**Title**

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**Authors**

Krupchyk, Katya
Uhlmann, Gunther

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# $L^p$ BOUNDS ON EIGENFUNCTIONS FOR OPERATORS WITH DOUBLE CHARACTERISTICS

KATYA KRUPCHYK AND GUNTHER UHLMANN

Abstract. We obtain sharp $L^p$ bounds on the ground states for a class of semiclassical pseudodifferential operators with double characteristics and complex valued symbols, under the assumption that the quadratic approximations along the double characteristics are elliptic.

## 1. Introduction and statement of results

Starting with the celebrated works [9] and [18], the question of establishing precise $L^p$ estimates for eigenfunctions of elliptic self-adjoint operators on compact manifolds in the high energy limit has been of fundamental significance in the spectral theory and applications. Most of the works have been concerned with the case of the Laplace operator and we refer to [19], [20], [21], [6], for some of the recent contributions.

Turning the attention to the case of operators on $\mathbb{R}^n$, similar problems have been studied in [11] in the case of the harmonic oscillator $-\Delta+|x|^2$, as well as for more general Schrödinger operators, see also [10], [23], and [24].

The work [12] has introduced a semiclassical point of view, unifying and extending the results of [18] and [11] to more general semiclassical pseudodifferential operators. To motivate our result and to place it into its natural context, let us recall some of the estimates established in [12] and [17], specialized to the case of the semiclassical Schrödinger operator.

Let

$$P=-h^2\Delta+V \quad \text{on} \quad \mathbb{R}^n, \quad n\ge 2,$$

where $V\in C^\infty(\mathbb{R}^n,\mathbb{R})$ is such that

$$\begin{aligned} |\partial_x^\alpha V(x)| &\le C_\alpha \langle x\rangle^m, \quad \alpha\in\mathbb{N}^n,\\ V(x) &\ge \langle x\rangle^m/C, \quad |x|\ge C, \end{aligned}$$

for some $m>0$. When equipped with the domain $\mathcal{S}(\mathbb{R}^n)$, the operator $P$ is essentially self-adjoint on $L^2(\mathbb{R}^n)$ and the spectrum of $P$ is discrete, with eigenfunctions microlocalized to compact subsets of $T^*\mathbb{R}^n$, see [25, Theorems 6.4 and

6.7]. If $u \in L^2(\mathbb{R}^n)$, $\|u\|_{L^2} = 1$, is an eigenfunction of $P$, then according to [12, Theorem 6] and [17], we have

$$\|u\|_{L^\infty} \le \mathcal{O}(1)h^{-\frac{(n-1)}{2}} \tag{1.1}$$

and

$$\|u\|_{L^{\frac{2n}{n-2}}} \le \mathcal{O}(1)h^{-1/2}. \tag{1.2}$$

Interpolating between the bounds (1.1), (1.2), and the trivial $L^2$ bound, the full range of $L^p$ estimates is obtained. For $n \ge 3$, we have

$$\|u\|_{L^p} \le \mathcal{O}(1)h^{\frac{n}{p}-\frac{(n-1)}{2}}, \quad \frac{2n}{n-2} \le p \le \infty, \tag{1.3}$$

and

$$\|u\|_{L^p} \le \mathcal{O}(1)h^{\frac{n}{2p}-\frac{n}{4}}, \quad 2 \le p \le \frac{2n}{n-2}, \tag{1.4}$$

and for $n = 2$,

$$\|u\|_{L^p} \le \mathcal{O}(1)h^{\frac{1}{p}-\frac{1}{2}}, \quad 2 \le p \le \infty, \tag{1.5}$$

see Figure 1.

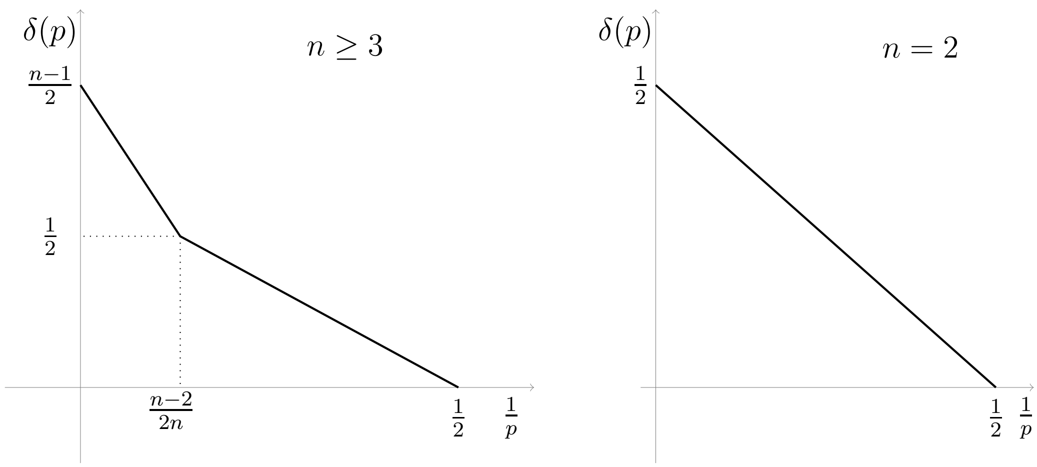


Figure 1. Estimates (1.3), (1.4) and (1.5) of [12] and [17] written in the form $\|u\|_{L^p} \le \mathcal{O}(1)h^{-\delta(p)}$.

The following example shows that the estimates (1.4) and (1.5) are sharp.

**Example.** Consider the quantum harmonic oscillator,

$$P = -h^2\Delta + |x|^2, \quad x \in \mathbb{R}^n, \quad n \ge 2.$$

The operator $P$, equipped with the domain,

$$\mathcal{D}(P) = \{u \in L^2(\mathbb{R}^n) : x^\alpha \partial_x^\beta u \in L^2(\mathbb{R}^n), |\alpha + \beta| \le 2\},$$

is self-adjoint with discrete spectrum given by

$$\lambda_\alpha(h) := (2|\alpha| + n)h, \quad \alpha \in \mathbb{N}^n.$$

The corresponding $L^2$ normalized eigenfunctions are of the form

$$u_\alpha(h)(x) = h^{-\frac{n}{4}} p_\alpha(x/h^{1/2}) e^{-\frac{|x|^2}{2h}},$$

where $p_\alpha$ are the Hermite polynomials of degree $|\alpha|$, see [25, Section 6.1]. A direct computation shows that

$$\|u_\alpha(h)\|_{L^p} = C_\alpha h^{\frac{n}{2p}-\frac{n}{4}}, \qquad 2 \le p \le \infty, \tag{1.6}$$

where

$$C_\alpha = \left( \int_{\mathbb{R}^n} |p_\alpha(x)|^p e^{-\frac{|x|^2 p}{2}} dx \right)^{1/p}.$$

It follows that the bounds (1.4) and (1.5) are saturated by the ground state eigenfunctions $u_\alpha(h)$, corresponding to $\lambda_\alpha(h) \le \mathcal{O}(h)$. Furthermore, (1.6) implies that the eigenfunctions $u_\alpha(h)$, corresponding to $\lambda_\alpha(h) \le \mathcal{O}(h)$, enjoy sharper bounds than those in (1.3), for $\frac{2n}{n-2} < p \le \infty$, $n \ge 3$.

The purpose of the present paper is to show that the sharp $L^p$ bounds of the form

$$\|u\|_{L^p} \le \mathcal{O}(1) h^{\frac{n}{2p}-\frac{n}{4}}, \qquad 2 \le p \le \infty, \tag{1.7}$$

continue to hold for ground states of a natural class of semiclassical pseudo-differential operators with complex valued symbols and double characteristics, approximated by complex harmonic oscillators near the double characteristics, see Figure 2.

