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On the Classification of Unstable First-Order Theories

by

Scott Mutchnik

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Thomas Scanlon, Chair
Professor Theodore Slaman
Associate Professor Pierre Simon

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Abstract

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Professor Thomas Scanlon, Chair

We discuss the classification of unstable theories in first-order logic.

In chapter 1, we initiate the study of a generalization of Kim-independence, *Conant-independence*, based on the notion of *strong Kim-dividing* of Kaplan, Ramsey and Shelah. We introduce an axiom on stationary independence relations essentially generalizing the "freedom" axiom in some of the *free amalgamation theories* of Conant, and show that this axiom provides the correct setting for carrying out arguments of Chernikov, Kaplan and Ramsey on NSOP_1 theories relative to a stationary independence relation. Generalizing Conant's results on free amalgamation to the limits of our knowledge of the NSOP_n hierarchy, we show using methods from Conant as well as our previous work that any theory where the equivalent conditions of this local variant of NSOP_1 holds is either NSOP_1 or SOP_3 and is either simple or TP_2, and observe that these theories give an interesting class of examples of theories where Conant-independence is symmetric, including all of Conant's examples, the small cycle-free random graphs of Shelah and the (finite-language) ω-categorical Hrushovski constructions of Evans and Wong.

We then answer a question of Conant, showing that the generic functional structures of Kruckman and Ramsey are examples of non-modular free amalgamation theories, and show that any free amalgamation theory is NSOP_1 or SOP_3, while an NSOP_1 free amalgamation theory is simple if and only if it is modular.

Finally, we show that every theory where Conant-independence is symmetric is NSOP_4. Therefore, symmetry for Conant-independence gives the next known neostability-theoretic dividing line on the NSOP_n hierarchy beyond NSOP_1. We explain the connection to some established open questions.

In chapter 2, we exhibit a connection between geometric stability theory and the classification of unstable structures at the level of simplicity and the NSOP_1-SOP_3 gap. Particularly, we introduce generic expansions T^R of a theory T associated with a definable relation R

of T, which can consist of adding a new unary predicate or a new equivalence relation. When T is weakly minimal and R is a ternary fiber algebraic relation, we show that T^R is a well-defined NSOP_4 theory, and use one of the main results of geometric stability theory, the *group configuration theorem* of Hrushovski, to give an exact correspondence between the geometry of R and the classification-theoretic complexity of T^R. Namely, T^R is SOP_3, and TP_2 exactly when R is geometrically equivalent to the graph of a type-definable group operation; otherwise, T^R is either simple (in the predicate version of T^R) or NSOP_1 (in the equivalence relation version.) This gives us new examples of strictly NSOP_1 theories.

In chapter 3, we prove the following fact:

NSOP_1 is equal to NSOP_2.

This answers an open question, first formally posed by Džamonja and Shelah in 2004, but attested in notes of Shelah based on lectures delivered at Rutgers University in fall of 1997.

In chapter 4, we prove some results about the theory of independence in NSOP_3 theories that do not hold in NSOP_4 theories. We generalize Chernikov's work on simple and co-simple types in NTP_2 theories to types with NSOP_1 induced structure in N-ω-DCTP_2 and NSOP_3 theories, and give an interpretation of our arguments and those of Chernikov in terms of the characteristic sequences introduced by Malliaris. We then prove an extension of the independence theorem to types in NSOP_3 theories whose internal structure is NSOP_1. Additionally, we show that in NSOP_3 theories with symmetric Conant-independence, finitely satisfiable types satisfy an independence theorem similar to one conjectured by Simon for invariant types in NTP_2 theories, and give generalizations of this result to invariant and Kim-nonforking types.

In chapter 5, we show that approximations of strict order can calibrate the fine structure of genericity. Particularly, we find exponential behavior within the NSOP_n hierarchy from model theory. Let $\mathop{\smile\!\!\!\!\!\mid}^{\eth^0}$ denote forking-independence. Inductively, a formula $(n+1)$-$\eth$-*divides* over M if it divides by every $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$-Morley sequence over M, and $(n+1)$-$\eth$-*forks* over M if it implies a disjunction of formulas that $(n+1)$-$\eth$-divide over M; the associated independence relation over models is denoted $\mathop{\smile\!\!\!\!\!\mid}^{\eth^{n+1}}$. We show that a theory where $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$ is symmetric must be $\mathrm{NSOP}_{2^{n+1}+1}$. We then show that, in the classical examples of $\mathrm{NSOP}_{2^{n+1}+1}$ theories, $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$ is symmetric and transitive; in particular, there are strictly $\mathrm{NSOP}_{2^{n+1}+1}$ theories where $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$ is symmetric and transitive, leaving open the question of whether symmetry or transitivity of $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$ is equivalent to $\mathrm{NSOP}_{2^{n+1}+1}$.

To those whom we entrust to merge the syntax to the semantics.

Contents

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Introduction

In this dissertation, we investigate *first-order logic*. Able to express the conjunction and negation of statements and to quantify over all individuals in a given domain, first-order logic allows us to discuss some of the basic concepts of our natural language in a rigorous way. However, its power is not absolute: it cannot quantify over properties. This is a constraint we deliberately impose on ourselves. From the viewpoint of the methodology we will use in this thesis—that of *model theory*—conjunction and negation alone may limit the reach of our investigation. But so will a logic that is able to express too much; as Quine famously put it in [93], full second-order logic no longer falls within the scope of model theory, but is rather set theory in disguise[1]. In analyzing the limits of what a logical system can express, the richness of its structure comes into focus. On the other hand, a language that is able to say too much*from the point of view of our methodology* will, like Borges's Library of Babel[2], end up saying nothing at all.

In model theory, the topic of *classification theory* aims to make sense of the expressive limits of a mathematical structure's logical theory—especially within first-order logic. By *classifying* structures according to their logical complexity, it often seeks to better understand the semantic properties of a structure alongside those of the language itself. Historically, classification theory arose out Morley's celebrated categoricity theorem in [82]—which holds that a structure categorical in one uncountable cardinal is categorical in any uncountable cardinal—and was developed by Shelah ([97]) in the context of the problem of determining the number of non-isomorphic models of a first-order theory. With the depth of its implications for the semantics of models, it is surprising that so much of the classification itself can be stated in terms of the syntactic properties of formulas. For example, the cornerstone of this classification, stability, is just the absence of a formula with the order property: a definable relation $R(x, y)$ with $\{a_i\}$, $\{b_i\}$ so that $\models R(a_i, b_j)$ if and only if $i < j$. By the compactness of first-order logic, this can easily be restated without any mention of sequences within a model. This absence of order within the syntax is all that is needed to get an independence relation that is *stationary* and a geometric structure on stable theories that underlies Morley and Shelah's classification according to non-isomorphic models. This independence relation, forking-independence, is analogous to linear independence in vector spaces and algebraic independence in algebraically closed fields, which in fact coincide with forking-independence

[1]See [108], [107] for some elaboration on this point.
[2]from a short story in the collection, "The Garden of Forking Paths."

in those respective theories. It is the point of origin for the use of model theory in studying the geometry of a general mathematical structure, an approach that extends well beyond the stable case into surprisingly complex mathematical theories.

This is one of the main reasons for classification theory's contemporary salience: its power as a tool for analyzing the *semantics* of a mathematical structure, and analyzing it, specifically, through a geometric lens, even outside of any of the concrete contexts classically associated with geometry. It lies at the core of why model theory has been so aptly described as "algebraic geometry minus fields" ([47]). While classification theory has evolved from its roots within classical stability theory to the more imperial ambitions of a "geography of tame mathematics," the influence of stability theory is evident in the classification theory of unstable structures, particularly in its geometric content. Indeed, much of the elegance of classification theory lies in its interactions with the field of *neostability theory*. Neostability theory combines generalizations of stability theory with applications of stability theory, aiming to develop the algebro-geometric picture of model theory through combinatorial means. It includes *geometric stability theory*, exposited in [87], which looks in particular at the geometry imposed by closure operators on stable theories, often satisfying the predimension axioms. A central result of geometric stability theory, the group configuration theorem of Hrushovski ([48]), says that the incidence pattern of four lines in a projective plane, within the geometry imposed by the model-theoretic algebraic closure in a stable theory, always arises from a type-definable group. Despite referring only to stable theories, we will see that the group configuration theorem admits a precise correspondence to the classification theory of *unstable* theories—in fact, to classification theory more than one step beyond the level of stability.

Outside of geometric stability theory as classically construed, neostability theory interacts more directly with the classification of unstable structures. This classification proceeds though multiple *dividing lines* which measure the complexity of first-order theories, forming a "map" of the mathematical universe much of which is visualized at [30]. Like stability, many of these dividing lines have entirely syntactic statements in terms of the combinatorics of formulas. However, also in common with stability, some of the dividing lines for theories also carry geometric information, if not equivalent statements in terms of the geometric properties of their models. The original geometric structure on stable theories, forking-independence, is further developed by Kim and Pillay for simple theories, defined as theories none of whose formulas have the tree property. In [61], Kim shows that forking-independence in simple theories is symmetric—in fact, symmetry of forking-independence characterizes simple theories–and in [65], Kim and Pillay prove a generalization of stationarity for forking-independence to simple theories, where forking-independence is no longer stationary, but satisfies an amalgamation property, the *independence theorem*. Beyond even simplicity, the strong order property hierarchy of Shelah, defined in [97] for levels $n \geq 3$ and extended to $n = 1, 2$ with definitions first formally introduced in [40], gives a further set of syntactic dividing lines which approximate, at different levels, a definable linear order:

Definition 0.0.1. *A theory T is* NSOP_1 *if there does not exist a formula $\varphi(x, y)$ and tuples*

$\{b_\eta\}_{\eta \in 2^{<\omega}}$ *so that* $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n \in \omega}$ *is consistent for any* $\sigma \in 2^\omega$*, but for any* $\eta_2 \trianglerighteq \eta_1 \frown \langle 0 \rangle$*,* $\{\varphi(x, b_{\eta_2}), \varphi(x, b_{\eta_1 \frown \langle 1 \rangle})\}$ *is inconsistent. Otherwise it is* SOP_1*.*

Definition 0.0.2. *A theory* T *is* NSOP_2 *if there does not exist a formula* $\varphi(x, y)$ *and tuples* $\{b_\eta\}_{\eta \in 2^{<\omega}}$ *so that* $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n \in \omega}$ *is consistent for any* $\sigma \in 2^\omega$*, but for incomparable* η_1 *and* η_2*,* $\{\varphi(x, b_{\eta_1}), \varphi(x, b_{\eta_2})\}$ *is inconsistent. Otherwise it is* SOP_2*.*

Definition 0.0.3. *Let* $n \geq 3$*. A theory* T *is* NSOP_n *(that is, does not have the* n-strong order property*) if there is no definable relation* $R(x_1, x_2)$ *with no* n*-cycles, but with tuples* $\{a_i\}_{i \in \omega}$ *with* $\models R(a_i, a_j)$ *for* $i < j$*. Otherwise it is* SOP_n*.*

The geometric side of the strongest property, NSOP_1, was developed by Kaplan and Ramsey in their work on *Kim-independence* ([52]). In place of forking-independence, Kaplan and Ramsey develop a new notion of forking-independence "at a generic scale," showing, in conjunction with work of Chernikov and Ramsey in [28], that NSOP_1 is characterized by symmetry of this new independence relation as well as by the independence theorem for this new relation.

Beyond this, in [32], Conant implicitly invents the concept of forking-independence "at a *maximally* generic scale" by considering forking-independence relative to a synthetic relation of "free amalgamation" satisfying certain axioms. However, in the context of Conant's work, this new geometric structure turns out to be trivial, in the sense that it is submerged beneath the algebraic closure. Part of the definition of this independence relation also makes an appearance in the "strong Kim-dividing" of Kaplan, Ramsey, and Shelah [54], who do not study it as an independence relation. Only in this thesis is the relation reified as a geometric concept and shown to coincide with previously "ad hoc" relations on concrete structures, such as the "d-independence" of [41], [42]. Prior to this thesis, our knowledge of the geometric information conveyed by the strong order hierarchy remained limited beyond NSOP_1[3]. We will show that the semantic reach of classification theory and its accompanying model-theoretic geometry goes farther than previously thought.

On the other hand, another reason for the continued significance of classification theory can in one sense be thought of as independent of the semantics. The saturation of an ultraproduct of a structure by a regular ultrafilter depends only on its theory, and not on the particular model, yielding an order on first-order theories. This order, the *Keisler order*, yields an entirely different approach for classifying theories: rather than analyzing the internal structure of their models, we rank their complexity in comparison with other theories. While we have only begun to understand the interaction between Keisler's order and the classification-theoretic dividing lines (as Malliaris and Shelah point out, "Keisler's order is not simple, and simple theories may not be either," [80]), what we do know is significant. For example, in celebrated work of Malliaris and Shelah ([79]), it is shown that SOP_2 theories are maximal in Keisler's order. Though it is open whether maximality in

[3]But see [81], where the analysis of "higher formulas" in NSOP_2 and NSOP_3 theories illuminates the structure of these theories, as well as laying the groundwork for the theory of independence of NSOP_1 theories in [52].

Keisler's order gives a complete characterization of SOP_2, [40] [102] [81] do give us, under mild set-theoretic assumptions, a complete characterization of SOP_2, as the maximal class in the related *interpretability order*. Though the semantic side of classification theory lies closer to our actual methods than this comparative classification in terms of complexity orders, we highlight it in order to motivate one of the key results of this thesis.

Motivation

The semantic and geometric considerations discussed above, as well as the interactions with ultrafilters and comparative complexity, both offer compelling motivations to study classification theory. Yet the limits of our classification-theoretic knowledge extend to a more fundamental level: the identity of the classification-theoretic properties themselves. Among the properties of Shelah's strong order hierarchy, it remains open whether the tree property SOP_2 is the same as the order property SOP_3. Until recently, it was also open whether the tree properties SOP_2 and SOP_1 are equal. This question was first formally posed by Džamonja and Shelah in [40], but can be found in notes of Shelah on lectures at Rutgers University as far back as Fall of 1997 ([98]). Given the connection between NSOP_1 (the negation of SOP_1) and geometry, and the connection between SOP_2 and ultrafilters, finding that these classes line up would suggest that the interactions between structure and complexity are deeper than previously thought.

Arguably our most startling finding in this dissertation is that these classes do line up:

NSOP_1 is equal to NSOP_2.

There is precedent for results concerning the identity of classification-theoretic properties: for example, Shelah, in [97], shows that theories with both NSOP_2 and NTP_2 are simple, and that theories that are NSOP and lack the independence property are stable. It is also known that $\mathrm{NSOP}_2 = \mathrm{NTP}_1$ ([2]), that weak k-TP_1 implies SOP_1 ([64]), and that $\mathrm{TP}_1 = k - \mathrm{TP}_1$ ([28]). For a full overview of the implications and equivalences between dividing lines, see [30]. Our result that $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$ is of a different nature from the other known equivalences and implications, for two reasons.

First of all, the established identities are all of a *quantifier-free* kind. For example, an unstable formula, every boolean combination of which is NSOP, has the independence property. Likewise, if every boolean combination of a formula is NTP_1 and NTP_2, it is simple, some conjunction of instances of any k-TP_1 formula must be TP_1, and so on. By contrast, the equivalence of NSOP_1 and NSOP_2 for *theories* does not say that an SOP_1 *formula* must have a Boolean combination that is SOP_2. In fact, Ahn and Kim produce an SOP_1 formula, no conjunction of instances of which is SOP_2. Since a standard indiscernibility argument shows that a disjunction of NSOP_2 formulas is NSOP_2, and the negation of Ahn and Kim's formula does not create any nontrivial inconsistency, their proof can be modified to show that no boolean combination of their SOP_1 formula is SOP_2. In this sense, the equivalence of SOP_1 and SOP_2 is a result about the global structure of theories, even as it influences the local combinatorics of formulas—though see Chapter 4 of this thesis for a discussion of how the SOP_2 formula we obtain is related to the original SOP_1 formula. In the terminology of Juliette Kennedy, we see that the classification of structures exhibits a degree of "logical entanglement" ([56]) with first-order logic—with the existential quantifier—that is not apparent in prior results.

Kennedy defines *logical entanglement* in the context of formalizing our natural-language concepts of mathematical objects and conversely, attempting to understand these concepts outside of any formalism. She defines logical entanglement as "the fact that certain canonical mathematical objects are remarkably sensitive to slight perturbations of syntax and logic," and contrasts this with *formalism freeness*, "the idea that certain canonical concepts and constructions are stable across a variety of conceptually distinct formalisations." In Kennedy's own work on mathematical logic, or more specifically *logics*, plural, she exhibits both logical entanglement and formalism freeness with respect to *strengthenings* of first-order logic. She discusses her joint work with Magidor and Väänänen on the set-theoretic constructible universe in various logics, with the goal of "implementing" Gödel's program of finding an "absolute" or "formalism-independent" concept of definability. In [57], they show that when the first-order definability in the construction of the constructible universe L is replaced with many other logics that are significantly stronger than first-order, the same model L of set theory is obtained; However, adding the cofinality quantifier produces models of set theory that differ from L, despite the logic satisfying compactness and the Löwenhein-Skolem property for $\aleph_1$.

In model theory, formalism freeness and logical entanglement become significant in the opposite direction, where one weakens first-order logic rather than strengthening it. Kennedy cites AECs, where one attempts to eliminate logical syntax altogether, as an example of formalism freeness in model-theoretic practice, but even classical model theory makes recourse to formalism freeness. While the existential quantifier makes first-order logic into a richer object of study, an important tool for handling this additional complexity, especially in applied contexts, is often to eliminate it. Using quantifier elimination, model theorists can often reduce the logical problems posed by model theory to classically mathematical problems in the (quantifier-free) language of a particular mathematical structure, such as the algebraically closed field. For example, to show a particular theory, such as the algebraically closed fields with a generic additive subgroup ([37], [36]), is simple or NSOP_1, one simply gives a description of the types in terms of a natural quantifier-free language (such as the algebraically closed field structure with a predicate for the additive subgroup) and then finds a relation between structures in this language with the correct amalgamation properties. (See the "Kim-Pillay" characterization of simplicity and NSOP_1, [65], [52]). Implicitly, what one is doing is showing that the problem of finding this relation is the same, whether construed in the full first-order logic, or just in the quantifier-free language for the structure. As Kennedy observes in the general case of stability theory, where one no longer eliminates quantifiers in a particular language but still tends to refer only to the geometric structure of a stable theory (such as closure operators and forking-independence), the "framework... is spelled out with only passing reference to formal languages and their properties." Baldwin's remark about "focusing on a specific vocabulary, designed for the topic rather than a global framework," ([9]) cited by Kennedy as a description of model theorists' approach to logical foundations, applies even in the classical setting, where a well-chosen mathematical language can eliminate the first-order logic of a given theory.

Outside of the applied context of quantifier elimination, model theory has seen success in analyzing formulas at a local, combinatorial level, independently of any quantifiers. For example, Malliaris and Shelah give a structure theorem specializing the Szemerédi regularity lemma to a graph with a stable edge relation ([75]), and Malliaris reduces the saturation requirements of Keisler's order to the saturation of types in a single formula ([76]). Likewise, in our prior account of the model-theoretic dividing lines themselves, the existential quantifier has been submerged. As discussed above, instead of saying, for example, that a unstable theory either has the strong order property or the independence property, one could just as well have said that the same about the quantifier-free formulas of a model of that theory, or even rephrased the statement of this result to talk about the combinatorics of particular binary relations. By contrast, whether $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$ depends on whether these refer to properties of the quantifier-free formulas of a theory, or properties of the entire first-order theory. Under a suitable formulation of the classification-theoretic dividing lines to vary the underlying logic, one can imagine a parametrized account of classification theory much like Kennedy, Magidor and Väänänen's account of the constructible universe in [57], where NSOP_1 is distinct from NSOP_2 in a logic with only Boolean combinations, but the two merge together in full first-order logic. Classification theory, even at the level of which dividing lines are which, is deeply entangled with the existential quantifier—though not all of the equivalences or implications in the model-theoretic "map" can see this.[4]

Another key, and related, difference between this result and other classification-theoretic identities is a methodological one. We prove the result using techniques from stability theory—or in more contemporary terminology, using geometric techniques from neostability theory. Much like the stability theorist or simplicity theorist develops forking-independence as a way of analyzing theories geometrically, and the specialist in NSOP_1 theories characterizes them in terms of Kim-independence, we start by translating the syntactic property NSOP_2 into a semantic theory of independence. The resulting structure theory for NSOP_2 theories will be similar to that of the *free amalgamation theories* described by Conant in [32], where he shows modular free amalgamation theories are simple or SOP_3. Using a similar technique, we show that NSOP_2 theories are either NSOP_1 or SOP_3. Because NSOP_2 theories are already NSOP_3, they must be NSOP_1.

This proof represents the *on-again-off-again-ism*, the "localized, dynamic and transient use of metamathematical ideas in logico-mathematical practice," described by Kennedy in [56]. Citing Baldwin's comments in [9] that "we approach global mathematical issues not by

[4]In the other direction, Väänänen observed that in Henkin second-order logic, even the theory of equality is unstable ([56]). A very expressive logic may often be too expressive from the perspective of classification theory.

seeking a common foundation but by finding common themes and tools for various areas, not in terms of the topic studied, but in terms of common combinatorial and geometric features isolated by formalizations in each area," Kennedy emphasizes the ad-hoc, *instrumental* role of logical foundations within model theorists' mathematical epistemology. Importantly, the effect of foundations can go in the direction from semantics to syntax, or vice versa. As "an example in which *semantics* serves as a waystation to proving now a *syntactic* result," ([56]) Kennedy points to Zilber's proof in [114] of the categoricity of pseudo-exponential fields, which have a formal axiomatization in a strengthening of first-order logic but are proven categorical using entirely geometric considerations. Again, Kennedy notes that the geometric properties of stable theories "can, arguably, be listed without any reference to the syntax and semantics of first-order logic" ([56]) and this is also a possibility not just for Zilber's examples, but for constructions like forking-independence in simple theories and Kim-independence in NSOP_1 theories. However, this abstraction away from the syntactic is not absolute: an important property of the Kim-Pillay characterization of simple or NSOP_1 theories in terms of abstract independence relations ([65], [52]) is "strong finite character," which says that dependence must always be witnessed by a formula.

Like the case study of Zilber's result, our proof that $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$ exemplifies this semantics-to-syntax direction of on-again-off-again-ism, starting with one syntactic property, NSOP_2, and developing a structure theory in order to arrive at another syntactic property, NSOP_1. Even with precedent for semantic arguments in the analysis of formal syntax, the reach of on-again-off-again-ism within the classification-theoretic context surprised us. A ubiquitous tool for studying syntactic properties in classification theory is the existence of indiscernible sequences and indiscernible trees, which allow us to study formulas and their Boolean combinations at the combinatorial level; some important recent developments in the theory of tree indiscernibilities are due to [96] and [105]. While interactions between syntactic classification-theoretic properties on one side and their semantic implications for theories on the other are well-established, it was not initially clear to us that semantics, at the level of a geometric theory of independence, was relevant to the relationship between the syntactic properties themselves, which had no *explicit* reference to the global properties of the structure at all. Of course, the fact that every Boolean combination of Ahn and Kim's SOP_1 formula ([3]) is NSOP_2 suggested that any proof of the equivalence of NSOP_1 and NSOP_2 at the level of theories would require heavily semantic methods, cutting to the core of what Kennedy calls the "fragility of the syntax/semantic distinction" ([56]).

Outline

In Part I of this thesis, we investigate the connection between free amalgamation relations and classification theory, in particular the problems of whether NSOP_1 equals NSOP_3, and whether NSOP_n theories without TP_2 are simple for $n > 2$. In addition to the geometric independence relations arising from the classification theory itself, many classification-theoretically "tame" theories have additional abstract relations giving a canonical or "free" amalgamation construction for sets. As observed in [42] and [32], these free amalgamation relations have classification-theoretic implications: a modular free amalgamation theory, or a theory obtained by the Fraïssé-Hrushovski construction, must be simple or SOP_3. Moreover, a free amalgamation theory or theory obtained by the Fraïssé-Hrushovski construction must be NSOP_4, and simple or TP_2[5].

In Chapter 1, we observe that Conant's argument can be modified to show that any free amalgamation theory is NSOP_1 or SOP_3, and that, answering Question 7.19 of Conant in [32], there is a non-modular free amalgamation theory, which will be strictly NSOP_1; in fact an NSOP_1 free amalgamation theory will be strictly NSOP_1 (i.e. non-simple) if and only if it is non-modular. So we will have improved Conant's results to a true partial result on the NSOP_1 vs. NSOP_3 problem. In fact, we will be able to isolate two structural properties, with no known counterexamples among the NSOP_4 theories (which contain the NSOP_3 theories), such that a theory with both of these properties must be NSOP_1 or SOP_3. This explains the difficulty of the NSOP_1 vs. SOP_3 problem, and gives a potential strategy for a positive solution.

The two properties we isolate with no known NSOP_4 counterexamples, the strong witnessing property and symmetry for Conant-independence, turn out to be related by a generalization of Conant's free amalgamation axioms in [32]. These generalized axioms include some weak assumptions on an abstract stationary independence relation, together with a generalization of Conant's freedom axiom. They cover not only Conant's free amalgamation theories, but the ω-categorical Hrushovski constructions of [41], [42], which did not fit into Conant's original axioms in [32]. Following Lemma 7.6 of [32], we develop the theory of Kim-independence relative to Morley sequences in these stationary independence relations. By relativizing the arguments about Kim-independence in NSOP_1 theories from [52] to an independence relation with these properties, we show that symmetry of the *relative* notion of Kim-independence is equivalent to a relative version of Kim's lemma, even outside of NSOP_1. When these equivalent conditions hold for a stationary independence relation satisfying the generalized freedom axiom, a theory has the strong witnessing property, which is defined to be a generalization of Kim's lemma, as well as symmetric Conant-independence. Generalizing Conant's arguments in [32], these properties, as noted above, imply NSOP_1 or SOP_3, and the strong witnessing property implies TP_2 or simplicity. The fact that there are no known NSOP_4 theories without symmetric Conant-independence and the strong witnessing property gives evidence that all NSOP_4 theories, and thus all theories, must be either SOP_3 or NSOP_1, and that all NSOP_4 theories must be either TP_2 or simple. This would answer two open problems about equivalences between dividing lines.

Conant-independence, though it arises in many examples as Kim-independence relative to an abstract free amalgamation relation, is in fact an *absolute* independence relation:

Definition 0.0.4. *Let M be a model and $\varphi(x, b)$ a formula. We say $\varphi(x, b)$* Conant-divides *over M if for* every *invariant Morley sequence $\{b_i\}_{i\in\omega}$ over M starting with b, $\{\varphi(x, b)\}_{i\in\omega}$*

[5] [42] do not show simple or TP_2 for the Fraïssé-Hrushovski construction, but it follows from arguments in [32].

is inconsistent. We say $\varphi(x, b)$ Conant-forks *over* M *if and only if it implies a disjunction of formulas Conant-dividing over* M*. We say* a *is* Conant-independent *from* b *over* M*, written* $a \mathop{\smile\!\!\!\!\!\!\,|}^{K^*}_M b$*, if* $\mathrm{tp}(a/Mb)$ *does not contain any formulas Conant-forking over* M*.*

In analogy to Kaplan and Ramsey's development of Kim-independence in NSOP_1 theories as forking-independence "at a generic scale" ([52]), it is forking-independence at a *maximally* generic scale. It coincides with Kim-independence in NSOP_1 theories, and offers a way of extending the theory of independence in the NSOP hierarchy beyond NSOP_1. In the remainder of Chapter 1, we characterize Conant-independence in some examples of theories; in the free amalgamation theories of Conant ([32]), it is trivial. In the finite-language case of the Fraïssé-Hrushovski constructions of [41], [42], it coincides with d-independence, which was shown in [41] to coincide with forking-independence in the simple case but had no known pure model-theoretic definition outside of the simple case. We also characterize Conant-independence in the generic $< n$-cycle-free undirected graphs of [101], where it has a natural description as a symmetric independence relation. All of these examples, where Conant-independence is symmetric, are NSOP_4, the important examples strictly so (i.e. SOP_3.)

We conclude Chapter 1 by showing that any theory where Conant-independence is symmetric, just like any free amalgamation theory, must be NSOP_4. So $n = 4$ is the greatest n so that there are strictly NSOP_n theories where Conant-independence is symmetric. This leaves open the question of whether all NSOP_4 theories have symmetric Conant-independence, which would give us a true theory of independence for NSOP_4 theories generalizing the theory of Kim-independence in NSOP_1 theories. It also leaves open whether theories with symmetric Conant-independence must be either NSOP_1 or SOP_3, and either TP_2 or simple; if we add in the related strong witnessing property, both of these conclusions become true. If we can show NSOP_4 theories have symmetric Conant-independence, and answer either of these two questions about symmetric Conant-independence and dividing lines, we have solved one of the central classification-theoretic problems: the problem of whether *any* theory must be either NSOP_1 or SOP_3, and whether *any* NSOP_4 theory must be either TP_2 or simple. This suggests a connection between the study of classification-theoretic dividing lines *qua* dividing lines, and the geometric theory of independence, a connection we will revisit in Chapter 3.

In Chapter 2, we demonstrate a connection between geometric stability theory, in particular Hrushovski's group theorem [48], and the classification of unstable structures at the level of NSOP_1 vs. SOP_3. Given a definable ternary relation R in a weakly minimal theory T that is *fiber algebraic* ([29]), we define an expansion $T^{\neg R}$ which will be part of a more general construction:

Definition 0.0.5. *Let* T *be a theory with quantifier elimination, and let* R *be a relation definable in* T*. Let* E *be an additional binary relation symbol and* $\mathcal{L}^E = \mathcal{L} \cup \{E\}$*. Let* R *be a definable* n*-ary relation in* $\mathcal{L}$*. Define* T_R *to be the* $\mathcal{L}^E$*-theory consisting of the axioms for* T*, the requirement that* E *be an equivalence relation, and the axiom* $\forall \bar{x} \bigwedge_{1 \leq i \neq j \leq n}(x_i \neq x_j \wedge E(x_i, x_j)) \to R(\bar{x})$*.*

Then we use T^R *to denote the model companion of* T_R*, if it exists.*

The significance of this expansion will be the connection between its classification-theoretic properties and the group configuration theorem. The theory $T^{\neg R}$ will have the generalized free amalgamation properties of Chapter 1, so will be either NSOP_1, or both SOP_3 and TP_2 (under a different variation, it will be either simple, or SOP_3 and TP_2.) Whether it is NSOP_1 (usually, non-simple NSOP_1) or SOP_3 will depend on the geometric properties of R. For R to be a fiber-algebraic ternary relation means that it will be a reasonable candidate for being the graph of a group operation. By translating a failure of the independence theorem in $T^{\neg R}$ in terms of the group configuration, and then applying the group configuration theorem of [48], we show that $T^{\neg R}$ is SOP_3 rather than NSOP_1 exactly when R is geometrically equivalent to the graph of a type-definable group operation:

Theorem 0.0.1. *Let T be weakly minimal and let R be a ternary relation definable in T. Assume $\neg R$ is fiber-algebraic. Then T^R is* NSOP_1 *if and only if there is no set of parameters A over which R is definable, and (rank-one) group G type-definable (or definable, if T is strongly minimal) over A, so that the coordinates of a point of $\neg R$ generic (that is, of full rank) over A are individually interalgebraic with the coordinates of a point of the graph Γ^G of the multiplication in G generic over A. Otherwise, T^R is* TP_2*,* SOP_3*, and* NSOP_4*.*

This gives an exact correspondence between geometric stability theory and the classification theory of unstable structures. It also gives an application of geometric stability theory to the construction of new examples of non-simple NSOP_1 theories, an active area of recent research.

In Part II, consisting of Chapter 3, we prove the equivalence of NSOP_1 and NSOP_2 described earlier in this introduction. We reproduce the definitions here:

Definition 0.0.6. *A theory T is* NSOP_1 *if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \restriction n})\}_{n \in \omega}$ is consistent for any $\sigma \in 2^\omega$, but for any $\eta_2 \trianglerighteq \eta_1 \frown \langle 0 \rangle$, $\{\varphi(x, b_{\eta_2}), \varphi(x, b_{\eta_1 \frown \langle 1 \rangle})\}$ is inconsistent. Otherwise it is* SOP_1*.*

Definition 0.0.7. *A theory T is* NSOP_2 *if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \restriction n})\}_{n \in \omega}$ is consistent for any $\sigma \in 2^\omega$, but for incomparable η_1 and η_2, $\{\varphi(x, b_{\eta_1}), \varphi(x, b_{\eta_2})\}$ is inconsistent. Otherwise it is* SOP_2*.*

It follows from the definitions that all NSOP_1 theories are NSOP_2. Conversely,

Theorem 0.0.2. *All* NSOP_2 *theories are* NSOP_1*.*

The proof draws from techniques originally used to study Kim-independence in NSOP_1 theories (by Kaplan and Ramsey in [52]), forking and dividing in NTP_2 theories (by Chernikov and Kaplan in [27]), and the classification of free amalgamation theories, initiated by Evans and Wong ([42]) and Conant ([32]) and further developed in Chapter 1 of this thesis. The equivalence of NSOP_1 and NSOP_3 remains open, but our partial results on this problem observed in Chapter 1 form the backbone of the proof of this theorem.

In Part III of this thesis, we continue a theme we previewed in Chapter 1 with respect to NSOP_4, investigating the interactions between classification-theoretic properties more complex than NSOP_2 and the structure theory provided by model-theoretic independence relations. We will have shown that symmetry of Conant-independence implies NSOP_4, but in Chapter 4, we will be interested in the opposite direction: to what extent can classification-theoretic properties beyond NSOP_2 give us a geometric theory of independence extending that of Kim-independence in NSOP_1 theories? (Recall that [81] gives the beginnings of a structure theory for NSOP_3 theories, but in terms of higher formulas, rather than in terms of a global theory of independence generalizing stability-theoretic constructions). We will investigate this question in the case of NSOP_3 theories, though the fact that it is open whether all NSOP_3 theories are NSOP_1 leads to a paucity of concrete examples. Instead, we will prove some properties of independence in NSOP_3 theories that fail when the assumption NSOP_3 is relaxed to NSOP_4, and give concrete examples of NSOP_4 theories where these properties fail.

We will first generalize work of Chernikov in [26] on *simple types* in NTP_2 theories. Chernikov shows that in NTP_2 theories, simple types satisfy the dual property of being *co-simple*. As expected, in a suitable weakening of NTP_2 theories related to the NATP theories of introduced in Ahn and Kim ([3]) and developed by Ahn, Kim and Lee ([4]), an analogous result holds for "NSOP_1 types". This is expected, because this weakening of NTP_2 is a possible candidate for a class analogizing to NSOP_1 theories the relationship between NTP_2 and simple theories, as asked for in [69]. However, we prove a result on NSOP_3 theories that does not fit into this analogy. Instead of directly generalizing the definition of simple type, we consider a more natural and often weaker property of a type, requiring it have internally NSOP_1 structure:

Definition 0.0.8. *Let $p(x)$ be a partial n-type over M. Let $\mathcal{L}_p$ contain an m-ary relation symbol R_φ for each formula $\varphi(x_1, \dots, x_m) \in L(M)$ with $|x_i| = n$ for $i \leq n$. Then $\mathcal{M}_p$ is the $\mathcal{L}_p$-structure with domain $p(\mathbb{M}^n)$ and with $R_\varphi(p(\mathbb{M}^n)^m) = \varphi(\mathbb{M}^{mn}) \cap p(\mathbb{M}^n)^m$. The type $p(x)$ is internally* NSOP_1 *if $\mathcal{M}_p$ is* NSOP_1

We show that an internally NSOP_1 type in a NSOP_3 theory must be *co*-NSOP_1, generalizing the results of Chernikov ([26]) in a new direction. This will fail if we relax NSOP_3 to NSOP_4. We then give an interpretation of our results and those of Chernikov in terms of the *characteristic sequences* of [77], which we hope will prove illuminating for both of these results.

Using our result that internally NSOP_1 types are co-NSOP_1 in NSOP_3 theories, we then show a variant of the independence theorem between internally NSOP_1 types in NSOP_3 theories:

Theorem 0.0.3. *Let T be* NSOP_3*, and let p_1, p_2, p_3 be internally* NSOP_1 *types over M. Let $a_1 \equiv_M a_1' \subset p_1(\mathbb{M})$, $a_2 \subset p_2(\mathbb{M})$, $a_3 \subset p_3(\mathbb{M})$. If $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^*} a_2$, $a_1' \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^*} a_3$, $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^*} a_3$, there is* *some a_1'' with $a_1'' \models \mathrm{tp}(a_1/Ma_2) \cup \mathrm{tp}(a_1'/Ma_3)$. Moreover, a_1'' can be chosen with $a_2a_3 \mathop{\smile\!\!\!\!\!\!\,|}_M^{K^*} a_1''$, $a_2a_1'' \mathop{\smile\!\!\!\!\!\!\,|}_M^{K^*} a_3$ and $a_3a_1'' \mathop{\smile\!\!\!\!\!\!\,|}_M^{K^*} a_2$.*

This will again fail if we relax NSOP_3 to NSOP_4, and will also not follow just from co-NSOP_1. While a full independence theorem, by results of [52], will be impossible unless a theory is NSOP_1, this result will tell us that internally NSOP_1 types will fit together in an NSOP_3 theory the same way they fit together in an NSOP_1 theory. This is of interest to the problem of whether NSOP_2 is equivalent to NSOP_3, because it suggests we cannot start with NSOP_2 (equivalently NSOP_1) structures, put them together somehow in a way that gives a failure of NSOP_2 via a failure of the independence theorem, and get a theory that is still NSOP_3.

We then turn to NSOP_3 theories with symmetric Conant-independence. This is a natural assumption on NSOP_3 theories, because, as noted above, there are no known NSOP_4 theories without symmetric Conant-independence. We show that in an NSOP_3 theory with symmetric Comant-independence, finitely satisfiable types satisfy an variant of the independence theorem, similar to that proposed for NTP_2 theories in a question of Simon ([104]). Namely:

Theorem 0.0.4. *Let T be an NSOP_3 theory, and assume $\mathop{\smile\!\!\!\!\!\!\,|}^{K^*}$ is symmetric. Suppose p and q are M-finitely satisfiable (global) types with $p^\omega|_M = q^\omega|_M$, and let $a, b \supseteq M$ be small supersets of M with $a \mathop{\smile\!\!\!\!\!\!\,|}_M^{K} b$. Then there is $c \models p(x)|_a \cup q(x)|_b$ with $c \mathop{\smile\!\!\!\!\!\!\,|}_M^{K^*} ab$*

We also prove an extension of this result that, compared to the proof of this one, uses more of the force of symmetry for Conant-independence.

The similarity to possible properties of NTP_2 theories is surprising given the additional question of [26], of whether $\mathrm{NSOP}_3 \cap \mathrm{NTP}_2$ coincides with simplicity, which could have led us to think that NSOP_3 theories are very different from NTP_2 theories. This result is also of interest in light of the question from Chapter 1 of whether NSOP_3 theories with symmetric Conant-independence are NSOP_1, as developing a further structure theory for NSOP_3 theories with symmetric Conant-independence could give us insight on this question. As with the other two results of this chapter, there are NSOP_4 theories with symmetric Conant-independence in which our conclusion fails.

Finally, in section 5, we show that the interactions between the theory of independence and the levels of the NSOP_n hierarchy for $n > 4$ exhibit exponential behavior. Using the same idea of "independence at a maximally generic scale" that inspired Conant-independence, we introduce an infinite family of independence relations, the n-ð-*independence* relations $\mathop{\smile\!\!\!\!\!\!\,|}^{ð^n}$ (pronounced "eth" as in "father"):

Definition 0.0.9. *(1) Let $\mathop{\smile\!\!\!\!\!\!\,|}^{ð^0}$,* 0-ð-independence*, denote forking-independence over a model M.*

Inductively,

(2a) A formula $\varphi(x, b)$ $(n+1)$-ð-divides over a model M if, for any $\mathop{\smile\!\!\!\!\!\!\,|}^{ð^n}$-Morley sequence $\{b_i\}_{i<\omega}$ with $b_0 = b$, $\{\varphi(x, b_i)\}_{i<\omega}$ is inconsistent.

(2b) A formula $\varphi(x,b)$ $(n+1)$-ð-forks *over a model* M *if there are* $\varphi_i(x,b_i)$ $(n+1)$-ð-*dividing over* M *so that* $\models \varphi(x,b) \to \bigvee_{i=1}^{n} \varphi_i(x,b_i)$.

(2c) We say that a *is* $(n+1)$-ð-independent *from* b *over* M, *denoted* $a \mathop{\smile\!\!\!\!\!\!\;|}_M^{ð^{n+1}} b$, *if* $\mathrm{tp}(a/Mb)$ *contains no formulas* $(n+1)$-ð-*forking over* M.

These relations coincide in NSOP_1 theories, but reveal a fine structure to independence that turns out to be calibrated by the properties $\mathrm{NSOP}_{2^{n+1}+1}$ for $n \geq 1$. We show that for $n \geq 1$, $\mathop{\smile\!\!\!\!\!\!\;|}^{ð^n}$ is symmetric and transitive in the classical examples of $\mathrm{NSOP}_{2^{n+1}+1}$ theories, including some $\mathrm{SOP}_{2^{n+1}}$ examples. On the other hand, we also show that for $n \geq 1$, symmetry or transitivity of $\mathop{\smile\!\!\!\!\!\!\;|}^{ð^n}$ implies $\mathrm{NSOP}_{2^{n+1}+1}$. In analogy to $k = 4$ being the least value of k so that there is a strictly NSOP_k theory with symmetric Conant-independence, we will have shown, in summation, that:

Theorem 0.0.5. *The least value of* k *so that there is a strictly* NSOP_k *theory where* $\mathop{\smile\!\!\!\!\!\!\;|}^{ð^n}$ *is symmetric is* $k = 2^{n+1}+1$. *Moreover, the least value of* k *so that there is a strictly* NSOP_k *theory where* $\mathop{\smile\!\!\!\!\!\!\;|}^{ð^n}$ *is transitive is* $k = 2^{n+1}+1$.

This leaves open the question of whether symmetry, or perhaps transitivity, of $\mathop{\smile\!\!\!\!\!\!\;|}^{ð^n}$ is equivalent to $\mathrm{NSOP}_{2^{n+1}+1}$ for $n \geq 1$, which would give a true theory of independence for $\mathrm{NSOP}_{2^{n+1}+1}$.

We assume a knowledge of basic model theory, including stability theory and simplicity theory. Two excellent expositions of simplicity theory are given in [110] and [62]. There are many good treatments of stability theory; for a more geometric perspective, see [87]. Additional background will be given in each chapter.

Part I

Chapter 1

Conant-independence and generalized free amalgamation

1.1 Introduction

One of the most rapidly evolving areas of model theory is the study of potentially non-NSOP_1 NSOP theories. Two cornerstone problems of this field include determining the status of the open regions of this part of the classification-theoretic map, and developing a theory of independence for these theories[1]. One of the main questions of the first program, asked by Džamonja and Shelah [40], of whether the class NSOP_2 coincides with the class NSOP_1, was recently answered in the affirmative by the author in Chapter 3. Yet the following question from [40] remains open:

Problem 1.1.1. *Is every* NSOP_3 *theory* NSOP_2 *(and therefore* NSOP_1*?).*

An additional open question ([30] [26]), involves the interactions of the NSOP_n hierarchy with NTP_2:

Problem 1.1.2. *Is the* NSOP_n *hierarchy strict within* NTP_2 *(including* NSOP_n *for* $n \geq 3$ *as well as* NSOP *itself?)*

Note that Shelah ([97]) showed that all NSOP_2 NTP_2 theories are simple. Partial results on these problems include work of Evans and Wong in [42] proving the ω-categorical Hrushovski constructions introduced in [41] are either simple or strictly NSOP_4, work of Conant in [32] proving modular *free amalgamation theories* are either simple or strictly NSOP_4 TP_2, and upcoming work of Kaplan, Ramsey and Simon ([95]) shows that all binary theories are either SOP_3 or NSOP_2, and either SOP_1 or simple (and therefore SOP_3 or simple.) Yet none of the previous literature explicitly treats general classes of theories that approach the

[1] For a somewhat different tradition in the theory of independence for potentially non-simple theories, with some overlap with the higher NSOP_n hierarchy including the modular free amalgamation theories from [32], see [84].

limits of our understanding of the NSOP_n hierarchy: potentially SOP_3, but also potentially *strictly* NSOP_1. (However, recent work of Johnson and Ye, introducing *curve-excluding fields* ([50]; see also [49]) known to be TP_2 and thus not simple but thought to be NSOP_4, deserves mention; see below.) One of the goals of this chapter is to show that the potentially *non-modular* free amalgamation theories are such a class (and that, answering a question of Conant in [32], non-modular free amalgamation theories exist), and to introduce some properties of theories, essentially generalizing the free amalgamation theories with no known NSOP_4 counterexamples, under the assumption of which the NSOP_1-SOP_3 dichotomy holds.

On the other hand, our understanding of *independence* in the NSOP region beyond NSOP_2 theories has remained thin to non-existent. Kaplan and Ramsey ([52]) have successfully introduced the concept of Kim-independence, or forking independence "at a generic scale," as the appropriate extension of forking-independence to the class NSOP_1. Yet to extend Kim-independence itself past NSOP_2 remains open. Stronger and often stationary abstract independence relations with no known concrete model-theoretic characterization are also abundant in the class NSOP. The theory of purely abstract independence relations is introduced by Adler in [1], where he outlines axioms these relations can satisfy to behave in certain ways like forking-independence in stable theories. In [36], D'Elbée proposes the problem of finding a model-theoretic definition of stronger "free amalgamation" relations alongside Kim-independence in NSOP_1 theories, such as the *strong independence* in the theory ACFG of algebraically closed fields with a generic additive subgroup; he also notes that relations with similar properties hold in the strictly NSOP_4 Henson graphs. Just as in the case of free amalgamation of generic functional structures in [71] or generic incidence structures in [33], d'Elbée observes that these stronger independence relations can be used to prove the equivalence of forking and dividing for complete types in many known NSOP_1 theories. Conant [32] introduces his formulation of free amalgamation based on concepts used to study the isometry groups of Urysohn spheres in [106], giving an abstract set of axioms for independence relations generalizing those found in homogeneous structures, such as those given by adding no new edges in the (simple) theory of the random graph or the (strictly NSOP_4) theory of the generic triangle-free random graph. Aside from the canonical coheirs introduced by the author in Chapter 3 to simulate the assumption of a stationary independence relation in the proof of NSOP_1 for NSOP_2 theories, our understanding of this phenomenon of "strong independence" is entirely synthetic. Yet theories exhibiting this phenomenon often come equipped with a weaker notion of independence, which we show to have a purely model-theoretic characterization as forking-independence "at a *maximally* generic scale" (in other words, the result of forcing Kim's lemma onto Kim-independence) extending that of Kim-independence in NSOP_1 theories. This notion is based on the concept of "strong Kim-dividing" introduced by Kaplan, Ramsey and Shelah in [54] in the context of "dual local character" in NSOP_1 theories. We show that NSOP_4 theories are the last class in the NSOP_n hierarchy where this notion of independence can be symmetric, providing the beginnings of a theory of independence beyond NSOP_1.

An outline of the chapter is as follows. In section 3, we introduce a weak set of axioms on stationary independence relations, essentially generalizing the "freedom" axiom in Conant's free amalgamation theories beyond the traditional homogeneous structures. It is not a true generalization of Conant's axioms, as Conant employs a non-standard definition of stationarity, yet these relations can be found in all of Conant's examples. We show that under these axioms, we can carry out arguments for NSOP_1 theories from Chernikov and Ramsey [28] and Ramsey and Kaplan [52] relative to an independence relation, even outside of the NSOP_1 context. Specifically, we prove the equivalence of a relative Kim's lemma, or minimality among invariant Morley sequences in the *dividing order* indroduced by [113], and symmetry for relative Kim-independence for a relation with these axioms. It follows that when the relative Kim-independence is symmetric, it is no longer a relative notion, but rather the absolute notion of forking-independence "at a maximally generic scale" that we call *Conant-independence*, after Conant's observation in [32] (Lemma 7.6) that Morley sequences in a free amalgamation relation can only witness dividing when the relation $A \mathop{\smile\!\!\!\!\!\mid}^a_C B$ defined by $\mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$ fails. A version of this was defined by the author in Chapter 3 as a candidate for Kim-independence in NSOP_2 theories, but we define it here in terms of invariant Morley sequences rather than coheir Morley sequences, as in the "strong Kim-dividing" of [54]. Using part of the proof from Chapter 3 of the equivalence of NSOP_1 and NSOP_2, in turn adapting many of the arguments from [32] on modular free amalgamation theories, we also show that when these equivalent "relative NSOP_1" conditions hold for a relation with our axioms, or more generally when we have symmetry for Conant-independence and a minimal Kim's lemma even without these axioms, a theory must be either NSOP_1 or NSOP_3; additionally, generalizing arguments from [32], it must be either simple or TP_2. Importantly, we do not know whether there is an NSOP_4 theory where either of these two symmetry or witnessing conditions fail.

In section 4, we extend Conant's result in [32] that modular free amalgamation theories must be either simple or SOP_3 to all free amalgamation theories, using the results of the previous section to show that free amalgamation theories must be either NSOP_1 or SOP_3. Accordingly, we show that Kruckman and Ramsey's example of the generic theory of a function from [71], when equipped with a nonstandard free amalgamation relation that actually falls under Conant's axioms, gives an example of a non-modular free amalgamation theory, answering the aformentioned question of Conant. As a corollary, we get a converse to Conant's result that a simple free amalgamation theory must be modular, showing that a modular NSOP_1 free amalgamation theory must be simple. In a personal communication, Conant noted to the author that Claim 1 of Theorem 7.7 of [32] was in error; by using an entirely different method in Chapter 3 for the analogous claim in our proof that a free amalgamation theory must be NSOP_1 or SOP_3, we recover Conant's theorem that a modular free amalgamation theory must be simple or SOP_3.

In section 5, we give some examples of theories with a "relatively NSOP_1" stationary independence relation with our axioms, and characterize Conant-independence in these theories. We show that the finite-language case of the ω-categorical Hrushovski constructions of [41], which Conant notes are not necessarily free amalgamation theories in his sense, do satisfy this more general notion of free amalgamation, and that Conant-independence gives us a purely model-theoretic interpretation of the *d-independence* of [41] even outside of the simple case (where it coincides with forking-independence). We then give a similar analysis to the generic graphs without small cycles introduced in [101] as examples of strictly NSOP_4 theories. It appears that the curve-excluding fields introduced in recent work of Johnson and Ye ([50]; see also [49]) might also have a stationary independence relation with the required properties, with Conant-independence coinciding with algebraic indepenence in the sense of fields, suggesting that these fields must be either strictly NSOP_1 or, taking into account the next paragraph, strictly NSOP_4.

In section 6, we show that any theory where Conant-independence is symmetric must be NSOP_4. The original suggestion of a special significance for NSOP_4, in connection with free amalgamation, is due to Patel ([85]), who in unpublished work provided an argument for NSOP_4 for various examples that was later generalized, along with work from various other authors, by Conant in [32] (where a more complete historical background can be found.) By showing $n = 4$ is the least so that there is a strictly NSOP_n theory with symmetric Conant-indepednence, we give neostability-theoretic justification for this significance. We then pose some questions about symmetry for Conant-independence within the neostability hierarchy, highlighting some connections with established open problems on dividing lines as well as a potential characterization of NSOP_4 in terms of Conant-independence, similar to Kaplan and Ramsey's characterization of NSOP_1 in terms of Kim-independence.

1.2 Preliminaries

Notations are standard; M will denote a model while a, b, c, A, B, C will denote sets. A *global type* $p(x)$ is a complete type over the sufficiently saturated model $\mathbb{M}$. For $M \prec \mathbb{M}$, a global type $p(x)$ is *invariant* over M if whether $\varphi(x, b)$ belongs to p for $\varphi(x, y)$ a fixed formula without parameters depends only on the type of the parameter b over M and not on the specific realization of that type. A special subclass of types invariant over M is that of those *finitely satisfiable* over M, meaning any formula in the type is satisfied by some element of M. We say an infinite sequence $\{b_i\}_{i\in I}$, is an *invariant Morley sequence* over M if there is a fixed global type $p(x)$ invariant over M so that $b_i \models p(x)|_{M\{b_j\}_{j<i}}$ for $i \in I$. Invariant Morley sequences over M are indiscernible over M, and the EM-type of an invariant Morley sequence over M depends only on $p(x)$.

We recall Conant's definition of *free amalgamation theories* in [32], and define a few other properties of relations between sets. Many of these definitions come originally from Adler ([1]) and the axiom system itself resembles that of Ziegler and Tent in [106]. A theory is a *free amalgamation theory* if there is a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ between two sets over another set with the following properties:

Invariance: Whether $A \mathop{\smile\!\!\!\!\!\!|}_C B$ is an invariant of the type of ABC.

Monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A_0 \subseteq A$, $B_0 \subseteq B$, then $A_0 \mathop{\smile\!\!\!\!\!\!|}_C B_0$.

Full transitivity: For any A, if $D \subseteq C \subseteq B$ then $A \mathop{\smile\!\!\!\!\!\!|}_D B$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}_D C$ and $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

Full existence: For any a, B and for C algebraically closed, there is some $a' \equiv_C a$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

Stationarity: For a, b, C algebraically closed with $C \subseteq a \cap b$, and for any $a' \equiv_C a$, if $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}_C b$ then $a' \equiv_b a$.

Freedom: For A, B, C, D with $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, if $C \cap AB \subseteq D \subseteq C$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_D B$.

Closure: For a, b, C algebraically closed with $C \subseteq a \cap b$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$, ab is algebraically closed.

Sometimes a relation is defined only between sets over a model, rather than over an arbitrary set. We define some additional properties that we will use in this case. As Conant's definition of stationarity is nonstandard, this includes the standard formulation of stationarity, which will apply to example 3.2.1 of [32], the random graphs, Henson graphs and Urysohn sphere.

Full stationarity: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$, $A' \mathop{\smile\!\!\!\!\!\!\mid}_M B$, and $A \equiv_M A'$, then $A \equiv_{MB} A'$.

Left extension: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $A \subseteq C$, there is some $B' \equiv_A B$ with $C \mathop{\smile\!\!\!\!\!\!\mid}_M B'$.

Right extension: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $B \subseteq C$, there is some $A' \equiv_B A$ with $A' \mathop{\smile\!\!\!\!\!\!\mid}_M C$.

We define $a \mathop{\smile\!\!\!\!\!\!\mid}^i_M b$ to mean that $\mathrm{tp}(a/Mb)$ extends to an M-invariant global type. The relation $a \mathop{\smile\!\!\!\!\!\!\mid}^a_M b$, denoting $\mathrm{acl}(aM) \cap \mathrm{acl}(bM) = M$ can be found in [32]; it is well-known to satisfy right (and left) extension.

We review the relevant regions of the generalized stability hierarchy. The following, which we take as the definition of simplicity, is well-known:

Definition 1.2.1. *We say* $\mathrm{tp}(a/bM)$ does not divide *over M, denoted $a \mathop{\smile\!\!\!\!\!\!\mid}^{div}_M b$, if there is no formula $\varphi(x, b) \in \mathrm{tp}(a/bM)$ and M-indiscernible sequence $\{b_i\}_{i \in I}$ starting with b so that $\{\varphi(x, b_i)\}_{i \in I}$ is inconsistent. A theory T is simple if $\mathop{\smile\!\!\!\!\!\!\mid}^{div}$ is symmetric.*

The properties NSOP_1 and NSOP_2 were introduced in [40]:

Definition 1.2.2. *A theory T is* NSOP_1 *if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n \in \omega}$ is consistent for any $\sigma \in 2^\omega$, but for any $\eta_2 \trianglerighteq \eta_1 \frown \langle 0 \rangle$, $\{\varphi(x, b_{\eta_2}), \varphi(x, b_{\eta_1 \frown \langle 1 \rangle})\}$ is inconsistent. Otherwise it is* SOP_1.

Definition 1.2.3. *A theory T is* NSOP_2 *if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n \in \omega}$ is consistent for any $\sigma \in 2^\omega$, but for incomparable η_1 and η_2, $\{\varphi(x, b_{\eta_1}), \varphi(x, b_{\eta_2})\}$ is inconsistent. Otherwise it is* SOP_2.

These two classes coincide; see Chapter 3.

Justifying the "order" terminology, the following family of classes was introduced in [101]:

Definition 1.2.4. *Let $n \geq 3$. A theory T is* NSOP_n *(that is, does not have the* n-strong order property*) if there is no definable relation $R(x_1, x_2)$ with no n-cycles, but with tuples $\{a_i\}_{i \in \omega}$ with $\models R(a_i, a_j)$ for $i < j$. Otherwise it is* SOP_n.

We will only concern ourselves with NSOP_n theories for $1 \leq n \leq 4$. Finally, [99] introduces the following notion, whose interaction with the NSOP_n hierarchy beyond NSOP_2 remains open:

Definition 1.2.5. *A theory T is* NTP_2 *(that is, does not have the* tree property of the second kind*) if there is no array $\{b_{ij}\}_{i,j\in\omega}$ and formula $\varphi(x,y)$ so that there is some fixed k so that, for all i, $\{\varphi(x,b_{ij})\}_{j\in\omega}$ is inconsistent, but for any $\sigma\in\omega^\omega$, $\{\varphi(x,b_{i\sigma(i)})\}_{i\in\omega}$ is consistent.*

Kaplan and Ramsey ([52]) extend the theory of forking-independence in simple theories to NSOP_1 theories. We give a brief overview, mostly by way of motivation:

Definition 1.2.6. *A formula $\varphi(x,b)$* Kim-divides *over M if there is an invariant Morley sequence $\{b_i\}_{i\in\omega}$ starting with b (said to* witness *the Kim-dividing) so that $\{\varphi(x,b_i)\}_{i\in\omega}$ is inconsistent. A formula $\varphi(x,b)$* Kim-forks *over M if it implies a (finite) disjunction of formulas Kim-dividing over M. We write $a\mathop{\smile\!\!\!\!\!\mid}^K_M b$, and say that a is* Kim-independent *from b over M if* $\mathrm{tp}(a/Mb)$ *does not include any formulas Kim-forking over M.*

Any NSOP_1 theory is characterized by the following variant of Kim's lemma for simple theories, as well as by symmetry of Kim-independence.

Fact 1.2.1. *([52]) Let T be* NSOP_1*. Then for any formula $\varphi(x,b)$ Kim-dividing over M, any invariant Morley sequence over M starting with b witnesses Kim-dividing of $\varphi(x,b)$ over M. Conversely, suppose that for any formula $\varphi(x,b)$ Kim-dividing over M, any invariant Morley sequence (even in a finitely satisfiable type) over M starting with b witnesses Kim-dividing of b over M. Then T is* NSOP_1.

It follows that Kim-forking coincides with Kim-dividing in any NSOP_1 *theory.*

Fact 1.2.2. *([28], [52]) The theory T is* NSOP_1 *if and only if $\mathop{\smile\!\!\!\!\!\mid}^K$ is symmetric.*

The following preorder restricts the *dividing order* of [113]. We are interested in the minimal class.

Definition 1.2.7. *Let $p(x), q(x)$ be two invariant global types extending a common type over a model M. Then $p(x)$* is greater than or equal to *$q(x)$* in the Kim-dividing order *if its invariant Morley sequences witness Kim-dividing of every formula the Kim-dividing over M of which is witnessed by invariant Morley sequences in $q(x)$.*

1.3 Generalized free amalgamation and relative NSOP_1

Throughout this section we assume unless otherwise noted a ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ between sets is defined over models and has invariance, monotonicity, full existence, and full stationarity. Following Definition 7.5 of [32], we first define special Morley sequences.

Definition 1.3.1. *Let $M\prec\mathbb{M}$. An $\mathop{\smile\!\!\!\!\!\mid}$-*Morley sequence *over M (in $q\in S(M)$) is an infinite sequence $\{a_i\}_{i\in I}$ so that $a_i\mathop{\smile\!\!\!\!\!\mid}_M a_{<i}$ for all $i\in I$ (so that $a_i\models q$ for all $i\in I$.)*

Note that $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequences of realizations of any type $p(x)$ over M will always exist, their type over M will depend only on $p(x)$, and they will be indiscernible over M with any two terms $\mathop{\smile\!\!\!\!\!\mid}^a$-independent.

We consider a new axiom on $\mathop{\smile\!\!\!\!\!\mid}$, motivated by the freedom axiom from [32] defined in section 2 and covering all of the examples from [32].

Generalized freedom: If $M \prec M' \prec \mathbb{M}$ and there is an $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence over M starting with a and indiscernible over M', then an $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence starting with a over M' is also an $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence over M.

(See [53], [39] for some results involving preservation of Morley sequences under change of base.)

Remark 1.3.1. If $\mathop{\smile\!\!\!\!\!\mid}$ additionally satisfies the freedom axiom, it also satisfies the generalized freedom axiom.

Proof. Since any two terms of $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequences over M starting with a will be $\mathop{\smile\!\!\!\!\!\mid}^a$-independent over M, the hypothesis of the generalized freedom axiom implies $M' \mathop{\smile\!\!\!\!\!\mid}^a_M a$. The rest is just the proof of Lemma 7.6 of [32]. By stationarity, it suffices to construct an $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence $\{a_i\}_{i\in\omega}$ starting with a over M' that remains an $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence over M. Suppose $a_0, \ldots, a_n$ already constructed. Choose a copy $a_{n+1} \mathop{\smile\!\!\!\!\!\mid}_{M'} a_0 \ldots a_n$ of a over M'. So $M' \cap a_0 \ldots a_{n+1} \subseteq M \subseteq M'$. Then by the freedom axiom, additionally $a_{n+1} \mathop{\smile\!\!\!\!\!\mid}_M a_0 \ldots a_n$. □

Example 1.3.2. In Examples 3.2.1(i-iii) of [32], the random graphs, Henson graphs and the Urysohn ball of radius 3, free amalgamation satisfies full stationarity and therefore satisfies the generalized freedom axiom.

Example 1.3.3. In the generic $(K_n + K_3)$-free graphs of [24] (the first of which is introduced in [67]), it follows from the discussion in Example 3.2.2 of [32] (namely the result of Patel [85] that the class of $(K_n + K_3)$-free graphs is closed under free amalgamation over an algebraically closed base; since the algebraic closure is distintegrated, this free amalgamation is itself algebarically closed) that isomorphic *algebraically closed* sets are elementarily equivalent. Since it is required for elementary equivalence that the sets be algebraically closed, the free amalgamation from this example only satisfies stationarity, rather than full stationarity. However, consider the fully stationary relation $A \mathop{\smile\!\!\!\!\!\mid}_M B$ defined by free amalgamation of $\mathrm{acl}(AM)$ and $\mathrm{acl}(BM)$ over M; we show that the generalized freedom axiom holds. Suppose the hypothesis holds, so $M' \mathop{\smile\!\!\!\!\!\mid}^a_M a$. Consider an $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence $\{a_i\}_{i\in\omega}$ over M' starting with a. Then $\{\mathrm{acl}(Ma_i)\}_{i\in\omega}$ can be seen to be $\mathop{\smile\!\!\!\!\!\mid}^a$-independent over M, and because $\mathrm{acl}(Ma_i)$ does not meet M' except in M, that $\{\mathrm{acl}(M'a_i)\}_{i\in\omega}$ are in free amalgamation (given by adding no new edges) over M' implies that the $\{\mathrm{acl}(Ma_i)\}_{i\in\omega}$ are in free amalgamation over M.

We consider Conant's other example from [32], the freely disintegrated ω-categorical Hrushovski constructions of [41], in Section 5, as part of larger general class of ω-categorical Hrushovski constructions that Conant notes in this example are *not* necessarily free amalgamation theories.

Example 1.3.4. In the strictly NSOP_1 theory ACFG of algebraically closed fields with a generic additive subgroup, the strong independence relation $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{st}}_M B$, introduced as part of a larger family in [37] and developed in [36], given by $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ACF}}_M B$ and $G(\mathrm{acl}(MAB)) = G(\mathrm{acl}(MA)) + G(\mathrm{acl}(MB))$, satisfies the generalized freedom axiom; note that Kim-independence $A \mathop{\smile\!\!\!\!\!\mid}^{K}_M B$ is given by the "weak independence" $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ACF}}_M B$ and $G(\mathrm{acl}(MA) + \mathrm{acl}(MB)) = G(\mathrm{acl}(MA)) + G(\mathrm{acl}(MB))$, and the hypothesis of this axiom in the NSOP_1 case is just Kim-independence. It is expected that all of the other examples from the literature of "strong independence" in NSOP_1 theories listed in [37] also satisfy this axiom.

