Title
On torsion theories and open classes of linear modular lattices

Journal
Communications in Algebra, ahead-of-print(ahead-of-print)

ISSN
0092-7872

Authors
González-Bayona, Francisco
Pardo-Guerra, Sebastián
Rincón-Mejía, Hugo Alberto
et al.

Publication Date
2023

DOI
10.1080/00927872.2023.2241080

Peer reviewed

On torsion theories and open classes of linear modular lattices

Francisco González-Bayona
elcorreodepancho@ciencias.unam.mx

Sebastián Pardo-Guerra
spardoguerra@eng.ucsd.edu

Hugo Alberto Rincón-Mejía
hurincon@gmail.com

Manuel Gerardo Zorrilla-Noriega
mgzn@hotmail.com

Abstract

Given a ring R, a bijection exists between torsion theories and idempotent radicals. Further, the class of hereditary torsion theories contains the skeleton of the open classes. In this work, we extend the notions and main results in R-Mod about torsion classes, torsion-free classes, and open classes to the category $\mathcal{L}_{\mathcal{M}}$ of linear lattices, whose objects are complete modular lattices and whose morphisms are linear morphisms.

Keywords: Complete modular lattices, linear morphism of lattices, big lattices, lattice preradical, idempotent lattice preradical, radical, torsion theories, TTF classes, open classes.
Mathematics Subject Classification: Primary: 18B35; Secondary: 06C99, 06B75.

1. Introduction

For every ring R, an R-module M induces a complete modular lattice $L(M)$, whose elements are the submodules of M and whose lattice operations of infimum and supremum are the intersection and sum of submodules, respectively. Further, each R-linear homomorphism $f : M \longrightarrow N$ induces a join-preserving function between the lattices $L(M)$ and $L(N)$, the "direct image under f", which, provided f is injective, is a *lattice morphism* (that is, it preserves infima and suprema).
However, T. Albu and M. Iosif in [2] introduce the concept of *linear morphisms* between two bounded modular lattices that, in contrast to usual lattice morphisms, summon the First Isomorphism Theorem for modules into a lattice framework. Moreover, the

authors show that the collection of bounded modular lattices equipped with linear morphisms forms a category. We shall restrict ourselves to complete modular lattices and denote the corresponding category by $\mathcal{L}_{\mathcal{M}}$. One has a functor from R-Mod to $\mathcal{L}_{\mathcal{M}}$ that assigns to each module M the complete modular lattice $L(M)$ and to each R-linear homomorphism a linear morphism in $\mathcal{L}_{\mathcal{M}}$ (again, the direct image). This functorial connection drove us to study the results concerning the closure properties of classes in R-Mod in their lattice counterpart category $\mathcal{L}_{\mathcal{M}}$.

The rest of the paper is organized as follows: Section 2 contains the notions and the definitions of linear morphisms and lattice preradicals. In Section 3 and Section 4, we study torsion theories in the category $\mathcal{L}_{\mathcal{M}}$ and show their connection with lattice preradicals. Section 5 describes the big lattice of open classes in $\mathcal{L}_{\mathcal{M}}$, and proves that its skeleton is contained in the big lattice of hereditary torsion classes.

2. Preliminaries

For a bounded lattice L, let us write 0_L (resp., 1_L) to mean the least (resp., greatest) element of L. If $a, b \in L$ are such that $a \leqslant b$, we write $b/a = \{x \in L \mid a \leqslant x \leqslant b\}$ for a general *interval*. The *initial interval* $b/0_L$ and the *quotient interval* $1_L/a$ are special cases.

We denote by $\mathcal{L}$ the class of all bounded modular lattices. This class forms a category when we consider as morphisms the usual lattice morphisms, this is, functions that respect the lattice operations of infimum and supremum. However, one can define different morphisms between lattices of this kind, thus giving rise to a new category.

Definition 2.1. [2, Definition 1.1] Let $L, L' \in \mathcal{L}$. The mapping $f : L \longrightarrow L'$ is called a **linear morphism** if there exists $k \in L$, called the kernel of f, and $a' \in L'$ such that the following two conditions hold:

1) $f(x) = f(x \vee k)$ for all $x \in L$.

2) The function f induces a lattice isomorphism $\overline{f} : 1_L/k \longrightarrow a'/0_{L'}$ such that $\overline{f}(x) = f(x)$ for all $x \in 1_L/k$.

Following the notation in [2, Proposition 2.2], we name $\mathcal{L}_{\mathcal{M}}$ the category of linear modular lattices, whose objects are complete modular lattices and whose morphisms are linear morphisms.

Observe that for $L, L' \in \mathcal{L}_{\mathcal{M}}$, L and L' are isomorphic in $\mathcal{L}_{\mathcal{M}}$ if and only if they are isomorphic as lattices.

Broadly, a preradical σ on a category $\mathcal{C}$ is a subfunctor of the identity functor $Id_{\mathcal{C}}$. This way, a preradical σ on $\mathcal{C}$ assigns to each object C a subobject $\sigma(C)$, and to each morphism $f : C \longrightarrow D$ a morphism $\sigma(f) : \sigma(C) \longrightarrow \sigma(D)$ in $\mathcal{L}_{\mathcal{M}}$ such that

$$\begin{array}{ccc} C & \xrightarrow{\ f\ } & D \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} \\ \sigma(C) & \xrightarrow{\sigma(f)} & \sigma(D) \end{array}$$

commutes. In particular, as in the category $\mathcal{L}_{\mathcal{M}}$ the subobjects of a lattice $L \in \mathcal{L}_{\mathcal{M}}$ correspond to initial intervals in L (see [2, Proposition 2.2(5)]), we have the following

Definition 2.2. A lattice preradical is a functor $r : \mathcal{L}_{\mathcal{M}} \longrightarrow \mathcal{L}_{\mathcal{M}}$ that satisfies the following two conditions:

1) $r(L)$ is an initial interval of L.

2) For any linear morphism $f : L \longrightarrow L'$, we have that $f(r(L)) \subseteq r(L')$. Furthermore, the restriction to $r(L)$ and corestriction to $r(L')$ of f is a linear morphism. Thus, $r(f) : r(L) \longrightarrow r(L')$ is defined as the restriction and corestriction of f, which is denoted as $f|$.

We denote by $\mathcal{L}_{pr}$ the class of lattice preradicals on $\mathcal{L}_{\mathcal{M}}$.

Theorem 2.3. *The following sentences are equivalent:*

1) $r \in \mathcal{L}_{pr}$

2) r *is an assignment from* $\mathcal{L}_{\mathcal{M}}$ *to* $\mathcal{L}_{\mathcal{M}}$ *such that:*

- *(i)* $r(L)$ *is an initial interval of* L *for each* $L \in \mathcal{L}_{\mathcal{M}}$.
- *(ii) For any linear morphism* $f : L \longrightarrow L'$ *we have that* $f(r(L)) \subseteq r(L')$.

Proof. Follows from [1, Lemma 3.2.].

(Alternatively, assume that (2) holds. Let $L, L' \in \mathcal{L}_{\mathcal{M}}$ and let $f : L \longrightarrow L'$ be a linear morphism. Have $i : r(L) \to L$ stand for the inclusion mapping, which is of course a linear morphism. Since $\mathcal{L}_{\mathcal{M}}$ is a category, the composite $f \circ i$, which is the restriction of f to $r(L)$, is also a linear morphism. But then the corestriction of $f \circ i$ to $r(L')$ is clearly a linear morphism. Therefore, (1) holds.) $\square$

Note that preradicals preserve isomorphisms: for $L, L' \in \mathcal{L}_{\mathcal{M}}$, if $L \overset{f}{\cong} L'$, then $r(L) \overset{f|}{\cong} r(L')$.

For a preradical r in $\mathcal{L}_{pr}$ and a lattice $L \in \mathcal{L}_{\mathcal{M}}$ we denote by X_L^r the element of L such that $r(L) = X_L^r / 0_L$.

Definition 2.4. For a class $\mathcal{C}$ in $\mathcal{L}_{pr}$, let us define the functions $\bigvee_{r \in \mathcal{C}} r$, $\bigwedge_{r \in \mathcal{C}} r : \mathcal{L}_{\mathcal{M}} \longrightarrow \mathcal{L}_{\mathcal{M}}$ through

$$\Big(\bigvee_{r\in\mathcal{C}} r\Big)(L) = {\scriptstyle (\bigvee_{r\in\mathcal{C}} X_L^r)/0_L}$$

and

$$\Big(\bigwedge_{r\in\mathcal{C}} r\Big)(L) = {\scriptstyle (\bigwedge_{r\in\mathcal{C}} X_L^r)/0_L}.$$

Proposition 2.5. *Let $\mathcal{C}$ be a class of lattice preradicals. Then $\bigvee_{r\in\mathcal{C}} r$ and $\bigwedge_{r\in\mathcal{C}} r$ are lattice preradicals.*

Proof. By definition, both functions send a lattice $L \in \mathcal{L}_{\mathcal{M}}$ to an initial interval of L. Given a linear morphism $f : L \longrightarrow L'$, one has that $f(r(L)) \subseteq r(L')$ for each $r \in \mathcal{C}$, so that $f(X_L^r) \leqslant X_{L'}^r$. Since linear morphisms distribute over arbitrary suprema (see [3, Lemma 0.6]), it follows that

$$f(\bigvee_{r\in\mathcal{C}} X_L^r) = \bigvee_{r\in\mathcal{C}} f(X_L^r) \leqslant \bigvee_{r\in\mathcal{C}} X_{L'}^r.$$

Therefore, as linear morphisms are order-preserving (see [3, Corollary 0.4]), $f((\bigvee_{r\in\mathcal{C}} r)(L)) \subseteq (\bigvee_{r\in\mathcal{C}} r)(L')$.

Similarly, as linear morphisms are order-preserving, it follows that $f(\bigwedge_{r\in\mathcal{C}} X_L^r) \leqslant f(X_L^s)$ for all $s \in \mathcal{C}$, and thus,

$$f(\bigwedge_{r\in\mathcal{C}} X_L^r) \leqslant \bigwedge_{r\in\mathcal{C}} f(X_L^r) \leqslant \bigwedge_{r\in\mathcal{C}} X_{L'}^r.$$

Hence, $f((\bigwedge_{r\in\mathcal{C}} r)(L)) \subseteq (\bigwedge_{r\in\mathcal{C}} r)(L')$.

Therefore, by Theorem 2.3, both operators are preradicals of lattices. □

For $r, s \in \mathcal{L}_{pr}$, we write $r \leqslant s$ if and only if $r(L) \subseteq s(L)$ for every $L \in \mathcal{L}_{\mathcal{M}}$. Thus ordered, $\mathcal{L}_{pr}$ is a big lattice, where the supremum and the infimum of a class $\mathcal{A}$ in $\mathcal{L}_{pr}$ are, respectively, $\bigvee_{r\in\mathcal{A}} r$ and $\bigwedge_{r\in\mathcal{A}} r$.

We say that a preradical is *idempotent* if $r(r(L)) = r(L)$ for all $L \in \mathcal{L}_{\mathcal{M}}$, and *radical* if $r(1_L/X_L^r) = X_L^r/X_L^r$ for all lattices $L \in \mathcal{L}_{\mathcal{M}}$.

