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DISCLAIMER

SOLUTION OF THE UNITARITY EQUATION WITH OVERLAPPING LEFT AND RIGHT CUTS: A TOOL FOR STUDY OF THE S* AND SIMILAR SYSTEMS.*

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ABSTRACT

The partial-wave unitarity condition is complicated by the presence of overlapping left and right branch cuts when the lowest exchanged mass is small in comparison to the direct channel mass. A coupled-channel ND^{-1} method for constructing unitary amplitudes with overlapping cuts is described. The study is motivated in part by the problem of analyzing the $\pi\pi - K\bar{K}$ system near the S* resonance.

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I. INTRODUCTION

We discuss the unitarity condition for coupled two-body channels in a definite angular momentum state. For simplicity in notation we take two channels, but our methods apply as well for any finite number. The square of the energy in the center-of-mass frame is denoted by s, and s_i is the threshold of the i-th channel, $s_1 \leqslant s_2$. As is appropriate in analytic S-matrix theory, we study a generalization of ordinary unitarity obtained by analytic continuation. If the masses of the particles in one channel are not too dissimilar to those in the other, generalized unitarity has the form

$$\frac{1}{2i}\left[T(s_+) - T(s_-)\right] = T(s_+)\rho(s)T(s_-),$$

$$s > s_1, \quad T(s_\pm) = \lim_{\varepsilon\to 0+} T(s\pm i\varepsilon), \tag{1.1}$$

where the 2×2 scattering matrix $T(s)$ is analytic in regions above and below the half-line $s > s_1$. The diagonal matrix of phase-space factors, $\rho(s)$, includes unit step functions θ which vanish below channel thresholds:

$$\rho(s) = \{\rho_i(s)\delta_{ij}\}, \quad \rho_i(s) = \theta(s - s_i)q_i(s). \tag{1.2}$$

For the case of spinless, equal-mass particles in channel i one has

$$q_i(s) = \left[\frac{s - s_i}{s}\right]^{\frac{1}{2}}. \tag{1.3}$$

Generalized unitarity (1.1) restricts the amplitudes $T_{12}(s)$ and $T_{22}(s)$ in the region $s_1 < s < s_2$ where channel 2 is closed, whereas ordinary unitarity refers only to open channels.

A complication arises if mass differences are large. Namely, the left cuts of some of the amplitudes overlap the half-line $s > s_1$. This occurs when the lowest mass in a cross channel is sufficiently small in comparison with the mass of the direct channel. The unitarity condition then becomes

$$\frac{1}{2i}\left[T(s_+) - T(s_-)\right] = T(s_+)\rho(s)T(s_-) + \Delta_L T(s), \qquad (1.4)$$

where $\Delta_L T(s)$ is the matrix of discontinuities of $T(s)$ over the left cuts (denoted collectively by L).

An example is the two-channel problem with $\pi\pi$ and $K\bar{K}$ channels in a definite isospin state, considered near the $K\bar{K}$ threshold where the 4π state has only a small production cross section. Under the assumption of Mandelstam analyticity, the partial-wave amplitude for $K\bar{K} \to K\bar{K}$ has a left cut beginning at the branch point $s = 4(m_K^2 - m_\pi^2)$. According to (1.4) the right cut begins at $s = 4m_\pi^2$, so that the two cuts overlap. The amplitudes for $\pi\pi \to \pi\pi$ and $\pi\pi \to K\bar{K}$ do not have overlapping cuts; their nearest left branch points are at $s = 0$. The possible importance of treating correctly the overlapping cuts in the phenomenology of the $\pi\pi - K\bar{K}$ system, especially near the S^* resonance, has been emphasized by Ynduráin [1,2,3], González-Arroyo [2,3], and coworkers [3]. Although the $\pi\pi - K\bar{K}$ system has been discussed

extensively [5], it appears that a full explication of the unitarity effects remains to be made. A similar situation of overlapping cuts occurs in the $N\bar{N}$ system, which is of high current interest in connection with baryonium states [6].

In studying systems with overlapping cuts, from either a dynamical or a phenomenological viewpoint, one encounters a generalization of the standard problem of partial-wave dispersion relations [7]. That is, given the left cut part of the T matrix,

$$B(s) = T(s) - \frac{1}{\pi}\int_{s_1}^{\infty} \frac{T(s'_+)\rho(s')T(s'_-)ds'}{s' - s}, \qquad (1.5)$$

determine the most general $T(s)$ having that left cut part and satisfying the augmented unitarity equation (1.4) as well as appropriate conditions of analyticity and asymptotic behavior. We shall provide a straight-forward solution of this problem, based on the matrix ND^{-1} method [8,9,10,11]. As in the usual ND^{-1} method, the problem is reduced to solving a linear integral equation for $N(s)$. It is gratifying to find that the equation is identical in form to the usual one. Only the derivation of the equation is altered. Being of Fredholm type under weak conditions on $B(s)$, the equation is amenable to numerical solution.

Our results are applicable in phenomenology as well as in dynamical schemes. In phenomenology the traditional approach to determination of $B(s)$ is to use crossing symmetry and experimental information on scattering in the cross channel. Such an approach determines the nearby singularities of $B(s)$ to a certain extent, but leaves the distant singularities to be represented by empirical parameters. A potentially more informative approach now under development is to use a new definition of Reggeon exchange, valid at low as well as high energy [12]. The Reggeon exchanges involve all partial waves in the cross channel, and form an important (possibly dominant) part of the analytically continued cross channel absorptive part. It is hoped that a model of B(s) based primarily on Reggeon exchanges will be realistic.

An ambitious scheme for construction of a crossing-symmetric unitary Regge theory, proposed in Refs. [12, 13] and extended in a forthcoming paper to allow coupled channels, requires solution of a generalization of the problem treated here. In a crossing-symmetric treatment of coupled $\pi\pi$ and $K\bar{K}$ channels, for instance, one must account for the 4π threshold at $s = 16\, m_\pi^2$ in the $K\bar{K}$ amplitude, which lies to the left of the beginning of the left cut at $s = 4(m_K^2 - m_\pi^2)$. As we shall show in a later paper, this complicated situation of overlapping cuts can be handled in a rather simple way by extending the present ND^{-1} method to allow a matrix of externally prescribed absorption parameters, in analogy to the work of Ref. [14]. In the crossing-symmetric theory the absorption parameters for the 4π state, etc., are obtained dynamically through crossing. The extended ND^{-1} method with absorption should also be useful in phenomenology, especially for study of absorption in the $N\bar{N}$ system. A correct treatment of overlapping cuts is conceivably important in assessing the effects of absorption on baryonium states predicted from crossed NN potentials [6].

Section II contains the general solution of the two-channel problem under rather weak conditions on B(s). It will be evident that the method works as well for n channels. The Castillejo-Dalitz-Dyson (CDD) ambiguity [7] is treated in detail, since a complete treatment for the coupled channel case has not been available in the literature. Recently Nenciu, Rasche, Stihi and Woolcock [15] criticized the ND^{-1} method, and suggested a method based on a pole approximation to B(s) as a replacement. We feel that the discussion of Sections II and III answers their criticisms, and shows that the method is both general and practical. In our experience the pole approximation has not been very useful, since in realistic models B(s) is not given in terms of poles, and to approximate it by poles with sufficient accuracy is rather awkward. We note, however, that the pole approximation can be used in the ND^{-1} scheme with overlapping cuts, and that it leads as usual to explicit analytic forms for the solution of the integral equation.

In Section IV we give an ND^{-1} method for a single-channel problem with absorption present at threshold; for example, $K\bar{K} \to K\bar{K}$. The absorption parameters are regarded as given, and left cuts may or may not overlap the absorption cut below threshold.

In Section V we discuss a special case of our problem solved recently by González-Arroyo [4]; namely, a two-channel problem in which only the element $B_{22}(s)$ of B(s) is non-zero. We reveal two new aspects of the González-Arroyo solution by deriving it from our formalism: (a) it necessarily entails CDD poles as defined in the two-channel formalism; if there is not at least one CDD pole, only the trivial solution in which $T_{11}(s) = T_{12}(s) = 0$ is obtained; (b) eventhough the González-Arroyo solution entails arbitrary rational functions, it is not the general solution of the problem with $B_{11}(s) = B_{12}(s) = 0$; rather, it corresponds to putting some elements of the CDD pole residue matrices equal to zero.

In Section VI we comment on a proposal of Yndurain for an explicit unitary parametrization of the T matrix with overlapping cuts.

Appendix A is concerned with asymptotic estimates of principal value integrals under conditions of logarithmic decrease of the density function. Appendix B contains the proof that the integral equation of Section II is of Fredholm type under conditions of logarithmic decrease of $B(s)$.

We hope to re-examine in a later paper the phenomenology of the $\pi\pi \to K\bar{K}$ system near the S^* resonance, using the methods described.

II. GENERAL SOLUTION FOR TWO-CHANNEL CASE

In this section we solve the two-channel problem, with two pseudoscalar mesons of mass m_i in the i-th channel. The phase space factors are as given in (1.3), with $s_i = 4m_i^2$. We make analyticity assumptions weaker than those implied by the Mandelstam representation, since the extra generality involves little effort.

Let us first recall the implications of the Mandelstam representation. The partial-wave amplitudes $T_{11}(s)$ and $T_{12}(s) = T_{21}(s)$ are analytic in the s-plane, each with cuts $(-\infty, 0]$, $[s_1, \infty)$, where $s_1 = 4m_1^2$. If $m_2^2 < 2m_1^2$, $T_{22}(s)$

is analytic in the plane with cuts $(-\infty, 4(m_2^2 - m_1^2)]$, $[s_1, \infty)$. If $m_2^2 \geqslant 2m_1^2$, we must regard $T_{22}(s)$ as sectionally analytic, since the cuts overlap and divide the plane in two:

$$T_{22}(s) = \begin{cases} T_{22}^{(+)}(s), & \text{Im}s > 0, \\ T_{22}^{(-)}(s), & \text{Im}s < 0, \end{cases} \tag{2.1}$$

where $T_{22}^{(+)}(s)$ and $T_{22}^{(-)}(s)$ are analytic in their respective half-planes. One has $T_{ij}(s) = T_{ij}(s^*)^*$, which for $i = j = 2$ means that $T_{22}^{(+)}(s) = T_{22}^{(-)}(s^*)^*$.

Our requirements on the T matrix, weaker with respect to analyticity, will be as follows:

(i) $T_{ij}(s) = T_{ji}(s)$

(ii) $T_{11}(s)$ and $T_{12}(s)$ are analytic in open neighborhoods Ω_{11}, Ω_{12} of the half-line $[s_1, \infty)$, as illustrated in Fig. 1.

(iii)

$$T_{22}(s) = \begin{cases} T_{22}^{(+)}(s), & s \in \Omega_{22}^{(+)}, \\ T_{22}^{(-)}(s), & s \in \Omega_{22}^{(-)}, \end{cases}$$

where $T_{22}^{(\pm)}(s)$ is analytic in $\Omega_{22}^{(\pm)}$. Here $\Omega_{22}^{(+)}$ is an open region of the upper half-plane with $[s_1-\varepsilon, \infty)$ as part of its boundary, and $\Omega_{22}^{(-)}$ is the complex conjugate of that region; see Fig. 1.

(iv) $T(s) = T(s^*)^*$

(v) $\Delta T(s) = \frac{1}{2i}\left[T(s_+) - T(s_-)\right] = T(s_+)\rho(s)T(s_-) + \Delta_L T(s)$, $s \geqslant s_1$,

$$\Delta_L T(s) = \begin{bmatrix} 0 & 0 \\ 0 & \theta(s_L - s)\phi(s) \end{bmatrix}, \quad s_1 < s < s_2,$$

where $\theta(s)$ is the unit step function and $\phi(s) = \Delta T_{22}(s)$, $s_1 \leqslant s \leqslant s_L$.

(vi) $|T(s_+)| \leqslant \kappa(\ell n s)^{-\alpha}$, $s \geqslant s_1$,

$$|T(s_+) - T(s'_+)| \leqslant \kappa(\ell n s)^{-\alpha} \left|\frac{s - s'}{s}\right|^{\mu}, \quad s' \geqslant s \geqslant s_1,$$

$$\alpha > 1, \quad 0 < \mu \leqslant 1/2. \qquad (2.2)$$

Here and in the following, κ represents a generic positive constant, which is understood to have different values in different equations. The inequalities (2.2 vi) apply to each element of the matrix $T(s)$ separately. The second of these inequalities follows from the stronger but more comprehensible requirement that $T(s_+)$ be Hölder-continuous for $s < r$ and continuously differentiable for $s > r$ with $|T'(s_+)| < \kappa s^{-1} \ell n^{-\alpha} s$, the point r being arbitrary.

We shall determine the entire class of T matrices satisfying conditions (2.2i) - (2.2vi) and having the same given left-hand cut term,

$$B(s) = T(s) - \frac{1}{\pi}\int_{s_1}^{\infty} \frac{T(s'_+)\rho(s')T(s'_-)ds'}{s' - s}. \qquad (2.3)$$

Note that property (2.2vi) ensures convergence of the integral in (2.3). The following conditions on $B(s)$ are a consequence of the conditions on $T(s)$ and the definition (2.3):

(i) $B_{ij}(s) = B_{ji}(s)$

(ii) $B_{11}(s)$, $B_{12}(s)$, and $B_{22}(s)$ are analytic in $\hat{\Omega}_{11} = \Omega_{11} \cup [s_1, \infty)$, $\hat{\Omega}_{12} = \Omega_{12} \cup [s_1, \infty)$, and $\hat{\Omega}_{22} = \Omega_{22}^{(+)} \cup \Omega_{22}^{(-)} \cup [s_L, \infty)$, respectively; see Fig. 2.

(iii) $B(s) = B^*(s^*)$

(iv) $|B(s)| \leqslant \kappa(\ell n s)^{-\alpha}$, $s \geqslant s_1$,

$$|B(s) - B(s')| \leqslant \kappa(\ell n s)^{-\alpha}\left|\frac{s - s'}{s}\right|^{\mu}, \quad s' \geqslant s \geqslant s_1. \qquad (2.4)$$

The property (2.4iv) is obtained from (2.3) with the help of Lemma 2 on asymptotic behavior of principal value integrals which is proved in Appendix A. The other properties of B follow immediately from (2.2).

