On Optimality of Linear Interference Alignment for the Three-User MIMO Interference Channel with Constant Channel Coefficients

Chenwei Wang, Tiangao Gou, Syed A. Jafar

Department of Electrical Engineering and Computer Science

University of California, Irvine, Irvine, CA 92697

E-mail : {chenweiw, tgou, syed}@uci.edu

Abstract

We investigate the optimality of linear interference alignment (allowing symbol extensions) for 3-user $M_T \times M_R$ MIMO interference channel where M_T and M_R denote the number of antennas at each transmitter and each receiver, respectively, and the *channel coefficients are held constant*. Recently, Wang et al. have conjectured that interference alignment based on linear beamforming using only proper Gaussian codebooks and possibly with symbol extensions, is sufficient to achieve the information theoretic DoF outer bound for all M_T, M_R values except if $|M_T - M_R| = 1$, $\min(M_T, M_R) \geq 2$. A partial proof of the conjecture is provided by Wang et al. for arbitrary M_T, M_R values subject to a final numerical evaluation step that needs to be performed for each M_T, M_R setting to complete the proof. The numerical evaluation step is also carried out explicitly by Wang et al. to settle the conjecture for all M_T, M_R values up to 10. For $|M_T - M_R| = 1$, $\min(M_T, M_R) \geq 2$, Wang et al. show that interference alignment schemes based on linear beamforming with proper Gaussian signaling and symbol extensions are not sufficient to achieve the information-theoretic DoF outer bonds. In contrast, in this note we show, for all M_T, M_R values up to 10, that interference alignment schemes based on linear beamforming over symbol extensions are enough to achieve the information theoretic DoF outer bounds for constant channels, if *asymmetric complex signaling* is utilized. Based on this new insight, we conjecture that linear interference alignment is optimal for achieving the information theoretic DoF outer bounds for all M_T, M_R values in the 3 user $M_T \times M_R$ MIMO interference channel with constant channel coefficients, except for the case $M_T = M_R = 1$ where it is known that either time/frequency-varying channels or non-linear (e.g., rational alignment) schemes are required.

1 Background

Recently, a new concept called *Subspace Alignment Chains* is introduced by Wang et al. in [1] to determine the degrees of freedom (DoF) of the three-user $M_T \times M_R$ MIMO interference channel where M_T and M_R denote the number of antennas at each transmitter and each receiver, respectively and $M_T \neq M_R$. As the length of the subspace alignment chains characterizes the DoF bottleneck, it is interesting to observe that the DoF value per user is a piecewise linear function of M_T, M_R. In [1], while the information theoretic DoF outer bound is presented for all M_T, M_R values, it is shown that for constant valued channels, linear interference alignment schemes employing the notion of subspace alignment chains are sufficient to achieve the DoF outer bound for all (M_T, M_R) pairs such that $2 \leq \min(M_T, M_R) < 10$, *except for* $|M_T - M_R| = 1$. For these exceptional cases, non-linear schemes such as the rational alignment scheme in [3], or even linear schemes over time-varying channels, can still achieve the optimal DoF value, such that we fully establish the DoF results. However, if we restrict ourselves to linear schemes as well as require channel coefficients to be constant valued over symbol extensions, can we still achieve the information theoretic DoF outer bound? In this note, we will show that the answer is yes, and the key of the solution relies on interference alignment with asymmetric complex signaling, which is first introduced in [4] for three-user SISO interference channel, and then applied to two-user X channel, cellular networks [4], compound broadcast channel [2], etc.

Remark: Note that the feasibility of linear interference alignment is already settled by Wang et al. in [1] for constant channel realizations under the constraint that *symbol extensions are not allowed*. While this formulation of the feasibility question without symbol extensions is originally proposed by Cenk et al. for analytical tractability and is the most commonly studied setting, including recent work by Bresler et al. and by Razaviyayn et al., clearly the assumption of no channel extensions is overly restrictive. Our goal here is to settle the linear inerference alignment feasibility question for constant channel realizations *allowing symbol extensions*. Clearly, while in the former formulation we are limited to only integer DoF values, the latter formulation incorporates fractional DoF values as well.