Lemma 2.6. *Let L be a lattice in $\mathcal{L}_{\mathcal{M}}$ and r a radical on $\mathcal{L}_{\mathcal{M}}$. If $a \in L$ is such that $a \leqslant X_L^r$, then $r(1_L/a) = X_L^r/a$.*

Proof. Firstly, consider the linear morphism $f : L \longrightarrow 1_L/a$ given by $f(x) = x \vee a$. By applying the preradical r to f we obtain that $f(X_L^r/0_L) = f(r(L)) \subseteq r(1_L/a) = X_{1_L/a}^r/a$, from which we get that $X_L^r = X_L^r \vee a = f(X_L^r) \leqslant X_{1_L/a}^r$, so that $X_L^r \leqslant X_{1_L/a}^r$.

Consider now the linear morphism $g : 1_L/a \longrightarrow 1_L/X_L^r$ given by $g(x) = x \vee X_L^r$. By applying r to g we obtain the linear morphism $g| : X_{1_L/a}^r/a \longrightarrow X_L^r/X_L^r$. This time we get that $X_{1_L/a}^r \vee X_L^r = X_L^r$, and thus that $X_{1_L/a}^r \leqslant X_L^r$.

Therefore, $r(1_L/a) = X_{1_L/a}^r/a = X_L^r/a$ □

Proposition 2.7. *Let $\{r_i\}_{i\in I}$ be a family of lattice preradicals.*

(i) If r_i is idempotent for each $i \in I$, then $\bigvee_{i\in I} r_i$ is idempotent.

(ii) If r_i is a radical for each $i \in I$, then $\bigwedge_{i\in I} r_i$ is a radical.

Proof. For (i), let $L \in \mathcal{L}_\mathcal{M}$ and $j \in I$. Then,

$$r_j(L) = r_j(r_j(L)) \leqslant (\bigvee_{i\in I} r_i)(r_j(L)) \leqslant (\bigvee_{i\in I} r_i)((\bigvee_{i\in I} r_i)(L)).$$

Therefore, $(\bigvee_{i\in I} r_i)(L) \leqslant (\bigvee_{i\in I} r_i)((\bigvee_{i\in I} r_i)(L))$, whence equality follows.

For (ii), let $L \in \mathcal{L}_\mathcal{M}$. Note that $X_L^{\bigwedge_{i\in I} r_i} = \bigwedge_{i\in I} X_L^{r_i}$. By Lemma 2.6,

$$(\bigwedge_{i\in I} r_i)({}^{1_L}/_{\bigwedge_{i\in I} X_L^{r_i}}) = {}^{(\bigwedge_{i\in I} X^{r_i}_{1_L/\bigwedge_{i\in I} X_L^{r_i}})}/_{\bigwedge_{i\in I} X_L^{r_i}} = {}^{(\bigwedge_{i\in I} X_L^{r_i})}/_{\bigwedge_{i\in I} X_L^{r_i}}.$$

(Alternatively, write $r = \bigwedge_{i\in I} r_i$ and let $j \in I$. Consider the linear morphism $\pi : {}^{1_L}/_{X_L^r} \longrightarrow {}^{1_L}/_{X_L^{r_j}}$ given by $\pi : x \mapsto x \vee X_L^{r_j}$. Applying r to π yields that

$$\pi(X^r_{1_L/X_L^r}/X_L^r) = \pi(r(1_L/X_L^r)) \subseteq r(1_L/X_L^{r_j}) \subseteq r_j(1_L/X_L^{r_j}) = X_L^{r_j}/X_L^{r_j},$$

so that $X^r_{1_L/X_L^r} \vee X_L^{r_j} = \pi(X^r_{1_L/X_L^r}) = X_L^{r_j}$, that is, $X^r_{1_L/X_L^r} \leq X_L^{r_j}$. Therefore, $X^r_{1_L/X_L^r} \leq \bigwedge_{i\in I} X_i^{r_i} = X_L^r$, which means that $r(1_L/X_L^r) = X_L^r/X_L^r$.) □

Write $\mathcal{L}_{id}$ for the class of idempotent lattice preradicals on $\mathcal{L}_\mathcal{M}$ and $\mathcal{L}_{rad}$ for the class of lattice radicals on $\mathcal{L}_\mathcal{M}$.

Corollary 2.8. *$\mathcal{L}_{id}$ (resp., $\mathcal{L}_{rad}$) is a big lattice, where the join (resp., meet) operation coincides with that in $\mathcal{L}_{pr}$.*

3. Preradicals and torsion theories on $\mathcal{L}_\mathcal{M}$

Definition 3.1. Let $\mathbb{T} \subseteq \mathcal{L}_\mathcal{M}$ be a class of linear modular lattices closed under isomorphisms. We say that $\mathbb{T}$ is a *pretorsion class* if the following holds:

1) For each $L \in \mathbb{T}$ one has that $1_L/a \in \mathbb{T}$ for every $a \in L$. In other words, $\mathbb{T}$ is closed under quotient intervals.

2) For each lattice $L \in \mathcal{L}_\mathcal{M}$ and any family of elements $\{x_i\}_{i\in I} \subseteq L$ with $x_i/0_L \in \mathbb{T}$, $\forall i \in I$, we have that ${}^{(\bigvee_{i\in I} x_i)}/_{0_L} \in \mathbb{T}$.

Remark 3.2. The class of all pretorsion classes in $\mathcal{L}_\mathcal{M}$, ordered by class inclusion, is a big lattice, where the meet operation is given by intersection of classes.

Lemma 3.3. *Let $\mathbb{T}$ be a pretorsion class in $\mathcal{L}_\mathcal{M}$, and let $f : L \longrightarrow L'$ be a linear morphism. If $x \in L$ is such that $x/0_L \in \mathbb{T}$ then ${}^{f(x)}/_{0_{L'}} \in \mathbb{T}$.*

Proof. Let us consider the composite of linear morphisms $x/0_L \overset{i}{\hookrightarrow} L \overset{f}{\longrightarrow} L'$ with kernel k. By the definition of linear morphism,

$$x/k \cong {}^{(f\circ i)(x)}/{}_{0_{L'}} = {}^{f(x)}/{}_{0_{L'}}.$$

As $\mathbb{T}$ is a class closed under quotient intervals, $x/k \in \mathbb{T}$. Therefore, ${}^{f(x)}/{}_{0_{L'}} \in \mathbb{T}$. □

Proposition 3.4. *Let r be a lattice preradical. Then*

$$\mathbb{T}_r = \{L \in \mathcal{L}_{\mathcal{M}} | r(L) = L\}$$

is a pretorsion class.

Proof. Clearly, $\mathbb{T}_r$ is a class closed under isomorphisms.

Suppose first that $L \in \mathbb{T}_r$ and let $a \in L$. Then, the function $L \overset{f}{\longrightarrow} 1_L/a$ defined by $f(x) = x \vee a$ is a linear morphism. Thus, by applying the lattice preradical r to f we obtain the linear morphism $r(L) \overset{f|}{\longrightarrow} r(1_L/a)$, so that $1_L/a = f(L) = f(r(L)) \subseteq r(1_L/a)$ which in turn implies that $1_L/a \in \mathbb{T}_r$.
Suppose now that for $L \in \mathcal{L}_{\mathcal{M}}$ there exists a set of elements $\{x_i\}_{i\in I} \subseteq L$ such that $x_i/0_L \in \mathbb{T}_r$ $\forall i \in I$. In order to prove that ${}^{(\bigvee_{i\in I} x_i)}/{}_{0_L} \in \mathbb{T}_r$, we will consider the linear morphism $\iota_i : x_i/0_L \longrightarrow {}^{(\bigvee_{i\in I} x_i)}/{}_{0_L}$ given by the inclusion mapping, for each $i \in I$. This way, by applying the lattice preradical r to each ι_i, we obtain that $x_i/0_L \overset{\iota_i|}{\longrightarrow} r({}^{(\bigvee_{i\in I} x_i)}/{}_{0_L})$, from which it follows that $x_i \leqslant X^r_{{}^{(\bigvee_{i\in I} x_i)}/{}_{0_L}}$. Since the last holds $\forall i \in I$, it then follows that $\bigvee_{i\in I} x_i \leqslant X^r_{{}^{(\bigvee_{i\in I} x_i)}/{}_{0_L}} \leqslant \bigvee_{i\in I} x_i$, and consequently $\bigvee_{i\in I} x_i = X^r_{{}^{(\bigvee_{i\in I} x_i)}/{}_{0_L}}$. Thus, ${}^{(\bigvee_{i\in I} x_i)}/{}_{0_L} \in \mathbb{T}_r$, and $\mathbb{T}_r$ is a pretorsion class. □

Proposition 3.5. *Let $\mathbb{T}$ be a pretorsion class. Then, $\mathbb{T}$ induces an idempotent lattice preradical $r_{\mathbb{T}}$ given as follows: for $L \in \mathcal{L}_{\mathcal{M}}$,*

$$r_{\mathbb{T}}(L) = {}^{(\bigvee_{x\in\mathbb{T}_L} x)}/{}_{0_L}$$

where $\mathbb{T}_L = \{x \in L | x/0_L \in \mathbb{T}\}$.

Proof. Note first that, for each $L \in \mathcal{L}_{\mathcal{M}}$, one has that $r_{\mathbb{T}}(L)$ defines an initial interval of L. Suppose now that $L \overset{f}{\longrightarrow} L'$ is a linear morphism. Then, for $x \in \mathbb{T}_L$, Lemma 3.3 gives that ${}^{f(x)}/{}_{0_{L'}} \in \mathbb{T}$, that is, $f(x) \in \mathbb{T}_{L'}$. Thus, we have that $\bigvee_{x\in\mathbb{T}_L} f(x) \leqslant \bigvee_{x\in\mathbb{T}_{L'}} x$. With this last in mind and the fact that linear morphisms preserve intervals and commute with arbitrary joins [3, Lema 0.6.], it follows that

$$f(r_{\mathbb{T}}(L)) = f\left({}^{(\bigvee_{x\in\mathbb{T}_L} x)}/{}_{0_L}\right) = {}^{f(\bigvee_{x\in\mathbb{T}_L} x)}/{}_{0_{L'}} = {}^{(\bigvee_{x\in\mathbb{T}_L} f(x))}/{}_{0_{L'}} \subseteq {}^{(\bigvee_{x\in\mathbb{T}_{L'}} x)}/{}_{0_{L'}} = r_{\mathbb{T}}(L'),$$

and thus, by Theorem 2.3, $r_{\mathbb{T}}$ is a lattice preradical.

Lastly, we shall verify that $r_{\mathbb{T}}$ is an idempotent lattice preradical. As $\mathbb{T}$ is a pretorsion class, for any lattice $L \in \mathcal{L}_{\mathcal{M}}$ it follows that $\bigvee \mathbb{T}_L \in \mathbb{T}_L$. In this way, one has that $r_{\mathbb{T}}(L) \in \mathbb{T}$. Consequently, $r_{\mathbb{T}}$ is an idempotent lattice preradical. □

Write $\mathcal{L}_{\mathcal{M}}$-*pret* for the big lattice of pretorsion classes in $\mathcal{L}_{\mathcal{M}}$.