Henceforth we suppose that a function $B(s)$, satisfying (2.4i) - (2.4iv), is given. We seek the most general $T(s)$ which gives that $B(s)$ through (2.3), and which satisfies (2.2i) - (2.2vi). Our analysis is based on the non-trivial theorem that any $T(s)$ satisfying conditions (2.2iv) - (2.2vi) has an ND^{-1} representation with appropriate properties. To be more exact, under those conditions there exists a 2×2 matrix $\mathcal{D}(s)$ such that [16, 17]

(i) $\mathcal{D}_{ij}(s)$ is analytic in the plane with cut $[s_i, \infty)$, and is defined by continuity on the cut. The function on the cut, $\mathcal{D}_{ij}(s_\pm)$, is Hölder-continuous on any finite interval.

(ii) $\mathcal{D}(s_-) = [1 + 2i\rho(s)T(s_+)]\mathcal{D}(s_+)$

(iii) $\mathcal{D}(s) = \mathcal{D}^*(s^*)$

(iv) $\mathcal{D}(s)$ is non-singular (has an inverse) at every finite point of the cut plane, including points $s_\pm$ on the cut.

(v) There are integers n_i such that the modified matrix

$$\tilde{\mathcal{D}}(s) = [s^{-n_1}\mathcal{D}_{.1}(s), \; s^{-n_2}\mathcal{D}_{.2}(s)]$$

tends to a finite, real, non-singular limit as $|s| \to \infty$:

$$\tilde{\mathcal{D}}(s) \to \tilde{\mathcal{D}}(\infty) = \tilde{\mathcal{D}}^*(\infty); \quad \det \tilde{\mathcal{D}}(\infty) \neq 0.$$

Here $\mathcal{D}_{\cdot j}(s)$ denotes the j-th column of $\mathcal{D}(s)$. (2.5)

The properties (2.5) clearly do not determine $\mathcal{D}(s)$ uniquely; at the least, one may interchange the columns of a given $\mathcal{D}(s)$, and multiply them by non-zero constants, thereby obtaining a new matrix which staisfies (2.5). Nevertheless, the (non-ordered) pair of integers n_1, n_2 is uniquely determined by the asymptotic behavior of $T(s)$, and $n_1 + n_2$ sets the degree of ambiguity in the determination of $T(s)$ from a given $B(s)$, as we shall explain presently.

In the single-channel case, $\mathcal{D}(s)$ is determined up to a constant multiplier and has the familiar form

$$\mathcal{D}(s) = A \exp\left[-\frac{s}{\pi}\int_{s_1}^{\infty}\frac{\delta(s')ds'}{s'(s'-s)}\right], \tag{2.6}$$

where A is an arbitrary real constant, and $\delta(s)$ is the phase shift, normalized so that $\delta(s_1) = 0$. In the many-channel case there is, in general, no closed expression for $\mathcal{D}(s)$. Rather, $\mathcal{D}(s)$ is obtained through solution of a certain Fredholm integral equation with a kernel constructed from $T(s_+)$. If $\delta(s)$ in (2.6) tends to a limit $\delta(\infty)$ and obeys the bounds

$$|\delta(s) - \delta(\infty)| \leqslant K(\ell n s)^{-\alpha}, \quad |\delta(s) - \delta(s')| \leqslant K(\ell n s)^{-\alpha}\left|\frac{s-s'}{s}\right|^{\mu},$$

$$s < s', \quad 0 < \mu < 1, \quad \alpha > 1, \tag{2.7}$$

then

$$\mathcal{D}(s_+) \sim s^{\delta(\infty)/\pi}, \quad s \to +\infty. \tag{2.8}$$

If $\delta(\infty) \geqslant \pi$, one has a Castillejo-Dalitz-Dyson (CDD) ambiguity in the determination of $T(s)$ from a given $B(s)$; cf. 7 . We shall find a similar ambiguity in the two-channel case if $n_1 + n_2 \geqslant 1$.

Let us write

$$T(s) = \left[T(s)\mathcal{D}(s)\right]\mathcal{D}^{-1}(s) = \mathcal{N}(s)\mathcal{D}^{-1}(s), \tag{2.9}$$

and compute the discontinutiy of $\mathcal{N}(s)$ from (2.5ii). If the unitarity equation (2.2v) holds, we have

$$\Delta\mathcal{N}(s) = \frac{1}{2i}\left[T(s_+)\mathcal{D}(s_+) - T(s_-)\mathcal{D}(s_-)\right]$$

$$= \frac{1}{2i}\left[T(s_+) - T(s_-)(1 + 2i\rho(s)T(s_+))\right]\mathcal{D}(s_+)$$

$$= \Delta_L T(s)\mathcal{D}(s_+) = \Delta_L T(s)\mathcal{D}(s), \quad s \geqslant s_1. \tag{2.10}$$

In the final step of this calculation we are able to replace $\mathcal{D}(s_+)$ by $\mathcal{D}(s)$ because of the form of $\Delta_L T(s)$ and the fact that the cut of $\mathcal{D}_{2j}(s)$ begins at $s = s_2$. We have $\Delta\mathcal{N}(s) = 0$, $s \geqslant s_1$, in the simpler case in which left- and right-hand cuts do not overlap. With overlapping cuts,

$$\Delta\mathcal{N}_{1j}(s) = 0,$$

$$\Delta\mathcal{N}_{2j}(s) = \theta(s_L - s)\phi(s)\mathcal{D}_{2j}(s),$$

$$j = 1, 2; \quad s \geqslant s_1. \tag{2.11}$$

The left-hand cut of each matrix element $\mathcal{N}_{ij}(s)$ does not overlap the right-hand cut of the corresponding $\mathcal{D}_{ij}(s)$.

We next consider the possible asymptotic behaviors of $\mathcal{D}(s)$ allowed by (2.5v) and for each type of behavior write a Cauchy representation for a matrix $D(s)$ closely related to $\mathcal{D}(s)$. The matrices $D(s)$ will subsequently be used to derive integral equations for $N(s) = T(s)D(s)$. We suppose initially that there is no bound state pole of $T(s)$, and also that neither column of $\mathcal{D}(s)$ tends to the null vector as $|s| \to \infty$; thus, $n_i \geqslant 0$. Presently we shall show that the vanishing of a column of $\mathcal{D}(s)$ at infinity is an exceptional case, not expected to occur in realistic models, provided that there are no bound states.

First take Case 1 : $n_1 = n_2 = 0$ in (2.5v). We define

$$D(s) = \mathcal{D}(s)\,\mathcal{D}^{-1}(\infty), \qquad N(s) = \mathcal{N}(s)\,\mathcal{D}^{-1}(\infty). \tag{2.12}$$

By properties (2.5ii), (2.5iii), and (2.2vi),

$$\mathrm{Im}D(s_+) = [D(s_+) - D(s_-)]/2i = -\rho(s)T(s_+)\,\mathcal{D}(s_+)\,\mathcal{D}^{-1}(\infty)$$

$$= -\rho(s)\,\mathcal{N}(s)\,\mathcal{D}^{-1}(\infty) = -\rho(s)N(s) = O(\ell n^{-\alpha}s). \tag{2.13}$$

It follows that $D(s)$ has the Cauchy representation

$$\text{(Case 1)}: \quad D(s) = 1 - \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\rho(s')N(s')ds'}{s' - s} \;. \tag{2.14}$$

We next consider Case 2: $n_1 + n_2 > 0$ in (2.5v). To define $D(s)$ in this case we first choose any polynomials of the form

$$P_j(s) = \begin{cases} \prod_{k=1}^{n_j} (s - s_{kj}), & s_{kj} < s_1, \text{ if } n_j > 0, \\ 1\,, & \text{if } n_j = 0, \end{cases} \tag{2.15}$$

where the real points s_{kj}, $k = 1, 2, \cdots, n_j$, $j = 1, 2$, are all distinct. Then $D(s)$ is defined by

$$D(s) = \mathcal{D}(s)\begin{bmatrix} P_1^{-1}(s) & 0 \\ 0 & P_2^{-1}(s) \end{bmatrix} \tilde{\mathcal{D}}^{-1}(\infty)\,, \tag{2.16}$$

and it has the Cauchy representation

$$D(s) = 1 - \sum_{j=1}^{2}\sum_{k=1}^{n_j} \frac{C^{(kj)}}{s_{kj} - s} - \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\rho(s')N(s')ds'}{s' - s}\,, \tag{2.17}$$

Since $D(\infty) = 1$ and $N(s) = O(\ell n^{-\alpha}s)$. The residue matrices $C^{(kj)}$ have components

$$C^{(kj)}_{\ell m} = \mathcal{D}_{\ell j}(s_{kj})\left[\prod_{\substack{p=1 \\ p\neq k}}^{n_j} (s_{kj} - s_{pj})\right]^{-1} \mathcal{D}^{-1}_{jm}(\infty)\,. \tag{2.18}$$

Henceforth we shall consolidate the indices k and j, and write (2.17) as

$$\text{(Case 2)}: \quad D(s) = 1 - \sum_{i=1}^{n_c} \frac{C_i}{\sigma_i - s} - \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\rho(s')N(s')ds'}{s' - s}\,. \tag{2.19}$$

Except for possible poles, the matrices $D(s)$ have the same properties (2.5) as $\mathcal{D}(s)$. Also, (2.10) implies that

$$\Delta N(s) = \Delta_L T(s) D(s), \quad s \geq s_1. \tag{2.20}$$

The poles in (2.19) are analogous to the familiar CDD poles of the single channel case; we shall again call them CDD poles. Because the pole positions σ_i are all distinct, the residue matrices C_i are singular:

$$\det C_i = 0. \tag{2.21}$$

That is seen from (2.18): the matrix $C^{(kj)}$ has rank 1, being a dyadic constructed from the vectors $\mathcal{D}_{.j}(s_{kj})$ and $\mathcal{D}^{-1}_{j.}(\infty)$.

We now turn to the derivation of the integral equation obeyed by $N(s) = T(s)D(s)$ for each of the two cases. The integral equation has a dual status. First, it is a necessary condition on the $N(s)$ associated with any given $T(s)$. Second, it is a means of constructing a properly analytic, unitary, and symmetric $T(s)$ from a given $B(s)$. In applications one usually thinks only of the second aspect, but for the general theory, especially for demonstrating the generality of the solution of the construction problem, it is necessary to consider both aspects. We begin by deriving the equation as a necessary condition on $N(s)$ for a given $T(s)$, and later treat the construction problem.

For a given $T(s)$ satisfying conditions (2.2) and such that Case 1 holds, we examine the matrix function

$$\Lambda(s) = \left[T(s) - B(s)\right] D(s) - \frac{1}{\pi}\int_{s_1}^{\infty} \frac{B(s')\rho(s')N(s')ds'}{s' - s}. \tag{2.22}$$

Since $B(s)\rho(s)N(s) = O(\ln^{-2\alpha} s)$, the integral converges. Notice that by (2.3) the difference $T(s) - B(s)$ is defined in the whole cut plane, eventhough $T(s)$ and $B(s)$ separately may not be, in view of our weak assumptions (2.2ii), (2.2iii) on the region of definition and analyticity of $T(s)$. Clearly, (2.22) defines a function $\Lambda_1(s) = \Lambda(s)$, analytic in the half-plane $\mathrm{Im}\, s > 0$, and another function $\Lambda_2(s) = \Lambda(s)$, analytic in $\mathrm{Im}\, s < 0$. We show that $\Lambda_2(s)$ is the analytic continuation of $\Lambda_1(s)$, and that in fact $\Lambda_1(s) = \Lambda_2(s) \equiv 0$. For $s < s_1$,

$$\Delta\Lambda(s) = \left[\Delta T(s) - \Delta B(s)\right] D(s) = 0. \tag{2.23}$$

For $s \geq s_1$,

$$\begin{aligned}\Delta\Lambda(s) &= \Delta N(s) - \Delta B(s) D(s_-) - B(s_+)\Delta D(s) - B(s)\rho(s)N(s) \\ &= \Delta_L T(s)D(s) - \Delta_L T(s)D(s) + B(s)\rho(s)N(s) - B(s)\rho(s)N(s) \\ &= 0.\end{aligned} \tag{2.24}$$

The structure of $\Delta_L T(s)$, assumed in (2.2v), and the θ functions in $\Delta D(s) = -\rho(s)N(s)$, allowed us to replace $D(s_-)$ by $D(s)$ and $B(s_+)$ by $B(s)$ in (2.24). In the case without overlapping cuts, the terms $\Delta N(s)$ and $-\Delta B(s)D(s_-)$ are separately zero; here they are non-zero but fortunately cancel. We see that $\Lambda(s)$ is analytic in the entire plane. Also, it vanishes at infinity, as is clear from (2.3), (2.14), and (2.22). Thus, $\Lambda(s) \equiv 0$, and (2.14) may be substituted into (2.22) to yield the integral equation

$$N(s) = B(s) + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{B(s) - B(s')}{s - s'}\rho(s')N(s')ds'. \qquad (2.25)$$

Thanks to the θ function in $\rho(s')$, the domains of the first and second rows of $N(s')$ in the integral are $[s_1, \infty)$ and $[s_2, \infty)$, respectively. Consequently, each $N_{ij}(s)$ is in a region of analyticity over the domain in which Eq. (2.25) is to be solved, as is seen from (2.11).

The derivation of the integral equation proceeds similarly in Case 2. The only change required is to account for the poles of D(s). Referring to (2.22) and (2.19) we see that $\Lambda(s)$ inherits the poles and in fact

$$\Lambda(s) = \sum_{i=1}^{n_c} \frac{1}{s - \sigma_i}\left[T(\sigma_i) - B(\sigma_i)\right]C_i. \qquad (2.26)$$

This equation yields the result

$$N(s) = B(s) + \sum_{i=1}^{n_c} \frac{1}{s - \sigma_i}\left[R_i + B(s)\right]C_i + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{B(s) - B(s')}{s - s'}\rho(s')N(s')ds', \qquad (2.27)$$

$$R_i = T(\sigma_i) - B(\sigma_i). \qquad (2.28)$$

Henceforth we refer to the general equation (2.27), which includes (2.25) as the special case with $C_i = 0$.

