the dual basis of $\operatorname{Hom}_k\big(\bigwedge^m \mathfrak{g}, k\big)$ corresponding to $\xi_{i_1} \wedge \cdots \wedge \xi_{i_m}$ in the basis of $\bigwedge^m \mathfrak{g}$. Looking at the cochain complexes and descriptions of the maps above together with the known bases of the graded chain complexes, we get the following precise descriptions of the of the graded cohomology spaces $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$:

$$\begin{aligned} H^{-1,2} &= k[e_1, e_4], \\ H^{-2,3} &= \ker \begin{pmatrix} 1 & 0 \end{pmatrix} = k[e_3], \\ H^{-3,5} &= k[e_{1,2}, e_{1,3}, e_{4,2}, e_{4,3}], \\ H^{-4,7} &= \frac{k[e_{1,4,2}, e_{1,4,3}]}{\operatorname{im}\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = k[e_{1,4,2}], \\ H^{-5,8} &= k[e_{1,2,3}, e_{4,2,3}], \\ H^{-6,10} &= k[e_{1,4,2,3}]. \end{aligned} \tag{3.19}$$

For $f \in \operatorname{Hom}_k\big(\bigwedge^p \mathfrak{g}, k\big)$ and $g \in \operatorname{Hom}_k\big(\bigwedge^q \mathfrak{g}, k\big)$, we recall from (3.7) that the cup product in cohomology is induced by: $f \cup g \in \operatorname{Hom}_k\big(\bigwedge^{p+q} \mathfrak{g}, k\big)$ defined by

$$(f \cup g)(x_1 \wedge \cdots \wedge x_{p+q}) = \sum_{\substack{\sigma \in S_{p+q} \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(p+q)}} \operatorname{sign}(\sigma) f(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) g(x_{\sigma(p+1)} \wedge \cdots \wedge x_{\sigma(p+q)}).$$

We will now find all the cup products in (3.18) by working with our given bases and the (3.7).

We will start by finding

$$H^{-1,2} \otimes H^{-2,3} \xrightarrow{\ \cup\ } H^{-3,5}.$$

Looking at (3.19), we need to describe the maps $e_1 \cup e_3$ and $e_4 \cup e_3$ on the basis of $\operatorname{gr}^3 \bigwedge^2 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_2, \xi_1 \wedge \xi_3, \xi_4 \wedge \xi_2, \xi_4 \wedge \xi_3)$. In the case of $e_1 \cup e_3$, (3.7) simplifies to

$$(e_1 \cup e_3)(x_1 \wedge x_2) = \sum_{\sigma \in S_2} \operatorname{sign}(\sigma) e_1(x_{\sigma(1)}) e_3(x_{\sigma(2)}),$$

which is zero on all of $\mathcal{B}$ except $\xi_1 \wedge \xi_3$ with $\sigma = (1)$, where we get (using that $\operatorname{sign}\big((1)\big) = 1$)

$$(e_1 \cup e_3)(\xi_1 \wedge \xi_3) = 1.$$

Hence $e_1 \cup e_3 = e_{1,3}$. In the case of $e_4 \cup e_3$, (3.7) simplifies to

$$(e_4 \cup e_3)(x_1 \wedge x_2) = \sum_{\sigma \in S_2} \operatorname{sign}(\sigma) e_4(x_{\sigma(1)}) e_3(x_{\sigma(2)}),$$

which is zero on all of $\mathcal{B}$ except $\xi_4 \wedge \xi_3$ with $\sigma = (1)$, where we get (using that $\operatorname{sign}\big((1)\big) = 1$)

$$(e_4 \cup e_3)(\xi_4 \wedge \xi_3) = 1.$$

Hence $e_4 \cup e_3 = e_{4,3}$. Looking at (3.19), we see that $e_{1,3}$ and $e_{4,3}$ are the second and forth basis elements of $H^{-3,5}$, so the above calculation caries over to the above cup product in cohomology.

We will now describe

$$H^{-1,2} \otimes H^{-3,5} \xrightarrow{\cup} H^{-4,7}.$$

Looking at (3.19), we need to describe the maps

$$e_1 \cup e_{1,2}, e_1 \cup e_{1,3}, e_1 \cup e_{4,2}, e_1 \cup e_{4,3},$$

$$e_4 \cup e_{1,2}, e_4 \cup e_{1,3}, e_4 \cup e_{4,2}, e_4 \cup e_{4,3}$$

on the basis of $\operatorname{gr}^4 \bigwedge^3 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_4 \wedge \xi_2, \xi_1 \wedge \xi_4 \wedge \xi_3)$. In any case with repeat numbers, it is clear from (3.7) (and the fact that there are no repeats in $\mathcal{B}$) that the cup product will be zero, so we only need to consider $e_1 \cup e_{4,2}$, $e_1 \cup e_{4,3}$, $e_4 \cup e_{1,2}$ and $e_4 \cup e_{1,3}$. In all cases of $e_i \cup e_{j,k}$, (3.7) simplifies to

$$(e_i \cup e_{j,k})(x_1 \wedge x_2 \wedge x_3) = \sum_{\substack{\sigma \in S_3 \\ \sigma(2) < \sigma(3)}} \operatorname{sign}(\sigma) e_i(x_{\sigma(1)}) e_{j,k}(x_{\sigma(2)} \wedge x_{\sigma(3)}),$$

i.e., the sum is over $\sigma \in \{(1), (1,2), (1,3,2)\}$. When $(i,j,k) = (1,4,2)$ or $(i,j,k) = (1,4,3)$, this sum is zero on all of $\mathcal{B}$ except $\xi_i \wedge \xi_j \wedge \xi_k$ with $\sigma = (1)$, since σ needs to fix 1 and $\mathcal{B}$ uses the same ordering. Here we get (using that $\operatorname{sign}\big((1)\big) = 1$)

$$(e_i \cup e_{j,k})(\xi_i \wedge \xi_j \wedge \xi_k) = 1.$$

Hence $e_1 \cup e_{4,2} = e_{1,4,2}$ and $e_1 \cup e_{4,3} = e_{1,4,3}$. When $(i,j,k) = (4,1,2)$ or $(i,j,k) = (4,1,3)$, the sum is zero on all of $\mathcal{B}$ except $\xi_j \wedge \xi_i \wedge \xi_k$ with $\sigma = (1,2)$, since the order of the first and second elements of (i,j,k) are swapped compared to in $\mathcal{B}$. Here we get (using that $\operatorname{sign}\big((1,2)\big) = -1$)

$$(e_i \cup e_{j,k})(\xi_i \wedge \xi_j \wedge \xi_k) = -1.$$

Hence $e_4 \cup e_{1,2} = -e_{1,4,2}$ and $e_4 \cup e_{1,3} = -e_{1,4,3}$. Looking at (3.19), we see that $e_{1,4,3}$ reduces to zero in $H^{-4,7}$, while $e_{1,4,2}$ is part of the basis. So in the cup product on the cohomology, the only nontrivial products are $e_1 \cup e_{4,2} = e_{1,4,2}$ and $e_4 \cup e_{1,2} = -e_{1,4,2}$.

At this point, it should be clear how to skip some of the details, so we will proceed with less justification than above.

Now consider

$$H^{-1,2} \otimes H^{-5,8} \xrightarrow{\cup} H^{-6,10}.$$

Looking at (3.19), the only nontrivial maps we need to describe are $e_1 \cup e_{4,2,3}$ and $e_4 \cup e_{1,2,3}$ on the basis of $\mathrm{gr}^6 \bigwedge^4 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_4 \wedge \xi_2 \wedge \xi_3)$. For $e_1 \cup e_{4,2,3}$ to be non-zero, we need $\sigma \in S_4$ that fixes 1 and satisfies $\sigma(2) < \sigma(3) < \sigma(4)$, which is only true for $\sigma = (1)$. For $e_4 \cup e_{1,2,3}$ to be non-zero, we need $\sigma \in S_4$ that swaps 1 and 2 and satisfies $\sigma(2) < \sigma(3) < \sigma(4)$, which is only true for $\sigma = (1,2)$. Since $\mathrm{sign}\big((1)\big) = 1$ and $\mathrm{sign}\big((1,2)\big) = -1$, we get that $e_1 \cup e_{4,2,3} = e_{1,4,2,3}$ and $e_4 \cup e_{1,2,3} = -e_{1,4,2,3}$. Looking at (3.19), we see that $e_{1,4,2,3}$ it the basis elements of $H^{-6,10}$, so the above calculation caries over to the above cup products in cohomology.

Continue with

$$H^{-2,3} \otimes H^{-3,5} \xrightarrow{\cup} H^{-5,8}.$$

Looking at (3.19), the only nontrivial maps we need to describe are $e_3 \cup e_{1,2}$ and $e_3 \cup e_{4,2}$ on the basis of $\mathrm{gr}^5 \bigwedge^3 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_2 \wedge \xi_3, \xi_4 \wedge \xi_2 \wedge \xi_3)$. For $e_3 \cup e_{1,2}$ or $e_3 \cup e_{4,2}$ to be non-zero, we need $\sigma \in S_3$ that satisfies $\sigma(1) = 3$ (putting ξ_3 first) and $\sigma(2) < \sigma(3)$, which is only true for $\sigma = (1,3,2)$. Since $\mathrm{sign}\big(((1,3,2)\big) = 1$, we get that $e_3 \cup e_{1,2} = e_{1,2,3}$ and $e_3 \cup e_{4,2} = e_{4,2,3}$. Looking at (3.19), we see that $e_{1,2,3}$ and $e_{4,2,3}$ are the basis elements of $H^{-5,8}$, so the above calculation caries over to the above cup products in cohomology.

Continue with

$$H^{-2,3} \otimes H^{-4,7} \xrightarrow{\cup} H^{-6,10}.$$

Looking at (3.19), the only map we need to describe is $e_3 \cup e_{1,4,2}$ on the basis of $\mathrm{gr}^6 \bigwedge^4 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_4 \wedge \xi_2 \wedge \xi_3)$. For $e_3 \cup e_{1,4,2}$ to be non-zero, we need $\sigma \in S_4$ that satisfies $\sigma(1) = 4$ (putting ξ_3 first) and $\sigma(2) < \sigma(3) < \sigma(4)$, which is only true for $\sigma = (1,4,3,2)$. Since $\mathrm{sign}\big(((1,4,3,2)\big) = -1$, we get that $e_3 \cup e_{1,4,2} = -e_{1,4,2,3}$. Looking at (3.19), we see that $e_{1,4,2,3}$ it the basis elements of $H^{-6,10}$, so the above calculation caries over to the above cup product in cohomology.

Finally, consider

$$H^{-3,5} \otimes H^{-3,5} \xrightarrow{\cup} H^{-6,10}.$$

Looking at (3.19), the only nontrivial maps we need to describe are $e_{1,2} \cup e_{4,3}$ and $e_{1,3} \cup e_{4,2}$ (getting the rest by graded-commutativity) on the basis of $\mathrm{gr}^6 \bigwedge^4 \mathfrak{g}$, i.e., on $\mathcal{B} = (\xi_1 \wedge \xi_4 \wedge \xi_2 \wedge \xi_3)$. For $e_{1,2} \cup e_{4,3}$ to be non-zero, we need $\sigma \in S_4$ that satisfies

- $\{\sigma(1), \sigma(2)\} = \{1, 3\}$ (putting ξ_1 and ξ_2 in $e_{1,2}$),
- $\{\sigma(3), \sigma(4)\} = \{2, 4\}$ (putting ξ_4 and ξ_3 in $e_{4,3}$),
- $\sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(4)$,

which is only true for $\sigma = (2, 3)$. Since $\mathrm{sign}\big((2, 3)\big) = -1$, we get that $e_{1,2} \cup e_{4,3} = -e_{1,4,2,3}$. For $e_{1,3} \cup e_{4,2}$ to be non-zero, we need $\sigma \in S_4$ that satisfies

- $\{\sigma(1), \sigma(2)\} = \{1, 4\}$ (putting ξ_1 and ξ_3 in $e_{1,3}$),
- $\{\sigma(3), \sigma(4)\} = \{2, 3\}$ (putting ξ_4 and ξ_2 in $e_{4,2}$),
- $\sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(4)$,

which is only true for $\sigma = (2, 4, 3)$. Since $\mathrm{sign}\big((2, 4, 3)\big) = 1$, we get that $e_{1,3} \cup e_{4,2} = e_{1,4,2,3}$. Looking at (3.19), we see that $e_{1,4,2,3}$ it the basis elements of $H^{-6,10}$, so the above calculation caries over to the above cup products in cohomology. Also, since $H^{-3,5} = H^2$, we get, by graded commutativity of the cup product, that $e_{4,2} \cup e_{1,3} = (-1)^{2\times 2} e_{1,3} \cup e_{4,2} = e_{1,4,2,3}$ and $e_{4,3} \cup e_{1,2} = (-1)^{2\times 2} e_{1,2} \cup e_{4,3} = -e_{1,4,2,3}$.

In conclusion, all the non-trivial and non-zero cup products (up to graded commutativity) are:

$$\begin{aligned}
e_1 \cup e_3 &= e_{1,3}, \\
e_4 \cup e_3 &= e_{4,3}, \\
e_1 \cup e_{4,2} &= e_{1,4,2}, \\
e_4 \cup e_{1,2} &= -e_{1,4,2}, \\
e_1 \cup e_{4,2,3} &= e_{1,4,2,3}, \\
e_4 \cup e_{1,2,3} &= -e_{1,4,2,3}, \\
e_3 \cup e_{1,2} &= e_{1,2,3}, \\
e_3 \cup e_{4,2} &= e_{4,2,3}, \\
e_3 \cup e_{1,4,2} &= -e_{1,4,2,3}, \\
e_{1,3} \cup e_{4,2} &= e_{1,4,2,3}, \\
e_{1,2} \cup e_{4,3} &= -e_{1,4,2,3}.
\end{aligned} \tag{3.20}$$

Now, since the spectral sequence collapses on the first page, all of the above work on the cup product of the Lie algebra cohomology transfers to the cup product on $H^*(I,k)$ as described above.

Remark 3.20. Let $\mathbb{F}_p[\varepsilon]$ denote the dual numbers ($\varepsilon^2 = 0$), where ε sits in grade -2. The above cup product calculations show that $H^*(I_{\mathrm{SL}_2(\mathbb{Q}_p)}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon] \cong H^*(I_{\mathrm{GL}_2(\mathbb{Q}_p)}, \mathbb{F}_p)$ as algebras, where I_G is the pro-p Iwahori subgroup of G. To see this, note that the both algebras are 12 dimensional, and on $H^*(I_{\mathrm{SL}_2(\mathbb{Q}_p)}, \mathbb{F}_p)$, and we know from (3.13) that

$$\begin{aligned}
&H^0 = \mathbb{F}_p, \\
&H^1 = H^{-1,2} = \mathbb{F}_p[e_1, e_3] = \mathbb{F}_p[x_1, x_2], \\
&H^2 = H^{-3,5} = \mathbb{F}_p[e_{1,2}, e_{3,2}] = \mathbb{F}_p[y_1, y_2], \\
&H^3 = H^{-4,7} = \mathbb{F}_p[e_{1,3,2}] = \mathbb{F}_p[z],
\end{aligned}$$

where we write $x_1 = e_1, x_2 = e_3, y_1 = e_{1,2}, y_2 = e_{3,2}, z = e_{1,3,2}$, and from (3.14) that

$$\begin{aligned}
x_1 \cup y_2 &= z, & x_2 \cup y_1 &= -z, \\
y_2 \cup x_1 &= (-1)^{1\times 2} x_1 \cup y_2 = z, & y_1 \cup x_2 &= (-1)^{1\times 2} x_2 \cup y_1 = -z.
\end{aligned}$$

are the only non-trivial and non-zero cup products. Now $\mathbb{F}_p[\varepsilon] \cong \mathbb{F}_p \oplus \mathbb{F}_p\varepsilon$ and

$$\begin{aligned}
&H^*(I_{\mathrm{SL}_2(\mathbb{Q}_p)}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon] \\
&= \big(\mathbb{F}_p \oplus \mathbb{F}_p[x_1, x_2] \oplus \mathbb{F}_p[y_1, y_2] \oplus \mathbb{F}_p[z]\big) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon] \\
&\cong \mathbb{F}_p[\varepsilon] \oplus \mathbb{F}_p[\varepsilon][x_1, x_2] \oplus \mathbb{F}_p[\varepsilon][y_1, y_2] \oplus \mathbb{F}_p[\varepsilon][z] \\
&\cong \underbrace{\mathbb{F}_p}_{H^{0,0}} \oplus \underbrace{\mathbb{F}_p\varepsilon}_{H^{-1,2}} \oplus \underbrace{\mathbb{F}_p x_1 \oplus \mathbb{F}_p x_2}_{H^{-1,2}} \oplus \underbrace{\mathbb{F}_p x_1\varepsilon \oplus \mathbb{F}_p x_2\varepsilon \oplus \mathbb{F}_p y_1 \oplus \mathbb{F}_p y_2}_{H^{-3,5}} \\
&\quad \oplus \underbrace{\mathbb{F}_p y_1 \oplus \mathbb{F}_p y_2}_{H^{-5,8}} \oplus \underbrace{\mathbb{F}_p y_1\varepsilon \oplus \mathbb{F}_p y_2\varepsilon}_{H^{-5,8}} \oplus \underbrace{\mathbb{F}_p z}_{H^{-4,7}} \oplus \underbrace{\mathbb{F}_p z\varepsilon}_{H^{-6,10}},
\end{aligned}$$

and the map

$$\begin{aligned}
H^*(I_{\mathrm{SL}_2(\mathbb{Q}_p)}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon] &\to H^*(I_{\mathrm{GL}_2(\mathbb{Q}_p)}, \mathbb{F}_p) \\
1 &\mapsto 1, \\
\varepsilon &\mapsto e_3, \\
x_1 &\mapsto e_1, \\
x_2 &\mapsto e_4, \\
x_1\varepsilon &\mapsto e_{1,3}, \\
x_1\varepsilon &\mapsto e_{4,3}, \\
y_1 &\mapsto e_{1,2}, \\
y_2 &\mapsto e_{4,2}, \\
y_1\varepsilon &\mapsto e_{1,2,3}, \\
y_2\varepsilon &\mapsto e_{4,2,3}, \\
z &\mapsto e_{1,4,2}, \\
z\varepsilon &\mapsto e_{1,4,2,3},
\end{aligned}$$

is an isomorphism of algebras (cf. (3.19)) since the above and (3.20) gives us (writing $\times$ for the product in the algebra on the left)

$$\begin{aligned}
x_1 \times \varepsilon &= x_1\varepsilon & e_1 \cup e_3 &= e_{1,3}, \\
x_2 \times \varepsilon &= x_2\varepsilon & e_4 \cup e_3 &= e_{4,3},
\end{aligned}$$

$$
\begin{aligned}
x_1 \times y_1 &= z & e_1 \cup e_{4,2} &= e_{1,4,2}, \\
x_2 \times y_1 &= -z & e_4 \cup e_{1,2} &= -e_{1,4,2}, \\
x_1 \times y_1\varepsilon &= z\varepsilon & e_1 \cup e_{4,2,3} &= e_{1,4,2,3}, \\
x_2 \times y_1\varepsilon &= -z\varepsilon & e_4 \cup e_{1,2,3} &= -e_{1,4,2,3}, \\
\varepsilon \times y_1 = (-1)^{1\times 2} y_1 \times \varepsilon &= y_1\varepsilon & e_3 \cup e_{1,2} &= e_{1,2,3}, \\
\varepsilon \times y_2 = (-1)^{1\times 2} y_2 \times \varepsilon &= y_2\varepsilon & e_3 \cup e_{4,2} &= e_{4,2,3}, \\
\varepsilon \times z = (-1)^{1\times 3} z \times \varepsilon &= -z\varepsilon & e_3 \cup e_{1,4,2} &= -e_{1,4,2,3}, \\
y_1 \times x_2\varepsilon &= -z\varepsilon & e_{1,2} \cup e_{4,3} &= -e_{1,4,2,3}, \\
y_2 \times x_1\varepsilon &= z\varepsilon & e_{4,2} \cup e_{1,3} &= e_{1,4,2}. \qquad \triangle
\end{aligned}
$$

3.5 $I \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$

In this section we will describe the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{SL}_3(\mathbb{Q}_p)$.