Theorem 3.6. *The big lattices $\mathcal{L}_{id}$ and $\mathcal{L}_{\mathcal{M}}$-pret are isomorphic.*

Proof. On the one hand, by Proposition 3.4, we have a mapping that assigns to each lattice preradical r the pretorsion class $\mathbb{T}_r$. In particular, when restricted to idempotent preradicals, we get a function $\varphi : \mathcal{L}_{id} \to \mathcal{L}_{\mathcal{M}}$-*pret* such that $\varphi(r) = \mathbb{T}_r$. On the other hand, by Proposition 3.5 we have a mapping $\psi : \mathcal{L}_{\mathcal{M}}$-*pret* $\to \mathcal{L}_{id}$ such that $\psi(\mathbb{T}) = r_{\mathbb{T}}$. Thus, we shall show that the maps ψ and φ are inverse to each other.

Let $r \in \mathcal{L}_{id}$. Then, for each $L \in \mathcal{L}_{\mathcal{M}}$,

$$(\psi \circ \varphi)(r)(L) = r_{\mathbb{T}_r}(L) = {}^{(\bigvee_{x\in(\mathbb{T}_r)_L} x)}/_{0_L},$$

where $(\mathbb{T}_r)_L = \{x \in L \mid x/0_L \in \mathbb{T}_r\}$. As r is an idempotent lattice preradical, we have that $X_L^r \in (\mathbb{T}_r)_L$, which in turn implies that

$$r(L) = X_L^r/0_L \subseteq {}^{(\bigvee_{x\in(\mathbb{T}_r)_L} x)}/_{0_L} = r_{\mathbb{T}_r}(L).$$

Therefore, $r \leqslant r_{\mathbb{T}_r}$.

Now, let $L \in \mathcal{L}_{\mathcal{M}}$ and observe that $\bigvee_{x\in(\mathbb{T}_r)_L} x$ is an element in $(\mathbb{T}_r)_L$, that is, $r\big(r_{\mathbb{T}_r}(L)\big) = r_{\mathbb{T}_r}(L)$. With this in mind, by considering the linear morphism given by the inclusion mapping $r_{\mathbb{T}_r}(L) \overset{\iota}{\hookrightarrow} L$, and applying r to ι, we obtain that $r_{\mathbb{T}_r}(L) \subseteq r(L)$. Therefore, $r_{\mathbb{T}_r} \leqslant r$, so that $(\psi \circ \varphi)(r) = r$.

Suppose now that $\mathbb{T}$ is a pretorsion class. Then, we have that

$$L \in \mathbb{T} \Longleftrightarrow r_{\mathbb{T}}(L) = L \Longleftrightarrow L \in \mathbb{T}_{r_{\mathbb{T}}} = (\varphi \circ \psi)(\mathbb{T}).$$

We shall now verify that φ is order-preserving. Let r and r' be two idempotent lattice preradicals such that $r \leqslant r'$. Then, for $L \in \mathbb{T}_r$ we have that $L = r(L) \leqslant r'(L) \leqslant L$, which in turn implies that $L \in \mathbb{T}_{r'}$. Hence, $\varphi(r) \leqslant \varphi(r')$. Clearly, also ψ is order-preserving.

Therefore, φ and ψ are inverse order isomorphisms between $\mathcal{L}_{id}$ and $\mathcal{L}_{\mathcal{M}}$-*pret*. By [4, Ch. III, Prop 1.1], they are both lattice isomorphisms. □

Theorem 3.7. *For every lattice preradical r, there exists a largest idempotent lattice preradical $\widehat{r}$ lesser or equal than r.*

Proof. Set $\widehat{r} = r_{\mathbb{T}_r}$, so that, for each $L \in \mathcal{L}_{\mathcal{M}}$,

$$\widehat{r}(L) = {}^{(\bigvee_{x\in(\mathbb{T}_r)_L} x)}/_{0_L},$$

where $(\mathbb{T}_r)_L = \{x \in L | x/0_L \in \mathbb{T}_r\}$

We know that $\widehat{r}$ is idempotent (see Proposition 3.5).

Note that, as $\mathbb{T}_r$ is a pretorsion class, one has that $\widehat{r}(L) \in \mathbb{T}_r$, for every $L \in \mathcal{L}_{\mathcal{M}}$. Thus, if we consider the linear morphism defined by the inclusion mapping $\widehat{r}(L) \hookrightarrow L$, after applying r we obtain $\widehat{r}(L) \hookrightarrow r(L)$. Hence, $\widehat{r} \leqslant r$.

Let $t \in \mathcal{L}_{pr}$ be idempotent with $t \leqslant r$. Observe that $\mathbb{T}_t \subseteq \mathbb{T}_r$. Therefore, making use of Theorem 3.6,

$$t = r_{\mathbb{T}_t} \leqslant r_{\mathbb{T}_r} = \widehat{r}.$$

Hence, $t \leqslant \widehat{r}$. □

Definition 3.8. Let $\mathbb{F} \subseteq \mathcal{L}_{\mathcal{M}}$ be a class closed under isomorphisms. We say that $\mathbb{F}$ is a *pretorsion-free class* if the following conditions hold:

(i) For each lattice $L \in \mathbb{F}$ and element $x \in L$, one has that $(x/0_L) \in \mathbb{F}$. In other words, $\mathbb{F}$ is closed under initial intervals.

(ii) For any $L \in \mathcal{L}_{\mathcal{M}}$ and any family of elements $\{x_i\}_{i\in I} \subseteq L$ with $\{1_L/x_i\}_{i\in I} \subseteq \mathbb{F}$, we have that $1_L/\bigwedge_{i\in I} x_i \in \mathbb{F}$

Remark 3.9. The class of all pretorsion-free classes in $\mathcal{L}_{\mathcal{M}}$, ordered by class inclusion, is a big lattice, where the meet operation is given by intersection of classes.

Let 0 stand for the trivial lattice, that is, the lattice with only one element.

Proposition 3.10. *Let $r \in \mathcal{L}_{pr}$ be a preradical. Then, the class*

$$\mathbb{F}_r = \{L \in \mathcal{L}_{\mathcal{M}} | r(L) = 0\}$$

is a pretorsion-free class.

Proof. Clearly, $\mathbb{F}_r$ is a class closed under isomorphisms. Thus, let us first consider $L \in \mathbb{F}_r$ and $x \in L$. As $x/0_L$ is an initial interval of L, we have the inclusion mapping $\iota : x/0_L \hookrightarrow L$, from which we get $\iota| : r(x/0_L) \longrightarrow 0$ after applying the preradical r. Therefore, $r(x/0_L) = 0$, and thus, $x/0_L \in \mathbb{F}_r$.

Let now $L \in \mathcal{L}_{\mathcal{M}}$, and let $\{x_i\}_{i\in I}$ be a family of elements in L such that $\{1_L/x_i\}_{i\in I} \subseteq \mathbb{F}_r$. If for each $j \in I$ we consider the linear morphism $f_j : 1_L/\bigwedge_{i\in I} x_i \longrightarrow 1_L/x_j$, with correspondence rule $f_j(y) = y \vee x_j$, then after applying the preradical r we get $f_j| : r(1_L/\bigwedge_{i\in I} x_i) \longrightarrow 0$, from which we have that $X^r_{(1_L/\bigwedge_{i\in I} x_i)} \vee x_j = x_j$, that is, $X^r_{(1_L/\bigwedge_{i\in I} x_i)} \leqslant x_j$. Since this last holds for each $j \in I$, it follows that $r(1_L/\bigwedge_{i\in I} x_i) = 0$, and thus, $1_L/\bigwedge_{i\in I} x_i \in \mathbb{F}_r$. Therefore, $\mathbb{F}_r$ is a pretorsion-free class. □

Proposition 3.11. *Let $\mathbb{F}$ be a pretorsion-free class in $\mathcal{L}_{\mathcal{M}}$. Then, $\mathbb{F}$ gives rise to a radical $r_{\mathbb{F}}$ defined, in each $L \in \mathcal{L}_{\mathcal{M}}$, by*

$$r_{\mathbb{F}}(L) = (\bigwedge_{x\in\mathbb{F}_L} x)/0_L$$

where $\mathbb{F}_L = \{x \in L | 1_L/x \in \mathbb{F}\}$.

Proof. We claim that $r_{\mathbb{F}}$ is a lattice preradical. Indeed, first observe that $r_{\mathbb{F}}(L)$ is an initial interval of L, for each $L \in \mathcal{L}_{\mathcal{M}}$. Let now $f : L \longrightarrow L'$ be a linear morphism. For $x' \in \mathbb{F}_{L'}$, we can take the linear morphism $g : L' \longrightarrow 1_{L'}/x'$ such that $g : y \longmapsto y \vee x'$, and thus consider the composite of linear morphisms $L \xrightarrow{f} L' \xrightarrow{g} 1/x'$. From this composition, we get that

$$1_L/k_{g\circ f} \cong {}^{(g\circ f)(1_L)}/_{x'},$$

where $k_{g\circ f}$ is the kernel of the linear morphism $g\circ f$.

As $\mathbb{F}$ is closed under initial intervals and $1_{L'}/x' \in \mathbb{F}$, then ${}^{(g\circ f)(1_L)}/_{x'} \in \mathbb{F}$. Furthermore, since $\mathbb{F}$ is closed under isomorphisms, we have that $1_L/k_{g\circ f} \in \mathbb{F}$. This last means that $k_{g\circ f} \in \mathbb{F}_L$, which in turn implies that $\bigwedge_{x\in\mathbb{F}_L} x \leq k_{g\circ f}$. Also, note that $x' = (g\circ f)(k_{g\circ f}) = f(k_{g\circ f}) \vee x'$. Hence, if we set $X_L^{r_\mathbb{F}} = \bigwedge_{x\in\mathbb{F}_L} x$, we have that

$$f(X_L^{r_\mathbb{F}}) \leq f(k_{g\circ f}) \leq x',$$

as linear morphisms are order-preserving maps [2, Corollary 1.4]. Furthermore, since the latter holds for every $x' \in \mathbb{F}_{L'}$, we then have that $f(X_L^{r_\mathbb{F}}) \leq \bigwedge_{x'\in\mathbb{F}_{L'}} x' = X_{L'}^{r_\mathbb{F}}$. Therefore, by Theorem 2.3, $r_\mathbb{F}$ is a lattice preradical. Finally, by how $r_\mathbb{F}$ was constructed, it is straightforward that $r_\mathbb{F}$ is a radical. □

Write $\mathcal{L}_\mathcal{M}$-*prf* for the big lattice of pretorsion-free classes in $\mathcal{L}_\mathcal{M}$.

Theorem 3.12. *The big lattices $\mathcal{L}_{rad}$ and $\mathcal{L}_\mathcal{M}$-prf are anti-isomorphic.*

Proof. On the one hand, by Proposition 3.10, we have $\varphi : \mathcal{L}_{rad} \to \mathcal{L}_\mathcal{M}$-*prf* such that $\varphi(r) = \mathbb{F}_r$. On the other hand, by Proposition 3.11, we have $\psi : \mathcal{L}_\mathcal{M}$-*prf* $\to \mathcal{L}_{rad}$ such that $\psi(\mathbb{F}) = r_\mathbb{F}$. We shall show that φ and ψ are inverse to each other, and further, they are order-reversing.