We wish to show that even outside of the NSOP_1 context, the theory of Kim-forking from [52] characteristic of NSOP_1 theories can be developed relative to an independence relation $\mathop{\smile\!\!\!\!\!\mid}$ satisfying the generalized freedom axiom, though when the equivalent relative versions of NSOP_1 are satisfied, the relative version of Kim-independence becomes a new absolute independence relation. We first introduce the relative notion:

Definition 1.3.2. *Let $\varphi(x, b)$ be a formula. We say $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\mid}$-*Kim-divides *over a model M if $\{\varphi(x, b_i)\}_{i\in I}$ is inconsistent (so k-inconsistent for some k) for some (any) $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence $\{b_i\}_{i\in I}$ over M starting with b, and that it $\mathop{\smile\!\!\!\!\!\mid}$-*Kim-forks *over M if it implies a (finite) disjunction of formulas $\mathop{\smile\!\!\!\!\!\mid}$-Kim-dividing over M. We say a is $\mathop{\smile\!\!\!\!\!\mid}$-Kim-independent from b over M (written $a \mathop{\smile\!\!\!\!\!\mid}^{K^{\mathop{\smile\!\!\!\!\!\mid}}}_M b$) if a does not satisfy a formula of the form $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\mid}$-Kim-forking over M.*

A feature of stationarity is that we automatically get "Kim's lemma" (the analogue of Fact 1.2.1) for the class of $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequences taken *alone*, giving us equivalence of $\mathop{\smile\!\!\!\!\!\mid}$-Kim-forking and $\mathop{\smile\!\!\!\!\!\mid}$-Kim-dividing with no further assumptions.

Proposition 1.3.5. *For formulas, $\mathop{\smile\!\!\!\!\!\mid}$-Kim-forking coincides with $\mathop{\smile\!\!\!\!\!\mid}$-Kim-dividing. Therefore, $\mathop{\smile\!\!\!\!\!\mid}^{K^{\mathop{\smile\!\!\!\!\!\mid}}}$ satisfies right extension.*

Proof. The following is standard; see [52] for the application of this method to Kim-independence in NSOP_1 theories. Let $\models \varphi(x, b) \to \bigvee_{i=1}^{n} \psi_i(x, c_i)$ for $\psi_i(x, c_i)$ $\mathop{\smile\!\!\!\!\!\mid}$-Kim-dividing over M. By left extension (which follows from the assumptions) and monotonicity, whether or not a formula $\mathop{\smile\!\!\!\!\!\mid}$-Kim-divides over M does not change when adding unused parameters, so we can assume $c_i = b$ for $1 \leq i \leq n$. Then $\varphi(x, b)$ Kim-divides over M, for suppose otherwise. Let $\{b_i\}_{i\in\mathbb{N}}$ be an $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence over M starting with b; then there will be some a realizing $\{\varphi(x, b_i)\}_{i\in\mathbb{N}}$. So by the pigeonhole principle, there will be some $1 \leq j \leq n$ so that a realizes $\{\psi_j(x, b_i)\}_{i\in S}$ for $S \subseteq \mathbb{N}$ infinite. But by monotonicity and an automorphism, we can assume $\{b_i\}_{i\in S}$ is an $\mathop{\smile\!\!\!\!\!\mid}$-Morley sequence over M starting with b, contradicting $\mathop{\smile\!\!\!\!\!\mid}$-Kim-dividing of $\psi_i(x, b)$. □

Next, we introduce one possible formulation of NSOP_1 relative to $\mathop{\smile\!\!\!\!\!\mid}$ (see Fact 1.2.1).

Definition 1.3.3. *The relation* $\downarrow$ *satisfies the* relative Kim's lemma *if the type over* $\mathbb{M}$ *of any* $b \downarrow_M \mathbb{M}$ *is always minimal in the Kim-dividing order.*

Aside from the motivation by NSOP_1 theories (as well as similarity to a property of the canonical coheirs of Chapter 3), this is a natural assumption. Strictly NSOP_4 theories are often defined as the generic examples of structures avoiding a particular configuration, such as the Henson graph avoiding K_n or the ω-categorical Hrushovski constructions avoiding finite substructures of negative predimension. Free amalgamation-like relations in these examples will have the minimal amount of obstructions to consistency along an invariant Morley sequence, which is to say, obstructions (say, edges or relations) to the avoidance of a forbidden configuration. Using the generalized freedom axiom, arguments from [28], [52] can be carried out here, showing the equivalence of this assumption to symmetry of the relative Kim-indepndence (see Facts 1.2.1 and 1.2.2).

Theorem 1.3.6. *Suppose* $\downarrow$ *satisfies the generalized freedom axiom. Then* $\downarrow$ *satisfies the relative Kim's lemma if and only if* $\downarrow^{K^{\downarrow}}$ *is symmetric.*

Proof. We follow the proofs of Theorems 3.16 and 5.16 of [52], taking note of where the generalized freedom axiom applies in each direction; note that because the $\downarrow$-Morley sequences will go in the opposite direction of the configurations originally found in the proofs of the results on NSOP_1 theories, we will require densely ordered indiscernible sequences. We will also need to make some modifications to respect the Skolemization.

($\Rightarrow$) Suppose $\downarrow$ also satisfies the relative Kim's lemma. Then we have the following chain condition:

Claim 1.3.7. *(Chain Condition) Let* $a \downarrow_M^{K^{\downarrow}} b$*. Then there is some* $\downarrow$*-Morley sequence* $I = \{b_i\}_{i \in \mathbb{N}}$ *over* M *indiscernible ovr* Ma *starting with* b *so that* $a \downarrow_M^{K^*} I$*.*

Proof. The proof can be taken nearly word-for-word from Proposition 3.5.2 of Chapter 3, itself similar to the standard proof of the chain condition found in, say, [52]. By compactness, there is a Morley sequence $I = \{b_i\}_{i \in \mathbb{N}}$ over M starting with b and indiscernible over Ma. We must show that $a \downarrow_M^{K^{\downarrow}} I$. It suffices to show that $a \downarrow_M^{K^{\downarrow}} b_1 \dots b_k$ for any $k \geq 1$. But the concatenation $\{b_{ik} b_{ik+1} \dots b_{ik+(k-1)}\}_{i \in \omega}$ is still an invariant Morley sequence over M, so by the relative Kim's lemma and compactness and Ramsey there is an $\downarrow$-Morley sequence starting with $b_1 \dots b_k$ and indiscernible over a. The claim follows by Proposition 1.3.5. □

Now suppose for contradiction that $a \downarrow_M^{K^{\downarrow}} b$ but b is $\downarrow$-Kim-dependent on a over M. Let $\varphi(x, a) \in \mathrm{tp}(b/Ma)$ $\downarrow$-Kim-divide over M, and choose a Skolemization of T.

Claim 1.3.8. *There is a sequence* $\{c_{i,0}, c_{i,1}\}_{i \in \omega}$ *with* $c_{i,j} \equiv_M a$*, with* $c_{i,0} \equiv_{\mathrm{dcl}_{\mathrm{Sk}}(Mc_{<i,0}, c_{<i,1})} c_{i,1}$*, and a formula* $\varphi(x, y)$ *with* $\{\varphi(x, c_{i,0})\}_{i \in \omega}$ *consistent, but* $\{c_{i,1}\}_{i \in \omega}$*, read backwards, an* $\downarrow$*-Morley sequence; therefore,* $\{\varphi(x, c_{i,1})\}_{i \in \omega}$ *will be inconsistent.*

Proof. Because this configuration is obtained the same way as that from the proof of Proposition 5.13 of [52], but with the difference that, as in Chapter 3, the branches must form *special* Morley sequences–in this case $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequences instead of canonical Morley sequences–rather than any invariant Morley sequence, and with the additional difference that the Morley tree is extracted in the Skolemization, we only sketch the proof. The idea is to first build a very large tree with the following properties:

(1) The type over M of a non-leaf node, taken together with a leaf with that node on its path, is the same of that of ab.

(2) The branches at a given node form an $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence.

(3) Each node is $\mathop{\smile\!\!\!\!\!\!\mid}$-Kim independent from its branches, taken together.

This is by transfinite induction, or by induction and compactness. Suppose at a given stage, the tree I is already built. We want to find some $a'_\emptyset \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^{\mathop{\smile\!\!\!\!\!\!\mid}}} I$ so that the type of $a'_\emptyset$ with each leaf node over M is the same as that of ab over M. At the base case, this is just the assumption $a \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^{\mathop{\smile\!\!\!\!\!\!\mid}}} b$. At the successor stage, note that the root node is $\mathop{\smile\!\!\!\!\!\!\mid}$-Kim-independent from the rest of the tree and satisfies, with each leaf node, the type of ab over M, so we get $a'_\emptyset$ by right extension for $\mathop{\smile\!\!\!\!\!\!\mid}^{K^{\mathop{\smile\!\!\!\!\!\!\mid}}}$. Now use the chain condition to choose an $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence $\{I_i\}_{i<\kappa}$ starting with I and indiscernible over $Ma'_\emptyset$ so that $a'_\emptyset \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^{\mathop{\smile\!\!\!\!\!\!\mid}}} \{I_i\}_{i<\kappa}$, and reindex accordingly.

Now in the Skolemization, use Lemma 5.10 of [52] to extract a Morley tree (see Definition 5.7 of [52] from the non-leaf nodes and let $c_{i,0}$ be the node indexed by $\langle 0\rangle^n$, $c_{i,1}$ be the node indexed by $\langle 0\rangle^{n-1} \frown \langle 1\rangle$. If one does not wish to deal with Morley trees, one may also use the more elementary argument of Proposition 5.6 of [28]. Suppose the $c_{i,0} = c_{\lambda_i}$, indexed by nodes λ_i and $c_{i,1} = c_{\eta_i}$ indexed by nodes η_i with $\eta_j \wedge \lambda_j \rhd \lambda_i$ for $1 \leq i < j \leq n$ and $\lambda_i \trianglerighteq (\eta_i \wedge \lambda_i) \frown \langle 0\rangle, \eta_i \trianglerighteq (\eta_i \wedge \lambda_i) \frown \langle 1\rangle$, are already constructed. Then using the pigeonhole principle, choose nodes $\lambda_{n+1} = \lambda_n \frown \langle 0\rangle^{\kappa_1} \frown \langle 1\rangle$, $\eta_{n+1} = \lambda_n \frown \langle 0\rangle^{\kappa_2} \frown \langle 1\rangle$ for $\kappa_1 < \kappa_2 < \kappa$ so that the corresponding terms of the tree, which we then call $c_{n+1,0} = c_{\lambda_{n+1}}$ and $c_{n+1,1} = c_{\eta_{n+1}}$, have the same type over $\mathrm{dcl}_{\mathrm{Sk}}(Mc_{\leq n,0}, c_{\leq n,1})$. □

We now apply the generalized freedom axiom to carry out the argument for Proposition 3.14 of [52], the one underlying Kim's lemma in actual NSOP_1 theories, to contradict the relative Kim's lemma.

We can find $\{c_{i,0}, c_{i,1}\}_{i\in\mathbb{Q}^+}$ for $\mathbb{Q}^+ = \mathbb{Q} \cup \{\infty\}$, indiscernible over M in the Skolemization with the same properties. Let $M' = \mathrm{dcl}_{\mathrm{Sk}}(M\{c_{i,0}, c_{i,1}\}_{i\in\mathbb{Q}})$, and $p(y) = \mathrm{tp}(c_{\infty,0}/M') = \mathrm{tp}(c_{\infty,1}/M')$.

Claim 1.3.9. *There is an $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence over M of realizations of $p(y)$.*

Proof. By compactness, it suffices to show the same replacing $p(y)$ with its restriction to $M_j = \mathrm{dcl}_{\mathrm{Sk}}(M\{c_{i,0}, c_{i,1}\}_{i<j})$ for some $j \in \mathbb{Q}$. But this is just $\mathrm{tp}(c_{j+1,1}/M_j)$, and $\{c_{k,1}\}_{j<k\leq j+1}^{\infty}$, read backwards, is an $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence starting with $c_{j+1,1}$ indiscernible over M_n. □

Now just as in the proof of Proposition 3.14 of [52], the consistency of $\{\varphi(x, c_{i,0})\}_{i\in\mathbb{Q}^+}$ gives an M'-finitely satisfiable extension of p not witnessing the Kim-dividing of $\varphi(x, c_{\infty,0})$

over M', namely the limit type of $\{c_{i,0}\}_{i\in\mathbb{Q}}$ over M. But by the generalized freedom axiom and Claim 1.3.9, a $\downfree$-Morley sequence starting with $c_{\infty,1}$ over M' will remain an $\downfree$-Morley sequence over M, so will witness the Kim-dividing of $\varphi(x, c_{\infty,1})$ over M, and thus over M', by the inconsistency of $\{\varphi(x, c_{i,1})\}_{i\in\mathbb{Q}}$. Since $c_{\infty,1} \equiv_M c_{\infty,0}$, this contradicts the relative Kim's lemma.

($\Leftarrow$) Suppose the relative Kim's lemma fails; then we find a, b, M' with $\mathrm{tp}(a/M'b)$ finitely satisfiable and thus invariant over M', so a fortiori $a \downfree^{K^{\downfree}}_{M'} b$, but b $\downfree$-Kim-dependent on a over M'. Let $\varphi(x, c)$ be a formula that $\downfree$-Kim divides over M, and let $q(y)$ be an M-invariant extension of $\mathrm{tp}(c/M)$ whose invariant Morley sequences do not witness this Kim-dividing. Choose a Skolemization of T.

Claim 1.3.10. *We get the same configuration as in Claim 1.3.8: There is a sequence $\{c_{i,0}, c_{i,1}\}_{i\in\mathbb{Z}}$ with $c_{i,0} \equiv_{\mathrm{dcl}_{\mathrm{Sk}}(Mc_{<i,0},c_{<i,1})} c_{i,1}$, and a formula $\varphi(x, y)$ with $\{\varphi(x, c_{i,0})\}_{i\in\mathbb{Z}}$ consistent, but $\{c_{i,1}\}_{i\in\mathbb{Z}}$, read backwards, an $\downfree$-Morley sequence over M; therefore $\{\varphi(x, c_{i,1})\}_{i\in\mathbb{Z}}$ is inconsistent.*

Proof. Attempting the method of Proposition 3.15 of [52], we fail to respect the Skolemization, so we will instead construct a very large tree, either by transfinite induction or induction and compactness. See the proof of Theorem 3.5.3; the construction here will be similar but easier. The requirements of the tree will be that the paths, read in the direction of the root, will be invariant Morley sequences in $q(y)$ over M, while the branches at each node will form an $\downfree$-Morley sequence over M (read left to right as in the proof of Claim 1.3.8. At the successor stage, suppose the tree I is already constructed. Take an $\downfree$-Morley sequence $\{I^i\}_{i<\kappa}$ of copies of I over M, then choose a new root node realizing $q(y)|_{M\{I^i\}_{i<\kappa}}$, and reindex accordingly.

The claim follows as in the last paragraph of the proof of Claim 1.3.8. □

Now we follow Proposition 5.6 in Chernikov and Ramsey in [28], the result underlying the other direction of Kaplan and Ramsey's symmetry characterization of NSOP$_1$ in [52]. After choosing $\{c_{i,0}, c_{i,1}\}_{i\in\mathbb{Z}}$ as in Claim 1.3.10 to be indiscernible in the Skolemization, let us replace the index set $\mathbb{Z}$ with $\mathbb{Q}^+ = \mathbb{Q} \cup \{\infty\}$. Let $M' = \mathrm{dcl}_{\mathrm{Sk}}(M\{c_{i,0}, c_{i,1}\}_{i\in\mathbb{Q}})$. Let $a = c_{\infty,0}$ and by Ramsey, compactness and an automorphism, let $b \models \varphi(x, a)$ so that $\{c_{i,0}\}_{i\in\mathbb{Q}^+}$ is Mb-indiscernible. Then $\mathrm{tp}(a/M'b)$ is finitely satisfiable over M'. It remains to show that $\varphi(x, a)$ $\downfree$-Kim-divides over M'. By invariance and $a \equiv_{M'} c_{1,\infty}$, it is enough to show that $\varphi(x, c_{1,\infty})$ $\downfree$-Kim-divides over M'. Since $\{c_{i,1}\}_{i\in\mathbb{Q}^+}$ read backwards, is an $\downfree$-Morley sequence over M starting with $c_{1,\infty}$, so will witness Kim-dividing of $\varphi(x, c_{1,\infty})$, we only have to show the following claim, concluding by the generalized freedom axiom as in the other direction:

Claim 1.3.11. *There is an $\downfree$-Morley sequence over M of realizations of* $\mathrm{tp}(c_{1,\infty}/M')$

Proof. Exactly as in Claim 1.3.9. □

□

Example 1.3.12. In Examples 1.3.2 and 1.3.3, $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}$-Kim-independence coincides with $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}^a$. Clearly it implies $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}^a$. Now suppose $a \mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}^a_M b$. By extension for $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}^a$ we can assume a and b are algebraically closed sets, or models, containing M. Now for Example 1.3.2 that $a \mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}^{K^{\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}}}_M b$ follows from Lemma 7.6 of [32] (or the proof of Remark 1.3.1). For Example 1.3.3 (where $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}$ differs from free amalgamation in general), it follows from the discussion in that example.

Example 1.3.13. If T is NSOP$_1$, $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}$-Kim independence will always coincide with Kim-independence.

If $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}$ satisfies the generalized freedom axiom and the equivalent conditions of Theorem 1.3.6, then the in this case superficially relative notion of $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}$-Kim independence is not really a relative notion at all, but rather a new notion of independence with an intrinsically model-theoretic definition.

Definition 1.3.4. *Let M be a model and $\varphi(x, b)$ a formula. We say $\varphi(x, b)$* Conant-divides *over M if for* every *invariant Morley sequence $\{b_i\}_{i\in\omega}$ over M starting with b, $\{\varphi(x, b)\}_{i\in\omega}$ is inconsistent. We say $\varphi(x, b)$* Conant-forks *over M if and only if it implies a disjunction of formulas Conant-dividing over M. We say a is* Conant-independent *from b over M, written $a \mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}^{K^*}_M b$, if $\mathrm{tp}(a/Mb)$ does not contain any formulas Conant-forking over M.*

Note that Conant-dividing is just "strong Kim-dividing," Definition 5.1 of [54].

Corollary 1.3.13.1. *Suppose $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}$ satisfies the generalized freedom axiom. Then if $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}$-Kim independence is symmetric, it coincides with Conant-independence.*

Proof. By Theorem 1.3.6, $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}$-Kim-dividing coincides with Conant-dividing, so $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}$-Kim-forking coincides with Conant-forking. □

Conant-independence will coincide with Kim-independence in NSOP$_1$ theories and with $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}^a$ in free amalgamation theories (see below), so it is not readily apparent from these examples that Conant-independence is a new independence notion. Nonetheless, in section 5 we will discuss some interesting examples of strictly NSOP$_4$ theories not covered by these cases.

Note that a related notion called "Conant-independence" is defined using *finitely satisfiable* Morley sequences in Chapter 3, where it is shown to coincide with Kim-independence in NSOP$_2$ theories. Despite the fact that the choice between invariant or finitely satisfiable Morley sequences does not matter for Kim-independence in NSOP$_1$ theories (see Fact 1.2.1), it is not known when our notion of Conant-independence and the one from Chapter 3 coincide[2].

[2] We suspect that they do not, even in NSOP$_4$ theories. If Conant-independence with respect to finitely satisfiable Morley sequences coincided with the standard Conant-independence $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}^a$ in the triangle-free random graph, and there were also minimal finitely satisfiable types in restriction of the Kim-dividing order to finitely satisfiable types, then $\mathop{\smile\hskip-0.9em\raise0.6em\hbox{$|$}}^a$ would have to satisfy a "weak independence theorem" (see Proposition 6.10 of [52] for the original result, or proposition 3.5.5, whose proof is quoted below, for a result involving

Example 1.3.14. In an NSOP_4 theory, or even a theory with symmetric Conant-independence, the relation $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$ might satisfy the generalized freedom axiom but not the equivalent conditions of Theorem 1.3.6. Consider the theory of the generic K_3-free graph with two constants, c and d, for distinct vertices. This is NSOP_4, originally by work of Shelah ([101]). Declare $A \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}_M B$ if $A \cap B \subseteq M$ and a node a of $A\backslash M$ and a node b of $B\backslash M$ have an edge between them if and only if a connects to c (denoted by its constant), b connects to d (denoted by its constant) and a and b connect to no common vertices in M. This makes sense as a stationary relation, as the fact that a and b connect to no common vertices in M tells us that no triangles can be formed with vertices in M, and the fact that two vertices of $A\backslash M$ or $B\backslash M$ must connect to a common constant, so must not be connected themselves, if they are to recieve any new edges, tells us that no triangles can be formed without vertices in M. We will first show the generalized freedom axiom. Suppose there is an $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$-Morley sequence $\{A_i\}_{i\in\omega}$ of copies of a set A over M realizing the type of A over $M' \succ M$. Then clearly $A_i \cap M' \subseteq M$ and $A_i \cap A_j \subseteq M$ for $i \neq j$, and it remains to show that the edges between nodes a of $A_i\backslash M$ and b of $A_j\backslash M$ remain as selected according to $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$, when considered over M'. If there is no edge between a and b, then the conditions for $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$ to dictate this remain clearly remain true over M'. On the other hand, if a and b are related by an edge, then (assuming $i > j$) a must be connected to c and b to d, but they must still not be connected to any common nodes even in $M'\backslash M$, because then a triangle would be formed. So the conditions for an edge according to $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$ remain true over M' in this case, too.

However, $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$ does not satisfy the relative Kim's lemma (so neither is relative Kim-independence symmetric). Consider distinct disconnected vertices $\bar{c} = \{c_1, c_2\}$ outside of M, c_1 connected only to c and to no other vertices of M, c_2 connected only to d and to no other vertices of M. Consider the formula $\varphi(x, \bar{c}) =: x \neq c_1 \wedge x \neq c_2 \wedge xRc_1 \wedge xRc_2$. Clearly this does not Kim-dividide with respect to the standard free amalgamation given by adding no new edges. But it $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$-Kim-divides, as if $\bar{c}^1, \bar{c}^2$ begin an $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$-Morley sequence of copeies of $\bar{c}$ over M, then c_1^2 and c_2^1 are related by an edge, making it impossible for some other vertex to connect to both of them.

Therefore, we cannot get a witnessing lemma for stationary independence relations for NSOP_4 theories of the kind we obtained for NSOP_2 theories in Theorem 3.4.2 in order to prove the NSOP_1-SOP_3 dichotomy (so in that case, NSOP_1). Moreover, this example tells us that even though the equivalent conditions of Theorem 1.3.6 imply that $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$-Kim independence is just Conant-independence, in the statement of Theorem 1.3.6 we must still consider $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}$-Kim-independence itself and not just Conant-independence, as Conant-independence can be symmetric (perhaps due to the presence of a relation $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}_1$ with the generalized freedom axiom and the relative Kim's lemma, relative Kim-independence with respect to which will then be Conant-independence; see Example 1.3.2for the triangle-free random graph), even if

Conant-independence with respect to finitely satisfiable Morley sequences) with respect to those minimal finitely satisfiable types. But satisfying a weak independence theorem for $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}^a$ can be seen in this example to characterize free amalgamation (the standard one, with no new edges). We do not think that the invariant types given by free amalgamation in the triangle-free random graph are finitely satisfiable.

there is some relation $\mathop{\smile\!\!\!\!\!\!\,|}_2$ (which will then be different from $\mathop{\smile\!\!\!\!\!\!\,|}_1$, such as in the relation constructed in this example) with the generalized freedom axiom but without the relative Kim's lemma, so that $\mathop{\smile\!\!\!\!\!\!\,|}_2$-Kim independence, which will not be Conant-independence, remains asymmetric.

We conclude this section by isolating two model-theoretic assumptions, related by the generalized freedom axiom as in Theorem 1.3.6 and without any known NSOP_4 counterexamples, which together imply that a theory must be either NSOP_1 or SOP_3, and either TP_2 or simple.

Definition 1.3.5. *We say a theory T has the strong witnessing property if for $M \prec \mathbb{M}$ there is some $\mathbb{M}_1 \succ M$ so that for $b \subset \mathbb{M}_1$, $\mathrm{tp}(b/\mathbb{M})$ is an M-invariant extension of $\mathrm{tp}(b/M)$ minimal in the Kim-dividing order.*

Note that if $\mathop{\smile\!\!\!\!\!\!\,|}^i$ has left extension, then whether a formula $\varphi(x, b)$ Kim-divides over M is unchanged under adding or removing unused parameters, so under this assumption the strong witnessing property is satisfied as long as every type over M has an M-invariant extension minimal in the Kim-dividing order.

Theorem 1.3.15. *If a theory T satisfies the strong witnessing property and has symmetric Conant-independence, then it is either NSOP_1 or SOP_3.*

Proof. If $p(x)$ is a type over M, then define a *strong witnessing extension* of $p(x)$ to be a global extension $q(x)$ of $p(x)$ so that, for all tuples $b \in \mathbb{M}$ if $c \in \mathbb{M}$ with $c \models q(x)|_{Mb}$, then for any $a \in \mathbb{M}$ there is $a' \equiv_{Mc} a$ with $a' \in \mathbb{M}$ so that $\mathrm{tp}(a'c/Mb)$ extends to an M-invariant type minimal in the Kim-dividing order among M-invariant extensions of $\mathrm{tp}(a'c/M) = \mathrm{tp}(ac/M)$. By the strong witnessing property, strong witnessing types extending any $p(x)$ exist (see Lemma 3.5.4). Conant-dividing is the same as Kim-dividing witnessed by a Morley sequence in some (any) strong witnessing type, and Conant-forking is the same as Conant-dividing as in Proposition 1.3.5. Meanwhile, Conant-independence is symmetric by assumption, and the chain condition for Conant-independence with respect to Morley sequences in strong witnessing types is as in Claim 1.3.7. So the result follows nearly word-for-word from the proofs of Proposition 3.5.5 and the discussion in Chapter 3.6, just replacing any reference to the coheir notions of Conant-independence and Kim-dividing independence with the invariant notions, and replacing any reference to canonical coheirs and canonical Morley sequences with strong witnessing types and invariant Morley sequences in those types.

Note that the proof of Proposition 3.5.5 comes directly from the proof of the "weak independence theorem" (Proposition 6.10) of Kaplan and Ramsey in [52]. It plays the role in our argument that the freedom axiom plays in Theorem 7.17 of [32], showing that a modular free amalgamation theory must be either simple or SOP_3. The proof of that theorem serves as a basis for Chapter 3.6, which requires an new argument as in Claim 3.6.1 of that section or as in [74]. □

The following follows Theorem 7.7 of Conant ([32]), using a similar argument:

Theorem 1.3.16. *If a theory T satisfies the strong witnessing property and has symmetric Conant-independence, then it is either* TP_2 *or simple.*

Proof. If T is not simple then dividing-independence is not symmetric, so because Conant-independence is symmetric, there are $a \mathop{\smile\!\!\!\!\!\mid}_M^{K^*} b$ and indiscernible sequence $I = \{b_i\}_{i<\kappa}$ for very large κ_1, starting with b, and $\varphi(x, b) \in \mathrm{tp}(a/Mb)$ so that $\{\varphi(x, b_i)\}_{i \in I}$ is k-inconsistent for some k. Find a copy pver I, $I' \equiv_M I$, in $\mathbb{M}_1$ from the definition of the strong witnessing property, and let $q(\overline{y}) = \mathrm{tp}(I'/\mathbb{M}_1)$. By the pigenonhole principle we can assume that each term of I satisfies the same type $q_1(y)$ over $\mathbb{M}$. Now take an invariant Morley sequence in $q(\overline{y})$ over M. Taking the each copy of I as a row, which will then give us an inconsistent set of instances of $\varphi(x, y)$, we see also that the paths are consistent, being invariant Morley sequences in the type $q(y)$ which is minimal in the Kim-dividing order, with $\varphi(x, b)$ not Conant-dividing over M. $\square$

Corollary 1.3.16.1. *Suppose* $\mathop{\smile\!\!\!\!\!\mid}$ *satisfies the generalized freedom axiom and the relative Kim's lemma. Then T is either* NSOP_1 *or* SOP_3*, and is either simple or* TP_2*.*

Remark 1.3.17. Under the hypotheses of the corollary, applying the generalized freedom axiom rather than the chain condition in the proof of the "weak independence theorem" analogue, we get (by stationarity) a "base monotone" version of this result (see section 2 of [71] for related results on "base monotone" versions of independence): if $M' \mathop{\smile\!\!\!\!\!\mid}_M^{K^*} a$ and $M' \mathop{\smile\!\!\!\!\!\mid}_M^{K^*} b$ and $a \mathop{\smile\!\!\!\!\!\mid}_{M'} b$, then $a \mathop{\smile\!\!\!\!\!\mid}_M b$. When in addition $\mathop{\smile\!\!\!\!\!\mid}^{K^*} = \mathop{\smile\!\!\!\!\!\mid}^a$, note the resemblance to the case of the freedom axiom where $C = M'$, $D = M$ are models and $C \cap AB \subseteq C \cap \mathrm{acl}(AD)\mathrm{acl}(BD) = D \subseteq C$.

1.4 Non-modular free amalgamation theories

The following property of relations $\mathop{\smile\!\!\!\!\!\mid}$ between sets can be found in the "full transitivity" from section 2:

Definition 1.4.1. *The relation* $\mathop{\smile\!\!\!\!\!\mid}$ *has base monotonicity if $A \mathop{\smile\!\!\!\!\!\mid}_B C$ and $B \subseteq D \subseteq C$ then $A \mathop{\smile\!\!\!\!\!\mid}_D C$.*

This is Proposition 8.8 of [52]:

Fact 1.4.1. *An* NSOP_1 *theory is simple if and only if Kim-independence satisfies base monotonicity for $B = M \prec M' = D$ models.*

Conant asks ([32], Question 7.19) if any free amalgamation theory is *modular*:

Definition 1.4.2. *A theory is* modular *if* $\mathop{\smile\!\!\!\!\!\mid}^a$ *has base monotonicity.*

Answering this question, we give an example of a nonmodular free amalgamation theory. Kruckman and Ramsey ([71]) show that the empty theory in a language with a binary function symbol $f(x, y)$ has a strictly NSOP_1 model completion, where Kim-independence coincides with $\mathop{\smile\!\!\!\!\!\!|}\,^a$ and the algebraic closure coincides with closure under $f(x, y)$. This theory is therefore non-modular. To show that forking coincides with dividing for complete types over models, they introduce a relation of "free amalgamation" that we expect to satisfy the generalized freedom axiom (see example 1.3.4), but it is not a free amalgamation relation in the sense of [32]. We define a nonstandard relation $\mathop{\smile\!\!\!\!\!\!|}$, which will satisfy the free amalgamation axioms. Let T be the model completion of the empty theory in the language with a binary function symbol, with an additional constant symbol c. Define $A \mathop{\smile\!\!\!\!\!\!|}_C B$ to mean $A \cap B \subseteq C$ and, for $a \in A \backslash C$ and $b \in B \backslash C$, $f(b, a) = f(a, b) = c$. We show that $\mathop{\smile\!\!\!\!\!\!|}$ is a free amalgamation relation. Invariance through full transitivity are straightfoward. For full existence, we can enlarge a, B to their algebraic closure with C, which we assume is algebraically closed. We can easily find a structure in the language extending C where a and B embed disjointly over C, and where a point with coordinates properly in each of a and B will have image c. Full existence then follows from the fact that T is the model completion. If $C \subseteq a \cap b$, a, b, C algebraically closed, then $a \mathop{\smile\!\!\!\!\!\!|}_C b$ determines the isomorphism type of ab over C, so stationarity follows from quantifier elimination. For freedom, if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $C \cap AB \subseteq D \subseteq C$, then $A \cap B \subseteq C \cap AB \subseteq D$, while for $a \in A \backslash D \subseteq A \backslash C$, $b \in B \backslash D \subseteq B \backslash C$, $f(a, b) = f(b, a) = c$ as before. Finally, if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ for $C \subseteq A \cap B$ and A, B, C algebraically closed, then the closure under $f(x, y)$, and therefore the algebraic closure, of AB remains AB, yielding the closure axiom.

The existence of non-modular free amalgamation theories motivates the following generalization of Theorem 7.17 of [32] that modular free amalgamation theories are either simple or SOP_3:

Theorem 1.4.2. *Free amalgamation theories are either* NSOP_1 *or* SOP_3.

First, we observe that, justifying the terminology, Lemma 7.6 of Conant in [32] is essentially a characterization of Conant-independence:

Proposition 1.4.3. *Conant-independence in free amalgamation theories coincides with* $\mathop{\smile\!\!\!\!\!\!|}\,^a$ *over models.*

Proof. Clearly Conant-independence implies $\mathop{\smile\!\!\!\!\!\!|}\,^a$. Conversely, suppose $a \mathop{\smile\!\!\!\!\!\!|}^a_M b$. By extension for $\mathop{\smile\!\!\!\!\!\!|}\,^a$, we can assume that a and b are algebraically closed sets containing M. But then by Lemma 7.6 of [32], there will be an $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence over M (extending Definition 1.3.1 appropriately) starting with b and indiscernible over Ma, so $a \mathop{\smile\!\!\!\!\!\!|}^{K^*}_C b$. □

We also see that free amalgamation theories satisfy the strong witnessing property (Definition 1.3.5): a formula $\varphi(x, b)$ not Conant-dividing over M must have a realization a with $a \mathop{\smile\!\!\!\!\!\!|}^a_M b$, and then we can proceed as in the above proof. So the theorem follows from Theorem 1.3.15.

If a free amalgamation theory T is NSOP$_1$, then by Fact 1.2.1 and Proposition 1.4, $\mathop{\smile\!\!\!\!\!\!|}^{K} = \mathop{\smile\!\!\!\!\!\!|}^{K^*} = \mathop{\smile\!\!\!\!\!\!|}^{a}$. Thus the characterization of simple theories within the class NSOP$_1$ in Fact 1.4.1 gives us the following, extending Conant's result [32] that a simple free amalgamation theory is modular:

Proposition 1.4.4. *An* NSOP$_1$ *free amalgamation theory is modular if and only if it is simple.*

1.5 Some examples

We consider two examples of theories with relations satisfying the assumptions at the beginning of Section 3, as well as the generalized freedom axiom and the relative Kim's lemma, and characterize Conant-independence in these structures. Our purposes are twofold: to give a model-theoretic interpretation of certain tame independence relations in potentially strictly NSOP$_4$ theories, and to extend the concept of free amalgamation to examples not covered by Conant's work in [32].

Example 1.5.1. (Countably categorical Hrushovski constructions.) We consider the case of the examples of ω-categorical structure with a predimension introduced in section 3 of [41], which is developed in [42]. Let $\mathcal{L}$ be a language with finitely many relations ([42] only require finitely many relations of each arity, but we include this requirement so that the predimension function only takes a discrete set of values), and for each relation symbol R_i, let α_i be a non-negative real number associated to R_i. For A a finite $\mathcal{L}$-structure, define a *predimension* $d_0(A) = |A| - \sum_i \alpha_i |R_i(A)|$, with $R_i(A)$ the set of tuples of R_i with elements of A, and define the relation $A \leq B$ for A a finite $\mathcal{L}$-structure and B any $\mathcal{L}$-structure to mean that every finite superstructure of A within B has predimension greater than A. Let f be an increasing continuous positive real-valued function and let $\mathcal{C}_f$ be the class of finite $\mathcal{L}$-structures any substructure A of which satifies $d_0(A) \geq f(A)$. Assume that, if $B_1 \geq A \leq B_2$ belong to $\mathcal{C}_f$, then their evident "free amalgamation," by taking their disjoint union over A and adding no new edges, likewise belongs to $\mathcal{C}_f$. Then there is a $\mathcal{L}$-structure M every finite substructure of which belongs to $\mathcal{C}_f$ and so that if $B \geq A \leq M$ with B finite, then there is an embedding $\iota : B \to M$ over A so that $\iota(B) \leq M$. Let T be its (complete) theory. The theory T is ω-categorical, so has bounded algebraic closure, and isomorphic algebraically closed sets are elementarily equivalent. For M a model of T, and $A \subseteq B \subseteq M$ with A finite and B any set, A is algebraically closed in B if $A \leq B$, and M will always continue to have the property that if $B \geq A \leq M$ with B finite, then there is an embedding $\iota : B \to M$ over A so that $\iota(B) \leq M$.

Though T is not necessarily simple, [42] show that it is either strictly NSOP$_4$ or simple. However, it does have a natural notion of independence, even in the strictly NSOP$_4$ case. We first recall an additional property of an abstract relation $\mathop{\smile\!\!\!\!\!\!|}$ between sets:

Definition 1.5.1. *We say* $\mathop{\smile\!\!\!\!\!\!|}$ *has* finite character *if* $A \mathop{\smile\!\!\!\!\!\!|}_B C$ *holds whenever* $A \mathop{\smile\!\!\!\!\!\!|}_B C_0$ *holds for all finite* $C_0 \subseteq C$.

This notion of independence, called d-independence, is defined in [41]; it will coincide with forking-independence in the simple case. For finite A, B, denote $d(A/B) = d_0(\mathrm{acl}(AB)) - d_0(\mathrm{acl}(B))$ (recalling the bounded algebraic closure). This notion of relative dimension has a natural extension over infinite sets: for A a finite set and B any set, denote $d(A/B) = \min(\{d(A/B_0) : B_0 \subseteq B \text{ finite}\})$. We use the following notation for the relation referred to in [41] as *d-independence*: for a finite and B, C any sets, let $a \mathop{\smile\!\!\!\!\!\!|}^d_B C$ if and only if $d(a/BC) = d(a/B)$ and $\mathrm{acl}(aB) \cap \mathrm{acl}(CB) = \mathrm{acl}(B)$; for a, B, C finite this last condition will be redundant. In [41] it is shown that this has finite character and is symmetric, monotone and fully transitive where defined, so it extends naturally to a relation defined for a possibly infinite with the same properties. We claim that there is a natural relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfying the assumptions at the beginning of section 3 as well as the generalized freedom axiom and the relative Kim's lemma, and that Conant-independence coincides with $\mathop{\smile\!\!\!\!\!\!|}^d$ (so is in particular, symmetric).

We first observe a variant of property (P5) of [41] which, in place of a finitary analogue of the "independence theorem" holding only in the simple examples, constitutes a base-monotone version of the "weak independence theorem" with respect to free amalgamation analogous to those in [52] with respect to coheir-independence or Chapter 3 with respect to canonical coheirs. This is used implicitly in [42] to show NSOP_4, but we provide some justification.

(P5') Let $B_1 \geq A \leq B_2$ be finite algebraically closed sets such that B_1 and B_2 are *freely amalgamated* over A, which is to say $\mathrm{acl}(B_1B_2)$ is the disjoint union of B_1 and B_2 over A with no new relations. Let c_1, c_2 be finite with $c_1 \mathop{\smile\!\!\!\!\!\!|}^d_A B_1$, $c_1 \mathop{\smile\!\!\!\!\!\!|}^d_A B_2$ with $c_1 \equiv_A c_2$; then there is some c realizing $\mathrm{tp}(c_1/B_1) \cup \mathrm{tp}(c_2/B_2)$–with $c \mathop{\smile\!\!\!\!\!\!|}^d_A B_1B_2$ (which is not needed here)–and $\mathrm{acl}(cB_1)$ and $\mathrm{acl}(cB_2)$ freely amalgamated over $\mathrm{acl}(cA)$.

When we only require that $B_1 \mathop{\smile\!\!\!\!\!\!|}^d B_2$ rather than that they be freely amalgamated, this is shown in Theorem 3.6(ii) of [41] under assumptions on f, so we need only observe that this proof works for this partial result without the assumptions on f. As in that proof we can form the $\mathcal{L}$-structure $F = E_{12} \cup E_{13} \cup E_{23}$ with no new relations, and with compatible isomorphisms $\varphi_{12} : \mathrm{acl}(B_1B_2) \to E_{12}$, $\varphi_{j3} : \mathrm{acl}(c_jB_j) \to E_{j3}$, which will be a special case of the construction from that proof where the "underlying" predimension y is just the cardinality. Now by point (i) of that proof, which does not use the additional assumption on f required for simplicity, $E_{ij} \leq E$. The part of the proof where this additional assumption is required is point (ii), where it is shown that $F \in \mathcal{C}_f$; it must be shown that for each $D \subseteq F$, $d_0(D) \geq f(|D|)$. However, the assumption on f is only used when D is not contained in the union of two of the E_{ij} (where the requirement follows by closure under free amalgamation). But $F = E_{13} \cup E_{23}$ because $\mathrm{acl}(B_1B_2) = B_1 \cup B_2$. So embedding a copy of F over B_1B_2 (where B_1B_2 is identified by its image in $E_{12} \subset F$) so that it is algebraically closed will realize both types, and in a d-independent way by point (iii), which does not rely on the additional assumptions on f.

Now note that for $B_1 \geq A \leq B_2$ algebraically closed finite sets and c any finite set with $\mathrm{acl}(cB_1)$ and $\mathrm{acl}(cB_2)$ freely amalgamated over $\mathrm{acl}(cA)$, the type of cB_1 and cB_2 then completely determine the type of cB_1B_2 and in particular B_1B_2, so (P5') implies that B_1 and B_2 are freely amalgamated over A. This observation leads to the following definition: for M a model and b, c finite sets of parameters, say $a \mathop{\smile\!\!\!\!\!\!|}_M b$ if for any finite $A \leq M$ with $a \mathop{\smile\!\!\!\!\!\!|}^d_A M$, $b \mathop{\smile\!\!\!\!\!\!|}^d_A M$ (such an A always exists because d_0 only takes a discrete set of values; see Lemma 2.17(a)(ii) of [41]), $\mathrm{acl}(aA)$ and $\mathrm{acl}(bA)$ are freely amalgamated over a. For existence, by compactness, it suffices to show that for types $p(x)$ and $q(y)$ over M, finitely many finite $A_i \leq M$ such that $p(x)$ and $q(x)$ d-independently extend their restrictions to A_i, and finite $B \subseteq M$, there are realizations a of $p(x)|_B$ and b of $q(y)|_B$ so that $\mathrm{acl}(aA_i)$ and $\mathrm{acl}(bA_i)$ are freely amalgamated over A_i for each i. But take any $A \leq M$ containing each of the A_i and B and take realizations a of $p(x)|_A$ and b of $q(y)|_A$ so that $\mathrm{acl}(aA)$ and $\mathrm{acl}(bA)$ are freely amalgamated over A; then the free amalgamation conditions over the A_i, by the observation at the beginning of this paragraph, will be satisfied. Since by the quantifier elimination, this relation is clearly stationary, and it is monotone by the properties of free amalgamation for finite sets, it extends to a relation $a \mathop{\smile\!\!\!\!\!\!|}_M b$ for a, b potentially infinite.

We next show that if $M' \mathop{\smile\!\!\!\!\!\!|}^d_M a$ and $\{a_i\}_{i \in I}$ is an $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence starting with a over M', then it is an $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence over M. But this follows from the definitions, by the fact that if $A \leq M$ is finite with $a \mathop{\smile\!\!\!\!\!\!|}^d_A M$ and $M' \mathop{\smile\!\!\!\!\!\!|}^d_M a$, then $a \mathop{\smile\!\!\!\!\!\!|}^d_A M'$ (transitivity and symmetry). So we have the generalized freedom axiom. We can also carry out a similar proof for a set in place of M' (which we can assume to be algebraically closed and contain M), so $\mathop{\smile\!\!\!\!\!\!|}^d$ implies $\mathop{\smile\!\!\!\!\!\!|}$-Kim independence.

We next show that $\mathop{\smile\!\!\!\!\!\!|}$-Kim independence implies $\mathop{\smile\!\!\!\!\!\!|}^d$: if $\mathop{\smile\!\!\!\!\!\!|}^d$ fails, the proof from [41] that this implies dividing-independence (Lemma 2.19 (a) of [41]) relies on [65], which will tell us than any $\mathop{\smile\!\!\!\!\!\!|}^d$-independent sequence witnesses dividing. (This will actually cover both of the cases of that lemma). But $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequences are $\mathop{\smile\!\!\!\!\!\!|}^d$-independent sequences.

So $\mathop{\smile\!\!\!\!\!\!|}^d$ coincides with $\mathop{\smile\!\!\!\!\!\!|}$-Kim independence, which is then symmetric, and $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the generalized freedom axiom, so $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the relative Kim's lemma and $\mathop{\smile\!\!\!\!\!\!|}^d$ coincides with Conant-independence.

Example 1.5.2. (Random graphs without small cycles). Shelah introduces this example in Claim 2.8.5 of [101]. Let $n \geq 3$, and consider first the case where n is even. Then the theory of graphs without cycles of length not exceeding n has a model companion T, but it is not the model completion. The theory T does have quantifier elimination, however, in the graph language expanded by the definable partial function symbols F^k_m, for $k \leq m$ not more than $\frac{n}{2}$, sending vertices a and b of distance m to the kth vertex along the path between a and b; note that (particularly to the even case) any two vertices in T have a unique path of length at most $\frac{n}{2}$ between them. (We adopt the convention that paths cannot retrace themselves.) The algebraically closed sets are then the induced subgraphs any two vertices of which have distance within the subgraph not more than $\frac{n}{2}$, and are determined up to elementary equivalence by their type in the graph language. Shelah shows this theory is NSOP_4.

We define a stationary relation $\mathop{\smile\!\!\!\!\!\!|}$ over models, as follows. Define $a \mathop{\smile\!\!\!\!\!\!|}_M b$ if $a \mathop{\smile\!\!\!\!\!\!|}^a_M b$ and the algebraic closure of Mab is constructed as follows, noting that this construction is free of any choice and thus gives, by quantifier elimination, a relation with invariance and stationarity. Let the graph given by $\mathrm{acl}(Ma) \cup \mathrm{acl}(Mb)$ with no new edges be stage 0. At stage $n+1$, add formal vertices forming a unique path of length $\frac{n}{2}$ between any two vertices with no path of length at most $\frac{n}{2}$ between them in stage n. Note that this cannot create small cycles, as a small cycle can be assumed to either contain any new formal path or to not meet it at all, but it would be too small to contain two new formal paths (and an edge from the old graph) and if it contained only one formal path then that path would never have been placed based on our criteria. Since T is the model companion, we have existence, and we need to show monotonicity.

To that end, we claim that if $A \subseteq B$ with A algebraically closed in B (that is, each path in B of length $\frac{n}{2}$ between verticies of A goes through A), then when we apply the construction of adding a formal path of length $\frac{n}{2}$ between any two vertices without such a path within B, we have that:

(i) the restriction of this construction to A (looking only at A together with the new paths added between vertices of A) is the same as what we would have obtained by looking only at A originally and applying the same construction

(ii) this restriction of the construction to A remains algebraically closed within this construction applied to B.

For (i), the only reason we would have added a new path between vertices of A considered alone, but not within B, is that there is a path between those vertices of length $\frac{n}{2}$ within B that is not within A. But that does not happen, because A is algebraically closed in B. For (ii), any path of length at most $\frac{n}{2}$ between two nodes of the restriction of this construction to A, within the construction applied to B, cannot go through one of the new formal paths that are *not* between two vertices of A as then it would be too long. So it must stay within B together with the new formal paths between vertices of A, but the parts of the path within B will in fact be within A, since A is algebraically closed in B.

It follows by induction that if A is algebraically closed in B, then applying all of the stages of the construction to B and restricting it to A is the same as applying it to A originally. But if $A \mathop{\smile\!\!\!\!\!\!|}^a_M B$ and $A' \subseteq A$, $B' \subseteq B$, then $\mathrm{acl}(MA') \cup \mathrm{acl}(MB')$ is algebraically closed in $\mathrm{acl}(MA) \cup \mathrm{acl}(MB)$; for any path of length at most $\frac{n}{2}$ between two vertices of $\mathrm{acl}(MA') \cup \mathrm{acl}(MB')$, the parts within $\mathrm{acl}(MA)$ will be in $\mathrm{acl}(MA')$ and the parts within $\mathrm{acl}(MB)$ will be in $\mathrm{acl}(MB')$. Thus we have monotonicity for $\mathop{\smile\!\!\!\!\!\!|}$.

Now define the relation $A \mathop{\smile\!\!\!\!\!\!|}^{\frac{n}{4}}_M B$ to mean that $A \mathop{\smile\!\!\!\!\!\!|}^a_M B$ and for $a, b \in \mathrm{acl}(AM) \cup \mathrm{acl}(BM)$ so that there is no path of length at most $\frac{n}{4}$ between a and b in the graph $\mathrm{acl}(AM) \cup \mathrm{acl}(BM)$ with no new edges, a and b are of distance greater than $\frac{n}{4}$ apart. (Note the importance of this distance restriction in Shelah's proof of NSOP_4.) We claim that if $M' \mathop{\smile\!\!\!\!\!\!|}^{\frac{n}{4}}_M a$, then an $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence of copies of a over M' will remain so over M. To see this, note that, if $\bigcup_{i\in I}\mathrm{acl}(M'a_i)$ were a disjoint union over M' of formal copies of $\mathrm{acl}(M'a_i)$ over M' with no new edges, then $\bigcup_{i\in I}\mathrm{acl}(Ma_i)$ would be algebraically closed within $\bigcup_{i\in I}\mathrm{acl}(M'a_i)$, as otherwise a path of length at most $\frac{n}{2}$ between two of the $\mathrm{acl}(Ma_i)$ would have to pass through $M'\backslash M$, and it would be too short not to pass between one of the $\mathrm{acl}(Ma_i)$ and $M'\backslash M$ in no greater than $\frac{n}{4}$ steps. So by the above observation that the construction of repeatedly adding formal paths does not depend on the ambient graph in which a graph is assumed algebraically closed, an $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence of copies of a over M' will remain an $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence over M. As the same reasoning works for a set in place of M', we see that $\mathop{\smile\!\!\!\!\!\!|}^{\frac{n}{4}}$ implies $\mathop{\smile\!\!\!\!\!\!|}$-Kim independence.

We show the reverse implication, which will tell us additionally that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the relative Kim's lemma and $\mathop{\smile\!\!\!\!\!\!|}^{\frac{n}{4}}$ is Conant-independence. Suppose $a \mathop{\smile\!\!\!\!\!\!|}^{\frac{n}{4}}_M b$ is false, and $a \mathop{\smile\!\!\!\!\!\!|}^{a}_M b$. Then there is a path of length at most $\frac{n}{4}$ not passing through M between a vertex a of $\mathrm{acl}(Ma)$ and a vertex of $\mathrm{acl}(Mb)$. Let $\varphi(x,b)\in \mathrm{tp}(a/Mb)$ imply that there is such a path. (Note that for a path of length at most $\frac{n}{4}$ not to pass through M, it need only avoid the finitely many elements of M within distance $\frac{n}{4}$ of b.) Suppose $\{b_i\}_{i\in\omega}$ is an invariant Morley sequence with $\{\varphi(x,b_i)\}_{i\in\omega}$ consistent, realized by some a'. Then a' will lie on a path of length at most $\frac{n}{2}$ between vertices of $\mathrm{acl}(Mb_0)\cup\mathrm{acl}(Mb_1)$ avoiding M, so will belong to $\mathrm{acl}(Mb_0b_1)\backslash M$. Similarly, $a\in\mathrm{acl}(Mb_2b_3)\backslash M$. But the concatenation $\{b_{2i}b_{2i+1}\}_{i\in\omega}$ remains an invariant Morley sequence, so $b_0b_1 \mathop{\smile\!\!\!\!\!\!|}^{a}_M b_2b_3$, a contradiction.

Note that $\mathop{\smile\!\!\!\!\!\!|}^{\frac{n}{4}}$ does not coincide with $\mathop{\smile\!\!\!\!\!\!|}^{a}$, making this an interesting case of Conant-independence. To see this, consider an vertex a of distance $\frac{n}{2}$ from the model M and take some algebraically closed graph (that is, a graph with no two vertices farther than $\frac{n}{2}$ apart) $B\supset M$ containing M and a, then take two disjoint copies B_1 and B_2 of this graph over M, with a_1 and a_2 the copies of a over M, and no further edges. Then we can add an edge of length at most $\frac{n}{4}$ between a_1 and a_2 and not create any small cycles. Embed this into a larger model over M, and the images of B_1 and B_2 will be independent according to $\mathop{\smile\!\!\!\!\!\!|}^{\frac{n}{4}}$ but not $\mathop{\smile\!\!\!\!\!\!|}^{a}$.

The case where $n=2m+1$ is odd is different in that, while the quantifier elimination still holds in the language expended by the definable partial function symbols, two vertices can be of length $m+1$ apart and none of the partial function symbols can be defined there, in which case there are infinitely many paths of length $m+1$ between them. So defining $\mathop{\smile\!\!\!\!\!\!|}$ is easier: let $a \mathop{\smile\!\!\!\!\!\!|}_M b$ if $a \mathop{\smile\!\!\!\!\!\!|}^{a}_M b$ and any two vertices in $\mathrm{acl}(Ma)\cup\mathrm{acl}(Mb)$ that are not already of distance at most m apart within $\mathrm{acl}(Ma)\cup\mathrm{acl}(Mb)$ with no new edges will have distance $m+1$. Then a similar analysis holds.

1.6 Conant-independence in the NSOP_n hierarchy

We prove that symmetry of Conant-independence implies NSOP_4. We begin with the following fact, whose proof is essentially that of Proposition 3.5.1:

Fact 1.6.1. *For formulas, Conant-forking over M implies witnessing of Kim-dividing by any Morley sequence in a type finitely satisfiable over M.*

Proof. Let $\models \varphi(x,b) \to \bigvee_{i=1}^{n} \psi_i(x,c_i)$ for $\psi_i(x,c_i)$ Conant-dividing over M, so in particular Kim-dividing over M by any invariant Morley sequence in a finitely satisfiable type. By left extension and monotonicity for $\mathop{\smile\!\!\!\!\!\!|}^{u}$, whether or not a formula $\mathop{\smile\!\!\!\!\!\!|}$-Kim-divides over M does not change when adding unused parameters, so we can assume $c_i = b$ for $1 \leq i \leq n$. Then $\varphi(x,b)$ Kim-divides over M by any invariant Morley sequence in a finitely satisfiable type over M, for suppose otherwise. Let $\{b_i\}_{i\in\mathbb{N}}$ be an invariant Morley sequence in a finitely satisfiable type over M starting with b; then there will be some a realizing $\{\varphi(x,b_i)\}_{i\in\mathbb{N}}$. So by the pigeonhole principle, there will be some $1 \leq j \leq n$ so that a realizes $\{\psi_j(x,b_i)\}_{i\in S}$ for $S \subseteq \mathbb{N}$ infinite. But by monotonicity and an automorphism, we can assume $\{b_i\}_{i\in S}$ is an invariant Morley sequence in that same finitely satisfiable type over M starting with b, contradicting Conant-dividing of $\psi_i(x,b)$. □

The following uses similar Skolemization methods to Proposition 5.6 of Chernikov and Ramsey in [28], which generalize in a surprising way to indiscernible sequences ordered by a definable relation with no 4-cycles.

Theorem 1.6.2. *Any theory where Conant-forking is symmetric is* NSOP_4. *Thus $n = 4$ is the greatest n so that there are strictly* NSOP_n *theories with symmetric Conant-independence.*

Proof. Suppose a theory T has SOP_4; we show that Conant-independence cannot be symmetric. Let $R(x,y)$ be a definable binary relation with no 4-cycles, and let $\langle a_i\rangle_{i\in I}$ be an infinite sequence so that $R(a_i,a_j)$ for $i < j$. Fixing a Skolemization of T, we can assume that this sequence is indiscernible in that Skolemization and is of the form $\langle c_i\rangle_{i\in\omega} + \langle a_1\rangle + \langle b_1\rangle + \langle a_2\rangle + \langle b_2\rangle + \langle a_3\rangle + \langle c_i\rangle_{i\in\omega^*}$. Let $M = \mathrm{dcl}_{\mathrm{Sk}}(\langle c_i\rangle_{i\in\omega} + \langle c_i\rangle_{i\in\omega^*})$, $\overline{a} = a_1a_2a_3$, $\overline{b} = b_1b_2$; we show $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} \overline{b}$ but $\overline{b}$ is Conant-dependent on $\overline{a}$ over M. For the first part, clearly $\langle c_i\rangle_{i\in\omega} + \langle a_1\rangle + \langle b_1\rangle + \langle a_2\rangle + \langle b_2\rangle + \langle a_3\rangle + \langle c_i\rangle_{i\in\omega^*}$ is contained in a sequence, indiscernible in the Skolemization, of the form $\langle c_i\rangle_{i\in\omega} + \langle a_1\rangle + \langle b_1^i\rangle_{i\in\omega} + \langle a_2\rangle + \langle b_2^i\rangle_{i\in\omega^*} + \langle a_3\rangle + \langle c_i\rangle_{i\in\omega^*}$, with $b_j^0 = b_j$ for $j = 1,2$. But $\langle b_1^ib_2^i\rangle_{i\in\omega}$ is a coheir Morley sequence over M starting with $\overline{b}$ and indiscernible over $M\overline{a}$, so by Fact 1.6.1 we get $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} \overline{b}$. For the dependent direction, we show $R(a_1,y_1) \wedge R(y_1,a_2) \wedge R(a_2,y_2) \wedge R(y_2,a_3) \in \mathrm{tp}(\overline{b}/M\overline{a})$ Conant-divides over M. Let $\langle a_1^ia_2^ia_3^i\rangle_{i\in\omega}$ be an M-invariant Morley sequence starting with $\overline{a}$ and suppose $\{R(a_1^i,y_1) \wedge R(y_1,a_2^i) \wedge R(a_2^i,y_2) \wedge R(y_2,a_3^i)\}_{i\in\omega}$ were consistent, realized by $b_1'b_2'$. Then $\models R(a_2^1,b_2') \wedge R(b_2',a_3^0)$. Now $\models \exists x R(a_1^0,x) \wedge R(x,a_2^1)$, witnessed by b_1'. But $a_1^0 \equiv_M a_3^0$, so by invariance, $a_1^0 \equiv_{M\overline{a}^1} a_3^0$, and in particular $a_1^0 \equiv_{Ma_2^1} a_3^0$. So $\models \exists x R(a_3^0,x) \wedge R(x,a_2^1)$, witnessed, say, by b_1''. But $\models R(a_2^1,b_2') \wedge R(b_2',a_3^0) \wedge R(a_3^0,b_1'') \wedge R(b_1'',a_2^1)$, a 4-cycle, contradiction. □

Thus one of the three main classification-theoretic properties Conant proved for free amalgamation theories in [32]–they are either NSOP_1 or SOP_3, are either simple or TP_2, and are NSOP_4–holds solely under the assumpton of symmetric Conant-independence. So far, we are only able to prove the other two identities for theories with symmetric Conant-independence and an additional assumption about invariant types minimal in the Kim-dividing order (see section 3); these assumptions together generalize the free amalgamation theories and are not known to fail in any NSOP_4 theory, but it would be desirable if we had a criterion analogous to independence in the simple or NSOP_1 case that gave us all of the classification-theoretic properties of free amalgamation theories. Can we get those other two properties with just symmetry for Conant-independence alone?

Problem 1.6.3. *Must a theory with symmetric Conant-independence be either simple or* TP_2*? Must it be either* NSOP_1 *or* SOP_3*?*

We are also interested in extending the theory of Kim-independence beyond NSOP_1. Given that the class of strictly NSOP_4 theories is the most complicated classification-theoretic class where Conant-independence is symmetric, we may ask whether symmetry for Conant-independence characterizes NSOP_4 the same way symmetry for Kim-independence characterizes NSOP_1.

Problem 1.6.4. *In an* NSOP_4 *theory, is Conant-independence always symmetric?*

A positive answer to both the last problem and one of the two questions from the previous problem will solve some of the open regions of the classification-theoretic hierarchy, further underscoring the connections between classification theory and the theory of model-theoretic independence.

Chapter 2

Generic expansions and the group configuration theorem

2.1 Introduction

This chapter connects two subfields of model theory: geometric stability theory and the classification theory of *unstable* structures. Geometric stability theory, an excellent exposition of which is given in [87], relates pregeometries in stable theories, such as the pregeometry defined by algebraic closure on a strongly or weakly minimal structure, to the global structure of those theories. One of the most important theorems of geometric stability theory is the *group configuration theorem* of Hrushovski, which says that the incidence pattern of four lines in a projective plane, viewed entirely from within the geometric structure of the algebraic closure in a stable theory, must arise from a type-definable group:

Fact 2.1.1. *(Group Configuration Theorem, Hrushovski ([48])): Let T be a stable theory and a, b, c, x, y, z nonalgebraic tuples. Suppose, in the below Figure 1, that any three noncollinear points are independent, but any point is in the algebraic closure of any other two points on the same line. Then for some parameter set A independent from $abcxyz$, there is some connected group G type-definable over A so that, for a', c', x' independent generics of G over A and $b' = c' \cdot a'$, $x' = a' \cdot y'$ and $b' = z' \cdot y'$, each of a, b, c, x, y, z is individually interalgebraic over D with, respectively, a', b', c', x', y', z'.*

This result has been generalized to some unstable contexts, such as simple theories [13], o-minimal theories [86], and generically stable types [111]. In the following, we will show that the original group configuration theorem for *stable* theories has applications to classification theory outside of the stable or even simple context.

One central question in the classification theory of unstable structures, much of which was initiated alongside the classification of stable theories by Shelah [97], asks which classification-theoretic properties are equivalent and which are distinct. For example, until recently it was open whether the class NSOP_1 was equal to NSOP_2, and it remains open

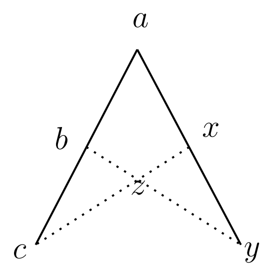

Figure 2.1: The basic case of the group configuration. Diagram based on [11].

whether NSOP_2 is equal to NSOP_3 ([40]); it is also open whether NSOP_n NTP_2 theories are simple for $n \geq 3$ ([26]). In the applied setting, there has also been interest in determining the classification-theoretic complexity of structures, including finding new examples of strictly NSOP_1 theories. Several new examples have recently been found using generic constructions, such as algebraically closed fields of prime characteristic with a generic additive subgroup ([37], [36]), generic incidence structures ([33]), generic expansions by Skolem functions ([71]), and the ω-free PAC fields ([28], further developed in [52]; see also [22], [21]), to give some examples. We will introduce the first examples, to our knowledge, where the classification-theoretic property NSOP_1 results from geometric stability theory, particularly the group configuration theorem. At the boundary of NSOP_1, the possible levels of classification-theoretic complexity have been characterized for theories with a suitable notion of *free amalgamation*. Evans and Wong ([42]) show that the ω-categorical Hrushovski constructions introduced by Evans ([41]) are either simple or SOP_3, and Conant ([32]) show that modular theories satisfying some abstract free amalgmation axioms are either simple, or both SOP_3 and TP_2; in Chapter 1, the author generalizes the work of Evans and Wong, and of Conant, to potentially strictly NSOP_1 theories, giving partial results on the equivalence of $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$ and NSOP_3 covering most of the known examples of NSOP_4 theories. We will introduce a family of structures defined by generic constructions, in particular the expansion of stable structures by new generic predicates or equivalence relations, whose complexity will be characterized by this $\mathrm{NSOP}_1 - \mathrm{SOP}_3$ dichotomy; *which* side of the dichotomy a structure in this family lies on will be characterized by the group configuration theorem.

The expansion of a theory by generic relations or function symbols was introduced by Winkler ([112]), and was studied by Chatzidakis and Pillay ([20]) in the case of a unary predicate, which was shown to preserve simplicity. Later, Kruckman and Ramsey ([71]) showed that expansions by generic function symbols, which covers generic equivalence relations considered as a unary function to a new sort, preserve the property NSOP_1. The construction is to start with a theory T, add symbols to the language but no new axioms to get the theory T_0, and take the model companion T' of T_0, which exists whenever T eliminates quantifiers and eliminates $\exists^\infty$. The setting for the correspondence between groups and classification theory will be the model companion T' of an expansion T_0 of a theory T. However, new axioms, and not just new symbols, will be added to form T_0. Allowing any axioms quickly becomes complicated, as one can encode, say, automorphisms; see [10], exposited in [88]; [59], [58], for some examples of the literature on the existence of model companions of theories with automorphisms, and [23] for a particularly interesting example. So instead of studying all possible new axioms, we add an n-ary relation R definable in a theory T with quantifier elimination, and add universal axioms of a particular form to get a new theory $T_0 = T_R$. Namely, for P a new unary relation symbol, we add $\forall \bar{x}(\bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j \wedge \bigwedge_{i=1}^{n} P(x_i) \to R(\bar{x}))$ to get T_R, or alternatively, for E a new binary relation symbol, we add that E is an equivalence relation and $\forall \bar{x} \bigwedge_{1 \leq i \neq j \leq n}(x_i \neq x_j \wedge E(x_i, x_j)) \to R(\bar{x})$ to get T_R. When T is nfcp, T_R in either case will then have a model companion $T' = T^R$.

