Title
Directly observable quantities for light fields in natural hydrosols

Author
Preisendorfer, Rudolph W

Publication Date
1958-06-01

Visibility Laboratory
University of California
Scripps Institution of Oceanography
San Diego 52, California

DIRECTLY OBSERVABLE QUANTITIES FOR LIGHT FIELDS IN NATURAL HYDROSOLS

R. W. Preisendorfer

1 June 1958
Index Number NS 714-100

Bureau of Ships
Contract NObs-72092

SIO REFERENCE 58-46

Approved:

Seibert Q. Duntley

Seibert Q. Duntley, Director
Visibility Laboratory

Approved for Distribution:

Roger Revelle

Roger Revelle, Director
Scripps Institution of Oceanog

Directly Observable Quantities for Light Fields in Natural Hydrosols*

Rudolph W. Preisendorfer

Scripps Institution of Oceanography, University of California

La Jolla, California

ABSTRACT

The increasing accuracy of experimental determinations of the optical absorption and scattering properties of oceans, lakes, and harbors has necessitated exact knowledge of the possible interrelations between these properties. The classical two-flow analysis of the light field has been an important source of such relations. While it continues to be a source of useful simple models for engineering calculations, it has become inadequate to collate the data of basic experimental research. The two-flow theory is reformulated to meet these experimental needs. The reformulation is based on the exact equations for the light field in scattering-absorbing media. The main results give useful relations between the magnitudes of the downwelling and upwelling irradiances, their depth-rates of change, and the scattering and absorption properties of natural hydrosols. The reformulations are applicable in general to arbitrarily stratified plane-parallel media.

* Contribution from the Scripps Institution of Oceanography, New Series No. ______________. This paper represents results of research which has been supported by the Bureau of Ships, U. S. Navy.

INTRODUCTION

Radiative transfer theory, which arose in the attempts of astrophysicists to understand the problem of the passage of light through stellar and planetary atmospheres, was subsequently directed by geophysicists to the problem of the passage of light through natural waters such as oceans and lakes. While both fields of study share a common interest in the theory of optical media that scatter and absorb light, their experimental procedures are vastly different: the geophysicist has easy access to his optical media and can amass first hand data from even the remotest reaches of a natural hydrosol; the astrophysicist has no such direct access and is forced to devise mathematical models, and limit his explorations to theoretical studies of such models and their observable consequences.

Thus when the geophysicists adopted the more useful of these models, they inherited, for better or worse, certain simplifying assumptions built into them, assumptions which had been explicitly and deliberately introduced to make them analytically tractable and of some practical worth. In the course of time, as the study of the light fields in natural hydrosols became progressively more exact and exhaustive, these adopted models, which were extremely valuable in planning and interpreting the early geophysical experiments, became progressively inadequate especially in their predictive and correlative powers. However, the formulas connected with these models had proved so convenient, and their simplicity so appealing, that their impending loss set off a search for more exact replacements which were to retain, whenever possible, the utility and simplicity of the classical formulas. The results of such a search are summarized in the discussion that follows. The model considered is the modern version of the two-flow theory originally devised by Schuster. In what follows we will use the general formulation of the two-flow equations, and require precise knowledge of the assumptions adopted in the classical formulations. This groundwork has been covered in adequate detail elsewhere[1], and the following discussion will draw freely from that source.

While the motivation and main emphasis of this study is in connection with that branch of radiative transfer known as hydrological optics, it would perhaps be of interest to observe that the results summarized below can actually be applied to the experimental study of the transfer of radiation through any arbitrarily stratified plane-parallel medium with arbitrary incident lighting and reflecting boundary conditions.

---

[1] R. Preisendorfer, "Unified Irradiance Equations," submitted for publication to J. Opt. Soc. Am.

THEORETICAL K-FUNCTIONS

In the present section we will prepare the way for the desired reformulations by exhibiting the two-flow theory as it has been developed and used in hydrological optics, and by isolating those formulas which have been of importance in checking and collating experimental data. The key notion in all that follows is that of the diffuse absorption coefficient k and its generalizations.