Let us first consider a radical r, and the radical given by $(\psi\circ\varphi)(r) = r_{\mathbb{F}_r}$. For any lattice $L \in \mathcal{L}_\mathcal{M}$ one has that $r(1_L/X_L^r) = 0$ since r is a radical. Thus, $\bigwedge\{x \in L | r(1_L/x) = 0\} \leq X_L^r$, so that $r_{\mathbb{F}_r}(L) \subseteq r(L)$. Conversely, let $x \in L$ such that $r(1_L/x) = 0$. Then, by applying the radical r to the linear morphism $f : L \longrightarrow 1_L/x$ given by $f(z) = z \bigvee x$, we get $f| : r(L) \longrightarrow 0$, from which it follows that $X_L^r \leq x$. Therefore, $r(L) \subseteq r_{\mathbb{F}_r}(L)$, and thus, $r = r_{\mathbb{F}_r} = (\psi\circ\varphi)(r)$.

Let now $\mathbb{F}$ be a pretorsion-free class. Then, for $L \in \mathcal{L}_\mathcal{M}$, we have that

$$L \in \mathbb{F}_{r_\mathbb{F}} \Longleftrightarrow r_\mathbb{F}(L) = 0 \Longleftrightarrow \bigwedge \mathbb{F}_L = 0_L \Longleftrightarrow 0_L \in \mathbb{F}_L \Longleftrightarrow L \in \mathbb{F},$$

because $\bigwedge \mathbb{F}_L \in \mathbb{F}_L$. Therefore, $(\varphi\circ\psi)(\mathbb{F}) = \mathbb{F}$.

Lastly, let us show that ψ is order-reversing. If $\mathbb{F} \subseteq \mathbb{F}'$ then for every lattice $L \in \mathcal{L}_\mathcal{M}$ we have that $\mathbb{F}_L \subseteq \mathbb{F}'_L$, thus, $\bigwedge_{x\in\mathbb{F}_L} x \geqslant \bigwedge_{x\in\mathbb{F}'_L} x$. Therefore, $r_{\mathbb{F}'}(L) \leqslant r_\mathbb{F}(L)$ for all $L \in \mathcal{L}_\mathcal{M}$, this is, $\psi(\mathbb{F}') = r_{\mathbb{F}'} \leqslant r_\mathbb{F} = \psi(\mathbb{F})$. Clearly, also φ is order-reversing.

Therefore, φ and ψ are inverse order anti-isomorphisms between $\mathcal{L}_{rad}$ and $\mathcal{L}_\mathcal{M}$-*prf*. By [4, Ch. III, Prop 1.1], they are both lattice anti-isomorphisms. □

Theorem 3.13. *For every lattice preradical r, there exists a least radical $\overline{r}$ greater or equal than r.*

Proof. Set $\overline{r} = r_{\mathbb{F}_r}$, so that, for $L \in \mathcal{L}_{\mathcal{M}}$, $\overline{r}(L) = (\bigwedge_{x \in (\mathbb{F}_r)_L} x)/0_L$, where $(\mathbb{F}_r)_L = \{x \in L | r(1_L/x) = 0\}$. We know that $\overline{r}$ is a radical (see Proposition 3.11). We claim that $r \leq \overline{r}$. Indeed, as $\mathbb{F}_r$ is a pretorsion-free class, we have that $r(1_L/X_L^{\overline{r}}) = 0$. Thus, for the linear morphism $f : L \longrightarrow 1_L/X_L^{\overline{r}}$ such that $f(y) = y \vee X_L^{\overline{r}}$, after applying r we get $f| : r(L) \longrightarrow X_L^{\overline{r}}/X_L^{\overline{r}}$. Therefore, $X_L^r \vee X_L^{\overline{r}} = X_L^{\overline{r}}$, that is, $r(L) \leqslant \overline{r}(L)$.

Now, if s is a radical such that $r \leqslant s$, then $\mathbb{F}_s \subseteq \mathbb{F}_r$. Keeping in mind the proof of Theorem 3.12, it follows that

$$\overline{r} = r_{\mathbb{F}_r} \leqslant r_{\mathbb{F}_s} = s.$$

Therefore, $\overline{r} \leqslant s$. □

Theorem 3.14. *Let $r \in \mathcal{L}_{pr}$. Then, the following hold:*

(i) If r is a radical, so is $\widehat{r}$.

(ii) If r is idempotent, so is $\overline{r}$.

Proof. (i) In order to show that $\widehat{r}(1_L/X_L^{\widehat{r}}) = 0$ for all $L \in \mathcal{L}_{\mathcal{M}}$, it suffices to verify that $1_L/X_L^{\widehat{r}}$ does not have any initial intervals in $\mathbb{T}_r$ other than the trivial one. Indeed, suppose that $y/X_L^{\widehat{r}}$ is any initial interval of $1_L/X_L^{\widehat{r}}$ belonging to $\mathbb{T}_r$. Let us consider the inclusion mapping $X_L^{\widehat{r}}/0_L \hookrightarrow y/0_L$. Then, after applying r, we get that $X_L^{\widehat{r}}/0_L \hookrightarrow r(y/0_L)$. From the last, we have that $X_L^{\widehat{r}} \leqslant X_{y/0_L}^r$. As r is a radical, by Lemma 2.6 we have that $y/X_L^{\widehat{r}} = r(y/X_L^{\widehat{r}}) = X_{y/0_L}^r/X_L^{\widehat{r}}$. Thus, $y = X_{y/0_L}^r$, and hence, $y/0_L \in \mathbb{T}_r$. Therefore, $y \leqslant X_L^{\widehat{r}}$, which in turn implies that $y/X_L^{\widehat{r}} = 0$.

(ii) In order to show that $\overline{r}$ is an idempotent lattice preradical, we need to prove that $\overline{r}(X_L^{\overline{r}}/0_L) = X_L^{\overline{r}}/0_L$ for each $L \in \mathcal{L}_{\mathcal{M}}$. To do that, by definition of $\overline{r}$, it suffices to show that $X_L^{\overline{r}}/0_l$ has no quotient intervals in $\mathbb{F}_r$ other than the trivial one. Thus, let us assume that $X_L^{\overline{r}}/y$ is a quotient interval of $X_L^{\overline{r}}/0_L$ lying in $\mathbb{F}_r$. Consider now the linear morphism $f : 1_L/y \longrightarrow 1_L/X_L^{\overline{r}}$ given by $f(z) = z \vee X_L^{\overline{r}}$. By applying r to f, and seeing as $1_L/X_L^{\overline{r}} \in \mathbb{F}_r$, we get $f| : X_{1_L/y}^r/y \longrightarrow X_L^{\overline{r}}/X_L^{\overline{r}}$, from which we see that $X_{1_L/y}^r \leqslant X_{1_L/y}^r \vee X_L^{\overline{r}} = X_L^{\overline{r}}$. Then, as $X_L^{\overline{r}}/y \in \mathbb{F}_r$ and $\mathbb{F}_r$ is closed under initial intervals, we have that $X_{1_L/y}^r/y \in \mathbb{F}_r$. Moreover, r being idempotent, $X_{1_L/y}^r/y = r(X_{1_L/y}^r/y) = 0$. Thus, $y = X_{1_L/y}^r$. This last implies that $1_L/y \in \mathbb{F}_r$ with which we have that $X_L^{\overline{r}} \leqslant y$. Therefore, $X_L^{\overline{r}}/y = 0$. □

Definition 3.15. Let $L, M, N \in \mathcal{L}_{\mathcal{M}}$. We say that the sequence of linear morphisms $L \xrightarrow{f} M \xrightarrow{g} N$ is *exact* if $f(1_L)$ is the kernel of g.

A sequence of more than two linear morphisms is *exact* if each consecutive pair of morphisms forms an exact sequence.

Definition 3.16. We say that a preradical $r \in \mathcal{L}_{pr}$ is *left exact* if for any exact sequence of linear morphisms

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0,$$

one has that

$$0 \longrightarrow r(L) \xrightarrow{f|} r(M) \xrightarrow{g|} r(N)$$

is exact.

Definition 3.17. Let $\mathcal{C} \subseteq \mathcal{L}_{\mathcal{M}}$ be a class closed under isomorphisms. We call $\mathcal{C}$ a *hereditary class* if for any lattice $L \in \mathcal{L}_{\mathcal{M}}$ and any elements $a \leqslant b \leqslant c$ in L with $c/a \in \mathcal{C}$, one has that $b/a \in \mathcal{C}$.

Lemma 3.18. *Let $f : L \longrightarrow L'$ be a linear morphism with kernel k. Then, the following holds:*

(i) For any $x, y \in L$, one has that $f(x) = f(y) \Leftrightarrow x \vee k = y \vee k$.

(ii) $f(k) = 0$ and k is the greatest element in L with this property. Thus, the kernel of a linear morphism is unique.

Proof. See [2, Proposition 1.3]. □

Definition 3.19. A linear morphism $f : L \longrightarrow M$ is a *linear monomorphism* if f induces, via corestriction, a lattice isomorphism $\overline{f} : L \longrightarrow f(1_L)/0_M$. Clearly, the sequence $0 \to L \xrightarrow{f} M$ is exact if, and only if, f is a linear monomorphism.

Theorem 3.20. *Let $r \in \mathcal{L}_{pr}$. Then, the following properties are equivalent:*

(i) r is left exact;

(ii) for any lattice $L \in \mathcal{L}_{\mathcal{M}}$ and elements $a \leq b$ in L, one has that $X^r_{a/0_L} = a \wedge X^r_{b/0_L}$;

(iii) r is idempotent and its corresponding pretorsion class $\mathbb{T}_r$ is a hereditary class.

Proof. *(i) $\Longrightarrow$ (ii)* The sequence

$$0 \longrightarrow a/0_L \xrightarrow{\iota} b/0_L \xrightarrow{f} b/a \longrightarrow 0,$$

where ι is the inclusion mapping and $f(x) = x \vee a$, is exact. Further, as r is left exact,

$$0 \longrightarrow X^r_{a/0_L}/0_L \xrightarrow{\iota|} X^r_{b/0_L}/0_L \xrightarrow{f|} X^r_{b/a}/a$$

is also an exact sequence, so that $X^r_{a/0_L}$ is the kernel of $f|$. Now, as $a \wedge X^r_{b/0_L} \leqslant a$, and linear morphisms are increasing mappings [2, Corollary 1.4], we have that

$$f|(a \wedge X^r_{b/0_L}) = f(a \wedge X^r_{b/0_L}) \leqslant f(a) = a \bigvee a = a.$$

Thus, $a \wedge X^r_{b/0_L} \leqslant X^r_{a/0_L}$ by Lemma 3.18-(ii). On the other hand, as $X^r_{a/0_L} \leqslant a$ and $X^r_{a/0_L} \leqslant X^r_{b/0_L}$, it follows that $X^r_{a/0_L} \leqslant a \wedge X^r_{b/0_L}$. Therefore, $X^r_{a/0_L} = a \bigwedge X^r_{b/0_L}$.

