Title

Advances in Doublet Mechanics: I. Multi-Scale Elastic Plane Wave Propagation

Authors

Granik, Vladimir

Ferrari, Mauro

Publication Date

1995-06-01

REPORT NO.
UCB/SEMM-95/03

STRUCTURAL ENGINEERING
MECHANICS AND MATERIALS

ADVANCES IN DOUBLET MECHANICS: I. MULTI-SCALE ELASTIC WAVE PROPAGATION

BY

V. T. GRANIK

AND

M. FERRARI

JUNE 1995

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

Advances in Doublet Mechanics:
I. Multi-scale Elastic Plane Wave Propagation

Vladimir T. Granik[a] **, Mauro Ferrari** [a,b]

[a] *Department of Civil Engineering, University of California, Berkeley, CA 94720*

[b] *Department of Civil Engineering and Department of Material Science and Mineral Engineering, University of California, Berkeley, CA 94720*

ABSTRACT

This paper addresses the propagation of elastic waves in media with an underlying discrete structure. The theory of doublet mechanics is employed, in view of its multi-scale nature, and capability to perform an analytical transition from the discrete to the continuum level. The three-dimensional doublet-mechanical field equations of elastodynamics are derived under some simplifying assumption, and the analysis is then focused on the cases of linear and planar arrangements, the latter with the further restriction of to continuum-level planar isotropy. Dispersion relations are established, that demonstrate the dispersivity and retardation of both P- and S-waves at all scales other than those for which the continuum approximation is valid. The compatibility of the doublet-mechanical analysis with solid-state physics and continuum mechanical approaches is demonstrated, and applications to crystals, granular media, and seismological problems are presented.

INTRODUCTION

Despite its extraordinary successes, continuum mechanics suffers the limitation of not being necessarily applicable to solids at all length scales of scientific or technological interests. As long as all dominant material features and inhomogeneities are dimensionally much smaller than the body or structure of interest, stress and deformation analyses may be successfully performed, at least in reversible cases, under the assumption of material homogeneity. Detailed analyses of the mechanical fields at and in the vicinity of material inhomogeneities, such as needed in fracture mechanics and micromechanics, are generally performed by imbedding said inhomogeneities in bodies and structures with macroscopically homogeneous properties. Recourse to homogeneized continuum representations of solid matter is commonly made also in cases where the material inhomogeneities introduce non-ignorable qualitative features in the mechanical fields. Cases of this type arise in many fields of engineering ans science.

One example for all is found in the field of mechanics of granular and particulate media, and is known as 'Flamant's paradox': An elastic half plane, subjected to compression by a vertical boundary point force, experiences compressive stresses everywhere, according to continuum elasticity. However, accordingly loaded granular regions exhibit tensile openings. This renders the homogenized continuum description questionable, for such cases, and necessitates the introduction of ad-hoc considerations and modeling assumptions. Other phenomena that are typical to granular media include: Dilatancy (Reynolds, 1885), liquefaction (Nemat-Nasser and Tobita, 1982), shear band formation (Tatsuoka et al., 1990), the conical transfers of surface loads into granular massives (Meek and Wolf, 1992), and the transition of granular flow from fluid-like to solid-like behavior (Campbell, 1993). For comprehensive review of issues that involve the complex nature of granular media, the reader is referred to (Hutter and Rajagopal, 1994),

Modeling paradoxes such as Flamant's essentially express the necessity for developments beyond Cauchy-type continuum mechanics, that afford mechanical analyses to be performed at several different dimensional scales, as advocated for instance by (Bažant, 1993), while preserving consistency and unity in the modeling approach. An additional feature that would be highly desirable is the capability of modeling both the continuum and the discrete aspects within the same theory, while mantaining rigorous consistency with established fields, such as continuum mechanics and solid-state physics. Such capability must not necessarily include multiscale effect, as non-scale features, such as coordination numbers, relative positions and orientations of granules in the packing, contact bonds are indeed significant, (Gray, 1968; Harr, 1977; Feda, 1982).

Considerable successes have been obtained in the formulation of generalized continuum theories, intended to allow the modeling of microscale phenomena, and in the various approaches in the discipline of homogenization. Generalized continuum mechanics, microstructural mechanics of granular media and homogenization methods are well-established and extensively researched fields, and no attempt is here made to review them exhaustively, other than to cite some basic references. For generalized continua, these include, (Cosserat, 1909, Eringen and Suhubi, 1964; Mindlin, 1964, 1968; Green and Rivlin, 1964a, 1964b; Palmov, 1964; Eringen, 1966; Novacki, 1970; Stojanoviċ, 1972; Picci and Saccomandi, 1990; Teodorescu and Soos, 1973; Mrithyumjaya and Pratar, 1993).

In what may be dubbed a non-axiomatic approach, the micromechanics of granular solids has proceeded from direct consideration of the real solid microstructure, without involving a priori any hypotheses and special tools, such as the vector-directors (Mogami, 1965; Horne, 1965, 1969; Matsuoka, 1974; Davis and Deresiewicz, 1977; Nemat-Nasser, 1982; Oda et al., 1982; Mehrabadi et al., 1982, 1983; Satake, 1978, 1982, 1993; Bathurst and Rothenburg, 1988; Chang and Misra, 1990; etc.). On this basis, continuous descriptions of discrete microstructures have been obtained. This has afforded advances in problems in granular mechanics, such as liquefaction (Nemat-Nasser and Tobita, 1982), dilatancy (Nemat-Nasser, 1982; Mehrabadi and Nemat-Nasser, 1983), constitutive relations (Satake, 1993), and so forth. The reader interested in granular mechanics is referred to the transactions of international symposiums and conferences of IUTAM (eds. Vermeer and Luger, 1982), ASME (ed. Mehrabadi, 1992), ASMGM (ed. Thornton, 1993), etc.

Guided by the objective of providing advances in multi-scale mechanical analyses, an entirely different approach was presented in (Granik and Ferrari, 1993a), that is based on a discrete representation of matter, i.e. views solids as collection of discrete points at possible finite distances. A full kinematic system was developed, that strains of the extensional, torsional and shear type at the level of pairs of particles, or 'doublets'. Being based on doublet-level quantities, the entire theory was then dubbed 'Doublet Mechanics'.

In (Granik and Ferrari, 1993a), the doublet-level microstresse were introduced, and a variational formulation of equilibrium was employed, that yields micro-level equations of equilibrium, as well as the transition between the conventional stresses and the microstructural geometry, fields and properties.

Doublet-level linear elastic constitutive equations were introduced, and an analytic solution was given to Flamant's paradox: Microstresses were proven to be tensile in certain regions, while the macroscopic level stresses coincided with Flamants'. (Note: a quantitative correction to the solution reported in (Granik and Ferrari, 1993a,b) will appear in (Nadeau et al., 1995)). Further work in doublet mechanics included the establishment of microstructurally-based failure theories

(Ferrari and Granik, 1994, 1995), and the extension to viscoelastic consitutions (Maddalena and Ferrari, 1995).

In this work, the first problems are solved with doublet mechanics that incorporate multi-scale effects. In particular, the plane propagation of elastic waves in granular media is researched, with reference a spatial arrangement of the particles that results in macroscopic-level isotropy in the plane of propagation and particle displacement. The topic is motivated by experiments that establish the dependence of wave frequencies and velocities depend on the parameters of granular microstructure, including porosity (Urick, 1948; Hampton, 1967), the coordination numbers and the contact bonds (Trent, 1989), the particle sizes and the interparticle distances (Iida, 1938; Matsukawa and Hunter, 1956). Theoretical approaches to the modeling of dispersion of plane seismological elastic waves have been proposed, that incorporate some characteristic of the earth's microstructure. Among these are the multilayered structure (Haskell, 1953), anisotropy (Crampin and Taylor, 1971), gravity (Ewing et al., 1957), the curvature and stratification (Sezawa, 1927) radial inhomogeneity (Saito, 1967), vertical discontinuities (Malischewsky, 1987), etc. Other than anisotropy, these properties induce surface wave dispersion. However, the quantitative results - when provided - are in strong disagreement with observations.

In this paper the theory of doublet mechanics is shown to predict strong dispersion, especially, at shorter wavelength. However, the paper nor the presented theory are dedicated to the detailed study of seismological phenomena: The emphasis is on the development of a full multiscale theory, applicable in principle to very different materials and microstructural dimensions. Consistently with this objective, the doublet mechanical is below shown to be fully compatible with crystal dynamics and continuum elastodynamics, yet intrinsically richer than the latter, in that it affords the modeling of dispersion and retardation phenomena. Both of these are scale-related, and disappear in the continuum or the infinite wavelength limit, as required.

The paper contains an extended summary of the governing equations and notations of doublet mechanics, which is to be employed for references for the forthcoming parts of this series. These include the study of the refraction phenomena of plane waves at the interface between a granular medium and a free surface (Zhang and Ferrari, 1995), where it is found that the critical angle of mode conversion is a scale-dependent quantity. Plane elastostatic solutions are to be presented in (Nadeau et al, 1995), which are the doublet-mechanical analogs of the celebrated continuum-level solutions of Kelvin, Eshelby, and the method of Airy's stress potentials. Thermodynamics and invariance arguments are employed in (Mon and Ferrari, 1995) to derive restrictions on the doublet-level constitutive forms.

GOVERNING EQUATIONS OF DOUBLET MICROMECHANICS

In this article the propagation of waves through granular media is studied, employing the general theory of Doublet Mechanics (DM). The salient features of DM are summarized next. For more details on DM, the reader is referred to (Granik and Ferrari, 1993a, 1993b; Ferrari and Granik, 1994, 1995; Maddalena & Ferrari, 1994).

To help with the visualization, a material model underlying doublet micromechanics may be introduced, that consists of a regular array H of N equal elastic spheres of diameter d the centers of which, or nodes, form a space Bravais lattice L. It is assumed that the number of particles N is large, and the diameter d is small but finite (Granik and Ferrari, 1993a).

The Bravais lattice L is fully defined by the stationary bundle T_n of n basic vectors $\vec{\zeta}_\alpha^{o}$ ($\alpha = 1, 2, \ldots, n$) where $n = m/2$ is the valence of the lattice, m is the coordination number of the array H, $\eta_\alpha \equiv |\vec{\zeta}^{o}_{\alpha}|$ is the internodal distance (the lattice constant) in an α-direction - a quantity that is generally different for different values of α. The unit lattice vectors are $\vec{\tau}^{o}_{\alpha} = \vec{\zeta}^{o}_{\alpha}/\eta_\alpha$. All the nodes of the Bravais lattice are arranged along straight lines, parallel to the n unit bundle directions $\vec{\tau}^{o}_{\alpha}$.

Along any positive direction $\vec{\tau}^{o}_{\alpha}$ in the granule array H, any particle $A \in H$ is adjacent to n other particles $B_\alpha \in H$ ($\alpha = 1, 2,..., n$). Any such pair $(A,B_\alpha) \in H$ is called a doublet. The vector $\vec{\zeta}^{o}_{\alpha}$ may is be termed a doublet axis, in that it joins the centers $a \in A$ and $b_\alpha \in B_\alpha$. The lattice constant $\eta_\alpha \geq d$ hence represents the length of the doublet (A,B_α) of an α-direction.

The particle $A \in (A,B_\alpha)$ is in contact with B_α if the doublet length $\eta_\alpha = d$. Such an α-doublet is considered is called a contact doublet, the α-direction being the contact direction. There may be s $(0 \leq s \leq n)$ contact directions in which all particles are in contact, and the underlying granule array may be called a regular s-contact n-valence array H_{sn}. If the contact number $s = n$, than the array H_{sn} becomes a regular completely contact array $H_{nn} = H_n$, or a regular n-valence packing, where all the adjacent particles are in contact with each other.

There are four regular packings (Deresiewicz, 1958): H_3 (simple cubic), H_4 (cubical-tetrahedral), H_5 (tetragonal-sphenoidal), and H_6 (face-centered, or pyramidal). If the particle array H_{sn} is not a packing, then in addition to the mentioned above, there may be other regular structures similar to the crystal ones: simple tetragonal, orthorombic face-centered, simple orthorombic, etc. (Cottrell, 1964). In these cases, the particles interact in the $n - s$ non-contact directions owing to intermediate substances (compliant inclusions) or spatial electrostatic forces binding atoms and molecules in crystals.

It should be noted that the doublet micromechanics, as presented in (Granik and

Ferrari, 1993a) is valid for all the regular arrays H_{sn} whether $s = n$ or $s \neq n$, if the forces of particle interactions are of a short-distance character. Contact forces such as friction, represent a particular case of such interactions.

