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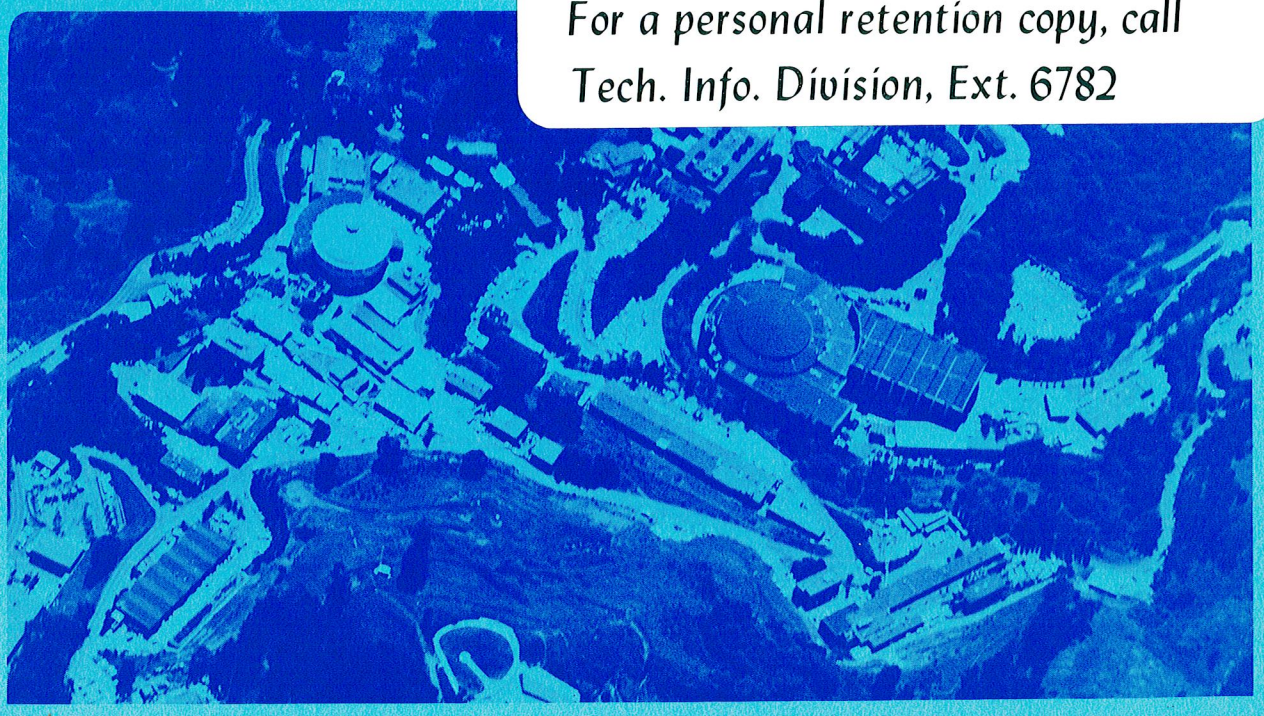
A STOCHASTIC METHOD FOR MODELING FLUID DISPLACEMENT IN PETROLEUM RESERVOIRS

C. Anderson and P. Concus

September 1980



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ABSTRACT

In the attempt to achieve optimal recovery of petroleum from a reservoir, it is usually necessary to model numerically the fluid displacements within the reservoir. These displacements often involve the propagation of steep fronts, such as those between different fluids or between regions of differing chemical concentrations. Such fronts generally pose difficulty for numerical methods, the overcoming of which has stimulated the development of new methods in recent years. We discuss our recent work on one such method, the random choice method, which has the inherent capability of following even perfectly sharp fronts. The use of the method is illustrated for multi-dimensional, two-phase, immiscible porous flow, including the effects of capillary pressure and of gravity.

INTRODUCTION

As part of the attempt to achieve optimal recovery of petroleum from underground reservoirs, mathematical models of fluid displacement in the porous medium of a reservoir have been formulated and solved numerically. As practical incentives for greater recovery increase so does the interest in better numerical solution methods, particularly because enhanced recovery techniques often give rise to complex mathematical problems.