Classical Two-flow Theory

The classical two-flow theory is embodied in the pair of equations,

$$\frac{dH^*(z,-)}{dz} = -[a^* + b^*]\,H^*(z,-) + b^* H^*(z,+) + f^\circ H^\circ(z,-)$$
$$-\frac{dH^*(z,+)}{dz} = -[a^* + b^*]\,H^*(z,+) + b^* H^*(z,-) + b^\circ H^\circ(z,-) \qquad (1)$$

Here $H^*(z,-)$ and $H^*(z,+)$ are the irradiances at depth z induced, respectively, by downwelling diffuse flux (-) and upwelling diffuse flux (+). The term, diffuse flux, refers to flux which has been scattered one or more times. $H^\circ(z,-)$ is the downwelling reduced irradiance at depth z and refers to flux which has not been scattered. a^* , b^* are the volume absorption and backward scattering coefficients for the diffuse flux. f° and b°

are the forward and backward scattering coefficients for the reduced flux. The quantities a^*, b^*, f°, and b° are assumed constant and known, $H^\circ(z,-)$ is assumed known for each depth z.

Diffuse Absorption Coefficient k

The parameter which plays the greatest single role in the classical two-flow theory is the so-called diffuse absorption coefficient k, defined as

$$k \equiv +[a^*(a^* + 2b^*)]^{\frac{1}{2}} = 2[a(a + b^*)]^{\frac{1}{2}} . \tag{2}$$

The alternate expression follows from the assumption that the diffuse flux is described by a uniform radiance distribution in the upper and lower hemispheres. a is the volume absorption coefficient, and is related to a^*, under the above assumption, by $a^* = 2a$. From a mathematical point of view, k is a natural consequence of the solution procedure for (1), which yields the general solutions:

$$\begin{aligned} H^*(z,-) &= g_- c_+ e^{kz} + g_+ c_- e^{-kz} - A(z) H^\circ(0,-), \\ H^*(z,+) &= g_+ c_+ e^{kz} + g_- c_- e^{-kz}. \end{aligned} \tag{3}$$

Here

$$g_- = \left(1 - \frac{a^*}{k}\right), \qquad g_+ = \left(1 + \frac{a^*}{k}\right),$$

and C_-, C_+ are constants of integration determined by the values $H^*(0,-)$, $H^*(z_1,+)$, and $H^\circ(0,-)$, where z_1 is the depth of the medium. A is a determinable function.

In order to fully understand the significance of the experimental K-functions discussed below, we underscore the fact that k, as defined in (2), and as used in the system (3), is a mathematical construct: it is devoid of any physical meaning as it emerges from the solution procedure. It has generally no basis in any realizable physical operation. Only in the case of an optically infinitely deep medium irradiated by a uniform radiance distribution may a physical interpretation be assigned to k. For, in this case, the system (3) reduces to

$$H(z,-) \equiv H^\circ(z,-) + H^*(z,-) = g_+ C_- e^{-kz} = H(0,-)\,e^{-kz}$$

$$H(z,+) \equiv H^\circ(z,+) + H^*(z,+) = g_- C_- e^{-kz} = H(0,+)\,e^{-kz} \tag{4}$$

$(H^\circ(z,+) = 0)$

In order to obtain the system (4), it was assumed that $H^\circ(z,-)$ — the reduced downwelling irradiance--is associated with an angularly uniform radiance distribution, and behaves with depth in the following manner:

$$H^\circ(z,-) = H^\circ(0,-)\,e^{-\alpha^\circ z},$$

where α^o is a constant. Under this assumption, it turns out that $A(z) = e^{-\alpha^o z}$. The final step to (4) was to deduce that $C_+ = 0$, and to recall that the observable irradiance $H(z,-)$ is the sum of $H^o(z,-)$ and $H^*(z,+)$. In the above case, and only that case, is it possible to interpret k as the decay-rate of the irradiances for both streams; and the system (4) suggests the following operational definition for k:

$$k = \frac{-1}{H(z,-)} \frac{dH(z,-)}{dz} = \frac{-1}{H(z,+)} \frac{dH(z,+)}{dz} . \qquad (5)$$

Even with this interpretation of k, its formal nature is inescapable; it is a rare medium, indeed, which exhibits a light field of exactly the structure described by the system (4).