While the validity of DM is apparent *a priori* for materials that exhibit a macroscopically evident granular lattice structure, it must be remarked that recent results - especially those concerning failure theories (Ferrari & Granik, 1994, 1995) - have established *a posteriori* its effectiveness for macroscopically continuous media. This suggests that DM be interpreted as a general model, the validity of which is to be verified for specific material classes, much in analogy with what is current practice with continuum mechanics (CM). An interpretive aide in this context is the standpoint that the nodes of DM actually be averages or representations of the complexity of actual particle or molecular interactions - a concept that is strongly related to the interpretation given by Clausius of the 'molecular theory of elasticity' (see Todhunter, 1886, article No. 1400).

The building block of CM is a mathematical fiction, the differential volume element. By contrast, the building block of DM is the deformable doublet (hence the name of the theory). Within DM, the particles in all doublets undergo translations and rotations that are independent to a certain degree. This gives rise to doublet microstrains of the axial, torsional, and shear type. Energy conjugates to these are the microstresses of the axial, torsional, and shear type.

The connection between the microstresses and the macrostresses, i.e. the conventional and the couple stresses of CM, follows as a natural boundary condition of the virtual work formulation. This permits the continuum-level representation of any phenomenon that is expressed at the doublet level. The converse is certainly false: By no method can doublet-level information by obtained in full from a macroscopic-level analysis. This follows from the fact that the basic field quantities of DM are the microstrains and the microstresses, while their macroscopic counterparts are just phenomenological descriptors.

The governing equations of the doublet micromechanics are given next. All vectors and tensors are considered in a rectangular Cartesian frame of reference $\{x_1, x_2, x_3\}$ with unit vectors $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$. By convention, Latin indices are in here assumed to take the values $1, 2, 3$, while Greek subscripts do not have vectorial nature, and are valued in the range [1,n]. The summation convention is enforced on Latin indices only.

The governing equations of doublet micromechanics are:

1. **Kinematic equations**, relating the doublet microstrains of elongation (compression) $\bar{\varepsilon}_\alpha$, torsion $\bar{\mu}_\alpha$, and shear $\bar{\gamma}_\alpha$ to the vector fields of the translations $\bar{u}$ and rotations $\bar{\phi}$

$$\varepsilon_\alpha = \sum_{\kappa=1}^{M} \frac{\eta_\alpha^{\kappa-1}}{\kappa!} \tau_{\alpha i}^{o} T_{\alpha(\kappa)} \partial^\kappa u_i , \tag{1}$$

$$\mu_\alpha = \sum_{\kappa=1}^{M} \frac{\eta_\alpha^{\kappa-1}}{\kappa!} \tau_{\alpha i}^{o} T_{\alpha(\kappa)} \partial^\kappa \phi_i , \tag{2}$$

$$\gamma_{\alpha i} = (\delta_{ij} - \tau_{\alpha i}^{o} \tau_{\alpha j}^{o}) \sum_{\kappa=1}^{M} \frac{\eta_\alpha^{\kappa-1}}{\kappa!} T_{\alpha(\kappa)} \partial^\kappa u_j$$

$$- (\phi_i + \frac{1}{2} \sum_{\kappa=1}^{M} \frac{\eta_\alpha^{\kappa}}{\kappa!} T_{\alpha(\kappa)} \partial^\kappa \phi_j) \tau_{\alpha p}^{o} \in_{ijp} . \tag{3}$$

Here $\bar{\tau}_\alpha^{o} = \tau_{\alpha i}^{o} \bar{e}_i$, $\in_{ijp}$ is the permutation symbol, δ_{ij} is the Kronecker delta (Sokolnikoff, 1951); other quantities are defined as follows:

$$\vec{u} = u_i \vec{e}_i \, , \ \vec{\phi} = \phi_i \vec{e}_i \, , \tag{4}$$

$$\vec{\varepsilon}_\alpha = \varepsilon_\alpha \vec{\tau}^{\,o}_\alpha = \varepsilon_\alpha \tau^o_{\alpha i} \vec{e}_i = \varepsilon_{\alpha i} \vec{e}_i \quad (\varepsilon_{\alpha i} = \varepsilon_\alpha \tau^o_{\alpha i}) \, , \tag{5}$$

$$\vec{\mu}_\alpha = \mu_\alpha \vec{\tau}^{\,o}_\alpha = \mu_\alpha \tau^o_{\alpha i} \vec{e}_i = \mu_{\alpha i} \vec{e}_i \quad (\mu_{\alpha i} = \mu_\alpha \tau^o_{\alpha i}) \, , \tag{6}$$

$$\vec{\gamma}_\alpha = \gamma_{\alpha i} \vec{e}_i \, , \tag{7}$$

$$\mathrm{T}_{\alpha(\kappa)} \equiv \tau^o_{\alpha k_1} \tau^o_{\alpha k_2} \cdots \tau^o_{\alpha k_\kappa} \, , \tag{8}$$

$$\partial^\kappa u_i \equiv \frac{\partial^\kappa u_i}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_\kappa}} \, , \quad \partial^\kappa \phi_i \equiv \frac{\partial^\kappa \phi_i}{\partial x_{k_1} \partial x_{k_2} \cdots \partial x_{k_\kappa}} \, . \tag{9}$$

The integer M indicates the selected approximation level, with M=1 representing the non-scaling variant of the theory. It is noted that a sign in (3) has been reversed, with respect to (Granik and Ferrari, 1993a).

2. **Dynamic equations**, connecting the microstresses of elongation (compression) $\vec{p}_\alpha$, torsion $\vec{m}_\alpha$, and shear $\vec{t}_\alpha$ to the volume force $\vec{F}$, the translation $\vec{u}$, and the bulk density of granular medium ρ:

(*A*) 'Force' equations:

$$\sum_{\alpha=1}^{n} \sum_{\kappa=1}^{M} (-1)^{\kappa+1} \frac{\eta_\alpha^{\kappa-1}}{\kappa!} \mathrm{T}_{\alpha(\kappa)} \partial^\kappa (p_{\alpha i} + t_{\alpha i}) + F_i = \rho \ddot{u}_i \, , \tag{10}$$

(*B*) 'Couple' equations:

$$\sum_{\alpha=1}^{n} [\epsilon_{ijq}\, \tau^{o}_{\alpha j}\, t_{\alpha q} + \sum_{\kappa=1}^{M} (-1)^{\kappa+1} \frac{\eta_\alpha^{\kappa-1}}{\kappa!}\, T_{\alpha(\kappa)}\, \partial^{\kappa}(m_{\alpha i} - \frac{1}{2}\eta_\alpha\, \epsilon_{ijq}\, \tau^{o}_{\alpha j}\, t_{\alpha q})] = 0 \,. \quad (11)$$

Here the free index $i = 1, 2, 3$. Other definitions are:

$$\vec{p}_\alpha = p_\alpha\, \vec{\tau}^{o}_{\alpha} = p_\alpha\, \tau^{o}_{\alpha i}\, \vec{e}_i = p_{\alpha i}\, \vec{e}_i \quad (p_{\alpha i} = p_\alpha\, \tau^{o}_{\alpha i}) \,, \quad (12)$$

$$\vec{m}_\alpha = m_\alpha\, \vec{\tau}^{o}_{\alpha} = m_\alpha\, \tau^{o}_{\alpha i}\, \vec{e}_i = m_{\alpha i}\, \vec{e}_i \quad (m_{\alpha i} = m_\alpha\, \tau^{o}_{\alpha i}) \,, \quad (13)$$

$$\vec{t}_\alpha = t_{\alpha i}\, \vec{e}_i \,, \quad (14)$$

$$\vec{F} = F_i\, \vec{e}_i \,, \quad (15)$$

$$\ddot{u}_i \equiv \partial^2 u_i / \partial t^2 \,, \quad (16)$$

where t is time. Again, signs in (10-11) have been reversed, with respect to (Granik and Ferrari, 1993a).

3. **Constitutive equations**, relating the microstresses to the microstrains and the increment of the temperature θ:

$$p_\alpha = \sum_{\beta=1}^{n} A_{\alpha\beta}\, \varepsilon_\beta + J_\alpha\, \theta \,, \quad (17)$$

$$m_\alpha = \sum_{\beta=1}^{n} E_{\alpha\beta}\, \mu_\beta \,, \quad (18)$$

$$t_{\alpha i} = \sum_{\beta=1}^{n} I_{\alpha\beta ij}\, \gamma_{\beta j} \,, \quad (19)$$

in which $A_{\alpha\beta}$, J_α , $E_{\alpha\beta}$, $I_{\alpha\beta ij}$ are the scalar and tensor micromoduli of thermoelasticity, $\theta = T - T_o$, T_o and T are the temperature of granular medium before and after deformation, respectively.

4. **The heat flow equation**, connecting the microstrains and the temperature increment:

$$\lambda_{ij} \frac{\partial^2 \theta}{\partial x_i \, \partial x_j} + T_o \sum_{\alpha = 1}^{n} J_\alpha \frac{\partial \varepsilon_a}{\partial t} - c_\varepsilon \frac{\partial \theta}{\partial t} + \mathrm{H} = 0 \ , \tag{20}$$

where $\lambda_{ij} = \lambda_{ji}$ are the components of the symmetric tensor of thermal conductivity, c_ε is the specific heat at constant microstrains ε_α, H is the density of internal heat sources (if any) in the granular medium.

Equation *(17)* and the term $T_o \sum_{\alpha = 1}^{n} J_\alpha \frac{\partial \varepsilon_a}{\partial t}$ in Eq. *(20)* relate the temperature field $\theta(x_i, t)$ to the fields of elongation microstrains $\varepsilon_\alpha(x_i, t)$ and microstresses $p_\alpha(x_i, t)$. Thus, the above governing equations represent a coupled systems of thermomechanical equation for granular media. The mechanical fields may be determined separately from the temperature field in the special cases of (i) isothermal and (ii) adiabatic processes when the heat equation *(20)* is replaced respectively by the equations (Granik and Ferrari, 1993a)

$$\text{(i)} \quad \theta = 0 \ , \tag{21}$$

$$\text{(ii)} \quad \theta = \frac{T_o}{c_\varepsilon} \sum_{\alpha = 1}^{n} J_\alpha \, \varepsilon_\alpha \ . \tag{22}$$

In these cases, the constitutive equations *(17)* reduce to the forms

$$\text{(i)} \quad p_\alpha = \sum_{\beta = 1}^{n} A_{\alpha\beta}\, \varepsilon_\beta , \qquad (23)$$

$$\text{(ii)} \quad p_\alpha = \sum_{\beta = 1}^{n} \acute{A}_{\alpha\beta}\, \varepsilon_\beta , \qquad (24)$$

where

$$\acute{A}_{\alpha\beta} = A_{\alpha\beta}\, \varepsilon_\beta + \frac{T_o}{c_\varepsilon} J_\alpha J_\beta . \qquad (25)$$

DYNAMIC SCALING EQUATIONS

Conventional studies of elastic wave propagation in particulate solids have neglected particle size and scaling effects. In well-developed branches of solid mechanics, such as soil dynamics (Prakash, 1981), seismology (Båth, 1968) and geophysics (Pilant, 1979), despite the macroscopic evidence of the particulate nature of the media under study, this approach has been justified by treating only dynamic phenomena with wavelenghts that are much larger than the particles' dimensions.

The same reluctance to address scaling effects is also pervasive in the literature on granular media, from the earliest theories (Takahashi and Sato, 1949, 1950; Gassmann, 1951) to contemporary models (Stout, 1989; Wijesinghe, 1989; Agarwal, 1992; Ostoja-Starzewski, 1992; Slade and Walton, 1993).

In view of its multi-scale nature, and its capability to bridge the discrete and the continuum viewpoints, Doublet Mechanics offers a natural framework for the discussion of scaling effects in the dynamics of particulate and granular media. The most general embodiment of the theory was presented in the previous section. A simplified version of the theory is now derived, with the purpose of studying mono- and bi-dimensional propagation phenomena. The assumptions on which the simplified theory is based are:

1. The dynamic process is *isothermal*, and Eqs. (*21*), (*23*) hold.

2. The volume forces vanish:

$$F_i = 0 \ . \tag{26}$$

3. The particle interactions are *longitudinal* (*central*), so that the shear and torsion microstresses vanish everywhere in the body:

$$m_\alpha = t_{\alpha i} \equiv 0 \ . \tag{27}$$

Identity (*27*) is equivalent to taking the micromoduli of elasticity in Eqs. (*18*) and (*19*) equal to zero: $E_{\alpha\beta} = I_{\alpha\beta ij} \equiv 0$. As shown in the work (Granik and Ferrari, 1993a), the granular medium with such properties is nonpolar: it bears only conventional (macro)stresses σ_{ij} and does not sustain couple (macro)stresses M_{ij} which are identically equal to zero in the volume V.