One area especially troublesome for numerical solution methods is that of following the propagation of steep fronts. These fronts typically may be those between regions of differing fluid saturation or chemical concentration. Even though steep fronts may not be present initially, they can develop naturally in time as a consequence of the inherent nonlinearities of fluid displacement in a porous medium. Specialized numerical methods are required if steep fronts are to be

followed stably and accurately, a task that may be crucial if, for example, reservoir dynamics depend strongly upon phenomena occurring at steep-front interfaces.

We discuss here the random choice method, a numerical method that is designed particularly for following steep fronts stably and accurately. The use of this method for porous flow problems is still under development, particularly for multi-dimensional problems. Some of our current work in this area is described below. Other work on the random choice method for porous flow problems can be found in [11] and [12], and work on specialized techniques for finite element methods in, for example, [9], [15], and [18].

FLUID DISPLACEMENT EQUATIONS

The simultaneous, immiscible flow of two incompressible fluids is considered in an isotropic, homogeneous porous medium, including the effects of gravity and of capillary pressure. For a region whose interior is free of sources and sinks, one is led to the equations [21]

$$\phi \frac{\partial s}{\partial t} + \underset{\sim}{q}\cdot\nabla f(s) - \gamma \frac{\partial}{\partial z} g(s) - \nabla\cdot[h(s)\nabla s] = 0 \tag{1}$$

$$\nabla\cdot\underset{\sim}{q} = Q \tag{2}$$

$$\underset{\sim}{q} = -\lambda(s)[\nabla p - \gamma \hat{g}(s)\underset{\sim}{e}_k]. \tag{3}$$

In these equations the quantity $s(\underset{\sim}{x},t)$ is the saturation of one of the fluids -- customarily the wetting one (the saturation of a fluid is the fraction of available pore volume occupied by that fluid); $1-s$ is then the saturation of the other fluid, the non-wetting one. There holds $0 \leq s \leq 1$. The quantities $\underset{\sim}{x}$ and t are the space and time variables, respectively, and $\underset{\sim}{q}(\underset{\sim}{x},t)$ is the total velocity (sum of the individual velocities of the two fluids). The coordinates $\underset{\sim}{x} = (x,y,z)$ are chosen so that gravity acts downward in the negative z direction, with $\underset{\sim}{e}_k$ the unit vector in the positive z direction. The quantity $p(\underset{\sim}{x},t)$ is the excess over gravitational head of the global pressure. (See the discussion below.) The quantity Q represents the sources and sinks of fluid (injection and producing wells) on the boundary of the domain. The porosity ϕ is taken, for convenience, to be constant in this study, and the coefficient γ of the gravitational term is $\gamma = g_0(\rho_w - \rho_n)$, where g_0 is the acceleration due to

gravity, and ρ_w and ρ_n are the densities of the wetting and non-wetting phases, respectively.

Eq. (1) is the saturation (Buckley-Leverett) equation, which for our problems of interest, those with steep fronts, is hyperbolic or nearly hyperbolic in nature for a given $\underset{\sim}{q}$ (the last term, which is the diffusive term, is assumed small). Eq. (2) arises from the incompressibility and (3) from Darcy's law. For a given s, (2) is of elliptic type.

The quantities $f(s)$, $g(s)$, $h(s)$, $\lambda(s)$, and $\hat{g}(s)$ are functions of the relative permeabilities and capillary pressure, which are empirically determined functions of saturation, and of the viscosities, which are assumed constant. Let λ_n and λ_w denote the phase mobilities (ratio of permeability to viscosity) of the non-wetting and wetting fluids, respectively; then for immiscible displacement

$$f(s) = \lambda_w/(\lambda_n + \lambda_w) \quad , \quad g(s) = \lambda_n f(s)$$

$$\hat{g}(s) = \lambda_n/(\lambda_n + \lambda_w) \quad , \quad h(s) = g(s)(-dp_c/ds)$$

$$\lambda(s) = \lambda_n + \lambda_w \ ,$$

where $p_c(s)$ is the capillary pressure. The quantity $(-dp_c/ds)$ is positive, λ_n and λ_w are non-negative, and $\lambda(s)$ is positive, bounded strictly away from zero. Of particular interest are the non-convexity of $f(s)$ and $g(s)$. For porous flow problems $f(s)$ has typically an S-shape with one inflection point, as depicted in Figure 1, and $g(s)$ has two inflection points, as depicted in Figure 2. The non-convexity of f and g implies that in the limiting case of zero capillary pressure, (1) with $\underset{\sim}{q}$ fixed is a hyperbolic equation that does not satisfy the strict nonlinearity condition, thus permitting weak solutions that are combinations of propagating shock and expansion waves in contact.