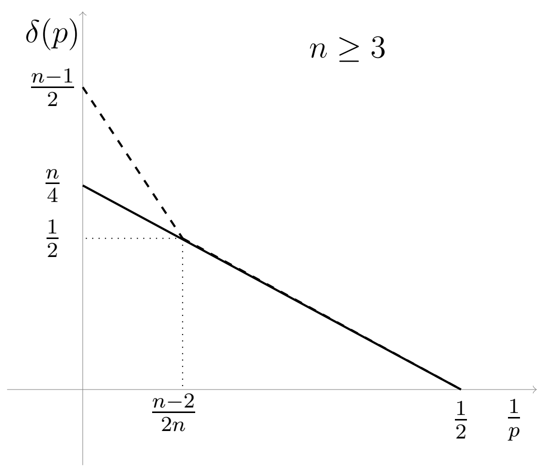


FIGURE 2. The solid lines correspond to the estimates (1.7), established in Theorem 1.1, and the dashed lines correspond to estimates (1.3) and (1.4) of [12].

Let us now describe precisely the class of operators that we are going to consider. We shall be concerned with operators of the form

$$P = \mathrm{Op}_h^w(p) \quad \text{on} \quad \mathbb{R}^n, \quad n \ge 2, \tag{1.8}$$

where $\mathrm{Op}_h^w(p)$ is the semiclassical Weyl quantization of the symbol $p = p(x,\xi)$,

$$(\mathrm{Op}_h^w(p)u)(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} p\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi. \tag{1.9}$$

Here $0 < h \le 1$ is the semiclassical parameter.

Let us state our assumptions on the symbol $p$. First we assume that $p \in C^\infty(\mathbb{R}^{2n};\mathbb{C})$ is such that

$$\partial^\alpha p \in L^\infty(\mathbb{R}^{2n}), \quad \alpha\in\mathbb{N}^{2n}, \quad |\alpha|\ge 2. \tag{1.10}$$

We assume that

$$\mathrm{Re}\, p(X) \ge 0, \quad X = (x,\xi)\in\mathbb{R}^{2n}, \tag{1.11}$$

and we also make the assumption of ellipticity at infinity for $\mathrm{Re}\, p$ in the sense that for some $C > 1$,

$$\mathrm{Re}\, p(X) \ge \frac{\langle X\rangle^2}{C}, \quad |X|\ge C. \tag{1.12}$$

Here $\langle X\rangle = \sqrt{1+|X|^2}$. Furthermore, let us assume that

$$(\mathrm{Re}\, p)^{-1}(0) = \{0\}. \tag{1.13}$$

Notice that (1.13) and (1.11) imply that

$$\nabla \mathrm{Re}\, p(0) = 0.$$

Next we assume that

$$\mathrm{Im}\, p(0) = \nabla \mathrm{Im}\, p(0) = 0,$$

so that $X = 0$ is a doubly characteristic point for the full complex valued symbol $p$. By Taylor's expansion, we write

$$p(X) = q(X) + \mathcal{O}(|X|^3), \quad \text{as} \quad |X|\to 0, \tag{1.14}$$

where

$$q(X) = \frac{1}{2}p''(0)X\cdot X,$$

and $p''$ is the Hessian of $p$. In view of (1.11), we know that $\mathrm{Re}\, q(X)\ge 0$, $X\in\mathbb{R}^{2n}$. Our final assumption is that the quadratic form $\mathrm{Re}\, q$ is positive definite, i.e.

$$\mathrm{Re}\, q(X) > 0, \quad 0\ne X\in\mathbb{R}^{2n}. \tag{1.15}$$

**Example.** As an example of an operator for which all the assumptions above are satisfied, let us consider a Schrödinger operator with a complex potential,

$$P = -h^2\Delta + V(x) + iW(x) \quad \text{on} \quad \mathbb{R}^n, \quad n\ge 2.$$

Here $V, W \in C^\infty(\mathbb{R}^n;\mathbb{R})$ are such that $\partial^\alpha V, \partial^\alpha W \in L^\infty(\mathbb{R}^n)$ for $|\alpha|\ge 2$. We assume that $V(x)\ge 0$ for $x\in\mathbb{R}^n$ and $V(x)\ge |x|^2/C$ for $|x|\ge C$. Furthermore, assume that $V^{-1}(0) = \{0\}$, $V''(0) > 0$, and $W(0) = \nabla W(0) = 0$.

Coming back to the operator $P$ in (1.8), we shall view it as a closed densely defined operator on $L^2(\mathbb{R}^n)$, equipped with the domain

$$\mathcal{D}(P) = \{u \in L^2(\mathbb{R}^n) : (-h^2\Delta + |x|^2)u \in L^2(\mathbb{R}^n)\}.$$

We notice that the inclusion map $\mathcal{D}(P) \hookrightarrow L^2(\mathbb{R}^n)$ is compact, and hence, the spectrum of $P$ is discrete.

Thanks to the works [15], [3], [7] and [8], we have complete asymptotic expansions for the eigenvalues of $P$ in an open disc $D(0, Ch)$, in fractional powers of $h$. Specifically, for any $C > 0$, there exists $h_0 > 0$ such that for all $0 < h \leq h_0$, the eigenvalues $\lambda_k$ of $P$ in $D(0, Ch)$ are given by

$$\lambda_k \sim h(\mu_k + h^{1/N_k}\mu_{k,1} + h^{2/N_k}\mu_{k,2} + \dots).$$

Here $\mu_k$ are the eigenvalues of $\mathrm{Op}_1^w(q)$ in $D(0, C)$, repeated with their algebraic multiplicity $N_k \in \mathbb{N}$.

The following is the main result of this paper, where we are concerned with estimates for eigenfunctions of $P$, corresponding to eigenvalues in the spectral region above.

**Theorem 1.1.** *Let $C > 0$ be fixed and let $\lambda \in \mathrm{Spec}(P)$ be such that $|\lambda| < Ch$. Assume that $u \in L^2(\mathbb{R}^n)$, $\|u\|_{L^2} = 1$, is such that*

$$(P - \lambda)u = 0 \quad \text{on} \quad \mathbb{R}^n, \quad n \geq 2.$$

*There exists $h_0 > 0$ such that for all $h \in (0, h_0]$, we have $u \in L^\infty(\mathbb{R}^n)$ and*

$$\|u\|_{L^\infty} \leq \mathcal{O}(1)h^{-\frac{n}{4}}. \tag{1.16}$$

*Hence, by interpolation,*

$$\|u\|_{L^p} \leq \mathcal{O}(1)h^{\frac{n}{2p}-\frac{n}{4}}, \qquad 2 \leq p \leq \infty. \tag{1.17}$$

The estimates (1.16) and (1.17) are sharp, since they are saturated by the ground states of the harmonic oscillator.

The case $n = 2$ of Theorem 1.1, when $P$ is self-adjoint, is a special case of the general results of [12] and [17].

