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Note

On a Functional Equation Related to Thurstone Models

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If f and g are nonvanishing characteristic functions the functional equation $g(s)\,g(t)\,g(-s-t) = f(as)\,f(at)\,f(-as-at)$ implies $g(s) = e^{ibs}f(as)$, i.e., f and g corresponding to probability distributions of the same type. It is shown here that when f and g are allowed to vanish this equation also has solutions in which f and g correspond to distributions of different types. The practical implication is that there are nonequivalent Thurstone models which cannot be discriminated by any choice experiment with three objects.

The functional equation

$$g(s)\,g(t)\,g(-s-t) = f(as)\,f(at)\,f(-as-at) \tag{1}$$

(where f and g are complex functions of a real variable) arises in connection with an identifiability problem in choice behavior described recently in this journal (Yellott, 1977). Briefly, if f and g are the characteristic functions of probability distributions F and G, it can be shown that two "Thurstone models" $\mathscr{T}_F$ and $\mathscr{T}_G$ (i.e., models identical to Thurstone's Case V except that distribution F (or G) is substituted for the normal) are equivalent for all choice experiments with three objects iff (1) is satisfied for some positive constant a and all real s and t. If f or g is assumed to be a *nonvanishing* characteristic function, the unique solution to (1) is readily shown to be

$$g(s) = e^{ibs}f(as) \tag{2}$$

where b is any real constant. This means that $F(x) = G(ax + b)$, and so when f is nonvanishing the predictions of Thurstone model $\mathscr{T}_F$ can be completely duplicated by another model $\mathscr{T}_G$ iff F and G are distributions of the same type, i.e., both normal, both exponential, or whatever. This turns out to be true in particular for the double exponential distribution function $F(x) = e^{-e^{-x}}$, which yields the Thurstone model corresponding to Luce's Choice Axiom.

However without the assumption that f or g is nonvanishing the argument used in the previous paper to establish the uniqueness of solution (2) does not go through, and

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consultation with J. Aczél indicated that it was not generally known whether any pair of characteristic functions that satisfy (1) must also satisfy (2).[1] (Examples of complex functions that satisfy (1) but not (2) are not hard to come by, but the question is whether they can also be characteristic functions.) Consequently in that paper I was forced to leave open the general question of whether two Thurstone models can be equivalent for experiments with three objects when F and G are distributions of different types.

The purpose of this note is to lay this problem to rest by exhibiting a pair of characteristic functions f and g that satisfy (1) but not (2). The probability distributions F and G corresponding to this pair of characteristic functions both satisfy the technical requirement for Thurstone models, i.e., the "difference distributions"

$$D_F(x) = \int_{-\infty}^{\infty} F(x+y)\, dF(y) \qquad \text{and} \qquad D_G(x) = \int_{-\infty}^{\infty} G(x+y)\, dG(y)$$

are continuous and strictly increasing, and so this example shows explicitly that two Thurstone models $\mathscr{T}_F$ and $\mathscr{T}_G$ can be completely equivalent for choice experiments with three objects even though their discriminal process distributions F and G are not of the same type, i.e., $F(x) \neq G(ax+b)$. Consequently it is not possible to strengthen the theorems in Yellott (1977), contrary to my conjecture in that paper.

Now for specifics. The characteristic function f of a probability distribution (i.e., cumulative distribution function) F is the Fourier-Stieltjes transform

$$f(s) = \int_{-\infty}^{\infty} e^{-i2\pi sx}\, dF(x) \tag{3}$$

(e.g., Feller (1966). Actually in probability theory it is customary to suppress the constant factor -2π, and so the characteristic functions tabled in probability texts correspond to $f(-s/2\pi)$. However a pair of characteristic functions defined according to (3) will satisfy functional equation (1) iff $f(-s/2\pi)$ and $g(-s/2\pi)$ do also, and for present purposes it is convenient to use definition (3) in order to take advantage of Bracewell's (1965) dictionary of transforms.) Recall that if f is absolutely integrable, the probability density function p_F corresponding to F (i.e., $p_F(x) = F'(x)$) is given by the inverse transform

$$p_F(x) = \int_{-\infty}^{\infty} e^{i2\pi sx} f(s)\, ds.$$

Now consider the functions

$$\begin{aligned} f(s) &= 1 - |s|, && s \in (-1, 1), \\ &= \tfrac{1}{2}(1 - |s-5|), && s \in (4, 6), \\ &= \tfrac{1}{2}(1 - |s+5|), && s \in (-6, -4), \\ &= 0, && \text{elsewhere}; \end{aligned} \tag{4}$$