The term 'diffuse absorption coefficient,' for k stems from the following three observations: First, k is defined in terms of coefficients a^*, b^* which are associated with the diffuse flux making up the light field. Second, in the extreme case where the medium exhibits only absorption, we have formally, $k = a^*$. And finally, if the medium exhibits no absorption, $k = 0$.

The R-infinity Formulas

In addition to the determination of the depth dependence of the values $H(z,-)$ and $H(z,+)$, the experimental determination of the ratio $H(z,+)/H(z,-)$ is of interest because knowledge of this ratio aids in the solution of underwater visibility and photography problems and certain aspects of marine biology problems.

The classical two-flow model yields several expressions for this ratio, the most important being:

$$R_\infty \equiv H(z,+)/H(z,-) = g_-/g_+ = (k - a^*)/(k + a^*) , \qquad (6)$$

which follows directly from (4). By using the definition of k, and some simple algebraic rearrangements, (6) may be cast into either of the following two equivalent forms:

$$R_\infty = \frac{a^* + b^* - k}{b^*} , \qquad (7)$$

$$R_\infty = \frac{b^*}{a^* + b^* + k} . \qquad (8)$$

The primary virtue of these formulas lies in their simple, explicit inclusion of b^*, whose magnitude may then be estimated if the remaining quantities are known.

Finally, by using the identity

$$\alpha^* = a^* + s^* = a^* + f^* + b^*,$$

we have:

$$R_\infty = \frac{\alpha^* - f^* - k}{b^*}, \tag{9}$$

$$R_\infty = \frac{b^*}{\alpha^* - f^* + k}. \tag{10}$$

In the above formulas, α^* is the volume attenuation coefficient for the diffuse flux. α^* is related to the basic volume attenuation coefficient α by the formula: $\alpha^* = 2\alpha$. α is the sum of the volume absorption coefficient a, and the volume total scattering coefficient s. If α is known, (9) and (10) may then be used to estimate f^*.

The Inequalities

One final set of relations which have consistently proved their usefulness in checking the experimentally obtained values of optical constants of turbid media are the inequalities

$$a^* \leq k < \alpha . \tag{11}$$

The left inequality follows from (6) by observing that R_∞, k, and a^* can never be negative. It also may be obtained from (2).

The right hand inequality is more difficult to establish. It can be "proved," for example, if one introduces the additional physical requirement that radiances can never be infinite in media that exhibit scattering.[2] It is on this basis that the left hand inequality is understood to hold.[+]

Observations

The increasing inadequacy of the two-flow model is perhaps most succinctly summarized in the proceding discussion of the basic inequalities of the theory. But this is a mathematical matter which may carry less weight than some more striking inadequacies

---

[2] R. Preisendorfer, "A Simple Model for Radiance Distribution in Natural Hydrosols," submitted for publication to J. Opt. Soc. Am.

[+] In media that exhibit no scattering, there can be no diffuse flux. The definition of k is so rigid, however, that a formal contradiction of the inequality $k < \alpha$ can be obtained by setting $\Delta = 0$. For then $b^* = 0$, and $k = a^* = 2a = 2\alpha$.

on the experimental front: the observed non-constancy of R_∞, the lack of an operational definition of k, except in extreme hypothetical cases; and the fact that the basic system (1) refers to non-observable quantities, namely $H^*(z,\pm)$. Most of the trouble stems from the fact that the angular structure of both the diffuse and reduced radiance distribution have, for all depths, been assigned a fixed uniform structure in both the upper and lower hemispheres. From this follows the constancy of all the absorption and scattering coefficient functions, and that of R_∞, which is in direct conflict with experimental observations.