Let us now describe the main idea of the proof of Theorem 1.1 and the plan of the paper. Heuristically, we expect the eigenfunctions $u$ of $P$, corresponding to eigenvalues $\lambda = \mathcal{O}(h)$, to be concentrated to the region where $p(x, \xi) = \mathcal{O}(h)$, so that $(x, \xi) = \mathcal{O}(h^{1/2})$. One wishes therefore to microlocalize $u$ by means of $h$-pseudodifferential operators of the form

$$\mathrm{Op}_h^w(\chi(X/h^{1/2})), \quad \chi \in C_0^\infty(\mathbb{R}^n). \tag{1.18}$$

Since the symbols $\chi(X/h^{1/2})$ are only regular on the scale $h^{1/2}$, we know from [25, Theorem 4.17] that the operators (1.18) belong to a calculus having no asymptotic expansion in powers of $h$. A suitable exotic $h^{1/2}$ calculus, involving two small parameters $0 < h \leq \widetilde{h} \ll 1$, was developed in [16], see also [4]. Here we shall not rely on this calculus explicitly but rather borrow some of its ideas and proceed as follows. First in Proposition 2.1 we establish a microlocalization of the ground state eigenfunctions of $P$ to a slightly larger region $(x,\xi) = \mathcal{O}(h^\delta)$, using the standard $h^\delta$–calculus with $0 < \delta < 1/2$. Secondly, using the sharp Gårding inequality, we get an a priori estimate for $P$, involving a microlocal cutoff, regular on the scale $(h/\widetilde{h})^{1/2}$, see Proposition 2.3. Using the a priori estimate, we obtain a uniform control in $L^2$ on

$$\mathrm{Op}_h^w(q^N(X\widetilde{h}^{1/2}/h^{1/2}))u,$$

where $q$ is the quadratic approximation of $p$ and $N$ large, see Proposition 2.4. The proof of Theorem 1.1 is concluded by a Sobolev embedding argument.

## 2. Proof of Theorem 1.1

2.1. **A rough microlocalization of the ground states.** To state our microlocalization result we have to introduce some notation. Let $m \geq 1$ be a $C^\infty$ order function on $\mathbb{R}^{2n}$, i.e. there exist $C_0 \geq 1$ and $N_0 > 0$ such that

$$m(X) \leq C_0 \langle X - Y\rangle^{N_0} m(Y), \quad X, Y \in \mathbb{R}^{2n}.$$

For $0 \leq \delta \leq \frac{1}{2}$, we consider the following symbol class,

$$\begin{aligned} S_\delta(m) = \{a(X;h) \in C^\infty(\mathbb{R}^{2n};\mathbb{C}) : \forall \alpha \in \mathbb{N}^{2n}, \exists C_\alpha > 0, \forall h \in (0,1], \\ \forall X \in \mathbb{R}^{2n}, |\partial_X^\alpha a(X;h)| \leq C_\alpha h^{-\delta|\alpha|} m(X)\}. \end{aligned}$$

We shall need the following composition formula for the Weyl quantization, see [5], [25], and [1]. If $a_1 \in S_{\delta_1}(m_1)$ and $a_2 \in S_{\delta_2}(m_2)$ with $0 \leq \delta_1, \delta_2 \leq 1/2$ and $\delta_1 + \delta_2 < 1$, then

$$\mathrm{Op}_h^w(a_1)\mathrm{Op}_h^w(a_2) = \mathrm{Op}_h^w(a_1\#a_2), \quad a_1\#a_2 \in S_{\max(\delta_1,\delta_2)}(m_1m_2), \tag{2.1}$$

and

$$(a_1\#a_2)(x,\xi) = e^{\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}(a_1(x,\xi)a_2(y,\eta))|_{\substack{y=x,\\ \eta=\xi}}$$

where

$$\sigma(D_x,D_\xi;D_y,D_\eta) = D_\xi \cdot D_y - D_x \cdot D_\eta.$$

By Taylor's formula, applied to $t \mapsto e^{\frac{iht}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}$, for any $N \in \mathbb{N}$, we have

$$\begin{aligned} (a_1\#a_2)(x,\xi) = \sum_{k=0}^N \frac{1}{k!}\Big(\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)\Big)^k (a_1(x,\xi)a_2(y,\eta))|_{\substack{y=x\\ \eta=\xi}} + \frac{1}{N!} \\ \times \int_0^1 (1-t)^N e^{\frac{iht}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}\Big(\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)\Big)^{N+1}(a_1(x,\xi)a_2(y,\eta))|_{\substack{y=x\\ \eta=\xi}} dt. \end{aligned} \tag{2.2}$$

It follows that

$$(a_1\#a_2)(x,\xi) - \sum_{k=0}^{N} \frac{1}{k!}\Big(\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)\Big)^k (a_1(x,\xi)a_2(y,\eta))|_{\substack{y=x\\ \eta=\xi}} \in h^{(N+1)(1-\delta_1-\delta_2)} S_{\max(\delta_1,\delta_2)}(m_1 m_2). \tag{2.3}$$

We shall also need the following formula from [13, p. 45], valid for $k = 1, 2, \dots,$

$$\sigma(D_x,D_\xi;D_y,D_\eta)^k (a_1(x,\xi)a_2(y,\eta))|_{\substack{y=x\\ \eta=\xi}} = \sum_{|\alpha|+|\beta|=k} (-1)^{|\alpha|} \frac{k!}{\alpha!\beta!} (\partial_\xi^\alpha \partial_x^\beta a_1(x,\xi))(\partial_x^\alpha \partial_\xi^\beta a_2(x,\xi)). \tag{2.4}$$

The main result of this subsection is as follows.

**Proposition 2.1.** *Let $C > 0$ be fixed and let $\lambda \in \mathrm{Spec}(P)$ be such that $|\lambda| < Ch$. Assume that $u \in L^2(\mathbb{R}^n)$, $\|u\|_{L^2} = 1$, is such that*

$$(P-\lambda)u = 0 \quad \text{on} \quad \mathbb{R}^n, \quad n \geq 2.$$

*Then there is $\psi \in C_0^\infty(\mathbb{R}^{2n}, [0,1])$ such that for any $0 < \delta < 1/2$, there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, we have*

$$u = \mathrm{Op}_h^w(\psi(X/h^\delta))u + Ru, \tag{2.5}$$

*where $R \in h^{M(1-2\delta)} S_\delta(\langle X\rangle^{-N})$ for any $M, N \in \mathbb{N}$.*

*Proof.* Let $\chi \in C_0^\infty(\mathbb{R}^{2n}, [0,1])$ be such that $\chi(X) = 1$ for $|X| \leq 1$ and supp $(\chi) \subset \{X \in \mathbb{R}^{2n} : |X| \leq 2\}$. Since $p$ is not elliptic near zero, to prove (2.5) we consider the symbol

$$\widetilde{p}(X;h) = p(X) - \lambda + h^{2\delta}\chi(X/h^\delta), \tag{2.6}$$

where $0 < \delta < 1/2$ is fixed, and construct a parametrix for the operator $\mathrm{Op}_h^w(\widetilde{p})$. In doing so we shall proceed similarly to the proof of the sharp Gårding inequality in [5].

First let us show that there is $C > 0$ such that

$$\mathrm{Re}\, p(X) \geq |X|^2/C, \quad X \in \mathbb{R}^{2n}. \tag{2.7}$$

Indeed, when $|X| \leq c_0$ with $c_0 > 0$ being a small but fixed constant, the estimate (2.7) follows from the quadratic approximation (1.14) together with (1.15). When $|X| \geq C_0$ with $C_0 > 0$ being a large but fixed constant, the estimate (2.7) follows from (1.12). Finally, when $c_0 \leq |X| \leq C_0$, using (1.11) and the fact that $\mathrm{Re}\, p$ vanishes only at $X = 0$, we conclude that $\mathrm{Re}\, p(X) \geq c > 0$, and hence, (2.7) follows.