The quantity $p(\underset{\sim}{x},t)$ is pressure, which here is the average of the individual phase pressures diminished by the gravitational head and adjusted to include the capillary pressure by means of the global pressure technique of [4]. Specifically, we take

$$p = p_A + g\rho_w z + \frac{1}{2}\int_0^s \frac{\lambda_w - \lambda_n}{\lambda_w + \lambda_n}\left(-\frac{dp_c}{ds}\right) ds \ ,$$

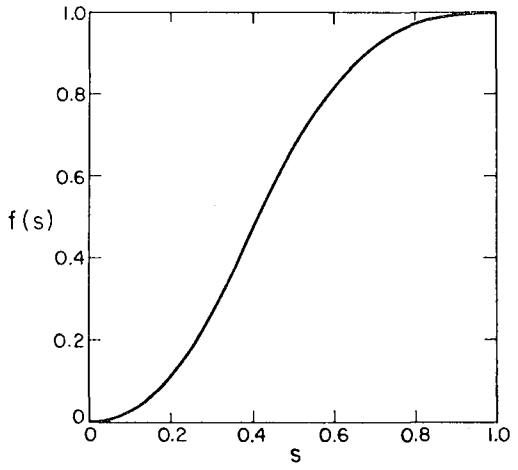

Figure 1

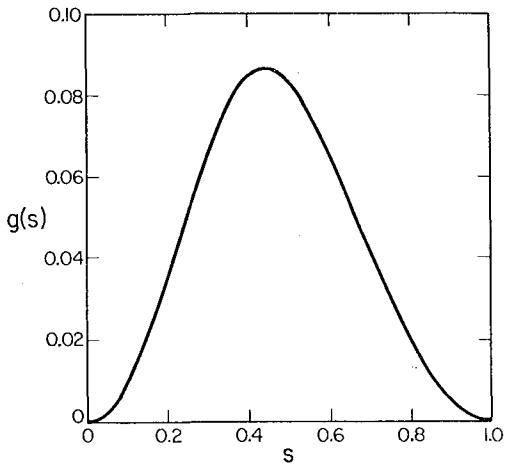

Figure 2

where p_A is the average of the pressure of the wetting and non-wetting fluids.

Eqs. (1),(2),(3) are to be solved for $s(\underset{\sim}{x},t)$ and $p(\underset{\sim}{x},t)$ given the initial value of s and suitable conditions on s and p on the boundary of the domain of interest. The boundary conditions on p typically ensure that $\underset{\sim}{q}$ can be determined uniquely from (2) and (3) when s is known. If the capillary pressure term in (1) is absent, s may be discontinuous, and the equation will be satisfied only in the weak sense. We consider solving (1),(2),(3) by the random choice numerical method.

RANDOM CHOICE METHOD

The random choice method is a numerical method that was developed for solving the hyperbolic equations of gas dynamics, for which solution discontinuities can occur. By using a sampling technique it controls numerical diffusion, even eliminates it altogether for one space dimension, and follows shock discontinuities sharply and accurately in a stable manner. The correct weak solution for a purely hyperbolic problem is obtained, corresponding to the limiting solution of parabolic problems as the dissipation approaches zero. A small amount of statistical uncertainty is introduced, which normally is totally acceptable within the context of the discretization truncation errors.

The random choice method is based on a mathematical construction of Glimm [10] that was developed into a practical and efficient computational algorithm by Chorin [5],[6]. The earlier study of Moler and Smoller [19] is also of interest. Several persons have since studied and extended the method for problems in gas dynamics and in combustion; see, for example, [7], [14], and [22]. The random choice method was first adapted to porous flow problems in [2], [3], and [8], and subsequently in [11] and [12], where situations involving front instability (viscous fingering) and specialized multi-dimensional front tracking techniques were considered.