The integral equation (2.27) is amenable to the Fredholm theory in an appropriate Banach space, as is shown in Appendix B. By the Fredholm Alternative Theorem [20], the integral equation has a unique solution in the space considered, provided that the corresponding homogeneous equation has no non-trivial solution in that space. We shall suppose that the homogeneous equation in fact does not have non-trivial solutions, since the contrary case has not arisen, as far as we know, in realistic physical models. It does arise in the anomalous event of an "extinct bound state" as discussed by Atkinson and Halpern [18]. The assumption that there is no solution of the homogeneous equation allows us to rule out the possibility that a column of $\mathcal{D}(s)$ vanishes at infinity, as promised above. If $\mathcal{D}_{.j}(s)$, the j-th column of $\mathcal{D}(s)$, tends to the null vector as $|s| \to \infty$, then it has a Cauchy representation

$$\mathcal{D}_{.j}(s) = -\frac{1}{\pi}\int_{s_1}^{\infty} \frac{\rho(s')\mathcal{N}_{.j}(s')ds'}{s' - s}. \qquad (2.29)$$

Owing to the lack of the usual unit matrix term on the right-hand side of (2.29), the corresponding integral equation for $\mathcal{N}_{.j}(s)$ is homogeneous:

$$\mathcal{N}_{.j}(s) = \frac{1}{\pi}\int_{s_1}^{\infty} \frac{B(s) - B(s')}{s - s'}\rho(s')\mathcal{N}_{.j}(s')ds'. \qquad (2.30)$$

Thus $\mathcal{N}_{.j}(s) = 0$ and $\mathcal{D}_{.j}(s) = 0$, contrary to the fact that $\mathcal{D}(s)$

is non-singular. The derivation of (2.30), carried out as before by showing that $\Lambda(s) = 0$, fails if $T(s)$ has a bound state pole. We defer the discussion of bound states.

Since we have ruled out the possibility that $\mathcal{D}_{.j}(s)$ vanishes at infinity, we may conclude that the matrix $N(s)$ for any $T(s)$ obeying (2.2) satisfies (2.27). Furthermore, the properties (2.2) and (2.5) guarantee that $N(s) = T(s)D(s)$ lies in the Banach space used in the Fredholm theory of Appendix B. Thus, for a given $T(s)$, the matrix $N(s) = T(s)D(s)$ coincides with the unique solution of the integral equation provided by Fredholm theory.

When $B(s)$ rather than $T(s)$ is given, there is no a priori certainty that a corresponding satisfactory $T(s)$ exists. By the preceeding remarks we do know that if such a $T(s)$ exists, it must be obtainable in the form $N(s)D(s)^{-1}$, where $N(s)$ is a Fredholm solution of (2.27) for some choice of the parameters C_i and R_i with an arbitrary choice of the σ_i; here $D(s)$ is given in terms of $N(s)$ by (2.19). We now show that the Fredholm solution $N(s)$ of (2.27) gives a $T(s)$ satisfying (2.2), provided that $\det D(s) \neq 0$ in the cut plane and that, when there are CDD poles, another minor condition holds [condition (2.34) below]. This assertion holds for any choice of the parameters consistent with restrictions already laid down. Those restrictions, we recall, are that all parameters be real, that $\sigma_i < s_1$, $\sigma_i \neq \sigma_j$, $\det C_i = 0$, and that the R_i be positive-definite, symmetric matrices. The positive-definite character of R_i follows from its definition and (2.3), since $T(s_+)\rho(s)T(s_-) = T(s_+)\rho(s)T(s_+)^\dagger$, where $\dagger$ denotes Hermitian adjoint. The matrices C_i and R_i entail only three real parameters

each, since C_i is singular and R_i is symmetric.

If $N(s)$ is a solution of (2.27) and $D(s)$ is given by (2.19), we may write $T(s) = N(s)D(s)^{-1}$, the proposed solution of our problem, in the form

$$T(s) = B(s) + \left[\sum_{i=1}^{n_c} \frac{1}{s - \sigma_i} R_i C_i + \frac{1}{\pi} \int_{s_1}^{\infty} \frac{B(s')\rho(s')N(s')ds'}{s' - s} \right] D^{-1}(s). \tag{2.31}$$

This expression is derived by recognizing a term $B(s)D(s)$ on the right-hand side of (2.27). Since $\det D(s)$ is non-vanishing, it is clear that $T(s)$ has analyticity in accord with conditions (2.2), provided that it has no pole at $s = \sigma_i$. To demonstrate absence of a pole we write $D^{-1}(s) = \mathrm{cof}\, D(s)/\det D(s)$ and show by calculation using $\det C_i = 0$ that

$$C_i \,\mathrm{cof}\, D(s) = O(1), \quad s \to \sigma_i,$$

$$\det D(s) \sim \frac{a_i}{s - \sigma_i} + O(1), \quad s \to \sigma_i, \tag{2.32}$$

$$a_i = \sum_{j\neq i} (C_{i11}C_{j22} + C_{i22}C_{j11} - C_{i12}C_{j21} - C_{j12}C_{i21})$$

$$+ C_{i11}(1 + I_{22}(\sigma_i)) + C_{i22}(1 + I_{11}(\sigma_i)) - I_{12}(\sigma_i)C_{i21} - I_{21}(\sigma_i)C_{i12}. \tag{2.33}$$

where $I_{ij}(s)$ denotes the integral that appears in $D_{ij}(s)$. Thus, formula (2.31) contains no pole provided that

$$a_i \neq 0 . \tag{2.34}$$

Condition (2.34) is the extra requirement for existence of a solution in the presence of CDD poles, mentioned above.

Having proved analyticity, we have yet to show that (2.31) is properly unitary and symmetric. To check unitarity, we first calculate $\Delta N(s) = \Delta(T(s)D(s))$ from (2.31):

$$\begin{aligned} \Delta N(s) &= B(s_+)\Delta D(s) + \Delta B(s) D(s_-) + B(s)\rho(s)N(s) \\ &= -B(s)\rho(s)N(s) + \Delta_L T(s)D(s) + B(s)\rho(s)N(s) \\ &= \Delta_L T(s)D(s). \end{aligned} \qquad (2.35)$$

The unitarity condition (2.2v) is now verified as follows:

$$\begin{aligned} T(s_+) - T(s_-) &= \left[N(s_+)D(s_+)^{-1}D(s_-) - N(s_-)\right] D(s_-)^{-1} \\ &= \left[N(s_+)D(s_+)^{-1}D(s_-) - N(s_+) + 2i\Delta_L T(s)D(s)\right] D(s_-)^{-1} \\ &= N(s_+)D(s_+)^{-1}\left\{D(s_-) - D(s_+)\right\} D(s_-)^{-1} + 2i\Delta_L T(s) \\ &= N(s_+)D(s_+)^{-1}\rho(s)N(s)D(s_-)^{-1} + 2i\Delta_L T(s) \\ &= T(s_+)\rho(s)T(s_-) + 2i\Delta_L T(s). \end{aligned} \qquad (2.36)$$

As before, the prefactors $\rho(s)$ and $\Delta_L T(s)$ allowed us, on occasion, to replace $s_\pm$ by s. This calculation reveals a situation not present in the case with non-overlapping cuts. Namely, $T(s)$ satisfies unitarity only if $N(s)$ satisfies the integral equation (2.27). In the non-overlapping case $N(s)D(s)^{-1}$ is unitary, but in general not symmetric, for an arbitrary real matrix $N(s)$ such that the integral in $D(s)$ is well-defined.

Symmetry of $T(s)$ in (2.31) is proved by the method of Bjorken and Nauenberg [10]. We examine the function

$$\Phi(s) = D^T(s)\left[T(s) - T^T(s)\right] D(s), \qquad (2.37)$$

where the superscript T denotes transposition. Because of the definition (2.3) and the assumed symmetry (2.3i) of $B(s)$, it is clear that $\Phi(s)$ is analytic in the upper and lower half planes, even though we have not assumed that $T(s)$ is analytic in a whole cut plane. We shall show that the discontinuity of $\Phi(s)$ over the real axis is zero, and that $\Phi(s)$ has no pole of $s = \sigma_i$. Since $\Phi(s)$ vanishes at infinity, it must then be identically zero. The symmetry of T will follow, since we have assumed that $D(s)$ is non-singular. For $s < s_1$, $\Delta\Phi(s)$ is obviously zero, since $\Delta D(s) = 0$ and $B(s) = B^T(s)$:

$$\begin{aligned} \Delta\Phi(s) &= D^T(s)\Delta(T(s) - T^T(s))D(s) \\ &= D^T(s)\Delta(B(s) - B^T(s))D(s) \\ &= 0\,, \qquad s < s_1. \end{aligned} \qquad (2.38)$$

For $s > s_1$ we apply (2.35) and find

$$\begin{aligned} \Delta\Phi(s) &= \Delta\left[D^T(s)N(s) - N^T(s)D(s)\right] \\ &= D^T(s_+)\Delta N(s) + \Delta D^T(s)N(s_-) - N^T(s)\Delta D(s) - \Delta N^T(s)D(s_-) \\ &= D^T(s)\Delta_L T(s)D(s) - N^T(s)\rho(s)N(s) + N^T(s)\rho(s)N(s) - D^T(s)\Delta_L T(s)D(s) \\ &= 0. \end{aligned} \qquad (2.39)$$

The CDD poles in the factors $D^T(s)$ and $D(s)$ of (2.37) cancel. That is seen by introducing (2.31) and invoking the symmetry of R_i; the sum of the pole terms is

$$\sum_i (s - \sigma_i)^{-2} C_i^T (R_i - R_i^T) C_i = 0. \tag{2.40}$$

To show that $T(s)$ of (2.31) satisfies the bounds (2.2vi), we refer to the Fredholm theory of Appendix B which shows that the solution $N(s)$ of the integral equation obeys bounds the same as those of $T(s)$. If $I(s)$ denotes the integral appearing in $D(s)$, then Lemma 2 of Appendix A shows that $I(s_+)$ also obeys bounds like (2.2vi). It follows that $|T(s_+)| \leqslant K(\ell n s)^{-\alpha}$. To verify the second inequality of (2.2vi) we write, with $s \leqslant s'$,

$$N(s)D(s_+)^{-1} - N(s')D(s'_+)^{-1}$$

$$= \left[N(s) - N(s')\right] D(s_+)^{-1} + N(s')D(s_+)^{-1} \left[D(s'_+) - D(s_+)\right] D(s'_+)^{-1}. \tag{2.41}$$

When there are no CDD poles it is obvious that the required bound is satisfied for each of the terms on the right side. With poles, the only additional task is to demonstrate local Hölder continuity near the poles. That is easily done with the help of (2.32) and assumption (2.34).

We have finished the proof that $T(s)$ constructed from a solution of the integral equation (2.27) satisfies all of the conditions(2.2), provided only that $\det D(s) \neq 0$ in the cut plane and $a_i \neq 0$.

The question arises of how to verify in practice the condition $\det D(s) \neq 0$. In the following section we describe a simple and practical method of verifying the condition, which involves computation in the physical region only. Fortunately, it is not necessary to search the complex plane for zeros of $\det D(s)$.

Suppose that we solve (2.27) with an arbitrary choice of the real, symmetric, positive-definite matrices R_i; let us denote these input parameters as R_i^{in}. If $T^{out}(s)$ denotes the amplitude constructed from (2.31) and the solution of (2.27), will it necessarily happen that $R_i^{out} = T^{out}(\sigma_i) - B(\sigma_i) = R_i^{in}$?

In general the answer is no, because it is always possible to change R_i^{in} without changing R_i^{out}. Since C_i is singular, it has a real left eigenvector v_i with eigenvalue zero:

$$v_i^T C_i = 0. \tag{2.42}$$

If we change R_i^{in} by adding to it the positive-definite symmetric dyadic $\lambda v_i v_i^T$, $\lambda > 0$, there is no resulting change in R_i^{out}, since R_i^{in} enters the equations for $N(s)$ and $T^{out}(s)$ only in the product $R_i^{in} C_i$. Furthermore, we may argue that in general

$$R_i^{out} = R_i^{in} + \mu v_i v_i^T, \tag{2.43}$$

where $\mu = \mu(R_i^{in})$ is a real scalar function of R_i^{in}. If we multiply (2.31) on the right by $D(s)$, and equate residues of the poles on either side of the equation, we find that

$$\left(R_i^{out} - R_i^{in}\right) C_i = 0. \tag{2.44}$$

Both columns of C_i are proportional to the same vector u_i, and $R_i^{out} - R_i^{in}$, being real and symmetric, has the representation

$$R_i^{out} - R_i^{in} = \sum_{j=1}^{2} \lambda_j w_j w_j^T . \tag{2.45}$$

By (2.42) and the orthogonality of the w_j we see that either $w_j^T u_i = 0$ or $\lambda_j = 0$ for each j, and that at most one of the λ_j is non-zero. If λ_1, say, is non-zero, then w_1 has the same direction as v_i (being orthogonal to u_i), and (2.43) follows. Since R_i^{out} is a non-linear function of R_i^{in} (in the domain where it is not a constant function) the function $\mu(R_i^{in})$ is not a simple one.

How many arbitrary parameters are associated with each CDD pole? To answer this question, we first note that the pole positions σ_i are not to be counted as free parameters. Suppose that we have constructed an amplitude $T(s)$ from (2.27) with input parameters σ_i, C_i, R_i. Recalling the derivation of (2.27), we see that the same $T(s)$ has a representation $T(s) = \hat{N}(s)\hat{D}(s)^{-1}$, where $\hat{D}(s)$ has new pole positions $\hat{\sigma}_i$ and new residues $\hat{C}_i$, and $\hat{N}(s)$ satisfies (2.27) with $(\sigma_i, C_i, R_i) \to (\hat{\sigma}_i, \hat{C}_i, \hat{R}_i)$. Thus, a change in pole positions σ_i may always be compensated by a change in C_i and R_i so as to yield the same amplitute $T(s)$. The essential parameters are three in C_i and three in R_i, but it must be remembered that there is a subspace in the space of R_i parameters on which $T(s)$ is constant; i.e., we may add any term of the form $\lambda v_i v_i^T$, $\lambda > 0$, to R_i without changing $T(s)$.