Now as a consequence of (2.7), we have

$$\mathrm{Re}\, p(X) + h^{2\delta}\chi(X/h^\delta) \geq \frac{h^{2\delta}}{C}\langle X\rangle^2, \quad X \in \mathbb{R}^{2n}. \tag{2.8}$$

Indeed, when $|X|/h^\delta \ge 1$, (2.8) follows from (2.7), and when $|X|/h^\delta \le 1$, the estimate (2.8) is a consequence of (1.11) and the fact that $\chi(X/h^\delta) = 1$ in this region.

Thus, since $0 < \delta < 1/2$, there exists $h_0 = h_0(\delta) > 0$ such that for $0 < h < h_0$ we have

$$\operatorname{Re}\widetilde{p}(X;h) \ge \frac{h^{2\delta}}{C}\langle X\rangle^2, \quad X \in \mathbb{R}^{2n}. \tag{2.9}$$

We shall next estimate $\partial^\alpha(1/\widetilde{p})$. To that end, we use Faà di Bruno's formula,

$$\partial^\alpha f^{-1} = f^{-1}\sum_{k=1}^{|\alpha|}\sum_{\alpha=\beta^1+\dots+\beta^k, |\beta^j|\ge 1} C_{\beta^1,\dots,\beta^k}\prod_{j=1}^k (f^{-1}\partial^{\beta^j} f), \tag{2.10}$$

for appropriate constants $C_{\beta^1,\dots,\beta^k}$, see [25, p.94]. Using (1.10), for $|\beta| \ge 2$, we get

$$|\partial^\beta \widetilde{p}(X;h)| \le C_\beta h^{\delta(2-|\beta|)}, \quad X \in \mathbb{R}^{2n}. \tag{2.11}$$

This estimate together with (2.9) implies that for $|\beta| \ge 2$,

$$\left|\frac{\partial^\beta \widetilde{p}}{\widetilde{p}}\right| \le C_\beta h^{-\delta|\beta|}\langle X\rangle^{-2}, \quad X \in \mathbb{R}^{2n}. \tag{2.12}$$

Let $|\beta| = 1$. Here we need the following gradient estimate. Let $f : \mathbb{R}^n \to \mathbb{R}$ be $C^2$ with $f'' \in L^\infty(\mathbb{R}^n)$, and $f \ge 0$, then

$$|\nabla f(x)|^2 \le 2\|f''\|_{L^\infty(\mathbb{R}^n)} f(x), \tag{2.13}$$

see [25, Lemma 4.31]. We have therefore,

$$|\partial^\beta(\operatorname{Re}\widetilde{p})| \le C(\operatorname{Re}\widetilde{p})^{1/2}, \quad |\beta| = 1, \tag{2.14}$$

with $C > 0$ independent of $h$.

Let us now estimate the gradient of $\operatorname{Im}\widetilde{p}$. By (1.10), (1.14) and (2.7), we get

$$|\operatorname{Im} p(X)| \le C|X|^2 \le C\operatorname{Re} p(X). \tag{2.15}$$

Treating the regions $|X|/h^\delta \le 1$ and $|X|/h^\delta \ge 1$ separately and using the estimate (2.7) in the latter region, for all $0 < h$ small enough, we see that

$$\operatorname{Re} p(X) \le C\operatorname{Re}\widetilde{p}(X;h). \tag{2.16}$$

Thus, it follows from (2.15) and (2.16) that

$$C\operatorname{Re}\widetilde{p}(X;h) - \operatorname{Im} p(X) \ge 0,$$

and therefore, using (2.13), we obtain that

$$\begin{aligned} |\partial^\beta \operatorname{Im}\widetilde{p}| &= |\partial^\beta \operatorname{Im} p| \le |\partial^\beta(C\operatorname{Re}\widetilde{p} - \operatorname{Im} p)| + C|\partial^\beta \operatorname{Re}\widetilde{p}| \\ &\le C(C\operatorname{Re}\widetilde{p} - \operatorname{Im} p)^{1/2} + C(\operatorname{Re}\widetilde{p})^{1/2} \le C(\operatorname{Re}\widetilde{p})^{1/2}, \quad |\beta| = 1. \end{aligned} \tag{2.17}$$

It follows from (2.14), (2.17) and (2.9) that for all $0 < h < 1$ small enough,

$$\left|\frac{\partial^\beta \widetilde{p}}{\widetilde{p}}\right| \le C|\widetilde{p}|^{-1/2} \le Ch^{-\delta}\langle X\rangle^{-1}, \quad |\beta| = 1. \tag{2.18}$$

Combining (2.12) and (2.18), we write

$$\left|\frac{\partial^\beta \widetilde{p}}{\widetilde{p}}\right| \le Ch^{-|\beta|\delta}\langle X\rangle^{-1}, \quad |\beta| \ge 1, \quad X \in \mathbb{R}^{2n}. \tag{2.19}$$

Letting $e(X;h) = 1/\widetilde{p}$, and using (2.10) together with (2.9) and (2.19), we obtain that

$$|\partial^\alpha e| \le C_\alpha h^{-2\delta-\delta|\alpha|}\langle X\rangle^{-2}, \quad |\alpha| \ge 0, \tag{2.20}$$

i.e. $h^{2\delta} e \in S_\delta(\langle X\rangle^{-2})$.

Using (2.2) with $N = 1$ and the fact that the Poisson bracket $\{e, \widetilde{p}\} = 0$, we get

$$(e\#\widetilde{p})(x,\xi) = 1 + \frac{1}{4}\int_0^1 (1-t) e^{\frac{iht}{2}\sigma(D_x,D_\xi;D_y,D_\eta)} (ih\sigma(D_x,D_\xi;D_y,D_\eta))^2 (e(x,\xi)\widetilde{p}(y,\eta))|_{\substack{y=x\\ \eta=\xi}} dt. \tag{2.21}$$

Next we would like to determine the symbol class of the integrand in (2.21) uniformly in $t$. To that end, in view of (2.4), we first conclude from (2.20) that

$$\partial_\xi^\alpha \partial_x^\beta e(x,\xi) \in h^{-4\delta} S_\delta(\langle X\rangle^{-2}), \quad |\alpha| + |\beta| = 2, \tag{2.22}$$

and from (2.6) and (1.10) that

$$\partial_y^\alpha \partial_\eta^\beta \widetilde{p}(y,\eta) \in S_\delta(1), \quad |\alpha| + |\beta| = 2. \tag{2.23}$$

Thus, using (2.4), (2.22) and (2.23), we get

$$h^2\sigma(D_x,D_\xi;D_y,D_\eta)^2(e(x,\xi)\widetilde{p}(y,\eta)) \in h^{2-4\delta} S_\delta(\langle X\rangle^{-2}). \tag{2.24}$$

Using the fact that

$$e^{\frac{iht}{2}\sigma(D_x,D_\xi;D_y,D_\eta)} : S_\delta(\langle X\rangle^{-2}) \to S_\delta(\langle X\rangle^{-2}),$$

see [25, Theorem 4.17], and (2.24), we obtain from (2.21) that

$$e\#\widetilde{p} = 1 + h^{2-4\delta} r, \quad r \in S_\delta(\langle X\rangle^{-2}).$$

Hence,

$$\mathrm{Op}_h^w(e)\mathrm{Op}_h^w(\widetilde{p}) = 1 + h^{2-4\delta}\mathrm{Op}_h^w(r), \tag{2.25}$$

where the operator $\mathrm{Op}_h^w(r) = \mathcal{O}(1) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded for all $0 < h$ small enough, see [25, Theorem 4.23]. As $0 < \delta < 1/2$, we have

$$\|h^{2-4\delta}\mathrm{Op}_h^w(r)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < 1/2,$$

for all $0 < h$ small enough and therefore, the inverse $(1 + h^{2-4\delta}\mathrm{Op}_h^w(r))^{-1}$ exists as an operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Next using that $1 + h^{2-4\delta} r \in S_\delta(1)$ and Beals's theorem for $S_\delta(1)$, see [25, p. 176 – 177], we see that $(1 + h^{2-4\delta}\mathrm{Op}_h^w(r))^{-1} := \mathrm{Op}_h^w(q)$ is a pseudodifferential operator with $q \in S_\delta(1)$.