1.1 Asymmetric Complex Signaling

Consider a point to point channel with a complex channel coefficient h and a complex noise term z. If we distinguish the real and imaginary parts of each complex dimension as two real dimensions, then the channel can be written as

$$\begin{bmatrix} y_R \\ y_I \end{bmatrix} = \begin{bmatrix} h_R & -h_I \\ h_I & h_R \end{bmatrix} \begin{bmatrix} x_R \\ x_I \end{bmatrix} + \begin{bmatrix} z_R \\ z_I \end{bmatrix} \tag{1}$$

where subscripts R and I denote the real and imaginary part of the complex number, respectively. That is, we convert the original complex scalar channel to a real MIMO channel. This operation does impose special channel structure, i.e., the 2×2 channel matrix in (1) is a rotation matrix. The conversion from symmetric to asymmetric complex signaling can be similarly applied to MIMO settings. Reference [4] provides a detailed exposition.

1.2 System Model and Metrics

The system model and metrics are identical to that in [1] with the restriction that the channel coefficients are held constant across symbol extensions. For the sake of completeness, we briefly summarize them here again.

We consider a fully connected three-user MIMO interference channel where there are M_T and M_R antennas at each transmitter and each receiver, respectively. Each transmitter sends one independent message to its own desired receiver. Denote by $\mathbf{H}_{ji}$ the $M_R \times M_T$ channel matrix from transmitter i to receiver j where $i, j \in \{1, 2, 3\}$. We assume that the channel coefficients are independently drawn from continuous distributions, and once drawn, they remain constant during the entire transmission. Global channel knowledge is assumed to be available at all nodes.

At time index $t \in \mathbb{Z}^+$, Transmitter i sends a complex-valued $M_T \times 1$ signal vector $X_i(t)$, or equivalently, a $2M_T \times 1$ real signal vector $\bar{X}_i(t)$, which satisfies an average power constraint $\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\|\bar{X}_i(t)\|^2] \leq \rho$ for T channel uses. At the receiver side, User j receives a $2M_R \times 1$ real signal vector $\bar{Y}_j(t)$ at time index t, which is given by:

$$\bar{Y}_j(t) = \sum_{i=1}^{3} \bar{\mathbf{H}}_{ji}\bar{X}_i(t) + \bar{Z}_j(t) \tag{2}$$

where $\bar{Z}_j(t)$ an $2M_R \times 1$ real column vector representing the i.i.d. real additive white Gaussian noise (AWGN) at Receiver j, each entry of which is an i.i.d. real Gaussian random variable with zero-mean and 1/2-variance. In addition, we use $\bar{\mathbf{H}}_{ji}$ to denote the $2M_R \times 2M_T$ channel matrix obtained from $\mathbf{H}_{ji}$ using asymmetric complex signaling.

Note that the user index k is interpreted modulo 3 so that, e.g., User 0 is the same as User 3.

2 DoF Achievability: Interference Alignment with Asymmetric Complex Signaling

For three-user $M_T \times M_R$ MIMO interference channel where $|M_T - M_R| = 1$ and $\min(M_T, M_R) \geq 2$, since we have shown the information theoretic DoF outer bound in [1], we only need to provide the achievable schemes in this section. The achievable schemes are based on the linear interference alignment scheme with *asymmetric complex signaling*. Due to the reciprocity of the linear scheme, without loss of generality, we only consider the case when $M_T = M_R - 1$, so that $M_T = M$ and $M_R = M + 1$.

Theorem 1 *For the 3-user $M_T \times M_R$ MIMO interference channel with constant channel coefficients and $\max(M_T, M_R) \leq 10$, linear beamforming schemes achieve the information theoretic DoF outer bound for all cases except $M_T = M_R = 1$.*

Since all other cases are established in [1], we will show the proof of the achievability in the next two subsections for all $(M, M+1)$ cases with $M + 1 \leq 10$. The proof argument requires a numerical validation step (to establish the linear independence of desired signal from interference in the almost surely sense) that needs to be performed for each M_T, M_R setting to complete the proof. We explicitly perform the numerical validation step for all M_T, M_R up to 10 to arrive at Theorem 1. In general, for arbitrary values of M_T, M_R, the reader can perform the numerical validation step to complete the proof. Based on all cases considered so far, we have the following conjecture.