III. BOUND STATES, LEVINSON'S THEOREM, AND A TEST FOR THE PRESENCE OF GHOST POLES.

Bound states seem not to occur in meson systems, but there is nevertheless a good technical reason to discuss them. The many-channel version of Levinson's theorem states that

$$\frac{1}{2i} \ln \det S(\infty) = - n_b + n_c , \tag{3.1}$$

where n_b is the number of bound state poles, n_c the number of CDD poles, and S the scattering matrix

$$S(s) = 1 + 2i\rho^{\frac{1}{2}}(s)T(s_+)\rho^{\frac{1}{2}}(s). \tag{3.2}$$

The quantity $\ln \det S(\infty)$ is defined by considering $\ln \det S(s)$ as a continuous function of s, with $\ln \det S(s_1) = 0$, and taking the increment between $s = s_1$ and $s = \infty$. Our interest in bound states and the Levinson relation stems from the circumstance that "ghost" poles (spurious poles of the T matrix lacking a physical interpretation) are counted in Levinson's relation just as though they were bound state poles. In a system with ghosts (3.1) is replaced by

$$\frac{1}{2i} \ln \det S(\infty) = - n_b - n_g + n_c , \tag{3.3}$$

where n_g is the number of ghost poles. In a calculation with the ND^{-1} method based on a specific model of $B(s)$, the number

$\ln \det S(\infty)$ is computed easily in conjunction with solution of the integral equation, n_c is an input parameter, and $n_b = 0$ is usually demanded by the physics of the situation. Thus, we can determine the number of ghosts from (3.3), rather than by searching the complex plane for zeros of $\det D(s)$. Should bound states be allowed in the problem, their location and number are easily determined by looking for zeros of $\det D(s)$ on a small interval of the real axis.

Levinson's relation is true for any amplitude $T(s)$ satisfying conditions (2.2), provided that the homogeneous form of Eq. (2.27) has no non-trivial solution (in the space considered in Appendix B). Of course, the latter condition is a restriction on $B(s)$ alone, and it seems invariably to be met in realistic models. It is understood that the poles of $T(s)$, n_b in number, are all simple poles with factorized residues (i.e., each residue matrix is of rank 1). A proof of Levinson's relation, valid under the conditions stated here, is given in section 5 of Ref. [11]. The proof as written applies when the poles of $T(s)$ are at real points $\hat{s} < s_1$. One may also have ghost poles at complex points in conjugate pairs $(\hat{s}, \hat{s}^*)$. An extension of the argument of Ref. [11] is required in that case.

IV. SINGLE-CHANNEL PROBLEM WITH OVERLAPPING CUTS AND ABSORPTION AT THRESHOLD

In some phenomenological studies it may be more practical to treat only one channel explicitly, accounting for coupled channels by empirical absorption parameters. A simple extension of the single-channel N/D method with absorption [7] allows one to handle processes such as $K\bar{K} \to K\bar{K}$ and $N\bar{N} \to N\bar{N}$ which have absorption at threshold

and overlapping cuts. The object is to construct unitary single-channel amplitudes of the form

$$T(s) = B_L(s) + \frac{1}{\pi}\int_{s_o}^{\infty} \frac{T(s'_+)q(s')T(s'_-)ds'}{s'-s} + \frac{1}{\pi}\int_{s_I}^{\infty}\frac{F(s')ds'}{s'-s}, \tag{4.1}$$

where the left cut part $B_L(s)$ and the absorption function $F(s)$ are given. We suppose that $B_L(s)$ has the properties of the function $B_{22}(s)$ of Section 2; (s_o in (4.1) is to be identified with s_2 in Section 2). The inelastic threshold s_I is assumed to be lower than the physical threshold s_o, and may be either to the left or to the right of the end of the left cut at s_L. With the channel considered labeled as the zeroth one, $F(s)$ is the inelastic part of the unitarity sum,

$$F(s) = \sum_{n\neq 0} T_{on}(s_+)\,\rho_n(s)T_{no}(s_-), \tag{4.2}$$

where the functions $\rho_n(s)$ contain step functions to account for the closing of channels. The sum over n may actually include integrals if states with more than two particles are involved. We have $F(s) \geqslant 0$ even for $s < s_o$; since the T matrix is real-analytic and symmetric, $T_{no}(s_-) = T_{on}(s_+)^*$ even below the threshold of channel 0. For $s \geqslant s_o$, $F(s)$ is expressed in terms of the usual elasticity function $\eta(s)$:

$$F(s) = \left[1 - \eta^2(s)\right]/4q(s), \quad s \geqslant s_o, \qquad (4.3)$$

$$T(s) = \left[\eta(s)e^{2i\delta(s)} - 1\right]/2iq(s), \quad s \geqslant s_o. \qquad (4.4)$$

We suppose that $F(s)$ satisfies bounds like those on $B(s)$ in (2.4iv). It then follows from (4.3) that $\left[1 - \eta(s)\right]/q(s)$ satisfies such bounds as well, and in particular that $\eta(s) \to 1$, $s \to s_{o+}$.

In the N/D method with absorption[7], the function $\mathcal{D}(s)$ is defined in terms of the real phase shift $\delta(s)$ of (4.4) by the expression (2.6). In the present extended method we use the same $\mathcal{D}(s)$, but use a $B(s)$ different from the usual one; namely,

$$B(s) = B_L(s) + \frac{1}{\pi}\int_{s_I}^{s_o} \frac{F(s')ds'}{s' - s} + \frac{1}{2\pi}\int_{s_o}^{\infty} \frac{1 - \eta(s')}{q(s')(s' - s)}\, ds' \; . \qquad (4.5)$$

In other words, we treat the part of the absorption cut between s_I and s_o just as though it were a left cut contribution. The derivation of the integral equation then proceeds in the same way as in Ref. 7. In the case without CDD poles the equation reads

$$\eta(s)n(s) = \mathrm{Re}B(s) + \frac{1}{\pi}\int_{s_o}^{\infty} \frac{\mathrm{Re}B(s) - \mathrm{Re}B(s')}{s - s'}\, q(s')n(s')ds' \qquad (4.6)$$

where $n(s) = -\,\mathrm{Im}D(s_+)/q(s)$. The amplitude is obtained in terms of $n(s)$ [which is not the same as the numerator function $N(s)$] by the formula

$$T(s) = B(s) + \frac{1}{\pi D(s)}\int_{s_o}^{\infty} \frac{\mathrm{Re}B(s')q(s')n(s')ds'}{s' - s}\, , \qquad (4.7)$$

$$D(s) = 1 - \frac{1}{\pi}\int_{s_o}^{\infty} \frac{q(s')n(s')ds'}{s' - s}\, . \qquad (4.8)$$

Each of the last two terms in (4.5), contributing to $\mathrm{Re}B(s)$, has a logarithmic singularity at $s = s_o$. The singularities of the two terms cancel, however, because $F(s)$ is Hölder-continuous and

$$F(s) = \frac{1 - \eta^2(s)}{4q(s)} \sim \frac{1 - \eta(s)}{2q(s)}, \quad s \to s_o+. \qquad (4.9)$$

As a result, $\mathrm{Re}B(s)$ is Hölder-continous for $s \geqslant s_o$, and the integral equation (4.6) is of Fredholm type on the space of Appendix B, provided that $\eta(s)$ has no zero. A solution of the integral equation gives an amplitude (4.7) that is properly analytic and satisfies unitarity in the form

$$\mathrm{Im}T(s_+) = T(s_+)\theta(s - s_o)q(s)T(s_-) + F(s) + \Delta_L T(s), \quad s > s_I. \qquad (4.10)$$

provided that $D(s)$ has no zero in the cut plane. As in Section III, a practical test for the presence of ghost zeros of $D(s)$ may be based on Levinson's relation, which in the present case holds in the form

$$\delta(\infty) = -\,\pi\left[n_b - n_c\right]. \qquad (4.11)$$

V. A SPECIAL CASE SOLVED BY GONZALEZ-ARROYO

We return to the two-channel problem of Section II and discuss a special case treated by González-Arroyo [4,3]; namely, the case in which the left-cut parts of T_{11} and T_{12} vanish, and non-relativistic kinematics hold:

$$B_{11}(s) = B_{12}(s) = B_{21}(s) = 0, \tag{5.1}$$

$$\rho_i(s) = \theta\,(s - s_i)(s - s_i)^{\frac{1}{2}}, \quad i = 1, 2. \tag{5.2}$$

Because $\rho_i(s)$ grows at infinity, we must assume that $B_{22}(s)$ vanishes more rapidly than does $B(s)$ of Section II. Instead of (2.4iv) we take

$$|B_{22}(s)| \leqslant Ks^{-\alpha}, \quad s \geqslant s_2,$$

$$|B_{22}(s) - B_{22}(s')| \leqslant Ks^{-\alpha} \left|\frac{s - s'}{s}\right|^{\mu}, \quad s' \geqslant s \geqslant s_2,$$

$$\frac{1}{2} < \alpha < 1, \quad 0 < \mu < 1/2. \tag{5.3}$$

For a given $B(s)$ satisfying (5.1), (5.3) we seek the general $T(s)$ satisfying (2.2i) - (2.2vi) and bounds such as (5.3) instead of (2.2vii). For such a $T(s)$ there is a $\mathcal{D}(s)$ satisfying (2.5i) - (2.5v) and a corresponding $D(s)$ as defined in (2.16), having the representation(2.19). Consequently, the integral equation (2.27) holds. The first row of the matrix equation is trivial, giving

$N_{ij}(s)$ explicitly as a function of CDD parameters:

$$N_{1j}(s) = \sum_{i=1}^{n_c} \frac{(R_i C_i)_{1j}}{s - \sigma_i}, \quad j = 1, 2. \tag{5.4}$$

From the second row of the integral equation we have

$$N_{21}(s) = \sum_{i=1}^{n_c} \left\{\left[R_i + B(s)\right] C_i\right\}_{21} \frac{1}{s - \sigma_i}$$

$$+ \frac{1}{\pi}\int_{s_2}^{\infty} \frac{B_{22}(s) - B_{22}(s')}{s - s'}\, q_2(s') N_{21}(s')\,ds' \tag{5.5}$$

and

$$N_{22}(s) = B_{22}(s) + \sum_{i=1}^{n_c} \left\{\left[R_i + B(s)\right] C_i\right\}_{22} \frac{1}{s - \sigma_i}$$

$$+ \frac{1}{\pi}\int_{s_2}^{\infty} \frac{B_{22}(s) - B_{22}(s')}{s - s'}\, q_2(s') N_{22}(s')\,ds'. \tag{5.6}$$

The integral in the D matrix elements (2.16) corresponding to (5.4) may be evaluated to obtain

$$D_{1j}(s) = \delta_{1j} + \sum_{i=1}^{n_c} \left\{(C_i)_{1j} + (R_i C_i)_{1j}\left[(s_1 - \sigma_i)^{\frac{1}{2}} + iq_1(s)\right]\right\} \frac{1}{s - \sigma_i} \tag{5.7}$$

We suppose as in Section II that the homogeneous version of the matrix integral equation for $N(s)$, equivalent in the present case to the homogeneous version of the scalar equation (5.5), has no non-trivial solution. Then if there are no CDD poles, the integral equation (5.5) for $N_{21}(s)$ is homogeneous and has only the trivial solution $N_{21}(s) = 0$. Without CDD poles we obtain only the trivial solution in which channel 2 is completely decoupled, and $T_{11}(s) = T_{12}(s) = T_{21}(s) = 0$. Thus, the solution of González-Arroyo necessarily entails CDD poles as defined in the two-channel formalism. Since González-Arroyo reduced his problem to a one-channel case by a special device, this fact was not previously apparent.

To reduce the problem to a one-channel case through our formalism, we circumvent solution of Eq. (5.5), and require solution of (5.6) alone. Accordingly, we suppose that the solution of (5.5), in a Banach space appropriate to conditions (5.3), is given; see the remark of the end of Appendix B. The solution $N_{22}(s)$ obeys conditions like (5.3); of course, the same is true of the solution $N_{21}(s)$ of (5.5). Because the inhomogeneous term in (5.5) is $O(s^{-1})$, it is possible to show that $N_{21}(s) = O(s^{-1})$ and $D_{21}(s_+) = O(s^{-\frac{1}{2}})$. The proof is done by showing that the integral operator "improves" the asymptotic behavior of $N_{21}(s)$. That is, if $N_{21}(s) = O(s^{-\alpha})$ then the integral is $O(s^{-2\alpha+\frac{1}{2}})$. By iteration of this argument one eventually gets $N_{21}(s) = O(s^{-1})$.

We exploit the symmetry of the T matrix, writing $T(s) = \hat{T}(s)/\det D(s)$ and

$$\hat{T}_{12}(s) = -N_{11}(s)D_{12}(s) + N_{12}(s)D_{11}(s) =$$

$$\hat{T}_{21}(s) = -N_{22}(s)D_{21}(s) + N_{21}(s)D_{22}(s). \qquad (5.8)$$

From (5.4) and (5.7) we may compute $\hat{T}_{12}(s)$; it is just a rational function, since the terms from the imaginary parts of $D_{12}(s)$ and $D_{11}(s)$ cancel. With that observation and a knowledge of $N_{22}(s)$ and $D_{22}(s)$ we can use (5.8) to find the general form of $N_{21}(s)$ and $D_{21}(s)$. The rational function $\hat{T}_{12}(s)$ is

$$\hat{T}_{12}(s) = -N_{11}(s)\mathrm{Re}D_{12}(s) + N_{12}(s)\mathrm{Re}D_{11}(s)$$

$$= -\sum_i \frac{(R_iC_i)_{11}}{s-\sigma_i} \sum_j \left[(C_j)_{12} + (s_1-\sigma_j)^{\frac{1}{2}}(R_jC_j)_{12}\right]\frac{1}{s-\sigma_j}$$

$$+ \sum_i \frac{(R_iC_i)_{12}}{s-\sigma_i}\left[1 + \sum_j \left[(C_j)_{11} + (s_1-\sigma_j)^{\frac{1}{2}}(R_jC_j)_{11}\right]\frac{1}{s-\sigma_j}\right]$$

$$= \frac{\Phi(s)}{P(s)}, \qquad s \geqslant s_1, \qquad (5.9)$$

where

$$P(s) = \prod_{i=1}^{n_c} (s-\sigma_i), \qquad (5.10)$$

and $\Phi(s)$ is a polynomial of degree not greater than n_c-1. The second order poles, corresponding to $i = j$ in the sums of (5.9), cancel because of the condition $\det C_i = 0$.