It follows from (2.25) that for all $0 < h$ small enough, we have

$$\mathrm{Op}_h^w(q)\mathrm{Op}_h^w(e)\mathrm{Op}_h^w(\widetilde{p}) = 1.$$

Using the composition formula (2.1), we see that

$$\mathrm{Op}_h^w(q)\mathrm{Op}_h^w(e)h^{2\delta} = \mathrm{Op}_h^w(\widetilde{q}), \quad \widetilde{q} \in S_\delta(\langle X\rangle^{-2}).$$

This together with (2.6), and the fact that $(P - \lambda)u = 0$ implies that

$$u = \mathrm{Op}_h^w(\widetilde{q})\mathrm{Op}_h^w(\chi(X/h^\delta))u. \tag{2.26}$$

Let $\psi \in C_0^\infty(\mathbb{R}^{2n}, [0, 1])$ be such that $\psi = 1$ near supp $(\chi)$ and

$$\text{supp } (\psi) \subset \{X \in \mathbb{R}^{2n} : |X| \le 3\}.$$

Then it follows from (2.26) that

$$u = \mathrm{Op}_h^w(\psi(X/h^\delta))u + Ru,$$

where

$$R = (1 - \mathrm{Op}_h^w(\psi(X/h^\delta)))\mathrm{Op}_h^w(\widetilde{q})\mathrm{Op}_h^w(\chi(X/h^\delta)).$$

Here we notice that

$$\chi(X/h^\delta) \in S_\delta(\langle X\rangle^{-N}), \quad \forall N \in \mathbb{N}, \quad \text{and} \quad 1 - \psi(X/h^\delta) \in S_\delta(1).$$

Since supp $(1 - \psi) \cap$ supp $(\chi) = \emptyset$, it follows from (2.2) that

$$R \in h^{M(1-2\delta)} S_\delta(\langle X\rangle^{-N}),$$

for any $N, M \in \mathbb{N}$. The proof is complete. □

2.2. **Applying Gårding's inequality.** We shall need the following version of the sharp Gårding inequality, see [22] and [2].

**Theorem 2.2.** *Let $a(x, \xi; h) \in C^\infty(\mathbb{R}^{2n})$ be such that $a \ge 0$ on $\mathbb{R}^{2n}$ and $\partial^\alpha a \in L^\infty(\mathbb{R}^{2n})$ for all $|\alpha| \ge 2$. Then there exist $C > 0$, depending only on $\|\partial^\alpha a\|_{L^\infty}$, $|\alpha| \ge 2$, and $h_0 > 0$ such that*

$$(\mathrm{Op}_h^w(a)u, u)_{L^2(\mathbb{R}^n)} \ge -Ch\|u\|^2_{L^2(\mathbb{R}^n)},$$

*for all $0 < h \le h_0$ and $u \in L^2(\mathbb{R}^n)$.*

We shall now establish a suitable a priori estimate for the operator $P = \mathrm{Op}_h^w(p)$. To that end, we let $0 < \widetilde{h}$ be sufficiently small but independent of $h$. We shall view $\widetilde{h}$ as a second semiclassical parameter. In order to relate the $h$–Weyl quantization and $\widetilde{h}$–Weyl quantization, following [16], we set

$$x = \sqrt{\varepsilon}\widetilde{x}, \quad \xi = \sqrt{\varepsilon}\widetilde{\xi}, \quad y = \sqrt{\varepsilon}\widetilde{y}, \quad \varepsilon = h/\widetilde{h}.$$

We obtain that

$$(\mathrm{Op}_h^w(a)u)(x) = \varepsilon^{-\frac{n}{4}}(\mathrm{Op}_{\widetilde{h}}^w(\widetilde{a})\widetilde{u})(\widetilde{x}),$$

where

$$\widetilde{a}(\widetilde{x},\widetilde{\xi}) = a(\sqrt{\varepsilon}\widetilde{x},\sqrt{\varepsilon}\widetilde{\xi}), \quad \widetilde{u}(\widetilde{x}) = \varepsilon^{\frac{n}{4}}u(\sqrt{\varepsilon}\widetilde{x}). \tag{2.27}$$

Letting

$$U : u(x) \mapsto \widetilde{u}(\widetilde{x}) = \varepsilon^{\frac{n}{4}}u(\sqrt{\varepsilon}\widetilde{x}), \tag{2.28}$$

one can easily see that $U$ is unitary on $L^2(\mathbb{R}^n)$, and we have

$$\mathrm{Op}_h^w(a) = U^{-1}\mathrm{Op}_{\widetilde{h}}^w(\widetilde{a})U. \tag{2.29}$$

We have the following consequence of Theorem 2.2.

**Proposition 2.3.** *Let $C>0$ and let $|\lambda| < Ch$. Let $\chi \in C_0^\infty(\mathbb{R}^{2n},[0,1])$ be such that $\chi(X) = 1$ for $|X| \leq 1$ and supp $(\chi) \subset \{X \in \mathbb{R}^{2n} : |X| \leq 2\}$. Then there exist $\widetilde{C} > 0$ and $\widetilde{h}_0 > 0$ such that*

$$\mathrm{Re}\,((P-\lambda)u,u)_{L^2(\mathbb{R}^n)} + \varepsilon(\mathrm{Op}_h^w(\chi(X/\sqrt{\varepsilon}))u,u)_{L^2(\mathbb{R}^n)} \geq \frac{\varepsilon}{\widetilde{C}}\|u\|^2_{L^2(\mathbb{R}^n)}, \tag{2.30}$$

*for all $0 < h \leq \widetilde{h} \leq \widetilde{h}_0$ and $u \in L^2(\mathbb{R}^n)$. Here $\varepsilon = h/\widetilde{h}$.*

*Proof.* To establish (2.30), using (2.29), we pass to the $\widetilde{h}$–Weyl quantization and get

$$P - \lambda + \varepsilon\mathrm{Op}_h^w(\chi(X/\sqrt{\varepsilon})) = \varepsilon U^{-1}\mathrm{Op}_{\widetilde{h}}^w(\widetilde{p})U, \tag{2.31}$$

where

$$\widetilde{p}(X;\varepsilon) = \frac{1}{\varepsilon}p(\sqrt{\varepsilon}X) - \frac{\lambda}{\varepsilon} + \chi(X). \tag{2.32}$$

Let us show that there is $C>0$ such that uniformly in $\varepsilon > 0$, we have

$$\frac{1}{\varepsilon}\mathrm{Re}\,p(\sqrt{\varepsilon}X) + \chi(X) \geq 1/C, \quad X \in \mathbb{R}^{2n}. \tag{2.33}$$

Indeed, when $|X| \leq 1$, the estimate (2.33) follows from (1.11) and the fact that $\chi(X) = 1$ here. When $|X| \geq 1$, (2.33) is implied by (2.7).