(ii) $\Longrightarrow$ (i) Let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an exact sequence in $\mathcal{L}_\mathcal{M}$. We will show that

$$0 \longrightarrow X^r_L/0_L \xrightarrow{f|} X^r_M/0_M \xrightarrow{g|} X^r_N/0_N$$

is exact. Now, as $X^r_L \leqslant 1_L$, by [2, Corollary 1.4] we have that

$$g|\big(f|(X^r_L)\big) = g\big(f(X^r_L)\big) = (g \circ f)(X^r_L) \leqslant (g \circ f)(1_L) = 0_N.$$

Further, for any $y \in X^r_M/0_M$ with $g|(y) = 0_N$, the exactness of the sequence defined by f and g implies that $y \leqslant f(1_L)$. Hence, $y \leqslant f(1_L) \wedge X^r_M$. Observe that, putting $a = f(1_L)$ and $b = 1_M$, the hypothesis implies that

$$X^r_{f(1_L)/0_M} = f(1_L) \bigwedge X^r_M.$$

Also, as f is a linear monomorphism, there exist isomorphisms, given by appropriate restrictions and corestrictions of f, between L and $f(1_L)/0_M$ and thus between $r(L)$ and $r\big(f(1_L)/0_M\big)$, so that $X^r_{f(1_L)/0_M} = f(X^r_L)$. This way, we reach that $y \leqslant f|(X^r_L)$. Therefore, $f|(X^r_L)$ is the greatest element in $X^r_M/0_M$ satisfying $g|\big(f|(X^r_L)\big) = 0_N$; this is, $f|(X^r_L)$ is the kernel of $g|$.

The remaining exactness follows from [2, Corollary 1.6]. Indeed, having kernel zero, f is injective. Then, $f|$ is injective, so its kernel is zero.

(ii) $\Longrightarrow$ *(iii)* Let $L \in \mathcal{L}_\mathcal{M}$. Since $X^r_L \leqslant 1_L$, applying the hypothesis to this pair yields that $X^r_{X^r_L/0_L} = X^r_L \wedge X^r_L = X^r_L$, whence $r(r(L)) = r(L)$.

Now, if $a \leqslant b \leqslant c$ are any elements in L such that $c/a \in \mathbb{T}_r$, by applying the hypothesis, this time to the elements $b \leqslant c$ within the lattice c/a, one gets that

$$X^r_{b/a} = b \wedge X^r_{c/a} = b \wedge c = b.$$

Hence, $b/a \in \mathbb{T}_r$, and thus, $\mathbb{T}_r$ is a hereditary class.

(iii) $\Longrightarrow$ *(ii)* On the one hand, we have that $X^r_{a/0_L} \leqslant a \wedge X^r_{b/0_L}$. On the other hand, as $\mathbb{T}_r$ is a hereditary class and $X^r_{b/0_L}/0_L = r(b/0_L) \in \mathbb{T}_r$, then ${a \wedge X^r_{b/0_L}}/0_L \in \mathbb{T}_r$. Thus, when applying r to the inclusion mapping ${a \wedge X^r_{b/0_L}}/0_L \overset{\iota}{\hookrightarrow} a/0_L$, we obtain ${a \wedge X^r_{b/0_L}}/0_L = r({a \wedge X^r_{b/0_L}}/0_L) \overset{\iota|}{\hookrightarrow} X^r_{a/0_L}/0_L$, which implies that $a \wedge X^r_{b/0_L} \leqslant X^r_{a/0_L}$. Therefore, $a \bigwedge X^r_{b/0_L} = X^r_{a/0_L}$. □

By Theorem 3.20(ii), the class of left exact preradicals is closed under infima, so this class is a big lattice. Then, bearing in mind Theorem 3.20(iii), appropriate restrictions of the bijections in Theorem 3.6 yield

Corollary 3.21. *The big lattice of left exact preradicals in $\mathcal{L}_{pr}$ is isomorphic to the big lattice of hereditary pretorsion classes in $\mathcal{L}_\mathcal{M}$.*

Definition 3.22. A linear morphism $f : L \longrightarrow M$ is a *linear epimorphism* if $f(1_L) = 1_M$, so that its induced lattice isomorphism is of the form $\overline{f} : 1_L/k_f \longrightarrow M$. Clearly, the sequence $L \xrightarrow{f} M \rightarrow 0$ is exact if, and only if, f is a linear epimorphism.

By [2, Proposition 2.2(4)], epimorphisms in the category $\mathcal{L}_\mathcal{M}$ coincide with the surjective linear morphisms. These, in turn, coincide with the linear epimorphisms.

Theorem 3.23. *Let r be a lattice preradical. Then the following properties are equivalent:*

(i) r preserves epimorphisms;

(ii) r is a radical and $\mathbb{F}_r$ is closed under quotient intervals.

(iii) For any lattice $L \in \mathcal{L}_{\mathcal{M}}$ and elements $a \leqslant b$ in L, one has that $X^r_{1_L/b} = b \vee X^r_{1_L/a}$.

Proof. *(ii)* $\Longrightarrow$ *(i)* Let us first note that, whenever $L \xrightarrow{\varphi} M$ is an epimorphism, we can regard M as a quotient interval of L of the form $1_L/y$. Suppose now that r is a radical and $\mathbb{F}_r$ is closed under quotient intervals. Then, $1_L/X^r_L \in \mathbb{F}_r$ for each $L \in \mathcal{L}_{\mathcal{M}}$. Further, ${}^{1_L}/_{X^r_L \vee y} \in \mathbb{F}_r$ since $\mathbb{F}_r$ is closed under quotient intervals.

Let $1_L/y \xrightarrow{g} {}^{1_L}/_{y \vee X^r_L}$ such that $g(z) = z \vee X^r_L$. Then, after applying r to g, we get $r(1_L/y) \xrightarrow{g|} {}^{y \vee X^r_L}/_{y \vee X^r_L}$, from which it follows that $z \vee X^r_L = y \vee X^r_L$, $\forall z \in r(1_L/y)$. In particular, we have that $X^r_{1_L/y} \vee X^r_L = y \vee X^r_L$, so that $X^r_{1_L/y} \leqslant y \vee X^r_L$.

Thus, for the linear epimorphism $L \xrightarrow{\varphi} 1_L/y$ given by $\varphi(x) = x \vee y$, after applying the radical r we get $X^r_L/0_L \xrightarrow{\varphi|} r(1_L/y)$. It follows that $X^r_L \vee y = \varphi(X^r_L) \leqslant X^r_{1_L/y}$. Therefore,

$$\varphi|(X^r_L) = X^r_L \vee y = X^r_{1_L/y}.$$

Hence, it follows that $\varphi|$ is a surjective linear morphism; this is, an epimorphism in $\mathcal{L}_{\mathcal{M}}$.

(i) $\Longrightarrow$ *(iii)* Consider the linear epimorphism $1_L/a \xrightarrow{\pi} 1_L/b$ such that $\pi : x \mapsto x \vee b$. Applying r we get $r(1_L/a) \xrightarrow{\pi|} r(1_L/b)$, which by hypothesis is also a linear epimorphism. This means that $X^r_{1_L/a} \vee b = X^r_{1_L/b}$.

(iii) $\Longrightarrow$ *(ii)* Let $L \in \mathcal{L}_{\mathcal{M}}$. Applying the hypothesis to the couple $0_L \leqslant X^r_L$ we get that $X^r_{1_L/X^r_L} = X^r_L \vee X^r_L = X^r_L$ and therefore r is a radical.

Also, if $L \in \mathbb{F}_r$ and $b \in L$, applying the hypothesis to the elements $0_L \leqslant b$ we obtain that

$$X^r_{1_L/b} = b \vee X^r_L = b \vee 0_L = b.$$

Therefore, $1_L/b \in \mathbb{F}_r$ and thus $\mathbb{F}_r$ is closed under quotient intervals. □

A class in $\mathcal{L}_{\mathcal{M}}$ is said to be *cohereditary* if and only if it is closed under isomorphisms and under quotient intervals.

By Theorem 3.23(iii), the class of epimorphism-preserving lattice preradicals is closed under suprema, and thus is a big lattice, which, in view of Theorem 3.23(ii), is anti-isomorphic to the big lattice of cohereditary pretorsion-free classes in $\mathcal{L}_{\mathcal{M}}$.

Definition 3.24. Let $\mathcal{C}$ be a class in $\mathcal{L}_{\mathcal{M}}$. We say that $\mathcal{C}$ is *closed under extensions* if for any $L, M, N \in \mathcal{L}_{\mathcal{M}}$, and any exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $L, N \in \mathcal{C}$, one has that $M \in \mathcal{C}$.

We saw in Theorem 3.6 that there exists a bijection between the class of pretorsion classes in $\mathcal{L}_{\mathcal{M}}$, and the class of idempotent preradicals in $\mathcal{L}_{pr}$. Next we show another bijection —a restriction of the first one—, now between the class of pretorsion classes which are closed under extensions, and the class of idempotent radicals in $\mathcal{L}_{pr}$.

Theorem 3.25. *Let $\mathbb{T}$ be a pretorsion class, and let r be the idempotent lattice preradical corresponding to $\mathbb{T}$. Then, $\mathbb{T}$ is a class closed under extensions if, and only if, r is a radical.*

Proof. $\Longrightarrow$) Suppose that r is not a radical. Then, there exists a lattice $L \in \mathcal{L}_{\mathcal{M}}$ for which $r(1_L/X_L^r) \neq 0$. Thus, $X_L^r < X_{1_L/X_L^r}^r$. This way, we can form the exact sequence

$$0 \longrightarrow X_L^r/0_L \longrightarrow X_{1_L/X_L^r}^r/0_L \longrightarrow r(1_L/X_L^r) \longrightarrow 0.$$

As the preradical r is idempotent by hypothesis, we have that both of $X_L^r/0_L$ and $r(1_L/X_L^r)$ belong to $\mathbb{T}$. Moreover, since $\mathbb{T}$ is closed under extensions, one has that $X_{1_L/X_L^r}^r/0_L \in \mathbb{T}$. This last contradicts the fact that X_L^r is the greatest element in L such that the initial interval $X_L^r/0_L$ in $\mathbb{T}$.

$\Longleftarrow$) Let $L \in \mathcal{L}_{\mathcal{M}}$ and $a \in L$. Assume that

$$0 \longrightarrow a/0_L \longrightarrow L \longrightarrow 1_L/a \longrightarrow 0$$

is an exact sequence with $a/0_L, 1_L/a \in \mathbb{T}$. Then, after applying r to the inclusion mapping $a/0_L \hookrightarrow L$, we get $a/0_L \hookrightarrow X_L^r/0_L$, so that $a \leqslant X_L^r$. As r is a radical, by Lemma 2.6 we have that $1_L/a = r(1_L/a) = X_L^r/a$. Therefore, $X_L^r = 1_L$, that is, $L \in \mathbb{T}$. □

Dually, we have the following

Theorem 3.26. *Let $\mathbb{F}$ be a pretorsion-free class, and let r be the radical corresponding to $\mathbb{F}$. Then, $\mathbb{F}$ is a class closed under extensions if, and only if, r is idempotent.*

Proof. $\Longrightarrow$) Suppose that r is not idempotent. Then, there exists a lattice $L \in \mathcal{L}_{\mathcal{M}}$ such that $X_{X_L^r/0_L}^r < X_L^r$. With this in mind, we can form the exact sequence

$$0 \longrightarrow X_L^r/X_{X_L^r/0_L}^r \longrightarrow 1_L/X_{X_L^r/0_L}^r \longrightarrow 1_L/X_L^r \longrightarrow 0.$$

Now, as by hypothesis r is a radical, it follows that $r(X_L^r/X_{X_L^r/0_L}^r) = 0 = r(1_L/X_L^r)$, from which we have that both of $X_L^r/X_{X_L^r/0_L}^r$ and $1_L/X_L^r$ belong to $\mathbb{F}$. Further, since $\mathbb{F}$ is closed under extensions, $1_L/X_{X_L^r/0_L}^r \in \mathbb{F}$, which contradicts the fact that X_L^r is the least element in L such that $1_L/X_L^r \in \mathbb{F}$. Therefore, r is an idempotent lattice preradical.