Equation (5.8) may be construed as a Riemann-Hilbert boundary value problem for determination of $D_{21}(s)$. Since $N_{2j}(s) = -\mathrm{Im}D_{2j}(s)/q_2(s)$, the real part of (5.8) reads

$$q_2(s)\,\frac{\Phi(s)}{P(s)} = -\,\mathrm{Im}D_{21}(s_+)\,\mathrm{Re}D_{22}(s_+) + \mathrm{Re}D_{21}(s_+)\,\mathrm{Im}D_{22}(s_+)$$

$$= \left[D_{22}(s_+)D_{21}(s_-) - D_{22}(s_-)D_{21}(s_+)\right]/2i \; . \qquad (5.11)$$

By the rearrangement displayed in the second line of (5.11), the Riemann-Hilbert problem [17] is transformed to an inhomogeneous Hilbert problem [17]:

$$D_{21}(s_-) = \left[\frac{D_{22}(s_-)}{D_{22}(s_+)}\right] D_{21}(s_+) + \frac{2iq_2(s)\Phi(s)}{P(s)D_{22}(s_+)} \; . \qquad (5.12)$$

To solve the Hilbert problem we invoke the ubiquitous phase integral,

$$d(s) = \exp -\left[\frac{s}{\pi}\int_{s_2}^{\infty} \frac{\delta(s')ds'}{s'(s'-s)}\right] \; , \qquad (5.13)$$

$$e^{-2i\delta(s)} = D_{22}(s_+)/D_{22}(s_-) \; . \qquad (5.14)$$

Notice that $\delta(s)$ is the phase shift for the amplitude $N_{22}(s)/D_{22}(s)$, which obeys elastic unitarity, and is not to be confused with the channel 2 scattering amplitude $T_{22}(s)$. It is easy to see that $D_{22}(s)/d(s)$, being real for $s > s_2$, is a rational function with poles only at $s = \sigma_i$; we may write

$$D_{22}(s) = \frac{\Psi(s)}{P(s)}\,d(s) \qquad (5.15)$$

where $\Psi(s)$ is a polynomial of degree n_z equal to the number of zeros of $D_{22}(s)$. Nothing prevents $D_{22}(s)$ from having zeros, in

general, since poles of $N_{22}(s)/D_{22}(s)$ are not poles of the full T matrix. We have $\delta(\infty) = \pi(n_c - n_z)$, since $D_{22}(s_+) \sim 1$ and $d(s_+) \sim \delta(\infty)/\pi$, $s \to +\infty$. Now substitute $d(s_-)/d(s_+)$ for $D_{22}(s_-)/D_{22}(s_+)$ in (5.12) and use (5.15) to obtain

$$\frac{1}{2i}\left[D_{21}(s_+)/d(s_+) - D_{21}(s_-)/d(s_-)\right] =$$

$$-\,\frac{q_2(s)}{|d(s_+)|^2}\,\frac{\Phi(s)}{\Psi(s)} \; , \quad s \geq s_2 \; . \qquad (5.16)$$

Thus we have the discontinuity of $D_{21}(s)/d(s)$ over the cut $[s_2, \infty)$, and it is real as required. In addition, we know that $D_{21}(s)$ is analytic in the plane with cut $[s_2, \infty)$, except for simple poles at $s = \sigma_i$, and that it vanishes at infinity: $D_{21}(s_+) = O(s^{-\frac{1}{2}})$. Since $d(s) \sim s^{n_c - n_z}$ at infinity, $D_{21}(s)/d(s)$ obeys an unsubtracted dispersion relation if $n_c \geq n_z$. The right hand side of (5.16) is $O(s^{-\frac{1}{2}+n_z-n_c})$ since degree $(\Phi) \leq n_c - 1$. For $n_c \geq n_z$ we have the representation

$$D_{21}(s) = d(s)\left[-\frac{1}{\pi}\int_{s_2}^{\infty} \frac{q_2(s')}{|d(s'_+)|^2}\,\frac{\Phi(s')}{\Psi(s')}\,\frac{ds'}{s'-s} + \sum_{i=1}^{n_c} \frac{(C_i)_{21}}{d(\sigma_i)}\,\frac{1}{s-\sigma_i}\right] \; . \qquad (5.17)$$

Note that $d(s)$ may be redefined through multiplication by a constant, but that (5.17) is invariant to such a change $(d \to \lambda d,\ \Psi \to \lambda^{-1}\Psi)$. For $n_z - n_c = n_s \geq 1$, we must introduce n_s subtractions, and replace (5.17) by the formula

$$D_{21}(s) = Q(s)d(s)\left\{ \sum_{i=1}^{n_s} \frac{D_{21}(t_i)}{d(t_i)} \quad \frac{1}{Q'(t_i)(s - t_i)} \right.$$

$$\left. - \frac{1}{\pi} \int_{s_2}^{\infty} \frac{q_2(s')}{|d(s'_+)|^2} \quad \frac{\Phi(s')}{\Psi(s')} \quad \frac{ds'}{Q(s')(s' - s)} + \sum_{i=1}^{n_c} \frac{(C_i)_{21}}{Q(\sigma_i)d(\sigma_i)} \frac{1}{s - \sigma_i} \right\}, \tag{5.18}$$

where $Q(s)$ is a polynomial with distinct roots $t_i < s_2$, none of the t_i coinciding with a σ_j. The function $N_{21}(s) = - \text{Im}D_{21}(s_+)/q_2(s)$ may be computed from (5.8), (5.9) or by taking the discontinuity of (5.17) or (5.18). By either method we find

$$N_{21}(s) = N_{22}(s) \left[\frac{D_{21}(s)}{D_{22}(s)}\right] + \frac{\Phi(s)}{P(s)D_{22}(s)} . \tag{5.19}$$

The representation (5.17) of $D_{21}(s)$ is determined by $(C_i)_{21}$ and the functions $\Phi(s)$, $D_{22}(s)$. The latter are in turn determined by the matrices C_i and R_i, through (5.9), (5.6), and (2.16). Thus, we have determined $D_{21}(s)$ in terms of the input parameters C_i, R_i without having solved the integral equation (5.5), provided that $n_c - n_z \geqslant 0$. On the other hand, we can assert that $N_{21}(s)$ as determined by (5.19) and (5.17) in fact solves the integral equation, since the matrix $D(s)$ constructed from a solution of (2.27) satisfies all the requirements that led to the unique function (5.17).

If $n_c - n_z < 0$ (which implies that $n_z \geqslant 2$), then $D_{21}(z)$ is not determined uniquely by the above considerations, because of the unknown subtraction constants $D_{21}(t_i)/d(t_i)$ in (5.18).

Consequently, we cannot be sure that the corresponding $N_{21}(s)$ satisfies (5.5). Nevertheless, we can demonstrate that $T(s)$ constructed from (5.18), (5.19) and the other previously determined elements of $D(s)$ and $N(s)$ actually is a solution of our problem for arbitrary subtraction constants (provided, as usual, that $\det D(s) \neq 0$ in the cut plane). It then follows that $N_{21}(s)$ constructed from (5.19), (5.18) with arbitrary subtraction constants satisfies (5.5), but with a value of the parameter $(R_i C_i)_{21}$ which may only be computed <u>a posteriori</u> as $\left\{ [T(\sigma_i) - B(\sigma_i)] \, C_i \right\}_{21}$ from the $T(s)$ constructed. To show that $T(s)$ (constructed with (5.18), (5.19) and arbitrary subtraction constants) is a solution of our problem, we have only to verify unitarity, since proper analyticity is evident, and symmetry of $T(s)$ was ensured through the use of (5.8). Unitarity follows from the calculation (2.36) if (2.35) holds. The first row of (2.35) is trivial because of (5.4), and we have $\Delta N_{22}(s) = (\Delta_L T(s) D(s))_{22} = \Delta_L T_{22}(s) D_{22}(s)$ as is usual for a single-channel N/D problem. To finish the proof of (2.35) one has only to show that $\Delta N_{21}(s) = \Delta_L T_{22}(s) D_{21}(s)$, and that is easily done by (5.19) and (5.18). For $s > s_2$, $N_{21}(s) = 0$ because $N_{21}(s)$ is real, being the discontinuity of the product of two real-analytic functions displayed in (5.18). For $s < s_2$, (5.19) gives

$$\Delta N_{21}(s) = \Delta N_{22}(s) \left[\frac{D_{21}(s)}{D_{22}(s)}\right] = \Delta_L T_{22}(s) D_{21}(s) . \tag{5.20}$$

To make contact with the solution of González-Arroyo, we look at the K matrix [19],

$$K(s) = N(s)\left[\mathrm{Re}D(s)\right]^{-1}, \quad s \geqslant s_2. \tag{5.21}$$

The solution in question is such that the element $K_{22}(s)$ is equal to the K matrix for the "decoupled" channel 2 problem, namely $N_{22}(s)/\mathrm{Re}D_{22}(s)$:

$$K_{22}(s) = \frac{(N_{22}/\mathrm{Re}D_{22})\mathrm{Re}D_{11} - (N_{21}/\mathrm{Re}D_{22})\mathrm{Re}D_{12}}{\mathrm{Re}D_{11} - (\mathrm{Re}D_{21}/\mathrm{Re}D_{22})\mathrm{Re}D_{12}}$$

$$= N_{22}(s)/\mathrm{Re}D_{22}(s). \tag{5.22}$$

Condition (5.22) can be met in only two ways: either $\mathrm{Re}D_{12}(s) = 0$ or $N_{21}(s)\mathrm{Re}D_{22}(s) - N_{22}(s)\mathrm{Re}D_{21}(s) = 0$. The latter equation implies that $K_{12}(s) = K_{21}(s) = 0$, however, from which it follows that $T_{12}(s) = T_{21}(s) = 0$; i.e., that the solution is trivial. We must take $\mathrm{Re}D_{12}(s) = 0$, and by (5.7) we see that the González-Arroyo solution corresponds to a particular choice of CDD parameters such that

$$(C_i)_{12} = 0, \quad (R_iC_i)_{12} = 0. \tag{5.23}$$

With $\mathrm{Re}D_{12}(s) = 0$ one has

$$K_{12}(s) = K_{21}(s) = \frac{N_{12}(s)}{\mathrm{Re}D_{22}(s)}, \tag{5.24}$$

and

$$K_{11}(s) = \frac{1}{\mathrm{Re}D_{11}(s)}\left[N_{11}(s) - N_{12}(s)\frac{\mathrm{Re}D_{21}(s)}{\mathrm{Re}D_{22}(s)}\right]. \tag{5.25}$$

According to (5.4) and (5.15), we may write (5.24) in the form

$$K_{12}(s) = \frac{\Psi_{12}(s)}{\Psi(s)}\,\frac{1}{\mathrm{Re}\,d(s)}, \tag{5.26}$$

where $\Psi_{12}(s) = N_{12}(s)P(s)$ is a polynomial of degree not greater that $n_c - 1$. González-Arroyo has

$$K_{12}(s) = \left[1 - iq_2(s)K_{22}(s)\right]\frac{\chi^{(o)}(s)}{d(s_+)}$$

$$= \frac{D_{22}(s_+)}{\mathrm{Re}D_{22}(s)}\,\frac{\chi^{(o)}(s)}{d(s_+)}$$

$$= \frac{\chi^{(o)}(s)}{\mathrm{Re}\,d(s)}, \tag{5.27}$$

where $\chi^{(o)}(s)$ is a rational function which is $O(s^{n_c-1-n_z})$ at infinity and has poles at the zeros of $D_{22}(s)$ (i.e., of $\Psi(s)$), in agreement with our function $\Psi_{12}(s)/\Psi(s)$ of (5.26). The argument of Ref. 4 seems to allow poles of $\chi^{(o)}(s)$ at other points as well, but our generally valid expression (5.26) shows that additional poles are not possible: we have $\chi^{(o)}(s) = \Psi_{12}(s)/\Psi(s)$ with poles only at the zeros of $D_{22}(s)$.

Next let us evaluate (5.25) using expression (5.17). With the help of (5.15) and (5.14) we get

$$K_{11}(s) = \left\{N_{11}(s) + \frac{\Psi_{12}(s)}{\Psi(s)}\left[\frac{P}{\pi}\int_{s_2}^{\infty}\frac{q_2(s')}{|d(s'_+)|^2}\,\frac{\Phi(s')}{\Psi(s')}\,\frac{ds'}{s'-s} - \sum_{i=1}^{n_c}\frac{(C_i)_{21}}{d(\sigma_i)}\,\frac{1}{s-\sigma_i} + \frac{q_2(s)\tan\delta(s)\Phi(s)}{|d(s_+)|^2\Psi(s)}\right]\right\}\frac{1}{\mathrm{Re}D_{11}(s)}, \quad s \geqslant s_2. \tag{5.28}$$

The corresponding formula in Ref. 4 is

$$K_{11}(s) = h^{(o)}(s) + \frac{P}{\pi}\int_{s_2}^{\infty} \frac{q_2(s')}{|d(s'_+)|^2} \left[\frac{\Psi_{12}(s')}{\Psi(s')}\right]^2 \frac{ds'}{s' - s}$$

$$+ \frac{q_2(s)\tan\delta(s)}{|d(s_+)|^2} \left[\frac{\Psi_{12}(s)}{\Psi(s)}\right]^2 , \quad s \geq s_2, \qquad (5.29)$$

where $h^{(o)}(s)$ is a rational function which has poles at the zeros of $\Psi(s)$ and is $0(s^{-1})$ at infinity. In order that the terms proportional to $\tan\delta(s)$ in (5.28) and (5.29) agree, it is necessary that $\Phi(s) = \Psi_{12}(s)$ and $\mathrm{Re}D_{11}(s) = 1$. According to (5.9) and the condition $\mathrm{Re}D_{12}(s) = 0$ already imposed, $\Phi(s) = \Psi_{12}(s)$ follows from $\mathrm{Re}D_{12}(s) = 1$. By (5.7) the latter is true if and only if

$$(C_i)_{11} = 0, \quad (R_iC_i)_{11} = 0. \qquad (5.30)$$

With $\Phi(s) = \Psi_{12}(s)$ we still have to resolve the discrepancy between the integrals that appear in (5.28) and (5.29). Consider the function

$$f(s) = \frac{\Psi_{12}(s)}{\Psi(s)} \frac{1}{\pi}\int_{s_2}^{\infty} \left[\frac{q_2(t)}{|d(t_+)|^2} \frac{\Psi_{12}(t)}{\Psi(t)}\right] \frac{dt}{t - s} . \qquad (5.31)$$