The question then arises: is there some procedure which retains the conceptual simplicity of the classical two-flow analysis and at the same time incorporates an accurate representation of the effects of the depth dependence of the observable radiance distributions? The answer is yes. The details are summarized below.

EXPERIMENTAL K-FUNCTIONS

Distribution Functions

The distribution functions for the downwelling and upwelling streams are defined by:

$$D(z,-) = \frac{h(z,-)}{H(z,-)} \quad , \tag{12}$$

$$D(z,+) = \frac{h(z,+)}{H(z,+)} \quad . \tag{13}$$

The quantities $h(z,-)$ and $h(z,+)$ are the scalar irradiances induced, respectively, by the down-and upwelling streams at depth z. They may be measured by suitably shielded spherical collectors, whereas, the ordinary irradiances $H(z,-)$ and $H(z,+)$ are measured by flat plate collectors. The ratio of a hemispherical scalar irradiance to an irradiance provides a simple means of characterizing the directional distribution of flux in each stream. But their usefulness extends considerably beyond this service, as we shall see below.

Exact Equations for the Two-Flow Theory

In what follows, $H(z,-)$ and $H(z,+)$ are the undecomposed downwelling and upwelling irradiances: the irradiances actually measured by horizontal flat plate collectors. The exact equations governing these are.

$$\frac{dH(z,-)}{dz} = -\left[a(z,-)+b(z,-)\right]H(z,-) + b(z,+)H(z,+),$$

$$-\frac{dH(z,+)}{dz} = -\left[a(z,+)+b(z,+)\right]H(z,+) + b(z,-)H(z,-). \tag{14}$$

The other functions appearing in (14) are the absorption, $a(z,\pm)$, and backward scattering, $b(z,\pm)$, functions for the respective streams, their exact definitions and properties are given in reference 1. For brevity, source terms have been omitted from the system (14); that is, terms which describe self-luminous sources of radiant flux distributed throughout the medium. Their inclusion requires only slight changes in the following results.

K-Functions and Reflectance Functions

In (5) the diffuse absorption coefficient was expressed as the logarithmic derivative of the irradiances of each flow. It was shown that this simple operational characterization was possible only in a very special set of circumstances in the classical two-flow theory. It turns out, however, that such an operation is the natural way to characterize the decay (and growth) rate of each stream in real media. Each stream in general has its own depth-rate of change. These rates of change are not completely independent, as we shall see below, and they are generally different in magnitude. Thus we are led to define,

$$K(z,-) = \frac{-1}{H(z,-)} \frac{dH(z,-)}{dz} ,$$

$$K(z,+) = \frac{-1}{H(z,+)} \frac{dH(z,+)}{dz} . \tag{15}$$

Just as the K-functions for each stream are found to change with depth, so does the irradiance ratio,

$$R(z,-) = \frac{H(z,+)}{H(z,-)} , \tag{16}$$

and hence its reciprocal,

$$R(z,+) = \frac{H(z,-)}{H(z,+)} , \tag{17}$$

exhibit change with depth. The expression (16) is the exact experimental counterpart to R_∞ , and may be thought of as the reflectance of the hypothetical plane surface at depth z , with respect to downwelling flux. A similar interpretation exists for (17).

From (14) and the preceding definitions:

$$K(z,-) = a(z,-) + b(z,-) - b(z,+)\,R(z,-)\,, \tag{18}$$

$$-K(z,+) = a(z,+) + b(z,+) - b(z,-)\,R(z,+)\,, \tag{19}$$

which show how the experimental K-functions are linked to the absorption and scattering properties of the medium. From these, we immediately obtain the exact counterparts to (7) and (8):

$$R(z,-) = \frac{a(z,-) + b(z,-) - K(z,-)}{b(z,+)}\,, \tag{20}$$

$$R(z,+) = \frac{b(z,-)}{a(z,+) + b(z,+) + K(z,+)}\,. \tag{21}$$

While the overall resemblance to (7) and (8) is quite evident, care should be taken to distinguish the relatively subtle roles now played by the functions associated with each flow.