When I is the pro-p Iwahori subgroup in $\mathrm{SL}_3(\mathbb{Q}_p)$, we know by Section 3.1 that we can take it to be of the form

$$
I = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}^{\det=1} \subseteq \mathrm{SL}_3(\mathbb{Z}_p),
$$

and, by Section 3.1, we have an ordered basis

$$
\begin{gathered}
g_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & & \\ p & 1 & \\ & & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix}, \\
g_4 = \begin{pmatrix} \exp(p) & & \\ & \exp(-p) & \\ & & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & & \\ & \exp(p) & \\ & & \exp(-p) \end{pmatrix}, \\
g_6 = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad g_7 = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}.
\end{gathered} \tag{3.21}
$$

Here we write any zeros as blank space in matrices, to make the notation easier to read for the bigger matrices.

3.5.1 Finding the commutators $[\xi_i, \xi_j]$

Now

$$g_1^{x_1} g_2^{x_2} g_3^{x_3} g_4^{x_4} g_5^{x_5} g_6^{x_6} g_7^{x_7} g_8^{x_8} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= \exp(px_4), \\ a_{12} &= x_7 \exp(px_4), \\ a_{13} &= x_8 \exp(px_4), \\ a_{21} &= px_2 \exp(px_4), \\ a_{22} &= px_2 x_7 \exp(px_4) + \exp\big(p(x_5 - x_4)\big), \\ a_{23} &= px_2 x_8 \exp(px_4) + x_6 \exp\big(p(x_5 - x_4)\big), \\ a_{31} &= px_1 \exp(px_4), \\ a_{32} &= px_1 x_7 \exp(px_4) + px_3 \exp\big(p(x_5 - x_4)\big), \\ a_{33} &= px_1 x_8 \exp(px_4) + px_3 x_6 \exp\big(p(x_5 - x_4)\big) + \exp(-px_5). \end{aligned} \tag{3.22}$$

Writing $g_{ij} = [g_i, g_j]$ and $\xi_{ij} = [\xi_i, \xi_j]$, we are now ready to find $x_1, \ldots, x_8$ such that $g_{ij} = g_1^{x_1} \cdots g_8^{x_8}$ for different $i < j$. (In the following we use that $\frac{1}{p-1} = 1 + p + p^2 + \cdots$ and $\log(1-p) = -p - \frac{p^2}{2} - \frac{p^3}{3} - \cdots$.) Also, except in the first case, we will note that $x_k \in p\mathbb{Z}_p$ implies that the coefficient on ξ_k in ξ_{ij} is zero.

We now list all non-identity commutators $g_{ij} = [g_i, g_j]$ and find $\xi_{ij} = [\xi_i, \xi_j]$ based on these. (For $g_{ij} = 1_3$ it is clear that $x_1 = \cdots = x_8 = 0$, and thus $\xi_{ij} = 0$.)

$g_{14} = \begin{pmatrix} 1 & & \\ & 1 & \\ p\big(1 - \exp(-p)\big) & & 1 \end{pmatrix}$**:** Comparing g_{14} with (3.22), we see that $x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{31} = px_1 = p\big(1 - \exp(-p)\big) = p^2 + O(p^3)$, which implies that $x_1 = p + O(p^2)$. Hence $\sigma(g_{14}) = \pi \,.\, \sigma(g_1)$, which implies that $\xi_{14} = 0$.

$g_{15} = \begin{pmatrix} 1 & & \\ & 1 & \\ p\big(1 - \exp(-p)\big) & & 1 \end{pmatrix}$**:** Since $g_{15} = g_{14}$, the above calculation shows that $\xi_{15} = 0$.

$g_{16} = \begin{pmatrix} 1 & & \\ -p & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{16} with (3.22), we see that $x_1 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{21} = px_2 = -p$, which implies that $x_2 = -1$. Hence $\sigma(g_{16}) = -\sigma(g_2)$, which implies that $\xi_{16} = -\xi_2$.

$g_{17} = \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix}$: Comparing g_{17} with (3.22), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_5 = x_6 = 0$. This leaves $a_{32} = px_3 = p$, which implies that $x_3 = 1$. Hence $\sigma(g_{17}) = \sigma(g_3)$, which implies that $\xi_{17} = \xi_3$.

$g_{18} = \begin{pmatrix} 1-p & & p \\ & 1 & \\ -p^2 & & 1+p+p^2 \end{pmatrix}$: Comparing g_{18} with (3.22), we see that $x_2 = x_7 = 0$, and thus also $x_3 = x_6 = 0$ and $x_4 = x_5$. Using

$$a_{11} = \exp(px_4) = 1-p,$$

$$a_{13} = x_8 \exp(px_4) = x_8(1-p) = p,$$

$$a_{31} = px_1 \exp(px_4) = px_1(1-p) = -p^2,$$

we get that

$$\begin{aligned} x_4 &= \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p), \\ x_8 &= \frac{p}{1-p} = p + O(p^2), \\ x_1 &= \frac{-p^2}{p(1-p)} = -p + O(p^2). \end{aligned}$$

Hence $\sigma(g_{18}) = -\pi\,.\,\sigma(g_1) - \sigma(g_4) - \sigma(g_5) + \pi\,.\,\sigma(g_8)$, which implies that $\xi_{18} = -(\xi_4 + \xi_5)$.

$g_{23} = \begin{pmatrix} 1 & & \\ & 1 & \\ -p^2 & & 1 \end{pmatrix}$: Comparing g_{23} with (3.22), we see that $x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{31} = px_1 = -p^2$, which implies that $x_1 = -p$. Hence $\sigma(g_{23}) = -\pi\,.\,\sigma(g_1)$, which implies that $\xi_{23} = 0$.

$g_{24} = \begin{pmatrix} 1 & & \\ p\big(1-\exp(-2p)\big) & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{24} with (3.22), we see that $x_1 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{21} = px_2 = p\big(1-\exp(-2p)\big) = p\big(1 - \big(1 + (-2p) + O(p^2)\big)\big) = 2p^2 + O(p^3)$, which implies that $x_2 = 2p + O(p^2)$. Hence $\sigma(g_{24}) = 2\pi\,.\,\sigma(g_1)$, which implies that $\xi_{24} = 0$.

$g_{25} = \begin{pmatrix} 1 & & \\ p\big(1-\exp(p)\big) & 1 & \\ & & 1 \end{pmatrix}$**:** Except a factor -2 in the exponential, which clearly does not change the final result, we have the same calculation as for g_{24}. Thus $\xi_{25} = 0$.

$g_{27} = \begin{pmatrix} 1-p & p & \\ -p^2 & 1+p+p^2 & \\ & & 1 \end{pmatrix}$**:** Comparing g_{27} with (3.22), we see that $x_1 = x_8 = 0$, and thus also $x_3 = x_6 = 0$, so $x_5 = 0$. Using

$$a_{11} = \exp(px_4) = 1-p,$$

$$a_{12} = x_7 \exp(px_4) = x_8(1-p) = p,$$

$$a_{21} = px_2 \exp(px_4) = px_2(1-p) = -p^2,$$

we get that

$$\begin{aligned} x_4 &= \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p), \\ x_7 &= \frac{p}{1-p} = p + O(p^2), \\ x_2 &= \frac{-p^2}{p(1-p)} = -p + O(p^2). \end{aligned}$$

Hence $\sigma(g_{27}) = -\pi \,.\, \sigma(g_2) - \sigma(g_4) + \pi \,.\, \sigma(g_7)$, which implies that $\xi_{27} = -\xi_4$.

$g_{28} = \begin{pmatrix} 1 & & \\ & 1 & p \\ & & 1 \end{pmatrix}$**:** Comparing g_{28} with (3.22), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = 0$. This leaves $a_{23} = x_6 = p$. Hence $\sigma(g_{28}) = \pi \,.\, \sigma(g_6)$, which implies that $\xi_{28} = 0$.

$g_{34} = \begin{pmatrix} 1 & & \\ & 1 & \\ & p\big(1-\exp(p)\big) & 1 \end{pmatrix}$**:** Comparing g_{34} with (3.22), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_5 = x_6 = 0$. This leaves $a_{32} = px_3 = p\big(1-\exp(p)\big) = p\big(1 - \big(1 + p + O(p^2)\big)\big) = -p^2 + O(p^3)$, which implies that $x_3 = -p + O(p^2)$. Hence $\sigma(g_{34}) = -\pi \,.\, \sigma(g_3)$, which implies that $\xi_{34} = 0$.

$g_{35} = \begin{pmatrix} 1 & & \\ & 1 & \\ & p\big(1-\exp(-2p)\big) & 1 \end{pmatrix}$**:** Except a factor -2 in the exponential, which clearly does not change the final result, we have the same calculation as for g_{34}. Thus $\xi_{35} = 0$.

$g_{36} = \begin{pmatrix} 1 & & \\ & 1-p & p \\ & -p^2 & 1+p+p^2 \end{pmatrix}$: Comparing g_{36} with (3.22), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$. Using

$$a_{22} = \exp(px_5) = 1 - p,$$

$$a_{23} = x_6 \exp(px_5) = x_6(1-p) = p,$$

$$a_{32} = px_3 \exp(px_5) = px_3(1-p) = -p^2,$$

we get that

$$x_5 = \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p),$$
$$x_6 = \frac{p}{1-p} = p + O(p^2),$$
$$x_3 = \frac{-p^2}{p(1-p)} = -p + O(p^2).$$

Hence $\sigma(g_{36}) = -\pi \,.\, \sigma(g_3) - \sigma(g_5) + \pi \,.\, \sigma(g_6)$, which implies that $\xi_{36} = -\xi_5$.

$g_{38} = \begin{pmatrix} 1 & -p & \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{38} with (3.22), we see that $x_1 = x_2 = x_4 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{12} = x_7 = -p$. Hence $\sigma(g_{38}) = -\pi \,.\, \sigma(g_3)$, which implies that $\xi_{38} = 0$.

$g_{46} = \begin{pmatrix} 1 & & \\ & 1 & \exp(-p)-1 \\ & & 1 \end{pmatrix}$: Comparing g_{46} with (3.22), we see that $x_1 = x_2 = x_4 = x_7 = x_8 = 0$, and thus also $x_3 = x_5 = 0$. This leaves $a_{23} = x_6 = \exp(-p) - 1 = -p + O(p^2)$. Hence $\sigma(g_{46}) = -\pi \,.\, \sigma(g_6)$, which implies that $\xi_{46} = 0$.

$g_{47} = \begin{pmatrix} 1 & \exp(2p)-1 & \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{47} with (3.22), we see that $x_1 = x_2 = x_4 = x_8 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{12} = x_7 = \exp(2p) - 1 = 2p + O(p^2)$. Hence $\sigma(g_{47}) = 2\pi \,.\, \sigma(g_7)$, which implies that $\xi_{47} = 0$.

$g_{48} = \begin{pmatrix} 1 & & \exp(p)-1 \\ & 1 & \\ & & 1 \end{pmatrix}$: Comparing g_{48} with (3.22), we see that $x_1 = x_2 = x_4 = x_7 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{13} = x_8 = \exp(p) - 1 = p + O(p^2)$. Hence $\sigma(g_{48}) = \pi \,.\, \sigma(g_8)$, which implies that $\xi_{48} = 0$.

$g_{56} = \begin{pmatrix} 1 & & \\ & 1 & \exp(2p)-1 \\ & & 1 \end{pmatrix}$**:** Except a factor -2 in the exponential, which clearly does not change the final result, we have the same calculation as for g_{46}. Thus $\xi_{56} = 0$.

$g_{57} = \begin{pmatrix} 1 & \exp(-p)-1 & \\ & 1 & \\ & & 1 \end{pmatrix}$**:** Except a factor -2 in the exponential, which clearly does not change the final result, we have the same calculation as for g_{47}. Thus $\xi_{57} = 0$.

$g_{58} = \begin{pmatrix} 1 & & \exp(p)-1 \\ & 1 & \\ & & 1 \end{pmatrix}$**:** Since $g_{58} = g_{48}$, the above calculation shows that $\xi_{58} = 0$.

$g_{67} = \begin{pmatrix} 1 & & -1 \\ & 1 & \\ & & 1 \end{pmatrix}$**:** Comparing g_{67} with (3.22), we see that $x_1 = x_2 = x_4 = x_7 = 0$, and thus also $x_3 = x_5 = x_6 = 0$. This leaves $a_{13} = x_8 = -1$. Hence $\sigma(g_{67}) = -\sigma(g_8)$, which implies that $\xi_{67} = -\xi_8$.

Thus the non-zero commutators $[\xi_i, \xi_j]$ with $i < j$ are:

$$\begin{aligned} &[\xi_1, \xi_6] = -\xi_2, \quad [\xi_1, \xi_7] = \xi_3, \qquad [\xi_1, \xi_8] = -(\xi_4 + \xi_5), \\ &[\xi_2, \xi_7] = -\xi_4, \quad [\xi_3, \xi_6] = -\xi_5, \quad [\xi_6, \xi_7] = -\xi_8. \end{aligned} \tag{3.23}$$

3.5.2 Describing the graded chain complex, $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$

Looking at (3.3) (with $e = 1$ and $h = 3$), we see that

$$\begin{aligned} &\omega(g_1) = 1 - \frac{2}{3} = \frac{1}{3}, && \omega(g_2) = 1 - \frac{1}{3} = \frac{2}{3}, && \omega(g_3) = 1 - \frac{1}{3} = \frac{2}{3}, \\ &\omega(g_4) = 1, && \omega(g_5) = 1, && \omega(g_6) = \frac{1}{3}, \\ &\omega(g_7) = \frac{1}{3}, && \omega(g_8) = \frac{2}{3}. \end{aligned}$$

Hence

$$\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, I = \mathrm{span}_k(\xi_1, \dots, \xi_8) = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3,$$

where

$$\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{3}} = \mathrm{span}_k(\xi_1, \xi_6, \xi_7),$$

$$\mathfrak{g}^2 = \mathfrak{g}_{\frac{2}{3}} = \mathrm{span}_k(\xi_2, \xi_3, \xi_8),$$

$$\mathfrak{g}^3 = \mathfrak{g}_1 = \operatorname{span}_k(\xi_4, \xi_5).$$

See Remark 3.9 for more details.

Now we are ready to describe the graded chain complex

$$\operatorname{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = \bigoplus_{j_1+\cdots+j_n=j} \mathfrak{g}^{j_1} \wedge \cdots \wedge \mathfrak{g}^{j_n},$$

but we will skip the description of the bases this time. For a description of the basis, we refer to the supplemental files of [Kon22]. We list the grading of $\bigwedge^n \mathfrak{g}$ for all n.

$n = 0:$

$$\operatorname{gr}^j(k) = \begin{cases} k & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 1:$

$$\operatorname{gr}^j(\mathfrak{g}) = \begin{cases} \mathfrak{g}^3 & j = 3, \\ \mathfrak{g}^2 & j = 2, \\ \mathfrak{g}^1 & j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 2:$

$$\operatorname{gr}^j\Big(\bigwedge^2 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 6, \\ \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 5, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^3 \oplus \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 4, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 3, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 3:$

$$\operatorname{gr}^j\Big(\bigwedge^3 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 8, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 7, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 6, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 5, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 4, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 & j = 3, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 4:$

$$\operatorname{gr}^j\Big(\bigwedge^4 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 10, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 9, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 8, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 7, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 6, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 & j = 5, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 5:$

$$\mathrm{gr}^j\Big(\bigwedge^5 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 12, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 11, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 10, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 9, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 8, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 7, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 6:$

$$\mathrm{gr}^j\Big(\bigwedge^6 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 13, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 12, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & \\ \quad \oplus \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 11, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 10, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 & j = 9, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 7:$

$$\mathrm{gr}^j\Big(\bigwedge^7 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 14, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 13, \\ \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 & j = 12, \\ 0 & \text{otherwise.} \end{cases}$$

$n = 8:$

$$\mathrm{gr}^j\Big(\bigwedge^8 \mathfrak{g}\Big) = \begin{cases} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 & j = 15, \\ 0 & \text{otherwise.} \end{cases}$$

$n \geq 9:$

$$\mathrm{gr}^j\Big(\bigwedge^n \mathfrak{g}\Big) = 0 \text{ for all } j.$$

Table 3.5: Dimensions of $\mathrm{gr}^j\big(\bigwedge^n \mathfrak{g}\big)$.

$n \backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1		3	3	2												
2			3	9	9	6	1									
3				1	9	15	19	9	3							
4						3	11	21	21	11	3					
5								3	9	19	15	9	1			
6										1	6	9	9	3		
7													2	3	3	
8																1

We collect the above information about the dimensions of the chain complex of $\mathfrak{g}$ in Table 3.5, and note that we only need to consider non-zero (non-empty) entries of the table, when we calculate $H^{s,t} = H^{s,n-s}$ (where $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$). Also, recalling that

$$\mathrm{Hom}_k\Big(\bigwedge^n \mathfrak{g}, k\Big) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s\Big(\bigwedge^n \mathfrak{g}, k\Big),$$

we see that, with $j = -s$, we get the same table for dimensions of the graded hom-spaces in the cochain complex.

3.5.3 Finding the graded Lie algebra cohomology, $H^{s,t}(\mathfrak{g}, k)$

We will now go through all different graded chain complexes one by one, using that gr^j in the chain complex corresponds to gr^s with $s = -j$ in the cochain complex. We note that the graded chain complex corresponds to vertical downwards arrows in Table 3.5, while the cochain complex corresponds to vertical upwards arrows. And finally, we reiterate that $H^n = H^n(\mathfrak{g}, k)$ and $H^{s,t} = H^{s,t}(\mathfrak{g}, k)$ in the following.

Remark 3.21. We will repeatedly use that, if

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{n \times m}(a_1, \ldots, a_r, 0, \ldots, 0)$$

with $a_1, \ldots, a_r$ non-zero (in $\mathbb{F}_p$), then

$$\dim \ker(d) = m - r,$$

$$\dim \operatorname{im}(d) = r,$$

$$\dim \operatorname{coker}(d) = n - r,$$

as described in Section 3.1.3. △

In grade 0 we have the chain complex

$$0 \longrightarrow k \longrightarrow 0$$

which gives us the grade 0 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^0(k,k) \longleftarrow 0$$

So $H^0 = H^{0,0}$ with $\dim H^{0,0} = 1$.

In grade 1 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \longrightarrow 0$$

which gives us the grade -1 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-1}(\mathfrak{g},k) \longleftarrow 0$$

So $\dim H^{-1,2} = 3$ by Table 3.5.