$\Longleftarrow$) Let $L \in \mathcal{L}_{\mathcal{M}}$ and let $a \in L$. Assume that

$$0 \longrightarrow a/0_L \longrightarrow L \xrightarrow{\pi} 1_L/a \longrightarrow 0$$

is an exact sequence with $a/0_L, 1_L/a \in \mathbb{F}$. Thus, after applying r to π we get $\pi| : X_L^r/0_L \longrightarrow a/a$ so that $X_L^r \vee a = \pi(X_L^r) = a$, that is, $X_L^r \leqslant a$. Further, one has that

$$X_L^r/0_L = r(X_L^r/0_L) \subseteq r(a/0_L) = 0_L/0_L,$$

which implies that $X_L^r = 0_L$. Therefore, $L \in \mathbb{F}$, and hence, $\mathbb{F}$ is closed under extensions. □

We call a pretorsion class closed under extensions a *torsion class*. Likewise, we call a pretorsion-free class closed under extensions a *torsion-free class*. Thus, given an idempotent radical r, we can associate to r the torsion class $\mathbb{T}_r = \{L \in \mathcal{L}_{\mathcal{M}} | r(L) = L\}$, and the torsion-free class $\mathbb{F}_r = \{L \in \mathcal{L}_{\mathcal{M}} | r(L) = 0\}$. Also, for any torsion class $\mathbb{T}$ (resp., torsion-free class $\mathbb{F}$), $r_{\mathbb{T}}$ (resp., $r_{\mathbb{F}}$) is an idempotent radical.

Let $\mathcal{C}$ be a category with a zero object. For objects A, B of $\mathcal{C}$, let us write $Hom_{\mathcal{C}}(A, B) = 0$ to mean that the only morphism from A to B is the zero morphism, that is, the composite $A \to 0 \to B$.

Definition 3.27. Let $\mathcal{C}$ be a category with a zero object. A *torsion theory* for $\mathcal{C}$ is a pair $(\mathbb{T}, \mathbb{F})$ of classes of objects of $\mathcal{C}$ such that the following holds:

(i) For any $T \in \mathbb{T}$ and any $F \in \mathbb{F}$, one has that $Hom_{\mathcal{C}}(T, F) = 0$.

(ii) If $C \in \mathcal{C}$ is such that $Hom_{\mathcal{C}}(C, F) = 0$ for all $F \in \mathbb{F}$, then $C \in \mathbb{T}$.

(iii) If $C \in \mathcal{C}$ is such that $Hom_{\mathcal{C}}(T, C) = 0$ for all $T \in \mathbb{T}$, then $C \in \mathbb{F}$.

Remark 3.28. Observe that for each class $\mathbb{T}$ (resp., $\mathbb{F}$) of objects in $\mathcal{C}$ there is at most one class $\mathbb{F}$ (resp., $\mathbb{T}$) of objects in $\mathcal{C}$ such that $(\mathbb{T}, \mathbb{F})$ is a torsion theory.

Also, let $(\mathbb{T}_1, \mathbb{F}_1), (\mathbb{T}_2, \mathbb{F}_2)$ be torsion theories in a category $\mathcal{C}$ with a zero object. Then,

$$\mathbb{T}_1 \subseteq \mathbb{T}_2 \text{ if and only if } \mathbb{F}_2 \subseteq \mathbb{F}_1$$

Theorem 3.29. *There is a one-to-one correspondence between torsion theories in $\mathcal{L}_{\mathcal{M}}$ and idempotent radicals on $\mathcal{L}_{\mathcal{M}}$.*

Proof. Let us first assume that r is an idempotent radical on $\mathcal{L}_{\mathcal{M}}$. We claim that $(\mathbb{T}_r, \mathbb{F}_r)$ is a torsion theory. Indeed, given any linear morphism $f : T \longrightarrow F$, with $T \in \mathbb{T}_r$ and $F \in \mathbb{F}_r$, one has that $r(f) : T \longrightarrow 0$, so that $f = 0$. Now, let $L \in \mathcal{L}_{\mathcal{M}}$ such that $Hom_{\mathcal{L}_{\mathcal{M}}}(L, F) = 0$, for all $F \in \mathbb{F}_r$. Then, as r is a radical, $1_L/X_L^r \in \mathbb{F}_r$. Thus, the linear morphism $f : L \longrightarrow 1_L/X_L^r$ such that $f(y) = y \vee X_L^r$ is the zero morphism, so that $1_L \vee X_L^r = X_L^r$, from which it follows that $X_L^r = 1_L$. Therefore, $L \in \mathbb{T}_r$. Likewise, let $L \in \mathcal{L}_{\mathcal{M}}$ be such that $Hom_{\mathcal{L}_{\mathcal{M}}}(T, L) = 0$ for all $T \in \mathbb{T}_r$. As r is idempotent, $r(L) \in \mathbb{T}_r$. Hence, when considering the inclusion mapping $\iota : r(L) \longrightarrow L$, we get that $\iota = 0$, so that $r(L) = 0$. Therefore, $L \in \mathbb{F}_r$, showing that $(\mathbb{T}_r, \mathbb{F}_r)$ is a torsion theory.

We shall now show that for each torsion theory $(\mathbb{T}, \mathbb{F})$ in $\mathcal{L}_{\mathcal{M}}$, $\mathbb{T}$ is a torsion class and $\mathbb{F}$ is a torsion-free class. We will first consider the class $\mathbb{T}$. From Definition 3.27, it is straightforward to verify that $\mathbb{T}$ is closed under isomorphisms.
Let $T \in \mathbb{T}$ and $x \in T$, and assume that $f : 1_T/x \longrightarrow F$ is a linear morphism, for some $F \in \mathbb{F}$. Then, from the commutative diagram

$$\begin{array}{ccc} 1_T/x & \xrightarrow{f} & F \\ {\scriptstyle \_\vee x}\uparrow & \nearrow_{0} & \\ T & & \end{array}$$

we see that $f(1_T) = f(1_T \vee x) = 0_F$, so that $f = 0$. Thus, $\mathbb{T}$ is closed under quotient intervals. Let now $L \in \mathcal{L}_\mathcal{M}$ and let $\{x_i\}_{i\in I}$ be a family of elements in L such that $x_i/0_L \in \mathbb{T}$ for every $i \in I$. If $F \in \mathbb{F}$ and $f : (\bigvee_{i\in I} x_i)/0_L \longrightarrow F$ is a linear morphism, then for each $j \in I$, we have the diagram

$$\begin{array}{ccc} \bigvee_{i\in I} x_i/0_L & \xrightarrow{\ f\ } & F \\ {\scriptstyle \iota_j}\uparrow & \overset{0}{\nearrow} & \\ x_j/0_L, & & \end{array}$$

where ι_j is the inclusion mapping. As the diagram is commutative, $f(x_j) = (f \circ \iota_j)(x_j) = 0_F$. Furthermore, since the latter holds for each $j \in I$, and as linear morphisms commute with arbitrary joins (see [3, Lema 0.6]), we have that $f(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} f(x_i) = 0_F$. Therefore, $(\bigvee_{i\in I} x_i)/0_L \in \mathbb{T}$, showing that $\mathbb{T}$ is a pretorsion class. We lastly show that $\mathbb{T}$ is closed under extensions. Suppose that $0 \longrightarrow x/0_L \longrightarrow L \longrightarrow 1_L/x \longrightarrow 0$ is an exact sequence with $x/0_L, 1_L/x \in \mathbb{T}$. Given a linear morphism $f : L \longrightarrow F$, with $F \in \mathbb{F}$, we have the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\ f\ } & F \\ {\scriptstyle i}\uparrow & \overset{0}{\nearrow} & \\ x/0_L & & \end{array}$$

(where i is the inclusion mapping) from which it follows that $f(x) = (f \circ i)(x) = 0_F$, so that $x \leqslant k$, where k is the kernel of f. Considering this and the linear morphism $1_L/x \xrightarrow{\_\vee k} 1_L/k$, we have the commutative diagram

$$\begin{array}{ccccc} 1_L/k & \xrightarrow{\ \bar{f}\ } & f(1_L)/0_F & \xrightarrow{\ j\ } & F \\ {\scriptstyle \_\vee k}\uparrow & & \overset{0}{\nearrow} & & \\ 1_L/x & & & & \end{array}$$

where j is the appropriate inclusion mapping. Thus, $f(1_L) = j(\bar{f}(1_L \vee k)) = 0_F$. Hence, $L \in \mathbb{T}$, and thus, $\mathbb{T}$ is a torsion class.

We now consider the class $\mathbb{F}$. From Definition 3.27, it is straightforward to verify that $\mathbb{F}$ is closed under isomorphisms.
Let $F \in \mathbb{F}$ and let $x/0_F$ be an initial interval of F. Given a linear morphism $f : T \longrightarrow x/0_F$, with $T \in \mathbb{T}$, we have the commutative diagram

$$\begin{array}{ccc} x/0_F & \xrightarrow{\ i\ } & F \\ {\scriptstyle f}\uparrow & \overset{0}{\nearrow} & \\ T & & \end{array}$$

(where i is the inclusion mapping) from which we see that $f = 0$. Thus, $\mathbb{F}$ is closed under initial intervals. Let now $L \in \mathcal{L}_\mathcal{M}$ and let $\{x_i\}_{i\in I}$ be a family of elements of L such that $1_L/x_i \in \mathbb{F}$ for every $i \in I$. Given $T \in \mathbb{T}$, and a linear morphism $f : T \longrightarrow 1_L/\bigwedge_{i\in I} x_i$, for each $j \in I$ consider the commutative diagram

$$\begin{array}{ccc} 1_L/\bigwedge\limits_{i\in I} x_i & \xrightarrow{\_\vee x_j} & 1_L/x_j \\ {\scriptstyle f}\uparrow & \nearrow_{0} & \\ T & & \end{array}$$

from which it follows that $f(1_T) \vee x_j = x_j$. As the last holds for all $j \in I$, we have that $f(1_T) \leqslant \bigwedge_{i\in I} x_i$, and consequently, $f = 0$. Therefore, $1_L/\bigwedge_{i\in I} x_i \in \mathbb{F}$, showing that $\mathbb{F}$ is a pretorsion-free class. Lastly, we shall verify that $\mathbb{F}$ is closed under extensions. Suppose that $0 \longrightarrow x/0_L \longrightarrow L \longrightarrow 1_L/x \longrightarrow 0$ is an exact sequence with $x/0_L, 1_L/x \in \mathbb{F}$. If $f : T \longrightarrow L$ is a linear morphism, with $T \in \mathbb{T}$, then, from the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\_\vee x} & 1_L/x \\ {\scriptstyle f}\uparrow & \nearrow_{0} & \\ T & & \end{array}$$

we get that $f(1_T) \leqslant x$. Further, as $x/0_L \in \mathbb{F}$, we have the commutative diagram

$$\begin{array}{ccc} f(1_T)/0_L & \xrightarrow{\ i\ } & x/0_L \\ {\scriptstyle f\restriction}\uparrow & \nearrow_{0} & \\ T & & \end{array}$$

(where i is the inclusion mapping and $f \restriction$ stands for the corestriction of f) which shows that $f = 0$. Therefore, $L \in \mathbb{F}$, and hence, $\mathbb{F}$ is a torsion-free class.