For the fundamental equation of a single nonlinear conservation law,

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial z}\,\psi(s) = 0 \,, \tag{4}$$

to which (1) reduces in one space dimension in the absence of capillary pressure (q is constant for one space dimension), the random choice method advances a solution in time as follows. First the solution $s(z,t_j)$ at the initial time t_j is approximated by a piecewise-constant function on a spatial grid $z_i = i\Delta z$, $i = \ldots,-1,0,1,\ldots$, where the function is equal to $s_i^j = s(z_i,t_j)$ in the interval $z_i - \frac{1}{2}\Delta z < z \leqslant z_i + \frac{1}{2}\Delta z$. Then the (weak) solution of (4) is constructed analytically by the method of characteristics for the piecewise-constant initial data, and this solution is sampled at the new time to obtain the values for the new piecewise-constant approximation.

By choosing the time increments t sufficiently small so that the Courant-Friedrichs-Lewy condition $(\Delta t/\Delta z)\max|\psi'(s)| < 1$ is satisfied, the waves propagating from the discontinuities in the initial piecewise-constant approximation can be prevented from interacting during any given time step. This permits the solution of (4) to be carried out for a given time step by joining together the separate solutions of the Riemann problems, (4) with initial data

$$s(z,t_j) = \begin{cases} s_i^j \,, & z \leqslant z_i + \frac{1}{2}\Delta z \\ \\ s_{i+1}^j \,, & z > z_i + \frac{1}{2}\Delta z \,. \end{cases} \tag{5}$$

The practical success of the random choice method depends upon being able to solve these associated Riemann problems efficiently.

RIEMANN PROBLEMS

The function $\psi(s)$ in (4) is the linear combination of $f(s)$ and $g(s)$,

$$\psi(s) = [qf(s) - \gamma g(s)]/\phi \ ,$$

and may typically have either one or two inflection points. Because of these inflections the solution of (4),(5) can be more complex than if $\psi(s)$ had no inflections, for which case the solution would be either a propagating single discontinuity (shock) or centered expansion wave. The presence of the inflections permits solutions that can consist of combinations of one or more shocks and an expansion wave in contact.

If the gravity term $\gamma g(s)$ is small compared with the transport term $qf(s)$, then $\psi(s)$, like $f(s)$, has only one inflection. For $q > 0$, this case is covered in [8], where the problem without gravity is considered. If $q < 0$, then the reasoning used in [8] would apply, except that all waves would propagate in the negative instead of the positive z-direction.

For cases in which the effects of the gravity term become significant, $\psi(s)$ will have two inflections. A typical example with $\gamma > 0$ is depicted in Figure 3.

The Riemann problem solution is obtained by means of the following conditions which must hold along any curve of discontinuity of $s(z,t)$. Let $s_- = \lim_{z \to z_-} s(z,t)$ and $s_+ = \lim_{z \to z_+} s(z,t)$ be the limiting values from the left and right, respectively, at the discontinuity. Then there must hold [17],[20]

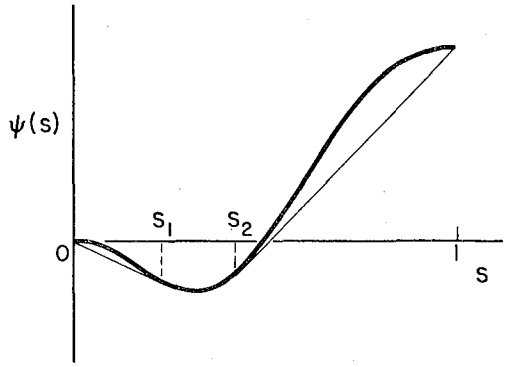

Figure 3

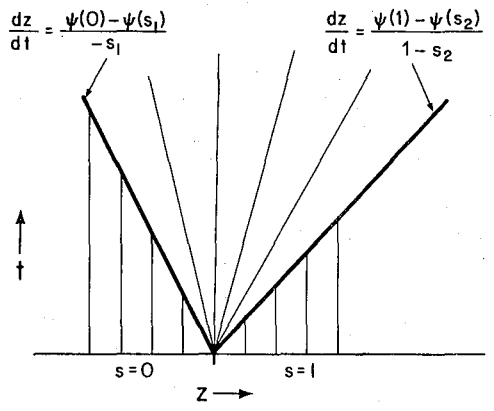

Figure 4

(a) Rankine-Hugoniot jump condition: The curve of discontinuity is a straight line with slope

$$\frac{dz}{dt} = \frac{\psi(s_+) - \psi(s_-)}{s_+ - s_-} \quad ;$$

(b) Generalized entropy condition:

$$\frac{\psi(s_+) - \psi(s_-)}{s_+ - s_-} \leqslant \frac{\psi(s_+) - \psi(s_-)}{s_+ - s_-}$$

for any s between s_+ and s_-.