The counterparts to (9) and (10) are obtained by making use of the general identity:+

$$\alpha(z,\pm) = a(z,\pm) + s(z,\pm) = a(z,\pm) + f(z,\pm) + b(z,\pm),$$

Thus, in general

$$R(z,-) = \frac{\alpha(z,-) - f(z,-) - K(z,-)}{b(z,+)}, \tag{22}$$

$$R(z,-) = \frac{b(z,-)}{\alpha(z,+) - f(z,+) + K(z,+)}. \tag{23}$$

The Basic Reflectance Relation

The general counterpart to (6) is singled out for special attention because it is the most useful of reflectance functions in practice. It relates the six direct observables: the two K-functions, the two distribution functions, the volume absorption function, and the reflectance function. It therefore may replace (6) by providing an exact formula to check the consistency of the experimentally determined values of these functions. Furthermore, from a theoretical point of view, it is closely related to the divergence relation for the light field, in fact directly derivable from it as shown below. Alternate derivations, of course, can

---

+ These are two equations. For one equation read all upper signs together, for the other, read all lower signs together.

be made from the system (14). The divergence relation for the light field in stratified media may be written in the form[3],

$$\frac{d\overline{H}(z,+)}{dz} = a(z)\,h(z) \; ; \tag{24}$$

where $\overline{H}(z,+) = H(z,+) - H(z,-)$, $h(z)$ is the scalar irradiance at depth z, and $a(z)$ is the value of the volume absorption function at depth z.

Rewriting this as

$$\frac{dH(z,+)}{dz} - \frac{dH(z,-)}{dz} = a(z)\left[h(z,+) + h(z,-)\right] ,$$

and dividing each side by, say $H(z,-)$, we have,

$$\frac{H(z,+)}{H(z,-)}\,\frac{1}{H(z,+)}\,\frac{dH(z,+)}{dz} - \frac{1}{H(z,-)}\,\frac{dH(z,-)}{dz} = a(z)\left[\frac{h(z,+)}{H(z,+)}\,\frac{H(z,+)}{H(z,-)} + \right.$$

Applying the appropriate definitions, this may be rewritten

$$-R(z,-)K(z,+) + K(z,-) = a(z)\left[D(z,+)R(z,-) + D(z,-)\right]$$

$$= a(z,+)R(z,-) + a(z,-) ,$$

---

[3] R. Preisendorfer, "The Divergence of the Light Field in Optical Media," submitted for publication to J. Opt. Soc. Am.

Solving for $R(z,-)$:

$$R(z,-) = \frac{K(z,-) - a(z,-)}{K(z,+) + a(z,+)} , \tag{25}$$

which is the desired experimental counterpart to ([illegible]).

The Inequalities

To obtain the counterparts to the classical inequalities (11), we encounter a reversal of difficulty: the counterpart to $k < \alpha$ is relatively simple to establish, and its validity is completely general; the counterpart to $\alpha^* \leq k$ requires additional assumptions, but of a kind which are a consequence of the generality of the present formulations rather than their shortcomings.

The first member of (14) may be written as[1]:

$$\frac{dH(z,-)}{dz} = -\alpha(z,-)H(z,-) + \int_{\Xi_-} N_*(z,\xi)\, d\Omega(\xi)$$

so that

$$K(z,-) = \alpha(z,-) - \frac{1}{H(z,-)} \int_{\Xi_-} N_*(z,\xi)\, d\Omega(\xi) .$$

Since the subtracted member of the right side is never negative, we have immediately:

$$K(z,-) \leq \alpha(z,-)$$

for all z .