Let us now prove that $r_{\mathbb{T}}$, the idempotent radical corresponding to the torsion class $\mathbb{T}$, coincides with $r_{\mathbb{F}}$, the idempotent radical corresponding to the torsion-free class $\mathbb{F}$. We shall use the tools developed in the proofs of Theorems 3.6 and 3.12. We have the torsion theories $(\mathbb{T},\mathbb{F})$ and $(\mathbb{T}_{r_{\mathbb{T}}},\mathbb{F}_{r_{\mathbb{T}}})$. Since $\mathbb{T} = \mathbb{T}_{r_{\mathbb{T}}}$, Remark 3.28 gives that $\mathbb{F}_{r_{\mathbb{T}}} = \mathbb{F} = \mathbb{F}_{r_{\mathbb{F}}}$. Thus, both being radicals, $r_{\mathbb{T}} = r_{\mathbb{F}}$. (Symmetrically, one can also consider the torsion theories $(\mathbb{T},\mathbb{F})$ and $(\mathbb{T}_{r_{\mathbb{F}}},\mathbb{F}_{r_{\mathbb{F}}})$ and obtain from them, since $\mathbb{F} = \mathbb{F}_{r_{\mathbb{F}}}$, that $\mathbb{T}_{r_{\mathbb{F}}} = \mathbb{T} = \mathbb{T}_{r_{\mathbb{T}}}$, so that, both being idempotent, $r_{\mathbb{T}} = r_{\mathbb{F}}$.)

We have assigned to each idempotent radical r a torsion theory $(\mathbb{T}_r,\mathbb{F}_r)$, and to each torsion theory $(\mathbb{T},\mathbb{F})$ an idempotent radical $r_{\mathbb{T}} = r_{\mathbb{F}}$. It is clear that these processes are mutually inverse. □

Remark 3.30. If $(\mathbb{T},\mathbb{F})$ is a torsion theory, then $\mathbb{F}_{r_{\mathbb{T}}} = \mathbb{F}_{r_{\mathbb{F}}} = \mathbb{F}$ and $\mathbb{T}_{r_{\mathbb{F}}} = \mathbb{T}_{r_{\mathbb{T}}} = \mathbb{T}$.

For a torsion theory $(\mathbb{T},\mathbb{F})$ in a category with a zero object, we call $\mathbb{T}$ *the torsion class of* $(\mathbb{T},\mathbb{F})$ and $\mathbb{F}$ *the torsion-free class of* $(\mathbb{T},\mathbb{F})$.

Remark 3.31. Let $\mathcal{C}$ be a class in $\mathcal{L}_{\mathcal{M}}$. Then, $\mathcal{C}$ is a torsion (resp., torsion-free) class if and only if $\mathcal{C}$ is the torsion (resp., torsion-free) class of a torsion theory. Indeed, for a given torsion class $\mathbb{T}$ (resp., torsion-free class $\mathbb{F}$), the proof of Theorem 3.29 yields that $(\mathbb{T}, \mathbb{F}_{r_{\mathbb{T}}})$ (resp., $(\mathbb{T}_{r_{\mathbb{F}}}, \mathbb{F})$) is a torsion theory.

(In a general category with a zero object, let us take this to be the definition of a torsion (resp., torsion-free) class.)

Definition 3.32. Given a class $\mathcal{C}$ within a category $\mathcal{D}$ with a zero object, we define the *left class* $L(\mathcal{C})$ and the *right class* $R(\mathcal{C})$, generated by $\mathcal{C}$, as follows:

$$L(\mathcal{C}) = \{F \in \mathcal{D} | Hom_{\mathcal{D}}(C, F) = 0, \forall C \in \mathcal{C}\};$$

$$R(\mathcal{C}) = \{T \in \mathcal{D} | Hom_{\mathcal{D}}(T, C) = 0, \forall C \in \mathcal{C}\}.$$

Remark 3.33. A pair $(\mathbb{T}, \mathbb{F})$ of classes of objects in a category with a zero object is a torsion theory if and only if $\mathbb{T} = R(\mathbb{F})$ and $\mathbb{F} = L(\mathbb{T})$.

Lemma 3.34. *Let $\mathcal{A}$ be a category with a zero object. The operators L and R, defined on the big lattice of classes of objects in $\mathcal{A}$, define a Galois connection.*

Proof. Let $\mathcal{C}, \mathcal{D}$ be classes in $\mathcal{A}$ such that $\mathcal{C} \subseteq \mathcal{D}$. Then, clearly, $L(\mathcal{D}) \subseteq L(\mathcal{C})$ and $R(\mathcal{D}) \subseteq R(\mathcal{C})$. Now, given $C \in \mathcal{C}$, one has that $Hom_{\mathcal{A}}(C, L) = 0$ for each $L \in L(\mathcal{C})$. Thus, $C \in R(L(\mathcal{C}))$, and hence, $\mathcal{C} \subseteq R(L(\mathcal{C}))$. Symmetrically, $\mathcal{C} \subseteq L(R(\mathcal{C}))$. □

As for any Galois connection, we have

Corollary 3.35. *For any class $\mathcal{C}$ in a category with a zero object, we have that $LRL(\mathcal{C}) = L(\mathcal{C})$ and $RLR(\mathcal{C}) = R(\mathcal{C})$.*

Corollary 3.36. *The operators RL and LR are closure operators on the big lattice of classes of objects in a category with a zero object (as is the case with any Galois connection). The closed classes, respectively, are the torsion classes and the torsion-free classes.*

Proof. We prove only the claim about closed classes. If $\mathcal{C} = RL(\mathcal{C})$ (resp., $\mathcal{C} = LR(\mathcal{C})$), then $(\mathcal{C}, L(\mathcal{C}))$ (resp., $(R(\mathcal{C}), \mathcal{C})$) is a torsion theory. Conversely, for a torsion theory $(\mathbb{T}, \mathbb{F})$, one has that $\mathbb{T} = R(\mathbb{F}) = RLR(\mathbb{F}) = RL(\mathbb{T})$ and that $\mathbb{F} = L(\mathbb{T}) = LRL(\mathbb{T}) = lR(\mathbb{F})$. □

Observe that $(R(L(\mathcal{C})), L(\mathcal{C}))$ is the torsion theory whose torsion class $R(L(\mathcal{C}))$ is the smallest torsion class containing $\mathcal{C}$. (Indeed, if $\mathbb{T}$ is a torsion class such that $\mathcal{C} \subseteq \mathbb{T}$, then $RL(\mathcal{C}) \subseteq RL(\mathbb{T}) = \mathbb{T}$.) We call it *the torsion theory generated by $\mathcal{C}$*. Similarly, $(R(\mathcal{C}), L(R(\mathcal{C})))$ is the torsion theory whose torsion-free class $L(R(\mathcal{C}))$ is the smallest torsion-free class containing $\mathcal{C}$. We call it *the torsion theory cogenerated by $\mathcal{C}$*.

Theorem 3.37. *If $\mathcal{C}$ is a cohereditary class in $\mathcal{L}_{\mathcal{M}}$, then*

$$R(L(\mathcal{C})) = \{L \in \mathcal{L}_{\mathcal{M}} | \forall x \in L, x < 1, \exists y > x \text{ with } y/x \in \mathcal{C}\}.$$

Proof. By definition, we have that $F \in L(\mathcal{C})$ if, and only if, $Hom_{\mathcal{L}_{\mathcal{M}}}(C,F) = 0$ for all $C \in \mathcal{C}$. We now show that the last holds if, and only if, F does not have nonzero initial intervals in $\mathcal{C}$. Indeed, for necessity, note that, if $x/0_F$ is a non-trivial initial interval of F lying in $\mathcal{C}$, then the inclusion mapping $x/0_F \hookrightarrow F$ is a nonzero linear morphism. For sufficiency, assume that $f : C \longrightarrow F$ is a non-trivial linear morphism for some $C \in \mathcal{C}$. Then, from f we get a lattice isomorphism $\overline{f} : 1_C/k \longrightarrow a/0_F$, which implies, $\mathcal{C}$ being cohereditary, that $a/0_F$ is a non-trivial initial interval of F lying in $\mathcal{C}$.

Now, we know that $T \in R(L(\mathcal{C}))$ if, and only if, $Hom_{\mathcal{L}_{\mathcal{M}}}(T,F) = 0$, $\forall F \in L(\mathcal{C})$. Let us verify that this holds if and only if T does not have quotient intervals lying in $L(\mathcal{C})$ other than the trivial one. For necessity, note that, if T has a non-trivial quotient interval $1_T/x$ in $L(\mathcal{C})$, then $\pi : T \longrightarrow 1_T/x$ such that $\pi : t \mapsto t \vee x$ is a non-trivial linear morphism. For sufficiency, suppose that $f : T \longrightarrow F$ is a non-trivial linear morphism, with $F \in L(\mathcal{C})$. Then the induced isomorphism $\overline{f} : 1_T/k \longrightarrow a/0_F$, and the fact that $L(\mathcal{C})$ is closed under initial intervals, yield that T has a non-trivial quotient interval in $L(\mathcal{C})$.

Considering the above, it follows that $T \in R(L(\mathcal{C}))$ if, and only if, any nonzero quotient interval of T contains a nonzero initial interval lying in $\mathcal{C}$. □

Corollary 3.38. *If r is an idempotent lattice preradical, then $\overline{r}$ is the idempotent radical that corresponds to the torsion theory generated by the class $\mathbb{T}_r$.*

Proof. By Theorem 3.13 and Theorem 3.14, $\overline{r}$ is the least idempotent radical greater or equal than r. Thus, the torsion class of its corresponding torsion theory is the smallest one containing the class $\mathbb{T}_r$. Hence, it is the torsion theory generated by $\mathbb{T}_r$.

□

Recall that an element a of a bounded lattice L is called *essential* (resp., *superfluous*) in L if and only if for every $b \in L$ such that $b \neq 0_L$ (resp., $b \neq 1_L$) it happens that $a \wedge b \neq 0_L$ (resp., $a \vee b \neq 1_L$).

For $L, M \in \mathcal{L}_{\mathcal{M}}$, we say that M is an *essential extension* (resp., a *superfluous cover*) of L if and only if L is isomorphic to an initial interval $x/0_M$ (resp., a quotient interval $1_M/x$) of M, where x is essential (resp., superfluous) in M.

Lastly, a class $\mathcal{C}$ in $\mathcal{L}_{\mathcal{M}}$ is said to be *closed under essential extensions* (resp., *closed under superfluous covers*) if and only if whenever $L \in \mathcal{C}$ and M is an essential extension (resp., a superfluous cover) of L, one has that $M \in \mathcal{C}$.

Theorem 3.39. *Let $\mathcal{C}$ be a class in $\mathcal{L}_{\mathcal{M}}$. Then,*

(i) If $\mathcal{C}$ is hereditary, then $L(\mathcal{C})$ is closed under essential extensions.