The main result of this chapter will be on the complexity of T^R, when T is weakly minimal (so nfcp, [43] as observed in [34]) and $\neg R$ is a ternary *fiber algebraic relation* ([29], Definition 3.1). Ternary fiber algebraic relations coincide with relations of rank ≤ 2 in the strongly minimal case, and in general they include all graphs of group operations on unary definable sets: this result says that T^R will be classification-theoretically complicated precisely when $\neg R$ *is* geometrically equivalent to the graph of a group operation:

Theorem 2.1.2. *Let T be weakly minimal and let R be a ternary relation definable in T. Assume $\neg R$ is fiber-algebraic. Then the equivalence relation version of T^R is* NSOP_1 *if and only if there is no set of parameters A over which R is definable, and (rank-one) group G type-definable (or definable, if T is strongly minimal) over A, so that the coordinates of a point of $\neg R$ generic (that is, of full rank) over A are individually interalgebraic with the coordinates of a point of the graph Γ^G of the multiplication in G generic over A. Otherwise, T^R is* TP_2 *and strictly* NSOP_4.

For the predicate version of T^R, this is the same, but replace "NSOP_1*" with "simple."*

So among ternary relations $\neg R$ that have no trivial reason *not* to be the graph of a group operation, classification theory at the level of $\text{NSOP}_1 - \text{SOP}_3$ gap measures exactly when $\neg R$ is equivalent to the graph of a (rank-one) group operation.

Geometric properties of a stable theory are known to be connected to classification-theoretic properties of its expansions. For example, [16] (including a result due to Hrushovski) and [109] relate the linearity of a theory T to the rank of certain generic expansions of that theory by a *model* (see also [14] for the relationship between the *dimensional order property* of a theory and the stability of its expansions by models), and [7] shows that the pregeometry of a strongly minimal set is trivial if and only if *arbitrary* expansions of that theory by a unary predicate are stable. The literature on expansions is vast–see [8] for an overview–and includes connections between properties of stable theories and simplicity of expansions (for example, nfcp within stable theories and simplicity in pseudo-algebraically closed expansions, [91]). Our result is the first that we know of to relate geometric stability theory to the classification of unstable expansions of stable theories at the level of simplicity and the NSOP_n hierarchy.

We give an outline of the chapter. In section 1, we define our setting for the generic expansion T^R of a theory T, associated with the definable relation R. The connection between the property nfcp from [97], which implies stability, and axiomatizability of generic expansions, was first demonstrated by Poizat ([89]) in his work on *belles paires* of models. This is generalized to the simple case in [12] using the weaker *wnfcp*, and generalized further using the nfcp in [68]; see [19], [7], [73], [15] for examples of connections of nfcp to more general (not necessarily generic) expansions, and [8] for an overview of the connections between expansions and nfcp. Generalizing the arguments of Poizat, we show that when T is nfcp, and R is a definable relation, both the predicate version and the equivalence relation version of T^R exist. We also give a converse, encoding a partial automorphism of a linear order with the construction for T^R when T is unstable and showing that the model companion cannot exist using the argument from [58]. This gives us a new characterization of nfcp of independent interest. When $\neg R$ is a fiber-algebraic ternary relation, we observe that T^R has quantifier elimination up to finite covers.[1]

In section 2, we consider relational expansions of NSOP_1 theories with quantifier elimination and general free amalgamation properties. In Chapter 1, the notion of *Conant-independence* was introduced as the extension of the *Kim-independence* from [52] beyond the NSOP_1 theories. Using results of Chapter 1, we characterize Conant-independence in these theories, and show that the theory is either NSOP_1, or both TP_2 and strictly NSOP_4; the underlying arguments for the classification-theoretic results come from Conant's work in [32], with a new lemma of the author from Chapter 3 (which can itself be proven using the proof of Proposition 3.14 of [52]; see footnote 1 of Chapter 4, and [74]). Meanwhile, the most general version of the result on Conant-independence will come from an improvement, very similar to [72], on the "algebraically reasonable chain condition" that was applied to NSOP_1 generic expansions in [71]. This gives us a general criterion for classifying expansions of NSOP_1 theories, which will be applied to the particular case of T^R where T is weakly minimal and $\neg R$ is a fiber-algebraic ternary relation.

Finally, in section 3, we prove our main result, Theorem 2.1.2.

Notations are standard. We use $\bar{x}, \bar{y}, \bar{z}$, etc. and $\bar{a}, \bar{b}, \bar{c}$ to denote tuples of variables or constants, and x, y, z, a, b, c to denote tuples or singletons depending on context.

2.2 The model companion

We define the general setting for this section and section 4. Throughout this chapter, the theory T will always have quantifier elimination in the language $\mathcal{L}$.

We start with the predicate version of this setting. Let $\delta(\bar{x})$ denote that the coordinates of $\bar{x}$ are distinct. Let P be an additional unary predicate symbol and $\mathcal{L}^P = \mathcal{L} \cup \{P\}$. Let R

[1] In fact, when $\neg R$ is a fiber-algebraic ternary relation, nfcp is not required for T^R to be well-defined: T^R is well-defined even when T only eliminates $\exists^\infty$. Because the weakly minimal case, where our main result holds, is already nfcp ([43], [32]), we relegate this to an appendix. We would like to acknowledge Gabriel Conant for drawing our attention to this.

be a definable n-ary relation in $\mathcal{L}$. Define T_R to be the $\mathcal{L}^P$-theory consisting of T together with the axiom $\forall \bar{x}(\delta(\bar{x}) \wedge \bigwedge_{i=1}^{n} P(x_i) \rightarrow R(\bar{x}))$. In words, T_R will be the theory of models of T together with a unary predicate P so that any n-tuple with distinct coordnates in P will belong to R. If T_R has a model companion, we denote it T^R; in analogy with [20], T^R will be the generic expansion of T by a unary predicate, subject to a universal constraint.

Now we define the equivalence relation version of the setting. Let E be an additional binary relation symbol and $\mathcal{L}^E = \mathcal{L} \cup \{E\}$. Let R be a definable n-ary relation in $\mathcal{L}$. Define T_R to be the $\mathcal{L}^E$-theory consisting of the axioms for T, the requirement that E be an equivalence relation, and the axiom $\forall \bar{x} \bigwedge_{1 \leq i \neq j \leq n}(x_i \neq x_j \wedge E(x_i, x_j)) \rightarrow R(\bar{x})$. In words, T_R will be the theory of models of T together with an equivalence relations E so that any n-tuple of distinct elements of the same equivalence class will belong to R. If T_R has a model companion, we denote it T^R.

The predicate and equivalence relation version of T^R will have the same properties, except that the equivalence relation version can be strictly NSOP$_1$, and the proofs for each version will be similar. When it is not clear from context, we will use $T_{R,P}$, T^R_P to denote the predicate version and $T_{R,E}$, T^R_E to denote the equivalence relation version.

We would like to know when T^R exists. In fact, we characterize the theories that can only interpret theories T, so that T^R always exists for any R definable in T. We need the following classification theoretic property, from [97]:

Definition 2.2.1. *A formula $\varphi(x, y)$ has* nfcp*, or the* non-finite cover property*, if there is some n so that any set $\{\varphi(x, b_i)\}_{i \in I}$ is consistent if and only if it is n-consistent. A theory is* nfcp *if every formula is nfcp.*

The following generalizes the direction (i) $\rightarrow$ (ii) of Theorem 6 of [90]; as expected, it uses the fact that consistency of a φ-type is definable in an nfcp theory.

Proposition 2.2.1. *Let T be an nfcp theory, and R an n-ary relation definable in T. Let $\mathcal{L}$ be the language $\mathcal{L}_0$ of T together with an additional symbol P for a unary relation. Let T_R be the theory in $\mathcal{L}$ of models $M \models T$ such that, for any n-tuple $\bar{a} \in P(M)$, $M \models R(\bar{a})$. Then the model companion T^R of T_R exists.*

Proof. (*Equivalence relation version.*)

The theory T_R is a consistent theory: isolate each element of a model in its own E-equivalence class. (This was the purpose of requiring that the x_i be distinct.) Since T_R is formed from the model-complete theory T by adding universal axioms, ascending chains of models of T are again models of T. It follows that existentially closed models of T_R exist, and it remains to show that the class of existentially closed models is axiomatizable. Let $\varphi(\bar{y}, \bar{x})$ be a quantifier-free $\mathcal{L}^E$-formula. We will show that there is an $\mathcal{L}^E$-formula $\rho(y)$ such that, for $M \models T_R$, $M \models \rho(\bar{m})$ if and only if there is some extension $N \supseteq M$ with $N \models T_R$ so that $N \models \varphi(\bar{m}, \bar{n})$ for some $\bar{n} \in N$ with $\bar{n} \cap M = \emptyset$. This will be enough for us, by the following claim:

Claim 2.2.2. *Suppose that for each quantifier-free $\mathcal{L}^E$-formula $\varphi(\bar{x}, \bar{y})$, there exists a $\mathcal{L}^E$-formula $\rho_\varphi(\bar{y})$ as above. Then the sentences $\forall y(\rho_\varphi(\bar{y}) \to \exists \bar{x}\varphi(\bar{y}, \bar{x}))$ where $\varphi(\bar{y}, \bar{x})$ ranges over the quantifier-free $\mathcal{L}^P$-formulas, will axiomatize when a model $M \models T_R$ is existentially closed.*

Proof. (Implicit in the proof of Theorem 2.4 of [20].) Clearly, an existentially closed model of T_R satisfies these sentences. Conversely, let $M \models T_R$ satisfy these sentences. We show that M is existentially closed. It suffices to show that for $\psi(\bar{y}; \bar{z}\bar{x})$ a quantifier-free $\mathcal{L}^P$-formula, $M \subseteq N \models T_R$, $\bar{m}, \bar{a} \in M$, $\bar{b} \in N$ with $\bar{b} \cap N = \emptyset$, and $N \models \psi(\bar{m}, \bar{a}, \bar{b})$, there is some $\bar{b}' \in M$ so that $M \models \psi(\bar{m}, \bar{a}, \bar{b}')$. We just apply the hypothesis to $\varphi(\bar{y}\bar{z}; \bar{x}) =: \psi(y, zx)$, noting $M \models \rho_\varphi(\bar{m}\bar{a})$ in this case. □

Our strategy will be as follows. The existence of $N \supseteq M$, with $\bar{n} \in N \models T_R$ and $\bar{n} \cap M = \emptyset$, so that $N \models \varphi(\bar{m}, \bar{n})$ will be equivalent to the consistency of a partial $\mathcal{L}$-type, consisting of instances of finitely many $\mathcal{L}$-formulas, where the parameters for those instances can be described in the language $\mathcal{L}^P$ in a way that is uniform in $\bar{m}$ and M. As the $\mathcal{L}$-formulas are nfcp, the consistency of this type is equivalent to n-consistency of the type, which can be expressed by an $\mathcal{L}^P$-formula.

We may assume that $\varphi(\bar{y}, \bar{x})$ is a formula of the form $\psi(\bar{y}, \bar{x}) \wedge \eta(\bar{y}, \bar{x})$ where

- $\psi(\bar{y}, \bar{x})$ is a $\mathcal{L}$-formula that implies $\bar{y}, \bar{x}$ is a tuple of distinct elements, and
- $\eta(\bar{y}, \bar{x})$ is a consistent boolean combination of instances of $E(x_i, y_j)$ that completely describes the restriction of the equivalence relation E to the variables $\bar{y}, \bar{x}$.

We will define a $\mathcal{L}$-formula $\tau(\bar{x}, \bar{y}, \bar{z})$. Here $\bar{z} = (z_1, \ldots, z_n, \bar{w}_1, \ldots, \bar{w}_n)$, and $\bar{w}_i$ is an $N+1$-tuple of variables, where $C_1, \ldots C_N$ is an enumeration of the equivalence classes on the variables $\bar{y}, \bar{x}$ described by the formula $\eta(\bar{y}, \bar{x})$ and containing variables from $\bar{y}$. Let $\tau(\bar{x}, \bar{y}, \bar{z})$ express that

(a) $\models \psi(\bar{y}, \bar{x})$

(b) Let $\bar{a}$ be an n-tuple with distinct coordinates drawn from $\bar{x}$, whose coordinates are required by $\eta(\bar{y}, \bar{x})$ to belong to the same equivalence class, not containing any of the $\bar{y}$. Then $\models R(\bar{a})$

(c) $\bigwedge_{i \leq |\bar{x}|, j \leq n} x_i \neq z_j$

(d) Let $\bar{a}$ be an n-tuple, consisting of $\bar{x}$-coordinates all required by $\eta(\bar{y}, \bar{x})$ to belong to the equivalence class represented by C_i for some fixed i, and $\bar{z}$ coordinates of the form $\bar{z}_j$ such that $\bar{w}_j$ consists of exactly i many distinct elements. (So the $\bar{w}_j$ encode the indices of the $C_1, \ldots C_N$). Then $\models R(\bar{a})$.

Note that $\tau(\bar{x}, \bar{y}, \bar{z})$ can indeed be chosen to be a $\mathcal{L}$-formula, and not a $\mathcal{L}^P$-formula, because the equivalence relation E itself is not referred to, only the requirements imposed by $\eta(\bar{y}, \bar{x})$. Note also that it can be chosen uniformly in M.

For $\bar{e} \in M$ with $|\bar{e}| = |\bar{y}|$, define a partial $\tau(\bar{x}, \bar{y}\bar{z})$-type $p(x, \bar{e})$ in the variables $\bar{x}$, with parameters in M, as follows:

Let $p(x, \bar{e})$ be the set of $\tau(\bar{x}, \bar{e}, \bar{b})$, where $(c_1, \ldots, c_n, \bar{d}_1, \ldots, \bar{d}_n) = \bar{b} \in M$, and for $j \leq n$, $\models E(c_j, e)$, where e is any element of $\bar{e}$ required by $\eta(\bar{y}, \bar{x})$ to belong to the equivalence class C_i on $\bar{y}, \bar{x}$, if and only if $\bar{d}_j$ consists of i many distinct elements, and $\models \wedge_{e\in\bar{e}} \neg E(c_j, e)$ if and only if $\bar{d}_j$ consists of $N+1$ many distinct elements.

In this form, it can be seen that for any k, there is an $\mathcal{L}^P$-formula $\rho_k(\bar{y})$ so that $\models \rho_k(\bar{e})$ if and only if $p(x, \bar{e})$ is k-consistent. But since $\tau(\bar{x}, \bar{y}\bar{z})$ is nfcp, there is some k so that a τ-type is consistent if and only if it is k-consistent. So there is $\rho(\bar{y})$ so that $\models \rho_k(\bar{e})$ if and only if $p(x, \bar{e})$ is consistent. Note that $\rho(\bar{y})$ can be chosen uniformly in M. We show $\rho(\bar{y})$ is as desired.

Note that $p(x, \bar{e})$ expressed the following conditions

(a′) $\models \psi(\bar{e}, \bar{x})$

(b′) Let $\bar{a}$ be an n-tuple with distinct coordinates drawn from $\bar{x}$, whose coordinates are required by $\eta(\bar{y}, \bar{x})$ to belong to the same equivalence class, not containing any of the $\bar{y}$. Then $\models R(\bar{a})$

(c′) $\bar{x} \cap M = \emptyset$

(d′) Let $\bar{a}$ be an n-tuple with distinct coordinates drawn from $M \cup \bar{x}$, consisting of $\bar{x}$-coordinates all required by $\eta(\bar{e}, \bar{x})$ to belong to the equivalence class of $e \in \bar{e}$, and elements of M belonging to the equivalence class of e. Then $\models R(\bar{a})$.

It remains to show that the following are equivalent:

(i) there exists $N \supseteq M$, with $\bar{n} \in N \models T_R$ and $\bar{n} \cap M = \emptyset$, so that $N \models \varphi(\bar{e}, \bar{n})$

(ii) $p(\bar{x}, \bar{e})$ is consistent

For (i $\Rightarrow$ ii), clearly $\bar{n} \models p(x, \bar{e})$. Conversely, let $M \prec N \models T$ be an $\mathcal{L}$-elementary of N, and $\bar{n} \models p(x, \bar{e})$. By (c'), $\bar{n} \cap M = 0$. Therefore, we can expand M to a $\mathcal{L}^E$-extension of M as follows: choose the finest equivalence relation on N extending that on M, so that $N \models \eta(\bar{e}, \bar{n})$. By (a') and the fact that $\varphi(\bar{y}, \bar{e}) = \psi(\bar{y}, \bar{x}) \wedge \eta(\bar{y}, \bar{x})$, $N \models \varphi(\bar{e}, \bar{n})$. To complete the proof of (ii) $\Rightarrow$ (i) it remains to show $N \models T^R$. Any elements of $\bar{n}$ equivalent to elements of M are equivalent to some $e \in \bar{e}$, while any elements of $N \backslash (M \cup \bar{n})$ are isolated. So to show show that $N \models T^R$, we just need to show $N \models R(\bar{a})$ for $\bar{a}$ a tuple of distinct elements taken from an equivalence class of some $a \in \bar{a}$ not equivalent to any element of M, or from an equivalence class of $e \in \bar{e}$. The first case follows from (b'), and the second from (d').

(*Predicate version.*) This is similar to the equivalence relation version, but less complicated, so we only give a sketch.

As in the equivalence relation version, it suffices to find $\rho(\bar{y})$. By the claim, it suffices to define those $\bar{m}$ so that $\varphi(\bar{m}, \bar{x})$ is realized in some model of T_R extending M, by a tuple with no coordinates in M. We may assume that $\varphi(\bar{y}, \bar{x})$ is of the form $\psi(\bar{y}, x_1, \ldots x_m) \wedge \bigwedge_1^k P(x_i) \wedge \bigwedge_{k+1}^m \neg P(x_i)$ for $\psi(\bar{y}, x_1, \ldots x_m)$ an $\mathcal{L}$-formula. Let $\tau(\bar{x}, \bar{y}, \bar{z})$ where $\bar{z} = (z_0, z_1, \ldots, z_n, \bar{w}_0, \bar{w}_1, \ldots, \bar{w}_n)$ and $\bar{w}_i = (u_i, v_i)$ be the $\mathcal{L}_0$-formula expressing the following: $\psi(\bar{y}, x_1, \ldots x_m)$ is true, $\bigwedge_{i \leq |\bar{x}|, j \leq n} x_i \neq z_j$, and for each of the n-tuples $\bar{a}$ of distinct elements whose coordinates are among the $x_1, \ldots x_k$ together with those z_i, $1 \leq i \leq n$ with $u_i = v_i$, $M \models R(\bar{a})$. For $\bar{e} \in M$, let the set $p(\bar{x}, \bar{y})$ consist of those $\bar{z}$-instances of $\tau(\bar{y}, \bar{x}, \bar{z})$ over M for which $z_i \in P(M)$ if and only if $u_i = v_i$. Then (i) $\varphi(\bar{e}, \bar{x})$ is realized in an extension of M by a tuple with no coordinates in M if and only if (ii) $p(\bar{x}, \bar{e})$ is consistent. For the direction (ii) $\Rightarrow$ (i), proceed as in the equivalence relation version, but choose $P(N) = P(M) \cup \bar{\bar{n}}$. Because T is nfcp, there is some k not depending on M so that, for $m \in M$, any family of $\bar{z}$-instances of $\tau(\bar{m}, \bar{y}, \bar{z})$ is consistent if it is k-consistent. But k-consistency of $p(\bar{m}, \bar{x})$ is expressible in $\mathcal{L}^P$.

□

Note that nfcp is preserved under interpretation. Therefore, if T is nfcp, any theory that is interpretable in T satisfies the conclusion of the previous proposition. We show a converse, which may be of independent interest. That is, if T is not nfcp, then it defines a theory T' such that, for some T'-definable relation R, T'^R is not well-defined. Since this converse is not necessary for our main results, we will focus on proving the predicate version. The following is Theorems II.4.2 and II.4.4 of [97]:

Fact 2.2.3. *A theory T is nfcp if and only if T is stable and T^{eq} eliminates $\exists^\infty$.*

Lemma 2.2.4. *A theory is nfcp if and only if it is stable and, for any formula $\varphi(x, y, z)$ (where x, y, z are tuples of variables) and $M \models T$, the set of $a \in M$ so that $\{\varphi(x, m, a) : m \in \mathbb{M}\}$ is consistent, is definable.*

Proof. ($\Rightarrow$) The property nfcp implies stability by the previous fact. It is immediate from nfcp that there is some k so that for $a \in \mathbb{M}$, $\{\varphi(x, m, a) : m \in \mathbb{M}\}$ is consistent if and only if it is is k-consistent, so consistency of this set is in fact definable.

($\Leftarrow$) By the previous fact, it suffices to show that T^{eq} eliminates $\exists^\infty$. Let E be a definable equivalence relation on tuples x and $\psi(x, a)$ a formula in T; it suffices to show that having infinitely many E-inequivalent realizations x of $\psi(x, b)$ is a definable property of b. For $\varphi(x, y, z) =: \psi(x, z) \wedge (\psi(y, z) \to \neg(xEy))$, and $b \in \mathbb{M}$, consistency of $\{\varphi(x, m, b) : m \in \mathbb{M}\}$ is the same thing as saying that, for any finite collection of realizations of $\psi(x, b)$, there is some realization that is E-inequivalent to any realization in this collection. This is of course the same thing as $\psi(x, b)$ having infinitely many E-inequivalent realizations. But by the assumption, consistency of $\{\varphi(x, m, b) : m \in \mathbb{M}\}$ is a definable property of b. □

The following generalizes the arguments from (iv) $\to$ (i) of Theorem 6 of [90], as well as Proposition 2.11 of [20].

Lemma 2.2.5. *Suppose that in T it fails that for any formula $\varphi(x, y, z)$, the set of a so that $\{\varphi(x, m, a) : m \in \mathbb{M}\}$ is consistent, is definable. Then there is a theory T' definable in T, and R definable in T', so that T'^R does not exist.*

Proof. (Predicate version) Our strategy will be to encode a definable family of families of sets, containing families that are n-consistent but not $n+1$ consistent for each n, as a unary predicate. Let T' be the theory of models $M \models T$ together with an additional sort S_1 for pairs $(m_1, m_3) \in M$ and another sort S_2 for singletons $m_2 \in M$, together with the same definable relations as in T^{eq}. (So S_1 can be identified with the set of pairs of elements of the home sort.) Let $\varphi(x, y, z)$ witness the failure of the above property, and let $R(x, y)$ express that if $x = (m_1, m_3) \in S_1$ and $y = m_2 \in S_2$ that $M \models \varphi(m_1, m_2, m_3)$. We show that T'^R is not well-defined.

Suppose it is well-defined. Let $M'_n \models T'$ and $b_n \in M'_n$ (the home sort) so that $\{\varphi(x, m, b_n) : m \in M'_n\}$ is n-consistent but not $n+1$-consistent. Expand M'_n to a model of T_R so that $\{\varphi(x, m, b_n) : m \in P(M'_n)\}$ is $n+1$-inconsistent. Find an extension $M_n \supseteq M'_n$ with $M_n \models T^R$; then for P_n the S_2-points of $P(M_n)$, $\{\varphi(x, m, b_n) : m \in P_n\}$ is n-consistent but not $n+1$-consistent. In fact, any $\lfloor \frac{n}{2} \rfloor$ formulas of $\{\varphi(x, m, b_n) : m \in P_n\}$ must have at least $\frac{n}{2}$ realizations, as the other formulas can be used to distinguish the realizations[2]. By the $n+1$-inconsistency, there can be no point of the form $(x, b_n) \in S_1$ in $P(M_n)$. So by compactness, we can find a model $M \models T^R$ and $b \in M$ so that $\{\varphi(x, m, b) : m \in M\}$ has infinitely many realizations, but there is no point of the form $(x, b) \in S_1$ in $P(M)$. This is a contradiction, since M is existentially closed; we can in fact find a point $a \notin M$ realizing $\{\varphi(x, m, b) : m \in M\}$ in an $\mathcal{L}$-elementary extension and label (a, b) with P to get a model of T_R.

(Equivalence relation version) This is essentially the same proof; just instead of considering the domain of P, get b_n, M_n and a particular equivalence class in place of $P(M_n)$ with the desired properties, then apply compactness so that there is b, M and a new equivalence class in place of $P(M)$ that gives us a contradiction.

□

To characterize nfcp *within general theories* in terms of generic structures, we show the following:

Lemma 2.2.6. *Suppose that T is unstable. Then there is a theory T' definable in T, and R definable in T', so that T'^R does not exist.*

One obstruction to the existence of a model companion is automorphisms of an ordered set. This strategy was used in [58] to show that the theory of a (necessarily SOP) unstable NIP structure with an automorphism did not have a model companion; then the result was improved to the general SOP case in [59]. We follow the arguments from these papers.

Proof. (Predicate version) Let $\varphi(x, y)$ be unstable, and suppose without loss of generality that there is an indiscernible sequence of singletons $\{a_i\}_{i \in \omega}$ so that $\models \varphi(a_i, a_j)$ if and only if $i \leq j$. Consider the theory T_* of models $M \models T$ expanded by a unary predicate P whose domain is linearly ordered by $\varphi(x, y)$ together with a binary relation $R(M) \subseteq M^2$ so that the restriction of R to $P(M) \times P(M)$ is the graph of a partial order-automrphism of $P(M)$ with respect to $\varphi(x, y)$. In a suitable power T' of T, the requirements on this structure can all be encoded by a universal axiom of the form $\forall \bar{x}(\delta(\bar{x}) \wedge \bigwedge_{i=1}^{n} P'(x_i)) \to R'(\bar{x})$ for R' some $\mathcal{L}$-definable relation and P' a unary predicate (in the power) representing the additional structure. Therefore, T'_R will be interdefinable with T_* in a natural way, so it suffices to show that T^* does not have a model companion.

[2]A similar observation was made in the proof of Theorem 7.3 of [26], also as a strategy of getting infinitely many realizations for a set of formulas.

Suppose this model companion T^* exists. We argue as in the proof of Theorem 3.1 of [58]. Let σ be the partial order-automorphism defined on P^2 by R. Let $\mathcal{L}_c^*$ be the language of T^* together with additional constant symbols $\{c_i\}_{i\in\omega+1}$. Let T_c^* be the $\mathcal{L}_c^*$-theory formed from T^* by additionally requiring that

(a) $\{c_i\}_{i\in\omega+1}$ be an $\mathcal{L}$-indiscernible $\varphi(x, y)$-increasing sequence within P, and

(b) $\sigma(c_i)$ is (defined and) equal to c_{i+1} for $i \in \omega$.

First, we see that $T_c^* \vdash \exists x(P(x) \wedge \varphi(c_0, x) \wedge \varphi(x, c_\omega) \wedge \sigma(x) = x)$.

To see this, let M be a model of T_c^*. In an $\mathcal{L}$-elementary extension M', find by indiscernibility some c greater, in the sense of $\varphi(x, y)$, than all of the c_i for $i \in \omega$, but less than any element of $P(M)$ greater than all of the c_i for $i \in \omega$. In other words, fill the right cut of $P(M)$ determined by the increasing sequence $\{c_i\}_{i\in\omega}$. Extend the additional structure only to declare that $c \in P(M')$ and $\sigma(c)$ is defined and equal to c; then since the cut determined by the increasing sequence $\{c_i\}_{i\in\omega}$ must be closed under σ where it is defined on $P(M)$, σ remains a partial order-automorphism on $P(M')$. So $M \models \exists x(P(x) \wedge \varphi(c_0, x) \wedge \varphi(x, c_\omega) \wedge \sigma(x) = x)$ by existential closedness.

Let $(T_c^*)_n$ be the theory T^* together with

$(a)_n$ $\{c_i\}_{i\in[n]\cup\{\omega\}}$ is an $\mathcal{L}$-indiscernible $\varphi(x, y)$-increasing sequence within P, and

$(b)_n$ $\sigma(c_i)$ is defined and equal to c_{i+1} for $i < n$.

By compactness, $(T_c^*)_n \vdash \exists x(P(x) \wedge \varphi(c_0, x) \wedge \varphi(x, c_\omega) \wedge \sigma(x) = x)$ for some n. For a contradiction, it remains to construct a model of $(T_c^*)_n$ where $\exists x(P(x) \wedge \varphi(c_0, x) \wedge \varphi(x, c_\omega) \wedge \sigma(x) = x)$ is not satisfied. But it is easy to construct a model of $(T_c^*)_n$ where, in addition to these requirements, $\sigma(c_n)$ is defined and equal to c_ω. And this cannot satisfy $\exists x(P(x) \wedge \varphi(c_0, x) \wedge \varphi(x, c_\omega) \wedge \sigma(x) = x)$.

(Equivalence relation version) Similar to the predicate version; in the above T'_R, the information of R, P is now encoded as a particular E-equivalence class, which we can distinguish by selecting a representative.

□

Combining Proposition 2.1 with lemmas 2.3 through 2.5, we characterize existence of these model companions as a classification-theoretic dividing line:

Theorem 2.2.7. *A theory T is* nfcp *if and only if for every theory T' definable (interpretable) in T and relation R definable in T', $(T')^R$ exists. Otherwise it is* fcp.

Poizat, in his analysis of belles paires in [90], treats only the stable case. Not only does our result generalize those of Poizat to T^R; it gives a full characterization of nfcp in terms of model companions, ruling out the unstable case.

We now return to the set-up for our main result, 2.1.2. We are interested in using the complexity of T^R to classify when $\neg R$ is geometrically equivalent to the graph of a group operation in the sense described in the statement of 2.1.2. (The negation is required for this construction to be nontrivial.) The following class of ternary relations, first defined in [29] for general n-ary relations, includes all relations without no trivial reason *not* to arise from a group in this sense.

Definition 2.2.2. *([29]) A definable ternary relation R is* fiber-algebraic *if whenever $\models R(\bar{a})$, any coordinate of $\bar{a}$ is algebraic over the other two.*

When $\neg R$ is a fiber-algebraic definable ternary relation, T^R is well-defined even when T eliminates $\exists^\infty$, and is not necessarily nfcp. Since the weakly minimal theories considered in our main result are nfcp ([43], as observed by [34]), we will show this in the appendix.

It is essential to our main result that T^R admit quantifier elimimination up to finite covers. When $\neg R$ is a fiber-algebraic ternary relation, this is easy.

Lemma 2.2.8. *Let the $\mathcal{L}$-definable ternary relation R be such that $\neg R$ fiber-algebraic, and let $C \subseteq A, B$ be substructures of models of T_R, algebraically closed in the sense of $\mathcal{L}$. Then there is a model D of T_R containing A and B as substructures, with $A \cap B = C$.*

Proof. We may assume that (the reducts to $\mathcal{L}$ of) A, B and C are substructures of some model $M \models T$, with $A \cap B = C$. In the predicate version, we expand M to a $\mathcal{L}$-structure extending A and B as follows: $P(M) = P(A) \cup P(B)$. In the equivalence relation version, we let $E(M)$ be the finest equivalence relation extending $E(A)$ and $E(B)$, so $A/E \cap B/E = C/E$ and each element of $M \backslash (A \cup B)$ is isolated. It remains to show $M \models T_R$.

We can assume without loss of generality that $\bar{m}$ is a triple with one coordinate in $B \backslash C$ and the other two in A and must show that $M \models R(\bar{m})$ if the coordinates of $\bar{m}$ belong to $P(M)$, or to a single $E(M)$-equivalence class. But this is clear, as the one coordinate cannot be algebraic over the other two.

(Note that when an element is isolated in its own E-class as an element of a set, when that set is identified with a subset of a model of T^R, it is not isolated within that model.) ☐

Proposition 2.2.9. *Let the $\mathcal{L}$-definable ternary relation R be such that $\neg R$ fiber-algebraic, and assume the equivalence relation version of T^R is well-defined. Let A and B be substructures of models of T^R, algebraically closed in the sense of $\mathcal{L}$. Then if $\mathrm{qftp}_{\mathcal{L}^{\mathcal{E}}}(A) = \mathrm{qftp}_{\mathcal{L}^{\mathcal{E}}}(B)$, $\mathrm{tp}_{\mathcal{L}^{\mathcal{E}}}(A) = \mathrm{tp}_{\mathcal{L}^{\mathcal{E}}}(B)$. The algebraic closure in the sense of $\mathcal{L}$ and $\mathcal{L}^E$ coincide, so T^R has quantifier elimination up to finite covers and the completions of T^R are determined by $E(\mathrm{acl}(\emptyset))$.*

The same holds for the predicate version, replacing $\mathcal{L}^E$ with $\mathcal{L}^P$

Proof. Follows from the previous lemma by the standard arguments. ☐

2.3 Conant-independence

We first give an overview of classification theory beyond the simple theories; some of this discussion will be for motivation. We will consider relational expansions of the NSOP_1 *theories* first formally introduced in [40], a class which contains all simple theories.

Definition 2.3.1. *A theory T is* NSOP_1 *if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n \in \omega}$ is consistent for any $\sigma \in 2^\omega$, but for any $\eta_2 \trianglerighteq \eta_1 \frown \langle 0 \rangle$, $\{\varphi(x, b_{\eta_2}), \varphi(x, b_{\eta_1 \frown \langle 1 \rangle})\}$ is inconsistent. Otherwise it is* SOP_1.

The main stability-theoretic tool for studying NSOP_1 theories is *Kim-independence*, a notion introduced by Kaplan and Ramsey ([52]) that coincides with forking-independence in simple theories. Recall that a global type $p(x)$ is *invariant* over a model M if whether $\varphi(x, b)$ belongs to p for $\varphi(x, y)$ a fixed formula without parameters depends only on the type of the parameter b over M and not on the specific realization of that type, and that an infinite sequence $\{b_i\}_{i\in I}$, is an *invariant Morley sequence* over M if there is a fixed global type $p(x)$ invariant over M so that $b_i \models p(x)|_{M\{b_j\}_{j<i}}$ for $i \in I$.

Definition 2.3.2. *A formula $\varphi(x, b)$* Kim-divides *over M if there is an invariant Morley sequence $\{b_i\}_{i\in\omega}$ starting with b (said to* witness *the Kim-dividing) so that $\{\varphi(x, b_i)\}_{i\in\omega}$ is inconsistent. A formula $\varphi(x, b)$* Kim-forks *over M if it implies a (finite) disjunction of formulas Kim-dividing over M. We write $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M b$, and say that a is* Kim-independent *from b over M if $\text{tp}(a/Mb)$ does not include any formulas Kim-forking over M.*

For example, in the algebraically closed fields with a generic additive subgroup G from [37], $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M B$ is given by the "weak independence" $A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\text{ACF}}_M B$ and $G(\text{acl}(MA)+\text{acl}(MB)) = G(\text{acl}(MA))+G(\text{acl}(MB))$. Analogously to simplicity, there is the following characterization of NSOP_1 theories:

Fact 2.3.1. *([52]) Let T be NSOP_1. Then for any formula $\varphi(x, b)$ Kim-dividing over M, any invariant Morley sequence over M starting with b witnesses Kim-dividing of $\varphi(x, b)$ over M. Conversely, suppose that for any formula $\varphi(x, b)$ Kim-dividing over M, any invariant Morley sequence over M starting with b witnesses Kim-dividing of b over M. Then T is NSOP_1.*

It follows that Kim-forking coincides with Kim-dividing in any NSOP_1 theory.

It is standard (Proposition 3.20 of [52]) that $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K$ satisfies extension: if $M \subseteq B \subseteq C$ and $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M B$ then there is $a' \equiv_B a$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M C$.

Fact 2.3.2. *([28], [52]) The theory T is NSOP_1 if and only if $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K$ is a symmetric relation over models.*

Definition 2.3.3. *Let $n \geq 3$. A theory T is NSOP_n (that is, does not have the* n-strong order property*) if there is no definable relation $R(x_1, x_2)$ with no n-cycles, but with tuples $\{a_i\}_{i\in\omega}$ with $\models R(a_i, a_j)$ for $i < j$. Otherwise it is SOP_n.*

Note that $\text{NSOP}_1 \subseteq \text{NSOP}_3$ and for $3 \leq n < m$, $\text{NSOP}_n \subsetneq \text{NSOP}_n$ ([101]); it is open whether the former inclusion is strict. We also have the following property extending simplicity:

Definition 2.3.4. *A theory T is NTP_2 (that is, does not have the* tree property of the second kind*) if there is no array $\{b_{ij}\}_{i,j\in\omega}$ and formula $\varphi(x, y)$ so that there is some fixed k so that, for all i, $\{\varphi(x, b_{ij})\}_{j\in\omega}$ is inconsistent, but for any $\sigma \in \omega^\omega$, $\{\varphi(x, b_{i\sigma(i)})\}_{(i)\in\omega}$ is consistent.*

The following generalization of Kim-independence beyond the NSOP_1 case, *Conant-independence*, was introduced in Chapter 1. There, any theory where *Conant-independence* is symmetric was shown to be NSOP_4, and Conant-independence was characterized in most of the known examples of NSOP_4 theories, leaving open the question of whether all NSOP_4 theories, in analogy with NSOP_1 theories and Fact 2.3.2, have symmetric Conant-independence.

Definition 2.3.5. *Let M be a model and $\varphi(x, b)$ a formula. We say $\varphi(x, b)$* Conant-divides *over M if for* every *invariant Morley sequence $\{b_i\}_{i\in\omega}$ over M starting with b, $\{\varphi(x, b_i)\}_{i\in\omega}$ is inconsistent. We say $\varphi(x, b)$* Conant-forks *over M if and only if it implies a disjunction of formulas Conant-dividing over M. We say a is* Conant-independent *from b over M, written $a \mathop{\smile\!\!\!\!\!\mid}^{K^*}_M b$, if $\mathrm{tp}(a/Mb)$ does not contain any formulas Conant-forking over M.*

Note that by Fact 2.3.1, Conant-independence really does coincide with Kim-independence in NSOP_1 theories.

Finally, an additional property is required to complete our classification-theoretic account of generic expansions. It is related to the *dividing order* from [113], and implies that Conant-forking coincides with Conant-dividing.

Definition 2.3.6. *We say a theory T has the* strong witnessing property *if for $M \prec \mathbb{M}$ there is some sufficiently saturated $\mathbb{M}_1 \succ M$ (lying in a, say, a very large elementary extension of $\mathbb{M}$) with the following property:*

For $b \subset \mathbb{M}_1$, $\mathrm{tp}(b/\mathbb{M})$ is an M-invariant type such that, if a Morley sequence in that type witnesses Kim-dividing of a formula $\varphi(x, b)$ over M, then any any Morley sequence in $\mathrm{tp}(b/M)$ witnesses Kim-dividing of $\varphi(x, b)$ over M.

The following is Theorem 1.3.15 (relying on the the arguments of [32] and Chapter 3), Theorem 1.3.16 (using the argument of [32]), and Theorem 1.6.2 (though the arguments of Theorem 4.4 of [32], based on arguments originally due to [85] will suffice in this case):

Fact 2.3.3. *Let T have the strong witnessing property, and let Conant-independence be symmetric over models. Then T is* NSOP_4 *either simple or* TP_2*, and either* NSOP_1 *or* SOP_3.

We now give a general context for relational expansions of NSOP_1 theories with free amalgamation. Let T be a theory with quantifier elimination in a language $\mathcal{L}$ and let $\mathcal{L}^*$ be a relational expansion of $\mathcal{L}$. Let T_* be a $\mathcal{L}^*$-theory expanding T. We assume that its model companion T^* exists. We assume (1), and either (2) or (2').

(1) Quantifier elimination up to finite covers: Let A, B be algebraically closed in the sense of T and have the same quantifier-free $\mathcal{L}^*$-type. Then they have the same $\mathcal{L}^*$-type.

(2) Let A, B be algebraically closed sets in T and $M \models T$ with $M \subseteq A, B$. Suppose $A \mathop{\smile\!\!\!\!\!\mid}^{K}_M B$, and expand $\mathrm{acl}(AB)$ to a $\mathcal{L}^*$-structure restricting to a model of $(T_*)_\forall$ (the theory of substructures of T_*) on A and B, and with no new relations from $\mathcal{L}^*$ other than those entirely lying in A or B (that is, with A and B *freely amalgamted* over M.) Then this expansion of $\mathrm{acl}(AB)$ is another model of $(T_*)_\forall$.

(2′) The language $\mathcal{L}^*$ consists of $\mathcal{L}$ together with an additional binary relation symbols E. Let A, B be algebraically closed sets in T and $M \models T$ with $M \subseteq A, B$. Suppose $A \mathop{\smile\!\!\!\!\!\mid}^K_M B$, and expand $\mathrm{acl}(AB)$ to a $\mathcal{L}^*$-structure restricting to a model of $(T_*)_\forall$ (the theory of substructures of T_*) on A and B, and such that $A/E \cap B/E = M/E$ and each element of $\mathrm{acl}(AB)/(A \cup B)$ is isolated in its own E-equivalence class. Then this expansion of $\mathrm{acl}(AB)$ is another model of $(T_*)_\forall$.

When R is an fiber-algebraic definable ternary relation in T, and T^R exists, T^R satisfies both (1), and either (2) (predicate version) or (2') (equivalence relation version). The property (1) is Proposition 2.2.9, and the property (2) follows from the proof of Lemma 2.2.8.

Example 2.3.4. In every known NSOP$_1$ theory including the simple theories, every type *over a set* has a nonforking extension; under this condition, [39] extend Kim-independence to types over arbitrary sets. (See Example C.2 of [72]). Then, defining the free amalgamation property (2), (2') over arbitrary sets analogously, quantifier elimination (1) follows from either of these properties.

We would like to characterize Conant-independence under these assumptions. The use of "base monotone" versions of the chain condition or the independence theorem in NSOP$_1$ theories to develop the theory of independence in generic expansions of those theories is not new; see [71] and [72]. While the "algebraically reasonable chain condition" from [71] suffices for the case where $\neg R$ is a fiber-algebraic definable ternary relation, we use the following result of [53] to indicate the full reach of the inheritance of Kim-independence as Conant-independence under expansions. Recall that a *Morley sequence* in any ternary relation $\mathop{\smile\!\!\!\!\!\mid}^*$ over A is an A-indiscernible sequence $\{b_i\}_{i\in I}$ so that $b_i \mathop{\smile\!\!\!\!\!\mid}^*_A b_{<i}$.

Fact 2.3.5. *([53], Proposition 6.5) Let T be* NSOP$_1$ *and $M' \mathop{\smile\!\!\!\!\!\mid}^K_M b$ with $M \prec M'$. Let $I = \{b_i\}_{i<\omega}$ be an invariant Morley sequence over M starting with b. Then we can find $I' \equiv_{Mb} I$ with $M' \mathop{\smile\!\!\!\!\!\mid}^K_M I$ and I' an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence over M'.*

We show that I' has the necessary "algebraic reasonability" properties. See Theorem 2.21 of [71] for a related result proven using similar techniques, and Theorem C.15 of [72] for a result that would work in place of these facts in the case where $\mathop{\smile\!\!\!\!\!\mid}^K$ is defined over sets (Example 2.3.4).

Fact 2.3.6. *In the above fact, let $I' = \{b'_i\}_{i<\omega}$. Then for any $i < \omega$ $\mathrm{acl}(MI') \cap \mathrm{acl}(b'_iM') = \mathrm{acl}(b'_iM)$.*

Proof. By compactness, it suffices to prove this when we replace ω with some large κ (say, $\kappa > 2^{|T|+|\mathrm{acl}(M'b)|}$). Assume without loss of generality that $i = 0$; that is, we show $\mathrm{acl}(MI') \cap \mathrm{acl}(b'_0M') = \mathrm{acl}(b'_0M)$. Suppose $\mathrm{acl}(Mb'_0, \ldots, b'_n)$ meets $\mathrm{acl}(M'b'_0)$ outside of $\mathrm{acl}(Mb'_0)$. Let $\{\bar{b}'_j\}_{j<\kappa} = \{b_{1+(jn)} \ldots b_{1+(jn+(n-1))}\}_{j<\kappa}$ be the concatenation into blocks of size n of the sequence $\{b'_i\}_{1\leq i<\kappa}$. Then $\mathrm{acl}(Mb'_0\bar{b}'_j)$ will, by indiscernibility of I' over M, meet $\mathrm{acl}(M'b'_0)$ outside of $\mathrm{acl}(Mb'_0)$. But the $\mathrm{acl}(Mb'_0\bar{b}'_j)$ will meet pairwise only in $\mathrm{acl}(Mb'_0)$. So it is impossible for each of the $\mathrm{acl}(Mb_0'\bar{b}_j')$ to meet $\mathrm{acl}(M'b_0')$ outside of $\mathrm{acl}(Mb_0')$, as κ is too large, a contradiction. □

We finally need the following fact, a strengthening of Kim's lemma, Fact 2.3.1:

Fact 2.3.7. *([53], Fact 5.1) Let T be* NSOP_1*, and let $\varphi(x,b)$ Kim-divide over M, and let $\{b_i\}_{i<\omega}$ be an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence starting with b. Then $\{\varphi(x,b_i)\}_{i<\omega}$ is inconsistent.*

We now characterize Conant-independence in T^*, when T^* satisfies both (1), and (2) or (2').

Proposition 2.3.8. *Let T be any* NSOP_1 *theory. Under assumptions (1) and (2) on the model companion T^* of an expansion–quantifier elimination up to finite covers, and the free amalgamation property–Conant-independence is the relation $\mathop{\smile\!\!\!\!\!\mid}^T$ over models of T^* inherited from the underlying Kim-independence of T (so in particular, is symmetric.) Moreover, T^* has the strong witnessing property.*

Under assumption (2'), the same is true, but where the relation $\mathop{\smile\!\!\!\!\!\mid}^T$ is defined so that $A \mathop{\smile\!\!\!\!\!\mid}^T_M B$ if and only if $A \mathop{\smile\!\!\!\!\!\mid}^K_M B$ in T, and $\mathrm{acl}(AM)/E \cap \mathrm{acl}(BM)/E = M/E$.

Proof. We first show one direction of the implication:

Claim 2.3.9. *In T^*, if $a \mathop{\smile\!\!\!\!\!\mid}^{K^*}_M b$ then $a \mathop{\smile\!\!\!\!\!\mid}^T_M b$.*

Proof. Suppose that $a \mathop{\smile\!\!\!\!\!\mid}^{K^*}_M b$ but $a \not\mathop{\smile\!\!\!\!\!\mid}^T_M b$. We first show the following claim:

Subclaim 2.3.10. *Let $\{b_i\}_{i<\omega}$ be an invariant Morley sequence over M in the sense of T^*. Then it is an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence in the sense of T.*

Proof. Invariant Morley sequences are not preserved under taking reducts, but invariant Morley sequences in a *finitely satisfiable* type are; we make use of this point.

It suffices to show that, if we assume that $\mathrm{tp}^{\mathcal{L}^*}(c/Md)$ extends to an M-invariant global type, then $c \mathop{\smile\!\!\!\!\!\mid}^K_M d$ in the sense of T. It follows from the assumption that there is an invariant Morley sequence $\{d_i\}_{i<\omega}$ over M starting with d in an M-*finitely satisfiable* type, that is indiscernible over Mc in the sense of T^*. It is then indiscernible over Mc in the sense of T, and is an M-invariant Morley sequence in an M-fintely satisfiable type in the sense of T. Therefore, by Fact 2.3.1, $c \mathop{\smile\!\!\!\!\!\mid}^K_M d$ in the sense of T. □

Now with (2) by $a \not\mathop{\smile\!\!\!\!\!\mid}^T_M b$ and 2.3.7, there is a formula $\varphi(x,b) \in \mathrm{tp}_{\mathcal{L}}(a/Mb)$ so that, for *any* $\mathop{\smile\!\!\!\!\!\mid}^K$ Morley sequence $\{b_i\}_{i\in\omega}$ in the sense of $\mathcal{L}$ starting with b, $\{\varphi(x,b_i)\}_{i\in\omega}$ is inconsistent. But by the subclaim, every invariant Morley sequence in the sense of $\mathcal{L}^*$ is in particular such a sequence. So $\varphi(x,b)$ Conant-divides over M, contradiction.

With (2'), we have the additional case that $\mathrm{acl}(aM)/E \cap \mathrm{acl}(bM)/E \neq M/E$. So in $(T^*)^{\mathrm{eq}}$, $\mathrm{acl}(aM) \cap \mathrm{acl}(bM) \neq M$, and there is a stable formula in $\mathrm{tp}(a/Mb)$ dividing over M. So it divides over M with respect to every invariant Morley sequence, and $a \not\mathop{\smile\!\!\!\!\!\mid}^{K^*}_M b$

□

The following will essentially give the other direction:

Claim 2.3.11. *Let $a \mathop{\smile\!\!\!\!\!\mid}^T_M b$ and $I = \{b_i\}_{i\in\omega}$ be a Morley sequence over M with respect to the free amalgamation given in (2) or (2'), invariant over M in the sense of $\mathcal{L}$ and starting with b. Then there is some $I' \equiv^{\mathcal{L}^*}_{Mb} I$ indiscernible in the sense of $\mathcal{L}^*$ over Ma (with $a \mathop{\smile\!\!\!\!\!\mid}^T_M I'$.)*

Proof. By the discussion following Fact 2.3.1, $\mathop{\smile\!\!\!\!\!\mid}^K$ and thus $\mathop{\smile\!\!\!\!\!\mid}^T$ satisfies the extension property. Noting that $\mathop{\smile\!\!\!\!\!\mid}^T$ is also symmetric, we can find some $\mathcal{L}$-elementary extension M' of M containing Ma so that $M' \mathop{\smile\!\!\!\!\!\mid}^T_M b$. So by replacing a with M', we may assume $a = M' \models T$ is an $\mathcal{L}$-elementary extension of M. (It could even have been an $\mathcal{L}^*$-elementary extension, but we do not need this.) Note that $M' \mathop{\smile\!\!\!\!\!\mid}^T_M b$ implies $M' \mathop{\smile\!\!\!\!\!\mid}^K_M b$ in the sense of T. So in the reduct to T, we can choose $I' = \{b'_i\}_{i\in I}$ as in Fact 2.3.5; that is, some $I' \equiv^{\mathcal{L}}_{Mb} I$ with $M' \mathop{\smile\!\!\!\!\!\mid}^K_M I$ and I' an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence over M', in the sense of T.

We will find an expansion of $\mathrm{acl}_{\mathcal{L}}(M'I')$ satisfying $(T^*)_\forall$ so that $\mathrm{acl}_{\mathcal{L}}(I'M) \equiv^{\mathcal{L}^*-\mathrm{qf}}_{Mb} \mathrm{acl}_{\mathcal{L}}(IM)$ and so that each $\mathrm{acl}_{\mathcal{L}}(b'_iM')$ realizes $\mathrm{qftp}^{\mathcal{L}^*}(\mathrm{acl}_{\mathcal{L}}(bM'))$. By the fact that T^* is the model companion of $(T^*)_\forall$, we can then take I' to lie in a monster model of T^*. Then by (1)–quantifier elimination up to finite covers–$I' \equiv^{\mathcal{L}^*}_{Mb} I$ and each b'_i realizes $\mathrm{tp}^{\mathcal{L}^*}(b/M')$. This will be enough, as we then can extract an M'-indiscernible sequence in the sense of $\mathcal{L}^*$ by Ramsey and compactness.

Since the $\mathrm{acl}(b'_iM')$ form an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence over M', if (2) holds, by repeated applications of (2) we can expand the structure on $\mathrm{acl}_{\mathcal{L}}(M'I')$ so that $\mathrm{qftp}^{\mathcal{L}^*}(\mathrm{acl}(b'_iM')) = \mathrm{qftp}^{\mathcal{L}^*}(\mathrm{acl}(bM'))$, and introduce no further relations. If (2') holds, by repeated applications of (2'), we can expand the structure on $\mathrm{acl}_{\mathcal{L}}(M'I')$ so that $\mathrm{qftp}^{\mathcal{L}^*}(\mathrm{acl}(b'_iM')) = \mathrm{qftp}^{\mathcal{L}^*}(\mathrm{acl}(bM'))$, and take the finest equivalence relation satisfying this requirement. In either case, (2) or (2') will have told us that $\mathrm{acl}_{\mathcal{L}}(M'I')$ has been expanded to a model of $(T_*)_\forall$. By construction, each $\mathrm{acl}_{\mathcal{L}}(b'_iM')$ realizes $\mathrm{qftp}^{\mathcal{L}^*}(\mathrm{acl}(bM'))$.

So it remains to show that $\mathrm{acl}_{\mathcal{L}}(I'M) \equiv^{\mathcal{L}^*-\mathrm{qf}}_{Mb} \mathrm{acl}_{\mathcal{L}}(IM)$. By 2.3.6, $\bigcup_{i=0}^{\omega} \mathrm{acl}_{\mathcal{L}}(M'b'_i) \cap \mathrm{acl}_{\mathcal{L}}(IM) = M$. Under (2), this means that, by not introducing any relations outside of the $\mathrm{acl}_{\mathcal{L}}(M'b'_i)$, we introduced no relations on $\mathrm{acl}_{\mathcal{L}}(IM)$ that were not already on one of the $\mathrm{acl}(Mb'_i)$. So the $\mathrm{acl}_{\mathcal{L}}(Mb'_i)$, which by construction have the same quantifier-free type in $\mathcal{L}^*$ as the $\mathrm{acl}_{\mathcal{L}}(Mb'_i)$, are in fact freely amalgamated over M. Therefore, $\mathrm{acl}_{\mathcal{L}}(I'M) \equiv^{\mathcal{L}^*-\mathrm{qf}}_{Mb} \mathrm{acl}_{\mathcal{L}}(IM)$. Under (2'), since $M' \mathop{\smile\!\!\!\!\!\mid}^T_M b$, $M'/E \cap \mathrm{acl}_{\mathcal{L}}(Mb'_i)/E = M/E$. So because $\mathrm{acl}_{\mathcal{L}}(Mb'_i)/E \cap \mathrm{acl}_{\mathcal{L}}(Mb'_j)/E \subseteq \mathrm{acl}_{\mathcal{L}}(M'b'_i)/E \cap \mathrm{acl}_{\mathcal{L}}(M'b'_j)/E = M'/E$, $\mathrm{acl}_{\mathcal{L}}(Mb'_i)/E \cap \mathrm{acl}_{\mathcal{L}}(Mb'_j)/E = M/E$. Moreover, $(\bigcup_{i=0}^{\omega} \mathrm{acl}_{\mathcal{L}}(M'b'_i)) \cap \mathrm{acl}_{\mathcal{L}}(IM) = M$ shows that, isolating each element of $\mathrm{acl}_{\mathcal{L}}(I'M')$ outside of $\bigcup_{i=0}^{\omega} \mathrm{acl}_{\mathcal{L}}(M'b'_i)$, we have isolated each element of $\mathrm{acl}_{\mathcal{L}}(I'M)$ outside of $\bigcup_{i=0}^{\omega} \mathrm{acl}_{\mathcal{L}}(Mb'_i)$. So again, the $\mathrm{acl}_{\mathcal{L}}(Mb'_i)$, are in fact freely amalgamated over M.

□

Now we show the strong witnessing property. A Morley sequence with respect to free amalgamation over M, invariant over M in the sense of $\mathcal{L}$, will by quantifier elimination also be invariant over M in the sense of $\mathcal{L}^*$. So it suffices to show that if such a Morley sequence witnesses dividing of an $\mathcal{L}^*$-formula $\varphi(x, b)$ over M, then every invariant Morley sequence in the sense of $\mathcal{L}^*$ witnesses dividing of $\varphi(x, b)$ over M. Suppose not. Then there is some invariant Morley sequence in the sense of $\mathcal{L}^*$ $J = \{b_i\}_{i<\omega}$ starting with b with $\{\varphi(x, b_i)\}_{i<\omega}$ consistent. By Ramsey and compactness, choose a realizing this set, so that J is indiscernible over Ma. Then $\mathrm{tp}(a/Mb)$ contains no formulas Conant-dividing over M. So by the proof of Claim 2.3.9, $a \mathop{\smile\!\!\!\!\!\!\mid}^T_M b$. By Claim 2.3.11, this contradicts the fact that $\varphi(x, b)$ divides with respect to a free amalgamation Morley sequence, invariant over M in the sense of $\mathcal{L}$.

By the strong witnessing property, Conant-forking coincides with Conant-dividing. Since the Morley sequences considered in Claim 2.3.11 are invariant over M in the sense of $\mathcal{L}^*$, it follows from that claim that $\mathop{\smile\!\!\!\!\!\!\mid}^T$ implies $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}$. With Claim 2.3.9, this gives $\mathop{\smile\!\!\!\!\!\!\mid}^T = \mathop{\smile\!\!\!\!\!\!\mid}^{K^*}$

□

By the previous proposition and 2.3.3, we get the following corollary.

Corollary 2.3.11.1. *Let T be any* NSOP_1 *theory. If the model companion T^* of an expansion satisfies both (1), and (2) or (2'), it is either simple or* TP_2*, and is either* NSOP_1 *or strictly* NSOP_4*; moreover, $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*} = \mathop{\smile\!\!\!\!\!\!\mid}^T$.*

If T^* is NSOP_1, this implies $\mathop{\smile\!\!\!\!\!\!\mid}^K = \mathop{\smile\!\!\!\!\!\!\mid}^{K^*} = \mathop{\smile\!\!\!\!\!\!\mid}^T$. On the other hand, if $\mathop{\smile\!\!\!\!\!\!\mid}^K = \mathop{\smile\!\!\!\!\!\!\mid}^T$, then $\mathop{\smile\!\!\!\!\!\!\mid}^K$ is symmetric, so T^* is NSOP_1 by Fact 2.3.2.

In the case where $T^* = T^R$, our main result will be to characterize the NSOP_1 case. We will use the Kim-Pillay characterization of NSOP_1, from [52], to obtain an abstract criterion for T^* to be NSOP_1 in terms of $\mathop{\smile\!\!\!\!\!\!\mid}^T$.

Fact 2.3.12. *Let T be any theory.*

(1a) Let $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}$ be symmetric and satisfy the independence theorem: for $a_1 \mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b_1$, $a_2 \mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b_2$, $b_1 \mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b_2$, and $a_1 \equiv^{\mathcal{L}^*}_M a_2$, there is some $a \mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b_1 b_2$ with $a \equiv_{Mb_i} a_i$ for $i = 1, 2$. Then T is NSOP_1.

(1b) If T is NSOP_1*, then $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*} = \mathop{\smile\!\!\!\!\!\!\mid}^K$ satisfies the independence theorem.*

(2) Let T be NSOP_1*. Then T is simple if and only if $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*} = \mathop{\smile\!\!\!\!\!\!\mid}^K$ satisfies base monotonicity: $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$ and $M \preceq M' \subseteq B$ implies $a \mathop{\smile\!\!\!\!\!\!\mid}^K_{M'} B$.*

Proof. (1a) follows from the definition of $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}$, and Theorem 9.1 of [52]. (We did not actually need symmetry here, and could have given a proof using 2.3.1.) (1b) is Theorem 6.5 of [52]. (2) is Proposition 8.8 of [52]. □

Note that, when T is stable and T^R is well-defined, $\mathop{\smile\!\!\!\!\!\!\mid}^T$ is base-monotone in the predicate version. So from Corollary 2.3.11.1 and Fact 2.3.12, we get the following:

Lemma 2.3.13. *Let T be weakly minimal and let R be a ternary relation definable in T. Assume $\neg R$ is fiber-algebraic. Then the equivalence relation version of T^R is* NSOP_1 *if and only if $\mathop{\smile\!\!\!\!\!\!\mid}^T$ satisfies the independence theorem. Otherwise, T^R is* TP_2 *and strictly* NSOP_4.

For the predicate version of T^R, this is the same, but replace "NSOP_1*" with "simple."*

We will use this criterion in the proof of our main result, Theorem 2.1.2, to translate the classification-theoretic properties of T^R into properties of T.

We conclude by showing that in the equivalence relation version, T^R, when NSOP$_1$, is usually strictly NSOP$_1$.

Proposition 2.3.14. *(Equivalence relation version) Let T be weakly minimal and let R be a ternary relation that is definable in T, and assume that $\neg R$ is fiber-algebraic. Suppose that in T there are a, b, A so that* $\operatorname{acl}(Aab) \neq \operatorname{acl}(Aa) \cup \operatorname{acl}(Ab)$ *(i.e. T has nontrivial pregeometry). Then T^R is not simple.*

Proof. Let $\mathop{\smile\!\!\!\!\!\!|}$ be forking-independence in T. First of all, let us choose $M \mathop{\smile\!\!\!\!\!\!|}_A ab$ so that M can be expanded to a model of T^R. By properties of forking in weakly minimal theories, we still have $\operatorname{acl}(Mab) \neq \operatorname{acl}(Ma) \cup \operatorname{acl}(Mb)$, so we can assume $M = A$ can be expanded to a model of T^R.

Now choose $M' \mathop{\smile\!\!\!\!\!\!|}_M a$ containing b so that M' can be expanded to a model of T^R elementarily extending M. Choose some $c \in \operatorname{acl}(Mab) \backslash \operatorname{acl}(Ma) \cup \operatorname{acl}(Mb)$. Again by the properties of forking, $c \in \operatorname{acl}(aM') \backslash \operatorname{acl}(aM) \cup M'$.

Now choose $d \mathop{\smile\!\!\!\!\!\!|}_M M'a$. So $a \mathop{\smile\!\!\!\!\!\!|}_M M'd$ by properties of forking.

Choose some $M'' \supseteq \operatorname{acl}(adM')$ with $M'' \models T$. Expand M'' to an $\mathcal{L}^E$-structure by defining E as follows:

- On M', E is defined so that $(M', E(M'))$ is a model of T^R elementarily extending M
- All elements of $M'' \backslash M'$ are isolated in their own E-class, except for a, d, which are in their own class of size 2.

Since there are no E-equivalence classes with three distinct elements that are not entirely inside M', which is a model of T^R, M'' is a model of T_R. So it can be identified with a substructure of a model of T^R.

Now $a \mathop{\smile\!\!\!\!\!\!|}_M dM'$ and $\operatorname{acl}(aM)/E \cap \operatorname{acl}(dM')/E = M/E$ by construction. So $a \mathop{\smile\!\!\!\!\!\!|}^T_M dM'$. On the other hand $c/E \in (\operatorname{acl}(aM')/E \cap \operatorname{acl}(dM')/E) \backslash M'/E$. So $a \not\mathop{\smile\!\!\!\!\!\!|}^T_{M'} dM'$ and $\mathop{\smile\!\!\!\!\!\!|}^T$ is not base monotone. So by Fact 2.3.12.2, T^R is not simple.

□

In fact, *in the equivalence relation version*, when T^R is simple, so T does not satisfy the hypothesis of this proposition (is geometrically trivial), $\mathop{\smile\!\!\!\!\!\!|}^E$ must be stationary. So *in the equivalence relation version*, if T^R is simple, T^R is stable.

2.4 The group configuration theorem

We now prove the main result of this chapter, 2.1.2. *Throughout this section, we assume the hypotheses of this theorem*: T is weakly minimal theory with quantifier elimination, R is a ternary relation definable in T, and $\neg R$ is fiber-algebraic. So because T is nfcp, Proposition 2.2.1 says that T^R exists.

We first state the basic amalgamation property for algebraically closed sets in stable theories. This is just the classical independence theorem in the model companion of theories with a generic predicate or equivalence relation.