4. The central interactions are *local*, i.e., the elongation microstress p_α in an arbitrary doublet (A,B_α) depends only on its elongation microstrain ε_α and is independent of microstrains ε_β ($\beta \neq \alpha$) in the other doublets (A,B_β) originated from the same particle A. Such an interaction arises, for instance, if two particles of any doublet are supposed to be rigid and bonded by a small elastic spring. This assumption of local interaction formally means that in the isothermal physical equation (*23*), the micromoduli of elasticity $A_{\alpha\beta} = A_\alpha \, \delta_{\alpha\beta}$ which, in turn, reduces the physical equations (*17*) to

$$p_\alpha = A_\alpha \, \varepsilon_\alpha \ . \tag{28}$$

5. The local interactions are *homogeneous*, i.e., all the micromoduli of elasticity $A_\alpha = A_o = constant$ for any $\alpha = 1, 2, ..., n$. In view of Eqs. (*5*) and (*12*), this assumption leads Eq. (*28*) to the form

$$p_{\alpha i} = A_o \, \varepsilon_{\alpha i} \,, \tag{29}$$

where A_o is microconstant of elasticity. Within this set of assumptions, it is noted that only this microconstant suffices to fully define the physical properties of the granular medium in question.

Assumption (*27*) turns the dynamic 'couple' equations (*11*) into identities and reduces the set of governing relations to the dynamic 'force' equations (*10*), kinematic relations (*1*), and physical dependencies (*29*). Substituting relations (*27*), (*29*), (*5*), and (*1*) into Eq. (*10*) and taking into account condition (*26*), the scaling dynamic equations are obtained in terms of the translations $u_i(x_j,t)$:

$$- A_o \sum_{\alpha = 1}^{n} \sum_{\kappa = 1}^{M} (-1)^{\kappa} \frac{\eta_{\alpha}^{\kappa-1}}{\kappa!} \, T_{\alpha(\kappa)} \, \tau^{o}_{\alpha i} \, \tau^{o}_{\alpha j} \sum_{\mu = 1}^{M} \frac{\eta_{\alpha}^{\mu-1}}{\mu!} \, T_{\alpha(\mu)} \, \partial^{\kappa+\mu} u_j = \rho \, \ddot{u}_i \,, \tag{30}$$

where the indices i, j = *1*, *2*, *3*. To simplify these equations Eq. (*30*) is first rewritten as:

$$- A_o \sum_{\alpha = 1}^{n} \tau^{o}_{\alpha i} \, \tau^{o}_{\alpha j} \sum_{\kappa = 1}^{M} \sum_{\mu = 1}^{M} (-1)^{\kappa} \frac{\eta_{\alpha}^{\kappa+\mu-2}}{\kappa!\mu!} \, T_{\alpha(\kappa+\mu)} \, \partial^{\kappa+\mu} u_j = \rho \, \ddot{u}_i \,. \tag{31}$$

Let us now take the sum $\kappa + \mu = \delta$. Since $1 \le \kappa \le M$ and $1 \le \mu \le M$, we have $2 \le \delta \le 2M$. For example, if $M = 1$ then $\delta = 2$; if $M = 2$ then $\delta = 2, 3, 4$; and so on. Denoting $R \equiv 2M$, we fulfill identical transformations of the internal double sum in Eq. (*31*):

$$\sum_{\kappa=1}^{M}\sum_{\mu=1}^{M}(-1)^{\kappa}\frac{\eta_{\alpha}^{\kappa+\mu-2}}{\kappa!\mu!}T_{\alpha(\kappa+\mu)}\,\partial^{\kappa+\mu}u_j \equiv \sum_{\delta=2}^{R}\sum_{\kappa=1}^{\delta-1}(-1)^{\kappa}\frac{\eta_{\alpha}^{\delta-2}}{\kappa!\,(\delta-\kappa)!}T_{\alpha(\delta)}\,\partial^{\delta}u_j \equiv$$

$$\sum_{\delta=2}^{R}\eta_{\alpha}^{\delta-2}\,T_{\alpha(\delta)}\,\partial^{\delta}u_j\sum_{\kappa=1}^{\delta-1}\frac{(-1)^{\kappa}}{\kappa!(\delta-\kappa)!}\ . \tag{32}$$

The last sum in (*32*) may be computed in terms of Newton's binomial formula

$$(a+b)^{\delta}=\sum_{\kappa=0}^{\delta}\frac{\delta!}{\kappa!(\delta-\kappa)!}\,a^{\kappa}\,b^{\delta-\kappa} \tag{33}$$

which at $a = -1$, $b = 1$ gives

$$\sum_{\kappa=0}^{\delta}(-1)^{\kappa}\frac{\delta!}{\kappa!(\delta-\kappa)!}\equiv 0\ . \tag{34}$$

Identity (*34*) may be rewritten as follows:

$$\sum_{\kappa=0}^{\delta}(-1)^{\kappa}\frac{\delta!}{\kappa!(\delta-\kappa)!}\equiv \delta!\sum_{\kappa=0}^{\delta}\frac{(-1)^{\kappa}}{\kappa!(\delta-\kappa)!}\equiv \delta!\sum_{\kappa=1}^{\delta-1}\frac{(-1)^{\kappa}}{\kappa!(\delta-\kappa)!}+S\equiv 0\ , \tag{35}$$

where $S = 2$ if δ is even, and $S = 0$ if δ is odd. Relation (*35*) yields

$$\sum_{\kappa=1}^{\delta-1}\frac{(-1)^{\kappa}}{\kappa!(\delta-\kappa)!}\equiv -\frac{S}{\delta!}\ . \tag{36}$$

On substituting identity (*36*) into (*32*) and then into (*31*), we finally obtain the following three *basic equations of scaling microdynamics*:

$$\rho\ddot{u}_i = 2\,A_o\sum_{\alpha=1}^{n}\tau_{\alpha i}^{o}\,\tau_{\alpha j}^{o}\sum_{\delta=2,\,4,\ldots}^{R}\frac{\eta_{\alpha}^{\delta-2}}{\delta!}\,T_{\alpha(\delta)}\,\partial^{\delta}u_j\ . \tag{37}$$

These equations include the scaling parameters η_α *in an explicit form*. In the first (or *nonscale*) approximation, i.e. for $M = 1$, the scaling parameters η_α vanish, and equations (*37*) reduce to:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = A_o \sum_{\alpha = 1}^{n} \tau^o_{\alpha i}\, \tau^o_{\alpha j}\, \tau^o_{\alpha k}\, \tau^o_{\alpha l} \frac{\partial^2 u_j}{\partial x_k\, \partial x_l}, \qquad (38)$$

where the definitions (*8*) and (*9*) were employed.

For clarity, the multi-scale Eq. (*37*) is written *in extenso* as:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = 2\, A_o \sum_{\alpha = 1}^{n} \tau^o_{\alpha i}\, \tau^o_{\alpha j} \Big(\frac{1}{2!} \tau^o_{\alpha k}\, \tau^o_{\alpha l} \frac{\partial^2 u_j}{\partial x_k\, \partial x_l} + \frac{\eta_\alpha^2}{4!} \tau^o_{\alpha k}\, \tau^o_{\alpha l}\, \tau^o_{\alpha p}\, \tau^o_{\alpha q} \frac{\partial^4 u_j}{\partial x_k\, \partial x_l\, \partial x_p\, \partial x_q} + \frac{\eta_\alpha^4}{6!} \tau^o_{\alpha k}\, \tau^o_{\alpha l}\, \tau^o_{\alpha p}\, \tau^o_{\alpha q}\, \tau^o_{\alpha r}\, \tau^o_{\alpha s} \frac{\partial^6 u_j}{\partial x_k\, \partial x_l\, \partial x_p\, \partial x_q\, \partial x_r\, \partial x_s} + \ldots \Big), \qquad (39)$$

or, in an explicit tensor notation,

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{\kappa = 2,\, 4,\, \ldots}^{R = 2M} C_{ijk_1 \ldots k_\kappa} \frac{\partial^\kappa u_j}{\partial x_{k_1} \cdots \partial x_{k_\kappa}}, \qquad (40)$$

where

$$C_{ijk_1 \ldots k_\kappa} \equiv 2\, A_o \sum_{\alpha = 1}^{n} \frac{\eta_\alpha^{\kappa - 2}}{\kappa!} \tau^o_{\alpha i}\, \tau^o_{\alpha j}\, \tau^o_{\alpha k_1} \cdots \tau^o_{\alpha k_\kappa} \qquad (41)$$

are tensors of rank $\kappa + 2$ that correspond to generalized moduli of elasticity. Formula (*41*) shows that these moduli comprise micro-level constitutive information (A_o), as well as microstructural parameters (η_α , $\tau^o_{\alpha i}$). In the first approximation, the scaling dynamic equations (*40*) and (*41*) are form-identical with the non-scaling equations of the conventional theory of anisotropic elasticity (Pilant, 1979)

$$\rho \frac{\partial^2 u_i}{\partial t^2} = C_{ijkl} \frac{\partial^2 u_j}{\partial x_k \, \partial x_l}. \tag{42}$$

Within the Doublet Mechanical context, however, the stiffness tensor is:

$$C_{ijkl} \equiv A_o \sum_{\alpha = 1}^{n} \tau^o_{\alpha i} \, \tau^o_{\alpha j} \, \tau^o_{\alpha k} \, \tau^o_{\alpha l} \, . \tag{43}$$

Eqs. (*42*), (*43*) are equivalent to Eq. (*38*). There is a major difference between the conventional understanding of (42) and its Doublet-Mechanical counterpart, where (43) holds: The tensor C_{ijkl} in (*43*) is invariant with respect to *any* permutation of its subscripts, including $C_{ijkl} = C_{ikjl}$ and thus possesses only *15* independent constants: C_{1231} , C_{1232} , C_{1233} , C_{1211} , C_{1212} , C_{1222} , C_{1311} , C_{1313} , C_{1333} , C_{2322} , C_{2323} , C_{2333} , C_{1111} , C_{2222} , C_{3333}. By contrast, the number of independent constants in conventional theory of anisotropic elasticity is *21*.

The difference of *6* constants for this case brings us back to the dawn of theory of elasticity when Stokes first raised the question: "Is elastic aelotropy (anisotropy) to be characterized by *21* constants or by *15*, and is elastic isotropy to be characterized by two constants or one?" (Love, 1944, page 13). Stokes and G. Green held *21* independent constants of anisotropy (the *multi-constant* theory) while Cauchy, Navier, Poisson and Saint Venant supposed this number to be *15* (the *rari-constant* theory). Based on equation (*43*), Stokes' question may answered as follows:

In order for an arbitrary anisotropic elastic granular medium to be completely characterized by *15* independent constants it is *sufficient* that the microforces of interaction between constitutive particles (granules, molecules, etc.) be *central*, *local*, and *homogeneous* in the sense specified above. It can easily be shown that the first two conditions are also *necessary*. The third condition is not necessary and was adopted here because it has eventually lead to the simplest form of *scaling* equations (*39*)-(*41*). In the case of *isotropy*, the elastic features are characterized by *one* macroconstant of elasticity (see below Eq. (*92*)). Using the above governing equations, it can also be shown that the tensor C_{ijkl} has *21* independent components if the interparticle microforces are only *central*. The basic equations of scaling microdynamics become then much more complicated than (*39*)-(*41*). The proofs of the above statements is omitted for brevity. The reader interested in the historical discussion of the multiconstancy versus rariconstancy is referred to (Todhunter, 1886, articles 921-934).

Equation (41) may be interpreted in two, quite different manners: In the first, the microstructural parameters A_o , η_α , $\tau^o_{\alpha i}$, are assumed known, and formula (*41*) enables one to compute the microstructural tensors $C_{ijk_1 \ldots k_\kappa}$ and then apply the dynamic equations in the explicit tensor notation (*40*). However, the computing the tensors $C_{ijk_1 \ldots k_\kappa}$ is not a necessary step, and may be avoided by inserting the known microstructural parameters directly into the dynamic equations (*39*).

Alternatively, if the microstructure of the granular body is unknown, the elastic moduli $C_{ijk_1 \ldots k_\kappa}$ must be considered as some macroscopic parameters of granular body to be determined by special macroscopic experiments. Nevertheless, based on the underlying theory, the experimental variables are only A_o , η_α , $\tau^o_{\alpha i}$.

Thus, using the doublet microstructural approach, we have first obtained the dynamic *scaling* equations (*39*)-(*41*). These equations not only enable us to study dynamic scaling problems for granular and particulate materials but also allow us to do this, if necessary, from two quite different viewpoints: microscopic (Eq. (*39*)) and macroscopic (Eqs. (*40*), (*41*)).