In grade 2 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathfrak{g}^2 \longrightarrow 0$$

since

$$\mathfrak{g}^1 \wedge \mathfrak{g}^1 \to \mathfrak{g}^2$$

$$\xi_1 \wedge \xi_6 \mapsto -[\xi_1, \xi_6] = \xi_2$$

$$\xi_1 \wedge \xi_7 \mapsto -[\xi_1, \xi_7] = -\xi_3$$

$$\xi_6 \wedge \xi_7 \mapsto -[\xi_6, \xi_7] = \xi_8.$$

This gives us the grade -2 cochain complex

$$0 \longleftarrow \operatorname{Hom}_k^{-2}\left(\bigwedge^2 \mathfrak{g}, k\right) \xleftarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \operatorname{Hom}_k^{-2}(\mathfrak{g}, k) \longleftarrow 0,$$

where

$$d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{3\times 3}(1,1,1).$$

So

$$\dim H^{-2,3} = \dim \ker(d) = 0,$$

$$\dim H^{-2,4} = \dim \operatorname{coker}(d) = 0.$$

In grade 3 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \xrightarrow{\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}^\top} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow[\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}]{} \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -3 cochain complex

$$0 \leftarrow \operatorname{Hom}_k^{-3}\left(\bigwedge^3 \mathfrak{g}, k\right) \xleftarrow{\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}} \operatorname{Hom}_k^{-3}\left(\bigwedge^2 \mathfrak{g}, k\right) \xleftarrow[\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}^\top]{} \operatorname{Hom}_k^{-3}(\mathfrak{g}, k) \leftarrow 0,$$

where

$$d_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 2}(1,1),$$

$$d_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{1\times 9}(1).$$

So

$$\begin{aligned} \dim H^{-3,4} &= \dim \ker(d_1) = 2 - 2 = 0, \\ \dim H^{-3,5} &= \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (9-1) - 2 = 6, \\ \dim H^{-3,6} &= \dim \mathrm{coker}(d_2) = 1 - 1 = 0. \end{aligned}$$

In grade 4 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow{d^\top} \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} \longrightarrow 0$$

which gives us the grade -4 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-4}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \xleftarrow{d} \mathrm{Hom}_k^{-4}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \longleftarrow 0$$

where

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 9}(1,1,1,1,1,1,0,0,0).$$

So

$$\begin{aligned} \dim H^{-4,6} &= \dim \ker(d) = 9 - 6 = 3, \\ \dim H^{-4,7} &= \dim \mathrm{coker}(d) = 9 - 6 = 3. \end{aligned}$$

In grade 5 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \xrightarrow{d_2^\top} \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} \xrightarrow{d_1^\top} \mathfrak{g}^2 \wedge \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -5 cochain complex

$$0 \leftarrow \operatorname{Hom}_k^{-5}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \xleftarrow{d_2} \operatorname{Hom}_k^{-5}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \xleftarrow{d_1} \operatorname{Hom}_k^{-5}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \leftarrow 0,$$

where

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{15\times 6}(1,1,1,1,1,1), \\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{3\times 15}(1,1,1). \end{aligned}$$

So

$$\begin{aligned} \dim H^{-5,7} &= \dim \ker(d_1) = 6 - 6 = 0, \\ \dim H^{-5,8} &= \dim \frac{\ker(d_2)}{\operatorname{im}(d_1)} = (15-3) - 6 = 6, \\ \dim H^{-5,9} &= \dim \operatorname{coker}(d_2) = 3 - 3 = 0. \end{aligned}$$

In grade 6 we have the chain complex

$$0 \longrightarrow \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} \xrightarrow{d_2^\top} \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} \xrightarrow{d_1^\top} \mathfrak{g}^3 \wedge \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -6 cochain complex

$$0 \leftarrow \operatorname{Hom}_k^{-6}\big(\textstyle\bigwedge^4 \mathfrak{g}, k\big) \xleftarrow{d_2} \operatorname{Hom}_k^{-6}\big(\textstyle\bigwedge^3 \mathfrak{g}, k\big) \xleftarrow{d_1} \operatorname{Hom}_k^{-6}\big(\textstyle\bigwedge^2 \mathfrak{g}, k\big) \leftarrow 0,$$

where

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{19\times 1}(1), \\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{11\times 19}(1,1,1,1,1,1,1,1,1,1,2). \end{aligned}$$

So

$$\begin{aligned} \dim H^{-6,8} &= \dim \ker(d_1) = 1 - 1 = 0, \\ \dim H^{-6,9} &= \dim \frac{\ker(d_2)}{\operatorname{im}(d_1)} = (19-11) - 1 = 7, \\ \dim H^{-6,10} &= \dim \operatorname{coker}(d_2) = 11 - 11 = 0. \end{aligned}$$

In grade 7 we have the chain complex

$$0 \to \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \overset{d_2^\top}{\to} \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \end{matrix} \overset{d_1^\top}{\to} \begin{matrix} \mathfrak{g}^1 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \\ \oplus\, \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \end{matrix} \to 0$$

which gives us the grade -7 cochain complex

$$0 \leftarrow \operatorname{Hom}_k^{-7}\left(\bigwedge^5 \mathfrak{g}, k\right) \overset{d_2}{\leftarrow} \operatorname{Hom}_k^{-7}\left(\bigwedge^4 \mathfrak{g}, k\right) \overset{d_1}{\leftarrow} \operatorname{Hom}_k^{-7}\left(\bigwedge^3 \mathfrak{g}, k\right) \leftarrow 0,$$

where

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{21\times 9}(1,1,1,1,1,1,1,1,1), \\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{3\times 21}(1,1,1). \end{aligned}$$

So

$$\begin{aligned} \dim H^{-7,10} &= \dim \ker(d_1) = 9 - 9 = 0, \\ \dim H^{-7,11} &= \dim \frac{\ker(d_2)}{\operatorname{im}(d_1)} = (21-3) - 9 = 9, \\ \dim H^{-7,12} &= \dim \operatorname{coker}(d_2) = 3 - 3 = 0. \end{aligned}$$

By [Fuk86, Chap 1 §3.6 and §3.7], we can now find the rest of the cohomology using a version of Poincaré duality for Lie algebra cohomology. But we keep the sketch work to make it clear that this works. We refer to [Kon22] for the calculations.

In grade -8 we get coboundary maps

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{21\times 3}(1,1,1), \\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 21}(1,1,1,1,1,1,1,1,1). \end{aligned}$$

So

$$\begin{aligned} \dim H^{-8,11} &= \dim \ker(d_1) = 3 - 3 = 0, \\ \dim H^{-8,12} &= \dim \frac{\ker(d_2)}{\operatorname{im}(d_1)} = (21-9) - 3 = 9, \\ \dim H^{-8,13} &= \dim \operatorname{coker}(d_2) = 9 - 9 = 0. \end{aligned}$$

In grade -9 we get coboundary maps

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{19\times 11}(1,1,1,1,1,1,1,1,1,1,1),\\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{1\times 19}(1). \end{aligned}$$

So

$$\begin{aligned} \dim H^{-9,13} &= \dim \ker(d_1) = 11 - 11 = 0,\\ \dim H^{-9,14} &= \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (19-1) - 11 = 7,\\ \dim H^{-9,15} &= \dim \mathrm{coker}(d_2) = 1 - 1 = 0. \end{aligned}$$

In grade -10 we get coboundary maps

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{15\times 3}(1,1,1),\\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{6\times 15}(1,1,1,1,1,1). \end{aligned}$$

So

$$\begin{aligned} \dim H^{-10,14} &= \dim \ker(d_1) = 3 - 3 = 0,\\ \dim H^{-10,15} &= \dim \frac{\ker(d_2)}{\mathrm{im}(d_1)} = (15-6) - 3 = 6,\\ \dim H^{-10,16} &= \dim \mathrm{coker}(d_2) = 6 - 6 = 0. \end{aligned}$$

In grade -11 we get coboundary maps

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 9}(1,1,1,1,1,1,0,0,0).$$

So

$$\begin{aligned} \dim H^{-11,16} &= \dim \ker(d) = 9 - 6 = 3,\\ \dim H^{-11,17} &= \dim \mathrm{coker}(d) = 9 - 6 = 3. \end{aligned}$$

In grade -12 we get coboundary maps

$$\begin{aligned} d_1 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{9\times 1}(1),\\ d_2 &\overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{2\times 9}(1,1). \end{aligned}$$

So

$$\dim H^{-12,17} = \dim \ker(d_1) = 1 - 1 = 0,$$

$$\dim H^{-12,18} = \dim \frac{\ker(d_2)}{\operatorname{im}(d_1)} = (9-2) - 1 = 6,$$

$$\dim H^{-12,19} = \dim \operatorname{coker}(d_2) = 2 - 2 = 0.$$

In grade -13 we get coboundary maps

$$d \overset{\mathsf{SNF}}{\sim} \mathrm{SNF}_{3\times 3}(1,1,1).$$

So

$$\dim H^{-13,19} = \dim \ker(d) = 3 - 3 = 0,$$

$$\dim H^{-13,20} = \dim \operatorname{coker}(d) = 3 - 3 = 0.$$

In grade 14 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -14 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-14}\big(\textstyle\bigwedge^7 \mathfrak{g}, k\big) \longleftarrow 0$$

So $\dim H^{-14,21} = 3$ by Table 3.5.

In grade 15 we have the chain complex

$$0 \longrightarrow \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^1 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^2 \wedge \mathfrak{g}^3 \wedge \mathfrak{g}^3 \longrightarrow 0$$

which gives us the grade -15 cochain complex

$$0 \longleftarrow \mathrm{Hom}_k^{-15}\big(\textstyle\bigwedge^8 \mathfrak{g}, k\big) \longleftarrow 0$$

So $H^8 = H^{-15,23}$ with $\dim H^{-15,23} = 1$ by Table 3.5.

Table 3.6: Dimensions of $E_1^{s,t} = H^{s,t} = \mathrm{gr}^s\, H^{s+t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$ case.

$t \backslash s$	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15
0	1															
1																
2		3														
3																
4																
5				6												
6					3											
7					3											
8						6										
9							7									
10																
11								9								
12									9							
13																
14										7						
15											6					
16												3				
17												3				
18													6			
19																
20																
21															3	
22																
23																1

Altogether, we see that

$$\begin{aligned}
H^0 &= H^{0,0}, \\
H^1 &= H^{-1,2}, \\
H^2 &= H^{-3,5} \oplus H^{-4,6}, \\
H^3 &= H^{-4,7} \oplus H^{-5,8} \oplus H^{-6,9}, \\
H^4 &= H^{-7,11} \oplus H^{-8,12}, \\
H^5 &= H^{-9,14} \oplus H^{-10,15} \oplus H^{-11,16}, \\
H^6 &= H^{-11,17} \oplus H^{-12,18}, \\
H^7 &= H^{-14,21}, \\
H^8 &= H^{-15,23}
\end{aligned} \tag{3.24}$$

with dimension as described in Table 3.6.

3.5.4 Describing the group cohomology, $H^n(I,k)$

We note that all differentials $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ in Table 3.6 has bidegree $(r, 1-r)$, i.e., they are all below the $(r,-r)$ arrow going r to the left and r up in the table, where $r \geq 1$. Looking at Table 3.6, this clearly means that all differentials for $r \geq 1$ are trivial, and thus the spectral sequence collapses on the first page. Hence $H^{s,t}(\mathfrak{g},k) = E_1^{s,t} \cong E_\infty^{s,t} = \operatorname{gr}^s H^{s+t}(I,k)$, and by (3.24) and Table 3.6 we get that

$$\dim H^n(I,k) = \begin{cases} 1 & n = 0, \\ 3 & n = 1, \\ 9 & n = 2, \\ 16 & n = 3, \\ 18 & n = 4, \\ 16 & n = 5, \\ 9 & n = 6, \\ 3 & n = 7, \\ 1 & n = 8. \end{cases} \tag{3.25}$$

Recalling that the spectral sequence is multiplicative, we also note, by Table 3.6, that $H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'}$ implies that the cup products

$$\operatorname{gr}^s H^n(I,k) \otimes \operatorname{gr}^{s'} H^{n'}(I,k) \to \operatorname{gr}^{s+s'} H^{n+n'}(I,k)$$

are trivial. But, since the spectral sequence collapses on the first page, we also have (3.24) for $H^n(I,k)$, and thus the cup product is trivial.

3.6 $I \subseteq \mathrm{GL}_3(\mathbb{Z}_p)$

In this section we will describe the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{GL}_3(\mathbb{Q}_p)$.

When I is the pro-p Iwahori subgroup in $\mathrm{GL}_3(\mathbb{Q}_p)$, we know by Section 3.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \subseteq \mathrm{GL}_3(\mathbb{Z}_p),$$

and, by Section 3.1, we have an ordered basis

$$\begin{gathered} g_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ p & & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & & \\ p & 1 & \\ & & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & p & 1 \end{pmatrix}, \\ g_4 = \begin{pmatrix} \exp(p) & & \\ & \exp(-p) & \\ & & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & & \\ & \exp(p) & \\ & & \exp(-p) \end{pmatrix}, \\ g_6 = \begin{pmatrix} \exp(p) & & \\ & \exp(p) & \\ & & \exp(p) \end{pmatrix}, \\ g_7 = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad g_9 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}. \end{gathered} \tag{3.26}$$

Since we just renamed some elements and added an element of the center of $\mathrm{GL}_3(\mathbb{Z}_p)$ when comparing to the ordered basis of $I \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$ from Section 3.5, it is clear from Equation (3.23) that the only non-zero commutators $[\xi_i, \xi_j]$ with $i < j$ are:

$$\begin{aligned} &[\xi_1, \xi_7] = -\xi_2, \quad [\xi_1, \xi_8] = \xi_3, \quad [\xi_1, \xi_9] = -(\xi_4 + \xi_5), \\ &[\xi_2, \xi_8] = -\xi_4, \quad [\xi_3, \xi_7] = -\xi_5, \quad [\xi_7, \xi_8] = -\xi_9. \end{aligned} \tag{3.27}$$

Looking at Section 3.5.3, we easily see that

$$\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{3}} = \operatorname{span}_k(\xi_1, \xi_7, \xi_8),$$

$$\mathfrak{g}^2 = \mathfrak{g}_{\frac{2}{3}} = \operatorname{span}_k(\xi_2, \xi_3, \xi_9),$$

$$\mathfrak{g}^3 = \mathfrak{g}_1 = \operatorname{span}_k(\xi_4, \xi_5, \xi_6).$$

This is enough to calculate the graded mod p cohomology of $\mathfrak{g}$, see [Kon22] for the details. We write the result in Table 3.7.

Table 3.7: Dimensions of $E_1^{s,t} = H^{s,t} = \operatorname{gr}^s H^{s+t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{GL}_3(\mathbb{Z}_p)$ case.

t \ s	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18
0	1																		
1																			
2		3																	
3																			
4				1															
5				6															
6					6														
7					3														
8						6													
9							13												
10								3											
11								12											
12									15										
13										7									
14										7									
15											15								
16												12							
17												3							
18													13						
19														6					
20															3				
21															6				
22																6			
23																1			
24																			
25																		3	
26																			
27																			1

3.7 $I \subseteq \mathrm{SL}_4(\mathbb{Z}_p)$ and $I \subseteq \mathrm{GL}_4(\mathbb{Z}_p)$

In this section we will briefly describe the problem with finding continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{SL}_4(\mathbb{Q}_p)$ and $\mathrm{GL}_4(\mathbb{Q}_p)$.

We leave all the calculations of commutators and p-valuations to the appendix, cf. Appendix A.1 and Appendix A.2, and note that the dimensions of the graded cohomology $H^{s,t}(\mathfrak{g}, k)$ for $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I$ are shown in Table 3.8 and Table 3.9.

Looking at Table 3.8, we see that it is no longer clear that the spectral sequence collapses on the first page. To see this, recall that all differentials on the first page are of the form $d_1^{s,t}\colon E_1^{s,t} \to E_1^{s+1,t}$, so we have maps like $d_1^{-5,7}\colon H^{-5,7}(\mathfrak{g}, k) \to H^{-4,7}(\mathfrak{g}, k)$ that are not obviously trivial, since $\dim_k H^{-5,7} = 4$ and $\dim_k H^{-4,7} = 4$. To figure out at what page the spectral sequence collapses in this case, one needs to look more carefully at how exactly the spectral sequence is obtained in [Sør21], which is much more complicated than what we have done so far.

3.8 $I \subseteq \mathrm{SL}_2(\mathcal{O}_F)$ for quadratic extensions $F/\mathbb{Q}_p$

In this section we will describe the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{SL}_2(F)$ for quadratic extensions $F/\mathbb{Q}_p$.

We write $F = \mathbb{Q}_p(\alpha)$, and we will focus on the cases $\alpha = i$ (when $p \equiv 3 \pmod 4$) and $\alpha = \sqrt{p}$.

When I is the pro-p Iwahori subgroup in $\mathrm{SL}_2(F)$, we know by Section 3.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1 + \varpi_F \mathcal{O}_F & \mathcal{O}_F \\ \varpi_F \mathcal{O}_F & 1 + \varpi_F \mathcal{O}_F \end{pmatrix}^{\det = 1} \subseteq \mathrm{SL}_2(\mathcal{O}_F),$$

where $\varpi_F = p$ when $F = \mathbb{Q}_p(i)$ and $\varpi_F = \sqrt{p}$ when $F = \mathbb{Q}_p(\sqrt{p})$. By Section 3.1, we have an ordered basis

$$\begin{gathered} g_1 = \begin{pmatrix} 1 & 0 \\ \varpi_F & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 \\ \varpi_F \alpha & 1 \end{pmatrix}, \\ g_3 = \begin{pmatrix} \exp(\varpi_F) & 0 \\ 0 & \exp(-\varpi_F) \end{pmatrix}, \quad g_4 = \begin{pmatrix} \exp(\varpi_F \alpha) & 0 \\ 0 & \exp(-\varpi_F \alpha) \end{pmatrix}, \\ g_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix}, \end{gathered} \tag{3.28}$$

since $1, \alpha$ is a $\mathbb{Z}_p$-basis of $\mathcal{O}_F$.

Table 3.8: Dimensions of $E_1^{s,t} = H^{s,t} = \operatorname{gr}^s H^{s+t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{SL}_4(\mathbb{Z}_p)$ case. This table only shows the graded dimensions of $H^0, \ldots, H^7$. We note that the graded dimensions of $H^8, \ldots, H^{15}$ can be found using Poincaré duality, which gives us $\dim_k H^{s,t} = \dim_k H^{-36-s,51-t}$ for (s,t) from the table.

t \ s	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18	−19
0	1																			
1																				
2		4																		
3																				
4			2																	
5				8																
6																				
7					4	4														
8						8														
9							20													
10								8												
11								20	1											
12									34											
13										16										
14										12	18									
15											26									
16											4	76								
17													39							
18													28	8						
19														68						
20															72					
21															12	68				
22																24	8			
23																	121			
24																		80		
25																			54	
26																				12

3.8.1 Finding the commutators $[\xi_i, \xi_j]$

Now

$$g_1^{x_1} g_2^{x_2} g_3^{x_3} g_4^{x_4} g_5^{x_5} g_6^{x_6} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Table 3.9: Dimensions of $E_1^{s,t} = H^{s,t} = \operatorname{gr}^s H^{s+t}(\mathfrak{g}, k)$ for the $I \subseteq \mathrm{GL}_4(\mathbb{Z}_p)$ case. This table only shows the graded dimensions of $H^0, \dots, H^7$. We note that the graded dimensions of $H^8, \dots, H^{15}$ can be found using Poincaré duality, which gives us $\dim_k H^{s,t} = \dim_k H^{-40-s,56-t}$ for (s,t) from the table.

$t \backslash s$	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18	−19	−20	−21	−22	−23
0	1																							
1																								
2		4																						
3																								
4			2																					
5				8	1																			
6																								
7					4	8																		
8						8																		
9							22																	
10								16																
11								20	1															
12									38	4														
13										24														
14										12	38													
15											26	8												
16											4	96	1											
17													73											
18													28	24										
19														80	18									
20															98									
21															16	144								
22																24	47							
23																	149	8						
24																		148						
25																		12	126					
26																			66	80				
27																				104	8			
28																					242			
29																						104		
30																							66	
31																								12

where

$$\begin{aligned} a_{11} &= \exp\bigl(\varpi_F(x_3 + \alpha x_4)\bigr), \\ a_{12} &= (x_5 + \alpha x_6)\exp\bigl(\varpi_F(x_3 + \alpha x_4)\bigr), \\ a_{21} &= \varpi_F(x_1 + \alpha x_2)\exp\bigl(\varpi_F(x_3 + \alpha x_4)\bigr), \\ a_{22} &= \varpi_F(x_1 + \alpha x_2)(x_5 + \alpha x_6)\exp\bigl(\varpi_F(x_3 + \alpha x_4)\bigr) + \exp\bigl(-\varpi_F(x_3 + \alpha x_4)\bigr). \end{aligned} \tag{3.29}$$

Writing $g_{ij} = [g_i, g_j]$ and $\xi_{ij} = [\xi_i, \xi_j]$, we are now ready to find $x_1, \dots, x_8$ such that $g_{ij} = g_1^{x_1} \cdots g_6^{x_6}$ for different $i < j$. (In the following we use that $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ and $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ for $x \in (\varpi_F)$.)