[1] Subsequently Z. Moszner (1977) has given a general characterization of the solutions to Eq. (1). In an adjoining note in this issue he shows how that translates into a characterization of the corresponding distribution functions.

$$
\begin{aligned}
g(s) &= 1 - |s|, && s \in (-1, 1),\\
&= (-i/2)(1 - |s - 5|), && s \in (4, 6),\\
&= (i/2)(1 - |s + 5|), && s \in (-6, -4),\\
&= 0, && \text{elsewhere.}
\end{aligned} \tag{5}
$$

(The graph of f consists of three triangles centered at 0, 5, and -5; each has width 2 and the heights are 1 at the origin, $\frac{1}{2}$ at ± 5. The graph of g consists of the same three triangles, but only the middle one lies in the real plane, while those centered at ± 5 lie in the imaginary plane, on opposite sides of the real axis.) We need to show that (i) f and g are characteristic functions, and (ii) they satisfy equation (1). To show the former, we invert both functions and obtain

$$p_F(x) = \text{Sinc}^2(x)[1 + \cos 10\pi x], \tag{6}$$

$$p_G(x) = \text{Sinc}^2(x)[1 + \sin 10\pi x], \tag{7}$$

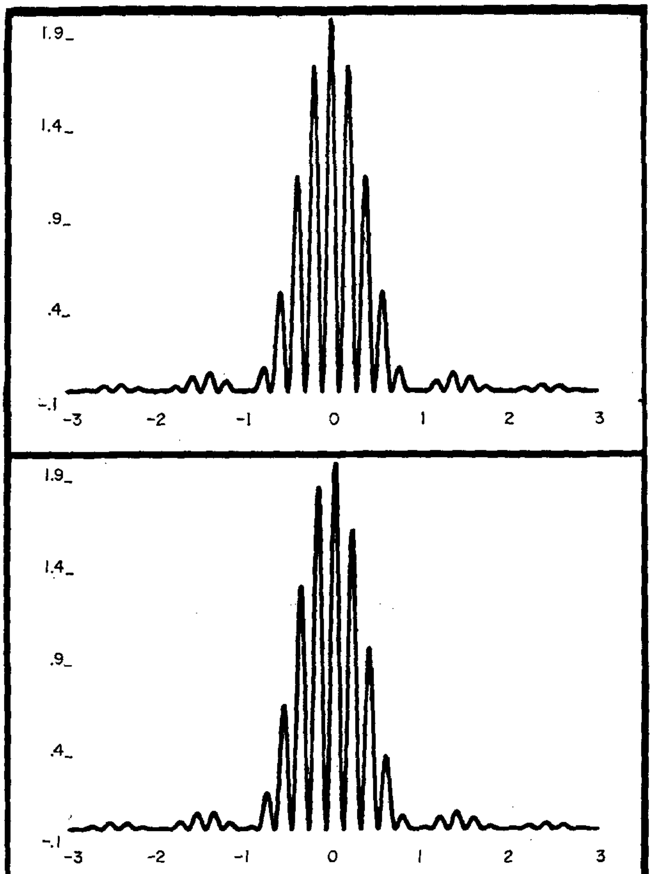

FIG. 1. Top panel shows the density function $\text{Sinc}^2(x)[1 + \cos 10\pi x]$. Bottom panel shows $\text{Sinc}^2(x)[1 + \sin 10\pi x]$.