For a particular $f(s)$ and $g(s)$ the Riemann problem solution is given explicitly in [1] for the $\psi(s)$ of Figure 3. In Figure 4 is depicted the solution of (4),(5) for $s_i^j = 0$ and $s_{i+1}^j = 1$. Figure 3 shows the concave hull of $\psi(s)$; the chord through $s = 0$ is tangent to $\psi(s)$ at $s = s_1$ and the chord through $s = 1$ is tangent at $s = s_2$. These values determine the shock propagation speeds. In Figure 4 are shown the two lines of discontinuity (shocks) that propagate, respectively, in the negative and positive z-direcitons from the initial discontinuity at $z = z_i + \frac{1}{2}\Delta z$. The characteristics from $z < z_i + \frac{1}{2}\Delta z$ on the initial line $t = t_j$ intersect the leftward travelling shock, and those from $z > z_i + \frac{1}{2}\Delta z$ intersect the rightward travelling shock. Between the two shocks is an expansion wave with a fan of characteristics emanating from the initial discontinuity.

If the sampled point is within the fan, then the corresponding value of s is determined by solving a single nonlinear equation to find the value of s between s_1 and s_2 at which $d\psi/ds$ is equal to the reciprocal of the slope of the characteristic through the point. If the sampled point is outside the fan, then either $s = 0$ or $s = 1$.

SAMPLING

The use of Riemann problem solutions to advance stepwise in time was introduced by Godunov [13]. It was the subsequent coupling with sampling, however, that contributed to the striking effectiveness of the random choice method in following sharp fronts. The sampled value of s in each interval at the new time is assigned as the new value of the piecewise constant approximation to s for that interval.

To yield an accurate representation of the solution the sampling sequence should be equidistributed [6],[7]. In [7] a deterministic van der Corput sampling sequence was proposed for use with the random choice method and was found to perform more favorably than previously used sequences with random components. The m^{th} number θ_m in the basic van der Corput sequence is given by

$$\theta_m = \sum_{k=0}^{M} i_k 2^{-(k+1)} ,$$

where the binary expansion for m is

$$m = \sum_{k=0}^{M} i_k 2^k .$$

A discussion of the method and extensions for use with multi-dimensional problems are given in [7].

SOLUTION PROCEDURE

Because of the numerous physical parameters, it is convenient to place (1),(2),(3) in non-dimensional form. To do so, we select the characteristic reference quantities: length x_0, mobility λ_0, source strength Q_0, capillary pressure p_{c_0}, and density ρ_0. The normalized non-dimensional quantities are then defined to be

$$\bar{\underset{\sim}{x}} = \underset{\sim}{x}/x_0 \ , \quad \bar{\underset{\sim}{q}} = \underset{\sim}{q}x_0/Q_0 \ , \quad \bar{t} = tQ_0/(\phi x_0^2) \ ,$$

$$\bar{\nabla} = x_0\nabla \ , \quad \bar{Q} = Qx_0^2/Q_0 \ , \quad \bar{\lambda}_w = \lambda_w/\lambda_0 \ , \quad \bar{\lambda}_n = \lambda_n/\lambda_0 \ ,$$

$$\bar{p} = p\lambda_0/Q_0 \ , \quad \bar{p}_c = p_c/p_{c_0} \ , \quad \bar{\rho}_w = \rho_w/\rho_0 \ , \quad \bar{\rho}_n = \rho_n/\rho_0 \ ,$$

$$\text{and} \quad \bar{h}(s) = h(s)/(\lambda_0 p_{c_0}) \ , \quad \bar{g}(s) = g(s)/\lambda_0 \ , \quad \bar{\lambda}(s) = \lambda(s)/\lambda_0 \ .$$

For problems without boundary sources, a characteristic pressure or velocity could be used for non-dimensionalization in place of Q_0 above.