We now list all non-identity commutators $g_{ij} = [g_i, g_j]$ and find $\xi_{ij} = [\xi_i, \xi_j]$ based on these. (For $g_{ij} = 1_2$ it is clear that $x_1 = \cdots = x_6 = 0$, and thus $\xi_{ij} = 0$.) To avoid confusion, we will do the $F = \mathbb{Q}_p(i)$ case (with $p \equiv 3 \pmod 4$) first, and then the $F = \mathbb{Q}_p(\sqrt{p})$ case afterwards. So for now $\varpi_F = p$ and $\alpha = i$. Also, note that we adapt the $O(p)$ notation to $\mathcal{O}_F$ in the obvious way.

$g_{13} = \begin{pmatrix} 1 & 0 \\ p\big(1-\exp(-2p)\big) & 1 \end{pmatrix}$: Comparing g_{13} with (3.29), we see that $x_3 = x_4 = x_5 = x_6 = 0$. This leaves $a_{21} = p(x_1 + ix_2) = p\big(1 - \exp(-2p)\big) = 2p^2 + O(p^3)$, which implies that $x_1, x_2 \in p\mathbb{Z}_p$. Hence $\xi_{13} = 0$.

$g_{14} = \begin{pmatrix} 1 & 0 \\ p\big(1-\exp(-2pi)\big) & 1 \end{pmatrix}$: Comparing g_{14} with (3.29), we see that $x_3 = x_4 = x_5 = x_6 = 0$. This leaves $a_{21} = p(x_1 + ix_2) = p\big(1 - \exp(-2pi)\big) = 2p^2 i + O(p^3)$, which implies that $x_1, x_2 \in p\mathbb{Z}_p$. Hence $\xi_{14} = 0$.

$g_{15} = \begin{pmatrix} 1-p & p \\ -p^2 & 1+p+p^2 \end{pmatrix}$: Comparing g_{15} with (3.29), we see that

$$\begin{aligned}
a_{11} &= \exp\big(p(x_3 + ix_4)\big) = 1 - p, \\
a_{12} &= (x_5 + ix_6)\exp\big(p(x_3 + ix_4)\big) = (x_5 + ix_6)(1-p) = p, \\
a_{21} &= p(x_1 + ix_2)\exp\big(p(x_3 + ix_4)\big) = p(x_1 + ix_2)(1-p) = -p^2,
\end{aligned}$$

and thus

$$\begin{aligned}
x_3 + ix_4 &= \frac{1}{p}\log(1-p) = \frac{1}{p}\big((-p) + O(p^2)\big) = -1 + O(p), \\
x_5 + ix_6 &= \frac{p}{1-p} = p + O(p^2), \\
x_1 + ix_2 &= \frac{-p^2}{p(1-p)} = -p + O(p^2).
\end{aligned}$$

Hence $x_1, x_2, x_4, x_5, x_6 \in p\mathbb{Z}_p$ and $x_3 \in -1 + p\mathbb{Z}_p$, which implies that $\xi_{15} = -\xi_3$.

$g_{16} = \begin{pmatrix} 1-pi & pi^2 \\ -p^2 i & 1+pi+p^2i^2 \end{pmatrix}$: Comparing g_{16} with (3.29), we see that

$$\begin{aligned}
a_{11} &= \exp\big(p(x_3 + ix_4)\big) = 1 - pi, \\
a_{12} &= (x_5 + ix_6)\exp\big(p(x_3 + ix_4)\big) = (x_5 + ix_6)(1-pi) = -p, \\
a_{21} &= p(x_1 + ix_2)\exp\big(p(x_3 + ix_4)\big) = p(x_1 + ix_2)(1-pi) = -p^2 i,
\end{aligned}$$

and thus

$$x_3 + ix_4 = \frac{1}{p}\log(1-pi) = \frac{1}{p}\big((-pi) + O(p^2)\big) = -i + O(p),$$
$$x_5 + ix_6 = \frac{-p}{1-p} = -p + O(p^2),$$
$$x_1 + ix_2 = \frac{-p^2 i}{p(1-pi)} = -pi + O(p^2).$$

Hence $x_1, x_2, x_3, x_5, x_6 \in p\mathbb{Z}_p$ and $x_4 \in -1 + p\mathbb{Z}_p$, which implies that $\xi_{16} = -\xi_4$.

$g_{23} = \begin{pmatrix} 1 & 0 \\ pi\big(1-\exp(-2p)\big) & 1 \end{pmatrix}$: Comparing g_{23} with g_{13}, it is not hard to see that $\xi_{23} = 0$.

$g_{24} = \begin{pmatrix} 1 & 0 \\ pi\big(1-\exp(-2pi)\big) & 1 \end{pmatrix}$: Comparing g_{24} with g_{14}, it is not hard to see that $\xi_{24} = 0$.

$g_{25} = \begin{pmatrix} 1-pi & pi \\ -p^2i^2 & 1+pi+p^2i^2 \end{pmatrix}$: Comparing g_{25} with g_{16}, it is not hard to see that $\xi_{25} = -\xi_4$.

$g_{26} = \begin{pmatrix} 1-pi^2 & pi^3 \\ -p^2i^3 & 1+pi^2+p^2i^4 \end{pmatrix}$: Comparing g_{26} with g_{15} (noting that $i^2 = -1$, so $1 - pi^2 = 1 + p$), it is not hard to see that $\xi_{25} = \xi_3$.

$g_{35} = \begin{pmatrix} 1 & \exp(2p)-1 \\ 0 & 1 \end{pmatrix}$: Comparing g_{35} with (3.29), we see that $x_1 = x_2 = x_3 = x_4 = 0$. This leaves $a_{12} = x_5 + ix_6 = \exp(2p) - 1 = 2p + O(p^2)$, which implies that $x_5, x_6 \in p\mathbb{Z}_p$. Hence $\xi_{35} = 0$.

$g_{36} = \begin{pmatrix} 1 & i\big(\exp(2p)-1\big) \\ 0 & 1 \end{pmatrix}$: Comparing g_{36} with g_{35}, it is not hard to see that $\xi_{36} = 0$.

$g_{45} = \begin{pmatrix} 1 & \exp(2pi)-1 \\ 0 & 1 \end{pmatrix}$: Comparing g_{45} with (3.29), we see that $x_1 = x_2 x_3 = x_4 = 0$. This leaves $a_{12} = x_5 + ix_6 = \exp(2pi) - 1 = 2pi + O(p^2)$, which implies that $x_5, x_6 \in p\mathbb{Z}_p$. Hence $\xi_{45} = 0$.

$g_{46} = \begin{pmatrix} 1 & i\big(\exp(2pi)-1\big) \\ 0 & 1 \end{pmatrix}$: Comparing g_{46} with g_{45}, it is not hard to see that $\xi_{45} = 0$.

Let's now do the calculations for $F = \mathbb{Q}_p(\sqrt{p})$, and note that we again adapt the $O(p)$ notation in the obvious way. Now $\varpi_F = \sqrt{p}$ and $\alpha = \sqrt{p}$.

$g_{13} = \begin{pmatrix} 1 & 0 \\ \sqrt{p}\big(1-\exp(-2\sqrt{p})\big) & 1 \end{pmatrix}$: Comparing g_{13} with (3.29), we see that $x_3 = x_4 = x_5 = x_6 = 0$. This leaves $a_{21} = \sqrt{p}(x_1 + \sqrt{p}x_2) = \sqrt{p}\big(1 - \exp(-2\sqrt{p})\big) = 2p + O(p^{3/2})$, which implies that $x_1 \in p\mathbb{Z}_p$ and $x_2 \in 2 + p\mathbb{Z}_p$. Hence $\xi_{13} = 2\xi_2$.

$g_{14} = \begin{pmatrix} 1 & 0 \\ \sqrt{p}\big(1-\exp(-2p)\big) & 1 \end{pmatrix}$**:** Comparing g_{14} with (3.29), we see that $x_3 = x_4 = x_5 = x_6 = 0$. This leaves $a_{21} = \sqrt{p}(x_1 + \sqrt{p}x_2) = \sqrt{p}\big(1-\exp(-2p)\big) = 2p\sqrt{p} + O(p^2)$, which implies that $x_1, x_2 \in p\mathbb{Z}_p$. Hence $\xi_{14} = 0$.

$g_{15} = \begin{pmatrix} 1-\sqrt{p} & \sqrt{p} \\ -p & 1+\sqrt{p}+p \end{pmatrix}$**:** Comparing g_{15} with (3.29), we see that

$$
\begin{aligned}
a_{11} &= \exp\big(\sqrt{p}(x_3 + \sqrt{p}x_4)\big) = 1-\sqrt{p}, \\
a_{12} &= (x_5 + \sqrt{p}x_6)\exp\big(\sqrt{p}(x_3 + \sqrt{p}x_4)\big) = (x_5 + \sqrt{p}x_6)(1-\sqrt{p}) = \sqrt{p}, \\
a_{21} &= \sqrt{p}(x_1 + \sqrt{p}x_2)\exp\big(\sqrt{p}(x_3 + \sqrt{p}x_4)\big) = \sqrt{p}(x_1 + \sqrt{p}x_2)(1-\sqrt{p}) = -p,
\end{aligned}
$$

and thus

$$
\begin{aligned}
x_3 + \sqrt{p}x_4 &= \frac{1}{\sqrt{p}}\log(1-\sqrt{p}) = \frac{1}{\sqrt{p}}\Big((-\sqrt{p}) - \frac{p}{2} + O(p^{3/2})\Big) = -1 - \frac{\sqrt{p}}{2} + O(p), \\
x_5 + \sqrt{p}x_6 &= \frac{\sqrt{p}}{1-\sqrt{p}} = \sqrt{p} + O(p), \\
x_1 + \sqrt{p}x_2 &= \frac{-p}{\sqrt{p}(1-\sqrt{p})} = -\sqrt{p} + O(p).
\end{aligned}
$$

Hence $x_1, x_5 \in p\mathbb{Z}_p$, $x_2, x_3, x_4 \in -1 + p\mathbb{Z}_p$ and $x_6 \in 1 + p\mathbb{Z}_p$, which implies that $\xi_{15} = -\xi_2 - \xi_3 - \frac{1}{2}\xi_4 + \xi_6$. (Here $-\frac{1}{2} = \frac{p-1}{2}$ since the characteristic of k is p.)

$g_{16} = \begin{pmatrix} 1-p & p\sqrt{p} \\ -p\sqrt{p} & 1+p+p^2 \end{pmatrix}$**:** Comparing g_{16} with (3.29), we see that

$$
\begin{aligned}
a_{11} &= \exp\big(\sqrt{p}(x_3 + \sqrt{p}x_4)\big) = 1-p, \\
a_{12} &= (x_5 + \sqrt{p}x_6)\exp\big(\sqrt{p}(x_3 + \sqrt{p}x_4)\big) = (x_5 + \sqrt{p}x_6)(1-p) = p\sqrt{p}, \\
a_{21} &= \sqrt{p}(x_1 + \sqrt{p}x_2)\exp\big(\sqrt{p}(x_3 + \sqrt{p}x_4)\big) = \sqrt{p}(x_1 + \sqrt{p}x_2)(1-p) = -p\sqrt{p},
\end{aligned}
$$

and thus

$$
\begin{aligned}
x_3 + \sqrt{p}x_4 &= \frac{1}{\sqrt{p}}\log(1-p) = \frac{1}{\sqrt{p}}\big((-p) + O(p^{3/2})\big) = -\sqrt{p} + O(p), \\
x_5 + \sqrt{p}x_6 &= \frac{p\sqrt{p}}{1-p} = p\sqrt{p} + O(p^2), \\
x_1 + \sqrt{p}x_2 &= \frac{-p\sqrt{p}}{\sqrt{p}(1-p)} = -p + O(p^{3/2}).
\end{aligned}
$$

Hence $x_1, x_2, x_3.x_5, x_6 \in p\mathbb{Z}_p$ and $x_4 \in -1 + p\mathbb{Z}_p$, which implies that $\xi_{16} = -\xi_4$.

$g_{23} = \begin{pmatrix} 1 & 0 \\ p\big(1-\exp(-2\sqrt{p})\big) & 1 \end{pmatrix}$**:** Comparing g_{23} with g_{13}, it is not hard to see that $\xi_{23} = 0$.

$g_{24} = \begin{pmatrix} 1 & 0 \\ p\big(1-\exp(-2p)\big) & 1 \end{pmatrix}$**:** Comparing g_{24} with g_{14}, it is not hard to see that $\xi_{24} = 0$.

$g_{25} = \begin{pmatrix} 1-p & p \\ -p^2 & 1+p+p^2 \end{pmatrix}$**:** Comparing g_{25} with (3.29), we see that

$$\begin{aligned} a_{11} &= \exp\big(\sqrt{p}(x_3+\sqrt{p}x_4)\big) = 1-p, \\ a_{12} &= (x_5+\sqrt{p}x_6)\exp\big(\sqrt{p}(x_3+\sqrt{p}x_4)\big) = (x_5+\sqrt{p}x_6)(1-p) = p, \\ a_{21} &= \sqrt{p}(x_1+\sqrt{p}x_2)\exp\big(\sqrt{p}(x_3+\sqrt{p}x_4)\big) = \sqrt{p}(x_1+\sqrt{p}x_2)(1-p) = -p^2, \end{aligned}$$

and thus

$$\begin{aligned} x_3+\sqrt{p}x_4 &= \frac{1}{\sqrt{p}}\log(1-p) = \frac{1}{\sqrt{p}}\big((-p)+O(p^{3/2})\big) = -\sqrt{p}+O(p), \\ x_5+\sqrt{p}x_6 &= \frac{p}{1-p} = p+O(p^2), \\ x_1+\sqrt{p}x_2 &= \frac{-p^2}{\sqrt{p}(1-p)} = -p^{3/2}+O(p^2). \end{aligned}$$

Hence $x_1, x_2, x_3.x_5, x_6 \in p\mathbb{Z}_p$ and $x_4 \in -1+p\mathbb{Z}_p$, which implies that $\xi_{25} = -\xi_4$.

$g_{26} = \begin{pmatrix} 1-p\sqrt{p} & p^2 \\ -p^2\sqrt{p} & 1+p\sqrt{p}+p^3 \end{pmatrix}$**:** Comparing g_{26} with (3.29), we see that

$$\begin{aligned} a_{11} &= \exp\big(\sqrt{p}(x_3+\sqrt{p}x_4)\big) = 1-p\sqrt{p}, \\ a_{12} &= (x_5+\sqrt{p}x_6)\exp\big(\sqrt{p}(x_3+\sqrt{p}x_4)\big) = (x_5+\sqrt{p}x_6)(1-p\sqrt{p}) = p^2, \\ a_{21} &= \sqrt{p}(x_1+\sqrt{p}x_2)\exp\big(\sqrt{p}(x_3+\sqrt{p}x_4)\big) = \sqrt{p}(x_1+\sqrt{p}x_2)(1-p\sqrt{p}) = -p^2\sqrt{p}, \end{aligned}$$

and thus

$$\begin{aligned} x_3+\sqrt{p}x_4 &= \frac{1}{\sqrt{p}}\log(1-p\sqrt{p}) = \frac{1}{\sqrt{p}}\big((-p\sqrt{p})+O(p^2)\big) = -p+O(p^{3/2}), \\ x_5+\sqrt{p}x_6 &= \frac{p^2}{1-p\sqrt{p}} = p^2+O(p^{5/2}), \\ x_1+\sqrt{p}x_2 &= \frac{-p^2\sqrt{p}}{\sqrt{p}(1-p\sqrt{p})} = -p^2+O(p^{5/2}). \end{aligned}$$

Hence $x_1, x_2, x_3, x_4, x_5, x_6 \in p\mathbb{Z}_p$, which implies that $\xi_{26} = 0$.

$g_{35} = \begin{pmatrix} 1 & \exp(2\sqrt{p}) - 1 \\ 0 & 1 \end{pmatrix}$**:** Comparing g_{35} with (3.29), we see that $x_1 = x_2 = x_3 = x_4 = 0$. This leaves $a_{12} = x_5 + ix_6 = \exp(2\sqrt{p}) - 1 = 2\sqrt{p} + O(p)$, which implies that $x_5 \in p\mathbb{Z}_p$ and $x_6 \in 2 + p\mathbb{Z}_p$. Hence $\xi_{35} = 2\xi_6$.

$g_{36} = \begin{pmatrix} 1 & \sqrt{p}\big(\exp(2\sqrt{p}) - 1\big) \\ 0 & 1 \end{pmatrix}$**:** Comparing g_{36} with g_{35}, it is not hard to see that $\xi_{36} = 0$.

$g_{45} = \begin{pmatrix} 1 & \exp(2p) - 1 \\ 0 & 1 \end{pmatrix}$**:** Comparing g_{45} with (3.29), we see that $x_1 = x_2 = x_3 = x_4 = 0$. This leaves $a_{12} = x_5 + ix_6 = \exp(2p) - 1 = 2p + O(p^2)$, which implies that $x_5, x_6 \in p\mathbb{Z}_p$. Hence $\xi_{45} = 0$.

$g_{46} = \begin{pmatrix} 1 & \sqrt{p}\big(\exp(2p) - 1\big) \\ 0 & 1 \end{pmatrix}$**:** Comparing g_{46} with g_{45}, it is not hard to see that $\xi_{45} = 0$.

In summary, the only non-zero commutators $[\xi_i, \xi_j]$ with $i < j$ are

$$\begin{aligned} &[\xi_1, \xi_5] = -\xi_3, \quad [\xi_1, \xi_6] = -\xi_4, \\ &[\xi_2, \xi_5] = -\xi_4, \quad [\xi_2, \xi_6] = \xi_3, \end{aligned} \tag{3.30}$$

when $p \equiv 3 \pmod 4$ and $F = \mathbb{Q}_p(i)$, and

$$\begin{aligned} &[\xi_1, \xi_3] = 2\xi_2 && [\xi_1, \xi_5] = -\xi_2 - \xi_3 - \frac{1}{2}\xi_4 + \xi_6, \\ &[\xi_1, \xi_6] = -\xi_4, && [\xi_2, \xi_5] = -\xi_4, \\ &[\xi_3, \xi_5] = 2\xi_6, \end{aligned}$$

when $F = \mathbb{Q}_p(\sqrt{p})$. To make the $F = \mathbb{Q}_p(\sqrt{p})$ case easier to work with, we make a base change $\xi_5' = \xi_5 - \frac{1}{2}\xi_6$ and $\xi_i' = \xi_i$ for $i \neq 5$, which gives us commutators

$$\begin{aligned} &[\xi_1', \xi_3'] = 2\xi_2' \quad && [\xi_1', \xi_5'] = -\xi_2' - \xi_3' + \xi_6', \\ &[\xi_1', \xi_6'] = -\xi_4', && [\xi_2', \xi_5'] = -\xi_4', \\ &[\xi_3', \xi_5'] = 2\xi_6'. \end{aligned} \tag{3.31}$$

3.8.2 Finding the cohomology

Looking at (3.3) in the $p \equiv 3 \pmod 4$ and $F = \mathbb{Q}_p(i)$ case (with $e = 1$ and $h = 2$), we see that

$$\omega(g_1) = 1 - \frac{1}{2} = \frac{1}{2}, \qquad \omega(g_2) = 1 - \frac{1}{2} = \frac{1}{2},$$

$$\omega(g_3) = 1, \qquad \omega(g_4) = 1,$$
$$\omega(g_5) = \frac{1}{2}, \qquad \omega(g_6) = \frac{1}{2}.$$

Hence

$$\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I = \operatorname{span}_k(\xi_1, \dots, \xi_6) = \mathfrak{g}^1 \oplus \mathfrak{g}^2,$$

where

$$\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{2}} = \operatorname{span}_k(\xi_1, \xi_2, \xi_5, \xi_6),$$
$$\mathfrak{g}^2 = \mathfrak{g}_1 = \operatorname{span}_k(\xi_3, \xi_4).$$

In the $F = \mathbb{Q}_p(\sqrt{p})$ case (with $e = 2$ and $h = 2$) (3.3) gives us

$$\omega(g_1) = \frac{1}{2}\Big(1 - \frac{1}{2}\Big) = \frac{1}{4}, \qquad \omega(g_2) = \frac{1}{2}\Big(1 - \frac{1}{2}\Big) = \frac{1}{4},$$
$$\omega(g_3) = \frac{1}{2}, \qquad \omega(g_4) = \frac{1}{2},$$
$$\omega(g_5) = \frac{1}{4}, \qquad \omega(g_6) = \frac{1}{4}.$$

Hence

$$\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} I = \operatorname{span}_k(\xi'_1, \dots, \xi'_6) = \mathfrak{g}^1 \oplus \mathfrak{g}^2,$$

where

$$\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{4}} = \operatorname{span}_k(\xi'_1, \xi'_2, \xi'_5, \xi'_6),$$
$$\mathfrak{g}^2 = \mathfrak{g}_{\frac{1}{2}} = \operatorname{span}_k(\xi'_3, \xi'_4).$$

See Remark 3.9 for more details.