Fact 2.4.1. *([20], [71]). Let T be a stable theory with quantifier elimination in the language $\mathcal{L}$, and let $\mathcal{L}^*$ be an expansion of $\mathcal{L}$ by a predicate symbol P or a binary relation symbol E. For $M \models T$ and $1 \leq i \neq j \neq k \leq 3$, $j < k$, let $p_i(X_j, X_k, X_{jk})$ be quantifier-free $\mathcal{L}^*(M)$-types over M consistent with $\mathrm{diag}_T(M)$, so that for $A'_i, A'_j, A'_{jk} \models p_i(X_j, X_k, X_{jk})$, A'_i, A'_j, A'_{jk} are algebraically closed sets in the sense of $\mathcal{L}$, $A'_{jk} = \mathrm{acl}_{\mathcal{L}}(A'_j A'_k) \backslash (A'_j \cup A'_k)$, $A'_j \mathop{\smile\!\!\!\!\!\!\mid}_M A'_k$ (forking-independence in the sense of T), and if the expansion is by E, $A'_j/E \cap A'_k/E = M/E$. Assume compatibility of these pairs: for $1 \leq i \neq j \neq k \leq 3$, $p_i|_{X_j} = p_j|_{X_k}$. Then in a monster model of T, there are forking-independent A_1, A_2, A_3 over M, and there is an interpretation of P or E on $\mathrm{acl}_{\mathcal{L}}(A_1 A_2 A_3)$ so that for $1 \leq i \neq j \neq k \leq 3$, $j < k$, $\mathrm{acl}_{\mathcal{L}}(A_j A_k) \models p_i$, and moreover $A_1/E \cap \mathrm{acl}_{\mathcal{L}}(A_2 A_3)/E = M/E$ when the expansion is by E.*

Proof. In the expansion by P, this is just the content of 2.7 of [20] (which was proven for simple theories). In the expansion by E, this is the content of Theorem 4.5 of [71] (which was proven for NSOP_1 theories), where L' in the statement of that theorem is taken to be L together with a symbol for a unary function to a new sort. □

Lemma 2.4.2. *In the previous fact, if the expansion is by P, the interpretation of P can be chosen to contain no points of $\mathrm{acl}_{\mathcal{L}}(A_1 A_2 A_3) \backslash \bigcup_{1 \leq j < k \leq 3} \mathrm{acl}_{\mathcal{L}}(A_j A_k)$. If the expansion is by E, each point of that set can be assumed isolated in its own E-class.*

Proof. If we change the interpretation of P or E outside of $\mathrm{acl}_{\mathcal{L}}(A_j A_k)$, so that this requirement is met, this does not change the fact that $\mathrm{acl}_{\mathcal{L}}(A_j A_k) \models p_i$, nor that $A_1/E \cap \mathrm{acl}_{\mathcal{L}}(A_2 A_3)/E = M/E$.

(Note that in [20], this is part of the proof of the previous fact, while in [52], it is stated in the proof that the map can indeed be defined arbitrarily outside of $\bigcup_{1 \leq j < k \leq 3} \mathrm{acl}_{\mathcal{L}}(A_j A_k)$.) □

We first prove the following lemma, reducing the classification-theoretic properties of the expansion T^R to the structure of the original weakly minimal theory T. As usual in a stability-theoretic context, independence, denoted $\mathop{\smile\!\!\!\!\!\!\mid}$, is forking-independence in the sense of T.

Lemma 2.4.3. *The theory T^R is SOP_3 if and only if in T, there are algebraically closed $A \subseteq A_1, A_2, A_3$, the A_i independent over A, and a_1, a_2, a_3 with $\models \neg R(a_1, a_2, a_3)$, so that for $1 \leq i, j, k \leq 3$ distinct, $a_i \in \mathrm{acl}(A_j A_k) \backslash (A_i \cup A_j)$*

Proof. ($\Rightarrow$) First suppose T^R is SOP_3. So $\mathop{\smile\!\!\!\!\!\!\mid}^T$, by Lemma 2.3.13, does not satisfy the independence theorem. So for $1 \leq i, j, k \leq 3$ distinct, $j < k$, there are compatible types $p_i(X_j, X_k)$ in T^R of $\mathop{\smile\!\!\!\!\!\!\mid}^T$-independent pairs over some $M \models T^R$ that do not have a common realization by A_1', A_2', A_3' with $A_1' \mathop{\smile\!\!\!\!\!\mid}^T A_2'A_3'$. Let $M \models T$, $A_1, A_2, A_3 \supseteq M$, together with an interpretation of P or E on $\mathrm{acl}(A_1A_2A_3)$ be obtained as in the previous lemma, from the types corresponding to $p_i(X_j, X_k)$ by quantifier elimination up to finite covers (Proposition 2.2.9). In the rest of this proof, let i, j, k range over distinct $1 \leq i, j, k \leq 3$

Suppose that $\mathrm{acl}(A_1A_2A_3)$ together with this interpretation $P(\mathrm{acl}(A_1A_2A_3))$ or $E(\mathrm{acl}(A_1A_2A_3))$ of P or E satisfies $(T_R)_\forall$ (is a substructure of a model of T_R). Then A_1, A_2, A_3 could be identified by a common realization A_1', A_2', A_3' of the $p_i(X_j, X_k)$ with $A_1' \mathop{\smile\!\!\!\!\!\mid}^T A_2'A_3'$, a contradiction. So $\mathrm{acl}(A_1A_2A_3)$ together with the additional structure does not satisfy $(T_R)_\forall$.

To witness this, there are distinct a_1, a_2, a_3 belonging to $P(\mathrm{acl}(A_1A_2A_3))$, or belonging to the same $E(\mathrm{acl}(A_1A_2A_3))$-equivalence class, so that $\models \neg R(a_1, a_2, a_3)$. Relabeling, it suffices to show that each of the three a_i belongs to precisely one of the three $\mathrm{acl}(A_jA_k)$. Because each of those pairs do satisfy the quantifier-free type of a model of $(T_R)_\forall$, a_1, a_2, a_3 cannot all belong to the same $\mathrm{acl}(A_jA_k)$. Because on $\mathrm{acl}(A_1A_2A_3) \setminus \bigcup_{1 \leq j < k \leq 3} \mathrm{acl}(A_jA_k)$, there are no points of P or each point is isolated in its own E-class, none of a_1, a_2, a_3 belong to $\mathrm{acl}(A_1A_2A_3) \setminus \bigcup_{1 \leq j < k \leq 3} \mathrm{acl}(A_jA_k)$. Finally, it remains to show that no two of a_1, a_2, a_3 can belong to $\mathrm{acl}(A_iA_j)$, while a third belongs to a different $\mathrm{acl}(A_iA_k)$ but not to A_i. Because in T, $A_j \mathop{\smile\!\!\!\!\!\mid}_{A_i} A_k$, the third cannot be algebraic over the other two, as then it would belong to $\mathrm{acl}(A_iA_j) \cap \mathrm{acl}(A_iA_k) = A_i$. But then $\models R(a_1, a_2, a_3)$ must hold, as one of the a_i is not algebraic over the other two.

($\Leftarrow$) Now assume the second condition. By Lemma 2.3.13, it suffices to show that the independence theorem fails for $\mathop{\smile\!\!\!\!\!\mid}^T$. By taking some $M \models T$, that can be expanded to a model of T^R, independently from $A_1A_2A_3$ over A, and replacing A_i with $\mathrm{acl}(MA_i)$, we can assume $A = M$ is a model of T that can be expanded to a model $(M, E(M))$ or $(M, P(M))$ of T^R. In the equivalence relation case, fix some $m \in M$, and expand each of the $\mathrm{acl}(A_iA_j)$ so that the additional structure extends that on M, and a_k lies in the same equivalence class as m, while each point of $\mathrm{acl}(A_iA_j) \setminus M$ besides a_k is isolated in its own class. In the predicate case, instead add no point of $\mathrm{acl}(A_iA_j) \setminus M$ to the intepretation of P, besides a_k. Because a_k is not algebraic over M, either of these constructions give a model of $(T_R)_\forall$. So these expansions of $\mathrm{acl}(A_iA_j)$ determine, by the quantifier elimination up to finite covers in T^R, $\mathcal{L}^E$ or $\mathcal{L}^P$-types $p_k(X_i, X_j)$ over the expansion of M in T^R for $i < j$. Because $a_k \notin A_i \cup A_j$, no nontrivial new structure was added to one of the A_i in any pair, other than that on M. So these types agree on the X_i, by the quantifier elimination up to finite covers. And by construction, each is realized by a $\mathop{\smile\!\!\!\!\!\mid}^T$-independent pair. So a failure of the independence theorem for $\mathop{\smile\!\!\!\!\!\mid}^T$ would be implied, if we can show that these types cannot by jointly realized in T^R by a triple that is forking-independent in the sense of T.

We claim that an obstruction to this joint realization would occur if

$$\mathrm{tp}_{\mathcal{L}}(A_1A_2A_3/M) \vdash \forall x_1x_2x_3 \bigwedge_{j \neq i \neq k, j < k} \varphi_i(X_j, X_k, x_i) \rightarrow \neg R(x_1, x_2, x_3)$$

for $\varphi_i(A_j, A_k, x_i)$ a $\mathcal{L}$-formula isolating $\mathrm{tp}_{\mathcal{L}}(a_i/A_jA_k)$. Indeed, a joint realization in T^R of the $p_k(X_i, X_j)$ that is a forking-independent triple in the sense of T over M, A'_1, A'_2, A'_3, must satisfy $\mathrm{tp}_{\mathcal{L}}(A_1A_2A_3/M)$, by stationarity. Therefore, it must satisfy the formula on the right. But because the A'_1, A'_2, A'_3 jointly realize the $p_k(X_i, X_j)$, for i, j, k there must be some $a'_i \models \varphi_i(A'_j, A'_k, x_i)$ belonging to the E-class of m, or to the interpretation of P. So a'_1, a'_2, a'_3 must all belong to the same equivalence class or to the interpretation of P. But by the formula on the right, $\models \neg R(a'_1, a'_2, a'_3)$. This contradicts the axioms of T_R.

So it remains to show that

$$A_1A_2A_3 \models \forall x_1x_2x_3 \bigwedge_{j\neq i\neq k, j<k} \varphi_i(X_j, X_k, x_i) \to \neg R(x_1, x_2, x_3)$$

For i, j, k, suppose a''_i satisfies $\varphi_i(A_j, A_k, x_i)$ and let σ_i be an automorphism of $\mathrm{acl}(A_jA_k)$ over A_jA_k sending a_i to a''_i. The independence theorem in T^A, Theorem 3.7 of [20] does not say these automorphisms have a common extension–only that some conjugates of these automorphisms do. But the proof of that theorem does in fact show that compatible automorphisms of the algebraic closures of pairs in an independent triple, indeed have a common extension. Since this is not stated explicitly, we review the proof of everything we need; we work in T. For our purposes, it suffices to show for each i, j, k that σ_i as above, so an automorphism of $\mathrm{acl}(A_jA_k)$ over A_jA_k with $\sigma_i(a_i) = a''_i$, can be chosen so that it extends to an automorphism $\tilde{\sigma}_i$ of the monster model $\mathbb{M} \models T$ that is the identity on $\mathrm{acl}(A_iA_j)$ and $\mathrm{acl}(A_iA_k)$. Indeed, then we can compose all three of the $\tilde{\sigma}_i$ together, to get an automorphism extending each of the σ_i. Because $a_i \equiv^{\mathcal{L}}_{A_jA_k} a''_i$, we will get the desired automorphism $\tilde{\sigma}_i$ of $\mathbb{M}$ over $\mathrm{acl}(A_iA_j)\mathrm{acl}(A_iA_k)$ with $\tilde{\sigma}_i(a_i) = a''_i$, as long as the orbit of a_i over A_jA_k is the same as that of a_i over $\mathrm{acl}(A_iA_j)\mathrm{acl}(A_iA_k)$. Now the latter orbit, in the sense of T^{eq}, belongs to $\mathrm{dcl}(\mathrm{acl}(A_iA_j)\mathrm{acl}(A_iA_k)) \cap \mathrm{acl}(A_jA_k)$. Now recall the claim of Theorem 3.7 of [20], namely that $\mathrm{dcl}(\mathrm{acl}(AB)\mathrm{acl}(AC)) \cap \mathrm{acl}(BC) = \mathrm{dcl}(BC)$ for A, B, C independent sets in a stable theory. This claim implies that $\mathrm{dcl}(\mathrm{acl}(A_iA_j)\mathrm{acl}(A_iA_k)) \cap \mathrm{acl}(A_jA_k) = \mathrm{dcl}(A_jA_k)$. But since the orbit of a_i over $\mathrm{acl}(A_iA_j)\mathrm{acl}(A_iA_k)$ is then in $\mathrm{dcl}(A_jA_k)$, all of the conjugates of a_i over A_jA_k must belong to the orbit of a_i over $\mathrm{acl}(A_iA_j)\mathrm{acl}(A_iA_k)$, so the orbit of a_i over $\mathrm{acl}(A_iA_j)\mathrm{acl}(A_iA_k)$ must coincide with the orbit of a_i over A_jA_k.

So there is an automorphism σ of $\mathbb{M}$ extending $\sigma_1, \sigma_2, \sigma_3$. So $\models \neg R(a_1, a_2, a_3)$ implies $\models \neg R(\sigma(a_1), \sigma(a_2), \sigma(a_3))$, so $\models \neg R(\sigma_1(a_1), \sigma_2(a_2), \sigma_3(a_3))$, so $\models \neg R(a''_1, a''_2, a''_3)$.

□

This is the main technical lemma required for the group configuration theorem. To prove it, we use properties of forking in weakly minimal theories throughout.

Lemma 2.4.4. *In the previous lemma, we can further require that* $U(A_i/A) = 1$ *for* $1 \leq i \leq 3$, *in the sense of* T.

Proof. Throughout this proof we refer to T and use the notation of the previous lemma. It suffices to find some $A \subseteq D \subset A_1 \cup A_2 \cup A_3$ with $U(A_i/D) = 1$ so that the second condition of that lemma is satisfied replacing each A_i with $\mathrm{acl}(A_iD)$. We do this by handling A_1, A_2 and A_3 successively.

We begin with the following observation: relative to a given set, if $b_1, \dots, b_n$ is an independent sequence and $a \in \mathrm{acl}(b_1, \dots b_n)$, there is some *least* $S \subseteq \{b_1, \dots, b_n\}$ so that $a \in \mathrm{acl}(S)$. Because if $S_1, S_2 \in \mathrm{acl}(b_1, \dots b_n)$ are two *minimal* such sets, then they are independent over $S_1 \cap S_2$, so $a \in \mathrm{acl}(S_1) \cap \mathrm{acl}(S_2) = \mathrm{acl}(S_1 \cap S_2)$, contradicting minimality.

Choose $1 \leq i, j, k \leq 3$ distinct. We can assume $A_i = \mathrm{acl}_A(b_1, \dots, b_n)$ for $b_1, \dots, b_n$ independent over A. The $b_1, \dots, b_n$ are thus independent over A_j and A_k since $A_j \downarrow_M A_i$, $A_k \downarrow_M A_i$. By the observation above, let $S_j, S_k \subseteq \{b_1, \dots, b_n\}$ be respectively the least so that $a_j \in \mathrm{acl}(A_k S_j)$ and $a_k \in \mathrm{acl}(A_j S_k)$. We claim that $S_j \cap S_k \neq \emptyset$. Otherwise, as A_i, A_j, A_k are independent over M, and $b_1, \dots, b_n$ are an independent subset of A_k, $A_k S_j \downarrow_A A_j S_k$. Therefore, $a_j \in \mathrm{acl}(A_k S_j) \downarrow_{A_j A_k} \mathrm{acl}(A_j S_k) \ni a_k$, so $a_j \downarrow_{A_j A_k} a_k$. Now $a_j \in \mathrm{acl}(A_i A_k) \backslash A_k$ and $A_i \downarrow_{A_k} A_j$, so $a_j \notin \mathrm{acl}(A_j A_k)$. But because $\models \neg R(a_1, a_2, a_3)$, a_j is algebraic over $\mathrm{acl}(A_j A_k) a_k \supseteq A a_i a_k$. So a_j and a_k are dependent over $\mathrm{acl}(A_j A_k)$, contradicting $a_j \downarrow_{A_j A_k} a_k$. This proves our claim that $S_j \cap S_k \neq \emptyset$.

Now let $A' = \mathrm{acl}_A(\{b_1, \dots b_n\} \backslash \{b\})$ for some $b \in S_j \cap S_k$. Then $U(A_i/A') = 1$. By choice of S_j and S_k, $a_k \in \mathrm{acl}(A_i A_j) \backslash (A_i \cup \mathrm{acl}(A' A_j))$ and $a_j \in \mathrm{acl}(A_i A_k) \backslash (A_i \cup \mathrm{acl}(A' A_k))$. By the same reasoning used to show $a_j \notin \mathrm{acl}(A_j A_k)$ above, $a_i \notin \mathrm{acl}(A_i A_j) \cup \mathrm{acl}(A_i A_k)$, so $a_i \in \mathrm{acl}(\mathrm{acl}(A' A_j) \mathrm{acl}(A' A_k)) \backslash (\mathrm{acl}(A' A_j) \cup \mathrm{acl}(A' A_k))$.

So replace A with $A' \subseteq A_i$ A_j with $\mathrm{acl}(A' A_j)$ and A_k with $\mathrm{acl}(A' A_k)$. Now repeat what we have done for A_i for each of A_j and A_k.

□

We are now in a position to prove Theorem 2.1.2. First, suppose G is a rank-one connected group type-definable over a parameter set A defining R, which we can assume to be algebraically closed. Let (a, b, c) be a generic of the graph of its operation. Then a and b are independent generics of G over A and $c = ab$. Then (as in the construction of a group configuration from an actual group; see [11] for an overview) we can find independent algebraically closed sets A_1, A_2, A_3 containing A with $a_i \in \mathrm{acl}(A_j A_k) \backslash (A_i \cup A_j)$; just find independent generics d_1, d_2, d_3 of G over A so that $a = d_1 d_2^{-1}$, $b = d_2 d_3^{-1}$ and $c = d_1 d_3^{-1}$. Now note that, by replacing a_1, a_2, and a_3 by elements individually interalgebraic over A, we preserve $a_i \in \mathrm{acl}(A_j A_k) \backslash (A_i \cup A_j)$, so by Lemma 2.4.3, T^R is not simple.

In the other direction, suppose T^R is not simple. Then we get a_i, A_i as in Lemmas 2.4.3, 2.4.4. To summarize, we have A_1, A_2, A_3, a_1, a_2, a_3 of rank one over A, $a_i \in \mathrm{acl}(A_j A_k) \backslash (A_i \cup A_j)$, A_1, A_2, A_3 forming an independent triple over A, and a_1, a_2, a_3 forming a *dependent* triple over A, since $\models \neg R(a_1, a_2, a_3)$.

Since they are all of rank one over A, we know from the properties of forking in weakly minimal theories that a_1, a_2, a_3 together with, for i, j, k distinct, each of a_i, A_j, A_k, form the lines of a group configuration (recall Figure 1 above). The conclusion follows from the group configuration theorem, Theorem 2.1.1.

Example 2.4.5. We given an example of a simple unstable theory T of SU-rank 1 and ternary relation R definable in T satisfying the group condition of Theorem 1.3, but with T^R still simple. Let T_0 be the theory of two-sorted structures consisting of a vector space

V over a finite field and a two-to-one map F from a set S to V, with a symmetric ternary relation U relating, for any three distinct fibers of F, exactly one point in each of the fibers. It has a model companion T which can be seen to be supersimple of SU-rank 1 with the evident quantifier elimination. Now let $R(x_1, x_2, x_3)$ be defined on S by $\neg(F(x_1) = F(x_2) + F(x_3) \wedge U(x_1, x_2, x_3)) \wedge \bigwedge_{1 \leq i,j, \leq 3} \neg(F(x_i) = F(x_j))$. The independence theorem still holds in T^R, which exists (see the appendix), for the relation $a \mathop{\smile\!\!\!\!\mid}^{a}_{M} b$ given by $\mathrm{acl}(aM) \cap \mathrm{acl}(bM) = M$.

However, there is a group configuration theorem for certain simple theories ([13]), and the left-to-right direction of Lemma 4.1 as well as Lemma 4.2 only require SU-rank one and not the additional assumption of stability, so when T^R exists, but is not simple, we may still get a characterization of R in terms of groups.

2.A Appendix: existence of T^R for fiber algebraic ternary relations

In Proposition 2.2.1, we show that T^R exists whenever T is nfcp. When T only eliminates $\exists^\infty$, T^R still exists in the case where $\neg R$ is an algebraic ternary relation. This may be useful for generalizing the results of this chapter to case of T unstable.

Proposition 2.A.1. *Let T eliminate the quantifier $\exists^\infty$and let R be the negation of an algebraic ternary relation definable in T. Then T^R exists.*

Proof. (Predicate version) Let $M \models T_R$ and let $\psi(\overline{y}, x_1, \ldots, x_n)$ be a formula of $\mathcal{L}_0$. As in the proof of Theorem 2.4 of [20], we can assume that $\psi(\overline{y}, x_1, \ldots, x_n)$ implies that $x_1, \ldots, x_n$ are distinct, and it suffices to find, for any $1 \leq k \leq n$, some $\mathcal{L}^P$-formula $\rho(\overline{y})$ independently of M so that $M \models \rho(\overline{m})$ if and only if there is some $\overline{a} \in N$ for $N \models T_R$ an extension of M such that $\overline{a}$ does not meet M and such that $N \models \psi(\overline{m}, \overline{a}) \wedge \bigwedge_1^k P(a_i) \wedge \bigwedge_{k+1}^n \neg P(a_i)$; then the $\forall y(\rho(\overline{y}) \rightarrow \exists \overline{x} \psi(\overline{y}, \overline{x}))$ will still axiomatize when M is existentially closed.

We can assume that for $1 \leq i \leq j \leq k$ and $\sigma \in S_3$, there is a constant $k_{ij\sigma}$ so that for any $\overline{m}$, $\psi(\overline{m}, \overline{a})$ implies that $\neg R(\sigma(a_i, a_j, x))$ has exactly $k_{ij\sigma}$ solutions, since every formula $\psi(\overline{y}, x_1, \ldots, x_n)$ can be written as a disjunction of formulas with this property. Consider the condition $\rho(\overline{m})$ on $\overline{m} \in M$ requiring that for $1 \leq i \leq j \leq k$ and $\sigma \in S_3$ there are some $0 \leq l_{ij\sigma} \leq k_{ij\sigma}$ and $e^1_{ij\sigma}, \ldots, e^{l_{ij\sigma}}_{ij\sigma} \in M \backslash P(M)$, distinct for fixed $ij\sigma$ so that the following condition $\tau(\overline{m}, \overline{e})$ holds:

There is some $M \prec_{\mathcal{L}} N$ and $a_1 \ldots a_n \in N \backslash M$ and, for $1 \leq i \leq j \leq k$ and $\sigma \in S_3$, $f^1_{ij\sigma} \ldots f^{k_{ij\sigma} - l_{ij\sigma}}_{ij\sigma} \in N \backslash M$, distinct for fixed $ij\sigma$ and distinct from all of the $a_1, \ldots, a_k$, so that $N \models \psi(\overline{m}, \overline{a})$, for all $1 \leq i_1, i_2, i_3 \leq k$, $N \models R(a_{i_1}, a_{i_2}, a_{i_3})$, and for fixed $1 \leq i \leq j \leq k$ and $\sigma \in S_3$, $N \models \neg R(\sigma(a_i, a_j, a))$ for a any of the $e^1_{ij\sigma}, \ldots, e^{l_{ij\sigma}}_{ij\sigma}$ or $f^1_{ij\sigma} \ldots f^{k_{ij\sigma} - l_{ij\sigma}}_{ij\sigma}$.

It follows from the following claim, used implicitly in [20], that $\tau(\overline{m}, \overline{e})$ is a definable condition in $\mathcal{L}$:

Claim 2.A.2. *For $\varphi(\overline{x}, \overline{y})$ any $\mathcal{L}$-formula, the set of $\overline{a} \in M$ so that there is $\overline{b}$ in an elementary extension of M not meeting M with $\models \varphi(\overline{b}, \overline{a})$ is definable in $\mathcal{L}$*

Proof. By elimination of $\exists^\infty$, we can apply Lemma 2.3 of [20], which says that the set of $\overline{a} \in M$, so that there is $\overline{b}$ in an elementary extension of M not meeting $\mathrm{acl}(a)$ with $\models \varphi(\overline{b}, \overline{a})$, is definable. But it is well-known that for any a, b and $a \in M$ there is always $b' \equiv_a b$ with $\mathrm{acl}(b) \cap M = \mathrm{acl}(a)$. So the set we have defined is in fact our desired set. □

Because $\tau(\overline{m}, \overline{e})$ can be expressed definably in $\mathcal{L}$, $\rho(\overline{m})$ can be expressed definably in $\mathcal{L}^P$. We claim that $\rho(\overline{m})$ is as desired. First suppose there is some $\overline{a} \in N$ for $N \models T_R$ an extension of M such that $\overline{a}$ does not meet M and such that $N \models \psi(\overline{m}, \overline{a}) \wedge \bigwedge_1^k P(a_i) \wedge \bigwedge_{k+1}^n \neg P(a_i)$. Then we can let $e^1_{ij\sigma}, \dots, e^{l_{ij\sigma}}_{ij\sigma}$ enumerate the solutions in M to $\neg R(\sigma(a_i, a_j, x))$–note that they must belong to $M \backslash P(M)$–and let $f^1_{ij\sigma} \dots f^{k_{ij\sigma} - l_{ij\sigma}}_{ij\sigma}$ enumerate the solutions in $N \backslash M$ to $\neg R(\sigma(a_i, a_j, x))$–note that they must be distinct from the $a_1, \dots, a_k$. Now suppose $\rho(m)$ holds. It remains to expand N to a model of T_R extending M so that $N \models \bigwedge_1^k P(a_i) \wedge \bigwedge_{k+1}^n \neg P(a_i)$. Note that the a_i are distinct; add just the $a_1, \dots, a_k$ to the domain of P, and no other new elements, to form $P(N)$. We must show that $(N, P(N)) \models T_R$; that is, for a triple $\overline{n} \in P(N)$, $N \models R(\overline{n})$. This is clearly the case if all of the coordinates over $\overline{n}$ lie in N/M, and also if they all lie in M, as we assume that for $1 \leq i_1, i_2, i_3 \leq k$, $N \models R(a_{i_1}, a_{i_2}, a_{i_3})$. If two of the coordinates of $\overline{n}$ are a_i and a_j for $1 \leq i \leq j \leq k$, and another coordinate lies in M, then $N \models R(\overline{n})$ still holds as all $k_{ij\sigma}$ of the solutions to any of the $\neg R(\sigma(a_i, a_j, x))$ must either be one of the $f^1_{ij\sigma} \dots f^{k_{ij\sigma} - l_{ij\sigma}}_{ij\sigma}$ that are not in $M \cup \overline{a}$ or one of the $e^1_{ij\sigma}, \dots, e^{l_{ij\sigma}}_{ij\sigma}$ that are not in $P(N)$. Finally, if exactly two of the coordinates of $\overline{n}$ lie in M, then $N \models R(\overline{n})$ as the other coordinate cannot be algebraic over the two that belong to M.

(Equivalence relation version) Similar.

□

Part II

Chapter 3

On NSOP$_2$ theories

3.1 Introduction

One of the most exciting areas of research in modern model theory is the classification along various dividing lines of non-simple but otherwise tame theories, especially NSOP$_n$ theories for $1 \leq n \leq 3$. The first two of these properties, introduced in [40], require the nonexistence of certain trees:

Definition 3.1.1. *A theory T is* NSOP$_1$ *if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n \in \omega}$ is consistent for any $\sigma \in 2^\omega$, but for any $\eta_2 \trianglerighteq \eta_1 \frown \langle 0 \rangle$, $\{\varphi(x, b_{\eta_2}), \varphi(x, b_{\eta_1 \frown \langle 1 \rangle})\}$ is inconsistent. Otherwise it is* SOP$_1$.

Definition 3.1.2. *A theory T is* NSOP$_2$ *if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n \in \omega}$ is consistent for any $\sigma \in 2^\omega$, but for incomparable η_1 and η_2, $\{\varphi(x, b_{\eta_1}), \varphi(x, b_{\eta_2})\}$ is inconsistent. Otherwise it is* SOP$_2$.

The property NSOP$_3$ is introduced in [101] as part of a family of notions NSOP$_n$ for $n \geq 3$:

Definition 3.1.3. *A theory T is* NSOP$_n$ *(that is, does not have the* n-strong order property*) if there is no definable relation $R(x_1, x_2)$ with no n-cycles, but with tuples $\{a_i\}_{i \in \omega}$ with $\models R(a_i, a_j)$ for $i < j$. Otherwise it is* SOP$_n$.

Fact 3.1.1. ([101], [40]) *Simple theories are* NSOP$_1$*, and* NSOP$_n$ *theories are* NSOP$_m$ *for $n \leq m$.*

In [102] it is shown that T^*_{feq}, the model companion of the theory of parametrized equivalence relations, is NSOP$_1$ but not simple; a limited number of further examples have since been found by various authors. Yet the main problem, posed by Džamonja and Shelah in [40], has remained unsolved:

Problem 3.1.2. *Are all* NSOP$_3$ *theories* NSOP$_2$*? Are all* NSOP$_2$ *theories* NSOP$_1$*?*

In this chapter we answer the latter question in the positive:

Theorem 3.1.3. *All* NSOP_2 *theories are* NSOP_1.

One reason for the significance of this problem comes from Shelah and Usvyatsov's proposal in [102] to characterize classes of theories both internally in terms of the structure of their sufficiently saturated models, and externally in terms of orders on theories. The NSOP_2 theories have a deep external characterization: under the generalized continuum hypothesis, Džamonja and Shelah [40] show that maximality in the order $\triangleleft^*$, an order related to the Keisler order, implies a combinatorial property related to SOP_2, which Shelah and Usvyatsov then show in [102] to be the same as SOP_2; later, Malliaris and Shelah in [81] show the equivalence between SOP_2 and $\triangleleft^*$-maximality under the generalized continuum hypothesis. On the other hand, NSOP_1 theories can be characterized internally not only in terms of trees, but through the theory of independence, in analogy with stability theory. It is well known that simple theories are characterized as those theories where forking and dividing behave in certain ways as they do in stable theories; for example, symmetry of forking characterizes simple theories. In [52], Kaplan and Ramsey show that *Kim-forking*, or forking witnessed by invariant Morley sequences, is the correct way of extending the theory of forking to NSOP_1 theories from simple theories. By relaxing the requirement of base monotonicity, they extend the Kim-Pillay characterization of simple theories in terms of the existence of abstract independence relations to NSOP_1 theories, and, more concretely, characterize NSOP_1 theories by the symmetry of Kim-independence, by the independence theorem for Kim-independence, and by a variant of Kim's lemma in simple theories, asserting that *Kim-dividing* of a formula, rather than dividing, is witnessed by any invariant Morley sequence. Our result that NSOP_1 theories coincide with NSOP_2 theories therefore shows a surprising agreement between dividing lines related to Keisler's order and dividing lines related to independence.

We outline the chapter and give a word on the strategy for the proof. In section 3, we develop in general theories a version of a construction used by Chernikov and Kaplan in [27] to study forking and dividing in NTP_2 theories. In [1], Adler initiated the study of abstract relations between sets in a model, generalizing some of the properties of forking-independence, coheirs, and other concrete relations from model theory, and provided a set of potential axioms for these relations[1]. We notice that the construction of Chernikov and Kaplan can be relativized to relations between sets satisfying certain axioms, obtaining new

[1]Other than Adler's work in [1] and Conant's work on free amalgamation theories in [32], an additional observation which ultimately led us to the proof of this result is found in [36], where d'Elbée proposes the problem of explaining the apparent ubiquity of additional independence relations with no known concrete model-theoretic independence relations in NSOP_1 theories, such as *strong independence* existing alongside Kim-independence in the theory ACFG (introduced as part of a more general class in [37]) of algebraically closed fields with a generic additive subgroup. He observes that just as in the case of free amalgamation of generic functional structures in [71] or generic incidence structures in [33], these stronger independence relations can be used to prove the equivalence of forking and dividing for complete types in many known NSOP_1 theories. Before proving Theorem 1.6, we gave some very weak axioms (including *stationarity*, a feature of the examples considered by [36]) for abstract relations between sets over a model, which appeared

relations between sets from old ones, and iterate this construction to obtain a canonical class of coheirs in any theory.

In section 4, we study this canonical class of coheirs in NSOP_2 theories. Before the development of Kaplan and Ramsey's theory of Kim-independence in NSOP_1 theories in [52], Chernikov [25] proposed finding a theory of independence for NSOP_2 theories, and the proof of our main result comes from our efforts to answer this proposal. Just as in [27], Chernikov and Kaplan's construction gives maximal classes in the *dividing order* of Ben Yaacov and Chernikov [113], we show that in NSOP_2 theories our variant of this construction gives *minimal* classes in the restriction of this order to coheir Morley sequences, proving an analogue of Kim's lemma. As a by-product of this construction, we also initiate the theory of independence in a class related to the NATP theories of Ahn and Kim [3], the study of which was further developed by Ahn, Kim and Lee in [4], showing that under this assumption Kim-forking and Kim-dividing coincide for coheir Morley sequences. (See [69] for the question of finding an analogue for NSOP_1 theories of the role that NTP_2 theories play relative to simple theories, and developing Kim-independence in that analogue; that Kim-forking coincides with Kim-dividing for coheir Morley sequences in a related class gives us preliminary evidence that NATP completes this analogy.)

In section 5, we investigate behavior similar to NSOP_1 theories in NSOP_2 theories. We introduce the notion of *Conant-independence*, which will generalize the relation $A \mathop{\smile\!\!\!\!\!\mid}^{a}_{M} B$ defined by $\mathrm{acl}(MA) \cap \mathrm{acl}(MB) = M$ in the *free amalgamation theories* introduced by Conant [32] (based on concepts used to study the isometry groups of Urysohn spheres in [106]); see the following section. While it will end up coinciding with Kim-independence in our case, we studied a version of Conant-independence in a potentially strictly NSOP_1, potentially SOP_3 generalization of free amalgamation theories in Chapter 1. Conant-independence in NSOP_2 theories can be defined as Kim-independence relative to canonical Morley sequences, just as $\mathop{\smile\!\!\!\!\!\mid}^{a}$ is Kim-independence relative to free amalgamation Morley sequences (as in lemma 7.7 of [32]); it can also be defined by forcing Kim's lemma on Kim-independence, requiring a formula to divide with respect to *every* Morley sequence instead of just one, as suggested in tentative remarks of Kim in [60] in his discussion of *strong dividing* in *subtle theories*. We show that many of Ramsey and Kaplan's arguments on Kim-independence in NSOP_1 theories in [52] can be generalized to Conant-independence in NSOP_2 theories, including a chain condition, symmetry and a weak independnece theorem. (But as is apparent in [32] and Chapter 1, similar behavior can occur in a SOP_3 theory, which is why the following section is essential to the proof of our main result.)

In section 6, we conclude the proof of Theorem 3.1.3. One consequence of Conant's free amalgamation axioms (say, the freedom, closure and stationarity axioms, in Defintion 2.1 in [32]) is the following:

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to be very common in NSOP theories including strictly NSOP_1 theories and NSOP_4 theories, and proved that theories with such a relation could not be NSOP_2; instead of considering Morley sequences in canonical coheirs as in the below, we used $\mathop{\smile\!\!\!\!\!\mid}$-independent sequences for the abstract relation $\mathop{\smile\!\!\!\!\!\mid}$, in the sense of Definition 7.5 of [32]. Note also that the property *quasi-strong finite character* considered below is a property of the examples in [36].

Let $\mathop{\smile\!\!\!\!\!\!\mid}$ denote free amalgamation and $A_1 \mathop{\smile\!\!\!\!\!\!\mid}^a_M B$, $A_2 \mathop{\smile\!\!\!\!\!\!\mid}^a_M C$, and $B \mathop{\smile\!\!\!\!\!\!\mid}_M C$ with $A_1 \equiv_M A_2$. Then there is some $A \mathop{\smile\!\!\!\!\!\!\mid}^a_M BC$ with $A \equiv_{MB} A_1$ and $A \equiv_{MC} A_2$.

We will have shown in the prior section that Conant-independence is symmetric, and that a similar fact holds, roughly, when replacing free amalgamation with canonical coheirs and $\mathop{\smile\!\!\!\!\!\!\mid}^a$ with Conant-independence. Conant shows in [32] that *modular* free amalgamation theories must either be simple or SOP_3 (see [42] for a related result on countably categorical Hrushovski constructions), starting with a failure of forking-independence to coincide with $\mathop{\smile\!\!\!\!\!\!\mid}^a$ (because forking-independence cannot be symmetric unless a theory is simple) and using the above fact to build up a configuration giving SOP_3. Starting, analogously, with the assumption that an NSOP_2 theory T is SOP_1, so Kim-dividing independence is not symmetric and therefore fails to coincide with Conant-independence, we simulate Conant's construction of an instance of SOP_3. In short, we show that a NSOP_2 theory is either NSOP_1 or SOP_3. But a NSOP_2 theory is of course not SOP_3, so it must be NSOP_1.

3.2 Preliminaries

We let a, b, c, d, e, A, B, C denote sets, potentially with an enumeration depending on context, and x, y, z, X, Y, Z denote tuples of variables. We let $\mathbb{M}$ denote a sufficiently saturated model of a theory T and let M denote an elementary submodel. We write AB to denote the union (or concatenation) of the sets A and B, and write I, J, etc. for infinite sequences (or sometimes trees) of tuples or an infinite linearly ordered set.

Relations between sets

Roughly following the axioms for abstract independence relations in [1], as well as others that are standard in the literature, we define the following axioms for relations $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ between sets over a model:

Invariance: For all $\sigma \in \mathrm{Aut}(\mathbb{M})$, $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ implies $\sigma(A) \mathop{\smile\!\!\!\!\!\!\mid}_{\sigma(M)} \sigma(B)$.

Full existence: For $M \subseteq A, B \subseteq \mathbb{M}$, there is always some $A' \equiv_M A$ with $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$.

Left extension: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $A \subseteq C$, there is some $B' \equiv_A B$ with $C \mathop{\smile\!\!\!\!\!\!\mid}_M B'$.

Right extension: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $B \subseteq C$, there is some $A' \equiv_B A$ with $A' \mathop{\smile\!\!\!\!\!\!\mid}_M C$.

Left monotonicity: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $M \subseteq A' \subseteq A$, then $A' \mathop{\smile\!\!\!\!\!\!\mid}_M B$

Right monotonicity: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ and $M \subseteq B' \subseteq B$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_M B'$

(We will refer to the two previous properties, taken together, as monotonicity.)

Symmetry: If $A \mathop{\smile\!\!\!\!\!\!\mid}_M B$ then $B \mathop{\smile\!\!\!\!\!\!\mid}_M A$

Coheirs and Morley sequences

A global type p is a complete type over $\mathbb{M}$. For $p \in S(A)$ for $M \subseteq A$, we say p is *finitely satisfiable* over M or a *coheir extension* of its restriction to M if every formula in p is satisfiable in M. Global types p finitely satisfiable in M are *invariant* over M: whether $\varphi(x, b)$ belongs to p for φ a formula without parameters, depends only on the type of the parameter b over M. We write $a \mathop{\smile\!\!\!\!\!\!\mid}^u_M b$ to denote that $\mathrm{tp}(a/Mb)$ is finitely satisfiable in M. We let $a \mathop{\smile\!\!\!\!\!\!\mid}^h_M b$ denote $b \mathop{\smile\!\!\!\!\!\!\mid}^u_M a$. The relation $\mathop{\smile\!\!\!\!\!\!\mid}^u$ (over models) is well-known to satisfy all of the above properties other than symmetry. We say $\{b_i\}_{i \in I}$, for I potentially finite, is a *coheir sequence* over M if $b_i \mathop{\smile\!\!\!\!\!\!\mid}^u_M b_{<i}$ for $i \in I$. We say a coheir sequence $\{b_i\}_{i \in I}$, for I infinite, is moreover a *coheir Morley sequence* over M if there is a fixed global type $p(x)$ finitely satisfiable in M so that $b_i \models p(x)|_{\{Mb_j\}_{j<i}}$ for $i \in I$. The type of a coheir Morley sequence over M (indexed by a given set) is well-known to depend only on $p(x)$, and coheir Morley sequences are known to be indiscernible; the type of a coheir sequence over M depends only on the global coheirs over M extending the $\mathrm{tp}(b_i/Mb_{<i})$.

NSOP_1 **theories and Kim-dividing**

In this chapter we use nonstandard terminology: Kim-dividing, etc. are defined in terms of Morley sequences in *invariant* types over M rather than *finitely satisfiable* types over M in [52]. The reason why we do this is that $\mathop{\smile\!\!\!\!\!\!\mid}^u$ is known to satisfy left extension. This will do us no harm for our main result, though when we briefly consider Kim-forking in some NATP theories, we will note the nonstandard usage.

Definition 3.2.1. *A formula $\varphi(x, b)$* Kim-divides *over M if there is an coheir Morley sequence $\{b_i\}_{i \in \omega}$ starting with b so that $\{\varphi(x, b_i)\}_{i \in \omega}$ is inconsistent (equivalently, k-inconsistent for some k: any subset of size k is inconsistent). A formula $\varphi(x, b)$* Kim-forks *over M if it implies a (finite) disjunction of formulas Kim-dividing over M. We write $a \mathop{\smile\!\!\!\!\!\!\mid}^{Kd}_M b$, and say that a is* Kim-dividing independent *from b over M if $\mathrm{tp}(a/Mb)$ does not include any formulas Kim-dividing over M.*

The following follows directly from Proposition 5.2 of [28]; see also Proposition 3.22 of [52] (where the evident argument for the version for invariant types is given) and Theorem 5.16 of [52] for the full symmetry characterization of NSOP_1.

Fact 3.2.1. *Symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}^{Kd}$ implies* NSOP_1.

NSOP_2 **theories**

A characterization of SOP_2 as k-TP_1 was proven by Kim and Kim in [64], where they also introduce the notion of *weak* k-TP_1, prove that it implies SOP_1, and conjecture that it also implies SOP_2:

Definition 3.2.2. *The theory T has* weak k-TP_1 *if there exists a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in \omega^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \restriction n})\}_{n \in \omega}$ is consistent for any $\sigma \in \omega^\omega$, but for pairwise incomparable $\eta_1 \dots, \eta_k \in \omega^{<\omega}$ with common meet, $\{\varphi(x, b_{\eta_i})\}_{i=1}^k$ is inconsistent.*

Later, Chernikov and Ramsey, in Theorem 4.8 of [28], claim to show that weak k-TP_1 implies SOP_2, but their proof is incorrect; the embedded tree $\{b_\eta\}_{\eta \in \omega^{<\omega}}$ in the proof of that theorem is not actually strongly indiscernible over the parameter set C. In an earlier version of this chapter, we used this result. In this section, we will introduce an equivalent form of SOP_2 that will suffice for our argument, and use the same method as [28] to give a proof that will work to show this equivalence despite failing for weak k-TP_1.

Definition 3.2.3. *(Proposition 2.51, item IIIa, [102]). A list $\eta_1, \dots, \eta_n \in \omega^{<\omega}$ is a* descending comb *if and only if it is an antichain so that $\eta_1 <_{\mathrm{lex}} \dots <_{\mathrm{lex}} \eta_n$, and so that, for $1 \leq k < n$, $\eta_1 \wedge \dots \wedge \eta_{k+1} \lhd \eta_1 \wedge \dots \wedge \eta_k$.*

So for example, all descending combs of length n have the same quantifier-free type in the language $\{<_{\mathrm{lex}}, \vartriangleleft, \wedge\}$ as the descending comb $\langle 0\rangle^{n-1} \frown \langle 1\rangle, \dots \langle 1\rangle$; meanwhile, $\langle 00\rangle, \langle 01\rangle, \langle 10\rangle, \langle 11\rangle$ is an example of a lexicographically ordered antichain that is not a descending comb.

Definition 3.2.4. *(Definitions 11 and 12, [105]) For tuples $\overline{\eta}, \overline{\eta}' \in \omega^{<\omega}$ of elements of $\omega^{<\omega}$, we write $\overline{\eta} \sim_0 \overline{\eta}'$ to mean that $\overline{\eta}$ has the same quantifier-free type in the language $\{<_{\mathrm{lex}}, \vartriangleleft, \wedge\}$ as $\overline{\eta}'$. For $(b_\eta)_{\eta\in\omega^{<\omega}}$ a tree-indexed set of tuples and $\overline{\eta} = \eta_1, \dots, \eta_n \in \omega^{<\omega}$ an n-tuple of elements of $\omega^{<\omega}$, we write $b_{\overline{\eta}} =: b_{\eta_1} \dots b_{\eta_n}$, and call $(b_\eta)_{\eta\in\omega^{<\omega}}$* strongly indiscernible *over a set A if for all tuples $\overline{\eta}, \overline{\eta}' \in \omega^{<\omega}$ of elements of $\omega^{<\omega}$ with $\overline{\eta} \sim_0 \overline{\eta}'$, $b_{\overline{\eta}} \equiv_A b_{\overline{\eta}'}$.*

Fact 3.2.2. *(Theorem 16, [105]; see [96] for an alternate proof) Let $(b_\eta)_{\eta\in\omega^{<\omega}}$ be a tree-indexed set of tuples, and A a set. Then there is $(c_\eta)_{\eta\in\omega^{<\omega}}$ strongly indiscernible over A so that for any tuple $\overline{\eta} \in \omega^{<\omega}$ of elements of $\omega^{<\omega}$ and $\varphi(x) \in L(A)$, if $\models \varphi(b_{\overline{\eta}'})$ for all $\overline{\eta}' \sim_0 \eta$, then $\models \varphi(c_{\overline{\eta}})$.*

Definition 3.2.5. *The theory T has k-*DCTP$_1$ *if there exists a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta\in\omega^{<\omega}}$ so that $\{\varphi(x, b_{\sigma\restriction n})\}_{n\in\omega}$ is consistent for any $\sigma \in \omega^\omega$, but for any descending comb $\eta_1 \dots, \eta_k \in \omega^{<\omega}$, $\{\varphi(x, b_{\eta_i})\}_{i=1}^k$ is inconsistent.*

Lemma 3.2.3. *For any $k > 1$, a theory has* SOP$_2$ *if and only if it has k-*DCTP$_1$.

Proof. ($\Rightarrow$) The property 2-DCTP$_1$ follows directly from Fact 4.2, [28].

($\Leftarrow$) We follow the proof of theorem 4.8 of ([28]), which is incorrect for the claimed result. Let $\{b_\eta\}_{\eta\in\omega^{<\omega}}$ witness DCTP$_1$ with the formula $\varphi(x, y)$. By fact 3.2.2, we can assume $\{b_\eta\}_{\eta\in\omega^{<\omega}}$ is strongly indiscernible (as paths and descending combs are preserved under $\sim_0$-equivalence), and will produce a witness to SOP$_2$. Let $\eta_i = \langle 0\rangle^i \frown \langle 1\rangle$ (so that, say, $\eta_n, \dots, \eta_0$ will form a descending comb), and let n be maximal so that

$$\{\varphi(x, b_{\eta_i\frown\langle 0\rangle^\alpha}) : i < n, \alpha < \omega\}$$

is consistent; by consistency of the paths, n will be at least 1, and by inconsistency of descending combs of size k, n will be at most k. Let $C = \{b_{\eta_i\frown\langle 0\rangle^\alpha} : i < n-1, \alpha < \omega\}$. We see that, say, $\mu = \langle 0\rangle_{n-1}$ sits strictly above the meets of any two or more of the η_i for $i < n-1$ in the order $\vartriangleleft$, and is incomparable to and lexicographically to the left of η^{n-2} when $n > 1$, so the appropriately tree-indexed subset $\{c_\eta\}_{\eta\in\omega^{<\omega}}$ of $\{b_\eta\}_{\eta\in\omega^{<\omega}}$ consisting of those b_η with $\mu \trianglelefteq \eta$ (that is, where $c_\eta = b_{\mu\frown\eta}$) really is strongly indiscernible over C. By strong indiscernibility of $\{b_\eta\}_{\eta\in\omega^{<\omega}}$ and the fact that $\{\varphi(x, b_{\eta_i\frown\langle 0\rangle^\alpha}) : i < n, \alpha < \omega\}$ is consistent, $\{\varphi(x, c_{\langle 0\rangle\frown\langle 0\rangle^\alpha}) : i < n, \alpha < \omega\} \cup \{\varphi(x, c) : c \in C\}$ is consistent; let d realize it, and by Ramsey, compactness and an automorphism over C, we can assume $\{c_{\langle 0\rangle\frown\langle 0\rangle^\alpha}\}_{\alpha<\omega}$ is indiscernible over dC. On the other hand, for $p(y, \overline{z}) = \mathrm{tp}(d, \{c_{\langle 0\rangle\frown\langle 0\rangle^\alpha}\}_{\alpha<\omega}/C)$, we see that $p(y, \{c_{\langle 0\rangle\frown\langle 0\rangle^\alpha}\}_{\alpha<\omega}) \cup p(y, \{c_{\langle 1\rangle\frown\langle 0\rangle^\alpha}\}_{\alpha<\omega})$ is inconsistent, by strong indiscernibility of $\{b_\eta\}_{\eta\in\omega^{<\omega}}$ and inconsistency (by maximality of n) of $\{\varphi(x, b_{\eta_i\frown\langle 0\rangle^\alpha}) : i \leq n, \alpha < \omega\}$ (noting that, say, $p(y, \{c_{\langle 0\rangle\frown\langle 0\rangle^\alpha}\}_{\alpha<\omega})$ contains $\{b_{\eta_i\frown\langle 0\rangle^\alpha} : i < n-1, \alpha < \omega\}$). This is exactly what the "path collapse lemma," Lemma 4.6 of [28], tells us that we need to obtain SOP$_2$. $\square$

Though the proof of Theorem 4.8 of [28] is incorrect, that theorem (albeit, not a "local" version) will be a corollary of our main result, Theorem 3.1.3, and the result of [64] that weak k-TP_1 implies SOP_1. (Note that SOP_2 is just weak 2-TP_1).

Corollary 3.2.3.1. *(to Theorem 3.1.3) For any k, a theory has weak k-TP_1 if and only if it has SOP_2.*

3.3 Canonical coheirs in any theory

The following section will require no assumptions on T. Iterating a similar construction to the one used by Chernikov and Kaplan in [27] to prove the equivalence of forking and dividing for formulas in NTP_2 theories, we will contruct a canonical class of coheir extensions in any theory. This class will end up satisfying a variant of the "Kim's lemma for Kim-dividing" in NSOP_1 theories (Theorem 3.16 of [52]) when considered in a NSOP_2 theory.

Proposition 3.3.1. *Let T be any theory. Consider relations $\mathop{\smile\!\!\!\!\!\mid}$ between sets over a model that are stronger that $\mathop{\smile\!\!\!\!\!\mid}^h$, satisfy invariance, monotonicity, full existence and right extension, and satisfy the coheir chain condition: if $a \mathop{\smile\!\!\!\!\!\mid}_M b$ and $I = \{b_i\}_{i\in\omega}$ is a coheir Morley sequence starting with b, then there is some $I' \equiv_M I$ with $a \mathop{\smile\!\!\!\!\!\mid}_M I'$ and each term of I' satisfying $\mathrm{tp}(b/Ma)$. There is a weakest such relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{CK}}$.*

The "weakest" clause is not necessary for the main result, but we include it anyway to show our construction is canonical.

We start by relativizing the notions of Kim-dividing, Kim-forking, and *quasi-dividing* (Definition 3.2 of [27]) to an M-invariant ideal on the definable subsets of $\mathbb{M}$.

Definition 3.3.1. *Let $\mathcal{I}$ be an M-invariant ideal on the definable subsets of $\mathbb{M}$. A formula $\varphi(x, b)$* $\mathcal{I}$-Kim-divides *over M if there is a coheir Morley sequence $\{b_i\}_{i\in\omega}$ starting with b so that for some k, the intersection of some (any) k-element subset of $\{\varphi(\mathbb{M}, b_i)\}_{i\in\omega}$ belongs to $\mathcal{I}$. We say $\varphi(x, b)$* $\mathcal{I}$-Kim-forks *over $\mathbb{M}$ if it implies a (finite) disjunction of formulas $\mathcal{I}$-Kim-dividing over M. We say $\varphi(x, b)$* $\mathcal{I}$-quasi-divides *over M if there are $b_1, \dots, b_n$ with $b \equiv_M b_i$ so that $\cap_{i=1}^n \varphi(\mathbb{M}, b_i) \in \mathcal{I}$.*

We say $\varphi(x, b) \vdash^{\mathcal{I}} \psi(x, c)$ if $\varphi(\mathbb{M}, b) \backslash \psi(\mathbb{M}, c) \in \mathcal{I}$.

The proof of the following lemma is adapted straightforwardly from the proof of the "broom lemma" of Chernikov and Kaplan (Lemma 3.1 of [27])[2]. For the convenience of the reader we give a simplified proof of the modified version; note that this version is just a rephrasing in terms of ideals of Lemma 4.19 in [26]:

[2]Alex Kruckman, in a personal communication with the author, discussed an alternative to this proof for showing the properness of the ideal corresponding to the independence result of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{CK}}$, with the broom lemma as a corollary, which works for invariant Morley sequences as well as coheir Morley sequences; it is based on unpublished work of James Hanson on the concept of "fracturing," a generalization of quasi-forking and quasi-dividing.

Lemma 3.3.2. *("$\mathcal{I}$-broom lemma") Suppose*

$$\alpha(x, e) \vdash^{\mathcal{I}} \psi(x, c) \vee \bigvee_{i=1}^{N} \varphi_i(x, a_i)$$

with $\varphi_i(x, a_i)$ $\mathcal{I}$-Kim-dividing over M with respect to $P(x)$ and $c \mathop{\smile\!\!\!\!\!\!\;|}_M^u a_1 \dots a_N$. Then there are some $e_1, \dots e_m$ with $e_i \equiv_M e$ so that $\bigwedge_{i=1}^{m} \alpha(x, e_i) \vdash^{\mathcal{I}} \psi(x, c)$. In particular, $\mathcal{I}$-Kim-forking implies $\mathcal{I}$-quasi-dividing over M.

Proof. We need the following claim:

Claim 3.3.3. *Let $a^1, \dots, a^n$ begin a coheir Morley sequence in a global type q finitely satisfiable over M. Let $a \equiv_M a^i$ and let b be any tuple. Then there are $b^1, \dots, b^n$ so that $b^1a^1, \dots, b^na^n$ begin a coheir Morley sequence and $b^ia^i \equiv_M ba$. (The same is true for Coheir morley sequences themselves, rather than just their initial segments).*

Proof. Left extension for $\mathop{\smile\!\!\!\!\!\!\;|}^u$ gives a global type r finitely satisfiable over M extending both q and $\mathrm{tp}(ab/M)$. Now take a coheir Morley sequence in r and apply an automorphism. The parenthetical is similar. □

Now we can prove the lemma by induction on N. Write $\bigvee_{i=1}^{N-1} \varphi_i(x, a_i)$ as $\varphi(x, b)$, and let $a = a_N$. Let p be a global coheir extension of $\mathrm{tp}(c/Mba)$. Let $(a^i)_{i=1}^n$ be such that $a^i \mathop{\smile\!\!\!\!\!\!\;|}^u a^{i-1}, \dots, a^1$ and $a^i \equiv_M a$ for $1 \leq i \leq n$ and $\wedge_{i=1}^{n} \varphi_N(x, a^i) \vdash^{\mathcal{I}} \perp$. By the claim, find $b^1, \dots b^n$ so that $a^ib^i \mathop{\smile\!\!\!\!\!\!\;|}^u a^{i-1}b^{i-1} \dots a^1b^1$ and $a^ib^i \equiv_M ab$ for $1 \leq i \leq n$. Then we can assume $c \models p|_{Maba^1b^1 \dots a^nb^n}$. From $c \mathop{\smile\!\!\!\!\!\!\;|}_M^u a^1b^1 \dots a^nb^n$, together with $a^ib^i \mathop{\smile\!\!\!\!\!\!\;|}^u a^{i-1}b^{i-1} \dots a^1b^1$ for $1 \leq i \leq n$, it is easy to check $ca^{i+1}b^{i+1} \dots a^nb^n \mathop{\smile\!\!\!\!\!\!\;|}_M^u a^ib^i$ for $0 \leq i \leq n$, and therefore

$$cb^{i+1} \dots b^n \mathop{\smile\!\!\!\!\!\!\;|}_M^u b^i$$

for $0 \leq i < n$.

Now for $1 \leq i \leq n$ we have $ca^ib^i \equiv_M cab$. Let $e_ica^ib^i \equiv_M ecab$ for $1 \leq i \leq n$. Then

$$\bigwedge \alpha(x, e_i) \vdash^{\mathcal{I}} \psi(x, c) \vee \bigvee_{i=1}^{n} \varphi(x, b^i) \vee \bigwedge_{i=1}^{n} \varphi_N(x, a^i)$$

But by choice of the a^i,

$$\bigwedge \alpha(x, e_i) \vdash^{\mathcal{I}} \psi(x, c) \vee \bigvee_{i=1}^{n} \varphi(x, b^i)$$

Now for $1 \leq i \leq n$, because $b^i \equiv_M b$, $\varphi(x, b^i)$ will be of the form $\bigvee_{j=1}^{N-1} \varphi_j(x, a'_j)$ for $\varphi_j(x, a'_j)$ $\mathcal{I}$-Kim-dividing over M. So, as the first of n steps, we can apply $cb^2 \dots b^n \mathop{\smile\!\!\!\!\!\!\;|}^u b^1$ and the inductive hypothesis on N to find some conjunction $\beta(x, \overline{e})$ of conjugates of $\bigwedge \alpha(x, e_i)$ (which will therefore be a conjunction of conjugates of $\alpha(x, e)$) so that

$$\beta(x,\overline{e}) \vdash^{\mathcal{I}} \psi(x,c) \vee \bigvee_{i=2}^{n} \varphi(x,b^i)$$

Repeating $n-1$ more times, we are done.

□

We now begin our construction. The following terminology comes from the notion of strong finite character (used in e.g. [28]).

Definition 3.3.2. *Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant relation between sets over a model. We say that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies* quasi-strong finite character *if for p, q complete types over some model M, $\{a, b \models p(x) \cup q(y) : a \mathop{\smile\!\!\!\!\!\!|}_M b\}$ is type-definable.*

Definition 3.3.3. *Let $\mathop{\smile\!\!\!\!\!\!|}$ be an invariant relation between sets over a model satisfying monotonicity, right extension and quasi-strong finite character, and fix a complete type $P(x)$ over a model M.*

(1) A set of formulas $\{\varphi_i(x, b_i)\}_{i\in I}$ is $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$ *if there is no $a \models P(x)$ with $a \mathop{\smile\!\!\!\!\!\!|}_M \{b_i\}_{i\in I}$ and $\models \varphi_i(a, b_i)$ for all $i \in I$.*

(2) A formula $\varphi(x, b)$ $h^{\mathop{\smile\!\!\!\!\!\!|}}$-Kim-divides with respect to $P(x)$ *if there is a coheir Morley sequence $\{b_i\}_{i\in\omega}$ starting with b so that $\{\varphi(x, b_i)\}_{i\in\omega}$ is $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$.*

(3) A formula $h^{\mathop{\smile\!\!\!\!\!\!|}}$-Kim-forks *with respect to $P(x)$ if it implies a disjunction of formulas $h^{\mathop{\smile\!\!\!\!\!\!|}}$-Kim-dividing with respect to $P(x)$.*

(4) A formula $\varphi(x, b)$ $h^{\mathop{\smile\!\!\!\!\!\!|}}$-quasi-divides *over M with respect to $P(x)$ if there are $b_1, \ldots, b_n$ with $b_i \equiv_M b$ and $\{\varphi(x, b_i)\}_{i=1}^{n}$ $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$.*

Lemma 3.3.4. *(1) The sets defined by formulas $\varphi(x, b)$ so that $\{\varphi(x, b)\}$ is $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$ form an M-invariant ideal $\mathcal{I}^{\mathop{\smile\!\!\!\!\!\!|}}_{P(x)}$.*

(2) A set $\{\varphi_i(x, b_i)\}_{i\in I}$ is $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$ if and only if some finite subset is (so its conjunction defines a set in the ideal $\mathcal{I}^{\mathop{\smile\!\!\!\!\!\!|}}_{P(x)}$.)

Proof. For (1), it suffices to show (a) that if $\models \forall x(\varphi(x, b) \to \psi(x, c))$, and $\psi(x, c)$ is $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$, then $\varphi(x, b)$ is $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$, and (b) that if both $\varphi(x, b)$ and $\psi(x, c)$ are $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$ then so is $\varphi(x, b) \vee \psi(x, c)$. For (a), suppose otherwise; then there is some realization a of $P(x)$ with $\models \varphi(a, b)$ and $a \mathop{\smile\!\!\!\!\!\!|}_M b$. By right extension, we can assume $a \mathop{\smile\!\!\!\!\!\!|}_M bc$. But then $\models \psi(a, c)$, and by right monotonicity, $a \mathop{\smile\!\!\!\!\!\!|}_M c$, contradicting that $\psi(x, c)$ is $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$. For (b), suppose otherwise; then there is some realization a of $P(x)$ with $\models \varphi(a, b) \vee \psi(a, c)$ and $a \mathop{\smile\!\!\!\!\!\!|}_M bc$; without loss of generality, $\models \varphi(a, b)$, and by right monotonicity, $a \mathop{\smile\!\!\!\!\!\!|}_M b$, contradicting that $\varphi(x, b)$ is $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$. The proof of (a) also gives us the fact that a set $\{\varphi_i(x, b_i)\}_{i\in I}$ is $h^{\mathop{\smile\!\!\!\!\!\!|}}$-inconsistent with respect to $P(x)$ if some finite subset is (so its conjunction defines a set in the ideal $\mathcal{I}^{\mathop{\smile\!\!\!\!\!\!|}}_{P(x)}$). To complete

(2), we show the "only if" direction. If $\{\varphi_i(x,b_i)\}_{i\in I}$ is $h^{\smile\mkern-12mu\mid}$-inconsistent with respect to $P(x)$ then there is no realization a of $P(x)\cup\{\varphi_i(x,b_i)\}_{i\in I}$ with $a \mathop{\smile\mkern-12mu\mid}_M \{b_i\}_{i\in I}$. But the set of realizations a of $P(x)$ that satisfy $a \mathop{\smile\mkern-12mu\mid}_M \{b_i\}_{i\in I}$ is, by quasi-strong finite character, type-definable. So by compactness, there must be some finite $I_0 \subseteq I$ so there is no realization a of $P(x)\cup\{\varphi_i(x,b_i)\}_{i\in I_0}$ with $a \mathop{\smile\mkern-12mu\mid}_M \{b_i\}_{i\in I}$. But if there is a realization a of $P(x)\cup\{\varphi_i(x,b_i)\}_{i\in I_0}$ with $a \mathop{\smile\mkern-12mu\mid}_M \{b_i\}_{i\in I_0}$, then we can even get $a \mathop{\smile\mkern-12mu\mid}_M \{b_i\}_{i\in I}$ by right-extension, so $\{\varphi_i(x,b_i)\}_{i\in I_0}$ will be as desired. □

Corollary 3.3.4.1. *For all formulas, $h^{\smile\mkern-12mu\mid}$-Kim-forking with respect to $P(x)$ implies $h^{\smile\mkern-12mu\mid}$-quasi-dividing with respect to $P(x)$.*

Proof. By Lemma 3.3.4, $h^{\smile\mkern-12mu\mid}$-Kim-dividing with respect to $P(x)$ is just $\mathcal{I}^{\smile\mkern-12mu\mid}_{P(x)}$-Kim-dividing. Apply Lemma 3.3.2 to $\mathcal{I}^{\smile\mkern-12mu\mid}_{P(x)}$. □

Lemma 3.3.5. *If a formula $\varphi(x,b)$ is $h^{\smile\mkern-12mu\mid}$-inconsistent with respect to $P(x)$, then it is $h^{\smile\mkern-12mu\mid}$-inconsistent with respect to any complete type $Q(x,y)$ extending $P(x)$. So the same is true for $h^{\smile\mkern-12mu\mid}$-Kim-dividing and $h^{\smile\mkern-12mu\mid}$-Kim-forking.*

Proof. Suppose otherwise. Then there is a realization ac of $Q(x,y)\cup\{\varphi(x,b)\}$ with $ac \mathop{\smile\mkern-12mu\mid}_M b$. So by left monotonicity, $a \mathop{\smile\mkern-12mu\mid}_M b$, but a realizes $P(x)\cup\{\varphi(x,b)\}$, a contradiction. □

We are now in a position to study derived independence relations:

Definition 3.3.4. *Let $\mathop{\smile\mkern-12mu\mid}$ be an invariant relation between sets over a model satisfying monotonicity, right extension and quasi-strong finite character. Then we define $a \mathop{\smile\mkern-12mu\mid}'_M b$ to mean that* $\mathrm{tp}(a/Mb)$ *does not contain any formulas $h^{\smile\mkern-12mu\mid}$-Kim-forking with respect to* $\mathrm{tp}(a/M)$.