ELASTIC WAVES IN A LINEAR MONATOMIC LATTICE

Consider a linear monatomic lattice with the valence $n = 1$ and internodal distance $\eta_\alpha \equiv \eta = constant$ (Fig. 1). The only lattice direction corresponds to $\alpha = 1$ is taken to be parallel to the Cartesian axis $x_1 \equiv x$ so that the lattice axis $\bar{\tau}^o_\alpha = \bar{e}_1$. Since $\bar{\tau}^o_\alpha = \tau^o_{\alpha i}\,\bar{e}_i$, it follows

$$\tau^o_{11} = 1 \quad \text{and} \quad \tau^o_{\alpha i} = 0 \quad \text{if} \quad \alpha \neq 1 \text{ or } i \neq 1 \,. \tag{44}$$

Relations (*41*) and (*44*) cause the tensor $C_{ijk_1 \dots k_\kappa}$ to be

$$C_{ijk_1 \dots k_\kappa} = 2\,A_o\,\frac{\eta^{\kappa-2}}{\kappa!} \text{ if } i = j = k_1 = \dots = k_\kappa = 1 \text{ and } C_{ijk_1 \dots k_\kappa} = 0 \text{ otherwise.} \tag{45}$$

In view of (*45*), the dynamic equation (*40*) becomes

$$\rho\,\frac{\partial^2 u}{\partial t^2} = 2\,A_o \sum_{\kappa = 2,\,4,\,\dots}^{R\,=\,2M} \frac{\eta^{\kappa-2}}{\kappa!}\,\frac{\partial^\kappa u}{\partial x^\kappa} = \frac{2\,A_o}{\eta^2} \sum_{\kappa = 2,\,4,\,\dots}^{R\,=\,2M} \frac{\eta^\kappa}{\kappa!}\,\frac{\partial^\kappa u}{\partial x^\kappa}\,, \tag{46}$$

in which $u = u_1$ is a node translation along the axis x.

It is of interest now to demonstrate that the *differential* equation (46) actually contains the *difference* equation employed in solid state physics for wave propagation in a monoatomic lattice. To this end, the limit of the right-end side of (46) is taken for M → ∞, yielding

$$\sum_{\kappa = 2, 4,\ldots}^{\infty} \frac{\eta_\alpha^\kappa}{\kappa!} \frac{\partial^\kappa u}{\partial x^\kappa} \equiv (u_{p+1} + u_{p-1} - 2\, u_p)/2 \,, \tag{47}$$

where $u_p \equiv u \equiv u(x,t)$, $u_{p-1} \equiv u(x-\eta,t)$, $u_{p+1} \equiv u(x+\eta,t)$ denote the translations of an arbitrary *p*th node and its nearest neighbors to the left and the right, respectively (see Fig. 1). To prove (*47*) let the displacements u_{p-1} and u_{p+1} be expanded into Taylor's series about *x*:

$$u_{p-1} \equiv u(x-\eta,t) \equiv u + \sum_{\kappa = 1}^{\infty} (-1)^\kappa \frac{\eta_\alpha^\kappa}{\kappa!} \frac{\partial^\kappa u}{\partial x^\kappa} \,, \tag{48}$$

$$u_{p+1} \equiv u(x+\eta,t) \equiv u + \sum_{\kappa = 1}^{\infty} \frac{\eta_\alpha^\kappa}{\kappa!} \frac{\partial^\kappa u}{\partial x^\kappa} \,. \tag{49}$$

The sum of series (*48*) and (*49*) is

$$u_{p+1} + u_{p-1} \equiv 2\, (u + \sum_{\kappa = 2, 4,\ldots}^{\infty} \frac{\eta_\alpha^\kappa}{\kappa!} \frac{\partial^\kappa u}{\partial x^\kappa}) \,. \tag{50}$$

Since $u \equiv u_p$, relation (*50*) directly entails identity (*47*), as was to be shown. Substituting (*50*) into the differential equation (*46*) at $M \rightarrow \infty$, the difference equation of motion of an arbitrary *p*th node, or atom is obtained:

$$m \frac{\partial^2 u_p}{\partial t^2} = C\,(u_{p+1} + u_{p-1} - 2\,u_p)\,, \tag{51}$$

where $m = \rho\eta^3$ may be interpreted as the mass of an atom in the monatomic lattice, and $C = A_o\eta$ is the force constant. Eq. (*51*) may be found in textbooks on solid state physics (Kittel, 1953, Eq. (4.6)) and in the dynamics of atoms in crystals (Cochran, 1973, Eq. (3.1)).

It was thus shown that for a linear monatomic lattice, the general scaling differential equations (*39*)-(*41*) may be reduced to the particular form of scaling difference equation (*51*). Clearly, the transition from (*51*) to (*39-41*) is impossible.

Consider now a longitudinal wave

$$u_p = u_o\, exp[i(\omega t - kp\eta)] \tag{52}$$

traveling along the axis x to the right. Here, $i = \sqrt{(-1)}$, while u_o , ω , k are the wave amplitude, (angular) frequency, and wavenumber, respectively; the quantity $p\eta$ is a discrete analog of the continuous variable x. On substituting the function u_p from (52) into the dynamic equation (*51*) we find u_p to be a solution of (*51*) provided that

$$-\,m\omega^2 = C\,[exp(ik\eta) + exp(-ik\eta) - 2]\,. \tag{53}$$

Since $exp(ik\eta) \equiv cos(k\eta) + i\, sin(k\eta)$, Eq. (*53*) becomes

$$\omega = 2\,(C/m)^{1/2}\, sin(k\eta/2) \equiv \frac{2}{\eta}\,(A_o/\rho)^{1/2}\, sin(k\eta/2)\,. \tag{54}$$

In view of the identity $k = 2\pi/\lambda$, λ being the wavelength, formula (*54*) yields the expression for the phase velocity V_p :

$$V_p \equiv \frac{\omega}{k} = V_0 \; [sin(\pi\eta/\lambda)]/(\pi\eta/\lambda) \quad , \tag{55}$$

where $V_0 \equiv \surd(A_0/\rho)$ is the phase velocity for waves with infinite wavelength. The case of infinitely long waves may be alternatively obtained by restricting the general dynamic equation (*46*) to the first, non-scaling approximation ($M = 1$, $R = 2$):

$$\rho \frac{\partial^2 u}{\partial t^2} = A_0 \frac{\partial^2 u}{\partial x^2} \tag{56}$$

and considering the continuous analog to the longitudinal wave (*52*)

$$u = u_0 \; exp[i(\omega t - kx)] \; . \tag{57}$$

Inserting then (*57*) into (*56*), we obtain the frequency $\omega = k \surd(A_0/\rho)$ and the phase velocity $V_p \equiv \omega/k = \surd(A_0/\rho) \equiv V_0$. The quantity V_0 is known to be the velocity of sound in a classical - non-scaling - continuum (Cochran, 1973).

Formula (*54*) shows that the frequency ω rises to a maximum of $\frac{2}{\eta} (A_0/\rho)^{1/2}$ at $k = \pi/\eta$ and falls to zero at $k = 2\pi/\eta$. In fact, the function $\omega(k)$ in (*54*) is periodic with periodicity $k = 2\pi/\eta$. Therefore with no loss of generality the wavenumber k may be restricted to the range $\pm \pi/\eta$: if $0 < k \leq \pi/\eta$, the wave travels to the right, and if $-\pi/\eta \leq k < 0$, the wave travels to the left. Since $|k| \leq \pi/\eta$ and $\lambda \equiv 2\pi/k$, it follows that

$$\lambda \geq 2\eta \; , \tag{58}$$

i.e., in the monatomic structure, the *only* wave modes that may propagate are those with wavelengths λ no smaller than twice the internodal distance (or twice the doublet length) $2\ \eta$. Restriction (*58*) holds for all particulate materials.

According to (*54*), the phase velocity V_p depends on the wavelength λ. This means that longitudinal waves in the monatomic lattice are dispersive: the longer the wave, the faster it propagates. The velocity V_p reaches a maximum $V_{pmax} = V_o \equiv \sqrt{(A_o/\rho)}$ at a maximum wavelength $\lambda_{max} \rightarrow \infty$, and a minimum $V_{pmin} = 2\ V_o/\pi = 0.63662\ V_o$ at a minimum wavelength $\lambda_{min} = 2\ \eta$. This scaling effect may be called the *wave retardation* at short wavelengths.

It is noted that the retardation effect may only be described if the proper scaling framework is employed: Longitudinal waves in a monoatomic lattice, modeled by the conventional - non-scaling - wave equation (*56*) are found to be non dispersive. A similar conclusion will be shown to hold for plane elastic waves in the next section.

PLANE ELASTIC WAVES IN GRANULAR MEDIA

T h e o r y

Consider a plane wave

$$u_j = u_{jo}\ exp[i(\omega t - kx)]\ , \qquad (59)$$

traveling in an unbounded granular medium along the axis $x \equiv x_1$. The symbols i, ω, k are the same as in Eq. (*57*). It follows from expression (*59*) that

$$\rho \frac{\partial^2 u_j}{\partial t^2} = - \omega^2 u_{jo} \, exp[i(\omega t - kx)] \ , \qquad (60)$$

$$\frac{\partial^\kappa u_j}{\partial x_1^\kappa} \equiv \frac{\partial^\kappa u_j}{\partial x^\kappa} = (-1)^{\kappa/2} \, k^\kappa \, u_{jo} \, exp[i(\omega t - kx)] \ , \qquad (61)$$

$$\frac{\partial^\kappa u_j}{\partial x_2^\kappa} = \frac{\partial^\kappa u_j}{\partial x_3^\kappa} \equiv 0 \ , \qquad (62)$$

where $\kappa = 2, 4, 6$, etc. Substituting (60)-(62) into Eq. (40) and using(41), the following set of homogeneous algebraic equations obtains:

$$u_{jo} \, \omega^2 + u_{lo} \, V_o^2 \, F_{jl} = 0 \ , \qquad (63)$$

in which

$$F_{jl} \equiv F_{lj} = 2 \sum_{\kappa = 2, 4, \ldots}^{R = 2M} (-1)^{\kappa/2} \frac{k^\kappa}{\kappa!} \sum_{\alpha = 1}^{n} \eta_\alpha^{\kappa-2} \, \tau_{\alpha j}^o \, \tau_{\alpha l}^o \, (\tau_{\alpha 1}^o)^\kappa \ . \qquad (64)$$

We assume that the $x_3 \equiv z$-component of the translation $\bar{u} = u_j \bar{e}_j$ vanishes in all the medium in question, so that $u_3 \equiv 0$. With this, the medium is considered to be in a *plane-strain* state; therefore the indices $j, l = 1, 2$ everywhere in this section. The same simplification would obviously result from considering a trubi-dimensional medium, with translational symmetry of the hexagonal type. Under either assumption, the above set (63) reduces to a system of two homogeneous equations, which admits a nonzero solution if and only if its determinant is equal to zero (Sokolnikoff I. and E., 1941):

$$\begin{vmatrix} \omega^2 + V_o^2 F_{11}, V_o^2 F_{12} \\ V_o^2 F_{21}, \omega^2 + V_o^2 F_{22} \end{vmatrix} = 0 \qquad (65)$$

where $V_o \equiv \surd(A_o/\rho)$. The parameters F_{jl} are functions of the lattice directions $\tau^o_{\alpha j}$ and doublet lengths η_α and thus depend on the microstructure of granular medium. Therefore the solution of Eq. (65) also depends on the microstructure. We are going to get a numerical solution of Eq. (65) which demands to focus attention on a particular microstructure. So we will further consider the cubical-tetrahedral packing H_4 which in the plane $(x_1, x_2) \equiv (x, y)$ resembles a honeycomb pattern (Fig. 2).