The bracketed expression is $0(t^{-n_c+n_z-\frac{1}{2}})$ at infinity and we are assuming that $n_c \geq n_z$; hence the integral is $0(s^{-\frac{1}{2}})$. The factor in front of the integral is $0(s^{n_c-n_z-1})$. We may write a dispersion relation for $f(s)$ with n subtractions where

$$n = \begin{cases} 0, & n_c = n_z \ , \ n_z + 1 \ , \\ n_c - n_z - 1, & n_c - n_z > 1. \end{cases} \qquad (5.32)$$

Namely,

$$f(s) = \sum_{m=0}^{n-1} f^{(m)}(0)s^m + s^n\left\{\sum_{i=1}^{n_z} \frac{a_i}{s - t_i} + \frac{1}{\pi}\int_{s_2}^{\infty}\left[\frac{q_2(t)}{|d(t_+)|^2}\left(\frac{\Psi_{12}(t)}{\Psi(t)}\right)^2\right]\frac{dt}{t^n(t-s)}\right\}, \qquad (5.33)$$

where the sum over i is due to the poles of $1/\Psi(s)$ at points t_i, assumed distinct. The bracketed factor in the integrand is $0(t^{-3/2})$, so that we can remove all n subtractions in the integral by iteration of the identity

$$\frac{s^n}{(s - t)t^n} = \frac{s^{n-1}}{t^{n-1}}\left[\frac{1}{s - t} + \frac{1}{t}\right] . \qquad (5.34)$$

thus

$$\mathrm{Re}f(s) = \frac{\Psi_{12}(s)}{\Psi(s)} \frac{P}{\pi}\int_{s_2}^{\infty} \frac{q_2(t)}{|d(t_+)|^2} \frac{\Psi_{12}(t)}{\Psi(t)} \frac{dt}{t - s}$$

$$= \frac{P}{\pi}\int_{s_2}^{\infty} \frac{q_2(t)}{|d(t_+)|^2} \left[\frac{\Psi_{12}(t)}{\Psi(t)}\right]^2 \frac{dt}{t - s} + R(s), \qquad (5.35)$$

where

$$R(s) = \sum_{m=0}^{n-1} f^{(m)}(o)s^m + s^n \sum_{i=1}^{n_z} \frac{a_i}{s - t_i}$$

$$+ \sum_{m=0}^{n-1} s^m \frac{1}{\pi} \int_{s_2}^{\infty} \frac{q_2(t)}{|d(t_+)|^2} \left(\frac{\Psi_{12}(t)}{\Psi(t)}\right)^2 \frac{dt}{t^{m+1}} \quad . \qquad (5.36)$$

We see that the integrals in (5.28) and (5.29) differ by a rational function which has poles at the t_i and which in general is $O(s^{n-1})$ at infinity.

Finally, in order that (5.28) and (5.29) be compatible the rational function

$$N_{11}(s) + R(s) - \frac{\Psi_{12}(s)}{\Psi(s)} \sum_{i=1}^{n_c} \frac{(C_i)_{21}}{d(\sigma_i)} \frac{1}{s - \sigma_i} \qquad (5.37)$$

must have the properties required of $h^{(o)}(s)$. If $n_c = n_z$, $n_z + 1$, then $R(s) = O(s^{-1})$ and all terms in (5.37) are $O(s^{-1})$ at infinity as required. Otherwise, the second and third terms of (5.37) must cancel appropriately at infinity. González-Arroyo tacitly assumed, in fact, that $n_z = 0$. With that assumption we get a solution of his form when $n_c = 1$ and the CDD residues satisfy conditions (5.23) and (5.30). Even though the solution of González-Arroyo contains arbitrary rational functions, it is far from being the general solution of the problem posed.

VI. REMARK ON A UNITARY PARAMETRIZATION SUGGESTED BY YNDURAIN

We have shown how to construct properly analytic amplitudes satisfying the unitarity equation (2.2v), but the construction has the

disadvantage of requiring the solution of an integral equation. For phenomenology it would be useful to have a parametrization of $T(s)$, analogous to the usual K matrix parametrization, which would automatically satisfy (2.2v). Yndurain [1] has proposed a parametrization which has the required property in the region $s_1 \leqslant s \leqslant s_2$. Define $\bar{T}(s)$ such that $\bar{T}_{ij}(s) = T_{ij}(s)$ except for $i = j = 2$, and

$$\bar{T}_{22}(s) = T_{22}(s) - \frac{1}{\pi} \int_0^{s_L} \frac{\phi(s')ds'}{s' - s} \quad . \qquad (6.1)$$

There is nothing special about the lower limit 0 in the integral ; any lower limit less than s_1 will do. Define a matrix $\bar{M}(s)$, which is related to $\bar{T}(s)$ in the way that $M(s) = K^{-1}(s)$ is related to $T(s)$:

$$\bar{T}(s)^{-1} = \bar{M}(s) - i\rho(s). \qquad (6.2)$$

Now we may show that the unitarity condition (2.2v) is equivalent to the reality condition $\bar{M}(s) = \bar{M}(s)^*$ in the region $s_1 \leqslant s \leqslant s_2$. Let us consider the region $s \geqslant s_1$, suppose that $\bar{M}(s)$ is real in that region, and write

$$T(s) = \bar{T}(s) + \hat{T}(s),$$

$$\hat{T}(s) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\pi}\int_0^{s_L} \frac{\phi(s')ds'}{s' - s} \end{bmatrix} , \qquad (6.3)$$

Since reality of $\bar{M}(s)$ implies that

$$\Delta\bar{T}(s) = \bar{T}(s_+)\rho(s)\bar{T}(s_-) \tag{6.4}$$

we have

$$\Delta T(s) = \bar{T}(s_+)\rho(s)\bar{T}(s_-) + \Delta_L T(s). \tag{6.5}$$

Also,

$$T(s_+)\rho(s)T(s_-) = \bar{T}(s_+)\rho(s)\bar{T}(s_-) + U(s), \tag{6.6}$$

where

$$\begin{aligned} U_{ij} &= \bar{T}_{-i1}\rho_1\hat{T}_{+1j} + \bar{T}_{-i2}\rho_2\,\hat{T}_{+2j} \\ &+ \hat{T}_{-i1}\rho_1\bar{T}_{+1j} + \hat{T}_{-i2}\rho_2\bar{T}_{+2j} \\ &+ \hat{T}_{-i1}\rho_1\hat{T}_{+1j} + \hat{T}_{-i2}\rho_2\hat{T}_{+2j} \\ &= \rho_2\left[\bar{T}_{-i2}\hat{T}_{+2j} + \hat{T}_{-i2}\bar{T}_{+2j} + \hat{T}_{-i2}\hat{T}_{+2j}\right]\delta_{i2}\delta_{j2} \end{aligned} \tag{6.7}$$

Since $U(s) = 0$ for $s \leq s_2$, Eq. (6.5) is indeed the unitarity equation for $s_1 < s < s_2$.

For $s > s_2$, however, unitarity is not equivalent to $\bar{M}(s)$ being real, since $U_{22}(s) \neq 0$ in that region. Indeed, unitarity for $s > s_2$ is equivalent to $M(s) = K^{-1}(s)$ being real, where

$$T(s)^{-1} = M(s) + i\rho(s). \tag{6.8}$$

The relation between M and $\bar{M}$ is as follows:

$$\left[1 + (\bar{M}_{22} - i\rho_2)\hat{T}_{22}\right]\begin{bmatrix} M_{11} - i\rho_1 & M_{12} \\ M_{21} & M_{22} - i\rho_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left[1 + (\bar{M}_{22} - i\rho_2)\hat{T}_{22}\right](\bar{M}_{11} - i\rho_1) - \bar{M}_{12}^2\hat{T}_{22} & \bar{M}_{12} \\ \bar{M}_{12} & \bar{M}_{22} - i\rho_2 \end{bmatrix}. \tag{6.9}$$

Clearly $\bar{M}(s)$ does not change continuously into $M(s)$ at $s = s_2$; rather, at $s = s_2$ we have

$$M = \bar{M} - \frac{\hat{T}_{22}}{1 + \bar{M}_{22}\hat{T}_{22}}\begin{bmatrix} \bar{M}_{12}^{\,2} & \bar{M}_{12}\bar{M}_{22} \\ \bar{M}_{12}\bar{M}_{22} & \bar{M}_{22}^2 \end{bmatrix}. \tag{6.10}$$

The matrices $M(s)$ and $\bar{M}(s)$ are two different analytic functions which one would try to represent in terms of a few empirical parameters, so as to meet the following conditions:

(i) $\bar{M}(s) = \bar{M}(s)^*$, $s_1 < s < s_2$

(ii) $M(s) = M(s)^*$, $s > s_2$

(iii) $M(s)$ and $\bar{M}(s)$ are related by Eq. (6.9), $s > s_1$.

(iv) The analyticity properties of $M(s)$ and $\bar{M}(s)$ should reflect to a reasonable extent the correct analyticity properties of $T(s)$, especially the nearby singularities corresponding to the principal particle exchanges.

It seems rather difficult to satisfy all of these requirements simultaneously; in particular it seems hard to satisfy (iii) in such a way that (i) and (ii) would also hold.

We would expect Ynduràin's proposal to be rather limited in usefulness. The only alternative that we can think of, short of solving the integral equation (2.27), is to make a pole approximation for $B(s)$. As is well known, the kernel of the equation is then separable, and solution of the equation is reduced to quadratures and solution of algebraic equations. Unfortunately, for a realistic representation of $B(s)$ one usually needs so many poles that the resulting formulas are not very illuminating.

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APPENDIX A

ESTIMATES OF PRINCIPAL VALUE INTEGRALS

We are concerned with the asymptotic behavior and continuity properties of principal value integrals of the form

$$g(s) = P \int_{s_o}^{\infty} \frac{f(t)dt}{t - s} . \tag{A.1}$$

In the following a, θ, and δ are fixed positive constants, and κ is "some positive constant" which is understood to have different values in different inequalities.

<u>Lemma 1</u>: Suppose that $f(t)$ obeys the conditions

$$\text{(a)} \quad |f(t)| \leqslant \frac{\kappa}{t^a}\left[\frac{t - s_o}{t}\right]^{\theta}$$

$$\text{(b)} \quad |f(t_1) - f(t_2)| \leqslant \frac{\kappa}{t_1^a}\left[\frac{t_2 - s_o}{t_2}\right]^{\theta-\delta} \left|\frac{t_1 - t_2}{t_1}\right|^{\delta} ,$$

$$t_2 \geqslant t_1, \quad a + \delta < 1, \quad \theta > \delta . \tag{A.2}$$

Then the integral $g(s)$ of (A.1) is such that

$$\text{(a)} \quad |g(s)| \leqslant \frac{\kappa}{s^a}; \quad \text{(b)} \quad |g(s_1) - g(s_2)| \leqslant \frac{\kappa}{s_1^a}\left|\frac{s_1 - s_2}{s_1}\right|^{\delta}, \quad s_2 \geqslant s_1. \tag{A.3}$$

<u>Lemma 2</u>: Suppose that $f(t)$ obeys the conditions

$$\text{(a)} \quad |f(t)| < \frac{\kappa}{(\ln t)^a} \left[\frac{t - s_o}{t}\right]^{\theta};$$

$$\text{(b)} \quad |f(t_1) - f(t_2)| \leqslant \frac{\kappa}{(\ln t_1)^a} \left[\frac{t_2 - s_o}{t_2}\right]^{\theta-\delta} \left|\frac{t_1 - t_2}{t_1}\right|^{\delta} ,$$

$$t_2 \geqslant t_1, \quad a > 1, \quad \theta > \delta. \tag{A.4}$$

Then the integral $g(s)$ of (A.1) is such that

(a) $|g(s)| \leqslant \dfrac{\kappa}{(\ell n s)^{a-1}}$;

(b) $|g(s_1) - g(s_2)| \leqslant \dfrac{\kappa}{(\ell n s_1)^{a-1}} \left|\dfrac{s_1 - s_2}{s_1}\right|^{\delta}$, $s_2 \geqslant s_1$ (A.5)

(A.5)

We give a proof of Lemma 2: a proof of Lemma 1 follows the same lines , but is somewhat easier. To verify (A.5a) we write

$$g = P \int_4^{\mu s} + \int_{\mu s}^{\infty} = g_1 + g_2, \quad \mu > 1, \qquad \text{(A.6)}$$

and majorize g_2 immediately:

$$|g_2| \leqslant \kappa \int_{\mu s}^{\infty} \frac{dt}{\ell n^a t(t - s)} \leqslant \kappa \int_{\mu s}^{\infty} \frac{dt}{t\, \ell n^a t} \leqslant \frac{\kappa}{\ell n^{a-1} s} . \qquad \text{(A.7)}$$

For g_1 we use the identity

$$g_1(s) = \int_{s_o}^{\mu s} \frac{f(t) - f(s)}{t - s}\, dt + f(s) \ell n \left\{\frac{(\mu - 1)s}{s - s_o}\right\}. \qquad \text{(A.8)}$$

By introducing (A.4b) and (A.4a) in the first and second terms of (A.8), respectively, we see that $g_1(s)$ is bounded at small s, say $s < 2s_o$, and consequently (A.5a) holds at small s. For $s > 2s_o$, the logarithmic term in (A.8) clearly satisfies (A.5a). The other term is decomposed and bounded as follows, with $1/2 < \lambda < 1$:

$$\left(\int_{s_o}^{\lambda s} + \int_{\lambda s}^{\mu s}\right) \left|\frac{f(t) - f(s)}{t - s}\right| dt$$

$$\leqslant \kappa \int_{s_o}^{\lambda s} \frac{dt}{t^{\delta} \ell n^a t} \frac{1}{|\lambda s - s|^{1 - \delta}} + \kappa \int_{\lambda s}^{\mu s} \frac{dt}{s^{\delta} \ell n^a s} \frac{1}{|t - s|^{1 - \delta}}$$

$$\leqslant \frac{\kappa}{s^{1 - \delta}} \int_{s_o}^{\lambda s} \frac{dt}{t^{\delta} \ell n^a t} + \frac{\kappa}{\ell n^a s} \int_{\lambda}^{\mu} \frac{du}{|u - 1|^{1 - \delta}} \leqslant \frac{\kappa}{\ell n^a s} .$$

(A.9)

Thus (A.5a) is proved, and we see that the dominant part of $g(s)$ at large s is from the tail of the integral, $g_2(s)$.