The dimensionless parameters

$$\beta = \gamma\lambda_0 x_0/Q_0 \ , \quad \varepsilon = p_{c_0}\lambda_0/Q_0$$

that arise give a measure of the magnitudes of the gravity and capillary pressure terms. Eqs. (1), (2), and (3) become (omitting bar superscripts)

$$\frac{\partial s}{\partial t} + \underset{\sim}{q}\cdot\nabla f(s) - \beta \frac{\partial}{\partial z} g(s) - \varepsilon\nabla\cdot[h(s)\nabla s] = 0 \tag{6}$$

$$\nabla\cdot\underset{\sim}{q} = Q \tag{7}$$

$$\underset{\sim}{q} = -\lambda(s)[\nabla p - \beta\hat{g}(s)\underset{\sim}{e}_k] \ . \tag{8}$$

For our illustrative numerical examples we solve (6),(7),(8), with accompanying boundary conditions, on the square $0 \leq x \leq 1$, $0 \leq z \leq 1$. A uniform mesh ($\Delta x = \Delta z = 1/N$) is placed on the square and p is approximated by its value at the mesh points $x_i = i/N$, $z_j = j/N$, $i,j = 0,1,\ldots,N$. The saturation s is approximated at the staggered points $[(i+\frac{1}{2})/N,(j+\frac{1}{2})/N]$, $i,j = 0,1,\ldots,N-1$, for convenience in setting up discrete approximations to (7),(8) [2].

Let $s^{(n)}$ denote the approximate solution for the saturation at time $t = t_n$. To obtain $p^{(n)}$ and $\underset{\sim}{q}^{(n)} = (u^{(n)},w^{(n)})$, discrete approximations are solved to (7),(8) and their boundary conditions, using $s^{(n)}$ for s. In our current study (6) is advanced one step in time to obtain $s^{(n+1)}$ by solving successively

$$\frac{\partial s}{\partial t} + u^{(n)} f(s) = 0$$

$$\frac{\partial s}{\partial t} + w^{(n)} \frac{\partial}{\partial z} f(s) - \beta \frac{\partial}{\partial z} g(s) = 0$$

by the one-dimensional random choice method, and

$$\frac{\partial s}{\partial t} - \varepsilon \nabla \cdot [h(s) \nabla s] = 0$$

by a standard explicit method. Alternatives to the above operator splitting procedure are under study which have promise of greater accuracy in following undulating fronts and those propagating in directions unfavorably oriented with respect to the mesh. For one approach, see [12].

Results for two numerical examples are depicted in Figures 5 and 6. A source of non-wetting fluid ($s = 0$) is located at one vertex of the square and a sink is located at the diagonally opposite vertex. On the boundary, the normal component of $\underset{\sim}{q}$ and the normal derivative of s are taken to be zero. Initially the square is occupied entirely by wetting fluid ($s = 1$).

The (non-dimensional) mobilities are taken to be $\lambda_w = s^2$ and $\lambda_n = \alpha(1-s)^2$, where α is the ratio of viscosities of wetting to non-wetting fluids. Figures 1 and 2 depict $f(s)$ and $g(s)$ for this case with $\alpha = 0.5$. The parameter ε is taken to be zero (zero capillary pressure), for which the solution has a sharp propagating discontinuity front.

Figure 5 depicts the solution for the case $\beta = 2.5$ and $\alpha = 5$ with a source of (dimensionless) strength 5 at the upper left vertex and sink of equal magnitude at the lower right vertex. Since $\beta > 0$, this case corresponds to the injected fluid having smaller density than the fluid being displaced. The mesh spacing is $1/N = 1/40$ as indicated by the tick marks. For these parameter values gravity effects outweigh the transport effects somewhat, except near the source and sink.