where Sinc (x) is $\mathrm{Sin}(\pi x)/\pi x$. (See Fig. 1.) This inversion is straightforward if we write (4) and (5) in the form

$$f(s) = \mathrm{Tri}\,(s) * \left[\delta(s) + \frac{\delta(s-5)}{2} + \frac{\delta(s+5)}{2}\right],$$

$$g(s) = \mathrm{Tri}\,(s) * \left[\delta(s) - \frac{i\delta(s-5)}{2} + \frac{i\delta(s+5)}{2}\right],$$

where $\mathrm{Tri}\,(s) = 1 - |s|$ for $s \in (-1, 1)$, 0 elsewhere, $\delta(s)$ is the Dirac delta, and $*$ denotes convolution. Then (6) and (7) follow from that fact that $\mathrm{Tr}\,i(s)$ is the Fourier transform of $\mathrm{Sinc}^2\,(x)$, $\delta(s)$ the transform of the constant function 1, $\frac{1}{2}[\delta(s-5) + \delta(s+5)]$ the transform of $\cos 10\pi x$, and $(i/2)[\delta(s+5) - \delta(s-5)]$ the transform of $\sin 10\pi x$ (Bracewell, 1965). It is clear that (6) and (7) are both probability density functions, since both are nonnegative and integrate to 1. (This last is shown by the fact that $f(0) = g(0) = 1$.) It is also clear that the difference distributions D_F and D_G are continuous and strictly increasing, since they correspond to a continuous nonvanishing density. (Note that $D_F = D_G$, since $|f|^2 = |g|^2$.)

It remains then to show that (4) and (5) satisfy Eq. (1) (for $a = 1$). This is simply a matter of considering the possible values of s and t. Let I_0 denote the interval $(-1, 1)$, I_5 the interval $(4, 6)$, I_{-5} the interval $(-6, -4)$, and $R = (I_0 \cup I_5 \cup I_{-5})^c$. Then suppose

(1) s or $t \in R$. Then both sides of (1) vanish.

(2) Both s and $t \in I_0$. Then $-s-t$ is either in R, so that both sides of (1) vanish, or in I_0, in which case f and g agree factor by factor on both sides of (1).

(3) $s \in I_0$, $t \in I_5$. Then $-s-t$ is either in R or I_{-5}. In the first case both sides of (1) vanish. In the second case we have

$$\begin{aligned} g(s)\,g(t)\,g(-s-t) &= \mathrm{Tri}\,(s)(-i/2)\,\mathrm{Tri}\,(t-5)(i/2)\,\mathrm{Tri}\,(-s-t+5) \\ &= f(s)f(t)f(-s-t). \end{aligned}$$

so (1) is satisfied.

(4) $s \in I_0$, $t \in I_{-5}$. This is equivalent to case 3.

(5) $s \in I_5$, $t \in I_5$. Then $-s-t < -8$, i.e., $\in R$, so both sides vanish.

(6) $s \in I_{-5}$, $t \in I_{-5}$. This is equivalent to case 5.

(7) $s \in I_{-5}$, $t \in I_5$. Then $-s-t$ is either in R or I_0. In the latter case, $g(s)\,g(t) = -i^2 f(s)f(t)$ and $g(-s-t) = f(-s-t)$, so (1) is satisfied.

This completes the proof. It is worth noting that an infinite number of such f, g pairs could be constructed by centering the outside triangles at any $\pm v$ ($v \geqslant 5$), in which case the corresponding densities p_F and p_G would be $\mathrm{Sinc}^2\,(x)[1 + \cos 2\pi vx]$ and $\mathrm{Sinc}^2\,(x)[1 + \sin 2\pi vx]$, all of which yield Thurstone models. Thus there are an infinite number of pairs of nonequivalent Thurstone models that cannot be distinguished by three object choice experiments.

References

Bracewell, R. *The Fourier transform and its applications.* New York: McGraw–Hill, 1965.

Feller, W. *Introduction to probability theory and its applications*, Vol. II. New York: Wiley, 1966.

Moszner, Z. Sur l'équation $f(x) f(y) f(-x - y) = g(x) g(y) g(-x - y)$. *Aequationes Mathematicae*, in press, 1977, .

Yellott, J. I., Jr. The relationship between Luce's Choice Axiom, Thurstone's theory of comparative judgment, and the double exponential distribution. *Journal of Mathematical Psychology*, 1977, **15**, 109–146.

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