Conjecture 1 *For the 3-user $M_T \times M_R$ MIMO interference channel with constant channel coefficients, linear beamforming schemes achieve the information theoretic DoF outer bound for all cases except $M_T = M_R = 1$.*

Figure 1 is the updated version of the corresponding figure from [1] incorporating the results of Theorem 1.

2.1 Example: $(M_T, M_R) = (2, 3)$

We will first consider the 2×3 setting, and then generalize the scheme to other cases in the next subsection. Recall that since we utilize asymmetric complex signaling, each complex dimension is equivalent to two real dimensions.

Let us start with the 2×3 setting. Applying the invertible linear transformation introduced in [1], we can obtain the resulting channel connectivity as shown in Figure 2. In Figure 2 there

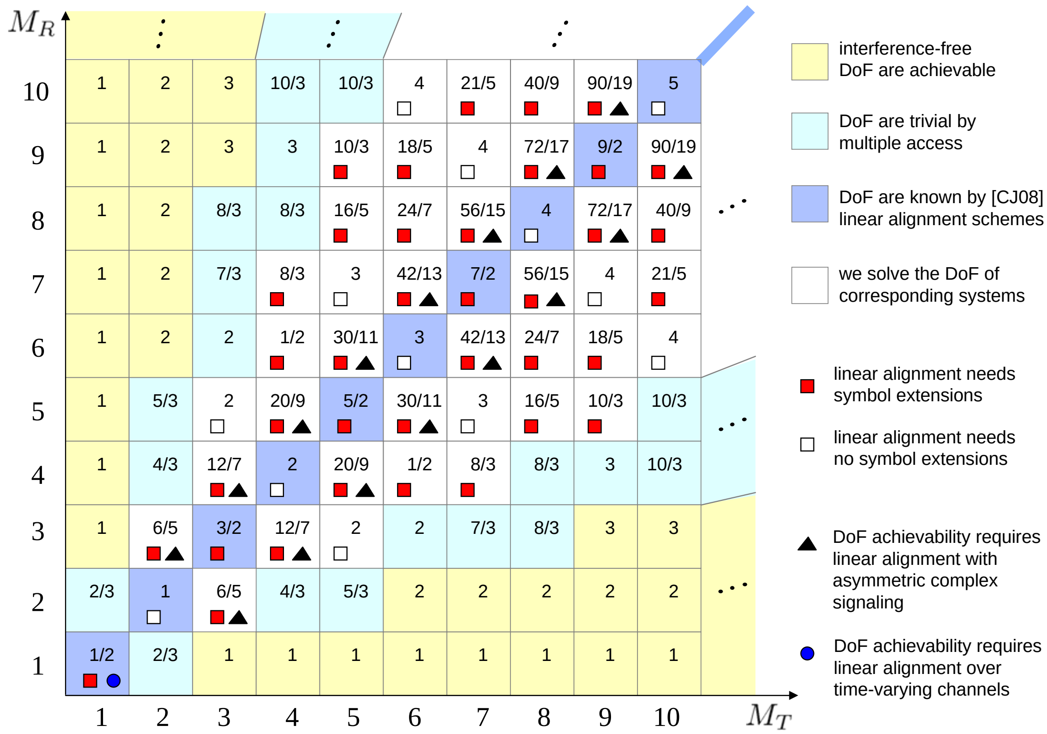

Figure 1: DoF per User Achieved by Linear Interference Alignment Schemes (without space-extensions) for the Three-User $M_T \times M_R$ MIMO Interference Channel

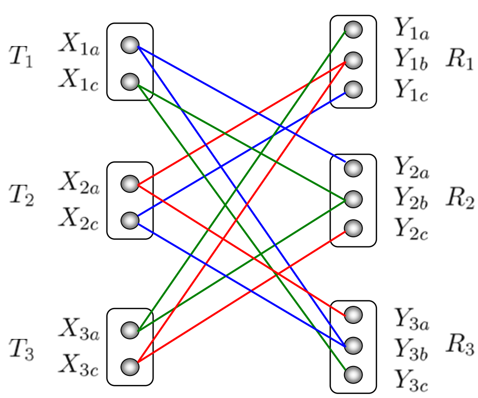

Figure 2: Normalizing the Interference-carrying Links of the 2×3 Setting to Identity Matrices

are three open chains, denoted as blue, green and red colors, each implying a subspace alignment chain with length 2. Because they are open loop, we can normalize the channel coefficient of each segment to be one. As a result, we can write the cross channel matrices as follows,

$$\mathbf{H}_{k(k+1)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{H}_{k(k-1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3}$$