It follows from (2.32) and (2.33) that for $0 < \widetilde{h}$ small enough,

$$\mathrm{Re}\,\widetilde{p}(X;\varepsilon) \geq 1/C, \quad X \in \mathbb{R}^{2n},$$

uniformly in $h$. Using (1.10), for $|\alpha| \geq 2$, we get

$$|\partial^\alpha\widetilde{p}(X;\varepsilon)| \leq \frac{(\sqrt{\varepsilon})^{|\alpha|}}{\varepsilon}|(\partial^\alpha p)(\sqrt{\varepsilon}X)| + |\partial^\alpha\chi(X)| \leq C_\alpha, \quad X \in \mathbb{R}^{2n},$$

uniformly in $\varepsilon \leq 1$. Applying Theorem 2.2 to $\mathrm{Re}\,\widetilde{p}$ in the $\widetilde{h}$–Weyl quantization, we obtain that there exist $\widetilde{C} > 0$ and $\widetilde{h}_0 > 0$ such that

$$\mathrm{Re}\,(\mathrm{Op}_{\widetilde{h}}^w(\widetilde{p})u,u)_{L^2(\mathbb{R}^n)} \geq \frac{1}{\widetilde{C}}\|u\|^2_{L^2(\mathbb{R}^n)}, \tag{2.34}$$

for all $0 < h \le \widetilde{h} \le \widetilde{h}_0$ and $u \in L^2(\mathbb{R}^n)$.

Using (2.31), (2.34) and the fact that $U$ is unitary on $L^2(\mathbb{R}^n)$, we obtain that

$$\begin{aligned}\mathrm{Re}\,((P-\lambda)u,u)_{L^2(\mathbb{R}^n)} + \varepsilon(\mathrm{Op}_h^w(\chi(X/\sqrt{\varepsilon}))u,u)_{L^2(\mathbb{R}^n)}\\ = \varepsilon \mathrm{Re}\,(\mathrm{Op}_{\widetilde{h}}^w(\widetilde{p})Uu,Uu)_{L^2(\mathbb{R}^n)} \ge \frac{\varepsilon}{\widetilde{C}}\|u\|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

for all $0 < h \le \widetilde{h} \le h_0$ and $u \in L^2(\mathbb{R}^n)$. This completes the proof. □

2.3. **Testing the a priori estimate.** In what follows we shall take $\widetilde{h} > 0$ sufficiently small but fixed, i.e. independent of $h$, so that Proposition 2.3 is valid. The dependence on the parameter $\widetilde{h}$ will therefore not be indicated explicitly.

The following result obtained by combining Proposition 2.1 and Proposition 2.3 is an essential step in the proof of Theorem 1.1.

**Proposition 2.4.** *Let $C > 0$ and let $\lambda \in \mathrm{Spec}(P)$ be such that $|\lambda| \le Ch$. Assume that*

$$(P-\lambda)u = 0 \quad on \quad \mathbb{R}^n, \quad n \ge 2,$$

*$u \in L^2(\mathbb{R}^n)$, $\|u\|_{L^2} = 1$. Set $q(X) = \frac{1}{2}p''(0)X\cdot X$. Then for every $N \in \mathbb{N}$, there exists $h_0 > 0$ such that for all $0 < h \le h_0$, we have*

$$\|\mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u\|_{L^2(\mathbb{R}^n)} \le \mathcal{O}_N(1), \quad \varepsilon = h/\widetilde{h}. \tag{2.35}$$

*Proof.* First using Proposition 2.1, we see that $\mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u \in L^2(\mathbb{R}^n)$ for any $N \in \mathbb{N}$. Thus, it follows from the a priori estimate (2.30) that there is $\widetilde{C} > 0$ such that

$$\begin{aligned}\mathrm{Re}\,((P-\lambda)\mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u, \mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u)_{L^2(\mathbb{R}^n)}\\ + \varepsilon(\mathrm{Op}_h^w(\chi(X/\sqrt{\varepsilon}))\mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u, \mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u)_{L^2(\mathbb{R}^n)}\\ \ge \frac{\varepsilon}{\widetilde{C}}\|\mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u\|^2_{L^2(\mathbb{R}^n)},\end{aligned} \tag{2.36}$$

for all $0 < h$ small enough and all $N \in \mathbb{N}$.

Let us start by estimating the second term in the left hand side of (2.36). Using (2.27), (2.29), and the fact that $U$ is unitary, we have

$$\begin{aligned}(\mathrm{Op}_h^w(\chi(X/\sqrt{\varepsilon}))\mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u, \mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u)_{L^2(\mathbb{R}^n)}\\ = (\mathrm{Op}_{\widetilde{h}}^w(\overline{q}^N(X))\mathrm{Op}_{\widetilde{h}}^w(\chi(X))\mathrm{Op}_{\widetilde{h}}^w(q^N(X))Uu,Uu)_{L^2(\mathbb{R}^n)} \le \mathcal{O}_N(1)\|u\|^2_{L^2(\mathbb{R}^n)},\end{aligned} \tag{2.37}$$

for all $0 < h$ small enough and all $N \in \mathbb{N}$. Here we have used the fact that $\chi$ has a compact support, and therefore,

$$\mathrm{Op}_{\widetilde{h}}^w(\overline{q}^N(X))\mathrm{Op}_{\widetilde{h}}^w(\chi(X))\mathrm{Op}_{\widetilde{h}}^w(q^N(X)) \in \mathrm{Op}_{\widetilde{h}}^w(S(1)),$$

so that

$$\mathrm{Op}_h^w(\overline{q}^N(X))\mathrm{Op}_h^w(\chi(X))\mathrm{Op}_h^w(q^N(X)) = \mathcal{O}_N(1) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is bounded, see [25, Theorem 4.23]

Let us consider the first term in the left hand side of (2.36) and show that

$$\mathrm{Re}\,((P-\lambda)\mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u, \mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u)_{L^2(\mathbb{R}^n)} \leq \mathcal{O}_N(h)\|u\|^2_{L^2(\mathbb{R}^n)}. \tag{2.38}$$

Since $(P-\lambda)u = 0$, we get

$$\begin{aligned}((P-\lambda)&\mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u, \mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))u)_{L^2(\mathbb{R}^n)} \\ &= (\mathrm{Op}_h^w(\overline{q}^N(X/\sqrt{\varepsilon}))[P, \mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))]u, u)_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Since $q$ is quadratic, by the composition formula for the Weyl quantization (2.2) we have

$$[\mathrm{Op}_h^w(q), \mathrm{Op}_h^w(q^N)] = \frac{h}{i}\mathrm{Op}_h^w(\{q, q^N\}) = 0. \tag{2.39}$$

Letting

$$r(X) = p(X) - q(X),$$

and using (2.39), we get

$$\mathrm{Op}_h^w(\overline{q}^N(X/\sqrt{\varepsilon}))[P, \mathrm{Op}_h^w(q^N(X/\sqrt{\varepsilon}))] = \frac{1}{\varepsilon^{2N}}\mathrm{Op}_h^w(\overline{q}^N(X))[\mathrm{Op}_h^w(r), \mathrm{Op}_h^w(q^N(X))].$$

We have

$$B := \mathrm{Op}_h^w(\overline{q}^N(X))[\mathrm{Op}_h^w(r), \mathrm{Op}_h^w(q^N(X))] \in hS_0(\langle X\rangle^{4N+2}), \tag{2.40}$$

as $r \in S_0(\langle X\rangle^2)$ in view of (1.10), and $q^N \in S_0(\langle X\rangle^{2N})$.