Lemma 3.3.6. *Suppose $\mathop{\smile\mkern-12mu\mid}$ is an invariant relation between sets over a model satisfying monotonicity, right extension, quasi-strong finite character, and full existence. Then so is $\mathop{\smile\mkern-12mu\mid}'$.*

Proof. Invariance is obviously inherited from $\mathop{\smile\mkern-12mu\mid}$. Quasi-strong finite character is by construction and right extension is also standard from the construction: if $a \mathop{\smile\mkern-12mu\mid}'_M b$ but, for some $c\in\mathbb{M}$ there is no $a' \equiv_{Mb} a$ with $a \mathop{\smile\mkern-12mu\mid}'_M bc$, then $\mathrm{tp}(a/Mb)$ must imply a disjunction of formulas with parameters in Mbc $h^{\smile\mkern-12mu\mid}$-Kim-forking with respect to $P(x)$; some formula in $\mathrm{tp}(a/Mb)$ must then imply this disjunction, which will then $h^{\smile\mkern-12mu\mid}$-Kim-fork with respect to $P(x)$, contradicting $a \mathop{\smile\mkern-12mu\mid}'_M b$. Right monotonicity is by definition. Left monotonicity is Lemma 3.3.5. It remains to show full existence; the proof is a straightforward generalization of the proof of Lemma 3.7 of [27]. By right extension, it suffices to show that $b \mathop{\smile\mkern-12mu\mid}'_M M$ for any tuple b (the "existence" property that is implied by full existence). Suppose otherwise; then $\mathrm{tp}(b/M)$ contains a formula $\varphi(x,m)$ for $m\in M$ that $h^{\smile\mkern-12mu\mid}$-Kim-forks over M. By Corollary 3.3.4.1, $\varphi(x,m)$ $h^{\smile\mkern-12mu\mid}$-quasi-divides over M. Since $m\in M$, this just means that $\varphi(x,m)\in\mathcal{I}^{\smile\mkern-12mu\mid}_{\mathrm{tp}(b/M)}$. But since $\varphi(x,m)\in\mathrm{tp}(b/M)$, this contradicts full existence for $\mathop{\smile\mkern-12mu\mid}$. □

The next observation is required to produce a relation with the coheir chain condition:

Lemma 3.3.7. *Let* $\mathop{\smile\!\!\!\!\!|}$ *be as in Lemma 3.3.6 and suppose* $a \mathop{\smile\!\!\!\!\!|}'_M b$. *Then for* $I = \{b_i\}_{i\in\omega}$ *a coheir Morley sequence starting with* b, *there is* $I' \equiv_M I$ *with* $a \mathop{\smile\!\!\!\!\!|}_M I'$ *and each term of* I' *satisfying* $\mathrm{tp}(b/Ma)$. *In particular,* $\mathop{\smile\!\!\!\!\!|}'$ *implies* $\mathop{\smile\!\!\!\!\!|}$, *so* $h^{\mathop{\smile\!\!\!\!\!|}}$*-Kim-forking implies* $h^{\mathop{\smile\!\!\!\!\!|}'}$*-Kim-forking.*

Proof. Suppose otherwise: then for $q = \mathrm{tp}(a, b/M)$, $\cup_{i\in\omega} q(x, b_i)$ is $h^{\mathop{\smile\!\!\!\!\!|}}$-inconsistent with respect to $\mathrm{tp}(a/M)$, so by part (2) of Lemma 3.3.4, some finite subset must be $h^{\mathop{\smile\!\!\!\!\!|}}$-inconsistent with respect to $\mathrm{tp}(a/M)$. This gives us a formula in $q(x, b)$ that $h^{\mathop{\smile\!\!\!\!\!|}}$-Kim divides with respect to $\mathrm{tp}(a/M)$, a contradiction. □

Note that $\mathop{\smile\!\!\!\!\!|}^h$ satisfies the assumptions of Lemma 3.3.6. Now define inductively, $\mathop{\smile\!\!\!\!\!|}^{(0)} = \mathop{\smile\!\!\!\!\!|}^h$, $\mathop{\smile\!\!\!\!\!|}^{(n+1)} = (\mathop{\smile\!\!\!\!\!|}^{(n)})'$. Let $\mathop{\smile\!\!\!\!\!|}^{\mathrm{CK}} = \bigcap_{i=0}^{\infty} \mathop{\smile\!\!\!\!\!|}^{(n)}$. Then because $h^{\mathop{\smile\!\!\!\!\!|}^{(n)}}$-Kim-forking implies $h^{\mathop{\smile\!\!\!\!\!|}^{(n+1)}}$-Kim-forking, and $a \mathop{\smile\!\!\!\!\!|}^{\mathrm{CK}}_M b$ means that $\mathrm{tp}(a/Mb)$ does not contain a $h^{\mathop{\smile\!\!\!\!\!|}^{(n)}}$-Kim-forking formula for any n, right extension and quasi-strong finite character are standard. Monotonicity and invariance follows from monotonicity and invariance of the $\mathop{\smile\!\!\!\!\!|}^{(n)}$. By right extension for $\mathop{\smile\!\!\!\!\!|}^{\mathrm{CK}}$, full existence for $\mathop{\smile\!\!\!\!\!|}^{\mathrm{CK}}$ would follow from the existence property $b \mathop{\smile\!\!\!\!\!|}^{\mathrm{CK}}_M M$ for any b, but this just follows from full existence for each of the $\mathop{\smile\!\!\!\!\!|}^{(n)}$. Finally, the coheir chain condition follows from Lemma 3.3.7 together with quasi-strong finite character for the $\mathop{\smile\!\!\!\!\!|}^{(n)}$ and compactness.

It remains to show that $\mathop{\smile\!\!\!\!\!|}^{\mathrm{CK}}$ is the weakest relation implying $\mathop{\smile\!\!\!\!\!|}^h$ and satisfying these properties. Let $\mathop{\smile\!\!\!\!\!|}$ be some other such relation and assume by induction that $\mathop{\smile\!\!\!\!\!|}$ implies $\mathop{\smile\!\!\!\!\!|}^{(n)}$ Assume $a \mathop{\smile\!\!\!\!\!|}_M b$; we show $a \mathop{\smile\!\!\!\!\!|}^{(n+1)}_M b$. Suppose otherwise; by right extension for $\mathop{\smile\!\!\!\!\!|}$, we can assume $\mathrm{tp}(a/Mb)$ contains a formula $\varphi(x, b)$ that $h^{\mathop{\smile\!\!\!\!\!|}^{(n)}}$-Kim-divides with respect to $\mathrm{tp}(a/M)$. Let $I = \{b_i\}_{i\in\omega}$ be a coheir Morley sequence starting with b witnessing this. Then by the coheir chain condition for $\mathop{\smile\!\!\!\!\!|}$, there is some a' with $a' \mathop{\smile\!\!\!\!\!|}_M I$, so in particular $a' \mathop{\smile\!\!\!\!\!|}^{(n)}_M I$ by induction, and with $a'b_i \equiv_M ab$ for $i \in \omega$, so in particular with a' satisfying $\{\varphi(x, b_i)\}_{i\in\omega}$, a contradiction.

This completes the proof of Proposition 3.3.1.

Remark 3.3.8. If $\mathbb{M}' \succ \mathbb{M}$ is a very large (sufficiently saturated) model, then $\mathop{\smile\!\!\!\!\!|}^{\mathrm{CK}}$ as computed in $\mathbb{M}'$ restricts to $\mathop{\smile\!\!\!\!\!|}^{\mathrm{CK}}$ as computed in $\mathbb{M}$. We can see that $\mathop{\smile\!\!\!\!\!|}^{\mathrm{CK}}$ has this property as it is true for $\mathop{\smile\!\!\!\!\!|}^h$ and is preserved by going from $\mathop{\smile\!\!\!\!\!|}^{(n)}$ to $\mathop{\smile\!\!\!\!\!|}^{(n+1)}$. However, it is also immediate that invariance, monotonicity, full existence and right extension, and the coheir chain condition are preserved on restriction.

3.4 Canonical coheirs in NSOP_2 theories

The goal of this section is to prove a version of "Kim's lemma for Kim-dividing" for canonical Morley sequences in NSOP_2 theories.

Lemma 3.4.1. *Let $p(x)$ be a type over M. Then it has a global extension $q(x)$ so that for all tuples $b \in \mathbb{M}$, if $c \models q|Mb$, then $b \mathop{\smile\!\!\!\!\!\!|}_{M}^{\mathrm{CK}} c$. So in particular, q is a global coheir of $p(x)$.*

Proof. In a very large $\mathbb{M}' \succ \mathbb{M}$, full existence and invariance for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{CK}}$, and an automorphism, gives us a realization c' of $p(x)$ with $\mathbb{M} \mathop{\smile\!\!\!\!\!\!|}_{M}^{\mathrm{CK}} c'$. Now take $q(x)$ to be $\mathrm{tp}(c'/\mathbb{M})$, and the lemma follows by monotonicity on the left. □

Definition 3.4.1. *We call $q(x)$ as in Lemma 3.4.1 a* canonical coheir*, and a coheir Morley sequence in it a* canonical Morley sequence.

Theorem 3.4.2. *Let T be* NSOP$_2$*. Suppose a canonical Morley sequence witnesses Kim-dividing of a formula $\varphi(x,b)$ over M. Then there is a finite bound (depending only on $\varphi(x,y)$ and the degree of Kim-dividing witnessed by the canonical Morley sequence) on the length of a coheir sequence $\{b_i\}_{i=1}^{n}$ over M of realizations of* $\mathrm{tp}(b/M)$ *so that $\{\varphi(x,b_i)\}_{i=1}^{n}$ is consistent. In particular, every coheir Morley sequence starting with b witnesses Kim-dividing of $\varphi(x,b)$ over M.*

To start, we introduce the notion of a *coheir tree* in a general theory T.

Definition 3.4.2. *Let p be any type over M. We say that a tree $(b_\eta)_{\eta \in \omega^{\leq n}}$ of realizations of p is a* coheir tree *in p if*

(1) for each $\mu \in \omega^{<n}$, $(\{b_\eta\}_{\eta \trianglerighteq \mu^\frown \langle i \rangle})_{i=0}^{\infty}$ (the sequence consisting of the subtrees above a fixed node) is a coheir Morley sequence over M.

(2) there are global coheir extensions $q_0, \ldots, q_n$ of p so that for each $\mu \in \omega^{n-m}$, $b_\mu \models q_m|_{\{b_\eta\}_{\eta \triangleright \mu}}$.

The key lemma of this section allows us to construct coheir trees in any theory so that sequences of nodes with common meet are canonical Morley sequences. Abusing the language by nodes, paths, etc. we often refer to the tuples which they index; the term "descending comb" will have a similar meaning in a tree of finite height or a set of subtrees as it does in $\omega^{<\omega}$.

Lemma 3.4.3. *Let $p(x)$ be any type over M. Let $q(x)$ be a canonical coheir extension of $p(x)$. Let $b_0, \ldots, b_n$ be a coheir sequence over M of realizations of p. Then there is a coheir tree indexed by $\omega^{\leq n}$, any path of which, read in the direction of the root, realizes* $\mathrm{tp}(b_0 \ldots b_n/M)$*, and descending comb of which, read in lexicographic order, begin a canonical Morley sequence in $q(x)$.*

Proof. We need the following claim:

Claim 3.4.4. *If $a \mathop{\smile\!\!\!\!\!\!|}_{M}^{\mathrm{CK}} b$ and I is a coheir tree in* $\mathrm{tp}(b/M)$*, then there is some $I' \equiv_M I$ with $a \mathop{\smile\!\!\!\!\!\!|}_{M}^{\mathrm{CK}} I'$ (so in particular $I' \mathop{\smile\!\!\!\!\!\!|}_{M}^{u} a$) each term of which satisfies* $\mathrm{tp}(b/Ma)$.

Proof. Let $I = \{b_\eta\}_{\eta \in \omega^{\leq n}}$; we find $I' = \{b'_\eta\}_{\eta \in \omega^{\leq n}}$ as desired. The proof is by downward induction on k: suppose $\{b'_\eta\}_{\eta \trianglerighteq \zeta_{n-k}}$ is already constructed, and we construct $\{b'_\eta\}_{\eta \trianglerighteq \zeta_{n-(k+1)}}$. First, $\{b'_\eta\}_{\eta \triangleright \zeta_{n-(k+1)}}$ comes directly from the chain condition. Second, by left extension for $\mathop{\smile\!\!\!\!\!\!|}\,^u$, find some copy J of $\{b'_\eta\}_{\eta \triangleright \zeta_{n-(k+1)}}$ over M with $J \mathop{\smile\!\!\!\!\!\!|}\,^u_M \{b'_\eta\}_{\eta \triangleright \zeta_{n-(k+1)}}$ and some arbitrary term of J satisfying the conjugate to $M\{b'_\eta\}_{\eta \triangleright \zeta_{n-(k+1)}}$ of $\mathrm{tp}(b_{\zeta_{n-(k+1)}}/M\{b_\eta\}_{\eta \triangleright \zeta_{n-(k+1)}})$ (that is, $q_{k+1}(x)|_{M\{b'_\eta\}_{\eta \triangleright \zeta_{n-(k+1)}}}$ from Definition 3.4.2). Then use the chain condition to find some $J' \equiv_{M\{b'_\eta\}_{\eta \triangleright \zeta_{n-(k+1)}}} J$ with $J' \equiv_{Ma} \{b'_\eta\}_{\eta \triangleright \zeta_{n-(k+1)}}$ and $a \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{CK}}_M \{b'_\eta\}_{\eta \triangleright \zeta_{n-(k+1)}} J'$. Finally, using monotonicity on the right, discard all the terms of J' other than the one corresponding to the chosen term of J, to obtain $b'_{\zeta_{n-(k+1)}}$. □

Now by induction, it suffices to show this for a coheir sequence $b_0 \dots, b_{n+1}$ assuming $I_n = (b_\eta)_{\eta \in \omega^{\leq n}}$ is already constructed for $b_0 \dots, b_n$. First, we find a long coheir sequence $\{I^i_n\}_{i=0}^{\alpha}$ of realizations of $\mathrm{tp}(I_n/M)$ so that each node of I^γ_n satisfies $q(x)|_{M\{I^i_n\}_{i<\gamma}}$; then having taken it long enough, we can find a coheir Morley sequence $\{I^i_n\}_{i=0}^{\omega}$ with the same property, preserving the condition on descending combs. (Any descending comb inside of these copies will either lie inside of one copy of I_n, so will of course begin a descending Morley sequence inside of $q(x)$ by the induction hypothesis, or will consist of a descending comb inside one copy I^i_n followed by an additional node of a later copy I^j_n for $i < j$, which will indeed continue the Morley sequence in $q(x)$ begun by the previous nodes.) Suppose $\{I^i_n\}_{i<\gamma}$ already constructed; taking $a = \{I^i_n\}_{i<\gamma}$ in the above claim and $b \models q(x)|_{M\{I^i_n\}_{i<\gamma}}$, we can choose I^γ_n to be the I' given by the claim.

Now let q_{n+1} be a global extension, finitely satisfiable in M, of $\mathrm{tp}(b_{n+1}/Mb_0 \dots b_n)$. Then we take $b \models q_{n+1}(x)|_{M\{I^i_n\}_{i=0}^{\infty}}$ as the new root, guaranteeing the condition on paths. Now reindex accordingly.

□

We can now prove Theorem 3.4.2. Let $q(x)$ be a canonical coheir extension of $\mathrm{tp}(b/M)$ and k the degree of Kim-dividing for $\varphi(x, b)$ witnessed by a canonical Morley sequence in $q(x)$. Let $\{b_i\}_{i=0}^{n}$ be a coheir sequence over M of realizations of $\mathrm{tp}(b/M)$ so that $\{\varphi(x, b_i)\}_{i=0}^{n}$ is consistent. Then the coheir tree given by the previous lemma gives the first $n+1$ levels of an instance of k-DCTP$_1$: the k-dividing witnessed by canonical Morley sequences in $q(x)$ gives the inconsistency condition for descending combs of size k, and the consistency of $\{\varphi(x, b_i)\}_{i=0}^{n}$ gives the consistency of the paths. So if n is without bound, we must have k-DCTP$_1$ for $\varphi(x, y)$ by compactness, and thus SOP$_2$ by lemma 3.2.3. This concludes the proof of 3.4.2.

We have some applications of this proof to a notion related to the NATP theories introduced by Ahn and Kim in [3], and studied in greater depth by Ahn, Kim and Lee in [4], assuming the NATP analogue of lemma 3.2.3. The result for NATP theories would be interesting because while NSOP$_1$ theories are NATP [3], as Ahn, Kim and Lee have shown in [4], there are examples of NATP SOP$_1$ theories. The following is the original definition from [3]:

Definition 3.4.3. *The theory T has* NATP (the negation of the antichain tree property) *if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n \in \omega}$ is* 2-*inconsistent for any $\sigma \in 2^\omega$, but for pairwise incompararable $\eta_1, \dots, \eta_l \in 2^{<\omega}$, $\{\varphi(x, b_{\eta_i})\}_{i=1}^{l}$ is consistent.*

In [4], Ahn, Kim and Lee define a theory to have k-ATP if the above fails replacing 2-inconsistency with k-inconsistency, and show that for any $k \geq 2$, a theory fails to be NATP (that is, has 2-ATP) if and only if it has k-ATP. That is, they show the analogue for NATP theories of results of Kim and Kim in [64] on NSOP_2 theories, but of not those claimed by Chernikov and Ramsey in [28], nor of the above Lemma 3.2.3. One might ask whether, for any k, the following definition is equivalent to the failure of NATP:

Definition 3.4.4. *The theory T has k-DCTP_2 if there exists a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma \upharpoonright n})\}_{n \in \omega}$ is k-inconsistent for any $\sigma \in 2^\omega$, but for any descending comb $\eta_1 \dots, \eta_l \in 2^{<\omega}$, $\{\varphi(x, b_{\eta_i})\}_{i=1}^{l}$ is consistent.*

If so, then the following applies to NATP theories:

Theorem 3.4.5. *Let T be a theory so that, for all $k \geq 2$, T does not have k-DCTP_2. Let M be any model and b any tuple. Then there is a global type extending $\mathrm{tp}(b/M)$, finitely satisfiable in M, so that for any formula $\varphi(x, y)$ with parameters in M, if coheir Morley sequences in this type do not witness Kim-dividing of $\varphi(x, b)$, no coheir Morley sequence over M starting with b witnesses Kim-dividing of $\varphi(x, b)$ over M.*

This follows from the same construction. The following corollary is standard; see Corollary 3.16 of [27] for a similar argument:

Corollary 3.4.5.1. *If, for all $k \geq 2$, T does not have k-DCTP_2, then Kim-forking (with respect to coheir Morley sequences) coincides with Kim-dividing (with respect to coheir Morley sequences).*

3.5 Conant-independence in NSOP_2 theories

We introduce a notion of independence which will generalize, in the proof of the main result of this chapter, the role played by $\mathop{\smile\!\!\!\!\!\mid}^{a}$ in the *free amalgamation theories* introduced in [32]. The notation $\mathop{\smile\!\!\!\!\!\mid}^{K^*}$ comes from the related notion of Kim-independence from [52], $\mathop{\smile\!\!\!\!\!\mid}^{K}$; a similar notion involving dividing with respect to *all* (invariant) Morley sequences is suggested in tentative remarks of Kim in [60].

Definition 3.5.1. *Let M be a model and $\varphi(x, b)$ a formula. We say $\varphi(x, b)$* Conant-divides *over M if for* every *coheir Morley sequence $\{b_i\}_{i \in \omega}$ over M starting with b, $\{\varphi(x, b_i)\}_{i \in \omega}$ is inconsistent. We say $\varphi(x, b)$* Conant-forks *over M if and only if it implies a disjunction of formulas Conant-dividing over M. We say a is* Conant-independent *from b over M, written $a \mathop{\smile\!\!\!\!\!\mid}^{K^*}_M b$, if $\mathrm{tp}(a/Mb)$ does not contain any formulas Conant-forking over M.*

Note that this definition differs from the standard definition of Conant-independence given in Chapter 1, in that it uses coheir Morley sequences rather than invariant Morley sequences. In [70] Alex Kruckman and the author show how to carry out this proof with the standard Conant-independence. We may also dualize Theorem 3.10 of [66].

Proposition 3.5.1. *In any theory T, Conant-forking coincides with Conant-dividing for formulas, and $\mathop{\smile\kern-0.9em\mid}^{K^*}$ has right extension.*

Proof. We see first of all that Conant-dividing is preserved under adding and removing unused parameters: it suffices to show that if $\models \forall x \varphi(x, a) \leftrightarrow \varphi'(x, ab)$ then $\varphi(x, a)$ Conant-divides over M if and only if $\varphi'(x, ab)$ Conant-divides over M. Let $\{a_i b_i\}_{i\in\omega}$ be a coheir Morley sequence starting with ab witnessing the failure of Conant-dividing of the latter; then $\{a_i\}_{i\in\omega}$ witnesses the failure of Conant-dividing of the former. Conversely, let $\{a_i\}_{i\in\omega}$ be a coheir Morley sequence starting with a witnessing the failure of Conant-dividing of $\varphi(x, a)$; then by Claim 3.3.3 and an automorphism there are $\{b_i\}_{i\in\omega}$ so that $\{a_i b_i\}_{i\in\omega}$ is a coheir Morley sequence starting with ab, and this will witness the failure of Conant-dividing of $\varphi'(x, ab)$. The result is now standard, following, say, the proof in [52] of the analogous fact for Kim-dividing under Kim's lemma. Suppose $\varphi(x, b)$ Conant-forks over M but does not Conant-divide over M; by the above we can assume it implies a disjunction of the form $\bigvee_{i=1}^{n} \varphi_i(x, b)$ where $\varphi_i(x, b)$ Conant-divides over M. Let $\{b_i\}_{i\in\omega}$ be a coheir Morley sequence starting with b witnessing the failure of Conant-dividing, so there is some a realizing $\{\varphi(x, b_i)\}_{i\in\omega}$. Then by the pigeonhole principle, there is some $1 \leq k \leq n$ so that a realizes infinitely many of the $\varphi_k(x, b_i)$. By an automorphism this contradicts Conant-dividing of $\varphi_k(x, b)$.

Right extension is standard and exactly as in Lemma 3.3.6: if $a \mathop{\smile\kern-0.9em\mid}_M^{K^*} b$ but there is no $a' \equiv_{Mb} a$ with $a' \mathop{\smile\kern-0.9em\mid}_M^{K^*} bc$, then $\mathrm{tp}(a/Mb)$ must imply a disjunction of formulas with parameters in Mbc Conant-forking over M; some formula in $\mathrm{tp}(a/Mb)$ must then imply this disjunction, which will then Conant-fork over M, contradicting $a \mathop{\smile\kern-0.9em\mid}_M^{K^*} b$. □

The following is immediate from Theorem 4.3:

Corollary 3.5.1.1. *Let T be* NSOP$_2$. *Then a formula Conant-divides (so Conant-forks) over M if and only if it Kim-divides with respect to some (any) canonical Morley sequence.*

We develop the theory of Conant-independence in NSOP$_2$ theories in analogy with the theory of Kim-independence in NSOP$_1$ theories.

Proposition 3.5.2. *(Canonical Chain Condition): Let T be* NSOP$_2$ *and suppose $a \mathop{\smile\kern-0.9em\mid}_M^{K^*} b$. Then for any canonical Morley sequence I starting with b, we can find some $I' \equiv_{Mb} I$ indiscernible over a; any such I' will satisfy $a \mathop{\smile\kern-0.9em\mid}_M^{K^*} I'$.*

Proof. This is similar to the proof of, say, the analogous fact about Kim-independence in NSOP$_1$ theories (Proposition 3.21 of [52]). The existence of such an I' follows from the previous corollary by Ramsey and compactness. To get $a \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} I'$, let $I' = \{b_i\}_{i\in\omega}$; it suffices to show $a \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} b_0 \dots b_{n-1}$ for any n. But $\{b_{in}b_{in+1}\dots b_{in+(n-1)}\}_{i\in\omega}$ is a coheir Morley sequence over M starting with $b_0 \dots b_{n-1}$, each term of which satisfies $\{b_0 \dots b_{n-1}/Ma\}$, so $a \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} b_0 \dots b_{n-1}$ follows. □

Theorem 3.5.3. *Let T be* NSOP$_2$. *Then Conant-independence is symmetric.*

Proof. Suppose otherwise, so for some $a, b \in \mathbb{M}$, $a \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} b$ but b is Conant-dependent on a over M. We use $a \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} b$ to build trees as in the proof of symmetry of Kim-independence for NSOP$_1$ theories (the construction is Lemma 5.11 of [52].) Specifically, what we want is, for any n, a tree $(I_n, J_n) = (\{a_\eta\}_{\eta\in\omega^{\leq n}}, \{b_\sigma\}_{\sigma\in\omega^n})$, infinitely branching at the first $n+1$ levels and then with each a_σ for $\sigma \in \omega^n$ at level $n+1$ followed by a single additional leaf b_σ at level $n+2$, satisfying the following properties:

(1) For $\eta \trianglelefteq \sigma$, $a_\eta b_\sigma \equiv_M ab$

(2) For $\eta \in \omega^{<n}$, the subtrees above η form a canonical coheir sequence indiscernible over a_η, so by Proposition 3.5.2, a_η is Conant-independent over M from those branches taken together.

Suppose (I_n, J_n) already constructed; we construct (I_{n+1}, J_{n+1}). We see that the root $a_\emptyset$ of (I_n, J_n) is Conant-independent from the rest of the tree, $(I_n, J_n)^*$: for $n = 0$ this is just the assumption $a \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} b$, where we allow $a_\emptyset b_\emptyset = ab$, while for $n > 0$ this is (2). So by extension we find $a'_\emptyset \equiv_{M(I_nJ_n)^*} a_\emptyset$ (so guaranteeing (1)), to be the root of (I_{n+1}, J_{n+1}), with $a \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} I_nJ_n$. Then by Proposition 3.5.2, find some canonical Morley sequence $\{(I_n, J_n)^i\}_{i\in\omega}$ starting with (I_n, J_n) indiscernible over $Ma'_\emptyset$, guaranteeing (2), and reindex accordingly.

Now let $\varphi(x, a) \in \mathrm{tp}(b/Ma)$ (so $\varphi(x, y)$ is assumed to have parameters in M) witness the Conant-dependence of b on a over M and let k be the (strict) bound supplied by Theorem 4.3. We show I_n gives the first $n+1$ levels of an instance of k-DCTP$_1$ for $\varphi(x, y)$, giving a contradiction to NSOP$_2$ by compactness and lemma 3.2.3. Consistency of the paths comes from (1). As for the inconsistency of a descending comb of size k, it follows from (2) (and the same reasoning as in the proof of Lemma 3.4.3) that a descending comb forms a coheir sequence, so the inconsistency follows by choice of k. □

Note that by constructing a tree of size κ and using an Erdős-Rado version of fact 3.2.2 (see Lemma 5.10 of [52] for a result of this kind for similar kind of indiscernible tree, itself based on Theorem 1.13 of [45]), we could have assumed the tree we constructed in the above proof to be *strongly indiscernible*. It follows that we could have only used that if a canonical Morley sequence witnesses Kim-dividing of a formula, then so does any coheir Morley sequence; the statement of Theorem 4.3 is somewhat stronger. (In fact, by using a local version of the chain condition–if $a \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} b$ and $\models \varphi(a, b)$, then there is some coheir Morley sequence $I = \{b_i\}_{i\in\omega}$ so that $b_i \equiv_M b$, $\models \varphi(a, b_i)$ for $i \in \omega$, and $a \mathop{\smile\!\!\!\!\!\!|}_M^{K^*} I$–we could have avoided Theorem 4.3 altogether up to this point, but we have not yet found a suitable replacement for the below "weak independence theorem" that does not require it. We leave the details to the reader.)

We next aim to prove a version of the "weak independence theorem." To formulate this, we need the following strengthening of Lemma 3.4.1:

Lemma 3.5.4. *Let $p(x)$ be a type over M. Then there is some global extension $q(x)$ of $p(x)$ so that, for all tuples $b \in \mathbb{M}$ if $c \in \mathbb{M}$ with $c \models q(x)|_{Mb}$, then for any $a \in \mathbb{M}$ there is $a' \equiv_{Mc} a$ with $a' \in \mathbb{M}$ so that $\mathrm{tp}(a'c/Mb)$ extends to a canonical coheir of $\mathrm{tp}(a'c/M) = \mathrm{tp}(ac/M)$. So in particular, $q(x)$ is a canonical coheir of $p(x)$.*

Proof. Working again in a very large $\mathbb{M}' \succ \mathbb{M}$, find $\mathbb{M}_1 \equiv_M \mathbb{M}$ with $\mathbb{M} \perp^{\mathrm{CK}}_M \mathbb{M}_1$ using full existence for $\perp^{\mathrm{CK}}$. Find a realization c'' of $p(x)$ in $\mathbb{M}_1$ and let $q(x)$ be its type over $\mathbb{M}$. Now suppose $b \in \mathbb{M}$ and $c \in \mathbb{M}$ with $c \models q(x)|_{Mb}$, and let $a \in \mathbb{M}$. Then there is some $a'' \in \mathbb{M}_1$ with $a''c'' \equiv_M ac$. Because $c'' \equiv_{Mb} c$, there is some $a' \in \mathbb{M}$ with $a''c'' \equiv_{Mb} a'c$. Together with $a''c'' \equiv_M ac$, it follows that $a' \equiv_{Mc} a$. And $\mathrm{tp}(a'c/Mb)$ extends to $\mathrm{tp}(a''c''/\mathbb{M})$, which it remains to show is canonical. But by right monotonicity, $\mathbb{M} \perp^{\mathrm{CK}}_M a''c''$, so the result follows by left monotonicity (see also the proof of Lemma 3.4.1). □

Definition 3.5.2. *We call $q(x)$ as in Lemma 3.5.4 a* strong canonical coheir*, and a coheir Morley sequence in it a* strong canonical Morley sequence.

The proof of the following is as in Proposition 6.10 of [52]:

Proposition 3.5.5. *(Weak Independence Theorem) Assume T is NSOP_2. Let $a_1 \perp^{K^*}_M b_1$, $a_2 \perp^{K^*}_M b_2$, $a_1 \equiv_M a_2$, and $\mathrm{tp}(b_2/Mb_1)$ extends to a strong canonical coheir $q(x)$ of $\mathrm{tp}(b_2/M)$. Then there exists a realization a of $\mathrm{tp}(a_1/Mb_1) \cup \mathrm{tp}(a_2/Mb_2)$ with $a \perp^{K^*}_M b_1b_2$.*

Proof. We start with the following claim, proven exactly as in [52] but with canonical rather than invariant Morley sequences:

Claim 3.5.6. *There exists some b_2' with $a_1b_2' \equiv_M a_2b_2$ and $a_1 \perp^{K^*}_M b_1b_2'$.*

Proof. It is enough by symmetry of $\perp^{K^*}$ to find b_2' with $a_1b_2' \equiv_M a_2b_2$ and $b_1b_2' \perp^{K^*}_M a_1$. If $p(x, a_2) = \mathrm{tp}(b_2/Ma_2)$ (leaving implied, throughout the proof of this claim, any parameters in M in types and formulas), then by $a_2 \perp^{K^*}_M b_2$ and symmetry we have $b_2 \perp^{K^*}_M a_2$, so because $a_1 \equiv_M a_2$ we know that $p(x, a_1)$ contains no formulas Conant-forking over M. It suffices to show consistency of

$$p(x, a_1) \cup \{\neg\varphi(x, b_1, a_1) : \varphi(x, y, a_1) \text{ Conant-forks over } M\}$$

Otherwise, by compactness and equivalence of Conant-forking with Conant-dividing, we must have $p(x, a_1) \vdash \varphi(x, b_1, a_1)$ for some $\varphi(x, y, z)$ with $\varphi(x, y, a_1)$ Conant-dividing over M. By symmetry, $b_1 \perp^{K^*}_M a_1$. So Proposition 3.5.2 yields a canonical Morley sequence $\{a_1^i\}_{i\in\omega}$ starting with a_1 and indiscernible over Mb_1. So

$$\bigcup_{i=0}^{\omega} p(x, a_1^i) \vdash \{\varphi(x, b_1, a_1^i)\}_{i\in\omega}$$

But because $p(x, a_1)$ contains no formulas Conant-dividing over M and $\{a_1^i\}_{i\in\omega}$ is a canonical Morley sequence, $\bigcup_{i=0}^{\infty} p(x, a_1^i)$ is consistent, so $\{\varphi(x, b_1, a_1^i)\}_{i\in\omega}$ and therefore $\{\varphi(x, y, a_1^i)\}_{i\in\omega}$ is consistent. But this contradicts the fact that $\varphi(x, y, a_1)$ Conant-divides over M. □

We now complete the proof of the proposition. Let $p_2(x, b_2) = \mathrm{tp}(a_2/Mb_2)$ (with parameters in M left implied); we have to show that $\mathrm{tp}(a_1/Mb_1) \cup p_2(x, b_2)$ has a realization a with $a \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^*} b_1b_2$. So for $b_2'' \equiv_{Mb_1} b_2$ with $b_2'' \models q(x)|_{Mb_1b_2'}$, it suffices to show that $\mathrm{tp}(a_1/Mb_1) \cup p_2(x, b_2'')$ has a realization a with $a \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^*} b_1b_2''$. Using $b_2'' \equiv_M b_2 \equiv_M b_2'$, we find b_1' with $b_1'b_2'' \equiv_M b_1b_2'$; using Lemma 3.5.4, we can assume $\mathrm{tp}(b_1'b_2''/Mb_1b_2')$ extends to a canonical coheir of its restriction to M. So $b_1'b_2'', b_1b_2'$ begins a canonical Morley sequence I over M, and by Proposition 3.5.2 and an automorphism, there is some $a \equiv_{Mb_1b_2'} a_1$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^*} I$ and therefore $a \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^*} b_1b_2''$, and with I indiscernible over Ma. By $a \equiv_{Mb_1} a_1$ we have that a realizes $\mathrm{tp}(a_1/Mb_1)$, and by $ab_2'' \equiv_M ab_2' \equiv_M a_1b_2' \equiv_M a_2b_2$ we have that a realizes $p_2(x, b_2'')$. □

3.6 NSOP$_2$ and NSOP$_1$ theories

We are now ready to prove that if T is NSOP$_2$, it is NSOP$_1$. The proof follows Conant's proof (Theorem 7.17 of [32]) that certain *free amalgamation* theories are either simple or SOP$_3$. As anticipated in Section 5, $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}$ will play the role of $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$, while (strong) canonical Morley sequences will play the role of Morley sequences in the free amalgamation relation. This makes sense, as Lemma 7.6 of [32] shows that $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$ is just Kim-independence with respect to Morley sequences in the free amalgamation relation, while Conant-independence in a NSOP$_2$ theory is Kim-independence with respect to canonical Morley sequences. Similarly to how Conant uses free amalgamation and $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$ to show that a (modular) free amalgamation theory is either simple or SOP$_3$, we will show by strong canonical types and $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}$ that if T is NSOP$_2$, then

T is either NSOP$_1$ or SOP$_3$

and therefore must be NSOP$_1$. (In Chapter 1, we generalize Conant's work by studying abstract independence relations in potentially strictly NSOP$_1$ or SOP$_3$ theories, finding a more general set of axioms for these relations than Conant's free amalgamation axioms under which the NSOP$_1$-SOP$_3$ dichotomy holds and showing relationships with Conant-independence for *invariant* rather than coheir Morley sequences–note that in Conant's free amalgamation theories, this is just $\mathop{\smile\!\!\!\!\!\!\mid}^{a}$.)

We begin our proof.

Assume T is NSOP$_2$ and suppose T is SOP$_1$. Obviously Kim-dividing independence, $\mathop{\smile\!\!\!\!\!\!\mid}^{Kd}$, implies $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}$; the reverse implication would imply that $\mathop{\smile\!\!\!\!\!\!\mid}^{Kd}$ is symmetric, contradicting SOP$_1$ by Fact 3.2.1. So there are $a \mathop{\smile\!\!\!\!\!\!\mid}_M^{K^*} b$ with a Kim-dividing dependent on b over M; let $r(x, y) = \mathrm{tp}(a, b/M)$, and let $\{b_i\}_{i\in\mathbb{N}}$ be a coheir Morley sequence over M starting with b such that $\{r(x,b_i)\}_{i\in\omega}$ is k-inconsistent for some k. The following corresponds to Claim 1 of the proof of Theorem 7.17 in [32], but requires a different argument; see also [74] and footnote 1 of Chapter 4, for another argument involving the proof of Proposition 3.14 of [52]:

Claim 3.6.1. *We can assume $k=2$. More precisely, there are $\tilde{a},\tilde{b}\in\mathbb{M}$ with $\tilde{a}\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M\tilde{b}$ and some coheir Morley sequence $\{\tilde{b}_i\}_{i\in\mathbb{N}}$ over M starting with $\tilde{b}$ such that, for $\tilde{r}(\tilde{x},\tilde{y})=:$ $\mathrm{tp}(\tilde{a},\tilde{b}/M)$, $\{\tilde{r}(x,\tilde{b}_i)\}_{i\in\omega}$ is 2-inconsistent.*

Proof. In particular there is no realization a' of $\{r(x,b_i)\}_{i<k}$ with $a'\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b_0\dots b_{k-1}$. Let k^* be the maximal value of k without this property, and $\tilde{b}=b_0\dots b_{k^*-1}$. Then $\{\tilde{b}_i\}_{i\in\omega}=\{b_{ik^*}\dots b_{ik^*+k^*-1}\}_{i\in\mathbb{N}}$ is a coheir Morley sequence starting with $\overline{b}$. Let $a'\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M\overline{b}$ realize $\{r(x,b_i)\}_{i<k^*}$, and let $r'(x,y)=\mathrm{tp}(a',\tilde{b}/M)$. Then by maximality and symmetry, there is no realization a'' of $r'(x,\tilde{b}_0)\cup r'(x,\tilde{b}_1)$ with $\tilde{b}_0\tilde{b}_1\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M a''$. So there is no coheir Morley sequence $\{a'_i\}_{i\in\mathbb{N}}$ starting with a', every term of which realizes $r'(x,\tilde{b}_0)\cup r'(x,\tilde{b}_1)$. But by $a'\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M\tilde{b}$, symmetry and Proposition 3.5.2, there is some $M\tilde{b}$-indiscernible canonical Morley sequence I starting with a so that $I\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M\tilde{b}$. So let $\tilde{a}$ be I and $\tilde{b}$ with $\overline{b}$. Since $\tilde{r}(\tilde{x},\tilde{b})=\mathrm{tp}(I/M\tilde{b})$ contains $\cup_{i=1}^n r'(x_i,\tilde{b})$, $\tilde{a}$ and $\tilde{b}$ are as desired. □

Now replace a with $\tilde{a}$ and b with $\tilde{b}$, as in claim 3.6.1; let $\rho(x,y)\in r(x,y)=\mathrm{tp}(a,b/M)$ be such that $\{r(x,b_i)\}_{i\in\omega}$ is 2-inconsistent, by compactness. We have $b_1\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b_0$, in analogy with Claim 2 of the proof of Theorem 7.17 of [32], because $b_1\mathop{\smile\!\!\!\!\!\!\mid}^{u}_M b_0$ and clearly $\mathop{\smile\!\!\!\!\!\!\mid}^{u}$ implies $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}$.

Fix a strong canonical coheir extension $q(x)$ of $p(x)=\mathrm{tp}(b/M)$. We wish to construct, by induction, a configuration $\{b^1_ib^2_i\}_{i\in\omega}$ with the following properties:

(1) For J_n the sequence beginning with b^2_i for $i<n$ and then continuing with b^1_i for $i\geq n$, J_n is a strong canonical Morley sequence in $q(x)$.

(2) For $i\leq j$, $b^1_ib^2_j\equiv_M b_0b_1$

(3) $b^0_1\dots b^1_n\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b^2_0\dots b^2_n$ for any $n\in\omega$.

Then by $a\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b$ (1) gives consistent sequences of instances of $r(x,y)$, while (2) gives inconsistent pairs by claim 3.6.1, so we can get an instance of SOP$_3$ from this configuration exactly as in the argument at the end of the proof of Theorem 7.17 in [32], which we will reproduce for the convenience of the reader.

We make repeated use of symmetry for $\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}$ throughout. Suppose $\{b^1_ib^2_i\}_{i\leq n}$ already constructed. We start by adding b^1_{n+1}, and then add b^2_{n+1}. If we take $b^1_{n+1}\models q(x)|_{Mb^1_0b^2_0\dots b^1_nb^2_n}$ then (1) and (2) are preserved up to this point, and (3) is preserved by the following claim (which also holds of Kim-independence in NSOP$_1$ theories):

Claim 3.6.2. *If $a\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b$ and $\mathrm{tp}(c/Mab)$ extends to an M-invariant type $q(x)$, then $ac\mathop{\smile\!\!\!\!\!\!\mid}^{K^*}_M b$.*

Proof. By Proposition 3.5.2, let $I=\{b_i\}_{i<\omega}$ be an Ma-indiscernible canonical Morley sequence over M starting with b. Choose $c^*\models q|_{MIa}$, so for $i<\omega$, $b_ia\equiv_{Mc^*}b_0a=ba$. Since

$I = \{b_i\}_{i<\omega}$ form a coheir Morley sequence with $b_i \equiv_{Mac^*} b$ for $i < \omega$, $ac^* \mathop{\smile\!\!\!\!\!\!\vert}_M^{K^*} b$ by 3.5.1, so $ac \mathop{\smile\!\!\!\!\!\!\vert}^{K^*} b$ as $c^* \models \mathrm{tp}(c/ab)$.

□

Now by $b_1 \mathop{\smile\!\!\!\!\!\!\vert}_M^{K^*} b_0$ and the fact that J_0 is still a (strong) canonical Morley sequence up to this point, we can find a realization $b_* \mathop{\smile\!\!\!\!\!\!\vert}_M^{K^*} b_0^1 \dots b_{n+1}^1$ of $\{t(b_i^1, y)\}_{i=1}^{n+1}$ for $t(x, y) = \mathrm{tp}(b_0 b_1/M)$ by Proposition 3.5.2 and an automorphism. Take $b^* \models q(x)|_{Mb_0^2 \dots b_n^2}$, so $b^* \equiv_M b_*$; then this together with (3) allows us to apply Proposition 3.5.5 to the conjugate q_1 of $\mathrm{tp}(b_0^1 \dots b_{n+1}^1/Mb_*)$ under an automorphism taking b_* to b^*, and $q_2 = \mathrm{tp}(b_0^1 \dots b_{n+1}^1/Mb_0^2 \dots b_n^2)$. This and an automorphism (over $b_0^2 \dots b_n^2$, taking the Conant-independent joint realization of q_1 and q_2 to $b_0^1 \dots b_{n+1}^1$) gives us our desired b_{n+1}^2 (as the image of b^* under this automorphism.)

Now having constructed the configuration, let a_n realize the consistent set of instances of $r(x, y)$ coming from J_n, and let $d_i = (b_i^1, b_i^2)$, $z = (y^1, y^2)$, $\phi(x, y) = \rho(x, y_1)$, $\psi(x, z) = \rho(x, y_2)$. As in the proof of Theorem 7.17 of [32], these satisfy the hypotheses of the following fact:

Fact 3.6.3. *(Corrected version of proposition 7.2, [32][3])*
Suppose there are sequences $\{a_i\}_{i<\omega}$, $\{d_i\}_{i<\omega}$, and $\phi(x, y)$, $\psi(x, y)$ so that
(i) $\models \varphi(a_i, d_j)$ for all $i < j$ and $\psi(a_i, d_j)$ for all $i \geq j$
(ii) for all $i < j$, $\varphi(x, b_i) \cup \psi(x, b_j)$ is inconsistent
Then T is SOP$_3$.

So T is SOP$_3$.

This concludes the proof of the main result of this chapter.

[3]Gabriel Conant, in a personal communication with the author ([31]), noted this correction to Proposition 7.2 of [32], and plans to publicize this in a future corrigendum. See also Observation 6.15 of [78] for an earlier version of this fact, which can also be used here.

Part III

Chapter 4

Properties of independence in NSOP_3 theories

4.1 Introduction

A central program in pure model theory is to develop the theory of independence, which originated within the stable theories, beyond stability and simplicity. This has been successful for the original notion of *forking-independence* within NTP_2 *theories*: for example, Chernikov and Kaplan, in [27], show that forking coincides with dividing in NTP_2 theories; Ben-Yaacov and Chernikov, in [113], give an independence theorem for forking-independence in NTP_2 theories that is improved by Simon in [104], and Chernikov, in [26], studies simple types in NTP_2 theories and gives a characterization of NTP_2 theories in terms of Kim's lemma. In a different direction, Kaplan and Ramsey in [52] extend the original theory of independence in simple theories to NSOP_1 *theories* by introducing the notion of *Kim-independence*, described as forking-independence "at a generic scale." Kaplan and Ramsey, in [52], show, using work of Chernikov and Ramsey in [28], that symmetry of Kim-independence characterizes the property NSOP_1; they also show that the independence theorem for Kim-independence characterizes NSOP_1. To give examples of further consequences of NSOP_1 for the theory of Kim-independence, Kaplan and Ramsey in [53] give a characterization of NSOP_1 in terms of transitivity, Kaplan, Ramsey and Shelah in [54] give a characterization in terms of local character; Dobrowolski, Kim and Ramsey in [39] and Chernikov, Kim and Ramsey in [6] study independence over arbitrary sets in NSOP_1 theories. Kruckman and Ramsey, in [71], prove an improved independence theorem, developed further by Kruckman, Tran and Walsberg in the appendix of [72]. Kim ([63]) initiates a theory of canonical bases. For extensions to positive logic, see [38], [51], [5]; see also [17] for extensions of Kim-independence to NTP_2 theories. Beyond NSOP_1 and NSOP_2, the author in Chapter 3 develops a theory of independence in NSOP_2 theories and uses this to show that every NSOP_2 theory is in fact NSOP_1, and Kim and Lee, in [66], use remarks by the author in Chapter 3 to develop Kim-forking and Kim-dividing in the NATP *theories* introduced by Ahn and Kim in [3] and further devloped by Ahn, Kim and Lee in [4], as well as the related N-k-DCTP$_2$ theories introduced by the author in Chapter 3.

However, much remains to be understood about the theory of independence in Shelah's strong order hierarchy, NSOP$_n$, for $n \geq 3$. In Chapter 1, the author relativizes the theory of Kim-indpendence in [28], [52] by developing a theory of independence relative to abstract independence relations generalizing the free amalgamation axioms of [32]; though the theories to which this result applies may be strictly NSOP$_4$ (NSOP$_4$ and SOP$_3$) as well as NSOP$_1$, NSOP$_4$ is not actually used in the result. The author also observes in the same chapter using the generalization in Chapter 3 of the arguments of [32] that theories possessing independence properties with no known NSOP$_4$ counterxamples–symmetric Conant-independence and the strong witnessing property that generalizes Kim's lemma–cannot be strictly NSOP$_3$. Conant-independence, which can be described as forking-independence at a *maximally* generic scale and is grounded in the *strong Kim-dividing* of [54], is introduced in that chapter (based on a similar notion with the same name developed in Chapter 3 to show the equivalence of NSOP$_1$ and NSOP$_2$) as a potential extension of the theory of Kim-independence beyond NSOP$_1$. There the author shows that a theory where Conant-independence is symmetric must be NSOP$_4$, and characterizes Conant-independence in most of the known examples of NSOP$_4$ theories, where it is symmetric. This leaves open the question of whether Conant-independence is symmetric in any NSOP$_4$ theory, a question intimately related to the question of whether any NSOP$_3$ theory is NSOP$_2$. In [55], Kaplan, Ramsey and Simon have recently shown that all binary NSOP$_3$ theories are simple, by developing a theory of independence for a class of theories containing all binary theories. In Chapter 5 the author develops the independence relations $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$, based on the same idea of forking-independence at a maximally generic scale, shows that any theory where $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$ is symmetric must be NSOP$_{2^{n+1}+1}$, and characterizes $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$ in the classical examples of NSOP$_{2^{n+1}+1}$ theories, leaving open the question of whether $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$ is symmetric in any NSOP$_{2^{n+1}+1}$ theory. (Demonstrating robustness of the result, the author proves a similar result for left and right transitivity.) In [81], Malliaris and Shelah initiate a structure theory for NSOP$_3$ theories, though instead of a theory of independence along the lines of forking-independence or Kim-independence, they show symmetric inconsistency for higher formulas, a result on sequences of realizations of two invariant types yielding inconsistent instances of two formulas, rather than any kind of indiscernible sequence witnessing the dividing of a single formula. Malliaris, in [78], also investigates the *graph-theoretic* depth of independence in NSOP$_3$ theories. The pressing question remains, for $n \geq 3$: using the assumption that T is NSOP$_n$ (and possibly some additional assumptions that are not already known to collapse NSOP$_n$ into NSOP$_1$), can we show any properties of T that fit into the program of generalizing the properties of independence in stable or simple theories, as was done for NSOP$_1$ and NSOP$_2$ theories?

The aim of this chapter is to show that this question is tractable for NSOP$_3$ theories, whose equivalence with NSOP$_1$ remains open. We prove three results on NSOP$_3$ theories, two about the NSOP$_1$ "building blocks" of NSOP$_3$ theories and the independence relations between them in the global NSOP$_3$ structure, and one about NSOP$_3$ theories with symmetric

Conant-independence. All three of these results truly use NSOP_3 in that they fail when the assumption is relaxed to NSOP_4 (and the first two results, though both concerning the NSOP_1 local structure, involve separate uses of NSOP_3 in a sense that will become apparent.) The first and third result will also appear similar to properties known or proposed for NTP_2 theories, in contrast to the open question of whether $\mathrm{NTP}_2 \cap \mathrm{NSOP}_n$ coincides with simplicity for $n \geq 3$, which would suggest that NSOP_n is much different from NTP_2.

We give an outline of the chapter.

In Section 3 we generalize work of Chernikov ([26]) on *simple types* in NTP_2 theories. As the property N-ω-DCTP_2 is a subclass of NATP which is one potential solution X to [69]'s proposed analogy "simple : NTP_2 :: NSOP_1 : X," (Chapter 3, [66]), it is to be expected that the analogous result for "NSOP_1 types" holds for N-ω-DCTP_2 theories. What is not predicted by this analogy is that the same result on NSOP_1 local structure holds in NSOP_3 theories. Instead of generalizing the definition of simple types, we introduce a definition schema for the *internal* properties of a (partial) type, which is more natural in that it refers to the global properties of a structure associated with that type. (We could also have generalized the defintion of simple types to NSOP_1 and gotten the same conclusion; see Remark 4.3.6.) We show that just as Chernikov implicitly showed in ([26]) for internally simple types in NTP_2 theories, the assumption of NSOP_3 controls how internally NSOP_1 types relate to the rest of the structure:

Theorem 4.1.1. *Let T be* NSOP_3*, and $p(x)$ an internally* NSOP_1 *type. Then $p(x)$ is co-*NSOP_1.

See Definitions 4.3.1 and 4.3.3. When T is only assumed to be NSOP_4, we give an internally simple type $p(x)$ for which this fails.

We then interpret the proof of this result as well the results of Chernikov in [26] (and their direct generalization to N-ω-DCTP_2) in terms of the *characteristic sequences* introduced by Malliaris in [77] to relate "classification-theoretic properties" of a theory to the "graph-theoretic properties" of hypergraphs, and used by Malliaris in [76] to study Keisler's order. Internally to a type $p(x)$, what the ambient theory perceives to be an instance of co-NSOP_1 (an instance of NSOP_1 with parameters realizing $p(x)$) is simply a definable hypergraph making no reference to consistency. Model-theoretic properties of a theory will give control of the graph-theoretic structure of hypergraphs definable in that theory, similarly to Shelah's classic result that an definable bipartite graph with the order property in an NSOP theory must even have the independence property. Applied in the case where the model-theoretic properties, such as simplicity and NSOP_1, are assumed of the internal structure on $p(x)$, this will illuminate the proof in [26] of co-simplicity in NTP_2 theories and our proof of co-NSOP_1 in N-ω-DCTP_2 and NSOP_3 theories.

In Section 4, we discuss how internally NSOP_1 types interrelate within the ambient structure of a NSOP_3 theory, showing that their behavior is similar to how they would interrelate in a globally NSOP_1 theory. By the Kim-Pillay characterization of NSOP_1, Theorem 9.1 of [52], for no reasonable notion of independence could a full independence theorem hold in an

SOP_1 (that is, non-NSOP_1) theory. However, we prove an independence theorem between internally NSOP_1 types in NSOP_3 theories:

Theorem 4.1.2. *Let T be NSOP_3, and let p_1, p_2, p_3 be internally NSOP_1 types over M. Let $a_1 \equiv_M a_1' \subset p_1(\mathbb{M})$, $a_2 \subset p_2(\mathbb{M})$, $a_3 \subset p_3(\mathbb{M})$. If $a_1 \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K^*}_M a_2$, $a_1' \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K^*}_M a_3$, $a_2 \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K^*}_M a_3$, there is some a_1'' with $a_1'' \models \mathrm{tp}(a_1/Ma_2) \cup \mathrm{tp}(a_1'/Ma_3)$. Moreover, a_1'' can be chosen with $a_2a_3 \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K^*}_M a_1''$, $a_2a_1'' \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K^*}_M a_3$ and $a_3a_1'' \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K^*}_M a_2$.*

Here $\mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K^*}$ is Conant-independence, Definition 4.2.3. Motivating this result, in an NSOP_1 theory, Conant-independence coincides with Kim-independence, $\mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K}$ and is symmetric; compare [52], Theorem 6.5, which characterizes NSOP_1. (Between tuples of realizations of two co-NSOP_1 types p_i, p_j it coincides with Kim-diving independence.) While in proving this result, we apply 4.1.1, it does not just follow from co-NSOP_1: we exhibit internally stable types p_1, p_2, p_3 in an NSOP_4 theory T for which this fails. This independence theorem for internally NSOP_1 types in NSOP_3 theories is not only of interest to the program of extending the theory of independence beyond NSOP_1 theories; it is also of interest to the question of whether NSOP_3 coincides with $\mathrm{NSOP}_2 = \mathrm{NSOP}_1$. One potential approach to building a strictly NSOP_3 theory (that is, one that is SOP_2) is by starting with NSOP_2 structures and somehow combining them to obtain a failure of NSOP_1 in the form of a failure of the independence theorem: this result says that it is impossible to obtain an NSOP_3 theory from such a construction. It may be of interest to ask whether there is any connection between this result on stability-theoretic independence and Theorem 7.7 of [78], which concerns the graph-theoretic depth of independence in NSOP_3 theories.

In section 5, we consider NSOP_3 theories where Conant-independence is symmetric. It is natural to assume this, as there is no known NSOP_4 theory where Conant-independence or Conant-dividing independence is not symmetric. Simon, in [104], proves an improved independence theorem for NTP_2 theories, Fact 4.5.1, and poses an existence question, Question 4.5.2, for invariant types with the same Morley sequence in NTP_2 theories; an independence theorem for forking-independence, for invariant types with the same Morley sequence in NTP_2 theories, would follow from a positive answer to this question, by Simon's result. In an NSOP_3 theory with symmetric Conant-independence, we prove a similar independence theorem for Conant-independence between *finitely satisfiable* types with the same Morley sequence:

Theorem 4.1.3. *Let T be an NSOP_3 theory, and assume $\mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K^*}$ is symmetric. Suppose p and q are M-finitely satisfiable (global) types with $p^\omega|_M = q^\omega|_M$, and let $a, b \supseteq M$ be small supersets of M with $a \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K}_M b$. Then there is $c \models p(x)|_a \cup q(x)|_b$ with $c \mathop{\smile\!\!\!\!\!\!\;|}\nolimits^{K^*}_M ab$.*

This fails when T is the model companion of triangle-free graphs, which is NSOP_4 with symmetric (indeed trivial) Conant-independence. We also give an extension of this result from finitely satisfiable types to Kim-nonforking types when Conant-dividing independence is symmetric, which has the advantage of exploiting the full force of symmetry for Conant-independence. While this result is again of interest to the question of extending the theory of independence beyond $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$, since there is precedent (see Chapter 3) for using facts about independence to prove the equivalence of classification-theoretic dividing lines, it is also of interest to another open question, whether an NSOP_3 theory with symmetric Conant-independence is NSOP_1.

One final remark: in Theorem 1.3.15, a strategy was suggested for proving the equivalence of NSOP_3 and NSOP_2 by proving two facts that have no known NSOP_4 counterexamples, symmetry for Conant-independence and the strong witnessing property, for all NSOP_4 or even all NSOP_3 theories. The results of this chapter suggest a different approach, via finding properties of independence in NSOP_3 theories that distinguish them from NSOP_4 theories.

4.2 Preliminaries

Notations are standard. We will need some basic defintions and facts about some standard relations between sets, as well as some facts about NSOP_1 and NSOP_3 theories.

Relations between sets

Adler, in [1], defines some properties of abstract ternary relations $A \mathop{\smile\hskip-0.9em^{|}\ }_M B$ between sets. In our case, we will assume M is a model, and we will only need to refer to a few of these properties by name:

Left extension: If $A \mathop{\smile\hskip-0.9em^{|}\ }_M B$ and $A \subseteq C$, there is some $B' \equiv_A B$ with $C \mathop{\smile\hskip-0.9em^{|}\ }_M B'$.

Right extension: If $A \mathop{\smile\hskip-0.9em^{|}\ }_M B$ and $B \subseteq C$, there is some $A' \equiv_B A$ with $A' \mathop{\smile\hskip-0.9em^{|}\ }_M C$.

Symmetry: If $A \mathop{\smile\hskip-0.9em^{|}\ }_M B$ then $B \mathop{\smile\hskip-0.9em^{|}\ }_M A$.

Chain condition with respect to invariant Morley sequences: If $A \mathop{\smile\hskip-0.9em^{|}\ }_M B$ and $I = \{B_i\}_{i<\omega}$ is an invariant Morley sequence over M (see below) with $B_0 = B$, then there is $I' \equiv_{MB} I$ indiscernible over MA with $A \mathop{\smile\hskip-0.9em^{|}\ }_M I'$.

We will refer to various relations between sets. For the convenience of the reader, here is an index of the notation to be used. Kim-independence and Kim-dividing, as well as Conant-independence and Conant-dividing, will be defined later in this section.

$a \mathop{\smile\hskip-0.9em^{|}\ }^{i}_{M} b$ if $\mathrm{tp}(a/Mb)$ extends to a global M-invariant type

$a \mathop{\smile\hskip-0.9em^{|}\ }^{u}_{M} b$ if $\mathrm{tp}(a/Mb)$ extends to a global M-finitely satisfiable type

$a \mathop{\smile\hskip-0.9em^{|}\ }^{f}_{M} b$ if a is forking-independent from b over M

$a \mathop{\smile\hskip-0.9em^{|}\ }^{K}_{M} b$ if a is Kim-independent from b over M

$a \mathop{\smile\hskip-0.9em^{|}\ }^{K^*}_{M} b$ if a is Conant-independent from b over M

$a \mathop{\smile\hskip-0.9em^{|}\ }^{Kd}_{M} b$ if $\mathrm{tp}(a/Mb)$ contains no formulas Kim-dividing over M

$a \mathop{\smile\hskip-0.9em^{|}\ }^{K^*d}_{M} b$ if $\mathrm{tp}(a/Mb)$ contains no formulas Conant-dividing over M.

We will use $\mathop{\smile\hskip-0.9em^{|}\ }^{K^+}$ and $\mathop{\smile\hskip-0.9em^{|}\ }^{K^+u}$ as ad-hoc notations in proofs; these will be defined in the course of those proofs.

We give an overview of some basic definitions. A *global type* $p(x)$ is a complete type over the sufficiently saturated model $\mathbb{M}$. For $M \prec \mathbb{M}$, a global type $p(x)$ is *invariant* over M if $\varphi(x, b) \in p(x)$ and $b' \equiv_M b$ implies $\varphi(x, b') \in p(x)$. One class of types invariant over M is the class of types that are *finitely satisfiable* over M, meaning any formula in the type is satisfied by some element of M. We say an infinite sequence $\{b_i\}_{i\in I}$, is an *invariant Morley sequence* over M (in the type $p(x)$) if there is a fixed global type $p(x)$ invariant over M so that $b_i \models p(x)|_{M\{b_j\}_{j<i}}$ for $i \in I$. If $p(x)$ is finitely satisfiable over M, we say $\{b_i\}_{i\in I}$ is a *coheir Morley sequence* or *finitely satisfiable Morley sequence* over M. Invariant Morley sequences over M are indiscernible over M, and the EM-type of an invariant Morley sequence over M depends only on $p(x)$. For $p(x), q(y)$ M-invariant types, $p(x) \otimes q(y)$ is defined so that $ab \models p(x) \otimes q(y)|_A$ for $M \subseteq A$ when $b \models q(y)|_A$ and $a \models p(x)|_{Ab}$.

Both $\mathop{\smile\!\!\!\!\!\mid}^i$ and $\mathop{\smile\!\!\!\!\!\mid}^u$ have right extension, but it is sometimes advantageous to work with coheir Morley sequences rather than general invariant Morley sequences because $\mathop{\smile\!\!\!\!\!\mid}^u$ is also known to have left extensions.

Kim-indepedence and NSOP$_1$

We assume knowledge of basic simplicity theory and the definition of forking-independence. An extension of the theory of independence from simple theories to NSOP$_1$ *theories* was developed by Kaplan and Ramsey in [52], via the definition of *Kim-independence*

Definition 4.2.1. *A theory T is* NSOP$_1$ *if there does not exist a formula $\varphi(x, y)$ and tuples $\{b_\eta\}_{\eta\in 2^{<\omega}}$ so that $\{\varphi(x, b_{\sigma\restriction n})\}_{n\in\omega}$ is consistent for any $\sigma \in 2^\omega$, but for any $\eta_2 \trianglerighteq \eta_1 \frown \langle 0\rangle$, $\{\varphi(x, b_{\eta_2}), \varphi(x, b_{\eta_1\frown\langle 1\rangle})\}$ is inconsistent. Otherwise it is* SOP$_1$.

Definition 4.2.2. *([52]) A formula $\varphi(x, b)$* Kim-divides *over M if there is an invariant Morley sequence $\{b_i\}_{i\in\omega}$ starting with b (said to* witness *the Kim-dividing) so that $\{\varphi(x, b_i)\}_{i\in\omega}$ is inconsistent. A formula $\varphi(x, b)$* Kim-forks *over M if it implies a (finite) disjunction of formulas Kim-dividing over M. We write $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$, and say that a is* Kim-independent *from b over M if $\mathrm{tp}(a/Mb)$ does not include any formulas Kim-forking over M.*

Kim-independence in NSOP$_1$ theories behaves, in many ways, like forking-independence in simple theories.

Fact 4.2.1. *([52]) Let T be* NSOP$_1$. *Then for any formula $\varphi(x, b)$ Kim-dividing over M, any invariant Morley sequence over M starting with b witnesses Kim-dividing of $\varphi(x, b)$ over M. Conversely, suppose that for any formula $\varphi(x, b)$ Kim-dividing over M, any invariant Morley sequence (even in a finitely satisfiable type) over M starting with b witnesses Kim-dividing of b over M. Then T is* NSOP$_1$.

It follows that Kim-forking coincides with Kim-dividing in any NSOP$_1$ *theory.*

Fact 4.2.2. *([28], [52]) The theory T is* NSOP$_1$ *if and only if $\mathop{\smile\!\!\!\!\!\mid}^K$ is symmetric.*

The *independence theorem* for Kim-independence in NSOP$_1$ theories generalizes that of [65] for simple theories, which in turn generalizes stationarity of forking-independence (the uniqueness of nonforking-extensions) in stable theories. Part of our argument for the results of section 4 will require re-proving the independence theorem in the context of co-NSOP$_1$ types. For motivation, we give the original statement:

Fact 4.2.3. *(*Independence theorem, [52].*) Let T be* NSOP_1*. Then if $a_1 \mathop{\smile\!\!\!\!\!\!|}^K_M b_1$, $a_2 \mathop{\smile\!\!\!\!\!\!|}^K_M b_2$, $b_1 \mathop{\smile\!\!\!\!\!\!|}^K_M b_2$, and $a_1 \equiv_M a_2$, there is some $a \mathop{\smile\!\!\!\!\!\!|}^K_M b_1 b_2$ with $a \equiv_{Mb_i} a_i$ for $i = 1, 2$.*

Conant-independence

Conant-independence was introduced in a modified form in Chapter 3 to show that NSOP_2 theories were NSOP_1. The standard version was defined in Chapter 1, based on Conant's implcit use of the concept in [32] to classify modular free amalgamation theories. It was proposed by the author of this chapter as an extension of Kim-independence beyond NSOP_1 theories.