To establish (A.5b) we split the integral as follows:

$$g = P \int_{s_o}^{2s_2} + \int_{2s_2}^{\infty} = g_1 + g_2. \qquad \text{(A.10)}$$

The bound of g_2 is easily obtained:

$$|g_2(s_1) - g_2(s_2)| \leqslant \kappa |s_1 - s_2| \int_{2s_2}^{\infty} \frac{dt}{\ell n^a t(t - s_1)(t - s_2)}$$

$$\leqslant \kappa |s_1 - s_2| \int_{2s_2}^{\infty} \frac{dt}{t^2 \ell n^a t} \leqslant \frac{\kappa |s_1 - s_2|}{s_2 \ell n^a s_2} \leqslant \frac{\kappa}{\ell n^a s_2} \left|\frac{s_1 - s_2}{s_1}\right|$$

$$\leqslant \frac{\kappa}{\ell n^a s_1} \left|\frac{s_1 - s_2}{s_1}\right|^{\delta} . \qquad \text{(A.11)}$$

Now put $s_2 = s_1(1 + b)$, and note that we may restrict attention to small b, say $b \leqslant 1/8$. For $b > 1/8$ the required bound is a direct consequence of (A.5a):

$$|g(s_1) - g(s_2)| \leqslant |g(s_1)| + |g(s_2)| \leqslant \frac{\kappa}{\ell n^{a-1} s_1}$$

$$\leqslant \frac{\kappa b^\delta}{\ell n^{a-1} s_1} = \frac{\kappa}{\ell n^{a-1} s_1} \left|\frac{s_1 - s_2}{s_1}\right|^\delta . \qquad (A.12)$$

Let us extend the domain of $f(t)$ to include the interval $s_o/2 \leqslant t \leqslant s_o$, putting $f(t) = 0$ on that interval. The property (A.4b) holds for the extended function; for $t_1 < s_o$ and $t_2 \geqslant s_o$,

$$|f(t_1) - f(t_2)| = |f(t_2)| \leqslant \frac{\kappa}{\ell n^a t_2}\left(1 - \frac{s_o}{t_2}\right)^\theta$$

$$\leqslant \frac{\kappa}{\ell n^a t_2}\left(1 - \frac{s_o}{t_2}\right)^{\theta-\delta}\left(1 - \frac{t_1}{t_2}\right)^\delta \leqslant \frac{\kappa}{\ell n^a t_1}\left[\frac{t_2 - s_o}{t_2}\right]^{\theta-\delta}\left[\frac{t_1 - t_2}{t_1}\right]^\delta . \qquad (A.13)$$

By (A.8) we have

$$g_1(s) = \int_{s_o/2}^{2s_2} \frac{f(t) - f(s)}{t - s}\, dt + f(s)\ell n\left(\frac{2s_2 - s}{s - s_o/2}\right) = h_1(s) + h_2(s), \qquad (A.14)$$

and

$$|h_2(s_1) - h_2(s_2)| \leqslant |f(s_1) - f(s_2)|\left|\ell n\left(\frac{2s_2 - s_1}{s_1 - s_o/2}\right)\right|$$

$$|f(s_2)| \cdot \left|\ell n\left(\frac{2s_2 - s_1}{s_1 - s_o/2}\right) - \ell n\left(\frac{2s_2\ s_2}{s_2 - s_o/2}\right)\right| . \qquad (A.15)$$

The logarithmic factor in the first term of (A.15) is clearly bounded by a constant for $b \leqslant 1/8$ and $s_1 \geqslant s_o$. For the second term we use the mean value theorem, noting that

$$\sup_{s_1 \leqslant s \leqslant s_2}\left|\frac{d}{ds}\ell n\left(\frac{2s_2 - s}{s - s_o/2}\right)\right| \leqslant \sup\left|\frac{1}{2s_2 - s}\right| + \sup\left|\frac{1}{s - s_o/2}\right|$$

$$= \frac{1}{s_2} + \frac{1}{s_1 - s_o/2} \leqslant \frac{\kappa}{s_1} . \qquad (A.16)$$

The difference of logarithms in (A.15) is then less than $\kappa b \leqslant \kappa b^\delta$, and the required bound of the increment of h_2 is obtained from (A.15). To estimate the increment of h_2 we break the integral into three parts,

$$h_1 = \int_{s_o/2}^{s_1(1-2b)} + \int_{s_1(1-2b)}^{s_1(1+2b)} + \int_{s_1(1+2b)}^{2s_2} = j_1 + j_2 + j_3 . \qquad (A.17)$$

The separate terms in the increment of j_2 are so small that we need not consider their difference:

$$|j_2(s_1) - j_2(s_2)| \leqslant |j_2(s_1)| + |j_2(s_2)|$$

$$\leqslant \frac{\kappa}{s_1^\delta\ \ell n^a s_1}\int_{s_1(1-2b)}^{s_1(1+2b)} dt\left[\frac{1}{|t - s_1|^\delta} + \frac{1}{|t - s_2|^\delta}\right]$$

$$= \frac{\kappa}{\ell n^a s_1}\int_{1-2b}^{1+2b} du\left[\frac{1}{|u - 1|^{1-\delta}} + \frac{1}{|u - 1 - b|^{1-\delta}}\right]$$

$$\leqslant \frac{\kappa b^\delta}{\ell n^a s_1} . \qquad (A.18)$$

Next we estimate

$$
\begin{aligned}
&j_1(s_2) + j_3(s_2) - j_1(s_1) - j_3(s_1) \\
&= \left[f(s_1) - f(s_2)\right]\left[\int_{s_o/2}^{s_1(1-2b)} + \int_{s_1(1+2b)}^{2s_2}\right]\frac{dt}{t-s} \\
&+ \left[\int_{s_o/2}^{s_1(1-2b)} + \int_{s_1(1+2b)}^{2s_2}\right] dt \left[f(t) - f(s_2)\right]\left[\frac{1}{t-s_2} - \frac{1}{t-s_1}\right] \\
&= k_1 + k_2 .
\end{aligned}
\tag{A.19}
$$

The part k_1 is easily disposed of:

$$
|k_1| = |f(s_1) - f(s_2)| \left|\ln\left(\frac{2s_2 - s_1}{s_1 - s_o/2}\right)\right| \leqslant \frac{\kappa}{\ln^a s_1} b^{\delta} . \tag{A.20}
$$

For the first integral in k_2 we need a further decomposition to handle the combination of two poles and a logarithm:

$$
|k_{21}| \leqslant \kappa |s_1 - s_2| \int_{s_o/2}^{s_1(1-2b)} \frac{dt}{t^{\delta}\ln^a t} \frac{1}{|t - s_2|^{1-\delta}} \frac{1}{|t - s_1|}
$$

$$
= \kappa b \int_{s_o/2s_1}^{1-2b} \frac{du}{u^{\delta}\ln^a s_1 u} \frac{1}{|u - 1 - b|^{1-\delta}} \frac{1}{|u - 1|}
$$

$$
\leqslant \kappa b \left[\int_{s_o/2s_1}^{1/2} \frac{du}{u^{\delta}\ln^a s_1 u} + \frac{1}{\ln^a s_1} \int_{1/2}^{1-2b} \frac{du}{|u - 1 - b|^{1-\delta}|u - 1|}\right] \tag{A.21}
$$

$$
\leqslant \kappa b \left[\frac{1}{s_1^{1-\delta}} \int_{s_o/2}^{s_1/2} \frac{dt}{t^{\delta}\ln^a t} + \frac{1}{\ln^a s_1} \int_{1/2}^{1-2b} \frac{du}{|u-1|^{2-\delta}}\right] \leqslant \frac{\kappa b}{\ln^a s_1}\, 1 + b^{\delta-1} \leqslant \frac{\kappa b^{\delta}}{\ln^a s_1} .
$$

To complete the proof of Lemma 2 we treat the second integral in k_2:

$$
|k_{22}| \leqslant \frac{\kappa |s_1 - s_2|}{s_2^{\delta}\ln^a s_2} \int_{s_1(1+2b)}^{2s_2} \frac{dt}{|t - s_2|^{1-\delta}} \frac{1}{|t - s_1|}
$$

$$
= \frac{\kappa |s_1 - s_2|}{s_2^{\delta} s_1^{1-\delta}\ln^a s_2} \int_{1+2b}^{2(1+b)} \frac{du}{|u - 1 - b|^{1-\delta}} \frac{1}{|u - 1|}
$$

$$
\leqslant \frac{\kappa b}{\ln^a s_2} \int_{1+2b}^{2(1+b)} \frac{du}{|u - 1 - b|^{2-\delta}} < \frac{\kappa b^{\delta}}{\ln^a s_1} . \tag{A.22}
$$

Notice that if two functions $f_1(t)$, $f_2(t)$ satisfy (A.4), then the product $f_1(t)f_2(t)$ satisfies (A.4) with the exponent a replaced by 2a. Consequently, when we estimate the integral in (2.3) using (2.2vi) and the definition (1.2) of $\rho(t)$, we find that it obeys conditions like (A.5) with $a = 2\alpha$, $\delta = \mu$. Since $2\alpha - 1 > \alpha$, we therby establish conditions (2.4iv) on $B(s)$.

APPENDIX B

FREDHOLM THEORY OF THE INTEGRAL EQUATION

We show that the integral equation (2.27) may be treated by Fredholm theory 20 , under conditions (2.4iv) on $B(s)$. We map the interval $[s_1, \infty)$ onto $(0, 1]$. The choice of the mapping is not crucial; we take $t = s_1/s$ for convenience. In a numerical calculation of the Fredholm solution it is usually best to choose the mapping $t(s)$ so as to make the integrand finite and non-zero at the point corresponding to $s = \infty$. We multiply the equation by $(\ln s)^{\alpha}$, $\alpha > 1$, and seek a solution $\phi(t) = (\ln s)^{\alpha} N(s)$ in a Banach space U consisting of real matrix functions $\phi(t)$ continuous on the closed interval $[0, 1]$ with norm

$$||\phi|| = \sup_{t,i,j} |\phi_{ij}(t)|. \tag{B.1}$$

Let us define the operator K by the formula

$$K\phi(t) = (\ell ns)^{\alpha} \int_{s_1}^{\infty} \frac{B(s) - B(s')}{s - s'} \rho(s') \frac{\phi(t')}{(\ell ns')^{\alpha}} ds'. \tag{B.2}$$

As we shall see presently, K maps U into itself if $K\phi(0)$ is defined to be zero. According to the Ascoli-Arzelà criterion [20], K is compact (completely continuous) if the sequence $\{K\phi_n(t)\}$ is bounded and equicontinuous, where $\{\phi_n(t)\}$ is any bounded sequence of functions in U.

Let $\{\phi_n(t)\}$ be a bounded sequence in U, $||\phi_n|| < \kappa$, and check boundedness of $||K\phi_n||$ as follows:

$$||K\phi_n|| \leqslant \sup_s \sum_{i,j} (\ell ns)^{\alpha} \int_{s_1}^{\infty} \left| \frac{B_{ij}(s) - B_{ij}(s')}{s - s'} \rho_j(s') \right| \frac{||\phi_n||}{(\ell ns')^{\alpha}} ds'. \tag{B.3}$$

An analysis like that in (A.6), (A.7), and (A.9) (but not requiring subtraction of a logarithm) shows that the integral in (B.3) is $O(\ell n^{-2\alpha+1} s)$, thus $||K\phi_n|| \leqslant \kappa$ since $\alpha > 1$. Incidentally we have shown that $K\phi(t) \to 0$, $t \to 0$. With the definition $K\phi(0) = 0$ the function $K\phi(t)$ is continuous on the closed interval $[0,1]$; K maps U into itself.

The requirement of equicontinuity of the functions $K\phi_n(t)$ is that for any $\varepsilon > 0$,

$$\max_{i,j} |[K\phi_n(t_1) - K\phi_n(t_2)]_{ij}| < \varepsilon \tag{B.4}$$

when $|t_1 - t_2| < \delta(\varepsilon)$, where δ is independent of n. With $f_n(t_1, t_2)$ defined as the left side of (B.4), $\{\phi_n\}$ any bounded sequence, and $s_1 \leqslant s_2$, we have

$$f_n(t_1,t_2) \leqslant \kappa \left|(\ell n s_1)^{\alpha} - (\ell n s_2)^{\alpha}\right| \sum_{i,j} \int_{s_1}^{\infty} \left| \frac{B_{ij}(s_2) - B_{ij}(s')}{s_2 - s'} \rho_j(s') \right| \frac{ds'}{(\ell ns')^{\alpha}}$$

$$+ \kappa (\ell n s_1)^{\alpha} \sum_{ij} \int_{s_1}^{\infty} \left| \frac{B_{ij}(s_1) - B_{ij}(s')}{s_1 - s'} - \frac{B_{ij}(s_2) - B_{ij}(s')}{s_2 - s'} \right| \rho_j(s') \frac{ds'}{(\ell n s')^{\alpha}} = g + h. \tag{B.5}$$

The right side of (B.5) is independent of n, and we have only to show that it vanishes with $|t_1 - t_2|$. The analysis of (A.10) - (A.22), simpler now because we needn't bother with subtraction of logarithmic terms, shows that the second term h in (B.5) has the bound

$$h(t_1, t_2) \leqslant \frac{\kappa}{(\ell n s_1)^{\alpha-1}} \left| \frac{s_1 - s_2}{s_1} \right|^{\mu} = \frac{\kappa}{(\ell n s_1)^{\alpha-1}} \left| \frac{t_1 - t_2}{t_2} \right|^{\mu}. \tag{B.6}$$

Also, we may bound the two terms in h separately to get

$$h(t_1, t_2) \leqslant \frac{\kappa}{(\ell n s_1)^{\alpha-1}} + \kappa \frac{(\ell n s_1)^{\alpha}}{(\ell n s_2)^{2\alpha-1}} \leqslant \frac{M}{(\ell n s_1)^{\alpha-1}} \tag{B.7}$$

For any $\varepsilon > 0$, let us divide the interval of t_2 into two parts, $t_2 \leqslant \eta(\varepsilon)$ and $t_2 > \eta(\varepsilon)$, where $\eta(\varepsilon)$ is chosen to make

$$\frac{M}{[\ell n(s_0/2\eta)]^{\alpha-1}} < \varepsilon/2 , \tag{B.8}$$

with M as in (B.7). Then if $t_2 \leqslant \eta(\varepsilon)$ and $|t_1 - t_2| < \eta(\varepsilon)$, we have by (B.7) that $h(t_1, t_2) < \varepsilon/2$. On the other hand if $t_2 > \eta(\varepsilon)$, we have by (B.6) that

$$h(t_1, t_2) \leqslant \frac{\kappa}{(\ell n s_o)^{\alpha-1}} \left| \frac{t_1 - t_2}{\eta(\varepsilon)} \right|^{\mu} < \varepsilon/2 \qquad \text{(B.9)}$$

for $|t_1 - t_2|$ less than some $\zeta(\varepsilon)$. Hence for $|t_1 - t_2| < \min\left[\eta(\varepsilon), \zeta(\varepsilon)\right] = \delta_1(\varepsilon)$ we have $h(t_1, t_2) < \varepsilon/2$. To majorize the first term g in (B.5) we apply the mean value theorem to the difference of logarithms and bound the integral as usual to obtain

$$g(t_1, t_2) \leqslant \frac{\kappa}{(\ell n s_1)^{\alpha}} \left| \frac{s_1 - s_2}{s_1} \right| . \qquad \text{(B.10)}$$

Alternatively, we may bound the two logarithmic terms separately and find

$$g(t_1, t_2) \leqslant \frac{\kappa}{(\ell n s_1)^{\alpha-1}} . \qquad \text{(B.11)}$$

The argument used above then shows that $g(t_1, t_2) < \varepsilon/2$ for $|t_1 - t_2|$ less than some $\delta_2(\varepsilon)$. For $|t_1 - t_2| < \min\left[\delta_1(\varepsilon), \delta_2(\varepsilon)\right] = \delta(\varepsilon)$ we have $f_n(t_1, t_2) < \varepsilon$, and the proof of equicontinuity and compactness of K is complete.