The contours in Figure 5 are curves of constant saturation from 0.1 to 0.6, in increments of 0.1, increasing away from the source (0,1). The contours for saturation 0.7 and greater generally lie directly underneath the 0.6 contour. (The advancing front as calculated by the random choice method is essentially a sharp discontinuity with a saturation value corresponding to s_2 in Figure 4.) The contour plotting program's interpolation procedure displaces, over a mesh interval, contours that

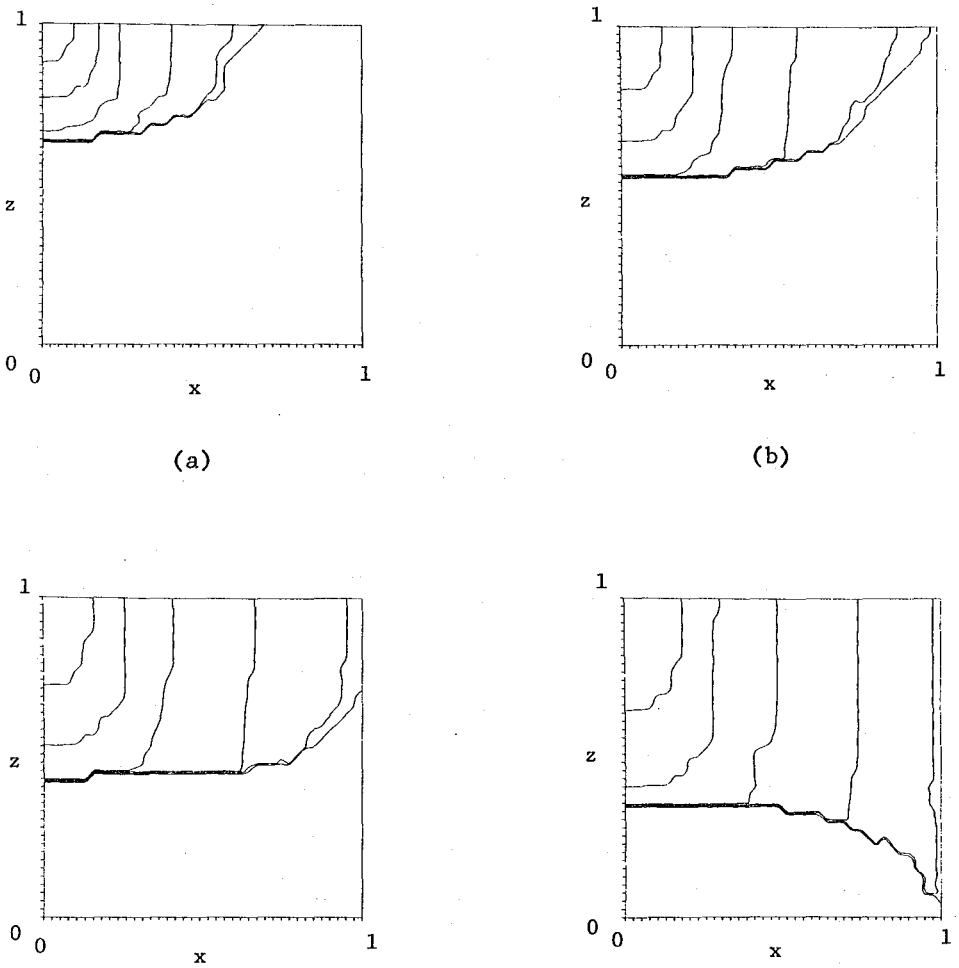

Figure 5

Saturation contours s = .1, .2, .3, .4, .5, .6
at times (a) t = .11, (b) t = .21, (c) t = .30, (d) t = .42
for first test problem.