The packing H_4 has in general the valence $n = 4$, i.e., four spatial directions that are determined by the following four unit vectors (see Fig. 2):

$$\bar{\tau}^o_1 = \bar{e}_1 , \quad \bar{\tau}^o_2 = \bar{e}_1 \cos\phi + \bar{e}_2 \sin\phi , \quad \bar{\tau}^o_3 = - \bar{e}_1 \cos\phi + \bar{e}_2 \sin\phi , \quad \bar{\tau}^o_4 = \bar{e}_3, \qquad (66)$$

with the structural angle $\phi = 60^0$. Since the translation $u_3 \equiv 0$ and we only consider the wave displacements u_1 and u_2 in the plane (x_1, x_2), the fourth microstructure direction $\bar{\tau}^o_4 = \bar{e}_3$, which is parallel to the $x_3 \equiv z$-axis and perpendicular to the plane (x_1, x_2), becomes insignificant and is further neglected. According to (66), the direction cosine matrix is

$$[\,\tau^o_{\alpha j}\,] = \begin{bmatrix} \tau^o_{11}, \tau^o_{12} \\ \tau^o_{21}, \tau^o_{22} \\ \tau^o_{31}, \tau^o_{32} \end{bmatrix} = \begin{bmatrix} 1,0 \\ 1/2, \surd 3/2 \\ -1/2, \surd 3/2 \end{bmatrix} \tag{67}$$

Eq. (*64*) takes the form

$$F_{jl} \equiv F_{lj} = \sum_{\kappa = 2,\, 4,\, \ldots}^{R\, =\, 2M} (-1)^{\kappa/2} \frac{2}{\eta^2} \frac{(k\eta)^\kappa}{\kappa!} a_{jl} , \tag{68}$$

where for $\alpha = 1, 2, 3$: $\eta_\alpha = \eta = constant$, and

$$a_{jl} \equiv a_{lj} = \sum_{\alpha = 1}^{n} \tau^o_{\alpha j} \tau^o_{\alpha l} (\tau^o_{\alpha 1})^\kappa . \tag{69}$$

According to (*67*) and (*69*), the parameters a_{jl} are determined by the formulas

$$a_{11} = 1 + 1/2^{\kappa+1} , \quad a_{22} = 3/2^{\kappa+1} , \quad a_{12} = a_{21} = 0 . \tag{70}$$

Substituting these expressions of a_{jl} into (*68*), we obtain

$$F_{11} = 2\, k^2 \sum_{\kappa = 2,\, 4,\, \ldots}^{R\, =\, 2M} (-1)^{\kappa/2} \frac{(k\eta)^{\kappa-2}}{\kappa!} [1 + 2^{-(\kappa+1)}] , \tag{71}$$

$$F_{22} = 6\, k^2 \sum_{\kappa = 2,\, 4,\, \ldots}^{R\, =\, 2M} (-1)^{\kappa/2} \frac{(k\eta)^{\kappa-2}}{\kappa!} 2^{-(\kappa+1)} , \tag{72}$$

The parameters $F_{12} \equiv F_{21} = 0$ and thus Eq. (65) splits up into two separate dispersion equations:

$$\omega^2 + V_o^2 F_{11} = 0 , \tag{73}$$

$$\omega^2 + V_o^2 F_{22} = 0 . \tag{74}$$

Eq. (73) involves the parameter F_{11} and according to (63) and (59) concerns the displacement u_1 that is parallel to the x_1-axis; Eq. (74) includes the parameter F_{22} and according to (63) and (59) concerns the displacement u_2 that is perpendicular to the x_1-axis. Since we consider the plane wave (59) traveling *along* the axis x_1 , Eq. (73) relates to *longitudinal* elastic displacements, or *P-waves*, and Eq. (74) refers to *transverse* elastic displacements, or *S-waves*. The fact that the P-waves and S-waves are described by separate equations means that these waves are not interconnected and propagate quite independently of each other: The characteristics of P-waves (amplitudes, frequencies, velocities) are independent of the same characteristics of S-waves - exactly as in the classical continuum isotropic elastic case (Kolsky, 1963).

The phase velocities V_{pp} and V_{ps} of the P- and S-waves, respectively, are found via (71-74) to be:

$$V_{pp} \equiv \frac{\omega}{k} = \frac{1}{2} V_o \left[\sum_{\kappa = 2, 4, \ldots}^{R = 2M} (-1)^{1+\kappa/2} (1 + 2^{\kappa+1}) \frac{\pi^{\kappa-2}}{\kappa!} \left(\frac{\eta}{\lambda} \right)^{\kappa-2} \right]^{1/2} , \tag{75}$$

$$V_{ps} \equiv \frac{\omega}{k} = \frac{\sqrt{3}}{2} V_o \left[\sum_{\kappa = 2, 4, \ldots}^{R = 2M} (-1)^{1+\kappa/2} \frac{\pi^{\kappa-2}}{\kappa!} \left(\frac{\eta}{\lambda} \right)^{\kappa-2} \right]^{1/2} \tag{76}$$

where the identity $k = 2\pi/\lambda$ has been taken into account, λ being the wavelength.

The group velocities V_g are related to the phase velocities V_p as (Brillouin, 1960):

$$V_g \equiv \frac{\partial \omega}{\partial k} = V_p - \lambda \frac{\partial V_p}{\partial \lambda} . \qquad (77)$$

By substituting the phase velocities V_{pp} and V_{ps} from (75-76) into (77) we obtain the group velocities V_{gp} and V_{gs} of the P- and S-waves, respectively:

$$V_{gp} = V_{pp} - \frac{1}{8 V_{pp}} \sum_{\kappa = 2, 4, \ldots}^{R = 2M} (-1)^{\kappa/2} (1 + 2^{\kappa+1}) \frac{\pi^{\kappa-2}}{\kappa!} \left(\frac{\eta}{\lambda}\right)^{\kappa-2} (\kappa - 2) , \qquad (78)$$

$$V_{gs} = V_{ps} - \frac{3}{8 V_{ps}} \sum_{\kappa = 2, 4, \ldots}^{R = 2M} (-1)^{\kappa/2} \frac{\pi^{\kappa-2}}{\kappa!} \left(\frac{\eta}{\lambda}\right)^{\kappa-2} (\kappa - 2) . \qquad (79)$$

The formulae (75-79) express the wave velocities to an arbitrary Mth approximation for M from 1 to ∞. In the first approximation, when $M = 1$, the above formulae yield the following expressions for the wave velocities:

$$V_{pp}^{(1)} = V_{gp}^{(1)} = V_{op} \equiv \frac{3}{2\sqrt{2}} V_o , \qquad (80)$$

$$V_{ps}^{(1)} = V_{gs}^{(1)} = V_{os} \equiv \frac{\sqrt{3}}{2\sqrt{2}} V_o , \qquad (81)$$

which are independent of the nondimensional scaling parameter η/λ. Alternatively, these expressions may be arrived at via two different assumptions:

(A) the constituent granules of the solid are *material points* (whose sizes are *infinitesimal*: $\eta = 0$) and all the waves may have arbitrary but *finite* lengths λ , or

(B) the constituent particles of the solid may have an arbitrary but *finite* size $\eta \neq 0$ and all the waves have an *infinite* length $\lambda \to \infty$.

The assumption (A) is adopted in classical theory of elasticity, while the restriction (B) is employed in the special discrete-continuum theories of wave propagation in granular media (Takahashi and Sato, 1949, 1950; Gassmann, 1951; etc.) and crystals (Cochran, 1973). It is clear that neither of these approaches permits the analysis of scaling effects.

The question of convergence of the series appearing in the wave velocity expressions (*75-79*) is considered next. Consider Eq. (*75*) first, in the limit as $M \to \infty$. Squaring both sides of the relation, an infinite alternating series is obtained on the right-hand side of (*75*). According to Leibnitz's theorem (Sokolnikoff I. and E., 1941), such series is convergent if its terms are such that

$$a_{\kappa+2} < a_{\kappa} , \tag{82}$$

$$\lim_{\kappa \to \infty} a_{\kappa} = 0 . \tag{83}$$

In (*82*), it has been taken into account that the term $a_{\kappa+2}$ follows a_{κ} because a subscript κ only runs through even values *2*, *4*, ... Relation (*75*) shows that

$$a_{\kappa} = (1 + 2^{\kappa+1}) \frac{\pi^{\kappa-2}}{\kappa!} \left(\frac{\eta}{\lambda}\right)^{\kappa-2} , \tag{84}$$

$$a_{\kappa+2} = (1 + 2^{\kappa+3}) \frac{\pi^{\kappa}}{(\kappa+2)!} \left(\frac{\eta}{\lambda}\right)^{\kappa} . \tag{85}$$

Substituting a_κ and $a_{\kappa+2}$ from (84) and (85) into (82), we transform it to the inequality

$$\pi^2 \left(\frac{\eta}{\lambda}\right)^2 < \frac{1+2^{\kappa+1}}{1+2^{\kappa+3}} (\kappa+1)(\kappa+2) . \tag{86}$$

According to (58), $max\ (\eta/\lambda) = 1/2$, and therefore the left-hand side of (86) has a *maximum* $[\pi^2 (\frac{\eta}{\lambda})^2] = \pi^2/4 = 2.467401$. On the other hand, under $\kappa = 2$, the right-hand side of (86) has a *minimum* $[\frac{1+2^{\kappa+1}}{1+2^{\kappa+3}} (\kappa+1)(\kappa+2)] = 3.272727 > 2.467401$. Thus inequality (86) and, consequently, condition (82) are satisfied.

We now turn to (85) and use Stirling's formula (Sokolnikoff I. and E., 1941)

$$\kappa! \simeq \sqrt{(2\pi\kappa)} \left(\frac{\kappa}{e}\right)^\kappa . \tag{87}$$

Substituting $\kappa!$ from (87) into (83) and taking account of the obvious inequalities $1 + 2^{\kappa+1} < 2^{\kappa+2}$, $\sqrt{(2\pi\kappa)} > 1$, $\eta/\lambda < 1$, we obtain

$$a_\kappa = \frac{(1+2^{\kappa+1})}{\sqrt{(2\pi\kappa)}} \frac{e^\kappa}{\kappa^\kappa} \frac{\pi^\kappa}{\pi^2} \left(\frac{\eta}{\lambda}\right)^{\kappa-2} < \frac{4}{\pi^2} \left(\frac{2\pi e}{\kappa}\right)^\kappa < \left(\frac{2\pi e}{\kappa}\right)^\kappa . \tag{88}$$

Since $\lim_{\kappa \to \infty} \left(\frac{2\pi e}{\kappa}\right)^\kappa = 0$, the term a_κ , by virtue of (88), obeys the condition (83). Thus the infinite series in (75) is convergent.

The convergence of the other three series in relations (76,78,79), with $M \to \infty$, is proven analogously. Because of convergence, the sums of these infinite alternating series may be determined by truncating them and computing the remainder, which is always less than the first truncated term a_{R+2} (Sokolnikoff I. and E., 1941).

Let us, for instance, calculate the remainder Δ for the infinite series in (*75*) by using formula (*85*) and replacing κ by R:

$$\Delta \le a_{R+2} \equiv (1 + 2^{R+3}) \frac{\pi^R}{(R+2)!} \left(\frac{\eta}{\lambda}\right)^R . \qquad (89)$$

Relation (*89*) shows that the term a_{R+2} attains a maximum when η/λ reaches a maximum which, by virtue of (*58*), is equal to $1/2$: then $max\ a_{R+2} = 2.497\ 10^{-20}$ at $R = 30$ and $M = 15$. In this case, the remainder Δ also attains a maximum that is no larger than $2.497\ 10^{-20}$, and hence $max\ \sqrt{\Delta} \le 1.580\ 10^{-10} < 10^{-9}$. Thus, if we truncate the infinite series in (*76*) at the *15*th term ($M = 15$, $R = 30$), we approximate the phase velocity V_{pp} to within one part in a billion.

The evaluations of other series involved in Eqs. (*76,78,79*), for the wave velocities V_{ps} , V_{gp} , V_{gs} , when $M \to \infty$, have been obtained analogously. These evaluations show that all the above velocities may be calculated with a relative error of about 10^{-9} by taking into account only *15* terms ($M = 15$) in each of the series.

The results of the calculations are presented in Figure 1, which compares the phase and group velocities of P- and S-waves in scaling elastic granular media (which have finite sizes of particles $d = \eta$) with the corresponding velocities V_{op} and V_{os} in conventional non-scaling elastic media (where $\eta = 0$) that are determined by classical theory of elasticity (see also Eqs. (*80-81*)).

Figure 1 shows a most interesting feature of the plane P- and S-waves in scaling granular media: The waves have *dispersion* because all their velocities depend on the wavelength λ and granule size η. In particular, the velocities diminish with the nondimensional parameter $l \equiv \lambda/\eta$ (see **Figure 3**). Thus, we again observe here the phenomenon of *wave retardation* at relatively short wavelengths which has been considered above in monatomic crystals.

It is further noted that, in the bands of relatively long waves when the parameter $l \equiv \lambda/\eta$ is large enough, i.e., $l \geq 15..20$, the velocities V_{pp} , V_{gp} are only slightly less than V_{op} , as well as the velocities V_{ps} , V_{gs} are somewhat less than V_{os}: The difference between these is no more than *2%* .

On the other hand, the shorter the waves, the more significant the wave retardation. In the wave band of short lengths that are close to their minimum - a double particle diameter (when $l = 2$) - the velocities reach the least values that are considerably less than V_{op} and V_{os} . In this band, classical theory of elasticity significantly overestimates the wave velocities: phase velocities up to *33%* for P-waves and *10%* for S-waves, and group velocities up to *89%* for P-waves and *29%* for S-waves.

DISCUSSION AND CONCLUSIONS

In the previous section, the doublet elastodynamics of the basal plane of the cubical-tetrahedral packing H_4 was considered, with the purpose of establishing the fact that elastic plane waves in an isotropic granular medium are dispersive. It was also thereby demonstrated that the capability of modeling dispersion is lost upon introducing the long-wavelength or the continuum approximations. In analogy with isotropic continuum elastodynamics, it was also shown that the longitudinal and shear waves are decoupled. To place this analogy in the proper perspective, however, it is remarked that the basal plane of H_4 is isotropic only at the non-scale approximation

(M=1). As a side remark, it is noted that eq. (42-43) do not allow for the propagation of nontrivial plane wave polarized in the normal direction (i.e., SH-waves).