Our hypothesis $B(s) = O([\ell n s]^{-\alpha})$, $\alpha > 1$, is close in some sense to being the weakest asymptotic condition on $B(s)$ that leads to a Fredholm equation in a classical Banach space of continuous functions. With $B(s) \sim (\ell n s)^{-1}$ the operator K is

non-compact in a space analogous to U, but may be regularized by extraction of a non-compact part in such a way that the problem is reduced to a regular Fredholm problem [21]. Under still weaker conditions on $B(s)$ a regularization is possible, but only at the expense of new arbitrary constants entering the equations [22].

Since Eq. (2.27) entails a compact operator, it may be solved numerically by various well-developed methods; see for instance the review of Atkinson [23] and the book of Anselone [24], which deals with the rigorous justification of discretization.

The operator of Eq. (5.6), multiplied by s^{α}, is compact on a Banach space V under conditions (5.2), (5.3) on $\rho(s)$ and $B_{22}(s)$. Here V consists of real continuous functions $\phi(t) = s^{\alpha} N_{22}(s)$ with

$$||\phi|| = \sup|\phi(t)| . \qquad \text{(B.12)}$$

The proof of compactness is the same as that above, but with the estimates of Lemma 1 of Appendix A replacing those of Lemma 2.

REFERENCES

1. F. J. Yndurâin, Nuc. Phys. B89 (1975) 317.

2. F. J. Yndurâin, Nuc. Phys. B88 (1975) 318.

3. M. Aguilar-Benitez et al., "Study of a $K^o_s K^o_s$ Threshold Enhancement Observed in the Final States $\bar{p}p \to K^o_s K^o_s +$ pions at 700 - 760 MeV/c", CERN preprint CERN/EP/PHYS 78-5, 1978.

4. A. Gonzâlez-Arroyo, "Extension of the K-Matrix Formalism to Amplitudes with Overlapping Left-hand Cuts", preprint

FTUAM/78-2, Universidad Autónoma de Madrid, 1978; "Analisis Multicanal de las Resonancias Escalares and Isoescalares", Ph.D. Thesis, Universidad Autónoma de Madrid, 1978.

5. H. Nielsen and G. C. Oades, Nuc. Phys. B55 (1973) 301; N. Hedegaard-Jensen, ibid. B77 (1974) 173; N. O. Johanneson and J. L. Petersen, ibid. B68 (1974) 397; T. Shimada, ibid. B100 (1975), 225; J. J. Brehm, ibid. B55 (1973) 167; S. M. Flatté, Phys. Lett. 63B (1976) 224; N. Kimura, T. Miyakoshi, and A. Kanazawa, "Bias-Free Pole Search on the S* Resonance", preprint EPHOU-78 IV 01, Hokkaido University, 1978.

6. I. S. Shapiro, Phys. Reports 35C (1978) 129; B. O. Kerbikov and I. S. Shapiro, "Quasinuclear Baryonium and Annihilation". preprint ITEP-159, Institute of Theoretical and Experimental Physics, Moscow, 1978; C. B. Dover and J. M. Richard, Brookhaven preprint BNL-24981, 1978; C. B. Dover and M. Goldhaber, Phys. Rev. 15D (1977) 1997; Yu. A. Simonov and J. A. Tjon, "Effects of Annihilation on Resonances in the $N\bar{N}$ System:, Utrecht preprint, 1978.

7. G. Frye and R. L. Warnock, Phys. Rev. 130 (1963) 478; R. L. Warnock, in Lectures in Theoretical Physics, edited by H. H. Aly (Wiley, New York, 1968).

8. R. Jost and R. Newton, Nuovo Cimento 1 (1955) 590.

9. J. D. Bjorken, Phys. Rev. Letters 4 (1960) 473.

10. J. D. Bjorken and M. Nauenberg, Phys. Rev. 121 (1961) 1250.

11. R. L. Warnock, Nuovo Cimento 50 (1967) 894 and erratum ibid. 52A (1967) 637.

12. R. L. Warnock, "A Program for Construction of a Regge Theory with Crossing Symmetry and Unitarity", Lawrence Berkeley Laboratory Preprint LBL-7982, 1978.

13. P. W. Johnson and R. L. Warnock, Phys. Rev. D15 (1977) 2354; D15 (1977) 2366; P. W. Johnson, R. L. Warnock, and M. Kaekebeke, ibid. D15 (1977) 482.

14. R. L. Warnock, Phys. Rev. 146 (1966) 1109.

15. G. Nenciu, G. Rasche, M. Stihi, and W. S. Woolcock, Helv. Phys. Acta 51 (1978) 608.

16. The existence of such a function $\mathcal{D}(s)$ was proved in Ref. [11], by reducing the question to the Hilbert problem in n unknown functions [17]. The conditions on T(s) assumed in Ref. [11] follow, as far as essentials are concerned, from (2.2). Property (2.5v) was obtained only under stronger conditions on T(s), however. As we shall show in a forthcoming paper, it is possible to simplify greatly the argument of Ref. [11], and at the same time prove the existence of a $\mathcal{D}(s)$ with properties (2.5) for any T(s) satisfying conditions (2.2).

17. N. I. Muskhelishvili, Singular Integral Equations (P. Noordhoff, Groningen, 1953).

18. D. Atkinson and M. Halpern, Phys. Rev. 149 (1966) 1133.

19. The definition (5.21) of K(s) agrees with the definition of Ref. 4 only for $s > s_2$. For $s < s_2$, the K(s) of Ref. 4 is the analytic continuation of (5.21). Notice that (5.21) is the boundary value of one analytic function for $s < s_2$, and that of another analytic function for $s > s_2$.

20. L. A. Liusternik and V. J. Sobolev, Elements of Functional Analysis (Ungar, New York, 1961).

21. P. W. Johnson and R. L. Warnock, J. Math. Phys. 16 (1975) 1880.

22. D. Atkinson and R. L. Warnock, Phys. Rev. D16 (1977) 1948.

23. K. E. Atkinson, A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind (Society for Industrial and Applied Mathematics, Philadelphia, 1976).

24. P. M. Anselone, Collectively Compact Operator Approximation Theory and Applications to Integral Equations, (Prentice-Hall, Englewood Cliffs, N. J., 1971).

FIGURE CAPTIONS

Fig. 1: Possible analyticity domains Ω_{11}, $\Omega_{22}^{(\pm)}$ of $T_{11}(s)$, $T_{22}(s)$, respectively.

Fig. 2: Possible analyticity domains $\hat{\Omega}_{11}$, $\hat{\Omega}_{22}$ of $B_{11}(s)$, $B_{22}(s)$, respectively.

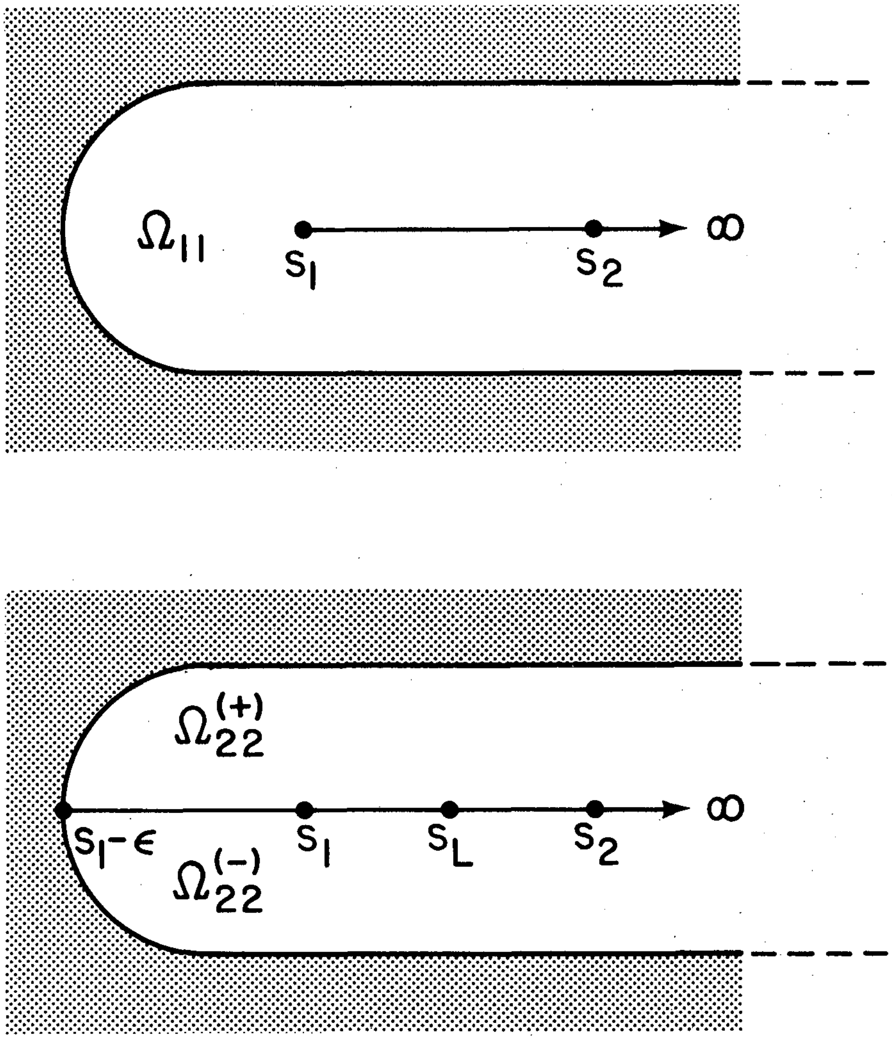

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Figure 1

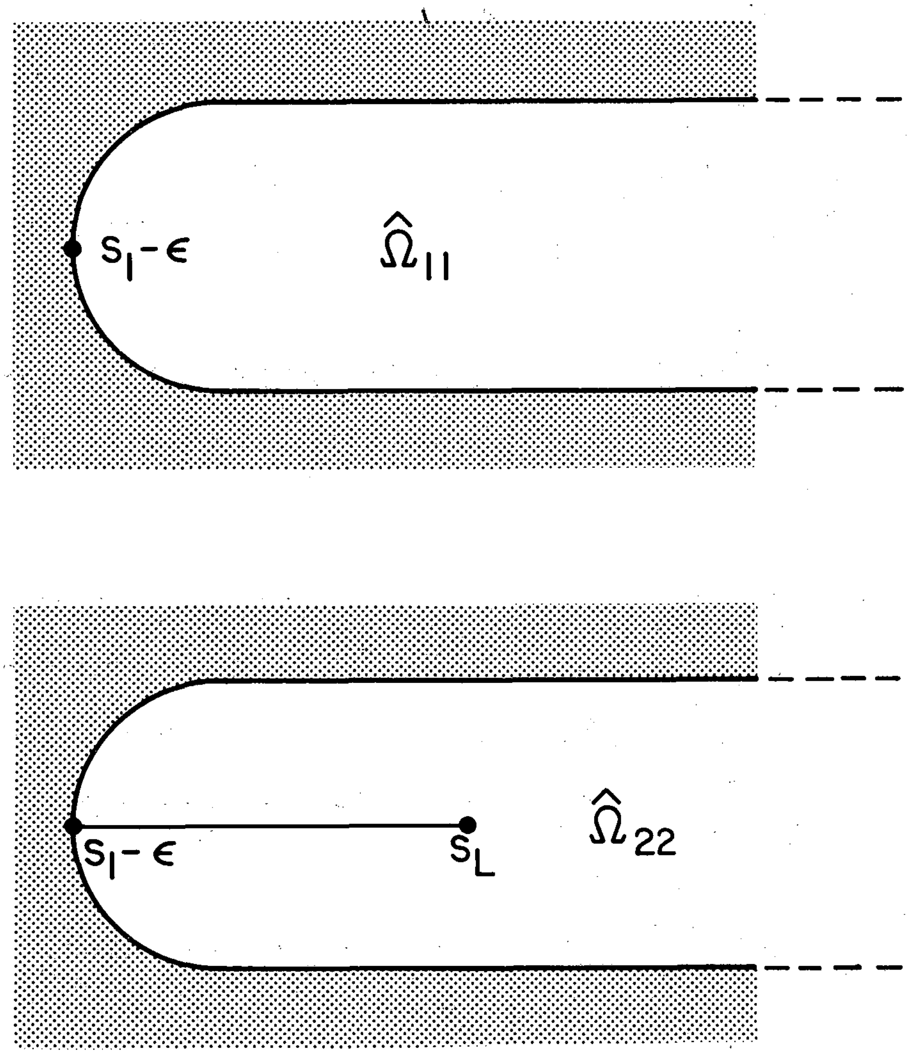

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Figure 2

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