Definition 4.2.3. *Let M be a model and $\varphi(x, b)$ a formula. We say $\varphi(x, b)$* Conant-divides *over M if for* every *invariant Morley sequence $\{b_i\}_{i\in\omega}$ over M starting with b, $\{\varphi(x, b)\}_{i\in\omega}$ is inconsistent. We say $\varphi(x, b)$* Conant-forks *over M if and only if it implies a disjunction of formulas Conant-dividing over M. We say a is* Conant-independent *from b over M, written $a \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M b$, if $\mathrm{tp}(a/Mb)$ does not contain any formulas Conant-forking over M.*

In Chapter 1 it is shown that if Conant-independence is symmetric in a theory T, T is NSOP_4. In the same chapter, Conant-indepedence is characterized for most of the known examples of NSOP_4 theories, where it is shown to be symmetric. It is open whether Conant-independence is symmetric in all NSOP_4 theories, or even all NSOP_3 theories. It is also open whether theories with symmetric Conant-independence display the classification-theoretic behavior characteristic of theories with a good notion of free amalgamation, first studied in [32] and later improved upon in Chapter 1: either NSOP_1 or SOP_3, and either simple or TP_2.

Classification theory

In this chapter, we will be interested in NSOP_3 theories and how they differ from NSOP_4 theories:

Definition 4.2.4. *Let $n \geq 3$. A theory T is* NSOP_n *(that is, does not have the* n-strong order property*) if there is no definable relation $R(x_1, x_2)$ with no n-cycles, but with tuples $\{a_i\}_{i\in\omega}$ with $\models R(a_i, a_j)$ for $i < j$. Otherwise it is* SOP_n*.*

We will need nothing about NSOP_4, other than that the below counterexamples to our results on NSOP_3 theories are NSOP_4, because they are free amalgamation theories; see [32], Theorem 4.4. We will need the following syntactic fact about NSOP_3, proven independently by Malliaris (Conclusion 6.15, [78]) and Conant ([32], Proposition 7.2 and proof of Theorem 7.17):

Fact 4.2.4. *Suppose there is an array $\{a_i, b_i\}_{i<\omega}$ and formulas $\varphi(x, y)$, $\psi(x, z)$ with*

(1) For $m < n$, $\{\varphi(x, b_i)\}_{i\leq m} \cup \{\psi(x, a_i)\}_{m<i\leq n}$ is consistent.

(2) For $i < j$, $\{\varphi(x, a_i), \psi(x, b_j)\}$ is inconsistent.

Then T is SOP_3*.*

Finally NTP_2 and N-ω-DCTP_1 will play a secondary role in this chapter, but we will discuss some results on these classes that motivate our main results on NSOP_3 theories.

Definition 4.2.5. *A theory T is* NTP$_2$ *(that is, does not have the* tree property of the second kind*) if there is no array $\{b_{ij}\}_{i,j\in\omega}$ and formula $\varphi(x,y)$ so that there is some fixed k so that, for all i, $\{\varphi(x,b_{ij})\}_{j\in\omega}$ is inconsistent, but for any $\sigma\in\omega^\omega$, $\{\varphi(x,b_{i\sigma(i)})\}_{i\in\omega}$ is consistent.*

The class NATP was introduced in [3] and further developed in [4] as a generalization of NTP$_2$; it has been proposed as one possible answer to a question of Kruckman [69], on what class can be viewed to generalize properties of NIP and NSOP$_1$ theories the same way NTP$_2$ theories generalize properties of NIP and simple theories. It is still open to what extent the analogy holds; for example, whether Kim-forking coincides with Kim-dividing in NATP theories, as forking coincides with dividing in NTP$_2$ theories. However, for N-ω-DCTP$_2$ *theories*, introduced in Chapter 3 and further developed in [66], the equivalence of Kim-forking and Kim-dividing was proven in [66] after being proven for coheir Kim-dividing and coheir Kim-forking in Chapter 3.

Definition 4.2.6. *(Proposition 2.51, item IIIa, [102]). A list $\eta_1,\ldots,\eta_n\in\omega^{<\omega}$ is a* descending comb *if and only if it is an antichain so that $\eta_1<_{\mathrm{lex}}\ldots<_{\mathrm{lex}}\eta_n$, and so that, for $1\leq k<n$, $\eta_1\wedge\ldots\wedge\eta_{k+1}\lhd\eta_1\wedge\ldots\wedge\eta_k$.*

Definition 4.2.7. *The theory T has k-*DCTP$_2$ *if there exists a formula $\varphi(x,y)$ and tuples $\{b_\eta\}_{\eta\in 2^{<\omega}}$ so that $\{\varphi(x,b_{\sigma\restriction n})\}_{n\in\omega}$ is k-inconsistent for any $\sigma\in 2^\omega$, but for any descending comb $\eta_1\ldots,\eta_l\in 2^{<\omega}$, $\{\varphi(x,b_{\eta_i})\}_{i=1}^{l}$ is consistent. If T does not have k-*DCTP$_2$ *for any k, it has* N-ω-DCTP$_2$.

4.3 Reflection principles for hypergraph sequences

Simple types were defined in [46]; then co-simple and NTP$_2$ types were defined in [26]. We define *co*-NSOP$_1$ types and give some equivalent definitions, similarly to Definition 6.7 of [26]. (When clear from context, when $p(x)$ is an n-type we refer to $p(\mathbb{M}^n)$ by $p(\mathbb{M})$.

Definition 4.3.1. *A partial type $p(x)$ over M is co-*NSOP$_1$ *if it satisfies one of the following equivalent conditions:*

(1) There does not exist a formula $\varphi(x,y)\in L(M)$ and tuples $\{b_\eta\}_{\eta\in 2^{<\omega}}$, $b_\eta\subset p(\mathbb{M})$ so that $\{\varphi(x,b_{\sigma\restriction n})\}_{n\in\omega}$ is consistent for any $\sigma\in 2^\omega$, but for any $\eta_2\trianglerighteq\eta_1\frown\langle 0\rangle$, $\{\varphi(x,b_{\eta_2}),\varphi(x,b_{\eta_1\frown\langle 1\rangle})\}$ is inconsistent.

(2, 2') There does not exist a formula $\varphi(x,y)\in L(M)$ and an array $\{c_{i,j}\}_{i=0,1,j<\omega}$, $c_{i,j}\subset p(\mathbb{M})$, so that $\{\varphi(x,c_{0,j})\}_{j<\omega}$ is consistent, $\{\varphi(x,c_{1,j})\}_{j<\omega}$ is k-inconsistent for some k (2-inconsistent), and $c_{0,j}\equiv_{Mc_{0,<j}c_{1,<j}}c_{1,j}$ for each $j<\omega$.

(3) Kim's lemma for Kim-dividing: For $M'\succeq M$, and $\varphi(x,y)\in L(M)$, if $\varphi(x,b)$ Kim-divides over M for $b\subset p(\mathbb{M})$, then for every M'-invariant Morley sequence $\{b_i\}_{i\in\omega}$ with $b_0=b$, $\{\varphi(x,b_i)\}_{i\in\omega}$ is inconsistent.

Proof. This is essentially proven in Chernikov and Ramsey, [28], and Kaplan and Ramsey, [52], so we will only give a sketch.

(1 ⇔ 2 ⇔ 2') Follows from the proof of Proposition 2.4 of [28] uses the proof of Proposition 5.6 of [28]) The part due to [28] shows that if there is a formula $\varphi(x,y) \in L(M)$ and tuples $\{b_\eta\}_{\eta\in 2^{<\omega}}$, $b_\eta \subset p(\mathbb{M})$ so that $\{\varphi(x, b_{\sigma|n})\}_{n\in\omega}$ is consistent for any $\sigma \in 2^\omega$, but for any $\eta_2 \trianglerighteq \eta_1 \frown \langle 0\rangle$, $\{\varphi(x, b_{\eta_2}), \varphi(x, b_{\eta_1\frown\langle 1\rangle})\}$ is inconsistent, then there is an array $\{c_{i,j}\}_{i=0,1,j<\omega}$, so that $\{\varphi(x, c_{0,j})\}_{j<\omega}$ is consistent, $\{\varphi(x, c_{1,j})\}_{j<\omega}$ is 2-inconsistent, and $c_{0,j} \equiv_{Mc_{0,<j}c_{1,<j}} c_{1,j}$ for each $j < \omega$. The other direction due to [52] shows that if there is $\varphi(x, y)$ and $\{c_{i,j}\}_{i=0,1,j<\omega}$, so that $\{\varphi(x, c_{0,j})\}_{j<\omega}$ is consistent, $\{\varphi(x, c_{1,j})\}_{j<\omega}$ is k-inconsistent for some k, and $c_{0,j} \equiv_{Mc_{0,<j}c_{1,<j}} c_{1,j}$ for each $j < \omega$, then there is a formula $\psi(x, y') \in L(M)$ (obtained as a conjunction of instances of φ) and tuples $\{b_\eta\}_{\eta\in 2^{<\omega}}$, $b_\eta \subset p(\mathbb{M})$ so that $\{\psi(x, b_{\sigma|n})\}_{n\in\omega}$ is consistent for any $\sigma \in 2^\omega$, but for any $\eta_2 \trianglerighteq \eta_1 \frown \langle 0\rangle$, $\{\psi(x, b_{\eta_2}), \psi(x, b_{\eta_1\frown\langle 1\rangle})\}$ is inconsistent. And if each $b_\eta \subseteq p(\mathbb{M})$ for $p(x)$ a fixed partial type, the former direction even shows that we can choose $c_{i,j} \subseteq p(\mathbb{M})$, and vice versa for the latter direction, proving the equivalence in the co-NSOP$_1$ case.

(3 ⇒ 2). This is basically the proof of Proposition 3.14 of [52]. Assume (2) is false; we show (3) is false. The equivalence (1 ⇔ 2) does not use anything about the fact that M is a model, and the failure of (1) to hold is preserved under expanding the language; therefore, we can fix a Skolemization T^{Sk} of T, and assume that $c_{0,j} \equiv^{L^{\mathrm{Sk}}}_{Mc_{0,<j}c_{1,<j}} c_{1,j}$ for each $j < \omega$. By Ramsey's theorem and compactness, we can choose $\{\bar{c}_j\}_{j<\omega}$ M-indiscnernible in T^{Sk}. Let $M' = \mathrm{dcl}_{\mathrm{Sk}}(\bar{c}_{<\omega}M)$. Choose non-principal ultrafilters U_i, $i = 0, 1$, containing $c_{i,<\omega}$, and let the global types $p_i(x) = \{\varphi(x, c) : \varphi(M, c) \in U_i\}$, so that each of the p_i are finitely satisfiable over M. It can be shown from $c_{0,j} \equiv^{L^{\mathrm{Sk}}}_{Mc_{0,<j}c_{1,<j}} c_{1,j}$ that $p_0|_M = p_1|_M$; let b realize this, so $b \subseteq p(\mathbb{M})$. Then $\{\varphi(x, b_i)\}_{i<\omega}$ will be consistent for $\{b_i\}_{i<\omega}$ a Morley sequence in p_0, but $\{\varphi(x, b'_i)\}_{i<\omega}$ will be consistent for $\{b'_i\}_{i<\omega}$ a Morley sequence in p_1, so Kim's lemma fails.

(2 ⇒ 3). This is the proof of Proposition 3.15 of [52]. We assume that (3) is false and show that (2) is false. Let $\varphi(x, b)$ for $b \subseteq p(\mathbb{M})$ Kim-divide over M', winessed by a Morley sequence in the M'-invariant type p_1. Let Morley sequences in the M'-invariant type p_0 fail to witness Kim-dividing of $\varphi(x, b)$ over M'. Find $\{c_{0,i}, c_{1,i}\}_{i\in\mathbb{Z}}$ so that $(c_{0,i}, c_{1,i})_{i\in\mathbb{Z}} \models (p_0 \otimes p_1)^{\mathbb{Z}}$. Then $\{c_{0,i}, c_{1,i}\}_{0\leq i<\omega}$ will be as desired.

□

While Chernikov ([52], Defintion 6.7) gives an additional characterization of co-simplicity in terms of symmetry for forking-independence, it requires additional elements of the base to belong to $p(\mathbb{M})$. Giving a characterization of co-NSOP$_1$ types in terms of symmetry for Kim-independence would be more complicated, because defining Kim-independence over arbitrary sets, rather than models, requires additional considerations; see [52]. However, co-NSOP$_1$ types over M do have a symmetry property over M, which will be useful in the sequel:

Proposition 4.3.1. *(Symmetry) Let $p(x)$ be a co-*NSOP$_1$ *type over M and $a \subset p(\mathbb{M})$. If $a \mathop{\smile\!\!\!\!\!\mid}^{Kd}_{M'} b$, then $b \mathop{\smile\!\!\!\!\!\mid}^{Kd}_{M} a$.*

Proof. Because b is not necessarily contained in $p(\mathbb{M})$ construction of the original tree must proceed like the proof of Theorem 3.5.3 (based in turn on the proof of Lemma 4.5.11 of [52]) in taking a specially chosen Morley sequence at each stage, rather than directly following the proof of Theorem 6.5 of [52]. Though the rest of the proof can be done as in Lemma 4.5.12 and Proposition 5.13 of [52], we give our own exposition, which only requires us to construct a tree of countable size rather than a much larger tree.

Suppose $p(\mathbb{M})$ is co-NSOP$_1$. We begin with the following claim (which we could have avoided by following Lemma 4.5.12 and Proposition 5.13 of [52]):

Claim 4.3.2. *Let $\varphi(x, c)$ Kim-divide over M for $\varphi(x, y) \in L(M)$ and $c \subseteq p(\mathbb{M})$. Then there is a bound depending only on $\varphi(x, y)$ and $\mathrm{tp}(c/M)$ on the size of a set $\{c_i\}_0^n$, $c_i \models \mathrm{tp}(c/M)$ for $0 \leq i \leq n$, so that there are M-finitely satisfiable types $p_0 \dots p_n$ so that $c_i \models p_i(x)|_{Mc_0 \dots c_{i-1}}$ for $0 \leq i \leq n$, and $\{\varphi(x, c_i)\}_{i=0}^n$ is consistent.*

Proof. This proceeds as in the direction (2 $\Rightarrow$ 3) of the previous definition (again, see [52], Proposition 3.15). Let $p_0, \dots, p_n$ be as in the claim, and let Morley sequences in the M-invariant type $q(x) \vdash \mathrm{tp}(c/M)$ witness Kim-dividing of $\varphi(x, c)$ over M: for $\bar{c}' \models q^\omega(x)$, $\{\varphi(x, c_i)\}_{i<\omega}$ is k-inconsistent for some fixed k. Find $\{c_{0,i}, c_{1,i}\}_{0 \leq i \leq n}$ so that $(c_{0,i}, c_{1,i})_{0 \leq i \leq n} \models (p_n \otimes q) \otimes \dots \otimes (p_0 \otimes q)$. Then $c_{0,n} \dots c_{0,0} \equiv_M c_0, \dots c_n$, so $\{\varphi(x, c_{0,i})\}_{i=0}^n$ is consistent. However, $c_{1,n}, \dots, c_{1,0} \models q^{(n+1)}(x)$, so $\{\varphi(x, c_{1,i})\}_{i=0}^n$ is k-inconsistent. Finally, $c_{0,0}c_{1,0} \dots c_{0,i-1}c_{1,i-1} \mathop{\smile\!\!\!\!\!\mid}^{i}_{M} c_{0,i}c_{1,i}$ for all $1 \leq i \leq n$, and $c_{0,i} \equiv_M c \equiv_M c_{1,i}$, so $c_{0,i} \equiv_{Mc_{0,0}c_{1,0} \dots c_{0,i-1}c_{1,i-1}} c_{1,i}$ for all $1 \leq i \leq n$.

Now if n is unbounded (k is fixed) this contradicts Defintion 4.3.1 (2), by compactness. $\square$

The following step, where we construct a tree, is where we must deviate from the proof of Theorem 5.16 of [52]. We use the notation $a \mathop{\smile\!\!\!\!\!\mid}^{K^+u}_{M} b$ to denote that there is a coheir Morley sequence $\{b_i\}_{i<\omega}$ over M with $b_0 = b$ that remains indiscernible over Ma. We prove some basic facts about this relation:

Claim 4.3.3. *Right extension: The relation $\mathop{\smile\!\!\!\!\!\mid}^{K^+u}$ satisfies right extension: if $a \mathop{\smile\!\!\!\!\!\mid}^{K^+u}_{M} b$, for any c there is some $a' \equiv_{Mb} a$ with $a' \mathop{\smile\!\!\!\!\!\mid}^{K^+u}_{M} bc$.*

Proof. Let $I = \{b_i\}_{i<\omega}$ be a Morley sequence in the M-finitely satisfiable type $q(x)$, $b_0 = b$, that remains indiscernible over Ma. By left extension for $\mathop{\smile\!\!\!\!\!\mid}^{u}$ there is some M-finitely satisfiable type $r(x, y)$ extending $q(x)$ and $\mathrm{tp}(bc/M)$. Then there are c_i', $i < \omega$ $c_0 = c$, so that $\{b_i c_i\}_{i<\omega}$ is a Morley sequence in $r(x, y)$. By Ramsey's theorem, compactness and an automorphism, a' can then be chosen so that $a' \equiv_{Mb} a$, indeed so that $a' \equiv_{MI} a$, and $\{b_i c_i\}_{i<\omega}$ is indiscernible over Ma. $\square$

Claim 4.3.4. *Chain condition: Let $I = \{b_i\}_{i \in \omega}$ be an M-finitely satisfiable Morley sequence indiscernible over Ma. Then $a \mathop{\smile\!\!\!\!\!\mid}^{K^+u}_{M} I$.*

Proof. By compactness there is $I' = \{b_i\}_{i \in \omega^2}$ so that $I'|_\omega = I$ and I' is indiscernible over Ma. Then $\{b'_{i\omega} \dots b'_{i\omega+j} \dots\}_{i<\omega}$ will be an M-finitely satisfiable Morley sequence starting with I' and indiscernible over Ma. $\square$

Assume for contradiction that $a \mathop{\smile\!\!\!\!\!\!|}^{Kd}_M b$ with $b \subseteq p(\mathbb{M})$, but $a \not\mathop{\smile\!\!\!\!\!\!|}^{Kd}_M b$. We find, for all n, a tree $(I_n, J_n) = (\{a_\eta\}_{\eta \in \omega^{\leq n}}, \{b_\sigma\}_{\sigma \in \omega^n})$, with the first $n+1$ levels I_n forming an infinitely branching tree, then with each a_σ for $\sigma \in \omega^n$ at level $n+1$ followed by a single additional leaf b_σ at level $n+2$, with the following two properties:

(1) For $\eta \trianglelefteq \sigma$, $|\sigma| = n$, $a_\eta b_\sigma \equiv_M ab$.

(2) For $\eta \in \omega^{<n}$, the subtrees at η form an M-finitely satisfiable Morley sequence indiscernible over a_η (so for I this sequence of subtrees, $a_\eta \mathop{\smile\!\!\!\!\!\!|}^{K^+u}_K I$.)

For $n = 0$, let $a_\emptyset = a$, $b_\emptyset = b$; then (2) follows from the fact that $\mathop{\smile\!\!\!\!\!\!|}^{Kd}$ easily implies $\mathop{\smile\!\!\!\!\!\!|}^{K^+u}$. Assume (I_n, J_n) already constructed; we construct (I_{n+1}, J_{n+1}). We see by (2) that for (I^*_n, J_n) the nodes of the tree excluding $a_\emptyset$, $a_\emptyset \mathop{\smile\!\!\!\!\!\!|}^{K^+u}_M I^*_n J_n$. By Claim 4.3.3, find $a'_\emptyset \equiv_{M I^*_n J_n} a_\emptyset$ with $a'_\emptyset \mathop{\smile\!\!\!\!\!\!|}^{K^+u}_M I_n J_n$, which will be the new root of (I_{n+1}, J_{n+1}). Then find some M-finitely satisfiable Morley sequence $\{(I_n, J_n)^i\}_{i \in \omega}$ starting with (I_n, J_n) indiscernible over $M a'_\emptyset$, giving the subtrees of (I_n, J_n) at $a'_\emptyset$. From $a_\emptyset J_n \equiv_M a'_\emptyset J_n \equiv_M a'_\emptyset J^i_n$, we will preserve (1) by indexing accordingly, and from choice of $\{(I_n, J_n)^i\}_{i \in \omega}$, we will preserve (2) as well.

We now find a contradiction to Definition 3.1.2; this is where, by constructing a much larger tree, we could have just followed Lemma 4.5.12 and Proposition 5.13 of [52]. By (1), the paths of each I_n are consistent: for $\sigma \in \omega^n$, $\{\varphi(x, a_\eta)\}_{\eta \trianglelefteq \sigma}$ is consistent, realized by b_σ. But by (2), for any k nodes $\eta_1, \ldots, \eta_n \in \omega^{<\omega}$, forming an antichain so that $\eta_1 <_{\mathrm{lex}} \ldots <_{\mathrm{lex}} \eta_k$, and so that, for $1 \leq i < k$, $\eta_1 \wedge \ldots \wedge \eta_{i+1} \triangleleft \eta_1 \wedge \ldots \wedge \eta_i$, $\{a_{\eta_i}\}_{i=1}^k$ form a sequence with $a_{\eta_i} \mathop{\smile\!\!\!\!\!\!|}^{i}_M a_{\eta_1} \ldots a_{\eta_{i-1}}$; by (1), $a_i \subseteq p(\mathbb{M})$. So for k the bound from Claim 4.3.2, and η_i with these conditions (forming a *descending comb*, Definition 3.2.3), for η_i for $1 \leq i \leq k$ satisfying the above property, $\{\varphi(x, a_{\eta_i})\}_{i=1}^k$ is k-inconsistent. So by compactness, we can find a tree $\{a_\eta\}_{\eta \in \omega^{<\omega}}$ with the same consistency and inconsistency properties for $\varphi(x, y)$ (consistency along the paths and inconsistency on descending combs of size k), and with $b_\eta \subseteq p(\mathbb{M})$.

We recall the following defintion and fact:

Definition 4.3.2. *(Definitions 11 and 12, [105]) For tuples $\overline{\eta}, \overline{\eta}' \in \omega^{<\omega}$ of elements of $\omega^{<\omega}$, we write $\overline{\eta} \sim_0 \overline{\eta}'$ to mean that $\overline{\eta}$ has the same quantifier-free type in the language $\{<_{\mathrm{lex}}, \triangleleft, \wedge\}$ as $\overline{\eta}'$. For $(b_\eta)_{\eta \in \omega^{<\omega}}$ a tree-indexed set of tuples and $\overline{\eta} = \eta_1, \ldots, \eta_n \in \omega^{<\omega}$ an n-tuple of elements of $\omega^{<\omega}$, we write $b_{\overline{\eta}} =: b_{\eta_1} \ldots b_{\eta_n}$, and call $(b_\eta)_{\eta \in \omega^{<\omega}}$* strongly indiscernible *over a set A if for all tuples $\overline{\eta}, \overline{\eta}' \in \omega^{<\omega}$ of elements of $\omega^{<\omega}$ with $\overline{\eta} \sim_0 \overline{\eta}'$, $b_{\overline{\eta}} \equiv_A b_{\overline{\eta}'}$.*

Fact 4.3.5. *(Theorem 16, [105]; see [96] for an alternate proof) Let $(b_\eta)_{\eta \in \omega^{<\omega}}$ be a tree-indexed set of tuples, and A a set. Then there is $(c_\eta)_{\eta \in \omega^{<\omega}}$ strongly indiscernible over A so that for any tuple $\overline{\eta} \in \omega^{<\omega}$ of elements of $\omega^{<\omega}$ and $\varphi(x) \in L(A)$, if $\models \varphi(b_{\overline{\eta}'})$ for all $\overline{\eta}' \sim_0 \eta$, then $\models \varphi(c_{\overline{\eta}})$.*

Now use Fact 4.3.5 to extract a strongly indiscernible tree $(c_\eta)_{\eta \in \omega^{<\omega}}$. Let $\{c_{j,i}\}_{j=0,1,i<\omega} = \{c_{\langle 0 \rangle^i \frown \langle j \rangle}\}$. Then $\{c_{j,i}\}_{j=0,1,i<\omega}$ is as in Defintion 3.1.2, contradiction.

□

We could likely have also proven Proposition 4.3.1 in the style of Definition 6.1 of [26]: use right extension to find an $\mathop{\smile\!\!\!\!\!\!|}^{Kd}$-Morley sequence of a over M, indiscernible over b, and then developed local character and Kim's lemma for $\mathop{\smile\!\!\!\!\!\!\,|}^{Kd}$-Morley sequences in the context of co-NSOP$_1$ types, [54] and [53]. Since these characterizations of co-NSOP$_1$ are not necessary for our main theorem on internally NSOP$_1$ types, we leave the details to the reader.

Notions such as co-simple and co-NSOP$_1$ types involve interaction of the types with the rest of the structure. In the other direction, there are the simple types defined in [46], the NIP and NTP$_2$ types defined in [26], and the fully stable types defined in [103]. We introduce a new schema for defining the local classification-theoretic properties of a type, which is in some sense more natural, because it depends only on the corresponding properties for a structure associated with the type.

Definition 4.3.3. *(1) Let $p(x)$ be a partial n-type over M. Let $\mathcal{L}_p$ contain an m-ary relation symbol R_φ for each formula $\varphi(x_1, \ldots, x_m) \in L(M)$ with $|x_i| = n$ for $i \leq n$. Then $\mathcal{M}_p$ is the $\mathcal{L}_p$-structure with domain $p(\mathbb{M}^n)$ and with $R_\varphi(p(\mathbb{M}^n)^m) = \varphi(\mathbb{M}^{mn}) \cap p(\mathbb{M}^n)^m$.*

(2) Let $\mathcal{P}$ be a property of theories. Then a partial type $p(x)$ is internally $\mathcal{P}$ *if the theory of $\mathcal{M}_p$ is $\mathcal{P}$.*

Remark 4.3.6. If $p(x)$ is not just a partial type, but a formula with parameters in M, then the theory of $\mathcal{M}_p$ clearly has quantifier elimination. In this case, for p to be internally simple, NIP, etc. is weaker than for it to be simple or NIP in the sense of [46], [26]. The case of a definable formula is in fact all we need to find counterexamples in NSOP$_4$ theories to our results on the internally NSOP$_1$ types of NSOP$_3$ theories. In the case of a general partial type, we could have have also considered the case where $\mathcal{P}$ is a property of formulas and all quantifier-free formulas of $\mathcal{L}_p$ have property $\mathcal{P}$. This defintion would also be weaker than the corresponding "external" property, and our results should go through even assuming only the quantifier-free version, by developing the theory of Kim-independence relative to only the quantifier-free formulas.

Theorem 6.17 of [26] says that simple types are co-simple; in fact, only internal simplicity is needed. By way of analogy, internally NSOP$_1$ types are co-NSOP$_1$ in ω-NDCTP$_2$ theories; see below. Beyond this analogy, we find that:

Theorem 4.3.7. *Let T be* NSOP$_3$*, and $p(x)$ an internally* NSOP$_1$ *type. Then $p(x)$ is co-*NSOP$_1$.

Example 4.3.8. Theorem 4.3.7 becomes false if we relax NSOP$_3$ to NSOP$_4$. Let T be the model companion of (undirected) triangle-free tripartite graphs, with the partition denoted by unary predicates P_1, P_2, P_3. Let M be a model and $p(x) = \{P_1(x) \vee P_2(x)\}$. Then T is NSOP$_4$, in fact a free amalgamation theory in the sense of Conant ([32]). Moreover, $p(x)$ is internally NSOP$_1$, in fact, internally simple. The associated theory has quantifier elimination in the language with unary predicates for P_1, P_2, P_m denoting xEm for each $m \in M$, and a binary relation symbol for the edge relation between elements of P_1 and elements of P_2. It is the model companion of graphs with interpretations for the unary predicates P_1, P_2 and P_m, so that P_1 and P_2 partition the graph and have no edges within them, there are no edges within P_m for any $m \in M$, P_{m_1} and P_{m_2} are disjoint for $m_1, m_2 \in M$ with $M \models m_1 E m_2$, and for $i = 1, 2$ P_m is disjoint from P_i when $M \models P_i(m)$. In this form, the theory associated to $p(x)$ can be easily seen to be simple (for example, check that the relation $A \cap B = C$ coincides with forking-independence).

However, $p(x)$ is not co-NSOP$_1$: let $\varphi(x, y) =: xEy_1 \wedge xEy_2$. For $\eta \in 2^{<\omega}$, choose $b_\eta = (b^1_\eta, b^2_\eta)$ with, for any $\eta, \nu \in 2^{<\omega}$, $b^i_\eta \in P_i$, $\models \neg b^i_\eta Em$ for $i = 1, 2$ and $m \in M$, and $\models b^1_\eta E b^2_\nu$ if and only if η and ν are incomparable. This is possible, as we create no triangles. But $\varphi(x, y)$, b_η witness the failure of definition 3.1.1.

We prove Theorem 4.3.7. Again following the arguments of Chernikov and Ramsey [28] and Kaplan and Ramsey [52], we start by carrying out the arguments of Definition 3.1, (1 $\Leftarrow$ 2' $\Leftarrow$ 3 (for 2-Kim-dividing)) *internally* to $\mathcal{M}_p$. Since the consistency in the defintion of (1) need not be witnessed by a realization of $p(x)$, we will no longer be dealing with actual consistency or inconsistency of instances of $\varphi(x, y)$, but rather the definable relations in $\mathcal{M}_p$ corresponding to this consistency, treated only as a definable hypergraph. This hypergraph will be part of the *characteristic sequence* of $\varphi(x, y)$, introduced by Malliaris in [77].

Suppose $p(x)$ is not co-NSOP$_1$. Let $\varphi(x, y) \in L(M)$, $b_\eta \subseteq p(\mathbb{M})$, $\eta \in 2^\omega$ be as in Defintion 3.1.1. By compactness, we can replace $2^{<\omega}$ with $2^{<\kappa}$, for large κ. Define $R_n(y_1, \ldots, y_n) =: R_{\exists x \varphi(x, y_1) \wedge \ldots \wedge \varphi(x, y_n)}(y_1, \ldots, y_n) \in \mathcal{L}_p$. Then $\mathcal{M}_p \models R_n(b_{\eta_1}, \ldots, b_{\eta_n})$ for $\eta_1 \lhd \ldots \lhd \eta_n \in 2^{<\kappa}$, but for any $\eta_2 \unrhd \eta_1 \frown \langle 0 \rangle$, $\mathcal{M}_p \models R_2(b_{\eta_2}, b_{\eta_1 \frown \langle 1 \rangle})$.

For a sequence of relations $\{R_n\}_{n<\omega}$ on a set, where R_n is an n-ary relation, call a sequence $\{a_i\}_{i \in I}$ a *clique* if for $i_1, \ldots, i_n \in I$, $(a_1, \ldots a_{i_n}) \in R_n$, and an n-*anticlique* if for $i_1, \ldots, i_n \in I$, $(a_1, \ldots a_{i_n}) \notin R_n$. Choose a Skolemization of $\mathcal{M}_p$. We show that there is an array $\{c_{i,j}\}_{i=0,1, j<\omega}$, $c_{i,j} \in \mathcal{M}_p$, so that $\{c_{0,j}\}_{j<\omega}$ is a clique, $\{c_{1,j}\}_{j<\omega}$ is a 2-anticlique, and $c_{0,j} \equiv^{\mathcal{L}_p^{\mathrm{Sk}}}_{c_{0,<j} c_{1,<j}} c_{1,j}$ for each $j < \omega$. We may follow the proof of [28], Proposition 5.6, that we cited in the direction (1$\Rightarrow$ 2') of Definition 3.1. We sketch the argument: we will draw the $c_{i,j}$ from $\{b_\eta\}_{\eta \in \omega^\kappa}$. Suppose that, for $1 \leq i \leq n$ $c_{i,0} = b_{\lambda_i}$, and $c_{i,1} = b_{\eta_i}$ are already chosen to satisfy these properties, with $\eta_j \wedge \lambda_j \rhd \lambda_i$ and $\lambda_i \unrhd (\eta_i \wedge \lambda_i) \frown \langle 0 \rangle, \eta_i \unrhd (\eta_i \wedge \lambda_i) \frown \langle 1 \rangle$, for $1 \leq i < j \leq n$. Then using the pigeonhole principle, choose nodes $\lambda_{n+1} = \lambda_n \frown \langle 0 \rangle^{\kappa_1} \frown \langle 1 \rangle$, $\eta_{n+1} = \lambda_n \frown \langle 0 \rangle^{\kappa_2} \frown \langle 1 \rangle$ for $\kappa_1 < \kappa_2 < \kappa$ so that $c_{n+1,0} = c_{\lambda_{n+1}}$ and $c_{n+1,1} = c_{\eta_{n+1}}$ are such that $c_{n+1,0} \equiv^{\mathcal{L}_p^{\mathrm{Sk}}}_{c_{0,\leq n} c_{1,\leq n}} c_{n+1,1}$.

We next find a model $\mathcal{M}$ of the theory of $\mathcal{M}_p$ and $\mathcal{M}$-invariant Morley sequences $\{b_i\}_{i<\omega}$ in the $\mathcal{M}$-invariant type p_0 and $\{b'_i\}_{i<\omega}$ in the $\mathcal{M}$-invariant type p_1, so that $b_0 = b'_0$, $\{b_i\}_{i<\omega}$ is a clique, and $\{b'_i\}_{i<\omega}$ is a 2-anticlique.[1] As in the proof of (3 $\Rightarrow$ 2) of Definition 3.1, we follow the proof of Proposition 3.14 of [52]. By Ramsey's theorem and compactness, we can choose $\{\bar{c}_j\}_{j<\omega}$ indiscernible in the theory of $\mathcal{M}_p^{\mathrm{Sk}}$. Let $\mathcal{M} = \mathrm{dcl}_{\mathrm{Sk}}(\bar{c}_{<\omega})$, and let $\mathcal{M}' \succ \mathcal{M}$ be sufficiently saturated. Choose non-principal ultrafilters U_i, $i = 0, 1$, containing $c_{i,<\omega}$, and let the global types $p_i(x) = \{\varphi(x, c) \in \mathcal{M}' : \varphi(\mathcal{M}, c) \in U_i\}$, so that each of the p_i are finitely

[1] It was observed by Hyoyoon Lee, Byunghan Kim, and the other participants of the Yonsei University logic seminar that the proof of Propsition 3.14 of [52] actually shows that in a SOP$_1$ theory, there is a formula that 2-Kim-divides for which Kim's lemma fails. This is the "internal" version of this observation.

satisfiable over $\mathcal{M}$. It can be shown from $c_{0,j} \equiv^{\mathcal{L}_p^{\mathrm{Sk}}}_{c_{0,<j}c_{1,<j}} c_{1,j}$ that $p_0|_{\mathcal{M}} = p_1|_{\mathcal{M}}$; let b realize this. Then we can choose b so that a p_0-Morley sequence $\{b_i\}_{i<\omega}$ with $b_0 = b$ is a clique, and a p_1-Morley sequence $\{b'_i\}_{i<\omega}$ with $b'_0 = b$ is a 2-anticlique.

Finally, we show, using the technique of Theorem 7.17 of [32], that the R_n have the compatible order property (SOP$_3$), Definition 3.10 of [77]. By compactness and Fact 4.2.4, the following will translate into an instance of SOP$_3$ in T, a contradiction. We find an array $c_0, \ldots, c_n, \ldots, d_0, \ldots, d_n, \ldots$, with the following properties:

(1) For $m < n$, $d_0, \ldots, d_m, c_{m+1}, \ldots, c_n$ form a Morley sequence in p_0, so a clique.

(2) For $m < n$, $c_m d_n$ begin a Morley sequence in p_1, so $\neg R_2(c_m, d_n)$.

(3) $c_0, \ldots, c_n, \ldots \mathop{\smile\!\!\!\!\!\!|}^K_{\mathcal{M}} d_0, \ldots, d_n, \ldots$

Suppose we have constructed $c_0, \ldots, c_n, d_0, \ldots, d_n$ satisfying these properties up to n. We find c_{n+1}, d_{n+1}. To find d_{n+1}, let $d'_{n+1} \models p_0(x)|_{\mathcal{M}d_0\ldots d_n}$ so $d'_{n+1} \mathop{\smile\!\!\!\!\!\!|}^K_{\mathcal{M}} d_0, \ldots, d_n$. By (3), $c_0, \ldots, c_n \mathop{\smile\!\!\!\!\!\!|}^K_{\mathcal{M}} d_0, \ldots, d_n$. Finally, if $d''_{n+1} \models p_1(x)|_{\mathcal{M}c_0\ldots c_n}$, by symmetry of Kim-independence, Fact 4.2.2, $c_0 \ldots c_n \mathop{\smile\!\!\!\!\!\!|}^K d''_{n+1}$. So by the independence theorem (fact 4.2.3) and an automorphism, there is $d_{n+1} \models p_0(x)|_{\mathcal{M}d_0\ldots d_n} \cup p_1(x)|_{\mathcal{M}c_0\ldots c_n}$ with $c_0, \ldots, c_n \mathop{\smile\!\!\!\!\!\!|}^K_{\mathcal{M}} d_0, \ldots, d_n d_{n+1}$. Finally choose $c_{n+1} \models p_0(x)|_{\mathcal{M}c_0,\ldots c_n d_0\ldots d_n}$. It remains to show that this preserves (3). This follows from the following claim:

Claim 4.3.9. *For any a, b, c, M, if $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$ and $\mathrm{tp}(c/Mab)$ extends to an M-invariant type $q(x)$, then $a \mathop{\smile\!\!\!\!\!\!|}^K_M bc$.*

This follows from Claim 4.5.13 below, using the fact that $\mathop{\smile\!\!\!\!\!\!|}^{K^+} = \mathop{\smile\!\!\!\!\!\!|}^K$ in the language of that claim (Kim's lemma, Fact 4.2.1, and compactness) and symmetry of $\mathop{\smile\!\!\!\!\!\!|}^K$ (Fact 4.2.2).

This concludes the proof of Theorem 4.3.7.

This proof can be viewed as an instance of a more general phenomenon. In this proof, the R_n are the restriction to $p_n(\mathbb{M})$ of the *characteristic sequence* of $\varphi(x, y)$, defined by Malliaris:

Definition 4.3.4. *([77] Let $\varphi(x, y)$ be a formula. The* characteristic sequence *of $\varphi(x, y)$ is the sequence of hypergraphs, on the vertices $\mathbb{M}^{|y|}$, defined by*

$$R_n(a_1, \ldots, a_n) =: (a_1, \ldots, a_n) \models \exists x \wedge_{i=1}^n \varphi(x, y_i)$$

On the other hand, within $\mathcal{M}_p$, the R_n are just a sequence of hypergraphs, and do not describe a pattern of consistency internally to $\mathcal{M}_p$. Nonetheless, by showing that a particular configuration, the *compatible order property*, arises among the R_n, we get a description of the complexity of $\varphi(x, y)$ *in the original theory* T. In [77], Malliaris introduces some hypergraph configurations corresponding, via the characteristic sequence, to consistency patterns (in the sense of [44]) in a first-order formula. We introduce some additional defintions to cover the case of the tree property, ω-DCTP$_2$, and SOP$_1$; the first of these comes from Observation 5.20 of [77].

Definition 4.3.5. *Let $R_\infty = (V, \{R_n\}_{n<\omega})$ be a sequence of hypergraphs on a common set of vertices V, where R_n is an n-ary edge relation Then R_∞ has*

(1) An $(\omega, \omega, 1)$*-array if there is an array* $\{b_{ij}\}_{i,j\in\omega}$ *so that there is some fixed* k *so that, for all* i, $\{b_{ij}\}_{j\in\omega}$ *is a clique, but for any* $\sigma \in \omega^\omega$, $\{b_{i\sigma(i)}\}_{i\in\omega}$ *is a* k*-anticlique. (Definition 3.4.2, [77]. By Claim 3.8, [77],* TP$_2$ *is equivalence to the presence of an* $(\omega, \omega, 1)$*-array in the characteristic sequence of a formula.*

(2) The compatible order property *if there are* $c_0, \dots, c_n, \dots, d_0, \dots, d_n, \dots$ *so that for* $m < n$, $d_0, \dots, d_m, c_{m+1}, \dots, c_n$ *form a clique, while for* $m < n$, $\neg R_2(c_m, d_n)$*. (Definition 3.10, [77]. In Conclusion 6.15 of [77],* SOP$_3$ *is shown to be equivalent to the compatible order property in the characteristic sequence of a formula.)*

(3) MTP *if there is some fixed* k *and parameters* $\{b_\eta\}_{\eta\in\omega^{<\omega}}$ *so that for each path* $\sigma \in \omega^\omega$, $\{b_{\sigma|_n}\}_{n\in\omega}$ *is a clique, but for each node* $\eta \in \omega^{<\omega}$, $\{b_{\eta\frown\langle n\rangle}\}_{n\in\omega}$ *is a* k*-anticlique. (In Observation 5.10 of [77], the failure of a formula to be simple is observed to be equivalent to* MTP *in the characteristic sequence.*

(4) MSOP$_1$ *if there are parameters* $\{b_\eta\}_{\eta\in 2^{<\omega}}$ *so that* $\{b_{\sigma|n}\}_{n\in\omega}$ *is a clique for any* $\sigma \in 2^\omega$*, but for any* $\eta_2 \trianglerighteq \eta_1 \frown \langle 0\rangle$, $\{b_{\eta_2}, b_{\eta_1\frown\langle 1\rangle}\}$ *is a 2-anticlique.*

(5) ω-MDCTP$_2$ *if for some fixed* k*, there are parameters* $\{b_\eta\}_{\eta\in 2^{<\omega}}$ *so that* $\{b_{\sigma|n}\}_{n\in\omega}$ *is a* k*-anticlique for any* $\sigma \in 2^\omega$*, but for any descending comb* $\eta_1 \dots, \eta_l \in 2^{<\omega}$, $\{x, b_{\eta_i}\}_{i=1}^l$ *is a clique.*

Remark 4.3.10. The letter M in MTP, MSOP$_1$ and ω-MDCTP$_2$ stands for *Malliaris*.

Note that these properties are all *graph-theoretic* in the sense of Malliaris, [77], referring only to incidence patterns of the edges, rather than their consistency. They are similar in this sense to stability or NIP, which make no reference to consistency but only ask for graph-theoretic configurations. In [97], Shelah shows the following:

Fact 4.3.11. *Let* $R(x, y)$ *be an unstable formula, and assume that all Boolean combinations of instances of* $R(x, y)$ *are* NSOP*. Then* $R(x, y)$ *has the independence property.*

Note that the form of this result is as follows: if a graph has one graph-theoretic configuration (instability), and an ambient model-theoretic tameness property (NSOP, indeed quantifier-free NSOP), then it has a more complicated graph-theoretic configuration (the independence property). It was Malliaris who first implicitly asked, in the context of strengthenings of the compatible order property (Remark 7.12, [76]), whether ambient classification-theoretic properties imply further graph-theoretic complexity gaps for hypergraphs. In the remainder of this section, we observe that model-theoretic tameness properties of hypergraph sequences that refer to consistency, such as simplicity and NSOP$_1$, provide additional information about their graph-theoretic structure, just as Shelah shows a gap between simplicity and independence in NSOP graphs. We then further observe that the connection between internal properties of types and external properties of theories, including the aforementioned work of Chernikov on co-simplicity ([26]), can be reinterpreted in terms of these graph-theoretic complexity gaps for model-theoretically tame hypergraphs.

Proposition 4.3.12. *Let* $R_\infty = (V, \{R_n\}_{n<\omega})$ *be sequence of hypergraphs on a common set of vertices* V*, where* R_n *is an* n*-ary edge relation.*

(1) If R_∞ is simple (in the hypergraph language) and has MTP*, it has an $(\omega, \omega, 1)$ array.*

(2) If R_∞ is NSOP$_1$ *and has* MSOP$_1$*, then it has ω-*MDCTP$_2$ *and the compatible order property.*

In fact, for (1), it suffices that no quantifier-free formula has the tree property, and similarly for (2) and SOP$_1$*.*

Example 4.3.13. If R_∞ is the model companion of the empty theory in the language of hypergraph sequences (or, say, the theory axiomatizing the basic properties of characteristic sequences; see [77], Observation 2.4), then it is a simple structure. But it has ω-MDCTP$_2$ and the compatible order property.

Proof. (Sketch) The argument for (1) is extracted from Chernikov's proof in [26] that simple types are co-simple. In particular, we notice that the proof Lemma 6.13 of [26] works when the rows are general indiscnernible sequences, not just Morley sequences, and relies only on the internal simplicity of a type, not the full definition of a simple type. Suppose R_∞ has MTP, but is simple as a structure in the language of hypergraph sequences. By the proof, which can be found in a standard reference on simplicity theory such as [62], that formulas with the tree property fail Kim's lemma for dividing, there is a model M of the theory of R_∞, some element b of the monster model, and some indiscernible k-anticlique $I = \{b_i\}_{i \in I}$ starting with $b_0 = b$, as well as a Morley sequence $J = \{b'_i\}_{i \in \omega}$ starting with $b'_0 = b$ and forming a clique. Now suppose, by induction, that for $i \leq n$ there are are $I^i = \{b^i_j\}_{j<\omega}$ with $I^i \equiv_M I$ and $b^i_0 = b'_i$ (so the I^i are anticliques), so that for $\sigma \in \omega^n$, $\{b^i_{\sigma(i)}\}_{i \leq n} \frown \{b'_i\}_{i \geq n+1}$ is a clique, and so that $I_{\leq n} \mathop{\smile\!\!\!\!\!\!|}_M b'_{\geq n+1}$. By properties of independence in simple theories, $I_{\leq n} b_{>n+1} \mathop{\smile\!\!\!\!\!\!|}_M b'_{n+1}$. By the chain condition, take $I^{n+1} = \{b^{n+1}_j\}_{j<\omega}$, $I^{n+1} \equiv_M I$, with $b^{n+1}_0 = b'_{n+1}$ so that I_{n+1} is indiscnernible over $I_{\leq n} b'_{>n+1}$, and with $I_{\leq n} b'_{>n+1} \mathop{\smile\!\!\!\!\!\!|}_M I_{n+1}$. This suffices for the induction. Now the existence of an $(\omega, \omega, 1)$-array follows.

For (2), suppose R_∞ is NSOP$_1$ in the hypergraph language, but has MSOP$_1$. Then as in the proof of Theorem 4.3.7, there is a model $\mathcal{M}$ of the theory of R_∞ and there are M-invariant Morley sequences $\{b_i\}_{i<\omega}$ in the M-invariant type p_0 and $\{b'_i\}_{i<\omega}$ in the M-invariant type p_1, so that $b_0 = b'_0$, $\{b_i\}_{i<\omega}$ is a clique, and $\{b'_i\}_{i<\omega}$ is a 2-anticlique. To show ω-MDCTP$_1$, it suffices to find a tree $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that the paths, read downward, are Morley sequences in p_1, and the descending combs are Morley sequences in p_0. Formally, the construction will follow Lemma 3.4.3. Say that a tree $\{c_\eta\}_{\eta \in 2^{\leq n}}$ is a *generic tree* if for $\eta \in 2^{<n}$ $c_{\trianglerighteq \eta \frown \langle 0 \rangle} \mathop{\smile\!\!\!\!\!\!|}^K_M c_{\trianglerighteq \eta \frown \langle 1 \rangle}$ (two subtrees at a node are Kim-independent), and $c_\eta \models p_1(x)|_{Mc_{\rhd \eta}}$ (each node satisfies the restriction of $p_1(x)$ to its subtrees.) We prove the following claim (corresponding to Claim 3.4.4):

Claim 4.3.14. *Let $\{c_\eta\}_{\eta \in 2^{\leq n}}$ be a generic tree, and A any set. Then there is some $\{c'_\eta\}_{\eta \in 2^{\leq n}} \equiv_M \{c_\eta\}_{\eta \in 2^{\leq n}}$ with $c'_\eta \models p_0(x)|_{MA}$ for each $\eta \in 2^{\leq n}$, and with $A \mathop{\smile\!\!\!\!\!\!|}^K_M \{c'_\eta\}_{\eta \in 2^{\leq n}}$.*

Proof. By induction on n, we may assume this is true for $\{c_\eta\}_{\eta \trianglerighteq \langle 0 \rangle}$ and $\{c_\eta\}_{\eta \trianglerighteq \langle 1 \rangle}$. Namely, fin $\{c''_\eta\}_{\eta \trianglerighteq \langle 0 \rangle} \equiv_M \{c_\eta\}_{\eta \trianglerighteq \langle 0 \rangle}$ with $c''_\eta \models p_0(x)|_{MA}$ for each $\langle 0 \rangle \trianglelefteq \eta \in 2^{\leq n}$ and $A \mathop{\smile\!\!\!\!\!\!|}^K_M \{c''_\eta\}_{\eta \trianglerighteq \langle 0 \rangle}$, and similarly, $\{c''_\eta\}_{\eta \trianglerighteq \langle 1 \rangle} \equiv_M \{c_\eta\}_{\eta \trianglerighteq \langle 1 \rangle}$ with $c''_\eta \models p_1(x)|_{MA}$ for each $\langle 1 \rangle \trianglelefteq \eta \in 2^{\leq n}$ and

$A \mathop{\smile\!\!\!\!\!\!\mid}^K_M \{c''_\eta\}_{\eta \trianglerighteq \langle 1 \rangle}$. Recall that $c_{\trianglerighteq \langle 0 \rangle} \mathop{\smile\!\!\!\!\!\!\mid}^K_M c_{\trianglerighteq \langle 1 \rangle}$ as $\{c_\eta\}_{\eta \in 2^{\leq n}}$ is a generic tree, so by the independence theorem and an automorphism, we can find $\{c'_\eta\}_{\eta \triangleright \langle\rangle} \equiv_M \{c_\eta\}_{\eta \triangleright \langle\rangle}$ with $c'_\eta \models p_0(x)|_{MA}$ for each $\langle\rangle \triangleleft \eta \in 2^{\leq n}$ and $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M \{c'_\eta\}_{\eta \triangleright \langle\rangle}$. Finally, by the independence theorem and an automorphism, find $c'_{\langle\rangle} \models p_0(x)|_{MA} \cup p_1(x)|_{M\{c'_\eta\}_{\eta \triangleright \langle\rangle}}$ so that $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M \{c'_\eta\}_{\eta \in 2^{\leq n}}$, as desired. □

By induction, we construct a *generic* tree $\{b_\eta\}_{\eta \in 2^{<\omega}}$ so that the paths, read downward, are Morley sequences in p_1, and the descending combs are Morley sequences in p_0. Suppose we have constructed $I = \{b_\eta\}_{\eta \in 2^{\leq n}}$ with these properties. By Claim 4.3.14, we can find $I^1 \equiv_M I \equiv_M I^2$ with $I^1 \mathop{\smile\!\!\!\!\!\!\mid}^K_M I^2$ and for $I^1 = \{b^1_\eta\}_{\eta \in 2^{\leq n}}$, $I^2 = \{b^2_\eta\}_{\eta \in 2^{\leq n}}$, $b^2_\eta \models p_0(x)|_{MI_1}$ for $\eta \in 2^{\leq n}$. The trees I_1 and I_2 of height n will be the subtrees of our new generic tree of height $n+1$. Finally, let $b_* \models q_1(x)|_{MI_1I_2}$ be the new root. Reindexing accordingly, we get a generic tree $\{b_\eta\}_{\eta \in 2^{\leq n+1}}$ so that the paths, read downward, are Morley sequences in p_1, and the descending combs are Morley sequences in p_0. This completes the induction.

Now the compatible order property comes from the proof of Theorem 4.3.7.

□

We connect this to the internal properties of types. We recall the defintion of co-simplicity from [26]:

Definition 4.3.6. *A type $p(x)$ over A is co-simple if there is no formula $\varphi(x, y)$, $k \geq 2$ and parameters $\{b_\eta\}_{\eta \in \omega^{<\omega}}$, $b_\eta \subseteq p(\mathbb{M})$ so that for each path $\sigma \in \omega^\omega$, $\{\varphi(x, b_{\sigma|_n})\}_{n \in \omega}$ is consistent, but for each node $\eta \in \omega^{<\omega}$, $\{\varphi(x, b_{\eta^\frown\langle n \rangle})\}_{n \in \omega}$ is k-inconsistent.*

Corollary 4.3.14.1. *(1) ([26], Theorem 6.17) In a* NTP$_2$ *theory, internally simple types are co-simple.*

(2) In an N-ω-DCTP$_2$ *theory or an* NSOP$_3$ *theory, internally* NSOP$_1$ *types are co-*NSOP$_1$

Proof. (1). If $p(x)$ is not co-simple then the restriction R_∞ of some charcateristic sequence to $p(\mathbb{M})$ has MTP. If $p(x)$ is internally NSOP$_1$, then R_∞ is *simple*, so by the previous proposition it has an $(\infty, \infty, 1)$ array. So T must have TP$_2$.

(2). If $p(x)$ is not co-NSOP$_1$ then the restriction R_∞ of some charcateristic sequence to $p(\mathbb{M})$ has MSOP$_1$. If $p(x)$ is internally NSOP$_1$, then R_∞ is NSOP$_1$, so by the previous proposition it has ω-MDCTP$_2$ and the compatible order property. So T must have ω-DCTP$_2$ and SOP$_3$. □

In other words, the fact that internally NSOP$_1$ types are co-NSOP$_1$ in NSOP$_3$ theories can be interpreted as saying that in an NSOP$_3$ theory, the graph-theoretic complexity of a characteristic sequence must be reflected in its model-theoretic complexity in the hypergraph language.

Remark 4.3.15. For R_∞ a hypergraph sequence, define $R^{(m)}_\infty$ to be the hypergraph sequence whose vertices are m-tuples of vertices of R_∞, and define $R^{(m)}_n((a^1_1, \ldots a^m_1), \ldots (a^1_n, \ldots a^m_m)) =: R_{mn}(a^1_1, \ldots a^m_1, \ldots a^1_n, \ldots a^m_m)$. For example, if R_∞ is the characteristic sequence of $\varphi(x, y)$, then $R^{(m)}_\infty$ is the characteristic sequence of $\wedge^m_{i=1}\varphi(x, y_i)$. If we consider hypergraphs up to the concatenation operation $R_\infty \mapsto R_\infty^{(m)}$, then we obtain additional information. For example, we can define $\mathrm{MSOP}_2 = \mathrm{MTP}_1$ to mean that there exists a binary (or infinitely branching; see [2], recounted in Fact 4.2 of [28]) tree whose paths are cliques and whose incomparable pairs are 2-anticliques. It follows from the proof of [97], III.7.7, III.7.11 that up to concatenation, a hypergraph sequence with MTP either has an $(\omega, \omega, 1)$ array or has MTP_1. So by Proposition 4.3.12.2, if R_∞ is NSOP_1, and has MTP, then up to concatenation it either has MTP or an $(\omega, \omega, 1)$ array.

It is also worth noting that if we define MATP to be the existence of a tree so that the antichains are cliques and the paths are k-anticliques, it follows from the proof of Theorem 4.8 of [3] that up to concatenation, an MSOP_1 hypergraph sequence has either MSOP_2 or MATP.

Although it is shown in Chapter 3 that NSOP_1 coincides with NSOP_2 for theories, we get additional graph-theoretic information when we assume only the quantifier-free formulas of R_∞ to be NSOP_2. Namely, if R_∞ has MSOP_1, it has MSOP_2 up to concatenation. This follows from the arguments in Chapter 3; we give an overview. If R_∞ has MSOP_1, there is a model M and two M-finitely satisfiable Morley sequences, one of which is a clique and one of which is a 2-anticlique. (That k may chosen to be 2 comes from [74], [52]). Now Lemma 3.4.3 says that for any coheir $p(x)$ over M and canonical coheir $q(x)$ over M, there is a tree whose paths are Morley sequences in $p(x)$ and whose descending combs are Morley sequences in $q(x)$. So if Morley sequences in $q(x)$ are anticliques and Morley sequences in $p(x)$ are cliques, the descending combs will be anticliques and the paths will be cliques. By the proof of Lemma 3.2.3 (SOP_2 = k-DCTP_1), such a tree, up to concatenation, gives an instance of MSOP_2.

So because there exists a finitely satisfiable Morley sequence that is a clique, either R_∞ has MSOP_2 and we are done, or there is also a *canonical* Morley sequence that is a clique. At this point, now that we have an M-finitely satisfiable Morley sequence that is a 2-anticlique and a canonical Morley sequence that is a clique, we can prove Kim's lemma for canonical Morley sequences, symmetry for Conant-independence, and the weak independence theorem for Conant-independence, all in the quantifier-free context as for NSOP_2 theories, and then use the technique of Conant ([32]) developed in Chapter 3.6 to show the compatible order property. But up to concatenation, the compatible order property implies MSOP_2 ([77], Observation 3.11).

There are SOP_1 formulas $\varphi(x, y)$ so that $\wedge_{i=1}^{n} \varphi(x, y_i)$ is NSOP_2; see [3], §6. Byunghan Kim asked, at the 2023 BIRS meeting on neostability theory, whether it can be shown that if a formula is SOP_1, a related formula is SOP_2. It follows from the above discussion that if a formula is SOP_1, SOP_2 must appear in the quantifier-free formulas of its characteristic sequence.

4.4 Independence of internally NSOP_1 types in NSOP_3 theories

In this section, we prove an extension of the independence theorem of Kaplan and Ramsey ([52]) to internally NSOP_1 types in NSOP_3 theories. We will use Theorem 4.3.7, namely that internally NSOP_1 types in NSOP_3 theories are co-NSOP_1.

While the theorem does not give $a_1'' \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_2a_2$, a_1'' can be chosen so that any two of a_1'', a_2, a_3 is Conant-independent from the third, somewhat similarly to Theorem 2.13 of [71].

Theorem 4.4.1. *Let T be NSOP_3, and let p_1, p_2, p_3 be internally NSOP_1 types over M. Let $a_1 \equiv_M a_1' \subset p_1(\mathbb{M})$, $a_2 \subset p_2(\mathbb{M})$, $a_3 \subset p_3(\mathbb{M})$. If $a_1 \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_2$, $a_1' \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_3$, $a_2 \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_3$, there is some a_1'' with $a_1'' \models \mathrm{tp}(a_1/Ma_2) \cup \mathrm{tp}(a_1'/Ma_3)$. Moreover, a_1'' can be chosen with $a_2a_3 \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_1''$, $a_2a_1'' \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_3$ and $a_3a_1'' \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_2$.*

It is of interest that the conclusion does not hold for NSOP_4 theories, nor does it follow from co-NSOP_1.

Example 4.4.2. Let T be the model companion of the theory of triangle-free tripartite graphs, with the partition denoted by $P_1(x), P_2(x), P_3(x)$ as in Example 4.3.8. Recall that T is NSOP_4, and T is a free amalgamation theory in the sense of [32], so $a \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M b$ if and only of $a \cap b \subseteq M$; see Proposition 1.4. Let $p_i(x) =: P_i(x)$ for $i = 1, 2, 3$. Then the $p_i(x)$ are internally stable–the structures $\mathcal{M}_{p_i}$ have quantifier elimination in the unary language of M-definable subsets of $P_i(x)$. Internally stable types are always co-NSOP_1: by the proof of Theorem 4.3.7, if an internally stable type p is not co-NSOP_1, then in the theory of $\mathcal{M}_p$, there is a hypergraph sequence $\{R_n\}$, a model $\mathcal{M}$, and Morley sequences $\{b_i\}_{i<\omega}$ and $\{b_i'\}_{i<\omega}$ with $b_0 \equiv_{\mathcal{M}} b_0'$ so that $\{b_i\}_{i<\omega}$ is a clique and $\{b_i'\}_{i<\omega}$ is an anti-clique. But this is impossible if the theory of $\mathcal{M}_p$ is stable, as $b_0 \equiv_{\mathcal{M}} b_0'$ implies $\{b_i\}_{i<\omega} \equiv_M \{b_i'\}_{i<\omega}$ when $\{b_i\}_{i<\omega}$ and $\{b_i'\}_{i<\omega}$ are Morley sequences in a stable theory.

However, the conclusion of theorem 4.4.1 does not hold. Let $a_1 \equiv_M a_1' \subseteq p_1(\mathbb{M})$, $a_2 \models p_2(\mathbb{M})$, $a_3 \models p_3(\mathbb{M})$ with $\models a_1Ea_2$, $\models a_1'Ea_3$, $\models a_2Ea_3$. Then $a_1 \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_2$, $a_1' \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_3$, $a_2 \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M a_3$, but $\mathrm{tp}(a_1/Ma_2) \cup \mathrm{tp}(a_1'/Ma_3)$ is inconsistent.

We prove theorem 4.4.1, beginning with some observations on co-NSOP_1 types. First of all, Conant-independence between co-NSOP_1 types is just Kim-dividing independence.

Claim 4.4.3. *If $b \subset p(\mathbb{M})$ for $p(x)$ a co-NSOP_1 type over M, then $a \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M b$ if and only if $a \mathop{\smile\!\!\!\!\!\!\mid}\,^{Kd}_M b$.*

Proof. If $a \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M b$, then $a \mathop{\smile\!\!\!\!\!\!\mid}\,^{Kd}_M b$ by Kim's lemma, Definition 3.1.3. Conversely, if $a \mathop{\smile\!\!\!\!\!\!\mid}\,^{Kd}_M b$, then in particular, by compactness we can choose an M-finitely satisfiable Morley sequence $\{b_i\}_{i<\omega}$, $b_0 = b$, that is indiscernible over Ma. But then, by Fact 4.5.7 below, formulas that do not Kim-divide over M by a some M-finitely satisfiable Morley sequence do not Conant-fork over M, so $a \mathop{\smile\!\!\!\!\!\!\mid}\,^{K^*}_M b$. □

Claim 4.4.4. *If $p(x), q(x)$ are co-*NSOP$_1$ *types over M and $a \subseteq p(\mathbb{M})$, $b \subseteq q(\mathbb{M})$, then $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M b$ if and only if $b \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M a$.*

This follows from Proposition 3.1.

Claim 4.4.5. *If $b \subset p(\mathbb{M})$ for $p(x)$ a co-*NSOP$_1$ *type over M, and $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M b$, then for any M-finitely satisfiable type $q(x)$, there is a Morley sequence $I = \{b_i\}_{i<\omega}$ in $q(x)$, $b_0 = b$, that is indiscernible over Ma, and any such Morley sequence will satisfy $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M I$.*

This is the "chain condition", Claim 4.3.4, together with compactness. We use Claims 4.4.3-4.4.5 throughout.

Third, we have the weak independence theorem between *two* co-NSOP$_1$ types, analogously to Proposition 6.1 of Kaplan and Ramsey, [52]:

Claim 4.4.6. *Let $p(x), q(x)$ be co-*NSOP$_1$ *types over M, and let $a \equiv_M a' \subseteq p(\mathbb{M})$, $b, c \subseteq q(\mathbb{M})$, $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M b$, $a' \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M c$, $c \mathop{\smile\!\!\!\!\!\!|}\,^{u}_M b$. Then there is $a'' \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M bc$ with $a'' \models \mathrm{tp}(a/Mb) \cup \mathrm{tp}(a'/Mc)$.*

Proof. The proof is similar to Proposition 6.1 of [52]. By Claims 4.4.4 and 4.4.5 and $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M b$, let $I = \{a_i\}_{i<\omega}$ be an M-invariant Morley sequence with $a_0 = a$ that is indiscernible over Mb. Again by Claims 4.4.4 and 4.4.5., $a' \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M c$ and $a' \equiv_M a$, for $r(x, y) = \mathrm{tp}(a', c)$, $\cup_{i<\omega} q(a_i, y)$ is consistent, so we can choose some $c' \models \cup_{i<\omega} r(a_i, y)$. By Ramsey's theorem, compactness and an automorphism, we can choose c' in particular so that I remains indiscernible over Mbc'. So $bc' \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M a$, and $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M bc'$, with $c'a \equiv_M ca'$.

Let $s(y)$ be an M-finitely satisfiable type extending $\mathrm{tp}(c/Mb)$, and let $c'' \models s(y)|_{Mbc'}$, so $c'' \equiv_{Mb} c$ and $c'' \mathop{\smile\!\!\!\!\!\!|}\,^{u} bc'$. As $c'' \equiv_M c$, choose b' with $b'c'' \equiv_M bc'$; by left extension for $\mathop{\smile\!\!\!\!\!\!|}\,^{u}$, b' can further be chosen with $b'c'' \mathop{\smile\!\!\!\!\!\!|}\,^{u} bc'$. Then $bc', b'c''$ begin an M-invariant Morley sequence J. As $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*} bc'$, there is an M-invariant Morley sequence $J' \equiv_{bc'} J$ indiscernible over Ma; using claim 4.4.5, $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*} J'$. Write $J' = (bc', b'''c''', \ldots)$. Then $c'''a \equiv_M c'a \equiv_M ca'$, $c''' \equiv_{Mb} c'' \equiv_{Mb} c$ and $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*} bc'''$. By an Mb-automorphism taking c'' to c, we obtain a' as desired.