This is enough to calculate the graded mod p cohomology of $\mathfrak{g}$, see [Kon22] for the details. We write the result in Table 3.11.

Table 3.10: Dimensions of $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$ for the $I \subseteq \mathrm{SL}_2(\mathcal{O}_F)$ case, where $F/\mathbb{Q}_p$ is a quadratic extension (either $F = \mathbb{Q}_p(i)$ or $F = \mathbb{Q}_p(\sqrt{p})$).

t \ s	0	−1	−2	−3	−4	−5	−6	−7	−8
0	1								
1									
2		4							
3									
4			4						
5				4					
6									
7					10				
8									
9						4			
10							4		
11									
12								4	
13									
14									1

Altogether, we see that

$$\begin{aligned} H^0 &= H^{0,0}, \\ H^1 &= H^{-1,2}, \\ H^2 &= H^{-2,4} \oplus H^{-3,5}, \\ H^3 &= H^{-4,7}, \\ H^4 &= H^{-5,9} \oplus H^{-6,10}, \\ H^5 &= H^{-7,12}, \\ H^6 &= H^{-8,14}, \end{aligned} \tag{3.32}$$

with dimension as described in Table 3.10 in both the $F = \mathbb{Q}_p(i)$ and the $F = \mathbb{Q}_p(\sqrt{p})$ case. I.e., the mod p cohomology does not depend on the field extension (among the above ones) in this case.

We note that all differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ in Table 3.10 have bidegree $(r, 1-r)$, i.e., they are all below the $(r, -r)$ arrow going r to the left and r up in the table, where $r \geq 1$. Looking at Table 3.10, this clearly means that all differentials for $r \geq 1$ are trivial, and thus the spectral sequence collapses on the first page. Hence $H^{s,t}(\mathfrak{g}, k) = E_1^{s,t} \cong E_\infty^{s,t} = \operatorname{gr}^s H^{s+t}(I, k)$, and by (3.32) and Table 3.10 we get that

$$\dim H^n(I,k) = \begin{cases} 1 & n = 0, \\ 4 & n = 1, \\ 8 & n = 2, \\ 10 & n = 3, \\ 8 & n = 4, \\ 4 & n = 5, \\ 1 & n = 6. \end{cases} \tag{3.33}$$

3.9 $I \subseteq \mathrm{GL}_2(\mathcal{O}_F)$

In this section we will describe the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{GL}_2(F)$ for quadratic extensions $F/\mathbb{Q}_p$.

We again write $F = \mathbb{Q}_p(\alpha)$ and focus on the cases $\alpha = i$ (when $p \equiv 3 \pmod 4$) and $\alpha = \sqrt{p}$.

When I is the pro-p Iwahori subgroup in $\mathrm{GL}_2(F)$, we know by Section 3.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1 + \varpi_F \mathcal{O}_F & \mathcal{O}_F \\ \varpi_F \mathcal{O}_F & 1 + \varpi_F \mathcal{O}_F \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_F),$$

where $\varpi_F = p$ when $F = \mathbb{Q}_p(i)$ and $\varpi_F = \sqrt{p}$ when $F = \mathbb{Q}_p(\sqrt{p})$. By Section 3.1, we have an ordered basis

$$\begin{gathered} g_1 = \begin{pmatrix} 1 & 0 \\ \varpi_F & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 \\ \varpi_F \alpha & 1 \end{pmatrix}, \\ g_3 = \begin{pmatrix} \exp(\varpi_F) & 0 \\ 0 & \exp(-\varpi_F) \end{pmatrix}, \quad g_4 = \begin{pmatrix} \exp(\varpi_F \alpha) & 0 \\ 0 & \exp(-\varpi_F \alpha) \end{pmatrix}, \\ g_5 = \begin{pmatrix} \exp(\varpi_F) & 0 \\ 0 & \exp(\varpi_F) \end{pmatrix}, \quad g_6 = \begin{pmatrix} \exp(\varpi_F \alpha) & 0 \\ 0 & \exp(\varpi_F \alpha) \end{pmatrix}, \\ g_7 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & \alpha \\ 1 & 1 \end{pmatrix}, \end{gathered} \tag{3.34}$$

since $1, \alpha$ is a $\mathbb{Z}_p$-basis of $\mathcal{O}_F$.

Since we just renamed some elements and added an element of the center of $\mathrm{GL}(\mathcal{O}_F)$ when comparing to the ordered basis of $I \subseteq \mathrm{SL}_2(\mathcal{O}_F)$ from Section 3.8, it is clear from Equation (3.30) and Equation (3.31) that the only non-zero commutators $[\xi_i, \xi_j]$ (with $i < j$) in $\mathfrak{g} = k \otimes \operatorname{gr} G$ are

$$\begin{aligned} &[\xi_1, \xi_7] = -\xi_3, \quad [\xi_1, \xi_8] = -\xi_4, \\ &[\xi_2, \xi_7] = -\xi_4, \quad [\xi_2, \xi_8] = \xi_3, \end{aligned} \tag{3.35}$$

when $p \equiv 3 \pmod 4$ and $F = \mathbb{Q}_p(i)$, and

$$\begin{aligned} &[\xi_1, \xi_3] = 2\xi_2 && [\xi_1, \xi_7] = -\xi_2 - \xi_3 - \frac{1}{2}\xi_4 + \xi_8, \\ &[\xi_1, \xi_8] = -\xi_4, && [\xi_2, \xi_7] = -\xi_4, \\ &[\xi_3, \xi_7] = 2\xi_8, \end{aligned}$$

when $F = \mathbb{Q}_p(\sqrt{p})$. To make the $F = \mathbb{Q}_p(\sqrt{p})$ case easier to work with, we again make a base change $\xi_7' = \xi_7 - \frac{1}{2}\xi_8$ and $\xi_i' = \xi_i$ for $i \neq 6$, which gives us commutators

$$\begin{aligned} &[\xi_1', \xi_3'] = 2\xi_2' \quad && [\xi_1', \xi_7'] = -\xi_2' - \xi_3' + \xi_8', \\ &[\xi_1', \xi_8'] = -\xi_4', && [\xi_2', \xi_7'] = -\xi_4', \\ &[\xi_3', \xi_7'] = 2\xi_8'. \end{aligned} \tag{3.36}$$

Looking at Section 3.8.2, we easily see that

$$\begin{aligned} \mathfrak{g}^1 &= \operatorname{span}_k(\xi_1, \xi_2, \xi_7, \xi_8), \\ \mathfrak{g}^2 &= \operatorname{span}_k(\xi_3, \xi_4, \xi_5, \xi_6), \end{aligned}$$

in both cases.

This is enough to calculate the graded mod p cohomology of $\mathfrak{g}$, see [Kon22] for the details. We write the result in Table 3.11.

Table 3.11: Dimensions of $E_1^{s,t} = H^{s,t}(\mathfrak{g}, \mathbb{F}_p)$ for the $I \subseteq \mathrm{GL}_2(\mathcal{O}_F)$ case, where $F/\mathbb{Q}_p$ is a quadratic extension (either $F = \mathbb{Q}_p(i)$ or $F = \mathbb{Q}_p(\sqrt{p})$).

$t \backslash s$	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12
0	1												
1													
2		4											
3			2										
4			4										
5				12									
6					1								
7					18								
8						12							
9						4							
10							28						
11								4					
12								12					
13									18				
14									1				
15										12			
16											4		
17											2		
18												4	
19													
20													1

Altogether, we see that

$$\begin{aligned}
H^0 &= H^{0,0}, \\
H^1 &= H^{-1,2} \oplus H^{-2,3}, \\
H^2 &= H^{-2,4} \oplus H^{-3,5} \oplus H^{-4,6}, \\
H^3 &= H^{-4,7} \oplus H^{-5,8}, \\
H^4 &= H^{-5,9} \oplus H^{-6,10} \oplus H^{-7,11}, \\
H^5 &= H^{-7,12} \oplus H^{-8,13}, \\
H^6 &= H^{-8,14} \oplus H^{-9,15} \oplus H^{-10,16}, \\
H^7 &= H^{-10,17} \oplus H^{-11,18}, \\
H^8 &= H^{-12,20}
\end{aligned} \tag{3.37}$$

with dimension as described in Table 3.11 in both the $F = \mathbb{Q}_p(i)$ and the $F = \mathbb{Q}_p(\sqrt{p})$ case. I.e., the mod p cohomology does not depend on the field extension (among the above ones) in this case.

We note that all differentials $d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+1-r}$ in Table 3.11 have bidegree $(r, 1-r)$, i.e., they are all below the $(r, -r)$ arrow going r to the left and r up in the table, where $r \geq 1$. Looking at Table 3.11, this clearly means that all differentials for $r \geq 1$ are trivial, and thus the spectral sequence collapses on the first page. Hence $H^{s,t}(\mathfrak{g}, k) = E_1^{s,t} \cong E_\infty^{s,t} = \operatorname{gr}^s H^{s+t}(I, k)$, and by (3.37) and Table 3.11 we get that

$$\dim H^n(I, k) = \begin{cases} 1 & n = 0, \\ 6 & n = 1, \\ 17 & n = 2, \\ 30 & n = 3, \\ 36 & n = 4, \\ 30 & n = 5, \\ 17 & n = 6, \\ 6 & n = 7, \\ 1 & n = 8. \end{cases} \tag{3.38}$$

3.10 Nilpotency index

Before ending this chapter with a brief discussion of future research directions, we will mention an interesting consequence of our above calculations.

Given any cohomology theory H (say over k), one can always think of the ring H^* with the cup product as $H^* = k \oplus H^+$, where $k = H^0$ and $H^+ = \bigoplus_{n>0} H^n$. Assuming that only finitely many H^n are non-zero and that each H^n is finite dimensional, one can note that H^+ must be nilpotent. Thus an interesting question becomes: what is the nilpotency index of H^+? I.e., what is the smallest positive integer m such that $(H^+)^m = 0$? In continuation of this, another slightly easier question to answer is, what is the nilpotency index of H^1? I.e., what is the smallest positive integer m such that $(H^1)^m = 0$.

We will now try to answer the above questions for the group cohomology $H^*(I, k)$ in each of the cases we have discussed in this chapter. Before beginning, recall that

$$H^{s,t} \cup H^{s',t'} \subseteq H^{s+s',t+t'} \tag{3.39}$$

by [Fuk86, Chap. 1 §3.7]. This will be useful for finding upper bounds for the nilpotency index. Also, note that we write $H^+ \cup H^+$ for the image of $\cup \colon H^+ \times H^+ \to H^+$, and similarly for H^1.

In the $I \subseteq \mathrm{SL}_2(\mathbb{Z}_p)$ case, we saw in (3.12) that the cup product is trivial except for $\cup \colon H^1 \times H^2 \to H^3$, so $H^1 \cup H^1 = 0$ and

$$H^+ \cup H^+ \neq 0, \qquad H^+ \cup H^+ \cup H^+ = 0.$$

In the $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ case, we completely described the (graded) cup product in (3.17), which should be enough to answer the questions. Looking at Table 3.4 and using (3.39), we see that an upper bound for H^1 is that

$$H^1 \cup H^1 \cup H^1 \neq 0, \qquad H^1 \cup H^1 \cup H^1 \cup H^1 = 0,$$

by starting with $H^{-1,2} \cup H^{-2,3} \subseteq H^{-3,5} \neq 0$ and then using that $H^{-3,5} \cup H^{-1,2} \subseteq H^{-4,7} \neq 0$ or $H^{-3,5} \cup H^{-2,3} \subseteq H^{-5,8} \neq 0$, and finally $H^{-4,7} \cup H^{-2,3} \subseteq H^{-6,10} \neq 0$ or $H^{-5,8} \cup H^{-1,2} \subseteq H^{-6,10} \neq 0$. The question is whether we can follow those steps with non-zero cup products. We note by (3.20) that

$$e_1 \cup e_3 = e_{1,3}, \qquad e_4 \cup e_3 = e_{4,3},$$

are the only non-zero cup product we can do in $H^1 = H^{-1,2} \oplus H^{-2,3}$. But $H^{-1,2} = k[e_1, e_4]$ and $H^{-2,3} = k[e_3]$ by (3.19), and we already noted in Section 3.4.3 that $e_{i_1,\dots,i_m} \cup e_{j_1,\dots,j_\ell}$ if $\{i_1,\dots,i_m\} \cap \{j_1,\dots,j_\ell\} \neq \emptyset$, so we clearly cannot cup with anything from H^1 without getting zero. Thus

$$H^1 \cup H^1 \neq 0, \qquad H^1 \cup H^1 \cup H^1 = 0.$$

Now, having only four possible numbers in the subscript and using the above equation, we note that we can only ever hope to have two cup products before getting zero (cf. (3.19)). By (3.20)

$$e_3 \cup (e_1 \cup e_{4,2}) = e_3 \cup e_{1,4,2} = -e_{1,4,2,3} \neq 0,$$

so

$$H^+ \cup H^+ \cup H^+ \neq 0, \qquad H^+ \cup H^+ \cup H^+ \cup H^+ = 0,$$

for $I \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$.

In the $I \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$ case, we have not described the cup product in detail, but we can tell purely from (3.39) and Table 3.6, that $H^1 \cup H^1 = 0$. Going through Table 3.6, we also note that an upper bound for H^+ is

$$H^+ \cup H^+ \cup H^+ \cup H^+ \neq 0, \qquad H^+ \cup H^+ \cup H^+ \cup H^+ \cup H^+ = 0,$$

which possibly can be achieved through (cf. [Kon22])

$$H^{-4,6} \cup H^{-6,9} \subseteq H^{-10,15}, \qquad H^{-10,15} \cup H^{-4,6} \subseteq H^{-14,21},$$
$$H^{-14,21} \cup H^{-1,2} \subseteq H^{-15,23}.$$

It still remains to check whether such a series of non-zero cup products exist, which we will not do here. (This would require a lot of extra computations by hand, or hopefully better automation for computing cup products than what has been achieved so far.)

Remark 3.22. To give estimates for the upper bounds of m such that $(H^+)^m \neq 0$, we can build a directed graph with nodes (s,t) for (s,t) such that $H^{s,t} \neq 0$ and arrows $(s,t) \to (s+s', t+t')$ (and $(s',t') \to (s+s', t+t')$) for (s',t') such that $H^{s',t'} \neq 0$ and $H^{s+s',t+t'} \neq 0$ (so that $H^{s,t} \cup H^{s',t'} \subseteq$

$H^{s+s',t+t'}$ has a chance of being non-zero). Then standard algorithms for finding the longest path in a directed graph can quickly give us a longest path as above.

Note that we sometime can be more restrictive than just finding the longest path by also considering dimension arguments. This is the case when working with $I \subseteq \mathrm{SL}_4(\mathbb{Z}_p)$ in the following, where $\dim_k H^{-5,7} = 4$, so if $H^{-5,7}$ is used in the sequence of cup products as part of the longest path more than 4 times, then we know the cup product must be zero. △

In the $I \subseteq \mathrm{GL}_3(\mathbb{Z}_p)$ case, we also have not described the cup product in detail, but we can tell purely from (3.39) and Table 3.7, that

$$H^1 \cup H^1 \cup H^1 \neq 0, \qquad H^1 \cup H^1 \cup H^1 \cup H^1 = 0,$$

is an upper bound, since $\dim_k H^{-3,4} = 1$, so $H^{-3,4}$ can only be used once in a non-zero cup product.This upper bound might be achieved through

$$H^{-3,4} \cup H^{-1,2} \subseteq H^{-4,6}, \qquad H^{-4,6} \cup H^{-1,2} \subset H^{-5,8},$$

but it still remains to check whether such a non-zero cup product exists. Going through Table 3.7, we also note that an upper bound for H^+ is

$$(H^+)^5 \neq 0, \qquad (H^+)^6 = 0,$$

which possibly can be achieved through (cf. [Kon22])

$$H^{-3,4} \cup H^{-1,2} \subseteq H^{-4,6}, \qquad H^{-4,6} \cup H^{-1,2} \subseteq H^{-5,8},$$
$$H^{-5,8} \cup H^{-3,4} \subseteq H^{-8,12}, \qquad H^{-8,12} \cup H^{-1,2} \subseteq HH^{-9,14},$$
$$H^{-9,14} \cup H^{-6,9} \subseteq H^{-15,23}.$$

Again it still remains to check whether such a series of non-zero cup products exist.

In the $I \subseteq \mathrm{SL}_4(\mathbb{Z}_p)$ case, we also have not described the cup product in detail, and we do not even fully know the cohomology of I in this case, but from (3.39) and Table 3.8, we at least know enough to see that

$$H^1 \cup H^1 \neq 0, \qquad H^1 \cup H^1 \cup H^1 = 0,$$

is an upper bound, since the dimension of the entries $E_r^{s,t}$ are non-increasing when r increases. This upper bound might be achieved through

$$H^{-1,2} \cup H^{-1,2} \subseteq H^{-2,4},$$

but it still remains to check whether such a non-zero cup product exists. Going through Table 3.8, we also note that an upper bound for H^+ is

$$(H^+)^7 \neq 0, \qquad (H^+)^8 = 0,$$

which possibly can be achieved through (cf. [Kon22])

$$H^{-1,2} \cup H^{-5,7} \subseteq H^{-6,9}, \qquad H^{-6,9} \cup H^{-5,7} \subseteq H^{-11,16},$$
$$H^{-11,16} \cup H^{-5,7} \subseteq H^{-16,23}, \qquad H^{-16,23} \cup H^{-5,7} \subseteq HH^{-21,30},$$
$$H^{-21,30} \cup H^{-7,10} \subseteq H^{-28,40}, \qquad H^{-28,40} \cup H^{-8,11} \subseteq H^{-36,51}.$$

Similarly, in the $I \subseteq \mathrm{GL}_4(\mathbb{Z}_p)$ case, we also have not described the cup product or fully know the cohomology of I in this case, but from (3.39) and Table 3.9, we at least know enough to see that

$$(H^1)^4 \neq 0, \qquad (H^1)^5 = 0,$$

is an upper bound, since the dimension of the entries $E_r^{s,t}$ are non-increasing when r increases. This upper bound might be achieved through

$$H^{-4,5} \cup H^{-1,2} \subseteq H^{-5,7}, \qquad H^{-5,7} \cup H^{-1,2} \subseteq H^{-6,9},$$
$$H^{-6,9} \cup H^{-1,2} \subseteq H^{-7,11},$$

but it still remains to check whether such a non-zero cup product exists. Going through Table 3.9, we also note that an upper bound for H^+ is

$$(H^+)^{10} \neq 0, \qquad (H^+)^{11} = 0,$$

which possibly can be achieved through (cf. [Kon22])

$$H^{-4,5} \cup H^{-1,2} \subseteq H^{-5,7}, \qquad H^{-5,7} \cup H^{-1,2} \subseteq H^{-6,9},$$
$$H^{-6,9} \cup H^{-1,2} \subseteq H^{-7,11}, \qquad H^{-7,11} \cup H^{-4,5} \subseteq HH^{-11,16},$$

$$H^{-11,16} \cup H^{-1,2} \subseteq H^{-12,18}, \qquad H^{-12,18} \cup H^{-5,7} \subseteq H^{-17,25},$$

$$H^{-17,25} \cup H^{-9,12} \subseteq H^{-26,37}, \qquad H^{-26,37} \cup H^{-5,7} \subseteq H^{-31,44},$$

$$H^{-31,44} \cup H^{-5,7} \subseteq H^{-36,51}.$$

In the $I \subseteq \mathrm{SL}_2(\mathcal{O}_F)$ (with $F/\mathbb{Q}_p$ quadratic) case, we also have not described the cup product in detail, but from (3.39) and Table 3.10, we at least know enough to see that

$$H^1 \cup H^1 \neq 0, \qquad H^1 \cup H^1 \cup H^1 = 0,$$

is an upper bound. This upper bound might be achieved through

$$H^{-1,2} \cup H^{-1,2} \subseteq H^{-2,4},$$

but it still remains to check whether such a non-zero cup product exists. Going through Table 3.10, we also note that an upper bound for H^+ is

$$(H^+)^4 \neq 0, \qquad (H^+)^5 = 0,$$

which possibly can be achieved through (cf. [Kon22])

$$H^{-3,5} \cup H^{-3,5} \subseteq H^{-6,10}, \qquad H^{-6,10} \cup H^{-1,2} \subseteq H^{-7,12},$$

$$H^{-7,12} \cup H^{-1,2} \subseteq H^{-8,14}.$$

In the $I \subseteq \mathrm{GL}_2(\mathcal{O}_F)$ (with $F/\mathbb{Q}_p$ quadratic) case, we also have not described the cup product in detail, but from (3.39) and Table 3.11, we at least know enough to see that

$$(H^1)^4 \neq 0, \qquad (H^1)^5 = 0,$$

is an upper bound. This upper bound might be achieved through

$$H^{-2,3} \cup H^{-1,2} \subseteq H^{-3,5}, \qquad H^{-3,5} \cup H^{-1,2} \subseteq H^{-4,7},$$

$$H^{-4,7} \cup H^{-1,2} \subseteq H^{-5,9},$$

but it still remains to check whether such a non-zero cup product exists. Going through Table 3.11, we also note that an upper bound for H^+ is

$$(H^+)^6 \neq 0, \qquad (H^+)^7 = 0,$$

which possibly can be achieved through (cf. [Kon22])

$$H^{-2,3} \cup H^{-1,2} \subseteq H^{-3,5}, \qquad H^{-3,5} \cup H^{-1,2} \subseteq H^{-4,7},$$

$$H^{-4,7} \cup H^{-1,2} \subseteq H^{-5,9}, \qquad H^{-5,9} \cup H^{-2,3} \subseteq H^{-7,12},$$

$$H^{-7,12} \cup H^{-1,2} \subseteq H^{-8,14}.$$

We collect our nilpotency index upper bounds in Table 3.12.