If we use asymmetric complex signaling to convert each complex dimension to two real dimensions, then the the cross channel matrices are given by:

$$\bar{\mathbf{H}}_{k(k+1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \bar{\mathbf{H}}_{k(k-1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

For simplicity, we still use $\mathbf{H}_{kk}$ to denote the direct 3×2 channel matrix of User k after the change of basis operations, and $\bar{\mathbf{H}}_{kk}$ to denote the associated 6×4 channel matrix utilizing asymmetric complex signaling. As described in [1], the achievable scheme for $M/N = 2/3$ is directly inherited from $M/N = 3/5$. For the 2×3 setting, our aim is to achieve 6/5 DoF per user, and this can be done by send 12 real symbols over 5 time slots, i.e., 10 real dimensions. With 5 symbol extensions, the effective channel matrix becomes

$$\mathbf{H}^{[ji]} = \mathbf{I}_5 \otimes \bar{\mathbf{H}}_{ji} \tag{5}$$

where $\mathbf{I}_5$ is the 5×5 identity matrix. To ensure each user can separate 12 real desired signal vectors from the interference in its 30 real dimensional signal space, the dimension of the space spanned by 24 real interference vectors cannot be more than 18. Therefore, at each receiver, we need to align 6 interference vectors. Note that in the signal space at each receiver, there is a 10 real dimensional subspace that can be accessed by two interferers. Then we can randomly choose 6 dimensional subspace in this common subspace as the subspace where the 6 interference vectors align. Mapping this 6 dimensional subspace back to the interferers determines the beamforming matrix at the transmitter. Since each transmitter interferes with 2 receivers, each of two unintended receivers will determine 6 beamforming vectors, for a total of 12 beamforming vectors per user.

With this intuitive understanding, on the symbol extended real channel, we write the 20×12 beamforming matrix of user i, $\mathbf{V}^{[i]}$, as $\mathbf{V}^{[i]} = [\mathbf{V}_{i,1} \; \mathbf{V}_{i,2}]$ where $\mathbf{V}_{i,1}$ and $\mathbf{V}_{i,2}$ are 20×6 matrices. Then at Receiver 1, we align the first 6 beams from Transmitter 2 with those from Transmitter 3. Mathematically, we have

$$\mathbf{H}^{[13]}\mathbf{V}_{3,1} = \mathbf{H}^{[12]}\mathbf{V}_{2,1} \Rightarrow \underbrace{\begin{bmatrix} \mathbf{H}^{[13]} & -\mathbf{H}^{[12]} \end{bmatrix}}_{\bar{\mathbf{A}}} \underbrace{\begin{bmatrix} \mathbf{V}_{3,1} \\ \mathbf{V}_{2,1} \end{bmatrix}}_{\bar{\mathbf{a}}} = \mathbf{0} \tag{6}$$

Since $\bar{\mathbf{A}}$ is a 30×40 matrix, $\bar{\mathbf{a}}$ can be obtained as 6 linearly independent vectors in the 10 dimensional null space of $\bar{\mathbf{A}}$. Notice that the 10 basis vectors of the null space of $\bar{\mathbf{A}}$ are columns of the matrix $\mathbf{I}_5 \otimes \mathbf{V}_1$ where $\mathbf{V}_1$ are a 8×2 matrix whose column vectors are the basis of the null space of the 6×8 matrix $[\mathbf{H}_{13} \quad - \mathbf{H}_{12}]$, i.e.,

$$[\mathbf{H}_{13} \quad - \mathbf{H}_{12}]\, \mathbf{V}_1 = \mathbf{0} \tag{7}$$

where

$$[\mathbf{H}_{13} \;\; - \mathbf{H}_{12}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{8}$$