By Proposition 2.1, there exists $\psi \in C_0^\infty(\mathbb{R}^{2n}, [0,1])$ such that for any $0 < \delta < 1/2$, we have for all $h > 0$ small enough,

$$u = \mathrm{Op}_h^w(\psi(X/h^\delta))u + Ru,$$

where $R \in h^{M_1(1-2\delta)}S_\delta(\langle X\rangle^{-M_2})$ for any $M_1, M_2 \in \mathbb{N}$. Thus,

$$\varepsilon^{-2N}BR \in h^{-2N+M_1(1-2\delta)}hS_\delta(\langle X\rangle^{4N+2-M_2}) \in hS_\delta(1),$$

provided we choose $M_1$ and $M_2$ so large that

$$M_1 \geq \frac{2N}{1-2\delta}, \quad M_2 \geq 4N+2.$$

Hence, the operator

$$\varepsilon^{-2N}BR = \mathcal{O}(h) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is bounded for $0 < h$ small enough.

Given $N \in \mathbb{N}$, let us choose $\delta$ so that

$$1/2 > \delta \geq \frac{2N}{4N+1},$$

and show that the operator

$$\varepsilon^{-2N} B\mathrm{Op}_h^w(\psi(X/h^\delta)) = \mathcal{O}(h) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \tag{2.41}$$

is bounded for $0 < h$ small enough. To that end, first letting $B = \mathrm{Op}_h^w(b)$, using the composition formula (2.3) and the fact that $\psi(X/h^\delta) \in S_\delta(\langle X\rangle^{-L})$ for any $L \in \mathbb{N}$, and (2.40), we write

$$\begin{aligned}\varepsilon^{-2N} b(x,\xi)\#\psi(x/h^\delta, \xi/h^\delta) \\ = \varepsilon^{-2N}\sum_{j=0}^{K-1}\frac{1}{j!}\frac{(ih)^j}{2^j}\sigma(D_x, D_\xi; D_y, D_\eta)^j (b(x,\xi)\psi(y/h^\delta, \eta/h^\delta))|_{y=x,\eta=\xi} + \widetilde{r},\end{aligned} \tag{2.42}$$

where

$$\widetilde{r} \in \varepsilon^{-2N} h^{K(1-\delta)} S_\delta(\langle X\rangle^{4N+2-L}),$$

for any $K \in \mathbb{N}$ and any $L \in \mathbb{N}$. Choosing

$$L \geq 4N+2 \quad \text{and} \quad K \geq 4N+2,$$

we conclude that the operator

$$\mathrm{Op}_h^w(\widetilde{r}) = \mathcal{O}(h) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is bounded for all $0 < h$ small enough.

To prove (2.41), let us determine the symbol class for the first term in the right hand side of (2.42), i.e.

$$\widetilde{b}(x,\xi) = \varepsilon^{-2N}\sum_{j=0}^{K-1}\frac{1}{j!}\frac{(ih)^j}{2^j}\sigma(D_x, D_\xi; D_y, D_\eta)^j (b(x,\xi)\psi(y/h^\delta, \eta/h^\delta))|_{y=x,\eta=\xi}. \tag{2.43}$$

Using the composition formula (2.3), (2.4), and the fact that $q$ is quadratic, we get

$$\begin{aligned} b(x,\xi) = \sum_{l=0}^{2N}\frac{(ih)^l}{2^l}\sum_{k=1}^{2N}\frac{(ih)^k}{2^k}\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{|\alpha|}}{\alpha!\beta!}\sum_{|\gamma|+|\delta|=l}\frac{(-1)^{|\gamma|}}{\gamma!\delta!}(\partial_\xi^\gamma\partial_x^\delta\overline{q}^N(x,\xi)) \\ \partial_x^\gamma\partial_\xi^\delta\big[(\partial_\xi^\alpha\partial_x^\beta r(x,\xi))(\partial_x^\alpha\partial_\xi^\beta q^N(x,\xi)) - (\partial_\xi^\alpha\partial_x^\beta q^N(x,\xi))(\partial_x^\alpha\partial_\xi^\beta r(x,\xi))\big].\end{aligned} \tag{2.44}$$

Hence, to estimate $\widetilde{b}$, we see using (2.43), (2.44), and (2.4) that we have to estimate the following terms,

$$\begin{aligned}\varepsilon^{-2N}h^{j+l+k-\delta j}\partial_\xi^\mu\partial_x^\nu\Big[(\partial_\xi^\gamma\partial_x^\delta\overline{q}^N)\partial_x^\gamma\partial_\xi^\delta\big[(\partial_\xi^\alpha\partial_x^\beta r)(\partial_x^\alpha\partial_\xi^\beta q^N) - (\partial_\xi^\alpha\partial_x^\beta q^N)(\partial_x^\alpha\partial_\xi^\beta r)\big]\Big] \\ (\partial_x^\mu\partial_\xi^\nu\psi)(X/h^\delta),\end{aligned} \tag{2.45}$$

where

$$\begin{aligned} j = 0, \ldots, K-1, l = 0, \ldots, 2N, k = 1, \ldots, 2N, \\ |\alpha| + |\beta| = k, |\gamma| + |\delta| = l, |\mu| + |\nu| = j. \end{aligned}$$

It follows from (2.45) that it is enough to estimate

$$\varepsilon^{-2N} h^{j+l+k-\delta j} \partial_X^\mu \Big[ (\partial_X^\gamma \overline{q}^N) \partial_X^\gamma \big[ (\partial_X^\alpha r)(\partial_X^\alpha q^N) \big] \Big], \tag{2.46}$$

on supp $(\psi(X/h^\delta))$, i.e. when $|X| \leq 3h^\delta$, with

$$|\alpha| = k, |\gamma| = l, |\mu| = j.$$

Using Leibniz's rule twice, we rewrite (2.46) as follows,

$$\varepsilon^{-2N} h^{j+l+k-\delta j} \sum_{\mu^1+\mu^2=\mu} C_{\mu^1,\mu^2} (\partial_X^{\mu^1+\gamma} \overline{q}^N) \Bigg( \sum_{\gamma^1+\gamma^2=\mu^2+\gamma} C_{\gamma^1,\gamma^2} (\partial_X^{\gamma^1+\alpha} r)(\partial_X^{\gamma^2+\alpha} q^N) \Bigg). \tag{2.47}$$

As $|\alpha| = k \geq 1$, we know that $|\gamma^1| + |\alpha| \geq 1$. Consider first the case $|\gamma^1| + |\alpha| = 1$. In this case

$$|\partial_X^{\gamma^1+\alpha} r| \leq \mathcal{O}(|X|^2),$$

since

$$r(X) = \mathcal{O}(|X|^3) \quad \text{near} \quad 0.$$

Therefore, using the fact that

$$|\partial_X^\beta q^N| \leq \begin{cases} \mathcal{O}(|X|^{2N-|\beta|}), & |\beta| \leq 2N, \\ 0, & |\beta| > 2N, \end{cases}$$

we estimate the absolute value of (2.47) in the case $|\gamma^1| + |\alpha| = 1$ by

$$\begin{aligned} &\leq \varepsilon^{-2N} h^{j+l+k-\delta j} \mathcal{O}(|X|^{4N-j-2l-k+|\gamma^1|}) \mathcal{O}(|X|^2) \\ &\leq \varepsilon^{-2N} \mathcal{O}(h h^{\delta(4N+1)} h^{(1-2\delta)(j+l+k-1)}) \leq \varepsilon^{-2N} \mathcal{O}(h h^{\delta(4N+1)}) \leq \mathcal{O}(h). \end{aligned} \tag{2.48}$$