Table 3.12: The upper bound for the nilpotency index of mod p cohomology for each pro-p Iwahori subgroup of SL_n and GL_n that we have found. Confirmed nilpotency indices are bolded, and pure upper bounds are not bolded.

n	2	3	4
$I \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$, H^1	**2**	**2**	3
$I \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$, H^+	**3**	5	8
$I \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$, H^1	**3**	4	5
$I \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$, H^+	**4**	7	11
$I \subseteq \mathrm{SL}_n(\mathcal{O}_F)$ (quadratic), H^1	3		
$I \subseteq \mathrm{SL}_n(\mathcal{O}_F)$ (quadratic), H^+	4		
$I \subseteq \mathrm{GL}_n(\mathcal{O}_F)$ (quadratic), H^1	5		
$I \subseteq \mathrm{GL}_n(\mathcal{O}_F)$ (quadratic), H^+	7		

3.11 Future work

In this section we will discuss some interesting future directions of research. We will assume for the whole section that $k = \mathbb{F}_p$.

3.11.1 Quaternion algebras

In this subsection, we will further assume that $p > 5$ is a prime of the form $p \equiv 3 \pmod 4$, so that $\mathbb{Q}_{p^2} = \mathbb{Q}_p(i)$ is the unique unramified quadratic extension of $\mathbb{Q}_p$, and $\mathbb{F}_{p^2} = \mathbb{F}_p[i]$ is the unique quadratic extension of $\mathbb{F}_p$.

Let D be the division quaternion algebra over $\mathbb{Q}_p$ and let $\tilde{G} = 1+\mathfrak{m}_D$ and $G = (1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}$, where $\mathrm{Nrd} = \mathrm{Nrd}_{D/\mathbb{Q}_p}$ is the norm form. By [Voi21, Thm. 12.1.5] we can assume that $i^2 = -1$ and $j^2 = p$ (i.e., we have a tower $D/\mathbb{Q}_p(i)/\mathbb{Q}_p$), and that $\mathcal{O}_D = \mathbb{Z}_p[i, j, k]$ (where $k = ij$) and $\mathfrak{m}_D = j\mathcal{O}_D = \mathcal{O}_D j$ (i.e., $\varpi_D = j$), which has $\mathbb{Z}_p$-basis p, pi, j, k, by [Voi21, Thm.13.1.6].

Now let $\sigma\colon \mathbb{Q}_p(i) \to \mathbb{Q}_p(i)$ be the complex conjugate and note that $\langle\sigma\rangle = \mathrm{Gal}\big(\mathbb{Q}_p(i)/\mathbb{Q}_p\big)$, so

$$D \cong \left\{ \begin{pmatrix} a+bi & c+di \\ p(c-di) & a-bi \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Q}_p \right\} \subseteq M_2\big(\mathbb{Q}_p(i)\big)$$

by [Voi21, Cor. 13.3.14]. Hence we have an embedding

$$\begin{aligned} D &\hookrightarrow M_2\big(\mathbb{Q}_p(i)\big) \\ a+bi+cj+dk &\mapsto \begin{pmatrix} a+bi & c+di \\ p(c-di) & a-bi \end{pmatrix} \end{aligned}$$

with

$$\mathrm{Nrd}(a+bi+cj+dk) = a^2+b^2-pc^2-pd^2 = \det\left(\begin{pmatrix} a+bi & c+di \\ p(c-di) & a-bi \end{pmatrix}\right).$$

We note furthermore that $\mathcal{O}_D = \mathbb{Z}_p[i] \oplus \mathbb{Z}_p[i]j$ and $\mathfrak{m}_D = \mathcal{O}_D j = p\mathbb{Z}_p[i] \oplus \mathbb{Z}_p[i]j$ gives us

$$\mathfrak{m}_D \cong \left\{ \begin{pmatrix} p(a+bi) & c+di \\ p(c-di) & p(a-bi) \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Q}_p \right\},$$

so $1+\mathfrak{m}_D \subseteq I_{\mathrm{GL}_2(\mathbb{Q}_p(i))}$, where we denote by I_G the (standard choice of) pro-p Iwahori subgroup of G (cf. Section 3.1.2). Altogether, we get a commutative diagram

$$\begin{array}{ccccc}
(1+\mathfrak{m}_D)^{\mathrm{Nrd}=1} & \hookrightarrow & I_{\mathrm{SL}_2(\mathbb{Q}_p(i))} & \hookleftarrow & I_{\mathrm{SL}_2(\mathbb{Q}_p)} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
(\mathcal{O}_D^\times)^{\mathrm{Nrd}=1} & \hookrightarrow & \mathrm{SL}_2\big(\mathbb{Z}_p[i]\big) & \hookleftarrow & \mathrm{SL}_2(\mathbb{Z}_p) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
(D^\times)^{\mathrm{Nrd}=1} & \hookrightarrow & \mathrm{SL}_2\big(\mathbb{Q}_p(i)\big) & \hookleftarrow & \mathrm{SL}_2(\mathbb{Q}_p) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
D^\times & \hookrightarrow & \mathrm{GL}_2\big(\mathbb{Q}_p(i)\big) & \hookleftarrow & \mathrm{GL}_2(\mathbb{Q}_p) \\
\big\uparrow & & \big\uparrow & & \big\uparrow \\
\mathcal{O}_D^\times & \hookrightarrow & \mathrm{GL}_2\big(\mathbb{Z}_p[i]\big) & \hookleftarrow & \mathrm{GL}_2(\mathbb{Z}_p) \\
\big\uparrow & & \big\uparrow & & \big\uparrow \\
1+\mathfrak{m}_D & \hookrightarrow & I_{\mathrm{GL}_2(\mathbb{Q}_p(i))} & \hookleftarrow & I_{\mathrm{GL}_2(\mathbb{Q}_p)}.
\end{array} \tag{3.40}$$

We saw in Remark 3.14 that $H^*(G,\mathbb{F}_p) \cong H^*(I_{\mathrm{SL}_2(\mathbb{Q}_p)},\mathbb{F}_p)$, and in Remark 3.20 we noted that $H^*(I_{\mathrm{SL}_2(\mathbb{Q}_p)},\mathbb{F}_p) \otimes_{F_p} \mathbb{F}_p[\varepsilon] \cong H^*(I_{\mathrm{GL}_2(\mathbb{Q}_p)},\mathbb{F}_p)$ (where $\varepsilon^2 = 0$), while [Sør21, Sect. 6.3] notes that $H^*(\tilde{G},\mathbb{F}_p) \cong H^*(G,\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon]$, so $H^*(\tilde{G},\mathbb{F}_p) \cong H^*(I_{\mathrm{GL}_2(\mathbb{Q}_p)},\mathbb{F}_p)$. (Recall that $G = (1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}$ and $\tilde{G} = 1+\mathfrak{m}_D$.) Furthermore, [Sør21, Sect. 6.3] argues that $H^*(\mathcal{O}_D^\times,\mathbb{F}_p) \cong H^*(\tilde{G},\mathbb{F}_p)^{\mathbb{F}_D^\times}$, using that we can factor $\mathcal{O}_D^\times$ as a semi-direct product $\tilde{G} \rtimes \mathbb{F}_D^\times$. Here the $\mathbb{F}_D^\times$-action on $H^*(\tilde{G}, \mathbb{F}_p)$ is understood and non-trivial, cf. [Hen07, Prop. 7 (b)]. An interesting question is, if the comparison between cohomology of the left side and right side of (3.40) can be somehow continued?

Remark 3.23. To see that $H^*(\tilde{G}, \mathbb{F}_p) \cong H^*(I_{\mathrm{GL}_2(\mathbb{Q}_p)}, \mathbb{F}_p)$ for $p \geq 5$ in general, and not just for $p \equiv 3 \pmod 4$, one can compare the basis and structure of $H^*(\tilde{G}, \mathbb{F}_p)$ described in [Rav77, Thm. 3.2] with the basis and structure we describe in (3.19) and (3.20). △

Another interesting direction of research is to note that we already have bijections between *certain* mod p representations of $D^\times$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ from the Jacquet-Langlands correspondence for GL_2 (cf. [JL70]), and we can ask whether there are similar relations in between the left and right side of the other rows of (3.40). Here we note that by [Tok15, Rem. 4.5] irreducible representations of $D^\times$ are trivial on $1 + \mathfrak{m}_D$, so we need something new if we want a correspondence between certain mod p representations of $G = (1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1}$ and $I_{\mathrm{SL}_2(\mathbb{Q}_p)}$ or between certain mod p representations of $\tilde{G} = 1 + \mathfrak{m}_D$ and $I_{\mathrm{GL}_2(\mathbb{Q}_p)}$.

Finally, although we already have isomorphisms $H^*(G, \mathbb{F}_p) \cong H^*(I_{\mathrm{SL}_2(\mathbb{Q}_p)}, \mathbb{F}_p)$ and $H^*(\tilde{G}, \mathbb{F}_p) \cong H^*(I_{\mathrm{GL}_2(\mathbb{Q}_p)}, \mathbb{F}_p)$, we note that these were obtained by concrete calculations, and we would really prefer to have canonical isomorphisms (possibly obtained by working with the corresponding row of (3.40)).

In pursuit of the canonical isomorphisms mentioned above, we note that one can show by explicit calculations (with bases) that the inclusions of (3.40) give inclusions

$$\mathrm{gr}(1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1} \hookrightarrow \mathrm{gr}\, I_{\mathrm{SL}_2(\mathbb{Q}_p(i))} \hookleftarrow \mathrm{gr}\, I_{\mathrm{SL}_2(\mathbb{Q}_p)},$$

$$\mathrm{gr}(1 + \mathfrak{m}_D) \hookrightarrow \mathrm{gr}\, I_{\mathrm{GL}_2(\mathbb{Q}_p(i))} \hookleftarrow \mathrm{gr}\, I_{\mathrm{GL}_2(\mathbb{Q}_p)},$$

where the pro-p Iwahori subgroups are graded as usual (start with $\mathrm{gr}\, I = \bigoplus_{\nu > 0} \mathrm{gr}_\nu I$ where $\mathrm{gr}_\nu I = I_\nu / I_{\nu+}$ and translate to $\mathrm{gr}^i I$), $1 + \mathfrak{m}_D$ is graded by $\mathrm{gr}^i(1 + \mathfrak{m}_D) = (1 + \mathfrak{m}_D^i)/(1 + \mathfrak{m}_D^{i+1})$, and $(1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1}$ is graded by $\mathrm{gr}^i(1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1} = (1 + \mathfrak{m}_D^i)^{\mathrm{Nrd}=1}/(1 + \mathfrak{m}_D^{i+1})^{\mathrm{Nrd}=1}$. These inclusions further translate to inclusions

$$\mathfrak{g}_{(1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}} \hookrightarrow \mathfrak{g}_{I_{\mathrm{SL}_2(\mathbb{Q}_p(i))}} \hookleftarrow \mathfrak{g}_{I_{\mathrm{SL}_2(\mathbb{Q}_p)}},$$

$$\mathfrak{g}_{(1+\mathfrak{m}_D)} \hookrightarrow \mathfrak{g}_{I_{\mathrm{GL}_2(\mathbb{Q}_p(i))}} \hookleftarrow \mathfrak{g}_{I_{\mathrm{GL}_2(\mathbb{Q}_p)}},$$

where $\mathfrak{g}_G = \mathbb{F}_p \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ is the Lazard Lie algebra of G. We note that these inclusions do not have the same images, but we noted in Remark 3.14 that $\mathfrak{g}_{(1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}} \cong \mathfrak{g}_{I_{\mathrm{SL}_2(\mathbb{Q}_p(i))}}$, so we might be able to come up with a canonical isomorphism through these somehow.

3.11.2 Central division algebras

Let D be the central division algebra over $\mathbb{Q}_p$ of dimension n^2 and invariant $\frac{1}{n}$. Recall the following setup from [Sør21, Sect. 6.3]: The valuation v_p on $\mathbb{Q}_p$ extends uniquely to a valuation $\tilde{v}\colon D^\times \to \frac{1}{n}\mathbb{Z}$ by the formula $\tilde{v}(x) = \frac{1}{n} v\big(\mathrm{Nrd}_{D/\mathbb{Q}_p}(x)\big)$, and the valuation ring $\mathcal{O}_D = \{x : \tilde{v}(x) > 0\}$ is the maximal compact subring of D. It is local with maximal ideal $\mathfrak{m}_D = \{x : \tilde{v}(x) > 0\}$ and residue field $\mathbb{F}_D \cong \mathbb{F}_{p^n}$. Furthermore, we can pick ϖ_D such that $\tilde{v}(\varpi_D) = \frac{1}{n}$, $\mathfrak{m}_D = \varpi_D \mathcal{O}_D = \mathcal{O}_D \varpi_D$ and $p = \varpi_D^n$.

When $p > n+1$, we also saw in [Sør21, Sect. 6.3] that $\tilde{G} = 1 + \mathfrak{m}_D$ has Lazard Lie algebra $\tilde{\mathfrak{g}} = \mathbb{F}_D \oplus \cdots \oplus \mathbb{F}_D$ concentrated in degrees $1, 2, \ldots, n$ with Lie bracket given by the formula

$$[x, y] = xy^{p^i} - yx^{p^j} \tag{3.41}$$

for $x \in \tilde{\mathfrak{g}}^i \cong \mathbb{F}_D$ and $y \in \tilde{\mathfrak{g}}^j$. Furthermore $G = (1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1}$ has Lazard Lie algebra $\mathfrak{g} = \mathbb{F}_D \oplus \cdots \oplus \mathbb{F}_D \oplus \mathbb{F}_D^{\mathrm{Tr}=0}$ concentrated in degrees $1, 2, \ldots, n$ with Lie bracket given by (3.41). (Note that one can easily check that $[x, y]$ has trace zero when $i + j = n$.) Here $\mathbb{F}_D^{\mathrm{Tr}=0}$ is the kernel of the trace $\mathrm{Tr}_{\mathbb{F}_D/\mathbb{F}_p}$ and $\mathfrak{g} \subseteq \tilde{\mathfrak{g}}$ is a codimension one Lie subalgebra.

In the previous subsection we focused on the case $n = 2$ (and $p \equiv 3 \pmod 4$), but one can ask if some of the ideas work in more general cases. For the remainder of this subsection we will focus on the case $n = 3$ and $p = 5$.

We note that $x^3 + 3x + 3$ is an irreducible polynomial in $\mathbb{F}_5[x]$, so $\mathbb{F}_D \cong \mathbb{F}_{5^3} \cong \mathbb{F}_p[\alpha]$ where $\alpha^3 = -3\alpha - 3 = 2\alpha + 2$. Now let $\xi_1 = 1, \xi_2 = \alpha, \xi_3 = \alpha^2$ be the basis of $\tilde{\mathfrak{g}}^1 \cong \mathbb{F}_D$, let $\xi_4 = 1, \xi_5 = \alpha, \xi_6 = \alpha^2$ be the basis of $\tilde{\mathfrak{g}}^2 \cong \mathbb{F}_D$, and let $\xi_7 = 1, \xi_8 = \alpha, \xi_9 = \alpha^2$ be the basis of

$\tilde{\mathfrak{g}}^3 \cong \mathbb{F}_D$. Using (3.41), we see that

$$\begin{aligned}
&[\xi_1, \xi_2] = 4\xi_4 + 3\xi_5 + 2\xi_6, \quad [\xi_1, \xi_3] = 3\xi_4 + 2\xi_5 + 4\xi_6, \\
&[\xi_1, \xi_5] = 4\xi_7 + 3\xi_8 + 2\xi_9, \quad [\xi_1, \xi_6] = 3\xi_7 + 2\xi_8 + 4\xi_9, \\
&[\xi_2, \xi_3] = 2\xi_4 + \xi_5 + 4\xi_6, \quad [\xi_2, \xi_4] = 4\xi_7 + \xi_8 + 2\xi_9, \\
&[\xi_2, \xi_5] = 3\xi_7 + \xi_8 + 4\xi_9, \quad [\xi_2, \xi_6] = 2\xi_8, \\
&[\xi_3, \xi_4] = 4\xi_7 + 2\xi_8 + 2\xi_9, \quad [\xi_3, \xi_5] = 3\xi_8, \\
&[\xi_3, \xi_6] = 3\xi_7 + 4\xi_9.
\end{aligned} \tag{3.42}$$

Here

$$\tilde{\mathfrak{g}}^1 = \operatorname{span}_{\mathbb{F}_5}(\xi_1, \xi_2, \xi_3), \quad \tilde{\mathfrak{g}}^2 = \operatorname{span}_{\mathbb{F}_5}(\xi_4, \xi_5, \xi_6), \quad \tilde{\mathfrak{g}}^3 = \operatorname{span}_{\mathbb{F}_5}(\xi_7, \xi_8, \xi_9),$$

so we order the basis by the index of the ξ_i's. Calculating the cohomology as in the previous sections with this information, we get Table 3.13.

Remark 3.24. Note that when calculating the cohomology here, we need to do all calculations modulo 5 since (3.42) do not lift to a Lie algebra over $\mathbb{Z}$ with these Chevalley constants. See [Kon22] for the details. △

Comparing Table 3.7 and Table 3.13, we see that $H^*(I, \mathbb{F}_5)$ for $I \subseteq \mathrm{GL}_3(\mathbb{Z}_p)$ and $H^*(1 + \mathfrak{m}_D, \mathbb{F}_5)$ have the same graded cohomology dimensions, and it would be interesting to investigate whether $H^*(I, \mathbb{F}_5) \cong H^*(1 + \mathfrak{m}_D, \mathbb{F}_5)$ as graded algebras. More generally, is $H^*(I, \mathbb{F}_p) \cong H^*(1 + \mathfrak{m}_D, \mathbb{F}_p)$ as graded algebras for $p \geq 5$?