□

Using this weak independence theorem, we can now show the full independence theorem between *two* co-NSOP$_1$ types.

Claim 4.4.7. *Let $p(x), q(x)$ be co-*NSOP$_1$ *types over M, and let $a \equiv_M a' \subseteq p(\mathbb{M})$, $b, c \subseteq q(\mathbb{M})$, $a \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M b$, $a' \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M c$, $c \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M b$. Then there is $a'' \mathop{\smile\!\!\!\!\!\!|}\,^{K^*}_M bc$ with $a'' \models \mathrm{tp}(a/Mb) \cup \mathrm{tp}(a'/Mc)$.*

Proof. We could have followed the proof of Theorem 6.5 of [52], but we offer our own exposition. We first show that it suffices to show consistency of $\mathrm{tp}(a/Mb) \cup \mathrm{tp}(a'/Mc)$ in the statement of Claim 4.4.9. Let $r(x)$ be an M-finitely satisfiable type extending $\mathrm{tp}(A/M)$. By Claims 4.4.4 and 4.4.5, we can find $I = \{a_i\}_{i<\omega}$ with $a_0 = a$ indiscernible over Mb and $I' = \{a'_i\}_{i<\omega}$ with $a'_0 = a'$ indiscernible over Mb, both M-finitely satisfiable Morley sequences in $r(x)$. Then $I \equiv_M I'$. So using the consistency, we can find $I'' \models (I/Mb) \cup \mathrm{tp}(I'/Mc)$, which by Ramsey's theorem and compactness, can be assumed indiscernible over Mbc. So $I'' \mathop{\smile\!\!\!\!\!\!|}^K_M bc$ by claim 4.4.4, and we can find a'' in this sequence as desired.

Therefore, suppose a, a', b, c are as in the statement of the claim, and $\mathrm{tp}(a/Mb) \cup \mathrm{tp}(b/Mc)$ is inconsistent. By compactness, there are some $\varphi(x, b) \in \mathrm{tp}(a/Mb)$ and $\psi(x, c) \in \mathrm{tp}(a/Mc)$ with $\varphi(x, b) \cup \psi(x, c)$ inconsistent. Let $s(y, z) = \mathrm{tp}(bc/M)$. We find $b_1, \ldots, b_n \ldots \subset q(\mathbb{M})$, $c_1, \ldots, c_n \subset q(\mathbb{M})$ with the following properties

(1) For $m < n$, $b_i \mathop{\smile\!\!\!\!\!\!|}^u_M b_{i-1} \ldots b_1$ for $i \leq m$, and $c_i \mathop{\smile\!\!\!\!\!\!|}^u_M c_{i-1} \ldots c_{m+1} b_m \ldots b_1$ for $m \leq i \neq n$. Thus by repeated applications of Claim 4.4.9, $\{\varphi(x, b_1), \ldots \varphi(x, b_m), \psi(x, c_{m+1}), \ldots, \psi(x, c_n)\}$ is consistent.

(2) For $i < j$, $b_i c_j \models s(y, z)$, so $\varphi(x, b_i)\psi(y, c_j)$ is inconsistent.

(3) $b_1 \ldots, b_n \ldots \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M c_1 \ldots, c_n \ldots$.

By Fact 4.2.4, this will give us a failure of co-NSOP$_1$, a contradiction. We use the technique of Conant, [32] (though it is not yet necessary to get SOP$_3$; this technique is similar to the "zig-zag lemma," Lemma 6.4, from the original proof of the independence theorem in [52]). Assume $b_1, \ldots, b_n, c_1, \ldots, c_n$ already constructed, satisfying these properties up to n (including $c_n \equiv_M c$). By repeated instances of Claim 4.4.9 (applied to $q(x), q(x)$), and $b \mathop{\smile\!\!\!\!\!\!|}^K_M c$, there is $b'_{n+1} \mathop{\smile\!\!\!\!\!\!|}^K c_1 \ldots c_n$ with $b'_{n+1} \models \cup_{i=1}^n s(y, c_i)$. Again by Claim 4.4.9, $b_n \ldots b_1 \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M c_1, \ldots c_n$, and an automorphism, we can additionally choose $b'_{n+1} = b_{n+1}$ so that $b_{n+1} \mathop{\smile\!\!\!\!\!\!|}^u_M b_1 \ldots b_n$ and $b_{n+1} \ldots b_1 \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M c_1, \ldots c_n$. Now choose $c_{n+1} \equiv_M c$ with $c_{n+1} \mathop{\smile\!\!\!\!\!\!|}^u c_1 \ldots c_n b_1 \ldots b_{n+1}$. We get $b_{n+1} \ldots b_1 \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M c_1, \ldots c_{n+1}$, by the proof of Claim 4.5.13. □

We first show that the "moreover" clause follows from the conclusion. Let $a_1 \equiv_M a'_1 \subseteq p_1(\mathbb{M})$, $a_2 \models p_2(\mathbb{M})$, $a_3 \models p_3(\mathbb{M})$ be as in the hypotheses of the theorem. As $a_2 \mathop{\smile\!\!\!\!\!\!|}^{K^*} a_3$, by Claims 4.4.3 and 4.4.5, there is an M-finitely satisfiable Morley sequence $I_2 = \{a_2^i\}_{i<\omega}$ with $a_2^0 = a_2$ that is indiscernible over Ma_3 and $I_2 \mathop{\smile\!\!\!\!\!\!|}^{K^*} a_3$. Likewise, there is an M-finitely satisfiable Morley sequence $I_3 = \{a_3^i\}_{i<\omega}$ with $a_3^0 = a_3$ that is indiscernible over MI_2 and with $I_2 \mathop{\smile\!\!\!\!\!\!|}^{K^*} I_3$. By $a_1 \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M a_2$ and an automorphism, we can find $a_1^* \equiv_{Ma_2} a_1$ with $a_1^* \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M I_2$ and I_2 indiscernible over Ma_1^*, so we can assume $a_1 \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M I_2$ and I_2 is indiscernible over Ma_1. Similarly, we can assume $a'_1 \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M I_3$ and I_3 is indiscnernible over Ma'_1. Fix an M-finitely satisfiable type $q(x)$ extending $\mathrm{tp}(a_1/M)$. Then there is a Morley sequence $I_1 = \{a_1^i\}_{i<\omega}$ in $q(x)$ with $a_1^0 = a_1$, $I_1 \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M I_2$ and I_1 indiscernible over I_2. There is also a Morley sequence $I'_1 = \{a_1^i\}_{i<\omega}$ in $q(x)$ with $a_1'^0 = a'_1$, $I'_1 \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M I_3$ and I'_1 indiscernible over I_3. Since the Morley sequences were chosen to be in the same M-finitely satisfiable type, $I_1 \equiv_M I'_1$. So applying the consistency part of the theorem, we can find $\{a_1''^i\}_{i<\omega} = I''_1 \models \mathrm{tp}(I_1/MI_2) \cup \mathrm{tp}(I'_1/MI_3)$. For $j, k, \ell < \omega$ $a_1''^j a_2^k \equiv_M a_1 a_2$, $a_1''^j a_3^\ell \equiv_M a'_1 a_3$, $a_2^k a_3^\ell \equiv_M a_2 a_3$. We apply the following case of Lemma 1.2.1 [26]:

Fact 4.4.8. *Let $I''_1 = \{a_1''^i\}_{i<\omega}$, $I_2 = \{a_2^i\}_{i<\omega}$, $I_3 = \{a_3^i\}_{i<\omega}$ be indiscernible sequences over M. Then there are mutually indiscernible $I'''_1 = \{a_1'''^i\}_{i<\omega}$, $I'''_2 = \{a_2'''^i\}_{i<\omega}$, $I'''_3 = \{a_3'''^i\}_{i<\omega}$, (i.e. each I'''_m indiscernible over $MI'''_{\neq m}$) so that for any formula $\varphi(\overline{x}, \overline{y}, \overline{z}) \in L(M)$, if for all*

$\overline{j},\overline{k},\overline{l}$ with $j_1 < \ldots < j_n < \omega$, $k_1 < \ldots < k_n < \omega$, $\ell_1 < \ldots < \ell_n < \omega$, $\models \varphi(\overline{a}_1^{\prime\prime\overline{j}}, \overline{a}_2^{\overline{k}}, \overline{a}_3^{\overline{\ell}})$, then for all such $\overline{j},\overline{k},\overline{l}$, $\models \varphi(\overline{a}_1^{\prime\prime\prime\overline{j}}, \overline{a}_2^{\prime\prime\prime\overline{k}}, \overline{a}_3^{\prime\prime\prime\overline{\ell}})$

Let $I_1''' = \{a_1'''^i\}_{i<\omega}$, $I_2''' = \{a_2'''^i\}_{i<\omega}$, $I_3''' = \{a_3'''^i\}_{i<\omega}$ be as in Fact 4.3. Then $a_2'''^0 a_3'''^0 \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} a_1'''^0$, $a_1'''^0 a_2'''^0 \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} a_3'''^0$, and $a_1'''^0 a_3'''^0 \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} a_2'''^0$, and $a_1'''^0 a_2'''^0 \equiv_M a_1 a_2$, $a_1'''^0 a_3'''^0 \equiv_M a_1' a_3$, $a_2'''^0 a_3'''^0 \equiv_M a_2 a_3$. So by an automorphism, we find a_1'' as desired in the "moreover" clause.

We finally show the actual consistency part of the theorem. Let $q_1(y,z) = \mathrm{tp}(a_2 a_3/M)$, $q_2(x,z) = \mathrm{tp}(a_1' a_3/M)$, $q_3(x,y) = \mathrm{tp}(a_1' a_2/M)$. By an automorphism, it sufficies to show that $q_1(y,z) \cup q_2(x,z) \cup q_3(x,y)$ is consistent. This will require NSOP$_3$; formally, we will use the technique of Evans and Wong, from Theorem 2.8 of [42]. Call $A \subset p_1(\mathbb{M})$, $B \subset p_2(\mathbb{M})$, $C \subset p_3(M)$ a *generic triple* if there are mutually indiscernible M-invariant Morley sequences $I_A = \{A_i\}_{i<\omega}$ with $A_0 = A$, $I_B = \{B_i\}_{i<\omega}$ with $B_0 = B$, $I_C = \{C_i\}_{i<\omega}$ with $C_0 = C$; note that it follows that A, B and C are pairwise Conant-independent.

Claim 4.4.9. *Let A, B, C be a generic triple, and $b \subseteq p_2(\mathbb{M})$ such that $A \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} b$. Then there is some $b' \equiv_{MA} b$ with $b' \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} B$ so that A, Bb', C form a generic triple.*

Proof. Let I_A, I_B, I_C be as in the definition of a generic triple.

Subclaim 4.4.10. *There is $b'' \equiv_M b$ and $I_{b''} = \{b_i''\}_{i<\omega}$ with $b_0'' = b''$ so that $\{B_i b_i''\}_{i<\omega}$ forms an invariant Morley sequence over M and $I_B \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} I_{b''}$.*

Proof. Let I_B be a Morley sequence in the M-invariant type $p(x)$. Chose an M-invariant type $q(x)$ extending $\mathrm{tp}(b/M)$. By an automorphism, there is $I_{b'} = \{b_i'\}_{i<\omega}$ so that, for $n < \omega$, $b_n' B_n \ldots b_0' B_0 \models (q(x) \otimes p(x))^{(n)}$. So $\{B_i b_i''\}_{i<\omega}$ is an M-invariant Morley sequence. Let $b_0'' = b''$; then $b'' \equiv_M b$. Finally, we show that $I_B \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} I_{b''}$. Suppose by induction that $B_0 \ldots B_n \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} b_0' \ldots b_n'$ Note that $\mathrm{tp}(B_{n+1}/MB_0 \ldots B_n b_0' \ldots b_n')$ extends to an M-invariant type, and $\mathrm{tp}(b_{n+1}'/MB_0 \ldots B_n B_{n+1} b_0' \ldots b_n')$ also extends to an M-invariant type. By the proof of Claim 4.5.13 below, we see that for any sets e, f, g of realizations of a common co-NSOP$_1$ type over M, if $e \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} f$ and $\mathrm{tp}(g/Mef)$ extends to an M-invariant type $q(x)$, then $e \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} fg$. So by two applications of this fact and symmetry (claim 4.4.4), $B_0 \ldots B_n B_{n+1} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} b_0' \ldots b_n' b_{n+1}'$. This completes the induction, from which it follows that $I_B \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} I_{b''}$. □

Let b'' be as in the subclaim. As $A \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} b$, for $p(X,y) = \mathrm{tp}(Ab/M)$, by claim 4.4.5 and an automorphism there is $A' \models \cup_{i<\omega} p(X, b_i'')$ with $A' \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} I_{b''}$. By claims 4.4.4 and 4.4.5, we can then find $\{A_i'\} = I_{A'} \equiv_M I_A$ indiscernible over $MI_{b''}$ with $A_0' = A'$ and $I_{A'} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} I_{b''}$. So we have $I_{A'} \equiv_M I_A$ and $I_{A'} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} I_{b''}$, $I_A \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} I_B$ by indiscernibility of I_B over I_A and claim 4.4.5, and $I_B \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_M^{K^*} I_{b''}$ by the subclaim. So by the independence theorem between two co-NSOP$_1$ types (Claim 4.4.7) and an automorphism, there is some $\{b_i^*\}_{i<\omega} = I_{b''}^*$ with $I_{b''}^* \equiv_{MI_B} I_{b''}$ and $I_{b''}^* I_A \equiv_M I_{b''} I_{A'}$. The sequence $\{b_i^*\}_{i<\omega} = I_{b''}^*$ will have the following three properties: $b_i^* \models p(A_j, y)$, so $b_i^* \equiv_{MA_j} b$, for $i, j < \omega$, $\{B_i b_i^*\}_{i<\omega}$ form an M-invariant Morley sequence, and $b_i^* \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K B_i$ for $i < \omega$. If we extract mutually indiscernible sequences from $I_A, I_B I_{b''}^*, I_C$, finding $\check{I}_A, \check{I}_B \check{I}_{b''}^*, \check{I}_C$ as in Fact 4.4.8, then $I_A I_B I_C \equiv_M \check{I}_A \check{I}_B \check{I}_C$, so we may assume $\check{I}_A = I_A$, $\check{I}_B = I_B$, $\check{I}_C = I_C$ and then $\check{I}_{b''}^* = \{\check{b}_i^*\}_{i<\omega}$ will also have these three properties that $\{b_i^*\}_{i<\omega} = I_{b''}^*$ has. Let $b' = \check{b}_0^*$. Then $b' \equiv_{MA} b$, $b' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K B$ and $I_A, I_B \check{I}_{b''}^*, I_C$ will be mutually indiscernible M-invariant Morley sequences, so A, Bb', C form a generic triple. □

Now we find $a^1, \ldots, a^n, \ldots \models \mathrm{tp}(a_1/M)$, $b^1, \ldots, b^n, \ldots \models \mathrm{tp}(a_2/M)$, and $c^1, \ldots, c^n, \ldots \models \mathrm{tp}(a_3/M)$ with the following properties:

(1) For $i < j$, $a^j c^i \models q_2(x, z)$, $a^i b^j \models q_3(x, y)$, $b^i c^j \models q_1(y, z)$.

(2) For $i < \omega$, $a^i \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K a^1 \ldots a^{i-1}$, $b^i \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K b^1 \ldots b^{i-1}$, and $c^i \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K c^1 \ldots c^{i-1}$

(3) For each $n < \omega$ $a^1 \ldots a^n$, $b^1 \ldots b^n$, $c^1 \ldots c^n$ form a generic triple.

Assume $a^1, \ldots, a^n$, $b^1, \ldots, b^n$, $c^1, \ldots, c^n$ already constructed, satisfying these properties up to n. As for $i \leq n$, $a^i \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K a^1 \ldots a^{i-1}$, we can find some $b \models \cup_{i=1}^n q_3(a^i, y)$ with $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^K a^1 \ldots a^n$, by, say, repeated applications of the independence theorem between two co-NSOP_1 types, Claim 4.4.7 (though could have stated the claim so that we need less than this). Then letting $a^1, \ldots, a^n = A$, $b^1, \ldots, b^n = B$, $c^1, \ldots, c^n = C$, we can choose $b_{n+1} = b'$ as in Claim 4.4.9, while will be as desired. Symmetrically, we find c_{n+1} and a_{n+1}.

Now let

$$\Phi(x^1, y^1, z^1; x^2, y^2, z^2) = q_2(x^2, z^1) \cup q_3(x^1, y^2) \cup q_1(y^1, z^2)$$

By (1) this has infinite chains, so by NSOP_3 and compactness it has a 3-cycle: some

$$\begin{gathered}(d^1, e^1, f^1, d^2, e^2, f^2, d^3, e^3, f^3) \\ \models \Phi(x^1, y^1, z^1; x^2, y^2, z^2) \cup \Phi(x^2, y^2, z^2; x^3, y^3, z^3) \cup \Phi(x^3, y^3, z^3; x^1, y^1, z^1)\end{gathered}$$

In particular, $(d^1, e^2, f^3) \models q_3(x, y) \cup q_1(y, z) \cup q_2(x, z)$, as desired. This concludes the proof of Theorem 4.2.

4.5 NSOP_3 theories with symmetric Conant-independence

In [104], Simon proves the following independence theorem for NTP_2 theories, using the independence theorem for NTP_2 theories of Ben Yaacov and Chernikov ([113]).

Fact 4.5.1. *Let T be NTP_2, and let $c \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^f ab$ and $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^f a$. Let $b' \equiv_M b$ with $b' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^f a$. Then there is some $c' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_M^f ab'$ with $c'a \equiv_M ca$ and $c'b' \equiv_M cb$.*

(In fact, Simon proves a more general version of this over extension bases.) He then poses the question

Question 4.5.2. *Suppose p and q are M-invariant types in an* NTP$_2$ *theory with $p^\omega|_M = q^\omega|_M$, and let $B, C \supseteq M$ be small supersets of M. For some/every $B' \equiv_M B$ so that $B' \mathop{\smile\!\!\!\!\!\!|}^f_M C$, is there $a \models p(x)|_{B'} \cup q(x)|_C$ with $a \mathop{\smile\!\!\!\!\!\!|}^f_M B'C$?*

This is true for simple theories by the independence theorem for simple theories ([65]), and for NIP theories because $p^\omega(x)|_M$ determines any invariant type $p(x)$ (Proposition 2.36 of [103]); fact 4.5.1 justifies the equivalence of "some" with "any" B'. We show that a similar property holds for finitely satisfiable types in NSOP$_3$ theories with symmetric Conant-independence:

Theorem 4.5.3. *Let T be an* NSOP$_3$ *theory, and assume $\mathop{\smile\!\!\!\!\!\!|}^{K^*}$ is symmetric. Suppose p and q are M-finitely satisfiable (global) types with $p^\omega|_M = q^\omega|_M$, and let $a, b \supseteq M$ be small supersets of M with $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$. Then there is $c \models p(x)|_a \cup q(x)|_b$ with $c \mathop{\smile\!\!\!\!\!\!|}^{K^*}_M ab$*

The "some" part, the analogue of a positive answer to Question 4.5.2, will be supplied by the symmetry of Conant-independence. Then the "every" part, correspnding to Fact 4.5.1, will follow from NSOP$_3$. Before proceeding, we will show this fails for NSOP$_4$ theories with symmetric Conant-independence.

Example 4.5.4. The model companion T of the theory of triangle-free graphs has NSOP$_4$ and symmetric Conant-independence; see Chapter 1. If p is a nonalgebraic M-finitely satisfiable type, $p|^\infty_M$ is determined by $p|_M$: By indiscernibility, $\neg x_i E x_j \in p^\omega(\bar{x})$ for $i < j$, as $\neg x_i E x_j \in p^\omega(\bar{x})$ for all $i < j$ is impossible.

Next, we claim that, if M is countable, for $p_0(x) \in S_1(M)$ the complete type over M containing $\neg p_0 E m$ for all $m \in M$, there are M-finitely satisfiable types p_1 and p_2 extending $p_0(x)$, and $xEb_i \in p_i(x)$ for $i = 1, 2$, $b_i \notin M$, so that there is no $m \in M$ with $b_1 E m \wedge b_2 E m$. Let $\{S_i\}_{i\in\omega}$ enumerate the set F of subsets of M defined by M-formulas in $p_0(x)$. We choose, by induction, disjoint anticliques A, B of M, both of which meet each of the S_i. Namely, we construct disjoint anticliques A_n, B_n for $n \in \omega$, so that $A_n \cap S_i \neq \emptyset$ and $B_n \cap S_i \neq \emptyset$ for $i \leq n$ and $A_i \subseteq A_j$ and $B_i \subseteq B_j$ for $i \leq j$, and take $A = \cup_{i=0}^\infty A_i$ and $B = \cup_{i=0}^\infty B_i$. Suppose A_n, B_n already constructed. Since S_{n+1} is defined by a conjunction of formulas of the form $x \neq m$ and $\neg x E m$ for $m \in M$, and M is a model of the model companion of the theory of triangle-free graphs, we can find distinct $a_{n+1}, b_{n+1} \in S_{n+1} \backslash A_n \cup B_n$ so that $\neg a_{n+1} E a$ for any $a \in A_n$, $\neg b_{n+1} E b$ for $b \in B_n$, and take $A_{n+1} = A_n \cup \{a_{n+1}\}$ and $B_{n+1} = B_n \cup \{b_{n+1}\}$. Now let U_1 be an ultrafilter containing $F \cup \{A\}$ and U_2 be an ultrafilter containing $F \cup \{B\}$. Let $p_i(x) = \{\varphi(x, b) \in L(\mathbb{M}) : \varphi(M, b) \in U_i\}$. Let $b_1 \in \mathbb{M}$ be such that, for $m \in M$, $b_1 E m$ if and only if $m \in A$, and similarly for b_2 and B. This is possible because A and B are anticliques. Then p_1, p_2, b_1, b_2 are as desired in the claim.

There is an invariant type q extending $\mathrm{tp}(b_1/M)$ so that, for $b_1' \models q(x)|_{Mb_2}$, $b_1' E b_2$; for example, we can require that $xEb \in q(x)$ if and only if $xEb \in \mathrm{tp}(b_1/M)$ or $b \models \mathrm{tp}(b_2/M)$. This gives a consistent type: let a_* be a node satisfying these relations in a graph extending $\mathbb{M}$; then there are no triangles involving a_*, a realization of $\mathrm{tp}(b_2/M)$ in $\mathbb{M}$, and an element of M, because we chose A and B to be disjoint; there are also no edges between realizations of $\mathrm{tp}(b_2/M)$ in $\mathbb{M}$, because B is nonempty and there are no triangles in $\mathbb{M}$, so there are no triangles involving a_* and two realizations of $\mathrm{tp}(b_2/M)$ in $\mathbb{M}$. Let $b_1' \models q(x)|_{Mb_2}$; then $b_1' \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^K b_2$ and $b_1' E b_2$.

But $p_1(x)|Mb_1' \cup p_2(x)|Mb_2$ is inconsistent.

We first study theories where Conant-independence is symmetric. Naïvely, one expects it to follow from compactness that $a \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{K^*}_M b$ implies the existence of an Ma-indiscernible M-invariant Morley sequence starting with b. This naïve argument fails, because the property of being an invariant Morley sequence of realizations of a fixed complete type over M is not type-definable. However, the following proposition about theories with symmetric Conant-independence is enough for our purposes:

Lemma 4.5.5. *Suppose $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{K^*}$ is symmetric, and let $I = \{a_i\}_{i\in\omega}$ be a coheir Morley sequence over M with $a_0 = a$ that is indiscernible over Mb. Then there is an M-invariant Morley sequence $J = \{b_i\}_{i\in\omega}$ with $b_0 = b$ that is indiscernible over Ma.*

Proof. The main claim of this proof is the following;

Claim 4.5.6. *There exists $b' \equiv_{MI} b$ with $b' \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^i b$ so that I remains indiscernible over bb'.*

Proof. We first show that $b \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{K^*}_M I$. We need the following fact:

Fact 4.5.7. *(Fact 1.6.1) Let $\{c_i\}_{i\in I}$ be a coheir Morley sequence over M with $c_0 = C$ so that $\{\varphi(x, c_i)\}_{i\in\omega}$ is consistent. Then $\varphi(x, c)$ does not Conant-fork over M.*

Now suppose $\varphi(x, \bar{a}) \in \mathrm{tp}(b/MI)$ for $\bar{a} = a_0 \dots a_n$. Then $\{\bar{a}_i\}_{i\in\omega}$ for $\bar{a}_i = a_{ni} \dots a_{ni+(n-1)}$ is a finitely satisfiable Morley sequence over M with $\bar{a}_0 = \bar{a}$ so that $\{\varphi(x, \bar{a}_i)\}_{i\in\omega}$ is consistent, so by the fact, $\varphi(x, \bar{a})$ does not Conant-fork over M and $b \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{K^*}_M I$ is as desired. (See the proof of Proposition 3.5.2, or Proposition 3.21 of [52].)

Let $q(\bar{x}, b) = \mathrm{tp}(I/Mb)$. By symmetry, $I \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{K^*}_M b$, so for every $\varphi(\bar{x}, b) \in q(\bar{x}, b)$, there is some $b' \equiv_M b$ with $b' \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^i_M b$ so that $\{\varphi(\bar{x}, b), \varphi(\bar{x}, b')\}$ is consistent. By compactness, the condition $x \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^i_M b$ is type-definable over Mb (contrast with the remark on invariant Morley sequences in the paragraph immediately preceding the proof of the proposition), so there is $b' \equiv_M b$ with $b' \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^i_M b$ so that $q(\bar{x}, b) \cup q(\bar{x}, b')$ is consistent. By an automorphism, we can assume $b' \equiv_{MI} b$, and by Ramsey's theorem and compactness (and an automorphism), we can assume I' is indiscernible over bb'.

□

We now show by induction that we can find b_i for $i < \kappa$, κ large, so that $b = b_0$, $b_i \equiv_{MI} b$, $b_i \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^i_M b_{<i}$, and I is indiscernible over $Mb_0 \dots b_\lambda$ for $\lambda \leq \kappa$. Suppose we have found b_i for $i < \lambda$ and we find b_λ: By the claim, there are $b'_{<\lambda} \equiv_{MI} b_\lambda$ with $b'_{<\lambda} \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^i_M b_{<\lambda}$ and I indiscernible over $b_{<\lambda} b'_\lambda$. Now let $b_\lambda = b'_0$.

Then by the Erdős-Rado theorem, we can find an MI-indiscernible invariant Morley sequence sequence J over M starting with B, which will in particular be Ma-indiscernible.

□

Remark 4.5.8. Conant-forking is often equal to Conant-dividing at the level of formulas; for example if $\perp\!\!\!\perp^i$ satisfies left extension, or T has the *strong witnessing property* that has no known counterexamples among the NSOP_4 theories (Definition 1.3.5). In particular, we know of no theories where $\perp\!\!\!\perp^{K^*}$ is symmetric and the relation $a \perp\!\!\!\perp^{K^*d}_M b$, defined to hold when $\mathrm{tp}(a/Mb)$ has no Conant-dividing formulas, is not symmetric. If we assume the symmetry of $\perp\!\!\!\perp^{K^*d}$ rather than $\perp\!\!\!\perp^{K^*}$, we can prove lemma 4.5.5 for I an invariant Morley sequence over M rather than a coheir Morley sequence over M; the only difference is that we no longer use Fact 4.5.7 on coheir Morley sequences. If we assume the symmetry of $\perp\!\!\!\perp^{K^*d}$ rather than $\perp\!\!\!\perp^{K^*}$ in Theorem 4.5.3, we can then prove the conclusion when p and q are assumed to be M-invariant types rather than M-finitely satisfiable types and $\perp\!\!\!\perp^{K^*}$ is replaced with $\perp\!\!\!\perp^{K^*d}$, getting something closer to the claim of Simon in [104]; the proof will be exactly the same as the below, except Fact 4.5.7 will not be used.

Lemma 4.5.9. *Assume $\perp\!\!\!\perp^{K^*}$ is symmetric. Let p and q be M-finitely satisfiable types with $p^\omega|_M = q^\omega|_M$, and let $a, b \supseteq M$ be small sets containing M. Then there is some M-invariant type $r \vdash \mathrm{tp}(b/M)$ so that, for any $a_1 \dots a_n$ with $a_i \models \mathrm{tp}(a/M)$ and $b_1, \dots b_m \models r^{(m)}(y)|_{Ma_1 \dots a_n}$, $p(x)|_{a_1,\dots,a_n} \cup q(x)|_{b_1,\dots,b_m}$ is consistent.*

Proof. (See also the proof of Proposition 3.5.5, or Proposition 6.10, [52].) Let $I \models p^\omega|_M = q^\omega|_M$. By an automorphism, there is an $|M| + |a|$-saturated model M' so that $I \models p^\omega|_{M'}$, and also by an automorphism, there is some $b' \equiv_M b$ with $I \models q^\omega|_{b'}$. By Ramsey's theorem and compactness, we can assume I is indiscernible over $M'b'$; now let $c = c_0$ for $I = \{c_i\}_{i \in \omega}$. By lemma 4.5.5, there is an M-invariant type $s(X, y) \vdash \mathrm{tp}(M'b'/M)$ and a Morley sequence $\{M'_i b'_i\}_{i \in I}$ with $M'_0 b'_0 = M'b'$ in $s(X, y)$ that is indiscernible over Mc. In particular, $p(x)|_{M'} \cup q(x)|_{b'_1,\dots,b'_m,\dots}$ is consistent, realized by c. Let $r(y) = s(X, y)|_y$; then for $b''_1, \dots b''_m \models r^{(m)}(y)|_{M'}$, $p(x)|_{M'} \cup q(x)|_{b''_1,\dots,b''_m}$ is consistent. Let $a_1, \dots, a_n$ have $a_i \models \mathrm{tp}(a/M)$ for $i \leq n$, and $b_1, \dots b_m \models r^{(m)}(y)|_{Ma_1,\dots,a_n}$. By $|M| + |a|$-saturation of M', there are $a'_1, \dots a'_n \in M'$ with $a'_1, \dots a'_n \equiv_M a_1, \dots a_n$. Let $b''_1, \dots b''_m \models r^{(m)}(y)|_{M'}$. Then $p(x)|_{a'_1,\dots,a'_n} \cup q(x)|_{b''_1,\dots,b''_m}$ is consistent. But by invariance, $a'_1 \dots a'_n b''_1 \dots b''_n \equiv_M a_1 \dots a_n b_1 \dots b_n$. So $p(x)|_{a_1,\dots,a_n} \cup q(x)|_{b_1,\dots,b_m}$ is consistent, as desired. ☐

We are now ready to prove Theorem 4.5.3. First of all, replacing p with p^ω and q with q^ω, we may assume that $p|_M = q|_M$ is the type of a coheir Morley sequence over M. Now assume $p(x)|_a \cup q(x)|_b$ is consistent, realized by a coheir Morley sequence I. It can be assumed indiscernible over ab by Ramsey's theorem and compactness, so $ab \perp\!\!\!\perp^{K^*}_M I$ by the paragraph immediately following Fact 4.5.7, and $I \perp\!\!\!\perp^{K^*}_M ab$ by symmetry. So it suffices to show $p(x)|_a \cup q(x)|_b$ is consistent. Suppose otherwise, so there are $\varphi(x, a) \in p(x)|_a$ and $\psi(x, b) \in q(x)|_b$ such that $\{\varphi(x, a), \psi(x, b)\}$ is inconsistent. Let $s(w, y) = \mathrm{tp}(a, b/M)$, and let $r(y)$ be as in lemma 4.5.9. Once again, we use Conant's technique, Theorem 7.17 of [32]. By induction, we will find $a_1, \dots, a_n, \dots, b_1, \dots, b_n, \dots$ so that

(1) For $i < j$, $a_j b_i \equiv_M ab$, so $\{\varphi(x, a_j), \psi(x, b_i)\}$ is inconsistent and $a_i \models \mathrm{tp}(a/M)$ for each $i \geq 1$.

(2) For $n \leq m$, $b_{n+1}, \dots b_m \models r^{(m-n)}(y)|_{Ma_1 \dots a_n}$, so $p(x)|_{a_1,\dots,a_n} \cup q(x)|_{b_{n+1},\dots,b_m}$ and in particular $\{\varphi(x, a_1), \dots \varphi(x, a_n), \psi(x, b_{n+1}) \dots \psi(x, b_m)\}$ is consistent by lemma 4.5.9.

Assume $a_1, \dots, a_n, b_1, \dots, b_n$ have already been constructed satisfying (1) and (2) up to n. Then $b_1, \dots, b_n$ begin an invariant Morley sequence in $\text{tp}(b/M)$, so because $a \mathop{\smile\!\!\!\!\!|}^K_M b$, $\cup_{i=1}^n s(w, b_i)$ is consistent, and we can take a_{n+1} to realize it. Then we can take $b_{n+1} \models r(y)|_{a_1,\dots,a_{n+1} b_1 \dots b_n}$.

By 4.2.4, properties (1) and (2) imply SOP$_3$–a contradiction. This proves Theorem 4.5.3.

Symmetry of $\mathop{\smile\!\!\!\!\!|}^{K^*}$ is not used directly in building the configuration satisfying (1) and (2); this is in contrast to Chapter 3, where the rows are required to be (coheir) Conant-independent throughout the construction. We now prove a version of Theorem 4.5.3 for Kim-nonforking types over M rather than finitely satisfiable or invariant types over M, that uses the full force of the assumption that the relevant independence relation, in this case $\mathop{\smile\!\!\!\!\!|}^{K^*d}$, is symmetric.

By remark 4.5.8, in an NSOP$_3$ theory where Conant-forking coincides with Conant-dividing and $\mathop{\smile\!\!\!\!\!|}^{K^*}$ is symmetric, Theorem 4.5.3 holds even if p and q are only assumed to be M-invariant types with $p^\omega|_M = q^\omega|_M$, rather than M-finitely satisfiable types. In this case, p and q are examples of types, so that for any small $A, B \supseteq M$, there are M-invariant Morley sequences $I = \{a_i\}_{i \in \omega}$ and $I' = \{b_j\}_{j \in \omega}$, so that $a_i \models p(x)|_A$ and $b_i \models q(x)|_B$ for $i \geq 0$, $I \equiv_M I$, and $I \mathop{\smile\!\!\!\!\!|}^K_M A$ (and $I \mathop{\smile\!\!\!\!\!|}^K_M B$). This assumption can be seen as an analogue of $p^\omega|_M = q^\omega|_M$ for Kim-nonforking types over M, and yields the conclusion of Theorem 4.5.3 with respect to $\mathop{\smile\!\!\!\!\!|}^i$:

Theorem 4.5.10. *Assume $\mathop{\smile\!\!\!\!\!|}^{K^*d}$ is symmetric and T is* NSOP$_3$. *Let $p(x)$ be an M-invariant type, $a, b \supseteq M$ be small supersets of M with $a \mathop{\smile\!\!\!\!\!|}^i_M b$ and I, J M-invariant Morley sequences in $p(x)$ indiscernible over a and b respectively, with $I \mathop{\smile\!\!\!\!\!|}^K_M a$. Then there is some $I'' \mathop{\smile\!\!\!\!\!|}^{K^*d}_M ab$ with $I'' \equiv_a I$ and $I'' \equiv_b I'$. If $\mathop{\smile\!\!\!\!\!|}^f$ (resp. $\mathop{\smile\!\!\!\!\!|}^K$) satisfies the chain condition, the assumption $a \mathop{\smile\!\!\!\!\!|}^i_M b$ can be relaxed to $a \mathop{\smile\!\!\!\!\!|}^f_M b$ (resp. $a \mathop{\smile\!\!\!\!\!|}^K_M b$).*

Note that $\mathop{\smile\!\!\!\!\!|}^f$ is known to satisfy the chain condition in NTP$_2$ theories (Proposition 2.8, [113]). It is not known whether there are non-simple examples of NSOP$_3$ NTP$_2$ theories. (Problem 3.16, [26]).

We start with the analogue of lemma 4.5.5.

Lemma 4.5.11. *Let M, I, J, a, b, be as in the statement of Theorem 4.5.10, and assume $\mathop{\smile\!\!\!\!\!|}^{K^*d}$ is symmetric. Let $p(X, y) = \text{tp}(I, a/M)$ and $q(X, z) = \text{tp}(J, b/M)$. Then there is some invariant type $r \models \text{tp}(b/M)$ so that for $a_1, \dots, a_n$ with $a_i \models \text{tp}(a/M)$ for $i < n$ beginning an invariant Morley sequence over M and $b_1, \dots b_m \models r^{(m)}(y)|_{Ma_1 \dots a_n}$, $\cup_{i=1}^n p(X, a_i) \cup \cup_{i=1}^m q(X, b_i)$ is consistent.*

Proof. Let $\{K^i\}_{i<\kappa}$ enumerate the invariant Morley sequences in $\text{tp}(a/M)$. Since $I \mathop{\smile\!\!\!\!\!|}^K_M a$, $\{p(X, a_i)\}_{i \in \omega}$ is consistent for $\{a_i\}$ any invariant Morley sequence in $\text{tp}(a/M)$, so by automorphisms, there are $\{K'^i\}_{i<\kappa}$ so that for $i < \kappa$ and $K'^i = \{a'^i_j\}_{j \in \omega}$, for $j \in \omega$, $a'^i_j \equiv_{MI} a$. Let

$K' = \cup_{i<\kappa} K'^i$. By another automorphism, find b' with $b'I \equiv bJ$. Then by Ramsey's theorem and compactness, I can be assumed indiscernible over $K'b'$. By remark 4.5.8, there is an M-invariant type $s(X, y) \vdash \mathrm{tp}(K'b'/M)$ and a Morley sequence $\{K'_i b'_i\}_{i<\omega}$ with $K'_0 b'_0 = M'b'$ in $s(X, y)$ that is indiscernible over MI.

In particular, for any $i \leq \kappa$, $\cup_{j<\omega} p(X, a^i_j) \cup \cup_{j<\omega} q(X, b'_j)$ is consistent, realized by I. Let $r(y) = s(X, y)|_y$; then for $b''_1, \ldots b''_m \models r^{(m)}(y)|_{K'}$, for any $i < \kappa$, $\cup_{j<\omega} p(X, a^i_j) \cup \cup_{j\leq m} q(X, b''_j)$ is consistent. Let $a_1, \ldots, a_n$ begin an invariant Morley sequence over M with $a_i \models \mathrm{tp}(a/M)$ for $i \leq n$, and $b_1, \ldots b_m \models r^{(m)}(y)|_{Ma_1,\ldots,a_n}$. Then there are $a'_1, \ldots a'_n \in K'$ with $a'_1, \ldots a'_n \equiv_M a_1, \ldots a_n$. Let $b''_1, \ldots b''_m \models r^{(m)}(y)|_{K'}$. Then $\cup_{j\leq n} p(X, a'_j) \cup \cup_{j\leq m} q(X, b''_j)$ is consistent. But by invariance, $a'_1 \ldots a'_n b''_1 \ldots b''_n \equiv_M a_1 \ldots a_n b_1 \ldots b_n$. So $\cup_{j\leq n} p(X, a_j) \cup \cup_{j\leq m} q(X, b_j)$ is consistent, as desired. □

We fix the auxilliary notation $a \mathop{\smile\!\!\!\!\!\mid}^{K^+}_M b$ to mean that there is an M-invariant Morley sequence $J = \{b_i\}_{i\in\omega}$ with $b_0 = b$ that is indiscernible over Ma. By Remark 4.5.8, if $\mathop{\smile\!\!\!\!\!\mid}^{K^*d}$ is symmetric then so is $\mathop{\smile\!\!\!\!\!\mid}^{K^+}$. We prove the following lemma about $\mathop{\smile\!\!\!\!\!\mid}^{K^+}$ and $\mathop{\smile\!\!\!\!\!\mid}^{K}$:

Lemma 4.5.12. *Let $d_0 \mathop{\smile\!\!\!\!\!\mid}^{K^+}_M c$ and $d_1 \mathop{\smile\!\!\!\!\!\mid}^{K}_M c$. Then there is $d'_1 \equiv_{Mc} d_1$ with $d_0 d'_1 \mathop{\smile\!\!\!\!\!\mid}^{K^+} c$ and $d'_1 \mathop{\smile\!\!\!\!\!\mid}^{i}_M d_0$.*

Proof. (See also the proof of Proposition 3.5.5, or Proposition 6.10, [52].) By $d_0 \mathop{\smile\!\!\!\!\!\mid}^{K^+}_M c$, let $I = \{c_i\}_{i\in I}$ with $c_0 = c$ be an M-invariant Morley sequence indiscernible over d_0. By $d_1 \mathop{\smile\!\!\!\!\!\mid}^{K}_M c$ and compactness, there is some d''_1 with $d''_1 c_i \equiv_M d_1 c_i$ for $i < \omega$. By Ramsey's theorem, compactness, and an automorphism, we can choose d''_1 so that I is indiscernible over $d_0 d''_1$, so $d_0 d''_1 \mathop{\smile\!\!\!\!\!\mid}^{K^+}_M I$: if I is a Morley sequence in the M-invariant type s, then by compactness, there is a Morley sequence $\{I_i\}_{i<\omega}$ in s^ω with $I_0 = I$ that is indiscernible over $d_0 d''_1$ (See also the paragraph immediately following Fact 4.5.7). So by the paragraph immediately preceding the statement of the lemma, $I \mathop{\smile\!\!\!\!\!\mid}^{K^+}_M d_0 d''_1$, and in particular there is $d'_0 d'_1 \mathop{\smile\!\!\!\!\!\mid}^{i}_M d_0 d''_1$ with $d'_0 d'_1 \equiv_{MI} d_0 d''_1$; by Ramsey's theorem, compactness, and an automorphism, we can choose $d'_0 d'_1$ so that I is indiscernible over $d'_0 d'_1 d_0 d''_1$. Then, again, $d'_0 d'_1 d_0 d''_1 \mathop{\smile\!\!\!\!\!\mid}^{K^+}_M I$, so in particular, $d_0 d'_1 \mathop{\smile\!\!\!\!\!\mid}^{K^+} c$; also, $d'_1 \equiv_{Mc} d''_1 \equiv_{Mc} d_1$. □

We are now ready to prove Theorem 4.5.10. Note that if we can find I'' so that $I'' \equiv_a I$ and $I'' \equiv_b I'$, then I'' can be chosen indiscernible over ab, so $ab \mathop{\smile\!\!\!\!\!\mid}^{K^*d}_M I''$, and $I'' \mathop{\smile\!\!\!\!\!\mid}^{K^*d}_M ab$. So it suffices to show $p(X, a) \cup q(X, b)$ is consistent. Suppose it is inconsistent. Then by compactness, there are some $\varphi(X, a) \in p(X, a)$ and $\psi(X, b) \in q(X, b)$ so that $\{\varphi(X, a), \psi(X, b)\}$ is inconsistent. Let $s(w, y) = \mathrm{tp}(a, b/M)$ and let $r(y)$ be as in Lemma 5.1. Let κ be large. By transfinite induction, we will find a_i, b_i, $i < \kappa$ so that

(1) For $i < j < \kappa$, $a_j b_i \equiv_M ab$, so $\{\varphi(x, a_j), \psi(x, b_i)\}$ is inconsistent and $a_i \models \mathrm{tp}(a/M)$.

(2) For $i < j_1 < \ldots < j_m < \kappa$, $b_{j_1}, \ldots b_{j_m} \models r^{(m)}(y)|_{Ma_{\leq i}}$ and $a_i \mathop{\smile\!\!\!\!\!\mid}^{i} a_{<i}$.

(3) For $i < \kappa$, $a_{\leq i} \mathop{\smile\!\!\!\!\!\mid}^{K^+}_M b_{\leq i}$.

Suppose a_i, b_i already constructed satisfying (1)-(3) for $i < \lambda$. We find a_λ and b_λ. By $a \mathop{\smile\!\!\!\!\!\!|}^i_M b$, or $a \mathop{\smile\!\!\!\!\!\!|}^f_M b$ or $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$ and the respective chain condition, since $\{b_i\}_{i<\lambda}$ is an invariant Morley sequence in $\mathrm{tp}(b/M)$, there is some $a_\lambda \models \cup_{i<\lambda} s(w, b_i)$ with $a_\lambda \mathop{\smile\!\!\!\!\!\!|}^K_M b_{<\lambda}$. By Lemma 4.5.12, a_λ can then additionally be chosen with $a_\lambda \mathop{\smile\!\!\!\!\!\!|}^i_M a_{<\lambda}$ and $a_\lambda a_{<\lambda} \mathop{\smile\!\!\!\!\!\!|}^{K^+}_M b_{<\lambda}$, as desired. We then choose $b_\lambda \models r(y)|_{a_{\leq\lambda} b_{<\lambda}}$, which will preserve (1) and (2); it remains to show (3). This will follow from the following claim, analogous to Claim 3.6.2:

Claim 4.5.13. *For any a, b, c, M, if $a \mathop{\smile\!\!\!\!\!\!|}^{K^+}_M b$ and $\mathrm{tp}(c/Mab)$ extends to an M-invariant type $q(x)$, then $a \mathop{\smile\!\!\!\!\!\!|}^{K^+}_M bc$. (This is true as long as $\mathop{\smile\!\!\!\!\!\!|}^{K^+}$ is symmetric.)*

Proof. It follows that $b \mathop{\smile\!\!\!\!\!\!|}^{K^+}_M a$, so let $I = \{a_i\}_{i\in\omega}$ be an Mb-indiscernible invariant Morley sequence over M with $a_0 = a$. By an automorphism, we can choose I so that $c \models q|MIb$. By Ramsey and compactness, we can further choose I indiscernible over Mbc, so $bc \mathop{\smile\!\!\!\!\!\!|}^{K^+}_M a$ and by symmetry of $\mathop{\smile\!\!\!\!\!\!|}^{K^+}$, $a \mathop{\smile\!\!\!\!\!\!|}^{K^+} bc$. □

Finally, by the Erdős-Rado theorem, we can find $\{a_i b_i\}_{i<\omega}$ indiscnernible over M, satisfying (1) and (2) (and (3)). Then $\{a_i\}_{i\in\omega}$ will be an invariant Morley sequence over M with $a_i \models \mathrm{tp}(a/M)$, so for $n \leq m$, by (2) and Lemma 4.5.11, $\cup_{i=1}^n p(X, a_i) \cup \cup_{i=n+1}^m q(X, b_i)$ and therefore $\{\varphi(x, a_1), \ldots \varphi(x, a_n), \psi(x, b_{n+1}) \ldots \psi(x, b_m)\}$ is consistent. This, together with (1), implies SOP_3 by fact 4.2.4 –a contradiction.

Chapter 5

On the properties $\mathrm{SOP}_{2^{n+1}+1}$

5.1 Introduction

This chapter is on Shelah's *strong order property* hierarchy, the properties SOP_n introduced in [101] and extended in [98], [40]. For $n \geq 3$, these are defined as follows:

Definition 5.1.1. *A theory T is* NSOP_n *(that is, does not have the* n-strong order property*) if there is no definable relation $R(x_1, x_2)$ with no n-cycles, but with tuples $\{a_i\}_{i\in\omega}$ with $\models R(a_i, a_j)$ for $i < j$. Otherwise it is* SOP_n.

For $1 \leq n \leq 4$, these properties have been developed to various degrees. In [52], Kaplan and Ramsey extend the theory of forking-independence in simple theories to *Kim-independence* in NSOP_1 *theories*, by modifying the definition of dividing to require an invariant Morley sequence to witness the dividing. There, a characterization of NSOP_1 is given in terms of symmetry for Kim-independence, using work of Chernikov and Ramsey in [28], and also in terms of the independence theorem for Kim-independence. Later work has continued the development of Kim-independence in NSOP_1 theories; for example, see Kaplan and Ramsey ([53]) for transitivity and witnessing; Kaplan, Ramsey and Shelah ([54]) for local character; Dobrowolski, Kim and Ramsey ([39]) and Chernikov, Kim and Ramsey ([6]) for independence over sets; Kruckman and Ramsey ([71]) and Kruckman, Tran and Walsberg ([72]) for improvements upon the independence theorem; Kim ([63]) for canonical bases; and Kamsma ([51]), Dobrowolski and Kamsma ([39]) and Dmitrieva, Gallinaro and Kamsma ([5]) for extensions to positive logic, as well as the examples, by various authors, of NSOP_1 theories in applied settings. See also Kim and Kim [64], Chernikov and Ramsey ([28]), Ramsey ([94]), and Casanova and Kim ([17]) for type-counting and combinatorial criteria for SOP_1 and SOP_2, and Ahn and Kim ([3]) for connections of SOP_1 and SOP_2 to the antichain tree property further developed by Ahn, Kim and Lee in [4].

The SOP_2 *theories* were characterized by Džamonja and Shelah ([40]), Shelah and Usvyatsov ([102]), and Malliaris and Shelah ([81]) as (under mild set-theoretic assumptions) the maximal class in the order $\triangleleft^*$, related to Keisler's order, and in celebrated work of Malliaris and Shelah ([79]), they were shown to be maximal in Keisler's order, in ZFC. Then in [83], it was shown that a theory is NSOP_2 if and only if it is NSOP_1, bringing together Kim-independence and Keisler's order. It remains open whether all NSOP_3 theories are NSOP_2

Generalizing work of Evans and Wong in [42], showing the ω-categorical Hrushovski constructions introduced in [41], which have a natural notion of free amalgamation, are either simple or SOP_3, and work of Conant in [32] showing that all modular free amalgamation theories are simple or SOP_3, the author in Chapter 1 isolates two structural properties, with no known NSOP_4 counterexamples, which generalize [42] and [32] and imply that a theory must be either NSOP_1 or SOP_3. As a consequence, all free amalgamation theories are NSOP_1 or SOP_3. Malliaris and Shelah ([81]) show symmetric inconsistency for *higher formulas* in NSOP_3 theories, and Malliaris ([78]) investigates the graph-theoretic depth of independence in relation to SOP_3. In [55], Ramsey, Kaplan and Simon show very recently that all binary NSOP_3 theories are simple, by giving a theory of independence for a class of theories containing all binary theories. Until recently, no consequences of NSOP_n were known for the program of further extending the theory of Kim-independence in $\mathrm{NSOP}_2 = \mathrm{NSOP}_1$ theories to NSOP_n for $n > 2$; then the author, in Chapter 4, shows that types in NSOP_3 theories with internally NSOP_1 structure satisfy Kim's lemma at an external level, as well as an independence theorem, and also shows that NSOP_3 theories with symmetric Conant-independence satisfy an independence theorem for finitely satisfiable types with the same Morley sequences, related to that proposed for NTP_2 theories by Simon ([104]).

Shelah, in [101], gives results on universal models in SOP_4 theories. Generalizing a line of argument from the literature originally used by Patel ([85]), Conant, in [32] (where an historical overview of this argument can be found), shows free amalgamation theories are NSOP_4. In Chapter 1, the author connects this result to a potential theory of independence in NSOP_4 theories, defining the relation of *Conant-independence*[1]:

Definition 5.1.2. *Let M be a model and $\varphi(x,b)$ a formula. We say $\varphi(x,b)$* Conant-divides *over M if for* every *invariant Morley sequence $\{b_i\}_{i\in\omega}$ over M starting with b, $\{\varphi(x,b)\}_{i\in\omega}$ is inconsistent. We say $\varphi(x,b)$* Conant-forks *over M if and only if it implies a disjunction of formulas Conant-dividing over M. We say a is* Conant-independent *from b over M, written $a \mathop{\smile\!\!\!\!\!\mid}^{K^*}_M b$, if $\mathrm{tp}(a/Mb)$ does not contain any formulas Conant-forking over M.*

By Kim's lemma (Theorem 3.16 of [52]), this coincides with Kim-independence in NSOP_1 theories. Conant-independence gives a plausible theory of independence for NSOP_4 theories:

Fact 5.1.1. *(Theorem 1.6.2) Any theory where Conant-independence is symmetric is* NSOP_4*, and there are strictly* NSOP_4 *(*NSOP_4 *and* SOP_3*) theories where Conant-independence is symmetric. Thus $n = 4$ is the largest value of n so that there are strictly* NSOP_n *theories where Conant-independence is symmetric.*

[1]This was originally introduced under a nonstandard definition in [83], to show NSOP_2 theories are NSOP_1.

In Chapter 1, the author characterizes Conant-independence in most of the known examples of NSOP_4 theories, where it is symmetric. This leaves open the question of whether Conant-independence is symmetric in all NSOP_4 theories, giving a full theory of independence for the class NSOP_4 theories.

However, to our knowledge, other than some examples ([101], [18]; see also [35]), little has been known about the properties SOP_n for $n \geq 5$.

The main result of this chapter will be to generalize the interactions betwen SOP_4 and Conant-independence to the higher levels of the SOP_n hierarchy. As with Conant-independence, we will move from the forking-independence "at a generic scale" considered by Kruckman and Ramsey's work on Kim-independence in ([52], where the phrase is coined), to forking-independence at a *maximally generic scale*, grounding our notion of independence in dividing with respect to *every* Morley sequence of a certain kind, rather than just some Morley sequence. There is precedent for this kind of definition in the "strong Kim-dividing" of Kaplan, Ramsey and Shelah in [54], defined in the context of "dual local character" in NSOP_1 theories and grounding the defintion of Conant-independence.

We will also turn our attention to the *fine structure* of the genericity in the sequences that witness dividing, taking into account the variation between different classes of Morley sequences. For Kim-independence in NSOP_1 theories, this fine structure is submerged: by Corollary 5.4 of [53], Kim-independence in NSOP_1 theories remains the same when one replaces invariant Morley sequences in the genericity with Kim-independence itself. In the examples of NSOP_4 theories where Conant-independence has been characterized, it can also be seen that Conant-independence remains the same when one replaces invariant Morley sequences in the definition (Definition 5.1.2) with Conant-nonforking Morley sequences; see remarks at the end of Section 2 of this chapter. However, in, say, strictly NSOP_5 theories, Conant-independence cannot be symmetric, but a symmetric notion of independence can be obtained in some examples by replacing the invariant Morley sequences with nonforking Morley sequences. More generally, we can iteratively obtain different levels of genericity, the independence relations $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ defined in Definition 5.2.3. The main result of this chapter will be to show, within the interaction between the layers $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ of genericity and the approximations SOP_k of strict order, the resonance of the exponential function $2^{n+1}+1$.

We show:

Theorem 5.1.2. *Let $n \geq 1$. If $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ is symmetric in the theory T, then T is $\mathrm{NSOP}_{2^{n+1}+1}$. Moreover, there exists an $\mathrm{SOP}_{2^{n+1}}$ theory in which $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ is symmetric. So $k = 2^{n+1}+1$ is the largest value of k so that there is an NSOP_k but SOP_{k-1} theory where $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ is symmetric.*

Similar results are proven for (left and right) transitivity. As with Conant-independence, this leaves open the question of whether $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ is symmetric in all $\mathrm{NSOP}_{2^{n+1}+1}$ theories, which would give a full theory of independence throughout the strict order hierarchy.

In Section 2, we define $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$, show some basic properties necessary for our main result, and give some connections with stability motivating the possibility that symmetry for $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ forms a hierarchy.

In section 3, we characterize $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ in the classical examples of $\mathrm{NSOP}_{2^{n+1}+1}$ theories, including the generic directed graphs without short directed cycles and undirected graphs without short odd cycles of [100], and the free roots of the complete graph of [18], developed in [35]. Though $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n} = \mathop{\smile\!\!\!\!\!\!|}^{a}$ will be trivial (and thus symmetric) in these classical examples, giving us the existence result of our main theorem, it is promising for the full characterization, Question 5.4.6, that the cycle-free examples and the free roots of the complete graph have $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n} = \mathop{\smile\!\!\!\!\!\!|}^{a}$ for different reasons. In the cycle-free examples, successive approximations of forking-independence tend towards larger graph-theoretic distances, while in the free roots of the complete graph, distances in successive approximations of forking-independence tend *away* from the extremes.

In section 4, we show that if $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ is symmetric in the theory T, then T is $\mathrm{NSOP}_{2^{n+1}+1}$, completing our main result. We pose the converse as an open question.

5.2 Definitions and basic properties

We recall the defintion of *forking-independence*:

Definition 5.2.1. *A formula* $\varphi(x, b)$ divides *over a model* M *if there is an* M*-indiscernible sequence* $\{b_i\}_{i<\omega}$ *with* $b_0 = b$ *and* $\{\varphi(x, b_i)\}_{i<\omega}$ *inconsistent. A formula* $\varphi(x, b)$ forks *over a model* M *if there are* $\varphi_i(x, b_i)$ *dividing over* M *so that* $\models \varphi(x, b) \to \bigvee_{i=1}^{n} \varphi_i(x, b_i)$*. We say that* a *is* forking-independent *from* b *over* M*, denoted* $a \mathop{\smile\!\!\!\!\!\!|}^{f}_{A} b$*, if* $\mathrm{tp}(a/Mb)$ *contains no formulas forking over* M*.*

We define the relations $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$, in analogy with the *Conant-independence* of Chapter 1. To give the definition, we need to generalize the notion of Morley sequence to any relation between sets over a model.

Definition 5.2.2. *Let* $\mathop{\smile\!\!\!\!\!\!|}$ *be a relation between sets over a model. An* $\mathop{\smile\!\!\!\!\!\!|}$*-Morley sequence over* M *is an* M*-indiscernible sequence* $\{b_i\}_{i<\omega}$ *with* $b_i \mathop{\smile\!\!\!\!\!\!|}_M b_0 \dots b_{i-1}$ *for* $i < \omega$*.*

Let $a \mathop{\smile\!\!\!\!\!\!|}^{u}_{M} b$ denote that $\mathrm{tp}(a/Mb)$ is M-finitely satisfiable; as elsewhere in the literature, a *finiteley satisfiable* or *coheir Morley sequence* will be a $\mathop{\smile\!\!\!\!\!\!|}^{u}$-Morley sequence.

Definition 5.2.3. *(1) Let* $\mathop{\smile\!\!\!\!\!\!|}^{\eth^0}$*,* 0-ð-independence*, denote forking-independence over a model* M*; a formula* 0-ð*-divides (*0-ð*-forks) over* M *if it divides (forks) over* M*.*

Inductively,

(2a) A formula $\varphi(x,b)$ $(n+1)$-ð-divides over a model M if, for any $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$-Morley sequence $\{b_i\}_{i<\omega}$ with $b_0=b$, $\{\varphi(x,b_i)\}_{i<\omega}$ is inconsistent.[2]

*(2b) A formula $\varphi(x,b)$ $(n+1)$-ð-*forks *over a model M if there are $\varphi_i(x,b_i)$ $(n+1)$-ð-dividing over M so that $\models \varphi(x,b)\rightarrow \bigvee_{i=1}^n \varphi_i(x,b_i)$.*

(2c) We say that a is $(n+1)$-ð-independent *from b over M, denoted $a \mathop{\smile\!\!\!\!\!\!|}^{\eth^{n+1}}_M b$, if* $\mathrm{tp}(a/Mb)$ *contains no formulas $(n+1)$-ð-forking over M.*

It will be useful for our main results to show that n-ð-forking coincides with n-ð-dividing in general, for $n>1$; this is not known to be the case for $n=1$, so this case will need to be handled separately in proving our main results.

Lemma 5.2.1. *(1) The relation $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ has right-extension for $n\geq 0$: if $a \mathop{\smile\!\!\!\!\!\!|}^{\eth^n}_M b$ then for any c there is $a\equiv_{Mb} a'$ with $a' \mathop{\smile\!\!\!\!\!\!|}^{\eth^n}_M bc$.*

(2) The relation $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ has left-extension for $n\geq 1$: if $a \mathop{\smile\!\!\!\!\!\!|}^{\eth^n}_M b$ then for any c there is $c\equiv_{Ma} c'$ with $ac' \mathop{\smile\!\!\!\!\!\!|}^{\eth^n}_M b$.

Proof. (1) This is known for $n=0$ and follows as in that case in the standard way for $n\geq 1$. Suppose that $a \mathop{\smile\!\!\!\!\!\!|}^{\eth^n}_M b$, but there were no such a'. Then by compactness, there would be some formulas $\varphi(x,b)\in \mathrm{tp}(a/Mb)$, and $\varphi_i(x,d_i)$ n-ð-forking over M for $d_i\subseteq Mbc$, so that $\models \varphi(x,b)\rightarrow \bigvee_{i=1}^n \varphi_i(x,d_i)$. By definition of n-ð-forking, for $1\leq i\leq n$, there are $\varphi_{ij}(x,b_{ij})$ n-ð-dividing over M so that $\models \varphi_i(x,d_i)\rightarrow \bigvee_{j=1}^{n_i}\varphi_{ij}(x,b_{ij})$. Then $\models \varphi(x,b)\rightarrow \bigvee_{i=1}^n\bigvee_{j=1}^{n_i}\varphi_{ij}(x,b_{ij})$, so $\varphi(x,b)$ n-ð-forks over M, contradicting $a \mathop{\smile\!\!\!\!\!\!|}^{\eth^n}_M b$.

(2) Suppose that $a \mathop{\smile\!\!\!\!\!\!|}^{\eth^n}_M b$. Let $M'\succ M$ be an $(|M|+|T|)^+$-saturated elementary extension of M. By (1) there is $a'\equiv_{Mb} a$ with $a' \mathop{\smile\!\!\!\!\!\!|}^{\eth^n}_M M'$. So by replacing a with a', we can assume that $b=M'$ is an $(|M|+|T|)^+$-saturated elementary extension of M.

We next show that, for any d with $\mathrm{tp}(d/M')$ containing no formulas n-ð-dividing over M, $d \mathop{\smile\!\!\!\!\!\!|}^{\eth^n}_M M'$. This argument is standard from the literature. Suppose otherwise. Then $\mathrm{tp}(d/M')$ contains a $\varphi(x,b)$ n-ð-forking over M for $b\subset M'$. So there are $\varphi(x,b_i)$ n-ð-dividing over M with $\models \varphi(x,b)\rightarrow \bigvee_{i=1}^n\varphi_i(x,b_i)$. By $(|M|+|T|)^+$-saturation of M' there are $b_1',\dots,b_n'\subset M'$ with $b_1'\dots b_n'\equiv_{Mb} b_1\dots b_n$. So $\varphi(x,b_i')$ n-ð-divide over M for $1\leq i\leq n$ and $\models \varphi(x,b)\rightarrow \bigvee_{i=1}^n\varphi_i(x,b_i')$. By the latter, that $\varphi(x,b)\in\mathrm{tp}(d/M')$ and $\mathrm{tp}(d/M')$ is a complete type over M' implies that there is some $1\leq i\leq n$ with $\varphi_i(x,b_i')\in \mathrm{tp}(d/M')$, a contradiction.

Now consider any c. It suffices to find $c'a'\equiv_M ca$ with $a'\equiv_{M'} a$ and $\mathrm{tp}(c'a'/M')$ containing no formulas n-ð-dividing over m. So by compactness, for $\psi(y,x)\in\mathrm{tp}(ca/M)$ and $\varphi(x,d)\in \mathrm{tp}(a/M')$ with $d\subseteq M'$, it suffices to find $c'a'$ with $\models \psi(c',a')\wedge\varphi(a',d)$ so that $\mathrm{tp}(c'a'/Md)$

[2] It is not immediate that this defintion is independent of adding or removing unusued parameters in b, though this is corrected by the definition of $n+1$-ð-forking. We fix the convention that a formula only has finitely many parameters. Fixing this convention, it will follow from the results of this section that n-ð-dividing of a formula $\varphi(x,b)$ is independent of adding or removing unused parameters in b for $n>1$; this is not known for $n=1$.

contains no formulas $\varphi'(y,x,e)$ n-ð-dividing over M with $e \subseteq d$. The formula $\exists y(y,x) \wedge \varphi(x,d)$ belongs to $\mathrm{tp}(a/M')$. So by $a \mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^n} M'$, $\exists y(y,x) \wedge \varphi(x,d)$ does not n-ð-divide over M. By definition of n-ð-dividing, there is an $\mathop{\smile\!\!\!\!\!\!|}\,^{ð^{n-1}}$-Morley sequence over M, $I = \{d_i\}_{i<\omega}$ with $d_0 = d$, so that $\{\exists y(y,x) \wedge \varphi(x,d_i)\}_{i<\omega}$ is consistent, realized by a'. By Ramsey's theorem, compactness, and an automorphism, we choose a' so that I is indiscernible over Ma'. In particular, $\models \exists y\psi(y,a')$, so choose c' so that $\models \psi(c',a')$. By another application of Ramsey's theorem, compactness and an automorphism, we can choose c' so that I is indiscernible over $Ma'c'$. It remains to show that $\mathrm{tp}(c'a'/Md)$ contains no formulas $\varphi'(y,x,e)$ n-ð-dividing over M with parameters $e \subseteq d$. For $i < \omega$ there are $e_i \subseteq d_i$ with $\{e_i\}_{i<\omega}$ $Ma'c'$-indiscernible and $e_0 = e$. By definition of $\mathop{\smile\!\!\!\!\!\!|}\,^{ð^{n-1}}$-Morley sequence, for $i < \omega$, $d_i \mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}} d_0 \dots d_{i-1}$. So it follows from the definition of $\mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}}$ that $e_i \mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}} e_0 \dots e_{i-1}$ (i.e. $\mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}}$ is monotone.) So $\{e_i\}_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}}$-Morley sequence over M. Let $\varphi'(y,x,e) \in \mathrm{tp}(c'a'/Md)$. Then by $Ma'c'$-indiscernibility, $\{\varphi'(y,x,e_i)\}_{i<\omega}$ is consistent, realized by $a'c'$. So $\varphi'(y,x,e)$ does not n-ð-divide over M.