Here we have used that $4N - j - 2l - k + |\gamma^1| \geq 0$ and $1/2 > \delta \geq \frac{2N}{4N+1}$. Similarly, using that

$$|\partial_X^{\gamma^1+\alpha} r| \leq \mathcal{O}(|X|) \quad \text{when} \quad |\gamma^1| + |\alpha| = 2,$$

and

$$|\partial_X^{\gamma^1+\alpha} r| \leq \mathcal{O}(1) \quad \text{when} \quad |\gamma^1| + |\alpha| \geq 3,$$

we obtain the estimate (2.48) also in the case when $|\gamma^1| + |\alpha| \geq 2$. Hence, we get

$$|\widetilde{b}(x,\xi)| \leq \mathcal{O}(h).$$

To estimate the derivatives $\partial_X^\rho \widetilde{b}(X)$, $|\rho|\ge 1$, arguing as above and using Leibniz's rule one more time, we conclude that we have to estimate

$$\varepsilon^{-2N} h^{j+l+k-\delta j-|\rho_2|\delta} \partial_X^{\rho_1+\mu}\Big[(\partial_X^\gamma \overline{q}^N)\partial_X^\gamma\left[(\partial_X^\alpha r)(\partial_X^\alpha q^N)\right]\Big], \tag{2.49}$$

on supp $(\psi(X/h^\delta))$, with

$$|\rho|=|\rho_1|+|\rho_2|, |\alpha|=k, |\gamma|=l, |\mu|=j.$$

Similarly to (2.47), we write (2.49) as follows,

$$\varepsilon^{-2N} h^{j+l+k-\delta j-|\rho_2|\delta} \sum_{\mu^1+\mu^2=\rho_1+\mu} C_{\mu^1,\mu^2}(\partial_X^{\mu^1+\gamma}\overline{q}^N) \Big(\sum_{\gamma^1+\gamma^2=\mu^2+\gamma} C_{\gamma^1,\gamma^2}(\partial_X^{\gamma^1+\alpha} r)(\partial_X^{\gamma^2+\alpha} q^N)\Big).$$

Therefore, using that $4N-|\rho_1|-j-2l-k+|\gamma^1|\ge 0$, we get

$$\begin{aligned}|\partial_X^\rho \widetilde{b}(x,\xi)| &\le \varepsilon^{-2N} h^{j+l+k-\delta j-|\rho_2|\delta}|\partial_X^{\gamma^1+\alpha} r|\mathcal{O}(|X|^{4N-|\rho_1|-j-2l-k+|\gamma^1|})\\ &\le h^{-\delta|\rho|}\mathcal{O}(h^{-2N} h h^{\delta(4N+1)} h^{(1-2\delta)(j+l+k-1)}) \le h^{-\delta|\rho|}\mathcal{O}(h),\end{aligned}$$

since $1/2>\delta\ge \frac{2N}{4N+1}$. Hence,

$$\widetilde{b}\in hS_\delta(1),$$

and thus, (2.41) and (2.38) follow.

The estimate (2.35) follows from (2.36), (2.37) and (2.38). The proof is complete. □

2.4. **Concluding the proof of Theorem 1.1.** Let $N\in\mathbb{N}$ be fixed. Then by Proposition 2.4 and scaling (2.29), we have

$$\|\mathrm{Op}_h^w(q^N(X))Uu\|_{L^2(\mathbb{R}^n)}\le \mathcal{O}(1), \tag{2.50}$$

for all $0<h$ small enough. Now it is convenient to make an additional scaling to pass to the case $\widetilde{h}=1$. By (2.29) and the homogeneity of $q^N$, we have

$$\mathrm{Op}_h^w(q^N)=\widetilde{h}^N V^{-1}\mathrm{Op}_1^w(q^N)V,$$

where

$$(Vu)(\widetilde{x})=(\widetilde{h})^{\frac{n}{4}}u(\sqrt{\widetilde{h}}\widetilde{x}).$$

Hence, in the remainder of the proof we may assume that $\widetilde{h}=1$.

We have $q^N(X)\in S_X^{2N}(\mathbb{R}^{2n})$. Here

$$S_X^m(\mathbb{R}^{2n})=\{a(X)\in C^\infty(\mathbb{R}^{2n};\mathbb{C}):\forall a\in\mathbb{N}^{2n},\exists C_\alpha>0, |\partial^\alpha a(X)|\le C_\alpha\langle X\rangle^{m-|\alpha|}\},$$

see [14, Section 23.1]. Using the fact that $\operatorname{Re} q(X)$ is a positive definite quadratic form, we get

$$|q^N(X)| \geq (\operatorname{Re} q(X))^N \geq |X|^{2N}/C, \quad X \neq 0.$$

It follows from [14, Theorem 25.1] that there is $b \in S_X^{-2N}(\mathbb{R}^{2n})$ such that

$$\operatorname{Op}_1^w(b)\operatorname{Op}_1^w(q^N) - I = R, \tag{2.51}$$

where the operator $R$ has a kernel in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$, and therefore,

$$R : \mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{S}(\mathbb{R}^{2n}). \tag{2.52}$$

Here $\mathcal{S}'(\mathbb{R}^{2n})$ is the space of tempered distributions.

Let $s \in \mathbb{R}$ and let

$$\mathcal{H}^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \operatorname{Op}_1^w((1 + |x|^2 + |\xi|^2)^{s/2})u \in L^2(\mathbb{R}^n)\}.$$

We know that

$$\operatorname{Op}_1^w(b) : L^2(\mathbb{R}^n) \to \mathcal{H}^{2N}(\mathbb{R}^n) \tag{2.53}$$

is bounded, see [14, Theorem 25.2]. It follows from (2.51), (2.50), (2.52) and (2.53) that

$$\|Uu\|_{\mathcal{H}^{2N}(\mathbb{R}^n)} \leq \|\operatorname{Op}_1^w(b)\operatorname{Op}_1^w(q^N)Uu\|_{\mathcal{H}^{2N}(\mathbb{R}^n)} + \|RUu\|_{\mathcal{H}^{2N}(\mathbb{R}^n)} \leq \mathcal{O}(1), \tag{2.54}$$

for all $0 < h$ small enough.

Choosing $N > n/4$ and using the fact that $\mathcal{H}^{2N}(\mathbb{R}^n) \subset H^{2N}(\mathbb{R}^n)$, the standard Sobolev space, together with the Sobolev embedding $H^{2N}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, we get

$$\|Uu\|_{L^\infty(\mathbb{R}^n)} \leq \mathcal{O}(1).$$

Hence, recalling (2.28), we obtain that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq \mathcal{O}(1)h^{-n/4}.$$

This completes the proof of Theorem 1.1.

**Remark 2.5.** *The estimate* (2.54) *also shows that for any $K \in \mathbb{N}$, there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, we have*

$$\left\| \left(\frac{x}{h^{1/2}}\right)^\alpha (h^{1/2}\partial_x)^\beta u(x) \right\|_{L^\infty(\mathbb{R}^n)} \leq \mathcal{O}_K(h^{-n/4}),$$

*for all $\alpha, \beta \in \mathbb{N}^n$, $|\alpha + \beta| \leq K$.*

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K. Krupchyk, Department of Mathematics, University of California, Irvine, CA 92697-3875, USA

*E-mail address*: katya.krupchyk@uci.edu

G. Uhlmann, Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA

*E-mail address*: gunther@math.washington.edu