Finally, whenever $0 \leq K(z,+)$, we have (see note 1, below) from (25):

$$a(z,-) \leq K(z,-) ,$$

which establishes the desired inequalities:

$$a(z,-) \leq K(z,-) \leq \alpha(z,-) , \tag{26}$$

or equivalently:

$$a(z) \leq \frac{K(z,-)}{D(z,-)} \leq \alpha(z) .$$

A corresponding set of inequalities for the upwelling stream may be obtained in a similar way:

$$a(z,+) \leq -K(z,+) \leq \alpha(z,+) \tag{27}$$

or equivalently:

$$a(z) \leq -\frac{K(z,+)}{D(z,+)} \leq \alpha(z) .$$

The right-hand side of (27), as that of (26), holds in general; the left side of (27) holds whenever $K(z,-) \leq 0$, a condition completely symmetric to the condition, $0 \leq K(z,+)$, used to establish the left side of (26).

Observations and Notes

1. The significance of the condition, $0 \leq K(z,+)$, used to establish the left side of (26), is quite important; and the adoption of this condition raises some interesting questions. First of all, we observe that if the condition holds, then the denominator of (25) is positive. Since $R(z,-)$ is positive, this then requires the numerator of (25) to be positive, from which the desired inequality follows. But one may ask: is this condition ever violated? In other words, can we ever have: $K(z,+) < 0$? Before answering this, we recall that $K(z,+) < 0$ means that $K(z,+)$ is negative, and physically this means that the function+ $H(\cdot,+)$ is <u>increasing</u> with <u>increasing</u> depth at z . A similar interpretation exists for the condition $K(z,-) < 0$. The preceding question may then be put very concretely as follows: As one measures upwelling and downwelling irradiances in a real stratified optical medium, is it ever possible to observe an <u>increase</u> in these irradiances as depth is increased? The answer is yes. There are two possible mechanisms which generally allow such a phenomenon to be observed.

\+ This symbol stands for the irradiance <u>function</u>; we are simply observing the need to distinguish between a function and its values. $H(\cdot,+)$ is the function, and $H(z,+)$ is its value at z.

The first mechanism is that associated with self-luminous sources within the medium. For example, light-giving organisms distributed in a horizontal layer of water clearly make it possible for increases of irradiance to be observed as the irradiance collectors approach the layer and pass downward through the layer. These layers can occur at quite large depths. Other examples are given by various physical emission processes, scattering with change in wavelength, et cetera. The latter mechanisms play a subordinate role in natural hydrosols, but in the atmosphere, they can be important.

The second mechanism is that of simple scattering processes: the redirection of radiant flux without change in wavelength. This results in an effective storage of radiant energy within scattering media. It seems plausible that if an increase of irradiance is induced by this mechanism, the increase should occur at depths near the surface of the medium. For in these regions of small depth the diffuse light field is still building up in magnitude. Generally, in optically deep scattering media, the downwelling diffuse radiant flux is zero at the surface, increases with increasing depth, reaches a maximum at some depth, and then falls off in a more or less exponential way ever afterward. But the diffuse irradiance $H^{*}(z,-)$ is not directly observed. Superimposed on it is the reduced irradiance $H^{o}(z,-)$, which clearly must decrease continuously with depth, starting right from the upper surface. Thus, whether or not the observable irradiance $H(z,\pm) = H^{\circ}(z,\pm) + H^{*}(z,\pm)$ exhibits any increase with increasing depth clearly depends upon the relative rates of change of each of its components.

A theoretical discussion of the conditions which govern the growth of the light field in stratified media is out of place here. We merely note in passing that many such conditions can be extracted from expressions like (14), (18), or (19) given above; furthermore, various approximate models of the light field such as the two-D theory discussed in reference 1 give very explicit, if only approximate, criteria for the growth of the light field. Finally, the possibility of the growth of radiance values has been predicted by a simple model for radiance distributions in natural hydrosols. This prediction has been verified by experiment. The details are given in reference 3.