Returing to the question of scale-dependence of isotropy, the generalized macromoduli of elasticity $C_{ijk_1...k_\kappa}$ are computed in accordance with Eq. (41), where of $\eta_\alpha = \eta = constant$, $\kappa = 2, 4, 6,...$. For ease of representation, the associated nondimensional macromoduli $\bar{C}_{ijk_1...k_\kappa}$ are introduced, that are defined as

$$\bar{C}_{ijk_1...k_\kappa} \equiv C_{ijk_1...k_\kappa} /(2 \cdot A_o \cdot \kappa!/\eta^{\kappa-2}) \equiv \sum_{\alpha = 1}^{n} \tau^o_{\alpha i}\, \tau^o_{\alpha j}\, \tau^o_{\alpha k_1} \cdots \tau^o_{\alpha k_\kappa} \quad (90).$$

In order to establish the conclusion, the case $i = j = k_1 = ... = k_\kappa = 1$, is considered for different values of κ and in different frames differing by the angle γ, i.e. under

$$\| \tau^o_{\alpha j} \| = \begin{Vmatrix} \tau^o_{11} & \tau^o_{12} \\ \tau^o_{21} & \tau^o_{22} \\ \tau^o_{31} & \tau^o_{32} \end{Vmatrix} = \begin{Vmatrix} cos\gamma & -\,sin\gamma \\ cos(60-\gamma) & sin(60-\gamma) \\ -\,cos(60+\gamma) & sin(60+\gamma) \end{Vmatrix} \quad (91)$$

The variation of the moduli with γ is plotted in Figure 3. It is noted that the non-scale macromodulus $\bar{C}_{1111}$, corresponding to $\kappa = 2$, is indeed independent of γ, i.e.isotropic in the plane.. On the contrary, the macromoduli $\bar{C}_{ijk_1...k_\kappa}$ for $\kappa = 4, 6, 8,...$ are anisotropic. It can be seen from Figure 3 that all $\bar{C}_{ijk_1...k_\kappa} \rightarrow 0$ as κ

$\longrightarrow \infty$ for any angle γ except for $\gamma = 0^\circ, 60^\circ, 120^\circ$ where $\bar{C}_{ijk_1 \ldots k_\kappa} \longrightarrow 1$, which are the angles that identify the directions of the doublets (see Fig. 2). It may then be concluded that in the first approximation, $\kappa = 2$, Eqs. (*40*) model the continuum-like behavior of solids, whereas in the other approximations, $\kappa = 4, 6, \ldots$, Eqs. (*40*) also reflect discrete-like features of the solid, in a manner that increases with κ.

In this sense, Doublet Mechanics may be concluded to be capable of modeling solids in view of their dual and to some extent contradictory discrete-continuous nature. The power of such dual-representation capability is evident in the discussion of isotropy: The basal plane of the cubic-tetrahedral arrangement is isotropic only in the continuum, non-scale approximation. Thus, isotropy is a scale-related notion - a fact that is of course physically evident, as no material may be argued to be isotropic at all dimensional scales, down to its most elementary component level. What is promising for the Doublet Mechanics approach is the fact that this theory is capable of modeling such observation, and recourse to different theories for different dimensional scales is avoided altogether.

According to (80-81), the velocity ratio is $C_1 \equiv \frac{V_{op}}{V_{os}} = \sqrt{3}$. By comparison with the well-known relation (Fung, 1977)

$$C_1 = \sqrt{[2(1 - \nu)/(1 - 2\nu)]}, \tag{92}$$

this leads to Poisson's constant $\nu = 1/4$. This value is conventionally assumed in seismology for the earth's crust (Leet, 1938; Macelwane, 1949; Båth, 1968). A theoretical validation of this assumption is furnished by Doublet Mechanics: Any isotropic tensor in accordance with relation (*43*) must have the form

$$C_{ijkl} = \lambda\, (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{93}$$

This differs from the conventional continuum elasticity tensor in that it is also endowed with the symmetry $C_{ijlk} = C_{ikjl}$, which is not necessary in continuum mechanics. In turn, the additional symmetry imposes equality of Lame's constants, i.e., $\lambda = \mu$, which reduces the number of independent moduli to only one. Of course, this conclusion holds only for materials with microstructure and properties that satisfy the assumptions that were employed for the derivation of (43). By employing the well-known relationship

$$\nu = \frac{\lambda}{2\,(\lambda + \mu)}\,. \tag{94}$$

it is then concluded that $\nu = 1/4$, which was to be proved.

The developed micromechanical elastodynamics theory is applicable not only to the unbounded isotropic *granular* solids but to any similar *particulate* media, i.e., any media with microstructure characterized *topologically* by matrix (67), *geometrically* by finite particle sizes (or finite central particle distances) η and *physically* by longitudinal particle interactions only (29).

The term *microstructure* is relative, in our context, in the sense that the wave dispersion and wave retardation depend on the wavelength λ and characteristic distance η not *separately*, but in a *nondimensional combination* $l \equiv \lambda/\eta$. Thus the internal size η si significant only in comparison with the wavelength λ. This may be arbitrary large and, consequently, the size η may also be arbitrary large provided the ratio l has finite values $l \geq 2$.

On these bases, the absolute dimension of the typical internal structure is not a determining factor concerning the applicability. To illustrate the point, three classes of materials are here briefly considered, that possess very different internal dimensional scales: 1) Crystals, with the interatomic spacing $\eta \sim 10^{-8}$ cm; 2) Granular materials, with the characteristic granule sizes $\eta \sim 10^{-1}...10^{2}$ cm (sand, gravel, rubble, boulders); 3) The earth's crust - with 'particle' sizes $\eta \sim 10^{6}$ cm (= *10* km).

1) **Crystals.** Phenomena of wave dispersion and retardation in linear monatomic crystals are well-known in solid state physics, where they are modeled by Eq. (51). Abundant experimental verifications of Eq. (53) is given e.g. in (Kittel, 1953; Cochran, 1973). In this paper, Eq. (51) obtained as a special case of Eqs. (*40*), (*41*). Thus, the experimental validation of Eqn. (51) offers elements for the indirect confirmation of the underlying Eqs. (*40*), (*41*) as well.

For bi- and tri- dimensional arrangements, the difference Eq. (*51*) is substituted respectively by two or three cognate equations, that describe small oscillations of the atom in two or three mutually perpendicular directions x, y, z. With a plane harmonic wave similar to (*52*), the dispersion relations is then found to be of the type $\omega_{ij}(k\eta)$ (i, j = *1*, *2* or *1*, *2*, *3*), which generalizes the above one-dimensional solution (*54*). The frequencies ω_{ij} are sinusoidal or quasi-sinusoidal functions of $k\eta$ in form (Cochran, 1973, Fig. 4.2), resulting in dispersion relations that follow closely the elastodynamics of linear crystals. In particular, the waves slow down as the wavelengths shorten.

These theoretical results obtained in solid state physics have been verified experimentally for many spatial crystal structures including face-centered cubic (FCC) ones such as aluminum, copper, and other metals. The FCC structure is equivalent to a regular pyramidal packing H_6 (Deresiewicz, 1958, pages 237-238) which has crystallographic planes that are identical with the basal plane of the cubical-tetrahedral packing H_4 considered in this section. Thus, the theoretical and experimental data on dynamics of spatial crystals also *indirectly* reconfirm the phenomena of wave dispersion and retardation, as modeled by the present approach.

Granular materials. The impact of particle sizes on wave propagation in elastic granular media has been *directly* studied in a few experimental works. The earliest of these (Iida, 1938) established that the phase velocities of P- and S-waves in dry sand depend on the particle diameter d and slightly rise as d increases. These data first indicated that wave dispersion was associated with size parameters, but was at variance with the phenomenon of wave retardation. The fact that the phase velocities of plane P-waves are sensitive to the granule sizes was also observed in other direct experiments with dry sand (Matsukawa and Hunter, 1956).

Several later studies brought *indirect* experimental data concerning the influence of particle sizes on wave velocities in granular massives. Among these, (Trent, 1989) dealt with two arrays of *270* like spheres which had diameter $d = 2$ mm and occupied equal volumes. The first array had *480* interparticle bonds, the second one *397*, i.e., *21%* less. By performing numerical experiments based on the so-called *distinct element method* (Cundall and Strack, 1979), it was established that the phase velocity of P-wave in the second array is *37%* less then in the first array (*1050* m/s versus *1440* m/s). There is only one difference in the two arrays which is responsible for the change in velocities - the difference in numbers N of the bonds. Meanwhile in any regular n-valence packing H_3 to H_6 (should they have equal volumes) the number N and diameter d are inversely correlated. So a decrease of bond numbers is, in general, equivalent to an increase of particle size. Therefore the retardation of wave observed in the second granular array tested may be attributed either *directly* to a reduced number of particle bonds or *indirectly* to an increase of particle diameters d and, consequently, to a decrease of the ratio $l \equiv \lambda/d$, λ being the wavelength. This means that the waves slow down as the scaling parameter l declines, a result that is in agreement with the quality of Figure 1.

The earth's crust. As discussed in (Granik and Ferrari, 1994c) the doublet-mechanical approach is applicable with the context of plane-wave seismology. In particular, comparisons were there made between classical experimental data (Leet, 1938, page 261; Miyabe, 1935, Pilant, 1979, page 254, Fig. 7-1), and scale-accounting doublet mechanical predictions that incorporate the effects of the particulate structure of the crust on the velocity of propagation of longitudinal waves. The results showed agreement in the prediction of the ratios of the velocities of waves with different wavelength to within 5% of the experimental values for three sets of experiments.

ACKNOWLEDGEMENT. The authors wish to express their gratitude to Ms. Miqin Zhang for her analysis and graphical display of the data. The support of the National Science Foundation with MF's National Young Investigator Award is gratefully acknowledged.

REFERENCES

Agarwal, T.K., and Ishibashi, I. (1992). "Anisotropic elastic constants of granular assembly from wave velocity measurements." *Advances in Micromechanics of Granular Materials: Proc., 2nd US/Japan Seminar on Micromechanics of Granular Materials*, Potsdam, N.Y., USA, 1991, H.H. Shen et al., eds., Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 51-60.

Båth, M. (1968). "Mathematical aspects of seismology." Elsevier Publishing Company, Amsterdam, The Netherlands.

Bathurst, R.J., and Rothenburg, L. (1988). "Micromechanical aspects of isotropic granular assemblies with linear contact interactions." *J. Appl. Mech. Trans. ASME*, 55, 17-23.

Bažant, Z.P. (1993). "Scaling laws in mechanics of failure." *J. Engrg. Mech.*, ASCE, 119(9), 1828-1844.

Bolt, B.A., and Dorman, J. (1961). "Phase and group velocities of Rayleigh waves in a spherical gravitating earth." *J. Geophys. Res.*, 66, 2965-2981.

Brillouin, L. (1960). "Wave propagation and group velocity." Academic Press, Inc., New York, N.Y.

Byerly, P. (1942). "Seismology." Prentice-Hall, Inc., New York, N.Y.

Campbell, C.S. (1993). "The transition from fluid-like to solid-like behavior in granular flows." *Powders & Grains 93: Proc., 2nd Int. Conf. on Micromechanics of Granular Media*, Association for the Study of Micromechanics of Granular Media (ASMGM), Birmingham, UK, 1993, C. Thornton, ed., A.A. Balkema, Rotterdam, The Netherlands, 289-294.

Chang, C.S., and Misra, A. (1990). "Packing structure and mechanical properties of granulates." *J. Engrg. Mech.*, ASCE, 116(5), 1077-1093.

Cochran, W. (1973). "The dynamics of atoms in crystals." William Clowes and Sons Ltd., London.

Cosserat, E., and Cosserat, F. (1909). "Thèorie des corps dèformables." Hermann, Paris.

Cottrell, A.H. (1964). "The mechanical properties of matter." John Wiley and Sons, Inc., New York, N.Y.

Crampin, S., and Taylor, D.B. (1971). "The propagation of surface waves in anisotropic media." *Geophys. J.*, 25, 71-87.

Cundall, P.A., and Strack, O.D.L. (1979). "A discrete numerical model for granular assemblies." *Geotechnique*, 29(1), 47-65.

Davis, R.A., and Deresiewicz, H. (1977). "A discrete probabilistic model for mechanical response of a granular medium." *Acta Mechanica*, 27, 69-89.

Deresiewicz, H. (1958). "Mechanics of granular matter." in: *Advances in Applied Mechanics*, Vol. 5, H.L. Dryden and Th. von Kàrmàn, eds., 1958, Academic Press Inc., Publishers, New York, N.Y., 233-306.