In a similar vein, we can recall that $\mathrm{Tr}_{\mathbb{F}_D \mathbb{F}_5}(x) = x + x^5 + x^{5^2}$ for $x \in \mathbb{F}_D \cong \mathbb{F}_{5^3}$, so

$$\mathrm{Tr}_{\mathbb{F}_D/\mathbb{F}_5}(1) = 3, \qquad \mathrm{Tr}_{\mathbb{F}_D/\mathbb{F}_5}(\alpha) = 0, \qquad \mathrm{Tr}_{\mathbb{F}_D/\mathbb{F}_5}(\alpha^2) = 4,$$

since $\alpha^3 = 2\alpha + 2$. Thus $\mathbb{F}_D^{\mathrm{Tr}=0}$ has basis $\alpha, 4 + 2\alpha^2$. Now let $\xi'_1 = 1, \xi'_2 = \alpha, \xi'_3 = \alpha^2$ be the basis of $\mathfrak{g}^1 \cong \mathbb{F}_D$, let $\xi'_4 = 1, \xi'_5 = \alpha, \xi'_6 = \alpha^2$ be the basis of $\mathfrak{g}^2 \cong \mathbb{F}_D$, and let $\xi'_7 = \alpha, \xi'_8 = 4 + 2\alpha$ be the basis of $\mathfrak{g}^3 \cong \mathbb{F}_D^{\mathrm{Tr}=0}$. Using (3.41), we see that

Table 3.13: Dimensions of $E_1^{s,t} = H^{s,t} = \mathrm{gr}^s H^{s+t}(\tilde{\mathfrak{g}}, \mathbb{F}_5)$ for $\tilde{G} = 1 + \mathfrak{m}_D$ in the $n = 3$ and $p = 5$ case.

$t \backslash s$	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18
0	1																		
1																			
2		3																	
3																			
4				1															
5				6															
6					6														
7					3														
8						6													
9							13												
10								3											
11								12											
12									15										
13										7									
14										7									
15											15								
16												12							
17												3							
18													13						
19														6					
20															3				
21															6				
22																6			
23																1			
24																			
25																		3	
26																			
27																			1

$$\begin{aligned}
&[\xi_1', \xi_2'] = 4\xi_4' + 3\xi_5' + 2\xi_6', && [\xi_1', \xi_3'] = 3\xi_4' + 2\xi_5' + 4\xi_6', \\
&[\xi_1', \xi_5'] = 3\xi_7' + \xi_8', && [\xi_1', \xi_6'] = 2\xi_7' + 2\xi_8', \\
&[\xi_2', \xi_3'] = 2\xi_4' + \xi_5' + 4\xi_6', && [\xi_2', \xi_4'] = \xi_7' + \xi_8', \\
&[\xi_2', \xi_5'] = \xi_7' + 2\xi_8', && [\xi_2', \xi_6'] = 2\xi_7', \\
&[\xi_3', \xi_4'] = 2\xi_7' + \xi_8', && [\xi_3', \xi_5'] = 3\xi_7', \\
&[\xi_3', \xi_6'] = 2\xi_8'.
\end{aligned} \tag{3.43}$$

Here

$$\mathfrak{g}^1 = \operatorname{span}_{\mathbb{F}_5}(\xi_1', \xi_2', \xi_3'), \quad \mathfrak{g}^2 = \operatorname{span}_{\mathbb{F}_5}(\xi_4', \xi_5', \xi_6'), \quad \mathfrak{g}^3 = \operatorname{span}_{\mathbb{F}_5}(\xi_7', \xi_8'),$$

so we order the basis by the index of the ξ_i''s. Calculating the cohomology as in the previous sections with this information, we get Table 3.14

Again, comparing Table 3.6 and Table 3.14, we see that $H^*(I, \mathbb{F}_5)$ for $I \subseteq \mathrm{SL}_3(\mathbb{Z}_p)$ and $H^*\big((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_5\big)$ have the same graded cohomology dimensions, and it would be interesting to investigate whether $H^*(I, \mathbb{F}_5) \cong H^*\big((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_5\big)$ as graded algebras. More generally, is $H^*(I, \mathbb{F}_p) \cong H^*\big((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\big)$ as graded algebras for $p \geq 5$?

Another interesting observation is that $H^*(\tilde{G}, \mathbb{F}_p) \cong H^*(G, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon]$ as graded algebras (with $\varepsilon^2 = 0$) by [Sør21, Sect. 6.3], so an interesting question is whether $H^*(I_{\mathrm{GL}_n(\mathbb{Q}_p)}, \mathbb{F}_p) \cong H^*(I_{\mathrm{SL}_n(\mathbb{Q}_p)}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon]$ as graded algebras.

Altogether, the above seems to hint at the following conjecture:

Conjecture 3.25. Let D be the central division algebra over $\mathbb{Q}_p$ of dimension n^2 and invariant $\frac{1}{n}$. Let $\mathcal{O}_D$ be the maximal compact (local) subring of D with maximal ideal $\mathfrak{m}_D$ and residue field $\mathbb{F}_D \cong \mathbb{F}_{p^n}$. If $p > n+1$ then

- $H^*\big(I_{\mathrm{GL}_n(\mathbb{Q}_p)}, \mathbb{F}_p\big) \cong H^*\big(1+\mathfrak{m}_D, \mathbb{F}_p\big)$ as graded algebras, and
- $H^*\big(I_{\mathrm{SL}_n(\mathbb{Q}_p)}, \mathbb{F}_p\big) \cong H^*\big((1+\mathfrak{m}_D)^{\mathrm{Nrd}=1}, \mathbb{F}_p\big)$ as graded algebras.

In particular, this implies by [Sør21, Sect. 6.3] that

$$H^*(I_{\mathrm{GL}_n(\mathbb{Q}_p)}, \mathbb{F}_p) \cong H^*(I_{\mathrm{SL}_n(\mathbb{Q}_p)}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\varepsilon]$$

as graded algebras, where $\mathbb{F}_p[\varepsilon]$ denote the dual numbers ($\varepsilon^2 = 0$). ○

Table 3.14: Dimensions of $E_1^{s,t} = H^{s,t} = \mathrm{gr}^s H^{s+t}(\mathfrak{g}, \mathbb{F}_5)$ for $G = (1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1}$ in the $n = 3$ and $p = 5$ case.

$t \backslash s$	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15
0	1															
1																
2		3														
3																
4																
5				6												
6					3											
7					3											
8						6										
9							7									
10																
11								9								
12									9							
13																
14										7						
15											6					
16												3				
17												3				
18													6			
19																
20																
21															3	
22																
23																1

3.11.3 Serre spectral sequence

Another interesting research direction is to try to work with the Serre spectral sequence in the following way.

Assume we have the "standard" setup with $\mathcal{G} = \mathrm{SL}_n$, $\mathcal{U}$ unipotent upper triangular matrices and $\mathcal{T}$ diagonal matrices with determinant 1. Let also $I \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$ be the pro-p Iwahori subgroup of $\mathrm{SL}_n(\mathbb{Q}_p)$ which is upper triangular and unipotent modulo p, and let

$$K := \ker\big(\mathrm{red}\colon \mathcal{G}(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{F}_p)\big),$$

where red: $\mathcal{G}(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{F}_p)$ is the reduction map. (Note that $I = \{g \in \mathcal{G}(\mathbb{Z}_p) : \mathrm{red}(g) \in \mathcal{U}(\mathbb{F}_p)\}$ in this case, cf. [CR16].) Then

$$I/K \cong \mathcal{U}(\mathbb{F}_p),$$

and thus we get the Serre spectral sequence

$$E_2^{i,j} = H^i\big(\mathcal{U}(\mathbb{F}_p), H^j(K, \mathbb{F}_p)\big) \Longrightarrow H^{i+j}(I, \mathbb{F}_p),$$

which is also a multiplicative spectral sequence. Since K is a uniformly powerful group (cf. [OS19, Prop. 7.6]), we know by [Laz65, p. 183] that

$$H^j(K, \mathbb{F}_p) \cong \bigwedge^j \mathrm{Hom}_{\mathbb{F}_p}(K, \mathbb{F}_p).$$

Now we can let $\mathrm{SL}_n(\mathbb{Z}_p)$ act by

$$(g \,.\, f)(x) = f(g^{-1}xg)$$

for $g \in \mathrm{SL}_n(\mathbb{Z}_p)$, $f\colon K \to \mathbb{F}_p$ and $x \in K$, and hope to split $\bigwedge^j \mathrm{Hom}_{\mathbb{F}_p}(K, \mathbb{F}_p)$ into a direct sum of Verma modules $\bigoplus_\lambda V(\lambda)$ for p-restricted λ (λ with $0 \leq \langle \lambda, \alpha^\vee \rangle \leq p-1$), similarly to what is done in [PT18] (as we used in Chapter 2). This description might be easier to generalize than what we have worked with in this chapter, but it is harder to get started with since the spectral sequence is more complicated. One can hope that the difference in the spectral sequence might make it so that it will always collapse on the second page (the starting page in this case).

Appendix A

Calculations

A.1 $I \subseteq \mathrm{SL}_4(\mathbb{Z}_p)$

In this section we will describe the work need to find the continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{SL}_4(\mathbb{Q}_p)$.

When I is the pro-p Iwahori subgroup in $\mathrm{SL}_4(\mathbb{Q}_p)$, we know by Section 3.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}^{\det=1} \subseteq \mathrm{SL}_4(\mathbb{Z}_p),$$

and, by Section 3.1, we have an ordered basis

$$
\begin{gathered}
g_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p & & & 1 \end{pmatrix}, \quad
g_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ p & & 1 & \\ & & & 1 \end{pmatrix}, \quad
g_3 = \begin{pmatrix} 1 & & & \\ p & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \\
g_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & p & & 1 \end{pmatrix}, \quad
g_5 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & p & 1 & \\ & & & 1 \end{pmatrix}, \quad
g_6 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & p & 1 \end{pmatrix}, \\
g_7 = \begin{pmatrix} \exp(p) & & & \\ & \exp(-p) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad
g_8 = \begin{pmatrix} 1 & & & \\ & \exp(p) & & \\ & & \exp(-p) & \\ & & & 1 \end{pmatrix}, \\
g_9 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \exp(p) & \\ & & & \exp(-p) \end{pmatrix}, \\
g_{10} = \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad
g_{11} = \begin{pmatrix} 1 & & & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad
g_{12} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \\
g_{13} = \begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad
g_{14} = \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad
g_{15} = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.
\end{gathered} \tag{A.1}
$$

Here we write any zeros as blank space in matrices, to make the notation easier to read for the bigger matrices.

Remark A.1. Note that the order is not going from the lower left corner to the upper right corner along "diagonals", which might be a more standard ordering to chose. The reason we choose this alternative order is to simplify some calculations. In particular, this order gives simpler a_{ij} below.△

A.1.1 Finding the commutators $[\xi_i, \xi_j]$

Now

$$
g_1^{x_1} g_2^{x_2} \cdots g_{15}^{x_{15}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},
$$

where

$$
\begin{aligned}
a_{11} &= \exp(px_7),\\
a_{12} &= x_{15}\exp(px_7),\\
a_{13} &= x_{13}\exp(px_7),\\
a_{14} &= x_{10}\exp(px_7),\\
a_{21} &= px_3\exp(px_7),\\
a_{22} &= px_{15}x_3\exp(px_7)+\exp\bigl(p(x_8-x_7)\bigr),\\
a_{23} &= px_{13}x_3\exp(px_7)+x_{14}\exp\bigl(p(x_8-x_7)\bigr),\\
a_{24} &= px_{10}x_3\exp(px_7)+x_{11}\exp\bigl(p(x_8-x_7)\bigr),\\
a_{31} &= px_2\exp(px_7), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.2)}\\
a_{32} &= px_{15}x_2\exp(px_7)+px_5\exp\bigl(p(x_8-x_7)\bigr),\\
a_{33} &= px_{13}x_2\exp(px_7)+px_{14}x_5\exp\bigl(p(x_8-x_7)\bigr)+\exp\bigl(p(x_9-x_8)\bigr),\\
a_{34} &= px_{10}x_2\exp(px_7)+px_{11}x_5\exp\bigl(p(x_8-x_7)\bigr)+x_{12}\exp\bigl(p(x_9-x_8)\bigr),\\
a_{41} &= px_1\exp(px_7),\\
a_{42} &= px_1x_{15}\exp(px_7)+px_4\exp\bigl(p(x_8-x_7)\bigr)\\
a_{43} &= px_1x_{13}\exp(px_7)+px_{14}x_4\exp\bigl(p(x_8-x_7)\bigr)+px_6\exp\bigl(p(x_9-x_8)\bigr),\\
a_{44} &= px_1x_{10}\exp(px_7)+px_{11}x_4\exp\bigl(p(x_8-x_7)\bigr)+px_{12}x_6\exp\bigl(p(x_9-x_8)\bigr)\\
&\qquad +\exp(-px_9).
\end{aligned}
$$

Furthermore, write $g_{i,j} = [g_i, g_j]$ and $\xi_{i,j} = [\xi_i, \xi_j]$. Then we are ready to find $x_1, \ldots, x_{15}$ such that $g_{i,j} = g_1^{x_1} \cdots g_{15}^{x_{15}}$ for different $i < j$. (In the following we use that $\frac{1}{p-1} = 1 + p + p^2 + \cdots$ and $\log(1-p) = -p - \frac{p^2}{2} - \frac{p^3}{3} - \cdots$.) Also, except in the first case, we will note that $x_k \in p\mathbb{Z}_p$ implies that the coefficient on ξ_k in $\xi_{i,j}$ is zero.

We now list all non-identity commutators $g_{i,j} = [g_i, g_j]$ and find $\xi_{i,j} = [\xi_i, \xi_j]$ based on these. (For $g_{i,j} = 1_4$ it is clear that $x_1 = \cdots = x_{15} = 0$, and thus $\xi_{i,j} = 0$.)

$g_{1,7} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p\bigl(1-\exp(-p)\bigr) & & & 1 \end{pmatrix}$: Comparing $g_{1,7}$ with (A.2), we see that $x_2 = x_3 = x_7 = x_{10} = x_{13} = x_{15} = 0$, and thus also $x_4 = x_5 = x_8 = x_{11} = x_{14} = 0$, which implies that

$x_6 = x_9 = x_{12} = 0$. This leaves $a_{41} = px_1 = p\big(1 - \exp(-p)\big) = p^2 + O(p^3)$, which implies that $x_1 = p + O(p^2)$. Hence $\sigma(g_{1,7}) = \pi\,.\,\sigma(g_1)$, which implies that $\xi_{1,7} = 0$.

$g_{1,9} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p\big(1-\exp(-p)\big) & & & 1 \end{pmatrix}$**:** Since $g_{1,9} = g_{1,7}$, the above calculation shows that $\xi_{1,8} = 0$.

$g_{1,10} = \begin{pmatrix} 1-p & & & p \\ & 1 & & \\ & & 1 & \\ -p^2 & & & 1+p+p^2 \end{pmatrix}$**:** Comparing $g_{1,10}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except $x_7 = x_8 = x_9$, for which we have $a_{11} = \exp(px_7) = 1-p$, and thus $x_7 = \frac{1}{p}\log(1-p) = -1+O(p)$. Hence $\xi_{1,10} = -\xi_7 - \xi_8 - \xi_9$.

$g_{1,11} = \begin{pmatrix} 1 & & & \\ -p & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$**:** Comparing $g_{1,11}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_3, for which we have $a_{21} = px_3 = -p$, and thus $x_3 = -1$. Hence $\xi_{1,11} = -\xi_3$.

$g_{1,12} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ -p & & 1 & \\ & & & 1 \end{pmatrix}$**:** Comparing $g_{1,12}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_2, for which we have $a_{31} = px_2 = -p$, and thus $x_2 = -1$. Hence $\xi_{1,12} = -\xi_2$.

$g_{1,13} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & p & 1 \end{pmatrix}$**:** Comparing $g_{1,13}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_6, for which we have $a_{43} = px_6 = p$, and thus $x_6 = 1$. Hence $\xi_{1,13} = \xi_6$.

$g_{1,15} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & p & & 1 \end{pmatrix}$**:** Comparing $g_{1,15}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_4, for which we have $a_{42} = px_4 = p$, and thus $x_4 = 1$. Hence $\xi_{1,15} = \xi_4$.

$g_{2,6} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -p^2 & & & 1 \end{pmatrix}$**:** Comparing $g_{2,6}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{2,6} = 0$.

$g_{2,7} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ p\big(1-\exp(-p)\big) & & 1 & \\ & & & 1 \end{pmatrix}$**:** Comparing $g_{2,7}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{2,7} = 0$.

$$g_{2,8} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ p\big(1-\exp(-p)\big) & & 1 & \\ & & & 1 \end{pmatrix}$$: Since $g_{2,8} = g_{2,7}$, the above shows that $\xi_{2,8} = 0$.

$$g_{2,9} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ p\big(1-\exp(p)\big) & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{2,9}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{2,9} = 0$.

$$g_{2,10} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & p \\ & & & 1 \end{pmatrix}$$: Comparing $g_{2,10}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{2,10} = 0$.

$$g_{2,13} = \begin{pmatrix} 1-p & & p & \\ & 1 & & \\ -p^2 & & 1+p+p^2 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{2,13}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except $x_7 = x_8$, for which we have $a_{11} = \exp(px_7) = 1-p$, and thus $x_7 = \frac{1}{p}\log(1-p) = -1 + O(p)$. Hence $\xi_{2,13} = -\xi_7 - \xi_8$.

$$g_{2,14} = \begin{pmatrix} 1 & & & \\ -p & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{2,14}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_3, for which we have $a_{21} = px_3 = -p$, and thus $x_3 = -1$. Hence $\xi_{2,14} = -\xi_3$.

$$g_{2,15} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & p & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{2,15}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_5, for which we have $a_{32} = px_5 = p$, and thus $x_5 = 1$. Hence $\xi_{2,15} = \xi_5$.

$$g_{3,4} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -p^2 & & & 1 \end{pmatrix}$$: Comparing $g_{3,4}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{3,4} = 0$.

$$g_{3,5} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ -p^2 & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{3,5}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{3,5} = 0$.

$$g_{3,7} = \begin{pmatrix} 1 & & & \\ p\big(1-\exp(-2p)\big) & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{3,7}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{3,7} = 0$.

$$g_{3,8} = \begin{pmatrix} 1 & & & \\ p\big(1-\exp(p)\big) & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{3,8}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{3,8} = 0$.

$$g_{3,10} = \begin{pmatrix} 1 & & & \\ & 1 & & p \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{3,10}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{3,10} = 0$.

$$g_{3,13} = \begin{pmatrix} 1 & & & \\ & 1 & p & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{3,13}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{3,13} = 0$.

$$g_{3,15} = \begin{pmatrix} 1-p & p & & \\ -p^2 & 1+p+p^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{3,15}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_7, for which we have $a_{11} = \exp(px_7) = 1-p$, and thus $x_7 = \frac{1}{p}\log(1-p) = -1 + O(p)$. Hence $\xi_{3,15} = -\xi_7$.

$$g_{4,7} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & p\big(1-\exp(p)\big) & & 1 \end{pmatrix}$$: Comparing $g_{4,7}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{4,7} = 0$.

$$g_{4,8} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & p\big(1-\exp(-p)\big) & & 1 \end{pmatrix}$$: Comparing $g_{4,8}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{4,8} = 0$.

$$g_{4,9} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & p\big(1-\exp(-p)\big) & & 1 \end{pmatrix}$$: Since $g_{4,9} = g_{4,8}$, the above shows that $\xi_{4,9} = 0$.

$$g_{4,10} = \begin{pmatrix} 1 & -p & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{4,10}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{4,10} = 0$.

$$g_{4,11} = \begin{pmatrix} 1 & & & \\ & 1-p & & p \\ & & 1 & \\ & -p^2 & & 1+p+p^2 \end{pmatrix}$$: Comparing $g_{4,11}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except $x_8 = x_9$, for which we have $a_{22} = \exp(px_8) = 1 - p$, and thus $x_8 = \frac{1}{p}\log(1-p) = -1 + O(p)$. Hence $\xi_{4,11} = -\xi_8 - \xi_9$.

$$g_{4,12} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -p & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{4,12}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_5, for which we have $a_{32} = px_5 = -p$, and thus $x_5 = -1$. Hence $\xi_{4,12} = -\xi_5$.

$$g_{4,14} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & p & 1 \end{pmatrix}$$: Comparing $g_{4,14}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_6, for which we have $a_{43} = px_6 = p$, and thus $x_6 = 1$. Hence $\xi_{4,14} = \xi_6$.

$$g_{5,6} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & -p^2 & & 1 \end{pmatrix}$$: Comparing $g_{5,6}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{5,6} = 0$.

$$g_{5,7} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & p\big(1-\exp(p)\big) & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{5,7}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{5,7} = 0$.

$$g_{5,8} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & p\big(1-\exp(-2p)\big) & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{5,8}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{5,8} = 0$.

$$g_{5,9} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & p\big(1-\exp(p)\big) & 1 & \\ & & & 1 \end{pmatrix}$$: Since $g_{5,9} = g_{5,7}$, the above shows that $\xi_{5,9} = 0$.

$$g_{5,11} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & p \\ & & & 1 \end{pmatrix}$$: Comparing $g_{5,11}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{5,11} = 0$.

$$g_{5,13} = \begin{pmatrix} 1 & -p & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{5,13}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{5,13} = 0$.

$$g_{5,14} = \begin{pmatrix} 1 & & & \\ & 1-p & p & \\ & -p^2 & 1+p+p^2 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{5,14}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_8, for which we have $a_{22} = \exp(px_8) = 1-p$, and thus $x_8 = \frac{1}{p}\log(1-p) = -1 + O(p)$. Hence $\xi_{5,14} = -\xi_8$.

$$g_{6,8} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & p\big(1-\exp(p)\big) & 1 \end{pmatrix}$$: Comparing $g_{6,8}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{6,8} = 0$.

$$g_{6,9} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & p\big(1-\exp(-2p)\big) & 1 \end{pmatrix}$$: Comparing $g_{6,9}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{6,9} = 0$.

$$g_{6,10} = \begin{pmatrix} 1 & & -p & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{6,10}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{6,10} = 0$.

$$g_{6,11} = \begin{pmatrix} 1 & & & \\ & 1 & -p & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$: Comparing $g_{6,11}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{6,11} = 0$.

$$g_{6,12} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1-p & p \\ & & -p^2 & 1+p+p^2 \end{pmatrix}$$: Comparing $g_{6,12}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_9, for which we have $a_{33} = \exp(px_9) = 1-p$, and thus $x_9 = \frac{1}{p}\log(1-p) = -1 + O(p)$. Hence $\xi_{6,12} = -\xi_9$.