Therefore, we let the two columns of $\mathbf{V}_1$ be its two orthogonal basis and it can be written as:

$$\mathbf{V}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T. \tag{9}$$

Then, we obtain three columns of $\bar{\mathbf{a}}$ as

$$\bar{\mathbf{a}} = (\mathbf{I}_5 \otimes \mathbf{V}_1)\mathbf{a} \tag{10}$$

where $\mathbf{a} = (a_{ij})$ is a 10×6 combining matrix. For simplicity, we let it have a block diagonal structure, i.e., $\mathbf{a} = \mathrm{diag}([\mathbf{a}_1 \ \ \mathbf{a}_2])$ where each block is a 5×3 matrix with i.i.d. randomly generated real entries.

Similarly, at Receiver 2, we align the first 6 beams from Transmitter 1 with the last 6 beams from Transmitter 3. Mathematically,

$$\mathbf{H}^{[21]}\mathbf{V}_{1,1} = \mathbf{H}^{[23]}\mathbf{V}_{3,2} \Rightarrow \begin{bmatrix}\mathbf{H}^{[21]} & -\mathbf{H}^{[23]}\end{bmatrix} \underbrace{\begin{bmatrix}\mathbf{V}_{1,1} \\ \mathbf{V}_{3,2}\end{bmatrix}}_{\bar{\mathbf{b}}} = \mathbf{0} \tag{11}$$

Then we can choose $\bar{\mathbf{b}}$ as

$$\bar{\mathbf{b}} = (\mathbf{I}_5 \otimes \mathbf{V}_2)\mathbf{b} \tag{12}$$

where $\mathbf{V}_2$ is a 8×2 matrix whose column vectors are the basis of the null space of the 6×8 matrix $[\mathbf{H}_{21} \ \ -\mathbf{H}_{23}]$, and we let $\mathbf{V}_2 = \mathbf{V}_1$. Also, $\mathbf{b} = (b_{ij}) = \mathrm{diag}([\mathbf{b}_1 \ \ \mathbf{b}_2])$ is a block diagonal 10×6 matrix, and each block is a 5×3 matrix with i.i.d. randomly generated real entries.

At Receiver 3, we align the last 6 beams from Transmitter 1 with those from Transmitter 2, hereby we have the equation:

$$\mathbf{H}^{[32]}\mathbf{V}_{2,2} = \mathbf{H}^{[31]}\mathbf{V}_{1,2} \Rightarrow \begin{bmatrix}\mathbf{H}^{[32]} & -\mathbf{H}^{[31]}\end{bmatrix} \underbrace{\begin{bmatrix}\mathbf{V}_{2,2} \\ \mathbf{V}_{1,2}\end{bmatrix}}_{\bar{\mathbf{c}}} = \mathbf{0} \tag{13}$$

Then we can choose $\bar{\mathbf{c}}$ as

$$\bar{\mathbf{c}} = (\mathbf{I}_5 \otimes \mathbf{V}_3)\mathbf{c} \tag{14}$$

where $\mathbf{V}_3$ is a 8×2 matrix whose column vectors are the basis of the null space of the 6×8 matrix $[\mathbf{H}_{32} \ \ -\mathbf{H}_{31}]$. Again we let $\mathbf{V}_3 = \mathbf{V}_1$, and $\mathbf{c} = \mathrm{diag}([\mathbf{c}_1 \ \ \mathbf{c}_2])$ is a block diagonal 10×6 matrix, and each block is a 5×3 matrix with i.i.d. randomly generated real entries.