Eringen, A.C., and Suhubi, E.S. (1964). "Nonlinear theory of simple micro-elastic solids (I and II)." *Int. J. Engng. Sci.*, 2, 189-203 and 389-404.

Eringen, A.C. (1966). "Linear theory of micropolar elasticity." *J. Math. Mech.*, 15, 909-924.

Ewing, M., Jardetzky, W.S., and Press, F. (1957). "Elastic waves in layered media." McGraw-Hill, New York, N.Y.

Feda, J. (1982). "Mechanics of particulate materials. The principles." Elsevier Scientific Publishing Co., Inc., New York, N.Y.

Ferrari, M., and Granik, V.T. (1994). "Doublet-based micromechanical approaches to yield and failure criteria." *Material Science and Engng.*, A175, 21-29.

Ferrari, M., and Granik, V.T. (1995). "Ultimate criteria for materials with different properties in tension and compression: A doublet-mechanical approach" *Material Science and Engng.* (in press).

Fung, Y.C. (1977). "A first course in continuum mechanics." Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Gassmann, F. (1951). "Elastic waves through a packing of spheres." *Geophysics*, 16, 673-685.

Granik, V.T., and Ferrari, M. (1993a). "Microstructural mechanics of granular media." *Int. J. Mechanics of Materials*, 15, 301-322.

Granik, V.T., and Ferrari, M. (1993b). "Multi-scale homogenization of granular media." *Composite Material Technology: Proc., 16th Annual Energy-Sources Technology Conf. ASME*, Houston, Texas, 1993, D. Hui and T.J.Kozik, eds., Vol. 53, ASME, New York, N.Y., 217-222.

Granik, V.T., and Ferrari, M. (1994). "Micromechanical theory of yield and failure criteria for isotropic materials in view of hydrostatic stress components." *SEMM Report 94-21*, Dept. Civ. Engng. Univ. Calif., Berkeley.

Gray, W.A. (1968). "The packing of solid particles." Chapman and Hall Ltd., London.

Green, A.E., and Rivlin, R.S. (1964a). "Simple force and stress multipoles." *Arch. Ratl. Mech. Analysis*, 16, 325-353.

Green, A.E., and Rivlin, R.S. (1964b). "Multipolar continuum mechanics." *Arch. Ratl. Mech. Analysis*, 17, 113-147.

Hampton, L. (1967). "Acoustic properties of sediments." *J. Acoust. Soc. Am.*, 42, 882-890.

Harr, M.E. (1977). "Mechanics of particulate media. A probabilistic approach." McGraw-Hill Inc., New York, N.Y.

Haskell, N.A. (1953). "The dispersion of surface waves in multilayered media."

Bulletin Seismol. Soc. Amer., 43, 17-34.

Horne, M.R. (1965). "The behavior of an assembly of rotund, rigid, cohesionless particles (I and II)." *Proc. Roy. Soc.*, A, 286, 62-97.

Horne, M.R. (1969). "The behavior of an assembly of rotund, rigid, cohesionless particles (III)." *Proc. Roy. Soc.*, Ser. A, 310, 21-34.

Hutter, K. and K. R. Rajagopal (1994) "On flows of granular media", *Continuum Mechanics and Thermodynamics,* v. 6, n. 2, pp. 81-140.

Iida, K. (1938). "The velocity of elastic waves in sand." *Bull. Earthquake Res. Inst.*, Japan, 16, 131-144.

Kittel, C. (1953). "Introduction to solid state physics." John Wiley and Sons, Inc., New York, N.Y.

Knopoff, L. (1972). "Observation and inversion of surface-wave dispersion." *Tectonophysics*, 13, 497-519.

Kolsky, H. (1963). "Stress waves in solids." Dover Publications, Inc., New York, N.Y.

Leet, L.D. (1938). "Practical seismology and seismic prospecting." D.Appleton-Century Company, Inc., New York, London.

Love, A.E.H. (1911). "Some problems of geodynamics." Cambridge University Press, London, 144-178.

Love, A.E.H. (1944). "A treatise on the mathematical theory of elasticity." Dover Publications, Inc., New York, N.Y.

Macelwane, J.B. (1949). "Introduction to theoretical seismology. Geodynamics." St. Louis University, Missouri, USA.

Maddalena, F. & M. Ferrari (1995) "Viscoelasticity of Granular Materials", *Mechanics of Materials*), vol.3, n. 20, pp. 241-250.

Malischewsky, P. (1987). "Surface waves and discontinuities." Elsevier Science

Publishers, Amsterdam, The Netherlands.

Matsukawa, E., and Hunter, A.N. (1956). "The variation of sound velocity with stress in sand." *Proc. Phys. Soc.*, London, B, 69, 847-848.

Matsuoka, H. (1974). "A microscopic study on shear mechanism of granular materials." *Soils an Foundations*, 14, 29-43.

Meek, J.W., and Wolf, J.P. (1992). "Cone models for homogeneous soil." *J. Geotech. Engrg.*, ASCE, 118, 667-685.

Mehrabadi, M.M., Nemat-Nasser, S., and Oda, M. (1982). "On statistical description of stress and fabric in granular materials." *Int. J. Numer. Anal. Methods Geomech.*, 6, 95-108.

Mehrabadi, M.M., and Nemat-Nasser, S. (1983). "Stress, dilatancy and fabric in granular materials." *Int. J. Mechanics of Materials*, 2, 155-161.

Mehrabadi, M.M., ed. (1992). "Mechanics of granular materials and powder system." *Proc., Annual Meeting ASME*, Anaheim, Calif., 1992, ASME, New York, N.Y.

Mindlin, R.D. (1964). "Microstructure in linear elasticity." *Arch. Ratl. Mech. Analysis*, 16, 51-78.

Mindlin, R.D. (1968). "Theories of elastic continua and crystal lattice theories." *Proc., Symposium on Mechanics of Generalized Continua*, Freudenstadt and Stuttgart, Germany, 1967, E. Kröner, ed., Springer-Verlag, Berlin, Heidelberg, New York, 312-320.

Miyabe, N. (1935). "Notes on the block structure of the earth's crust." *Bulletin Earthquake Res. Inst., Tokyo Imperial Univ.*, 13, 280-285.

Mogami, T. (1965). "A statistical approach to the mechanics of granular materials." *Soils and Foundations*, 5, 26-36.

Mrithyumjaya, R.K., and Pratar, R.M. (1993). "Rayleigh-type wave propagation on a micropolar cylindrical surface." *J. Appl. Mech.*, ASME, 60, 857-865.

Mon, K. and M. Ferrari "Advances in doublet mechanics. III: Doublet thermomechanics" (SEMM report 95-06, Department of Civil Engineering, University of California, Berkeley, CA. Also, submitted for publication)

Nadeau, J.C., A. H. Nashat and M. Ferrari "Advances in doublet mechanics. III: Consideration on plane elastostatics" (SEMM report 95-05, Department of Civil Engineering, University of California, Berkeley, CA. Also, submitted for publication)

Nemat-Nasser, S., and Tobita, Y. (1982). "Influence of fabric on liquefaction and densification potential of cohesionless sand." *Int. J. Mechanics of Materials*, 1, 43-62.

Nemat-Nasser, S. (1982). "Fabric and its influence on mechanical behavior of granular materials." *Proc., Symposium on Deformation and Failure of Granular Materials*, Delft, The Netherlands, 1982, P.A. Vermeer and H.J. Luger, eds., Int. Union of Theor. and Appl. Mechanics (IUTAM), A.A. Balkema, Rotterdam, The Netherlands, 37-42.

Nowacki, W. (1970). "Teoria sprezystości." Państwowe Wydawnictwo Naukowe, Warsza-wa, Poland. (Chapter 13, 825-827).

Oda, M., Nemat-Nasser, S., and Mehrabadi, M.M. (1982). "A statistical study of fabric in a random assembly of spherical granules." *Int. J. Numer. Anal. Methods Geomech.*, 6, 77-94.

Ostoja-Starzewski, M. (1992). "Random fields and processes in mechanics of granular materials." *Advances in Micromechanics of Granular Materials: Proc., 2nd US/Japan Seminar on Micromechanics of Granular Materials*, Potsdam, N.Y., USA, 1991, H.H. Shen et al., eds., Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 71-80.

Palmov, V.A. (1964). "Fundamental equations of theory of asymmetric elasticity." *Appl. Mech. Math.*, 28, 401-420.

Picci, E., and Saccomandi, G. (1990). "Symmetries and conservation laws in micropolar elasticity." *Int. J. Engng. Sci.*, 28, 557-562.

Pilant, W.L. (1979). "Elastic waves in the earth." Elsevier Scientific Publishing Co., Amsterdam, The Netherlands.

Prakash, S. (1981). "Soil dynamics." McGraw-Hill Inc., New York, N.Y.

Rayleigh, Lord (1885). "On waves propagated along the plane surface of an elastic solid." *Proc., London Math. Soc.*, 17, 4-11.

Reynolds, O. (1885). "On the dilatancy of media composed of rigid particles in contact." *Phil. Mag.*, Ser. 5, 20, 469-481.

Saito, M. (1967). "Excitation of free oscillations and surface waves by a point source in a vertically heterogeneous earth." *J. Geophys. Res.*, 72, 3689-3699.

Satake, M. (1978). "Constitution of mechanics of granular materials through the graph theory." *Proc., Continuum-Mechanical and Statistical Approaches in the Mechanics of Granular Materials*, Tokyo, 1978, S.C. Cowin and M. Satake, eds., Gakujutsu Bunken Fukyu-kai, Tokyo, Japan, 47-62.

Satake, M. (1982). "Fabric tensor in granular materials." *Proc., Symposium on Deformation and Failure of Granular Materials*, Delft, The Netherlands, 1982, P.A. Vermeer and H.J. Luger, eds., Int. Union of Theor. and Appl. Mechanics (IUTAM), A.A. Balkema, Rotterdam, The Netherlands, 63-68.

Satake, M. (1993). "Discrete-mechanical approach to granular media." *Powders & Grains 93: Proc., 2nd Int. Conf. on Micromechanics of Granular Media*, Association for the Study of Micromechanics of Granular Media (ASMGM), Birmingham, UK, 1993, C. Thornton, ed., A.A. Balkema, Rotterdam, The Netherlands, 3-9.

Sezawa, K. (1927). "Dispersion of elastic wave propagated on the surface of stratified bodies and on curved surfaces." *Bulletin Earthquake Res. Inst., Tokyo Imperial Univ.*, 3, 1-18.

Slade, R.E., and Walton, K. (1993). "Inter-granular friction and the mechanical and acoustic properties of granular media." *Powders & Grains 93: Proc., 2nd Int. Conf. on Micromechanics of Granular Media*, Association for the Study of Micromechanics of Granular Media (ASMGM), Birmingham, UK, 1993, C. Thornton, ed., A.A. Balkema, Rotterdam, The Netherlands,93-98.

Sokolnikoff, I.S., and Sokolnikoff, E.S. (1941). "Higher mathematics for engineers and physicists." McGraw-Hill Book Company, Inc., New York and London.

Sokolnikoff, I.S. (1951). "Tensor analysis." John Wiley and Sons, Inc., New York, N.Y.

Stojanoviċ, R. (1972). "On the mechanics of materials with microstructure." *Acta Mechanica*, 15, 261-273.

Stout, R.B. (1989). "Statistical model for particle-void deformation kinetics in granular materials during shock wave propagation." *Proc., Annual Meeting ASME Wave Propagation in Granular Media*, San Francisco, Calif., 1989, D. Karamanlidis and R.B. Stout, eds., ASME, New York, N.Y., 51-74.

Takahashi, T., and Sato, Y. (1949). "On the theory of elastic waves in granular substance." *Bull. Earthquake Res. Inst.*, Tokyo, 27, 11-16.

Takahashi, T., and Sato, Y. (1950). "On the theory of elastic waves in granular substance." *Bull. Earthquake Res. Inst.*, Tokyo, 28, 37-43.

Tatsuoka, F., Nakamura, S., Huang, C.-C., and Tani, K. (1990). "Strength anisotropy and shear band direction in plane strain tests of sand." *Soils and Foundations*, 10, 35-54.

Teodorescu, P.P., and Soȯs, E. (1973). "Discrete, quasi-continuous and continuous models of elastic solids." *Zeitschrift Angew. Math. Mech.*, 53, T33-T43.

Thornton, C., ed. (1993). "Powders and Grains 93." *Proc., 2nd Int. Conf. on Micromechanics of Granular Media*, Association for the Study of Micromechanics of

Granu-lar Media (ASMGM), Birmingham, UK, 1993, A.A. Balkema, Rotterdam, The Netherlands.

Todhunter, I, (1886) *A History of the Theory of Elasticity*, vol. 1, Cambridge University Press.

Trent, B.C. (1989). "Numerical simulation of wave propagation through cemented gra-nular material." *Proc., Annual Meeting ASME Wave Propagation in Granular Media*, San Francisco, Calif., 1989, D. Karamanlidis and R.B. Stout, eds., ASME, New York, N.Y., 9-15.