□

Proposition 5.2.2. *For $n \geq 2$, n-ð-forking coincides with n-ð-dividing.*

Proof. Exactly as in Fact 1.6.1 of Chapter 1, using right- and left-extension for $\mathop{\smile\!\!\!\!\!\!|}\,^{ð^{n-1}}$, and the standard arguments. Suppose $\varphi(x,b)$ n-ð-forks over M. Then $\varphi(x,b) \rightarrow \bigvee_{j=1}^N \varphi_j(x,c^j)$ for some $\varphi_j(x,c^j)$ n-ð-dividing over M. We show that $\varphi(x,b)$ n-ð-divides over M; suppose otherwise. Then by the definition of n-ð-dividing and compactness, there is an $\mathop{\smile\!\!\!\!\!\!|}\,^{ð^{n-1}}$-Morley sequence $\{b_i\}_{i<\kappa}$, for large κ, i.e. indiscernible over M with $b_i \mathop{\smile\!\!\!\!\!\!|}\,^{ð^{n-1}} b_{<i}$ for $i < \kappa$, with $b_0 = b$ and $\{\varphi(x,b_i)\}_{i<\kappa}$ consistent. By induction we find c_i^j, $1 \leq j \leq N$, $i < \kappa$, so that $\{c_i^j\}_{j=1}^N b_i \equiv_M \{c^j\}_{j=1}^N b$ and $\{c_i^j\}_{j=1}^N b_i \mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}} \{c_{<i}^j\}_{j=1}^N b_{<i}$ for $i < \kappa$. Suppose by induction that for $\lambda < \kappa$ we have found c_i^j, $1 \leq j \leq N$, $i < \lambda$, so that $\{c_i^j\}_{j=1}^N b_i \equiv_M \{c^j\}_{j=1}^N b$ and $\{c_i^j\}_{j=1}^N b_i \mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}} \{c_{<i}^j\}_{j=1}^N b_{<i}$ for $i < \lambda$. Then because $b_\lambda \mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}} b_{<\lambda}$, by right extension we could have chosen c_i^j, $1 \leq j \leq N$, $i < \lambda$ so that $b_\lambda \mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}} \{c_{<\lambda}^j\}_{j=1}^N b_{<\lambda}$. Now by left extension and an automorphism, find c_λ^j, $1 \leq j \leq N$, with $\{c_\lambda^j\}_{j=1}^N b_\lambda \equiv_M \{c^j\}_{j=1}^n b$ and $\{c_\lambda^j\}_{j=1}^N b_\lambda \mathop{\smile\!\!\!\!\!\!|}\,_M^{ð^{n-1}} \{c_{<\lambda}^j\}_{j=1}^n b_{<\lambda}$. This completes the induction. Now by the Erdős-Rado theorem and an automorphism, we can find $c_i'^j$ for $i < \omega$, $1 \leq j \leq N$, so that $\{c_i'^1 \dots c_i'^N b_i\}_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!|}\,^{ð^{n-1}}$-Morley sequence over M with $c_0'^1 \dots c_0'^N b_0 = c^1 \dots c^N b$. Now we give the standard argument to get a contradiction. Each $\{c_i'^j\}_{i<\omega}$ for $1 \leq j \leq N$ is an $\mathop{\smile\!\!\!\!\!\!|}\,^{ð^{n-1}}$-Morley sequence over M with $c_0'^j = c^j$. Let a realize $\{\varphi(x,b_i)\}_{i<\kappa}$. For each $i < \omega$, $\models \varphi(x,b_i) \rightarrow \bigvee_{j=1}^N \varphi_j(x,c_i'^j)$, so $\models \varphi_j(a,c_i'^j)$ for some $1 \leq j \leq n$. So there is some $1 \leq j \leq n$ so that $\models \varphi_j(a,c_i'^j)$ for infinitely many $i < \omega$. By an automorphism, $\{\varphi_j(x,c_i'^j)\}_{i<j}$ is consistent. But because $\{c_i'^j\}_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!|}\,^{ð^{n-1}}$-Morley sequence over M with $c_0'^j = c^j$, this contradicts n-ð-dividing of $\varphi_j(x,c^j)$ over M. □

While we will not use the results of the rest of this section in the sequel, they will give some additional motivation to the main results of this chapter and to the open question posed at the end. We will be interested in the question of when $\mathop{\smile\!\!\!\!\!\!\mid}^{\eth^n}$ has properties analogous to Kim-independence in NSOP_1, including transitivity ([53]) and especially symmetry ([52]). If the answer to Question 5.4.6 is positive, either of these properties will be equivalent to $\mathrm{NSOP}_{2^{n+1}+1}$, so in particular symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}^{\eth^n}$ will imply symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}^{\eth^{n+1}}$. However, even the question of whether symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}^{\eth^n}$ implies symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}^{\eth^{n+1}}$ is still open. But if we strengthen symmetry to an analogue of the stable forking conjecture, a linear hierarchy starts to emerge.

Definition 5.2.4. *A theory T satisfies the* stable n-ð-forking conjecture *if whenever $a \not\mathop{\smile\!\!\!\!\!\!\mid}^{\eth^n}_M b$, there is some $L(M)$-formula $\varphi(x, y)$, which is stable as an $L(M)$-formula, so that $\models \varphi(a, b)$ and $\varphi(x, b)$ n-ð-forks over M.*

Remark 5.2.3. Unlike the stable forking conjecture for simple theories, here we require only that $\varphi(x, y)$ be stable as a $L(M)$-formula, not an L-formula. It makes sense to make this allowance outside of the simple case, as the analogous conjecture about Kim-forking fails for, say, T^{feq} (see, e.g., [100], [102], [28], [52]), when the stability is as an L-formula, but the "stable Kim-forking conjecture" is open for NSOP_1 theoires when the stability is taken to be as an $L(M)$ formula.

Note that if a stable formula 1-ð-forks over M, it forks over M, so by basic stability theory divides with respect to any nonforking Morley sequence; that is, 1-ð-divides over M. If T satisfies the stable 1-ð-forking conjecture, it follows that if $\mathrm{tp}(a/Mb)$ contains no 1-ð-dividing formulas, $a \mathop{\smile\!\!\!\!\!\!\mid}^{\eth^1}_M b$.

The first part of this fact is well-known; the second is observed without proof for theories in [16], but the evident proof works equally well for formulas.

Fact 5.2.4. *If a formula $\varphi(x, y)$ is stable, it is without the tree property and is low: there is some k so that for $\{b_i\}_{i\in I}$ an indiscernible sequence, $\{\varphi(x, b_i)\}_{i\in I}$ is inconsistent if and only if it is k-inconsistent.*

The argument for the following is similar to the literature: see, for example, Theorem 5.16 of [52] or Theorem 3.5.3. The construction of the tree is lifted mostly word-for-word from the proof of Theorem 3.5.3, but we follow the local approach in the paragraph following the proof of that theorem.

Proposition 5.2.5. *If T satisfies the stable n-ð-forking conjecture, then $\mathop{\smile\!\!\!\!\!\!\mid}^{\eth^n}$ is symmetric.*

Proof. Assume $a \mathop{\smile\!\!\!\!\!\!\mid}^{\eth^n}_M b$ but $b \not\mathop{\smile\!\!\!\!\!\!\mid}^{\eth^n}_M a$. Let $\varphi(x, y)$ be a stable $L(M)$-formula with $\models \varphi(b, a)$ and $\varphi(x, a)$ a stable formula n-ð-forking over M; by Proposition 5.2.2 and the remarks after Remark 5.2.3, this n-ð-divides over M. We find, for any n, a tree $(I_n, J_n) = (\{a_\eta\}_{\eta\in\omega^{\leq n}}, \{b_\sigma\}_{\sigma\in\omega^n})$, infinitely branching at the first $n+1$ levels and then with each a_σ for $\sigma \in \omega^n$ at level $n+1$ followed by a single additional leaf b_σ at level $n+2$, satisfying the following properties:

(1) For $\eta \trianglelefteq \sigma$, $\models \varphi(b_\sigma, a_\eta)$.

(2) For $\eta \in \omega^{<n}$, the branches at η form an $\mathop{\smile\!\!\!\!\!\mid}^{\eth^{n-1}}$-Morley sequence over M indiscernible over Ma_η, so by Proposition 5.2.2 and the remarks after Remark 5.2.3, a_η is n-ð-independent over M from those branches taken together.

Suppose (I_n, J_n) already constructed; we construct (I_{n+1}, J_{n+1}). We see that the root $a_\emptyset$ of (I_n, J_n) is n-ð-independent from the rest of the tree, $(I_n, J_n)^*$: for $n = 0$ this is just the assumption $a \mathop{\smile\!\!\!\!\!\mid}^{\eth^n}_M b$, while for $n > 0$ this is (2). So by extension we find $a'_\emptyset \equiv_{M(I_nJ_n)^*} a_\emptyset$ (so guaranteeing (1)), to be the root of (I_{n+1}, J_{n+1}), with $a'_\emptyset \mathop{\smile\!\!\!\!\!\mid}^{K^*}_M I_nJ_n$. Then by applying $a \mathop{\smile\!\!\!\!\!\mid}^{K^*}_M I_nJ_n$ to the formulas giving (1), find some $\mathop{\smile\!\!\!\!\!\mid}^{\eth^{n-1}}$-Morley sequence $\{(I_n, J_n)^i\}_{i\in\omega}$ with $(I_n, J_n) \equiv_M (I_n, J_n)^i$ indiscernible over $Ma'_\emptyset$, guaranteeing (2) and preserving (1), and reindex accordingly.

By lowness and n-ð-dividing of $\varphi(x, a)$ over M, the successors to each node witness k-dividing of $\varphi(x, a)$ over M for some fixed k. This together with 1 gives the k-tree property for $\varphi(x, a)$, a contradiction. □

We now show the linear hierarchy obtained when symmetry of n-ð-independence is improved to the conclusion of the stable n-ð-forking conjecture.

Proposition 5.2.6. *If T satisfies the stable n-ð forking conjecture, then* $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n} = \mathop{\smile\!\!\!\!\!\mid}^{\eth^{n+1}}$ *(so* $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n} = \mathop{\smile\!\!\!\!\!\mid}^{\eth^m}$ *for $m \geq n$.)*

Proof. It suffices to show that for $\varphi(x, y)$ a stable formula, if $\varphi(x, b)$ n-ð-forks over M, so n-ð-divides over M, then it $n+1$-ð-divides over M. Let $\{b_i\}_{i<\omega}$ be an $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$-Morley sequence over M; by the definition of $n+1$-ð-dividing, it suffices to show that $\{\varphi(x, b_i)\}_{i<\omega}$ is k-inconsistent. Suppose otherwise. By compactness, extend $I = \{b_i\}_{i<\omega}$ to a $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$-Morley sequence $\{b_i\}_{i<\omega+\omega}$ over M. Then $\{b_i\}_{\omega\leq i<\omega+\omega}$ is a (nonforking) Morley sequence over MI that does not witness dividing of $\varphi(x, b_\omega)$. So it suffices to show that $\varphi(x, b_\omega)$ divides over MI anyway, contradicting the basic properties of stability. By the previous proposition, $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$ is symmetric, so $I \mathop{\smile\!\!\!\!\!\mid}^{\eth^n}_M b_\omega$. Note that $\varphi(x, b_\eta)$ divides over M. So by lowness, there is some k so that each $\mathop{\smile\!\!\!\!\!\mid}^{\eth^{n-1}}$-Morley sequence over M starting with b_ω witnesses k-dividing of $\varphi(x, b_\eta)$. Now for any formula in $\mathrm{tp}(b_\omega/MI)$, by $I \mathop{\smile\!\!\!\!\!\mid}^{\eth^n}_M b_\omega$ there is an $\mathop{\smile\!\!\!\!\!\mid}^{\eth^{n-1}}$-Morley sequence over M starting with b_ω, each term of which realizes this formula. In sum, for any formula in $\mathrm{tp}(b_\omega/MI)$, there is an M-indiscernible sequence of realizations of this formula witnessing the k-dividing of $\varphi(x, b_\omega)$ over M. So by compactness, $\varphi(x, b_\eta)$ k-divides over MI. □

In the next section, we will characterize $\mathop{\smile\!\!\!\!\!\mid}^{\eth^n}$ in the classical examples of $\mathrm{NSOP}_{2^{n+1}+1}$ theories, for $n \geq 1$; it will be trivial in these examples, so satisfies the stable n-ð-forking conjecture. Note that, if we start with the analogous stability assumption for Conant-independence (see Chapter 1), the proof of the previous propositions show that it coincides with n-ð-independence for $n \geq 1$, and is symmetric. There are no known counterexamples to the "stable Conant-forking conjecture" for NSOP_4 theories. [3]

5.3 Attainability/examples

In NSOP_1 theories, $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ is just Kim-independence for $n \geq 1$. Moreover, in the examples of NSOP_4 theories where Conant-independence has been characterized, it coincides with $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ and is symmetric (see the end of the previous section). We now give some proper examples. These examples will show the attainability of $\mathrm{SOP}_{2^{n+1}}$ as the bound on the levels of the SOP_k hierarchy where $\mathop{\smile\!\!\!\!\!\!|}^{\eth^n}$ can be symmetric or transitive.

Example 5.3.1. (Free roots of the complete graph) In [18], Casanovas and Wagner show that the theory T_n^- of metric spaces valued in the set $\{0, \ldots, n\}$ has a model companion T_n. (More precisely, this is interdefinable with the theory introduced in [18], but we use the language of metric spaces.) They show that this theory is ω-categorical, eliminates quantifiers, and has trivial algebraicity, and that it is NSOP but not simple. Later, Conant and Terry show in [35] that T_n is strictly NSOP_{n+1}. We want to show the following

Theorem 5.3.2. *In T_n, if $2^{k+1} \leq n$, $a \mathop{\smile\!\!\!\!\!\!|}^{\eth^k}_M b$ if and only if $a \mathop{\smile\!\!\!\!\!\!|}^{a}_M b =: a \cap b = M$. Therefore, there are $\mathrm{SOP}_{2^{k+1}}$ theories where $\mathop{\smile\!\!\!\!\!\!|}^{\eth^k}$ is symmetric.*

We first show the following lemma:

Lemma 5.3.3. *Let $C, \subseteq A, B$ be metric spaces valued in $\{0, \ldots, n\}$. Then there is a metric space D valued in $\{0, \ldots, n\}$ together with isometric embeddings $\iota_A : A \hookrightarrow D$ and $\iota_B : B \hookrightarrow D$ with $\iota_A|_C = \iota_B|_C$ and with, for $a \in A \backslash C$, $b \in B \backslash C$, and $d_{ab} =: d_D(\iota_A(a), \iota_B(b))$*

(a) $d_{ab} = m_{ab} =: \min_{c \in C}(d_A(a, c) + d_B(b, c))$ if $m_{ab} < n^* =: \lceil \frac{n}{2} \rceil$.

(b) $d_{ab} = m^{ab} =: \max_{c \in C}(|d_A(a, c) - d_B(b, c)|)$ if $m^{ab} > n^*$

(c) Otherwise, $d_{ab} = n^*$.

Proof. We may assume $A \cap B = C$ as sets. So it suffices to define a metric on $D = A \cup B$ extending that on A and B and satisfying (a), (b), (c). We claim that for all $a \in A \backslash C$, $b \in B \backslash C$, $m_{ab} \geq m^{ab}$, so only one of (a), (b), (c) may hold. Suppose otherwise. Then there are $c_*, c^* \in C$ with $d(a, c_*) + d(b, c_*) < |d(a, c^*) - d(b, c^*)|$. But because T_n has quantifier elimination and trivial algebraicity, the class of finite metric spaces valued in $\{0, \ldots, n\}$ has the strong amalgamation property, so there is a metric $d : \{a, b, c_*, c^*\}^2 \to \{0 \ldots n\}$ extending the metric on $\{a, c_*, c^*\}$ and $\{b, c_*, c^*\}$. Then $d(a, b) \leq d(a, c_*) + d(b, c_*) < |d(a, c^*) - d(b, c^*)| \leq d(a, b)$, a contradiction. So conditions (a), (b) and (c), together with

[3] Conant-independence is characterized for some classical examples of NSOP_4 theories in Chapter 1. For the Fraïssé-Hrushovski constructions of finite language, where the author has shown that Conant-independence coincides with d-independence, the proof of the stable forking conjecture for the simple case of these structures in [92], [41] should extend to the general case using this characterization.

the requirement of extending the metric on A and B, give a well-defined function $d : D^2 \to \{0 \dots n\}$, and it remains to show this is a metric.

Suppose first that $a \in A \backslash C$, $b \in B \backslash C$, and $c \in C$. Then d satisfies the triangle inequality on $\{a, b, c\}$ if $|d(a, c) - d(b, c)| \leq d(a, b) \leq d(a, c) + d(b, c)$. The first inequality is by (b), (c) and the second is by (a), (c).

Now suppose without loss of generality that $a_1, a_2 \in A \backslash C$ and $b \in B \backslash C$. It remains to show the triangle inequality on $\{a_1, a_2, b\}$. By the definition of n^*, this is immediately the case if $d(a_1, b) = d(a_2, b) = n^*$. Otherwise, without loss of generality there are three cases, where $d(a_1, b) < n^*$ and $d(a_2, b) > n^*$, where $d(a_1, b) < n^*$ and $d(a_2, b) \leq n^*$, and where $d(a_1, b) > n^*$ and $d(a_2, b) \geq n^*$.

In the case where $d(a_1, b) < n^*$ and $d(a_2, b) > n^*$, easily $d(a_1, b) \leq d(a_2, b) + d(a_1, a_2)$. To show $d(a_2, b) \leq d(a_1, a_2) + d(a_1, b)$, by the strong amalgamation property, there is some metric d' on D extending d on A and B. So

$$d(a_2, b) \leq d'(a_2, b) \leq d'(a_1, b) + d'(a_1, a_2) = d'(a_1, b) + d(a_1, a_2) \leq d(a_1, b) + d(a_1, a_2)$$

Note that (b) is used for the first inequality, and (a) is used for the last inequality. Finally, we show $d(a_1, a_2) \leq d(a_2, b) + d(a_1, b)$, so $d(a_2, b) \geq d(a_1, a_2) - d(a_1, b)$. Note that $d(a_1, b) = d(a_1, c) + d(b, c)$ for some $c \in C$. Then by (b), $d(a_2, b) \geq d(a_2, c) - d(c, b) \geq (d(a_1, a_2) - d(a_1, c)) - d(c, b) = d(a_1, a_2) - d(a_1, b)$.

In the case where $d(a_1, b) < n^*$ and $d(a_2, b) \leq n^*$, we first show $d(a_1, b) \leq d(a_2, b) + d(a_1, a_2)$ and $d(a_2, b) \leq d(a_1, b) + d(a_1, a_2)$. If additionally, $d(a_2, b) = n^*$, then the first of these inequalities is immediate, so it suffices to prove the second inequality, as by symmetry we will have then proven the first inequality when both $d(a_1, b) < n^*$ and $d(a_2, b) < n^*$. By (a), there is some $c \in C$ with $d(a_1, b) = d(a_1, c) + d(b, c)$, so by (a), (c), $d(a_2, b) \leq d(a_2, c) + d(c, b) \leq d(a_1, a_2) + d(a_1, c) + d(c, b) = d(a_1, a_2) + d(a_1, b)$. Finally, we show in this case that $d(a_1, a_2) \leq d(a_2, b) + d(a_1, b)$. For any $c \in C$, if $d(a_2, b) = n^*$ this must be because $m^{a_2 b} \leq n^*$, and if $d(a_2, b) < n^*$, then $d(a_2, b) = m_{ab} \geq m^{ab}$, so $d(a_2, b) \geq d(a_2, c) - d(c, b)$. So $d(a_1, a_2) \leq d(a_2, b) + d(a_1, b)$ follows exactly as in the case of $d(a_1, b) < n^*$ and $d(a_2, b) > n^*$.

Finally, if $d(a_1, b) > n^*$ and $d(a_2, b) \geq n^*$, then $d(a_1, a_2) \leq d(a_1, b) + d(a_2, b)$ is immediate. We next show $d(a_1, b) \leq d(a_2, b) + d(a_1, a_2)$ and $d(a_2, b) \leq d(a_1, b) + d(a_1, a_2)$. If $d(a_2, b) = n^*$, then the second inequality is immediate, so it suffices to prove the first inequality, as by symmetry we will have then proven the second inequality when both $d(a_1, b) > n^*$ and $d(a_2, b) > n^*$. By (a) there will be some $c \in C$ with either $d(a_1, b) = d(a_1, c) - d(b, c)$ or $d(a_1, b) = d(b, c) - d(a_1, c)$. In the case of $d(a_1, b) = d(a_1, c) - d(b, c)$ by (b), (c), $d(a_2, b) \geq d(a_2, c) - d(b, c) \geq d(a_1, c) - d(a_2, a_1) - d(b, c) = d(a_1, b) - d(a_2, a_1)$, proving the first inequality in this case. In the case of $d(a_1, b) = d(b, c) - d(a_1, c)$, again by (b), (c), $d(a_2, b) \geq d(b, c) - d(a_2, c) = d(a_1, c) + d(a_1, b) - d(a_2, c) = d(a_1, b) - (d(a_2, c) - d(a_1, c)) \geq d(a_1, b) - d(a_1, a_2)$, proving the first inequality in the other case.

□

Remark 5.3.4. Say that in the statement of Lemma 5.3.3, we let $n^* = \lceil \frac{n}{2} \rceil + 1$ instead of $\lceil \frac{n}{2} \rceil$. Then the above proof works: n^* can be any constant at least $\lceil \frac{n}{2} \rceil$, and the only place that $n^* \geq \lceil \frac{n}{2} \rceil$ is used is in the case where $a_1, a_2 \in A \backslash C$, $b \in B \backslash C$ and $d(a_1, b) = d(a_2, b) = n^*$. In the sequel, we will still let n^* denote $\lceil \frac{n}{2} \rceil$

Definition 5.3.1. *Let $C \subseteq A, B$ be subspaces of some fixed metric space with values in $\{0, \ldots, n^*\}$, with $A \cap B = C$.*

(1) A and B are freely amalgamated *over C if the inclusions ι_A and ι_B satisfy the conclusion of Lemma 5.3.3.*

(2a) For $k \leq n$, A and B have distance $\leq n$ *over C if for $a \in A \backslash C$, $b \in B \backslash C$, $d(a, b) \leq \max(k, m^{ab})$.*

(2b) For $k \geq 1$, A and B have distance $\geq n$ *over C if for $a \in A \backslash C$, $b \in B \backslash C$, $d(a, b) \geq \max(k, m_{ab})$.*

Lemma 5.3.5. *Let $C \subseteq A, B, D$ be subsets of a fixed metric space, and $1 \leq k_1 \leq n_* \leq n_2 \leq n$. Suppose $A \cup D$ and $B \cup D$ are freely amalgamated over D, and both A and B have distance $\geq k_1$ and $\leq k_2$ from D over C. Then A has distance $\geq \min(n^*, 2k_1)$ and $\leq \max(k_2 - k_1, n^*)$ from B over C, and moreover, D has distance $\geq k_1$ and $\leq k_2$ from $A \cup B$ over C.*

Proof. The second clause is obvious, so we prove the first; let d be the metric. First we show that A has distance $\geq \min(n^*, 2k_1)$ from B over C. Let $a \in A \backslash C$, $b \in B \backslash C$. It suffices to show that if $d(a, b) < \min(n^*, 2k_1)$ then there is some $c \in C$ with $d(a, c) + d(c, b) \leq d(a, b)$. Because $d(a, b) < n^*$, there is some $d \in D$ with $d(a, d) + d(b, d) = d(a, b) < 2k_1$. So either $d(a, d)$ or $d(b, d)$ must be less that k_1. Without loss of generality, $d(a, d) < k_1$. Then because D and A have distance $\leq k_1$ over C, there is some $c \in C$ with $d(a, d) = d(a, c) + d(c, d)$. Then $d(a, b) = d(a, c) + d(c, d) + d(b, d) \geq d(a, c) + d(c, b)$. Next we show that A has distance $\leq \max(k_1 - k_2, n^*)$ from B over C. Let $a \in A \backslash C$, $b \in B \backslash C$. It suffices to show that if $d(a, b) > \max(k_2 - k_1, n^*)$, there is some $c \in C$ with $d(a, b) \leq |d(a, c) - d(b, c)|$. Because $d(a, b) > n^*$, without loss of generality there is some $d \in D$ with $d(a, d) - d(b, d) = d(a, b) > k_2 - k_1$. So either $d(a, d) > k_2$ or $d(b, d) < k_1$. In the case where $d(a, d) > k_2$, since A has distance $\geq k_2$ from D over C, there is some $c \in C$ so that either $d(a, d) = d(a, c) - d(c, d)$ or $d(a, d) = d(c, d) - d(a, c)$. If $d(a, d) = d(a, c) - d(c, d)$, then $d(a, b) = d(a, c) - d(b, d) - d(c, d) \leq d(a, c) - d(b, c)$. If $d(a, d) = d(c, d) - d(a, c)$, then $d(a, b) = d(c, d) - d(a, c) - d(b, d) \leq d(c, b) - d(a, c)$. In the case where $d(b, d) < k_1$, since B has distance $\geq k_1$ from D over C, there is some $c \in C$ with $d(b, d) = d(b, c) + d(c, d)$. Then $d(a, b) = d(a, d) - d(b, c) - d(c, d) \leq d(a, c) - d(b, c)$. □

Lemma 5.3.6. *Let A and B have distance $\geq n^*$ and $\leq n^* + 1$ over C. Then $A \mathop{\smile\!\!\!\!\!\mid}^f_C B$.*

Proof. Again, let d be the ambient metric. We first show that $\mathrm{tp}(A/CB)$ does not contain any formulas dividing over C. Let $I = \{B_i\}_{i<\omega}$ be a C-indiscernible sequencene with $B_0 = B$. We can find a function $d^* : (A \cup I)^2 \to \{0, \ldots n\}$ so that the bijection from AB to AB_i given by enumeration is an isomorphism, and so that d^* agrees with d on I. If we show this is a metric, then by quantifier elimination, there is $I' \equiv_{CB} I$ indiscernible over A, showing that $\mathrm{tp}(A/CB)$ does not contain any formulas dividing over C since I was arbitrary. Without loss of generality, it suffices to show the triangle inequality for d^* on $\{b_0, b_1, a\}$ for $b_0 \in B_0$, $b_1 \in B_1$, $a \in A$, $b_0 \neq b_1$. Suppose first that $d^*(b_0, a), d^*(b_1, a) \in \{n^*, n^*+1\}$. Then since $d^*(b_0, b_1) \geq 1$, the triangle inequality is immediate in all directions. Otherwise, without loss of generality, $d^*(b_0, a)$ is either n^* or n^*+1, and $d^*(b_1, a)$ is either greater than n^*+1 and equal to $\max_{c \in C}(|d(b_1, c) - d(a, c)|)$ or less than n^* and equal to $\min_{c \in C}(d(b_1, c) + d(a, c))$. In either case, by Lemma 5.3.3 and Remark 5.3.4 applied to $C \subseteq A, I$, there is a metric on $A \cup I$ that agrees with d^* on $\{a, b_0, b_1\}$, so d^* satisfies the triangle inequality on $\{a, b_0, b_1\}$.

To show $A \mathop{\smile\!\!\!\!\!\!|}^{f}_{C} B$, we need the following claim:

Claim 5.3.7. *Let the relation* $A \mathop{\smile\!\!\!\!\!\!|}_{C} B$ *be defined to hold when* A *and* B *have distance* $\geq n^*$ *and* $\leq n^*+1$ *over* C *has right extension: if* $B \subseteq D$ *there is* $A' \equiv_B A$ *with* $A' \mathop{\smile\!\!\!\!\!\!|}_{C} D$.

Proof. By Lemma 5.3.3, we may find $A' \equiv_B A$ so that AB is freely amalgamated with D over B. So it suffices to show that if A is freely amalgamated with D over B and $A \mathop{\smile\!\!\!\!\!\!|}_{C} B$, then $A \mathop{\smile\!\!\!\!\!\!|}_{C} D$. We first show that A and D have a distance of $\geq n^*$ over C. Suppose $a \in A \backslash C$, $d \in D \backslash C$, and $d(a, d) < n^*$. Then there is some $b \in B$ (and we can assume $b \neq d$) so that $d(a, d) = d(b, d) + d(b, a)$ Then $d(b, a) < n^*$ so there is some $c \in C$ so that $d(b, a) = d(b, c) + d(a, c)$. So $d(a, d) = d(b, d) + d(b, c) + d(a, c) \geq d(d, c) + d(a, c)$, as desired. We now show that A and D have a distance of $\leq n^*+1$ over C. Suppose $a \in A \backslash C$, $d \in D \backslash C$, and $d(a, d) > n^*+1$. Then there is some $b \in B$ (and we can assume $b \neq d$) so that either $d(a, d) = d(b, d) - d(b, a)$ or $d(a, d) = d(b, a) - d(b, d)$. Assume first that $d(a, d) = d(b, d) - d(b, a)$. Then as $d(a, d) > n^*+1$, $d(b, a) < n^*$, so there is some $c \in C$ with $d(b, a) = d(b, c) + d(c, a)$. Then $d(a, d) = d(b, d) - d(b, c) - d(c, a) \leq d(d, c) - d(c, a)$. Now assume $d(a, d) = d(b, a) - d(b, d)$. Then as $d(a, d) > n^*+1$, $d(b, a) > n^*+1$, so there is some $c \in C$ with either $d(b, a) = d(b, c) - d(a, c)$ or $d(b, a) = d(a, c) - d(b, c)$. If $d(b, a) = d(b, c) - d(a, c)$, then $d(a, d) = (d(b, c) - d(b, d)) - d(a, c) \leq d(d, c) - d(a, c)$. If $d(b, a) = d(a, c) - d(b, c)$, then $d(a, d) = d(a, c) - (d(b, c) + d(b, d)) \leq d(a, c) - d(d, c)$. Either way, this is as desired. □

Then if A and B have distance $\geq n^*$ and $\leq n^*+1$ over C, we can assume B is an $|C|^+$-saturated model by the claim, so the it follows as in the second paragraph of the proof of Lemma 5.2.1.2 and the fact that $\mathrm{tp}(A/BC)$ does not divide over C that $A \mathop{\smile\!\!\!\!\!\!|}^{f}_{C} B$.

□

Now suppose $2^{k+1} \geq n$. First, $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{d}^k}$ always implies $\mathop{\smile\!\!\!\!\!\!|}^{a}$, as $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{d}^0}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{a}$ and if $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{d}^i}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{a}$ it is seen that $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{d}^{i+1}}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{a}$. We now show that $\mathop{\smile\!\!\!\!\!\!|}^{a}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{d}^k}$. Let $\mathop{\smile\!\!\!\!\!\!|}^{0} = \mathop{\smile\!\!\!\!\!\!|}^{a}$, and for $m \geq 1$, let $A \mathop{\smile\!\!\!\!\!\!|}^{m}_{C} B$ indicate that A has distance $\geq \min(2^m, n^*)$ and $\leq \max(n^*, n - (\sum_{i=0}^{m-1} 2^i)) = \max(n^*, n - (2^m - 1))$ from B over C. Then by repeated applications of Lemma 5.3.5, where $k_1 = 2^i$ and $k_2 = n - (\sum_{j=0}^{i-1} 2^j) = n - (2^i - 1)$ for $i \geq 0$, if $A \mathop{\smile\!\!\!\!\!\!|}^{i}_{C} B$ there is an $\mathop{\smile\!\!\!\!\!\!|}^{i+1}$-Morley sequence $\{B_j\}_{j<\omega}$ over C with $B_0 = B$ indiscernible over

A. Moreover, $\mathop{\smile\!\!\!\!\!\!|}^{k}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!|}^{\eth^0}$ and has right, and therefore left extension by Lemma 5.3.6 and Claim 5.3.7, as $2^k \geq n^*$. So by the proof of Proposition 5.2.2, and the fact that if $A \mathop{\smile\!\!\!\!\!\!|}_{C}^{k-1} B$ there is an $\mathop{\smile\!\!\!\!\!\!|}^{k}$-Morley sequence (and therefore an $\mathop{\smile\!\!\!\!\!\!|}^{\eth^0}$-Morley sequence) $\{B_j\}_{j<\omega}$ over C with $B_0 = B$ indiscernible over A, $\mathop{\smile\!\!\!\!\!\!|}^{k-1}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{\eth^1}$. Suppose inductively that $\mathop{\smile\!\!\!\!\!\!|}^{k-i}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{\eth^i}$ for $1 \leq i \leq k$. Then by Proposition 5.2.2 and the fact that if $A \mathop{\smile\!\!\!\!\!\!|}_{C}^{k-(i+1)} B$ there is an $\mathop{\smile\!\!\!\!\!\!|}^{k-i}$-Morley sequence (and therefore an $\mathop{\smile\!\!\!\!\!\!|}^{\eth^i}$-Morley sequence) $\{B_j\}_{j<\omega}$ over C with $B_0 = B$ indiscernible over A, $\mathop{\smile\!\!\!\!\!\!|}^{k-(i+1)}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{\eth^{i+1}}$. So $\mathop{\smile\!\!\!\!\!\!|}^{a} = \mathop{\smile\!\!\!\!\!\!|}^{0}$ implies and is therefore equal to $\mathop{\smile\!\!\!\!\!\!|}^{\eth^k}$.

This concludes the proof of Theorem 5.3.2, and our discussion of the free roots of the complete graph.

Example 5.3.8. (Model companion of directed graphs with no directed cycles of length $\leq n$.) In Example 2.8.3 of [101], Shelah shows that the theory of directed graphs with no directed cycles of length $\leq n$ has a strictly NSOP_{n+1} model companion T_n. (Note that if there is a directed cycle of length $\leq n$ that repeats vertices, there is one that does not repeat any vertices.) We show again that if $2^k \geq n$, then $\mathop{\smile\!\!\!\!\!\!|}^{\eth^k}$ coincides with $\mathop{\smile\!\!\!\!\!\!|}^{a}$. This will be for somewhat different reasons than Example 5.3.1, with successive approximations of forking-independence given by longer directed distances, rather than distances tending away from the extremes. We will use the same technique as in Theorem 5.3.2, but the proof will be more intuitive.

Shelah observes that T_n has quantifier elimination in the language with binary relation symbols $R_n(x, y)$ indicating a path of length n from x to y. We refine this quantifier elimination. Let $n^* = \lceil \frac{n}{2} \rceil$. Then there must be a path of length $\leq n^*$ between any two nodes by existential closedness: if there is not one, one can be added on without creating any cycles of length $\leq n$. Moreover, by the lack of cycles, if there is a path of $\leq \frac{n}{2}$ there must be such a path in only one direction, by the absence of directed cycles of length $\leq n$. So if n is even any pair of distinct nodes a, b has one of $2n^* = n$ types, depending on the length of a minimal path, which will be at most n^*, and the direction of that path; call this the "directed distance" between a and b. If n is odd, any pair of distinct nodes has one of $2n^* + 1 = n$ types, depending on the length of a minimal path, which will be at most n^*, and the direction of that path if its length is $< n^*$; if the length of a minimal path is n^*, a and b will have an bi-directed distance of n^*, and a and b will otherwise have a directed distance of $< n^*$.

The type of a set S of distinct nodes will be determined by the distances between any two elements of S. We give necessary and sufficient criteria for an assignment of distances to pairs of nodes in S to be consistent:

(1) If a directed path of length $k \leq \frac{n}{2}$ is indicated by a chain of directed distances from a and b with total length d (i.e. there are $a = a_0, a_1, \ldots, a_{m-1}, a_m = b$, with a directed distance of d_i from a_{i-1} to a_i and $d = \sum_{i=1}^{n} d_i \leq \frac{n}{2}$), then the directed distance from a to b is at most d.

(2) Chains of distances distances cannot indicate a directed cycle of length $\leq n$ (i.e. $a = a_0, a_1, \ldots, a_{m-1}, a_m = a$, with a directed (or bi-directed) distance of a_i from a_{i-1} to a_i and $d = \sum_{i=1}^{n} d_i \leq n$).

Clearly (1) and (2) are necessary. To show they are sufficient, assume distances are assigned to pairs in a set S; we will find a model M containing S realizing those assignments. For each pair $a, b \in S$, if a directed distance of d (or an bi-directed distance of $d = n^*$) is assigned from a to b, draw a path from a to b of length d as well as a path in the opposite direction of length $n+1-d$. We show that S together with these new vertices has no directed cycles of length $\leq n$. Suppose it contained such a cycle. That cycle can be partitioned into the paths added between elements of S. Suppose it contained the longer of the two paths, of length $n-d+1$, added from, say, b to a. It cannot contain the shorter path, of length d, between a and b, as then this cycle would be too long. But perhaps there another path that is even shorter than the directed distance d from a to b, formed out of paths of length less than $d \leq n^*$ between other vertices in S. This cannot happen, by (1). (This also handles the case where a and b have an bi-directed distance of n^* when n is odd.) Otherwise, all of the added paths between elements of S that make up the cycle of length $\leq n$, are the paths of length $< n^*$ going in the direction of the directed distances. But this cannot happen, by (2). Call the union of S with the additional paths T, which we have shown has no cycles of length $\leq n$, and find a model M containing T. Then any two nodes in S will have directed distance in M at most the assigned distance d, as we added the shorter path going in that direction, but no less than the assigned distance, as we added the longer paths of length $n-d$ in the other direction, so a new path of length $< d$ in the direction of the directed distance would create a cycle of length $\leq n$.

We next show, as an analogue of Lemma 5.3.3, that sets can be "freely amalgamated:" if $A \cap B = C$ with A and B given consistent assignments of distances agreeing on C, then there is an assignment of distances on C agreeing with that on A and B and so that for $a \in A \backslash C$, $b \in B \backslash C$, the distance between a and b is the least total length of a chain of directed distances of total length $\leq n^*$ between A and B going through C (i.e., without requiring a directed distance between a point of $A \backslash C$ and a point of $B \backslash C$ as one of the steps in the chain), going in the direction of that chain, and is otherwise of length n^*. We first show that the directed distances already within $A \cup B$ (i.e. not between a point of $A \backslash C$ and a point of $B \backslash C$) satisfy (1) and (2) (i.e., if $a \in A$ then $b \in A$, and if $a_i \in A$ then $a_{i+1} \in A$, and similarly for B.) For (1), we can assume without loss of generality that a and b are in A, and then the chain can be broken up into parts in A and parts in B each going between two nodes of C, but all of the parts in B can be presumed to be in $C \subseteq A$, by (1) on B; having assumed this, we can then apply 1 on A. For (2), a directed cycle of length $\leq n$ formed by chains of directed distances can similarly be broken up into parts in A and parts in B, but only one of the parts, say in A can have length $\geq n^*$, and all of the other parts, by (1) in A and B, can be presumed in $C \subseteq A$, contradicting (2) in A. So the direction of the chain of shortest total length $\leq n^*$ between a point of $A \backslash C$ and a point of $B \backslash C$ going through C is well-defined, by (2) for chains in $A \cup B$ going through C, so if such a chain exists we use it as the definition for the directed distance, and otherwise we choose a distance in an *arbitrary* direction of size n^*.

It remains to show (1) and (2) on the whole of $A \cup B$. For (1), if a, b are not both in A or both in B, then any directed distance of length $\leq n^*$ on a chain from a to b between a point of $A \backslash C$ and a point of $B \backslash C$ can be replaced with a chain going through C of the same total length in the same direction, by construction. So any chain of directed distances between a and b can be replaced by a chain of directed distances of the same total length between a and b and in the same direction, going through C, so the directed distance between a and b will be as short or shorter in that direction, by definition. To complete (1) on the whole of $A \cup B$, if a, b both belong to, say, A, then again by construction we can assume a chain of directed distances to be a chain going through C, and then use (1) for chains going through C. Finally, for (2), suppose first that the distances indicating a $\leq n$ cycle are either between two points of A or B or added between a point of $A \backslash C$ and a point of $B \backslash C$ because of a chain of the same length and the same direction going through C. Then those distances can be replaced with those chains, reducing (2) to the case of chains going through C, see above. Otherwise, one of the distances must be n^* between a point $a \in A \backslash C$ and a point $b \in B \backslash C$, added because there is no chain going through C of length $\leq n^*$. But the rest of the cycle must be a chain of distances of total length $\leq n^*$, each of which, being of length $< n^*$ can be replaced with chains going through C, and can thus be assumed a chain of length $< n^*$ going through C from a to b, a contradiction to the assumption on a and b that caused their distance to have length n^*.

Say that $C \subset A, B$, sitting in a fixed model, have distance $\geq k$ over C for $k \leq n^*$ if $A \cap B = C$ and any $a \in A \backslash C$ and $b \in B \backslash C$ have distance in the same length and direction as the directed chain of distances of minimal total length through C in $A \cup B$ if that length is $< k$, and otherwise have distance of length $\geq k$. We show an analogue of Lemma 5.3.5: if $C \subseteq A, B, D$ with $A \cup D$ and $B \cup D$ freely amalgamated over D as in the above discussion, and A and B both of distance $\geq k$ from D over C, then (a) A has distance $\geq \min(2k, n^*)$ from B over C and (b) $A \cup B$ has distance $\geq k$ from D over C. To show A, if a point $a \in A \backslash C$ and $b \in B \backslash C$ have distance of length $< \min(2k, n^*)$ (say, going from a to b), then there must be a chain of length $< \min(2k, n^*)$ going through D. This chain can be broken alternately into parts entirely in A and entirely in B. Using the fact that the chain has length $\leq n^*$, all of the parts except for the first and the last can be assumed entirely in D, and then replaced by (1) with a single distance between two points in D; that distance and the distance from the second point in D to B can then be replaced by a single distance, giving a chain of length 2, going from a to $d \in D$ and then to B. One of the two distances, say from a to d, must have length $< k$, so there must be a chain of distances with the same total length and direction from A to D, going through C in $A \cup D$. So we can replace the distance from a to d with a distance from a to c followed by from c to d, and can then replace the distance from c to d and from d to b with a single distance from c to b, yielding a chain of equal or shorter length from a to b going through C, as desired. Meanwhile, (b) is obvious.

Next, we show, if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ denotes a distance of $\geq n^*$ (so, A and B are freely amalgamated over C; note that this amalgamation is not unique), it implies forking-independence. We first show it implies dividing-independence: let $I = \{B_i\}_{i<\omega}$ with $B_0 = B$ be a C-indiscernible sequence. Assign distances on $A \cup I$ to give AB_i the same quantifier-free type as AB, and so the assignment agrees with the actual one on I. It suffices to show this satisfies (1) and (2). If $b \in B_i \backslash C$ and $a \in A \backslash C$, then if there is a chain of distances of length $\leq n^*$ in $A \cup I$ going through C between a and b, then by breaking this chain up into parts in A and parts in I, and replacing the parts in I with parts in B_i, there is a chain of distances of total length at least as short going in the same direction in $A \cup B_i$ through C. So the distance between a and b in our chosen assignment is the length and direction of the chain of least total length between a and b in $A \cup I$ going through C, using the actual assignments on A and I. For the other pairs $a \in A \backslash C$ and $b \in I \backslash C$, a distance of length n^* is chosen. But in constructing the free amalgam, we showed that an arbitrary choice of direction for distances of length n^* (though there will only be a choice when n is even) is allowed for pairs with no path of length $\leq n^*$ going through the base. So our chosen assignment is one instance of the construction of the free amalgam of A and I over C, so in fact satisfies (1) and (2).

It remains to show right-extension for $\mathop{\smile\!\!\!\!\!\!|}$, which will hold if it is transitive, that is for $C \subseteq A$ and $C \subseteq B \subseteq D$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_B D$ implies $A \mathop{\smile\!\!\!\!\!\!|}_C D$. Suppose that $a \in A \backslash C$, $d \in D \backslash C$ and a and d have a distance $< n^*$, say, going from a to d. Then there is, as above, a chain of distances going from, say a to $b \in B$ to d, of the same total length. There is also a chain going from a to $c \in C$ to d of the same total length as the distance between a and d, because that distance is also $< n^*$, so following that with the distance from $d \in D \subseteq B$ to $b \in B$, we get a chain going through C of the same total length as the distance from a to b, as desired.

Now let $\mathop{\smile\!\!\!\!\!\!|}^0 = \mathop{\smile\!\!\!\!\!\!|}^a$, $\mathop{\smile\!\!\!\!\!\!|}^i$ denote a distance of $\min(2^i, n^*)$ over the base. Exactly as in Example 5.3.1, we can then show that $\mathop{\smile\!\!\!\!\!\!|}^{\eth^k} = \mathop{\smile\!\!\!\!\!\!|}^a$ when $2^{k+1} \geq n$.

Example 5.3.9. (Model companion of undirected graphs without odd cycles of length $\leq n$, for n odd).

In [101], Shelah shows that the theory T_n of undirected graphs without odd cycles of length $\leq n$, for n odd, has a model companion and is strictly NSOP_{n+1}. (This theory is further developed in [24].) Again, this theory has quantifier elimination in the language with binary relations for paths of length k ([101]). Instead of directed distances in the previous example, we now have the minimal length of a path between two vertices, an undirected distance which will be $\leq n^* = \lceil \frac{n}{2} \rceil$. A similar analysis to example 5.3.8 will hold in this theory. Note, however, that because it is never true that $n = 2^k$, T_n in this case cannot be used to witness that there are SOP_{2^k} theories where $\mathop{\smile\!\!\!\!\!\!|}^{\eth^k}$ is symmetric.

5.4 Bounds for symmetry and transitivity

We now show that $\mathrm{SOP}_{2^{k+1}+1}$ is required for $\mathop{\smile\!\!\!\!\!\!|}^{\eth^k}$ to be symmetric. (See Theorem 1.6.2 for a related result on NSOP_4.) From this and the previous section will follow the second clause of this theorem:

Theorem 5.4.1. *Assume* $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\eth^n}$ *is symmetric for* $n \geq 1$. *Then* T *is* $\mathrm{NSOP}_{2^{n+1}+1}$. *Thus* $2^{n+1}+1$ *is the least* k *so that every theory where* $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\eth^n}$ *is symmetric is* NSOP_k.

We state the construction; fix a Skolemization of T. Suppose T is $\mathrm{SOP}_{2^{n+1}+1}$; we show $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\eth^n}$ is asymmetric. Let $R(x,y)$ witness this; then there is an indiscernible sequence $\{c^*_i\}_{i \in 3\mathbb{Z}}$ so that $\models R(c_i, c_j)$ for $i < j$, but there are no $(2^{n+1}+1)$-cycles. Let $M = \mathrm{dcl}_{\mathrm{Sk}}(\{c^*_i\}_{i \in \mathbb{Z}} \cup \{c^*_{2\mathbb{Z}+i}\}_{i \in \mathbb{Z}})$, and let $c_i = c^*_{\mathbb{Z}+i}$ for $i \in \mathbb{Z}$. For $k \geq 1$, let $R_k(x,y) =: \exists x_0 \ldots x_{n-2} R(x, x_0) \wedge \bigwedge_{i=0}^{n-3} R(x_i, x_{i+1}) \wedge R(x_{n-2}, y)$ (so $R_1(x,y) =: R(x,y)$ and $R_2(x,y) =: \exists x_0 R(x,x_0) \wedge R(x_0, y)$. We find instances of k-ð-dividing:

Lemma 5.4.2. *Let* $0 \leq k \leq n$. *Then*

$$R^k(y_0, \ldots, y_{2^k-1}, c_0, \ldots, c_{2^k}) =: \bigwedge_{i=0}^{2^k-1} R_{2^{n-k}}(c_i, y_i) \wedge R_{2^{n-k}}(y_i, c_{i+1})$$

k-ð-*divides (and therefore* k-ð-*forks) over* M.

Proof. By induction on k. For $k = 0$, we show that $R_{2^n}(c_0, y) \wedge R_{2^n}(y, c_1)$ divides over M, specifically by $\{c_{2i}c_{2i+1}\}_{i<\omega}$. That is, $\{R_{2^n}(c_{2i}, y) \wedge R_{2^n}(y, c_{2i+1})\}_{i<\omega}$ is inconsistent; suppose it is consistent, realized by c. Then $\models R_{2^n}(c, c_1) \wedge R(c_1, c_2) \wedge R_{2^n}(c_2, c)$. So there is a $(2^{n+1}+1)$-cycle, a contradiction.

Now suppose the statement holds for $0 \leq k \leq n-1$; we prove it is true for $k+1$. Let $\{c^i_j\}^{i<\omega}_{0 \leq j \leq 2^{k+1}}$ be a sequence with $c^0_j = c_j$ so that $\{R^{k+1}(y_0, \ldots, y_{2^{k+1}-1}, c^i_0, \ldots, c^i_{2^{k+1}})\}_{i<\omega}$ is realized by $c'_0, \ldots, c'_{2^{k+1}-1}$; by the definition of $(k+1)$-ð-dividing, it suffices to show that $\{c^i_j\}^{i<\omega}_{0 \leq j \leq 2^{k+1}}$ is not an $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\eth^k}$-Morley sequence over M. For $0 \leq i \leq 2^k - 1$, in particular

$$\models R_{2^{n-(k+1)}}(c^1_{2i}, c'_{2i}) \wedge R_{2^{n-(k+1)}}(c'_{2i}, c^0_{2i+1}) \wedge R_{2^{n-(k+1)}}(c^0_{2i+1}, c'_{2i+1}) \wedge R_{2^{n-(k+1)}}(c'_{2i+1}, c^1_{2(i+1)})$$

It follows that $\models R_{2^{n-k}}(c^1_{2i}, c^0_{2i+1}) \wedge R_{2^{n-k}}(c^0_{2i+1}, c^1_{2(i+1)})$ for $0 \leq i \leq 2^k - 1$. Therefore,

$$R^k(y_0, \ldots, y_{2^k-1}, c^1_0, \ldots, c^1_{2i}, \ldots, c^1_{2^{k+1}}) \in \mathrm{tp}(c^0_1 \ldots c^0_{2i+1} \ldots c^0_{2^{k+1}-1}/M c^1_0 \ldots c^1_{2i} \ldots c^1_{2^{k+1}})$$

Because $c^1_0 \ldots c^1_{2i} \ldots c^1_{2^{k+1}} \equiv_M c^0_0 \ldots c^0_{2i} \ldots c^0_{2^{k+1}} = c_0 \ldots c_{2i} \ldots c_{2^{k+1}} \equiv_M c_0 \ldots c_{2^k}$, and $R^k(y_0, \ldots, y_{2^k-1}, c_0, \ldots, c_{2^k})$ k-ð-divides over M by the induction hypothesis, $R^k(y_0, \ldots, y_{2^k-1}, c^1_0, \ldots, c^1_{2i}, \ldots, c^1_{2^{k+1}})$ k-ð-divides over M, so

$$c^0_1 \ldots c^0_{2i+1} \ldots c^0_{2^{k+1}-1} \not\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\eth^k}_M c^1_0, \ldots c^1_{2i} \ldots c^1_{2^{k+1}}$$

and therefore,

$$c^0_0 \ldots c^0_{2^{k+1}} \not\mathop{\smile\hspace{-0.9em}\mid\hspace{0.4em}}^{\eth^k}_M c^1_0 \ldots c^1_{2^{k+1}}$$

So $\{c^i_j\}^{i<\omega}_{0\leq j\leq 2^{k+1}}$ is not an $\mathop{\smile\!\!\!\!\!\vert}^{\eth^k}$-Morley sequence over M.

□

It follows from the case $k = n$ of Lemma 5.4.2 and an automorphism that

$$\{c_{2i-1}\}_{-2^{n-1}<i\leq 2^{n-1}} \mathop{\not\!\perp}^{\eth^n}_{M} \{c_{2i}\}_{-2^{n-1}\leq i\leq 2^{n-1}}$$

So we have obtained an instance of n-ð independence. When $n = 1$, so $c_{-1}c_1 \not\perp^{\eth^1}_M c_{-2}c_0c_2$, this is one direction of the asymmetry: $c_{-2}c_0c_2 \perp^{\eth^1}_M c_{-1}c_1$. To show this, we extend $\{c_i\}_{i\in\mathbb{Z}}$ to an M-indiscernible sequence $\{c_i\}_{i\in\mathbb{Q}}$. Then by construction, $\{c_{-(1+i)}c_{1+i}\}_{i\in[0,1)}$ (note $c_{-(1+0)}c_{1+0} = c_{-1}c_1$) is a finitely satisfiable Morley sequence over M, indiscernible over $Mc_{-2}c_0c_2$. So $c_{-2}c_0c_2 \perp^{\eth^1}_M c_{-1}c_1$ follows from the following fact, which is immediate from Fact 1.6.1 (this is just a standard application of left extension for finite satisfiability, as in the proof of Proposition 5.2.2):

Fact 5.4.3. *Let $\{b_i\}_{i<\omega}$ be a finitely satisfiable Morley sequence over M with $b_0 = b$ so that $\{\varphi(x, b_i)\}_{i<\omega}$ is consistent. Then $\varphi(x, b)$ does not 1-ð-fork over M.*

This concludes the NSOP_5 case. When $n \geq 2$, we may not be able to obtain the desired finitely satisfiable Morley sequence. However, unlike the case where $n = 1$, we would not need anything stronger than an $\perp^{\eth^{n-1}}$-Morley sequence to show that a formula does not n-ð-fork over M, as opposed to just n-ð-dividing over M, because n-ð-forking already coincides with n-ð-dividing (Proposition 5.2.2). Still, we do not show that $\{c_{2i}\}_{-2^n\leq i\leq 2^n} \perp^{\eth^n}_M \{c_{2i-1}\}_{-2^n<i\leq 2^n}$ using an explicit $\perp^{\eth^{n-1}}$-Morley sequence. Rather, let $m \leq 2^{n-1}$ be least such that

$$\{c_{2i-1}\}_{-m<i\leq m} \mathop{\not\!\perp}^{\eth^n}_{M} \{c_{2i}\}_{-m\leq i\leq m}$$

Then $m > 0$. Let $\bar{a} = \{c_{2i}\}_{-m<i<m}$, $\bar{b} = \{c_{2i-1}\}_{-m<i\leq m}$, and $\bar{c} = c_{-2m}c_{2m}$. Then $\bar{b} \not\perp^{\eth^n}_M \bar{a}\bar{c}$, $\bar{a} \perp^{\eth^n}_M \bar{b}$ by minimality of m and the fact that $\bar{a}\bar{b} \equiv_M \{c_{2i-1}\}_{-(m-1)<i\leq m-1}\{c_{2i}\}_{-(m-1)\leq i\leq m-1}$, and $\mathrm{tp}(ab/cM)$ is finitely satisfiable over M by construction. To show asymmetry of $\perp^{\eth^n}_M$, it remains to show that $\bar{a}\bar{c} \perp^{\eth^n}_M \bar{b}$. We use the proof technique from Claim 3.6.2. Let $\varphi(\bar{x}, \bar{z}, \bar{b}) \in \mathrm{tp}(\bar{a}\bar{c}/M\bar{b})$; we show it does not n-ð-fork over M, for which it suffices that it not n-ð-divide over M. More explicitly, $\models \varphi(\bar{a}, \bar{c}, \bar{b})$, so $\models \varphi(\bar{a}, \bar{y}, \bar{b}) \in \mathrm{tp}(\bar{c}/M\bar{a}\bar{b})$. By finite satisfiability, there is some $\bar{m} \in M$ so that $\models \varphi(\bar{a}, \bar{m}, \bar{b})$. So $\varphi(\bar{x}, \bar{m}, \bar{b}) \in \mathrm{tp}(\bar{a}/M\bar{b})$. Because $\bar{a} \perp^{\eth^n}_M \bar{b}$, there is then some $\perp^{\eth^{n-1}}$-Morley sequence $\{\bar{b}_i\}_{i<\omega}$ over M with $\bar{b}_0 = \bar{b}$ so that $\{\varphi(\bar{x}, m, \bar{b}_i)\}_{i<\omega}$ is consistent. A fortiori, $\{\varphi(\bar{x}, \bar{z}, \bar{b}_i)\}_{i<\omega}$ is consistent. So $\varphi(\bar{x}, \bar{z}, \bar{b})$ does not n-ð-divide over M, and as $\varphi(\bar{x}, \bar{z}, \bar{b}) \in \mathrm{tp}(\bar{a}\bar{c}/M\bar{b})$ was arbitrary, $\bar{a}\bar{c} \perp^{\eth^n}_M \bar{b}$. This concludes the proof of Theorem 5.4.1.

In all of the examples of the previous section, $\perp^{\eth^n} = \perp^a$ has the following properties:

Right transitivity: $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M} M'$ and $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M'} b$ implies $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M} b$
Left transitivity:
$M' \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M} b$ and $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M'} b$ implies $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M} b$
when $M \prec M'$ are models. We show:

Theorem 5.4.4. *Assume $\mathop{\smile\hspace{-0.9em}|}^{\eth^n}$ is left or right transitive for $n \geq 1$. Then T is $\mathrm{NSOP}_{2^{n+1}+1}$. Thus $2^{n+1}+1$ is the least k so that every theory where $\mathop{\smile\hspace{-0.9em}|}^{\eth^n}$ is right transitive is NSOP_k, and similarly for left transitivity.*

The proof is easier, as to show, say, right transitivity fails, we produce $a, b, M_0 \prec M_1 \prec \ldots \prec M_k$ so that $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M_i} M_{i+1}$ for $0 \leq i \leq k$, $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M_k} b$, but $a \mathop{\not\smile\hspace{-0.9em}|}^{\eth^n}_{M_0} b$; the dependency $a \mathop{\not\smile\hspace{-0.9em}|}^{\eth^n}_{M_0} b$ will be a slight modification of the above to produce models, but the instances of independence, $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M_i} M_{i+1}$, will arise directly from the construction, unlike in the proof for symmetry. Again, assume $R(x, y)$ gives an instance of $\mathrm{SOP}_{2^{n+1}+1}$. Choose a Skolemization of T and a T^{Sk}-indiscernible sequence $\{c^*_i\}_{\mathbb{Z}+\mathbb{Z}\times\mathbb{Z}+\mathbb{Z}}$ so that $\models R(c^*_i, c^*_j)$ for $i < j < \mathbb{Z}+\mathbb{Z}\times\mathbb{Z}+\mathbb{Z}$. Let $M_0 = \mathrm{dcl}_{\mathrm{Sk}}(\{c^*_i\}_{i\in\mathbb{Z}} \cup \{c^*_{\mathbb{Z}+\mathbb{Z}\times\mathbb{Z}+i}\}_{i\in\mathbb{Z}})$. For $i \in \mathbb{Z}\times\mathbb{Z}$, let $c'_i = c'_{\mathbb{Z}+i}$. Now $\mathbb{Z}\times\mathbb{Z}$ as a set of ordered pairs, ordered lexicographically, and define $c_i = \{c_{(i,j)}\}_{j\in\mathbb{Z}}$. Again, it follows from Lemma 5.4.2 that

$$\{c_{2i-1}\}_{-2^{n-1}<i\leq 2^{n-1}} \mathop{\not\smile\hspace{-0.9em}|}^{\eth^n}_{M_0} \{c_{2i}\}_{-2^{n-1}\leq i\leq 2^{n-1}}$$

To make the notation easier, let $a_i = c_{2(-2^{n-1}+1+i)-1}$ for $0 \leq i < 2^n$; in other words, a_i is the ith term of $\{c_{2i-1}\}_{-2^{n-1}<i\leq 2^{n-1}}$. Let $b_i = c_{2(-2^{n-1}+i)}$ for $0 \leq i \leq 2^n$. Let $a = \{a_i\}_{0\leq i<2^n}$ and $b = \{b_i\}_{0\leq i\leq 2^n}$. Then $a \mathop{\not\smile\hspace{-0.9em}|}^{\eth^n}_{M_0} b$. For the right transitivity case, it suffices to find $M_0 \prec M_1 \prec \ldots \prec M_k$ so that $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M_i} M_{i+1}$ for $0 \leq i \leq k = 2^n+1$, $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M_k} b$, despite having shown $a \mathop{\not\smile\hspace{-0.9em}|}^{\eth^n}_{M_0} b$. For $0 < i \leq 2^n+1$, let $M_i = \mathrm{dcl}_{\mathrm{Sk}}(\{b_0 \ldots b_{i-1}\})$. We show that for $0 < i < 2^n+1$, $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M_i} M_{i+1}$, and $a \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{M_k} b$. This follows directly from unwinding definitions and applying to $\{c^*_i\}_{i\in\mathbb{Z}+\mathbb{Z}\times\mathbb{Z}+\mathbb{Z}}$ the following claim:

Claim 5.4.5. *Let $\{e_i\}_{i\in I}$ be an indiscernible sequence in T^{Sk}, where I is a linear order. Let $J \subset I$ be a set with no greatest element. Let $I_1, I_2 \subseteq I$ be such that every element of I_2 is above every element of J, and no element of I_1 is between any two elements of I_2. Let $e_{I_2} = \mathrm{dcl}_{\mathrm{Sk}}(\{e_s\}_{s\in J\cup I_2})$, $e_J = \mathrm{dcl}_{\mathrm{Sk}}(\{e_s\}_{s\in J})$, and let $e_{I_1} = \{e_s\}_{s\in I}$. Then $e_{I_1} \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{e_J} e_{I_2}$ (in T).*

Proof. We may assume that any element of I_2 between two elements of I_2 is itself in I_2. Let I_- be the set of elements of I below all of the elements of I_2, and I_+ the set of elements of I above all of the elements of I_2. Extend $\{e_i\}_{i\in I}$ to $\{e_i\}_{i\in I_-+\omega\times I_2+I_+}$ (so that $e_{I_2} = \{e_s\}_{s\in J\cup\{0\}\times I_2}$) so that it is still indiscernible in T^{Sk}. For $i < \omega$, let $e^i_{I_2} = \mathrm{dcl}_{\mathrm{Sk}}(\{e_s\}_{s\in J\cup\{i\}\times I_2})$. Then $\{e^i_{I_2}\}_{i<\omega}$ is a finitely satisfiable Morley sequence over e_J with $e^0_{I_2} = e_{I_2}$, because every element of I_2 is above every element of J and J has no greatest element. Moreover, because no element of I_1 is in between any two elements of I_2, $\{e^i_{I_2}\}_{i<\omega}$ is indiscernible over $e_J e_{I_1}$. So $e_{I_1} \mathop{\smile\hspace{-0.9em}|}^{\eth^n}_{e_J} e_{I_2}$ by Fact 5.4.3. ☐

This completes the case of right transitivity. For left transitivity, we must find $M_0 = M^0 \prec M^1 \prec \ldots \prec M^k$ so that for $0 \leq i \leq 2^n$, $M^{i+1} \mathop{\smile\hskip -0.9em ^| \ }^{n\eth}_{M^i} b$, and $a \mathop{\smile\hskip -0.9em ^| \ }^{\eth^n}_{M^k} b$, despite having shown $a \not\mathop{\smile\hskip -0.9em ^| \ }^{\eth^n}_{M_0} b$. For $0 < i \leq 2^n$, let $M_i = \mathrm{dcl}_{\mathrm{Sk}}(\{a_0 \ldots a_{i-1}\})$. Then for $0 \leq i \leq 2^n$, $M^{i+1} \mathop{\smile\hskip -0.9em ^| \ }^{u}_{M^i} b$, and $a \mathop{\smile\hskip -0.9em ^| \ }^{u}_{M^k} b$. This completes the case of left transitivity and thus the proof of Theorem 5.4.4.

We conclude by asking whether the converse holds, giving us a theory of independence for $\mathrm{NSOP}_{2^{n+1}+1}$ theories:

Question 5.4.6. *Does* $\mathrm{NSOP}_{2^{n+1}+1}$ *imply symmetry of* $\mathop{\smile\hskip -0.9em ^| \ }^{\eth^n}$*? Does it imply transitivity of* $\mathop{\smile\hskip -0.9em ^| \ }^{\eth^n}$*?*

We close this dissertation with a quote from a great Armenian-American cultural critic:

See you in hell.
–Anna Khachiyan

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