$g_{7,10} = \begin{pmatrix} 1 & & & \exp(p)-1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$: Comparing $g_{7,10}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{7,10} = 0$.

$g_{7,11} = \begin{pmatrix} 1 & & & \\ & 1 & & \exp(-p)-1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$: Comparing $g_{7,11}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{7,11} = 0$.

$g_{7,13} = \begin{pmatrix} 1 & & \exp(p)-1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$: Comparing $g_{7,13}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{7,13} = 0$.

$g_{7,14} = \begin{pmatrix} 1 & & & \\ & 1 & \exp(-p)-1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$: Comparing $g_{7,14}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{7,14} = 0$.

$g_{7,15} = \begin{pmatrix} 1 & \exp(2p)-1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$: Comparing $g_{7,15}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{7,15} = 0$.

$g_{8,11} = \begin{pmatrix} 1 & & & \\ & 1 & & \exp(p)-1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$: Comparing $g_{8,11}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{8,11} = 0$.

$g_{8,12} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \exp(-p)-1 \\ & & & 1 \end{pmatrix}$: Comparing $g_{8,12}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{8,12} = 0$.

$g_{8,13} = \begin{pmatrix} 1 & & \exp(p)-1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$: Since $g_{8,13} = g_{7,13}$, the above shows that $\xi_{8,13} = 0$.

$$g_{8,14} = \begin{pmatrix} 1 & & & \\ & 1 & \exp(2p)-1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}:$$ Comparing $g_{8,14}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{8,14} = 0$.

$$g_{8,15} = \begin{pmatrix} 1 & \exp(-p)-1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}:$$ Comparing $g_{8,15}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{8,15} = 0$.

$$g_{9,10} = \begin{pmatrix} 1 & & & \exp(p)-1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}:$$ Since $g_{9,10} = g_{7,10}$, the above shows that $\xi_{8,15} = 0$.

$$g_{9,11} = \begin{pmatrix} 1 & & & \\ & 1 & & \exp(p)-1 \\ & & 1 & \\ & & & 1 \end{pmatrix}:$$ Since $g_{9,11} = g_{8,11}$, the above shows that $\xi_{9,11} = 0$.

$$g_{9,12} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \exp(2p)-1 \\ & & & 1 \end{pmatrix}:$$ Comparing $g_{9,12}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{9,12} = 0$.

$$g_{9,13} = \begin{pmatrix} 1 & & \exp(-p)-1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}:$$ Comparing $g_{9,13}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{9,13} = 0$.

$$g_{9,14} = \begin{pmatrix} 1 & & & \\ & 1 & \exp(-p)-1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}:$$ Comparing $g_{9,14}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$. Hence $\xi_{9,14} = 0$.

$$g_{11,15} = \begin{pmatrix} 1 & & & -1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}:$$ Comparing $g_{11,15}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_{10}, for which we have $a_{14} = x_{10} = -1$. Hence $\xi_{11,15} = -\xi_{10}$.

$$g_{12,13} = \begin{pmatrix} 1 & & & -1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}:$$ Since $g_{12,13} = g_{11,15}$, the above shows that $\xi_{12,13} = -\xi_{10}$.

$g_{12,14} = \begin{pmatrix} 1 & & & \\ & 1 & & -1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$: Comparing $g_{12,14}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_{11}, for which we have $a_{24} = x_{11} = -1$. Hence $\xi_{12,14} = -\xi_{11}$.

$g_{14,15} = \begin{pmatrix} 1 & & -1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$: Comparing $g_{14,15}$ with (A.2), we see that all x_i are in $p\mathbb{Z}_p$ except x_{13}, for which we have $a_{13} = x_{13} = -1$. Hence $\xi_{14,15} = -\xi_{13}$.

Thus the non-zero commutators $[\xi_i, \xi_j]$ with $i < j$ are:

$$\begin{aligned}
&[\xi_1, \xi_{10}] = -(\xi_7 + \xi_8 + \xi_9), && [\xi_1, \xi_{11}] = -\xi_3, && [\xi_1, \xi_{12}] = -\xi_2, \\
&[\xi_1, \xi_{13}] = \xi_6, && [\xi_1, \xi_{15}] = \xi_4, && [\xi_2, \xi_{13}] = -(\xi_7 + \xi_8), \\
&[\xi_2, \xi_{14}] = -\xi_3, && [\xi_2, \xi_{15}] = \xi_5, && [\xi_3, \xi_{15}] = -\xi_7, \\
&[\xi_4, \xi_{11}] = -(\xi_8 + \xi_9), && [\xi_4, \xi_{12}] = -\xi_5, && [\xi_4, \xi_{14}] = \xi_6, \\
&[\xi_5, \xi_{14}] = -\xi_8, && [\xi_6, \xi_{12}] = -\xi_9, && [\xi_{11}, \xi_{15}] = -\xi_{10}, \\
&[\xi_{12}, \xi_{13}] = -\xi_{10}, && [\xi_{12}, \xi_{14}] = -\xi_{11}, && [\xi_{14}, \xi_{15}] = -\xi_{13}.
\end{aligned} \tag{A.3}$$

A.1.2 Describing the graded chain complex, $\mathrm{gr}^j\left(\bigwedge^n \mathfrak{g}\right)$

Looking at (3.3) (with $e = 1$ and $h = 4$), we see that

$$\begin{aligned}
&\omega(g_1) = 1 - \frac{3}{4} = \frac{1}{4}, && \omega(g_2) = 1 - \frac{2}{4} = \frac{1}{2}, && \omega(g_3) = 1 - \frac{1}{4} = \frac{3}{4}, \\
&\omega(g_4) = 1 - \frac{2}{4} = \frac{1}{2}, && \omega(g_5) = 1 - \frac{1}{4} = \frac{3}{4}, && \omega(g_6) = 1 - \frac{1}{4} = \frac{3}{4}, \\
&\omega(g_7) = 1, && \omega(g_8) = 1, && \omega(g_8) = 1, && \omega(g_9) = 1, \\
&\omega(g_{10}) = \frac{3}{4}, && \omega(g_{11}) = \frac{2}{4} = \frac{1}{2}, && \omega(g_{12}) = \frac{1}{4}, \\
&\omega(g_{13}) = \frac{2}{4} = \frac{1}{2}, && \omega(g_{14}) = \frac{1}{4}, && \omega(g_{15}) = \frac{1}{4}.
\end{aligned}$$

Hence

$$\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \mathrm{gr}\, I = \mathrm{span}_k(\xi_1, \ldots, \xi_{15}) = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3 \oplus \mathfrak{g}^4,$$

where

$$\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{4}} = \mathrm{span}_k(\xi_1, \xi_{12}, \xi_{14}, \xi_{15}),$$

$$\mathfrak{g}^2 = \mathfrak{g}_{\frac{1}{2}} = \mathrm{span}_k(\xi_2, \xi_4, \xi_{11}, \xi_{13}),$$
$$\mathfrak{g}^3 = \mathfrak{g}_{\frac{3}{4}} = \mathrm{span}_k(\xi_3, \xi_5, \xi_6, \xi_{10}),$$
$$\mathfrak{g}^4 = \mathfrak{g}_1 = \mathrm{span}_k(\xi_7, \xi_8, \xi_9).$$

See Remark 3.9 for more details.

This is enough to calculate the graded mod p cohomology of $\mathfrak{g}$, see [Kon22] for the details. We write the result in Table 3.8.

A.2 $I \subseteq \mathrm{GL}_4(\mathbb{Z}_p)$

In this section we will briefly describe the work needed to find continuous group cohomology of the pro-p Iwahori subgroup I of $\mathrm{GL}_4(\mathbb{Q}_p)$.

When I is the pro-p Iwahori subgroup in $\mathrm{GL}_4(\mathbb{Q}_p)$, we know by Section 3.1 that we can take it to be of the form

$$I = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix} \subseteq \mathrm{GL}_4(\mathbb{Z}_p),$$

and, by Section 3.1, we have an ordered basis

$$
\begin{aligned}
&g_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p & & & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ p & & 1 & \\ & & & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & & & \\ p & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \\
&g_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & 1 & & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & p & 1 & \\ & & & 1 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & p & 1 \end{pmatrix}, \\
&g_7 = \begin{pmatrix} \exp(p) & & & \\ & \exp(-p) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad g_8 = \begin{pmatrix} 1 & & & \\ & \exp(p) & & \\ & & \exp(-p) & \\ & & & 1 \end{pmatrix}, \\
&g_9 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \exp(p) & \\ & & & \exp(-p) \end{pmatrix}, \quad g_{10} = \begin{pmatrix} \exp(p) & & & \\ & \exp(p) & & \\ & & \exp(p) & \\ & & & \exp(p) \end{pmatrix}, \\
&g_{11} = \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad g_{12} = \begin{pmatrix} 1 & & & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad g_{13} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \\
&g_{14} = \begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad g_{15} = \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad g_{16} = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.
\end{aligned}
\tag{A.4}
$$

Since we just renamed some elements and added an element of the center of $\mathrm{GL}_4(\mathbb{Z}_p)$ when comparing to the ordered basis of $I \subseteq \mathrm{SL}_4(\mathbb{Z}_p)$ from Appendix A.1, it is clear from Equation (A.3) that the only non-zero commutators $[\xi_i, \xi_j]$ with $i < j$ are:

$$
\begin{array}{lll}
[\xi_1, \xi_{11}] = -(\xi_7 + \xi_8 + \xi_9), & [\xi_1, \xi_{12}] = -\xi_3, & [\xi_1, \xi_{12}] = -\xi_2, \\
[\xi_1, \xi_{14}] = \xi_6, & [\xi_1, \xi_{16}] = \xi_4, & [\xi_2, \xi_{14}] = -(\xi_7 + \xi_8), \\
[\xi_2, \xi_{15}] = -\xi_3, & [\xi_2, \xi_{16}] = \xi_5, & [\xi_3, \xi_{16}] = -\xi_7, \\
[\xi_4, \xi_{12}] = -(\xi_8 + \xi_9), & [\xi_4, \xi_{13}] = -\xi_5, & [\xi_4, \xi_{15}] = \xi_6, \\
[\xi_5, \xi_{15}] = -\xi_8, & [\xi_6, \xi_{13}] = -\xi_9, & [\xi_{12}, \xi_{16}] = -\xi_{11}, \\
[\xi_{12}, \xi_{14}] = -\xi_{11}, & [\xi_{12}, \xi_{15}] = -\xi_{12}, & [\xi_{15}, \xi_{16}] = -\xi_{14}.
\end{array}
\tag{A.5}
$$

Looking at Appendix A.1, we easily see that

$$\mathfrak{g}^1 = \mathfrak{g}_{\frac{1}{4}} = \mathrm{span}_k(\xi_1, \xi_{13}, \xi_{15}, \xi_{16}),$$

$$\mathfrak{g}^2 = \mathfrak{g}_{\frac{1}{2}} = \mathrm{span}_k(\xi_2, \xi_4, \xi_{12}, \xi_{14}),$$
$$\mathfrak{g}^3 = \mathfrak{g}_{\frac{3}{4}} = \mathrm{span}_k(\xi_3, \xi_5, \xi_6, \xi_{11}),$$
$$\mathfrak{g}^4 = \mathfrak{g}_1 = \mathrm{span}_k(\xi_7, \xi_8, \xi_9, \xi_{10}).$$

This is enough to calculate the graded mod p cohomology of $\mathfrak{g}$, see [Kon22] for the details. We write the result in Table 3.9.

Appendix B

Other Research

In this chapter, I will give a very brief introduction to other research I have participated in. This is all joint research and it is beyond the scope of this dissertation. I refer to the papers [Dia+21a; Dia+21b; DKK20] for background and details. The research has been focused on two different, but closely related, subjects.

List-decodable mean estimation. In many statistical settings, including machine learning security and exploratory data analysis e.g. in biology, datasets contain arbitrary — and even adversarially chosen — outliers. The central question of the field of robust statistics is to design estimators tolerant to a small amount of unconstrained contamination (corrupted points).

The main question we have been researching is how to quickly find a robust estimator of the mean in the case where more than $\frac{1}{2}$ the points are corrupted. In this case a single accurate hypothesis is information-theoretically impossible, but one may be able to compute a small list of hypotheses with the guarantee that *at least one of them* is accurate. This relaxed notion of estimation is known as *list-decodable learning* in general, and *list-decodable mean estimation* in our more specialized case. In other words, we are giving an algorithm that solves the following problem "quickly".

Let D be a distribution with unknown mean μ and unknown bounded covariance $\Sigma \preceq \sigma^2 I$. Given a set $T \subset \mathbb{R}^d$ of size n and $\alpha \in (0, 1/2)$ such that an α-fraction of the points in T are i.i.d. samples from D. We want to output a list of candidate vectors $\{\widehat{\mu}_i\}_{i\in[s]}$ such that $s = \text{poly}(1/\alpha)$

(or optimally $O(1/\alpha)$) and with high probability $\min_{i\in[s]} \|\widehat{\mu}_i - \mu\|_2$ is small.

During our research we managed to give algorithms that solve the above list-decodable problem using $n = \Omega(d/\alpha)$ samples (optimal), $O(1/\alpha)$ hypotheses (optimal), and error and runtime as follows:

	Error	Time
[CMY20][a]	$O(\sigma/\sqrt{\alpha})$	$\widetilde{O}(nd/\alpha^C)$ $(C \geq 6)$
[DKK20]	$O(\sigma \log(1/\alpha)/\sqrt{\alpha})$	$\widetilde{O}(n^2 d/\alpha)$
[Dia+21b]	$O(\sigma/\sqrt{\alpha})$	$\widetilde{O}(nd/\alpha + 1/\alpha^6)$
	$O(\sigma\sqrt{\log(1/\alpha)/\alpha})$	$\widetilde{O}(nd/\alpha)$
[Dia+21a]	$O(\sigma \log(1/\alpha)/\sqrt{\alpha})$	$\widetilde{O}(n^{1+\varepsilon}d)$ ($\varepsilon > 0$ small)

[a] Concurrent work.

In summary, our most recent result (cf. [Dia+21a, Thm. 6]) — which is the best known currently in this setting — is:

Theorem B.1 (informal). *For any fixed constant $\varepsilon_0 > 0$, there is an algorithm* FastMultifilter *with the following guarantee. Let $\mathcal{D}$ be a distribution over $\mathbb{R}^d$ with unknown mean μ^* and unknown covariance Σ with $\|\Sigma\|_{\mathrm{op}} \leq \sigma^2$, and let $\alpha \in (0,1)$. Given α and a multiset of $n = \Omega(\frac{d}{\alpha})$ points on $\mathbb{R}^d$ such that an α-fraction are i.i.d. draws from $\mathcal{D}$,* FastMultifilter *runs in time $O(n^{1+\varepsilon}d)$ and outputs a list L of $O(\alpha^{-1})$ hypotheses so that with high probability we have*

$$\min_{\hat{\mu}\in L}\|\hat{\mu} - \mu^*\|_2 = O\Big(\frac{\sigma \log \alpha^{-1}}{\sqrt{\alpha}}\Big).$$

♣

Clustering well-separated mixture models. Mixture models are a well-studied class of generative models used widely in practice. Given a family of distributions $\mathcal{F}$, a mixture model $\mathcal{M}$ with k components is specified by k distributions $\mathcal{D}_1, \ldots, \mathcal{D}_k \in \mathcal{F}$ and non-negative mixing weights $\alpha_1, \ldots, \alpha_k$ summing to one, and its law is given by $\sum_{i\in[k]} \alpha_i \mathcal{D}_i$. That is, to draw a sample from $\mathcal{M}$, we first choose $i \in [k]$ with probability α_i, and then draw a sample from $\mathcal{D}_i$. When the weights are all equal to $\frac{1}{k}$, we call the mixture *uniform*. Mixture models, especially Gaussian mixture models, have been widely studied in statistics since pioneering work of Pearson in 1894, and more recently, in theoretical computer science.

A canonical learning task for mixture models is the *clustering problem*. Namely, given independent samples drawn from $\mathcal{M}$, the goal is to approximately recover which samples came from which component. To ensure that this inference task is information-theoretically possible, a common assumption is that $\mathcal{M}$ is "well-separated" and "well-behaved": for example, we may assume each component $\mathcal{D}_i$ is sufficiently concentrated (with sub-Gaussian tails or bounded moments), and that component means have pairwise distance at least Δ, for sufficiently large Δ. The goal is then to efficiently and accurately cluster samples from $\mathcal{M}$ with as small a separation as possible.

For this problem, we gave different algorithms for different settings, and managed to get different interesting results in each case in [Dia+21a]. In particular, see [Dia+21a, Cor. 6, 8, 9] for more details. Our main result can be considered to be:

Theorem B.2 (informal). *For any fixed constant $\varepsilon_0 > 0$, there is an algorithm with the following guarantee. Given a multiset of $n = \Omega(dk)$ i.i.d. samples from a uniform mixture model $\mathcal{M} = \sum_{i\in[k]} \frac{1}{k}\mathcal{D}_i$, where each component $\mathcal{D}_i$ has unknown mean μ_i, unknown covariance matrix Σ_i with $\|\Sigma_i\|_{\mathrm{op}} \leq \sigma^2$, and $\min_{i,i'\in[k],i\neq i'} \|\mu_i - \mu_{i'}\|_2 = \widetilde{\Omega}(\sqrt{k})\,\sigma$, the algorithm runs in time $O(n^{1+\varepsilon_0}\max(k,d))$, and with high probability correctly clusters 99% of the points.* ♣

Again, this is the best known currently in this setting.

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