2. For source-free stratified media, we can make several general observations about the relative magnitudes of the observable H-values and K-values. These have been of help in checking and collating experimental data. First of all, from the integrated relation (21), we deduce that

$$\bar{H}(z_1,-) + \bar{H}(z_2,+) = \int_{z_1}^{z_2} a(z)\, h(z)\, dz \geq 0,$$

so that

$$\bar{H}(z_2,-) \leq \bar{H}(z_1,-), \tag{28}$$

which demonstrates that the net downward irradiance function $\overline{H}(\cdot,-) \equiv$ $\equiv H(\cdot,-) - H(\cdot,+)$ never increases with depth. Here z_1, and z_2 are any two depths, z_2 being the greater. If the medium is, in particular, finitely deep with a bottom whose reflentance r is $0 \le r \le 1$, then (28) immediately implies that, for all depths z,

$$R(z,-) = \frac{H(z,+)}{H(z,-)} \le 1 . \tag{29}$$

If the medium is optically infinitely deep, we have a similar result. For in this case, we have for all depths z :

$$\overline{H}(z,-) = \int_z^{\infty} a(z')\, h(z')\, dz' \ge 0$$

so that

$$H(z,+) \le H(z,-),$$

from which (29) follows once more.

We can derive a correspondingly general inequality that must hold between $K(z,-)$ and $K(z,+)$. From (25):

$$K(z,-) - K(z,+)\, R(z,-) = a(z,-) + a(z,+)\, R(z,-) \ge 0 ;$$

whence

$$K(z,+)\,R(z,-) \leq K(z,-) \tag{30}$$

or equivalently:

$$\frac{dH(z,-)}{dz} \leq \frac{dH(z,+)}{dz} .$$

This relation throws some light on the questions raised above. Relation (30) shows that if $K(z,-)$ is negative, then necessarily $K(z,+)$ is negative too. Conversely, if $K(z,+)$ is positive, then so must $K(z,-)$ be positive. Finally, (30) hints at real situations in which $K(z,+)$ may well be negative while $K(z,-)$ is positive. Examples of each of these three situations are easily found.

3. The classical two-flow theory gives a convenient expression for k in terms of absorption and scattering coefficients. There is no correspondingly simple formula which individually characterizes $K(z,-)$ or $K(z,+)$, outside of (18) and (19). The closest counterpart to (2) is obtained by eliminating $R(z,-)$ from (20) and (21). The result is:

$$1 = \frac{b(z,-)}{K(z,-) - a(z,-)} + \frac{b(z,+)}{K(z,+) + a(z,+)} . \tag{31}$$

This may be verified to be the appropriate generalization of (2) by setting, as such a verification requires, $b(z,-) = b(z,+) = b^{*}$, $a(z,-) = a(z,+) = a^{*}$, and $K(z,-) = K(z,+) = k$. When this is done, (31) reduces to (2).

4. Since the experimental counterpart to R_{∞} generally varies with depth, it is of interest to characterize the variation in terms of the experimental K-functions. The desired formula follows immediately from the definition of $R(z,-)$:

$$\frac{dR(z,-)}{dz} = R(z,-)\left[K(z,-) - K(z,+)\right] . \tag{32}$$

From this we see that the constancy of $R(\cdot,-)$ is equivalent to the equality of $K(\cdot,-)$ and $K(\cdot,+)$. In other words, $R(\cdot,-)$ is constant over any interval (z_1,z_2) of depths when and only when $K(z,-) = K(z,+)$ for every depth z in the interval (z_1,z_2).