After aligning interference, we ensure that the dimension of the space spanned by interference is small enough. In order for each receiver to decode the desired message, it remains to be shown the desired signals and interference do not overlap at each receiver. Specifically, we need to see if the 30×30 matrix consisting of 12 desired real signal vectors and 18 effective real interference vectors has full rank. Due to the symmetry of the signaling, let us consider Receiver 1. The 30×30 matrix at Receiver 1 is given by:

$$\mathbf{G} = \begin{bmatrix}\mathbf{H}^{[11]}\mathbf{V}_{1,1} & \mathbf{H}^{[11]}\mathbf{V}_{1,2} & \mathbf{H}^{[12]}\mathbf{V}_{2,1} & \mathbf{H}^{[12]}\mathbf{V}_{2,2} & \mathbf{H}^{[13]}\mathbf{V}_{3,2}\end{bmatrix} \tag{15}$$

where $\mathbf{H}^{[13]}\mathbf{V}_{3,1}$ does not appear above because it already aligns with $\mathbf{H}^{[12]}\mathbf{V}_{2,1}$ at Receiver 1. Now let us substitute the channels matrices as well as beamforming matrices into the equation above, and rearrange the rows and columns, then we obtain:

$$\mathbf{G} = \begin{bmatrix} h_{12R}\mathbf{a}_1 & -h_{12I}\mathbf{a}_2 & h_{11R}\mathbf{b}_1 & -h_{11I}\mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_1 & \mathbf{0} \\ h_{12I}\mathbf{a}_1 & h_{12R}\mathbf{a}_2 & h_{11I}\mathbf{b}_1 & h_{11R}\mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_2 \\ h_{22R}\mathbf{a}_1 & -h_{22I}\mathbf{a}_2 & h_{21R}\mathbf{b}_1 & -h_{21I}\mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{c}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ h_{22I}\mathbf{a}_1 & h_{22R}\mathbf{a}_2 & h_{21I}\mathbf{b}_1 & h_{21R}\mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_2 & \mathbf{0} & \mathbf{0} \\ h_{32R}\mathbf{a}_1 & -h_{32I}\mathbf{a}_2 & h_{31R}\mathbf{b}_1 & -h_{31I}\mathbf{b}_2 & \mathbf{b}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ h_{32I}\mathbf{a}_1 & h_{32R}\mathbf{a}_2 & h_{31I}\mathbf{b}_1 & h_{31R}\mathbf{b}_2 & \mathbf{0} & \mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{16}$$

where $h_{ij} = [\bar{\mathbf{H}}_{11}]_{ij}$, $h_{ijR} = \Re\{h_{ij}\}$, $h_{ijI} = \Im\{h_{ij}\}$, and $\mathbf{0}$ is a 5×3 zero matrix.

It is straightforward to verify that the matrix $\mathbf{G}$ has full rank almost surely when we randomly pick the entries of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{c}_1$ and $\mathbf{c}_2$ from a continuous distribution.

2.2 General Cases: $(M_T, M_R) = (M, M+1)$

For the $M \times (M+1)$ setting, Theorem 1 implies that we can achieve $M(M+1)/(2M+1)$ DoF per user. Specifically, by utilizing asymmetric complex signaling, we will show that each user is able to send $2M(M+1)$ real symbols over $2(M+1)(2M+1)$ time slots, i.e., $2(M+1)(2M+1)$ real dimensions. The remaining work of the achievability is only to follow the achievable schemes that we introduced in Section 8.3.2 in [1]. Specifically, we randomly generate a $2M(2M+1) \times 2(M+1)(2M-1)$ matrix with real entries at each transmitter independently and multiply it to the equivalent channel seen at each transmitter, such that each transmitter effectively has a $2(M+1)(2M-1)$ real dimensional space. As a consequence, we have an effective $(2(M+1)(2M-1), 2(M+1)(2M+1))$ setting, for which interference alignment schemes introduced in Section 8.1.3 in [1] can be applied directly. Again, the only remaining task is to verify that the desired signal and interference are linearly independent almost surely. This is equivalent to the statement that the polynomial corresponding to the determinant of the matrix comprised of desired and interference vectors is not the zero polynomial. As usual, this is readily established if the polynomial evaluates to a non-zero value for any chosen realization of channel coefficients. Thus, the remaining step to complete the proof is simply to choose a channel realization and verify that the polynomial takes a non-zero value for this realization, thereby establishing that it is not the zero polynomial, and therefore that it must be non-zero for almost all channel realizations. Performing this step explicitly for all M_T, M_R upto 10, we establish the result of Theorem 1. In general, we conclude with the conjecture that in *all* $M \times (M+1)$ cases, the DoF outer bound value is tight and can be achieved with linear interference alignment with asymmetric complex signaling.

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