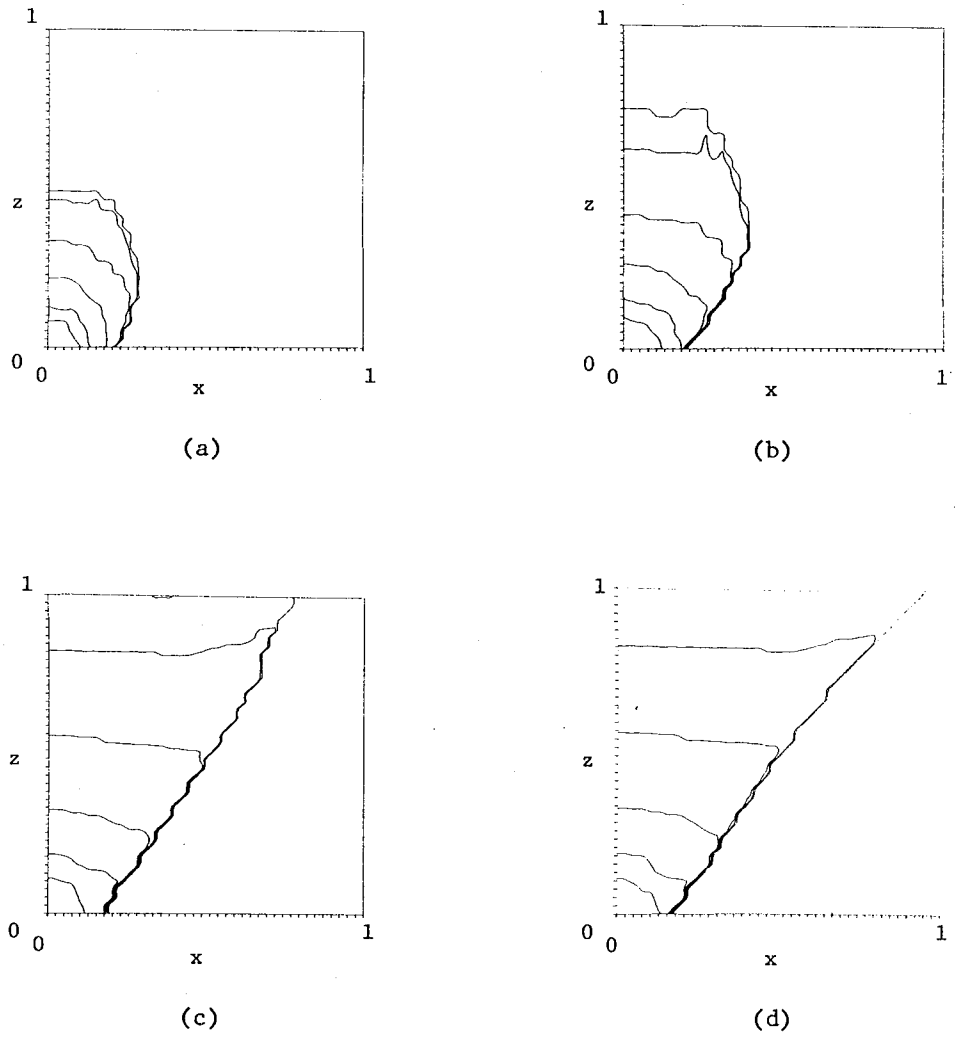

Figure 6

Saturation contours s = .1, .2, .3, .4, .5, .6
at times (a) t = .06, (b) t = .12, (c) t = .24, (d) t = .28
for second test problem.

should lie on top of each other, hence the drawing of contours .7, .8, and .9 was suppressed for clarity in the figures. The solution is depicted at several values of t, up to a time prior to breakthrough of the displacing fluid into the sink. The undulations in the advancing front, of the order of one mesh spacing, are due in part to the interpolation introduced by the contour plotting program, but mainly to the properties of the random choice method, of which they are typically representative. These statistical fluctuations do not cause numerical difficulty, however, even for the present case of $\varepsilon = 0$, for which the front is seen to move in a generally stable manner.

The example depicted in Figure 6 is for $\beta = 5$ and $\alpha = 2$ with a source of strength 5 at (0,0) and a sink of the same strength at (1,1). Here the less-dense fluid is injected at the bottom of the domain and advances buoyantly towards the top while progressing more slowly toward the sink. The contours and mesh spacing are as in Figure 5.

The time required to perform the complete calculation to breakthrough, including numerical solution of (7),(8) by a fast-Poisson preconditioned conjugate-gradient method, was approximately 3 minutes on the CDC 7600. This is several times slower than for a problem without gravity [2], in part because with $\beta \neq 0$ $\psi(s)$ generally has a different shape at each mesh point.

No attempt was made to assess the error in the computed solution for these examples except by verifying qualitative agreement with solutions obtained with coarser grids. The analytic solution for a test problem without gravity for which q does not vary with time [$\lambda(s) \equiv 1$] is compared with the random choice method numerical solution in [3]. Recently obtained experimental results for a one-dimensional Buckley-Leverett problem have indicated that the random choice method obtains more accurate solutions near a shock than do other commonly used methods, strikingly more accurate than the difference method that yielded incorrect weak solutions not satisfying the entropy condition [16].

ACKNOWLEDGMENTS

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