5. In this paragraph we will briefly discuss the connection between the k of the classical theory and the K-functions of the present work. First of all we observe the simple connection that exists between the scalar irradiances $h(z,\pm)$ and the irradiances $H(z,\pm)$ within the framework of the classical theory. Recall that both the diffuse and reduced radiance distributions in both the

upper and lower hemispheres are assumed uniform, therefore, for all depths z ,

$$D(z,\pm) = \frac{h(z,\pm)}{H(z,\pm)} = 2 .$$

This means that $h(z,\pm)$ and $H(z,\pm)$ differ by a fixed factor 2. Thus the operational definition (5) for k lets us conclude:

$$k = \frac{-1}{H(z,\pm)} \frac{dH(z,\pm)}{dz} = \frac{-1}{h(z,\pm)} \frac{dh(z,\pm)}{dz}$$

In other words, the classical theory says that k may be estimated equally well from measurements of scalar irradiances or ordinary irradiances. As demonstrated above, distinctions between h and H are necessary not only in theory but in careful experimental practice. So that when it becomes necessary to discuss the growth or decay of, say, $h(\cdot,-)$ with depth, its logarithmic derivative is generally considered distinct **from** that of $H(\cdot,-)$. A similar statement is true for $h(\cdot,+)$. Thus we are led to consider operations of the kind:

$$\frac{-1}{h(z,\pm)} \frac{dh(z,\pm)}{dz} ,$$

and to distinguish these from the operations

$$\frac{-1}{H(z,\pm)} \frac{dH(z,\pm)}{dz} = K(z,\pm),$$

we set

$$\frac{-1}{h(z,\pm)} \frac{dh(z,\pm)}{dz} = k(z,\pm).$$

As a mnemonic, we observe that the little k's go with little h's, and big K's go with big H's. In real media the connection between these is:

$$k(z,\pm) = K(z,\pm) - \frac{1}{D(z,\pm)} \frac{dD(z,\pm)}{dz}.$$

We may now state the connection we set out to establish: $k(\cdot,\pm)$ and $K(\cdot,\pm)$ are equal over some interval (z_1, z_2) of depths when and only when the distribution function $D(\cdot,\pm)$ is constant over that interval. This is precisely the situation that holds for optically infinitely deep media in the classical theory so that $k(\cdot,\pm) = K(\cdot,\pm)$. Furthermore, in this case, $R(\cdot,-) = R_\infty$, and thus is independent of depth. From (32) we may then conclude that $K(\cdot,-) = K(\cdot,+)$. The net conclusion is that for each depth z, $k(z,+)$, $k(z,-)$, $K(z,+)$, $K(z,-)$, which are generally four distinct quantities in real media, are constrained in the two-flow theory to be identical, their common value being k, as given by (2).

6. The K-functions discussed in this note are associated with a two-flow analysis of the light field. They are the little k's for the little h's, and the big K's for the big H's. In more detailed experimental studies of the light field, namely those that document the radiance distribution values $N(z,\theta,\phi)$ at each depth z in all directions (θ,ϕ), a corresponding K-function has been found extremely useful in graphical and tabular representations of these distributions. It is defined as:

$$K(z,\theta,\phi) = \frac{-1}{N(z,\theta,\phi)} \frac{dN(z,\theta,\phi)}{dz} . \tag{33}$$

No confusion should arise from the continued use of the letter K, (33) will always explicitly exhibit three variables, the other K's only one. We note in passing that this function, analogously to the other K-functions discussed above, has several interesting theoretical consequences in addition to its immediate experimental uses. However, a discussion of these matters would be out of place here.

To round out the discussion on the experimental K-functions, we note that all of the K-functions defined above fall into a specific class, each member of which is defined by an operation of the kind:

$$\frac{-1}{A} \frac{dA}{dz} ,$$

where A could be any of the functions: $H(\cdot, \pm)$, $h(\cdot, \pm)$, $N(\cdot, \theta, \phi)$. Some further possibilities for A are, $h(\cdot)$, $\bar{H}(\cdot, +)$. Finally, we observe, by means of the divergence relation (21), that the basic volume absorption function may be defined as the result of the operation,

$$\frac{1}{h} \frac{d\bar{H}}{dz},$$

on the two types of irradiances shown. On the basis of these examples, it appears that the most general notion of an experimental attenuation function (i.e., a general K-function) is definable by an operation of the kind,

$$\frac{1}{A} \frac{dB}{dz},$$

on any two <u>observable</u> irradiance-like quantities A and B.