Urick, R.J. (1948). "The absorption of sound in suspensions of irregular particles." *J. Acoust. Soc. Am.*, 20, 283-289.

Vermeer, P.A., and Luger, H.J., eds. (1982). "Deformation and failure of granular materials." *Proc., Symposium on Deformation and Failure of Granular Materials*, Delft, The Netherlands, 1982, Int. Union of Theor. and Appl. Mechanics (IUTAM), A.A. Balkema, Rotterdam, The Netherlands.

Wijesinghe, A.M. (1989). "A constitutive theory for stress wave propagation across closed fluid-filled rough fractures." *Proc., Annual Meeting ASME Wave Propagation in Granular Media*, San Francisco, Calif., 1989, D. Karamanlidis and R.B. Stout, eds., ASME, New York, N.Y., 75-90.

Zhang, M. and M. Ferrari (1995) 'Advances in doublet mechanics. II: Free-boundary reflection of P- and S- waves in granular media. (SEMM report 95-04, Department of Civil Engineering, University of California, Berkeley, CA. Also, submitted for publication)

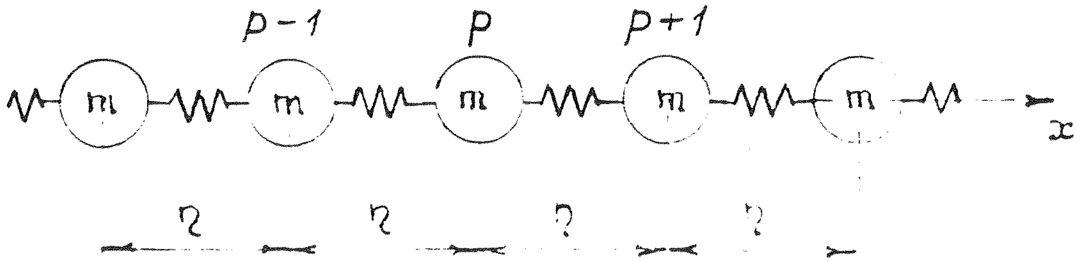


Fig.1. A linear monatomic lattice

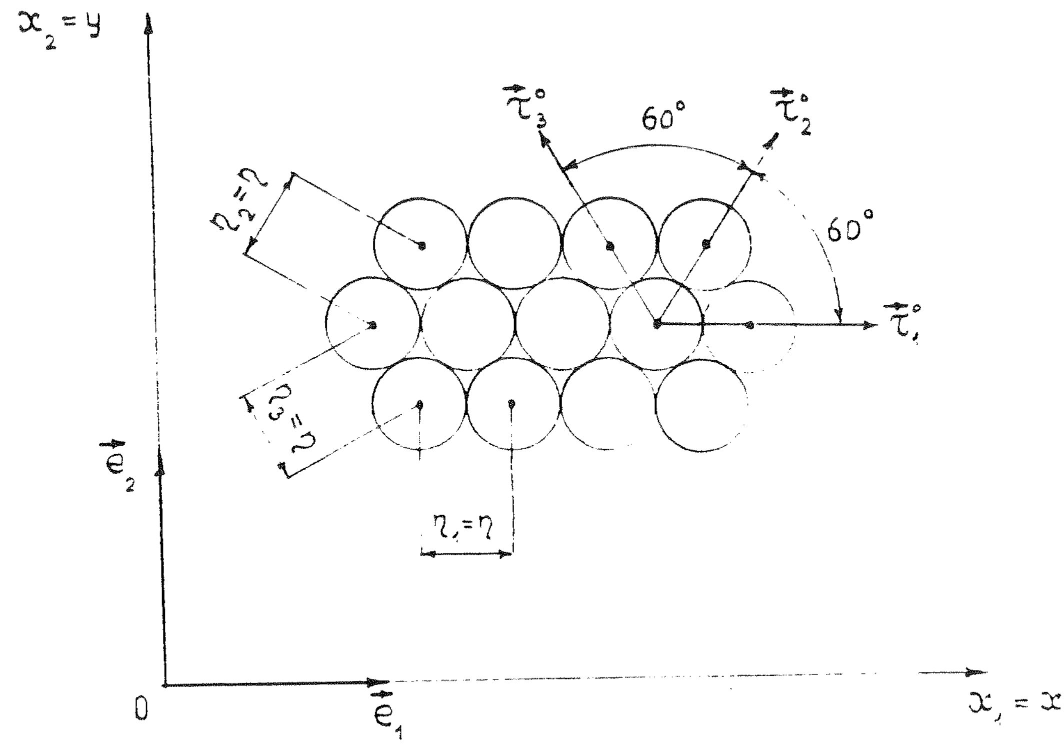
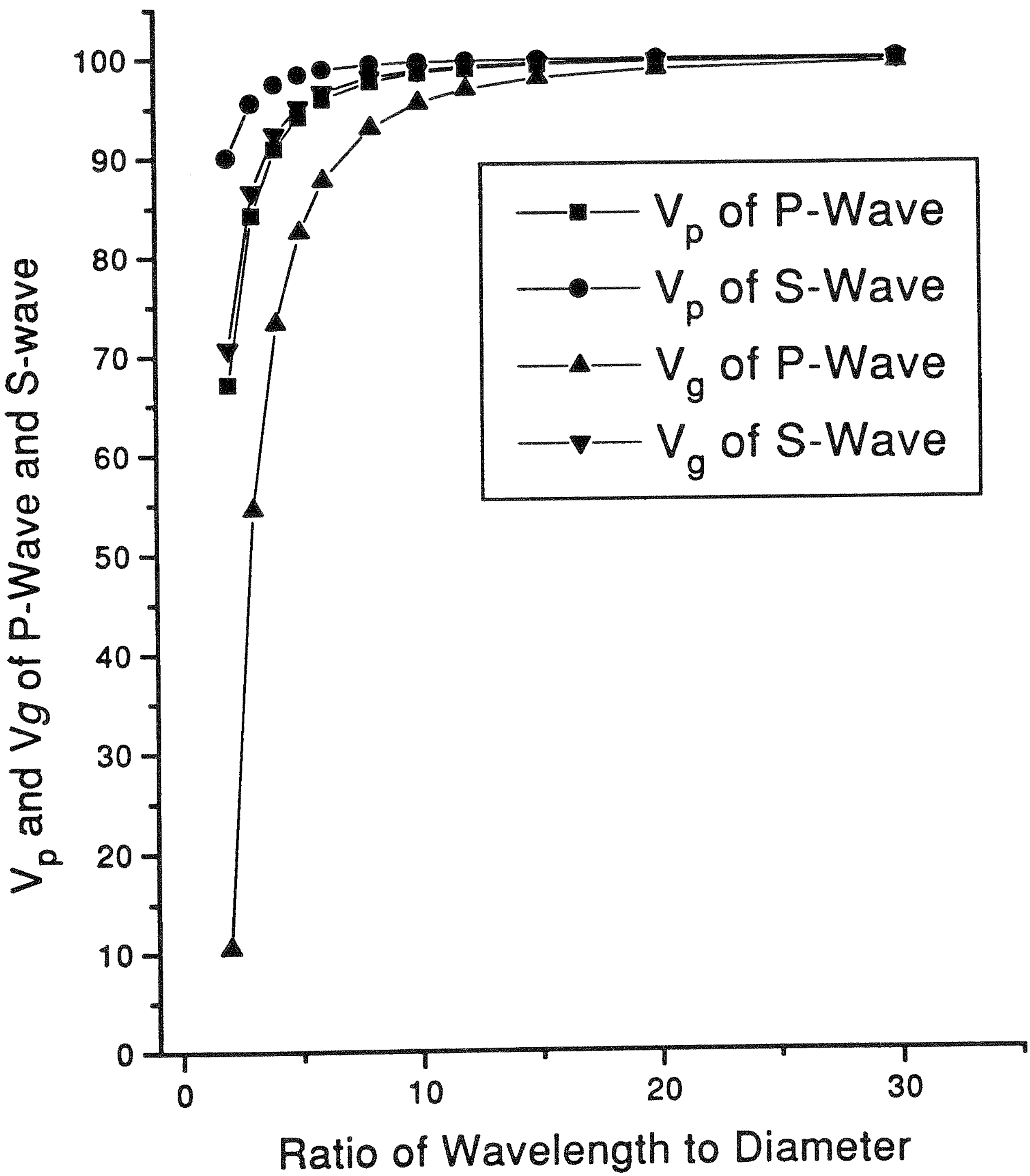
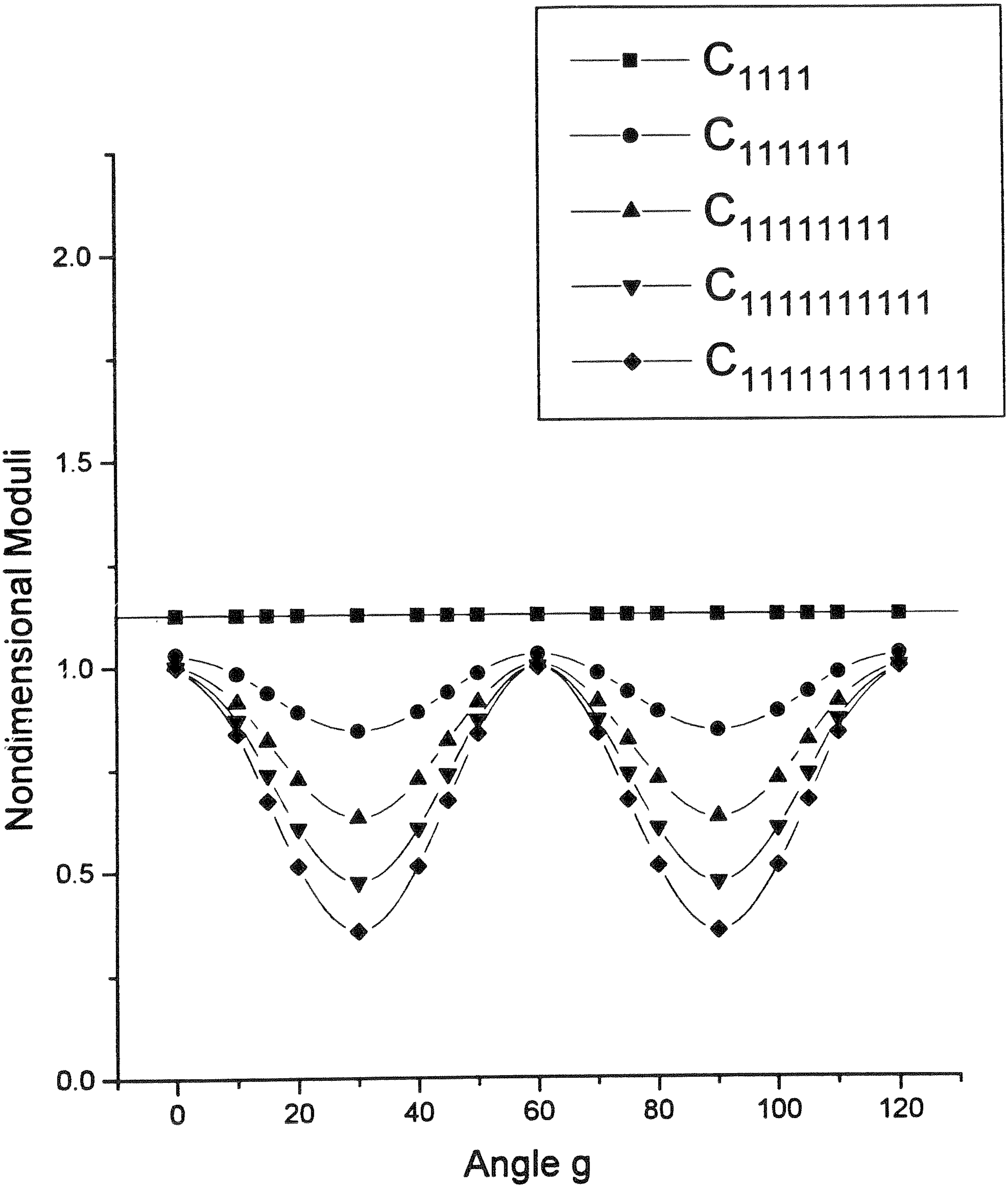


Fig. 2. A cubical-tetrahedral packing H_4



V_p and Vg of P-Wave and S-wave
Ratio of Wavelength to Diameter
V_p of P-Wave
V_p of S-Wave
V_g of P-Wave
V_g of S-Wave
0
10
20
30
40
50
60
70
80
90
100
0
10
20
30



Variations of Nondimensional Moduli of Elasticity $C_{ijk_1 \ldots k_k}$ with the Rotation of a Plane Frame of Reference by an Angle g