(ii) If $\mathcal{C}$ is cohereditary, then $R(\mathcal{C})$ is closed under superfluous covers.

Proof. *(i)* Let $L \in L(\mathcal{C})$ and let $M \in \mathcal{L}_{\mathcal{M}}$ be an essential extension of L. We may suppose that $L = 1_L/0_M \subseteq M$, where 1_L is an essential element of M. If $C \in \mathcal{C}$ and $f : C \longrightarrow M$ is a linear morphism, then f induces a lattice isomorphism $\overline{f} : 1_C/k \longrightarrow f(1_C)/0_M$ where

k is the kernel of f. Thus, there exists $x \in C$ such that $f(x) = f(1_C) \wedge 1_L$. Observe that the composite $x/0_C \overset{\iota}{\hookrightarrow} C \overset{f}{\rightarrow} M$, where ι is the inclusion mapping, takes values in L, seeing as linear morphisms preserve the order. Consider the corestriction $x/0_C \overset{(f\circ\iota)\upharpoonright}{\longrightarrow} L$ of $f \circ \iota$ to L, and note that, since $\mathcal{C}$ is a hereditary class, $x/0_C \in \mathcal{C}$. Hence, the linear morphism $(f \circ \iota) \upharpoonright$ is zero, so that $f(1_C) \wedge 1_L = f(x) = 0$. Therefore, $f(1_C) = 0$. Hence, $f = 0$, and thus, $M \in L(\mathcal{C})$.

(ii) Let $L \in R(\mathcal{C})$ and suppose that $M \in \mathcal{L}_\mathcal{M}$ is a superfluous cover for L, so that $L = 1_M/0_L \subseteq M$ and 0_L is superfluous in M. Given a lattice $C \in \mathcal{C}$ and a linear morphism $f : M \longrightarrow C$, we have a lattice isomorphism $\overline{f} : 1_M/k \longrightarrow f(1_M)/0_C$ where k is the kernel of f, induced by f. Consider the following composition of the restriction and corestriction of $\overline{f}$ followed by the inclusion mapping:

$$1_M/k \vee 0_L \overset{\overline{f}|}{\longrightarrow} f(1_M)/f(k \vee 0_L) \overset{\iota}{\hookrightarrow} 1_C/f(k \vee 0_L).$$

As $\mathcal{C}$ is cohereditary, we have that $1_C/f(k \vee 0_L) \in \mathcal{C}$. Likewise, $R(\mathcal{C})$ is cohereditary, so that $1_M/k \vee 0_L \in R(\mathcal{C})$. This way, the composite $\iota \circ (\overline{f}|)$ is the zero morphism. Therefore, $\overline{f}|(1_M) = \overline{f}|(k \vee 0_L)$, and thus, $1_M = k \vee 0_L$. It follows that $k = 1_M$, and thus, $f = 0$. Hence, $M \in R(\mathcal{C})$.

□

4. TTF classes.

Definition 4.1. A class $\mathbb{T} \subseteq \mathcal{L}_\mathcal{M}$ is a *TTF* class when it is both a torsion class and a torsion-free class. This happens precisely when there are classes $\mathbb{C}, \mathbb{F} \subseteq \mathcal{L}_\mathcal{M}$ such that $(\mathbb{C}, \mathbb{T})$ and $(\mathbb{T}, \mathbb{F})$ are both torsion theories.

Note that, if $\mathbb{T}$ is a TTF class, then the torsion theory $(\mathbb{T}, \mathbb{F})$ is hereditary[1]. In addition, a hereditary torsion class $\mathbb{T}$ is a TTF class if, and only if, for any lattice $L \in \mathcal{L}_\mathcal{M}$ and any subset of elements $\{x_i\}_{i\in I} \subseteq L$ with $\{1_L/x_i\}_{i\in I} \subseteq \mathbb{T}$, one has that $1_L/\bigwedge_{i\in I} x_i \in \mathbb{T}$.

Lemma 4.2. *Let* $\mathbb{T}$ *is a TTF class, with* $(\mathbb{C}, \mathbb{T})$ *and* $(\mathbb{T}, \mathbb{F})$ *torsion classes. If* $(\mathbb{C}, \mathbb{T})$ *is hereditary, then* $\mathbb{C} \subseteq \mathbb{F}$.

Proof. Let t be the idempotent radical corresponding to $\mathbb{T}$. Then, for $L \in \mathbb{C}$, we have that $t(L) \in \mathbb{C} \cap \mathbb{T} = \{0\}$. Thus, $L \in \mathbb{F}$. □

Definition 4.3. Let $(\mathbb{T}, \mathbb{F})$ be a torsion theory and let t be its associated idempotent radical. We say that $(\mathbb{T}, \mathbb{F})$ *splits* if, for each lattice $L \in \mathcal{L}_\mathcal{M}$, there exists $x \in L$ such that

$$X_L^t \vee x = 1_L \text{ and } X_L^t \wedge x = 0_L.$$

Lemma 4.4. *Let* $\mathbb{T}$ *be a TTF class, with* $(\mathbb{C}, \mathbb{T})$ *and* $(\mathbb{T}, \mathbb{F})$ *torsion classes. Then, the following statements are equivalent:*

[1]We say that a torsion theory is hereditary if and only if its torsion class is hereditary.

(i) $\mathbb{C} = \mathbb{F}$.

(ii) *For any lattice* $L \in \mathcal{L}_{\mathcal{M}}$, *we have that* $1_L = X_L^c \vee X_L^t$ *and* $0_L = X_L^c \wedge X_L^t$. *Here c and t are the idempotent radicals associated, respectively, to* $(\mathbb{C}, \mathbb{T})$ *and to* $(\mathbb{T}, \mathbb{F})$.

(iii) *The torsion theories* $(\mathbb{C}, \mathbb{T})$ *and* $(\mathbb{T}, \mathbb{F})$ *both split.*

(iv) *The torsion theory* $(\mathbb{C}, \mathbb{T})$ *is hereditary and splits.*

(v) *The torsion theory* $(\mathbb{C}, \mathbb{T})$ *is hereditary and the class* $\mathbb{F}$ *is TTF.*

Proof. (i) $\Longrightarrow$ *(ii).* We know that $\mathbb{F}_c = \mathbb{T}$, which is closed under quotient intervals. Thus, by Theorem 3.23, c preserves epimorphisms.

Let us now consider $L \in \mathcal{L}_{\mathcal{M}}$ and $h : L \longrightarrow {}^{1_L}/_{X_L^c \vee X_L^t}$ given by $h(y) = y \vee (X_L^c \vee X_L^t)$. As t is a radical, $t(1_L/X_L^t) = 0$, so that $1_L/X_L^t \in \mathbb{F} = \mathbb{C}$. Further, as $\mathbb{C}$ is cohereditary, ${}^{1_L}/_{X_L^c \vee X_L^t} \in \mathbb{C}$. This way, when applying c to the linear epimorphism h, we obtain $h| : X_L^c/0_L \longrightarrow {}^{1_L}/_{X_L^c \vee X_L^t}$, which again is an epimorphism. Therefore,

$$1_L = h|(X_L^c) = X_L^c \vee (X_L^c \vee X_L^t) = X_L^c \vee X_L^t.$$

Finally, as $\mathbb{C}$ and $\mathbb{T}$ are both hereditary, ${}^{X_L^c \wedge X_L^t}/_{0_L} \in \mathbb{C} \cap \mathbb{T} = \{0\}$. Hence,

$$X_L^c \wedge X_L^t = 0_L.$$

(ii) $\Longrightarrow$ *(i)* On the one hand, if $L \in \mathbb{C}$, then $X_L^c = 1_L$. Thus, $X_L^t = X_L^c \wedge X_L^t = 0_L$, which implies that $t(L) = 0$. Hence, $L \in \mathbb{F}$. On the other hand, if $L \in \mathbb{F}$, we have that $X_L^t = 0_L$, which implies that $X_L^c = X_L^c \vee X_L^t = 1_L$. Thus, $L \in \mathbb{C}$. Therefore $\mathbb{C} = \mathbb{F}$.

(i) $\Longrightarrow$ *(v)* As $\mathbb{C} = \mathbb{F}$, we have that $\mathbb{C}$ is a hereditary class, that is, $(\mathbb{C}, \mathbb{T})$ is hereditary. Moreover, $\mathbb{F}$ is TTF since $(\mathbb{C}, \mathbb{T}) = (\mathbb{F}, \mathbb{T})$ is a torsion theory.

(v) $\Longrightarrow$ *(i)* As $(\mathbb{C}, \mathbb{T})$ is hereditary by hypothesis, Lemma 4.2 yields that $\mathbb{C} \subseteq \mathbb{F}$. Now, if $F \in \mathbb{F}$ and $T \in \mathbb{T}$ is such that there exists a linear morphism $F \overset{g}{\longrightarrow} T$, then we get that $1_F/k \cong g(1_F)/0_T$. However, as by hypothesis $\mathbb{F}$ is TTF, it is cohereditary. Likewise, as $\mathbb{T}$ is TTF by general hypothesis, it is hereditary. Hence, we have that $1_F/k \in \mathbb{T} \cap \mathbb{F} = \{0\}$, which implies that $k = 1_F$, and consequently, $g = 0$. Therefore $F \in \mathbb{C}$ and thus, $\mathbb{F} \subseteq \mathbb{C}$.

(i) $\Longrightarrow$ *(iv)* As $\mathbb{C} = \mathbb{F}$, then $\mathbb{C}$ is a hereditary class, that is, $(\mathbb{C}, \mathbb{T})$ is hereditary. Further, as (i) implies (ii), for each lattice $L \in \mathcal{L}_{\mathcal{M}}$ we have that $1_L = X_L^c \vee X_L^t$ and $0_L = X_L^c \wedge X_L^t$, which implies that the torsion theory $(\mathbb{C}, \mathbb{T})$ splits.

(iv) $\Longrightarrow$ *(i).* As $(\mathbb{C}, \mathbb{T})$ is hereditary, by Lemma 4.2 we get that $\mathbb{C} \subseteq \mathbb{F}$. Let us show that $\mathbb{F} \subseteq \mathbb{C}$. Take $F \in \mathbb{F}$. Write c for be the idempotent radical associated to the torsion theory $(\mathbb{C}, \mathbb{T})$. As $(\mathbb{C}, \mathbb{T})$ splits, there exists $y \in F$ such that $X_F^c \vee y = 1_F$ and $X_F^c \wedge y = 0_F$. Then, we have the following cases:

(a) If $X_F^c = 0_F$, then $F \in \mathbb{T} \cap \mathbb{F} = \{0\}$ and, of course, $0 \in \mathbb{C}$.

(b) If $y = 0_F$, then $X_F^c = X_F^c \vee y = 1_F$, so that $F \in \mathbb{C}$.

(c) If $X_F^c \neq 0_F$ and $y \neq 0_F$, then X_F^c and y are independent elements as $X_F^c \wedge y = 0_F$. Further, as c is an idempotent lattice preradical we have that $X^c_{X_F^c/0_F} = X_F^c$. Thus, when applying [3, Proposition 1.3] to the join $X_F^c \vee y = 1_F$, we get that

$$X_F^c \vee X^c_{y/0_F} = X^c_{X_F^c/0_F} \vee X^c_{y/0_F} = X_F^c.$$

This last, in turn, implies that $X^c_{y/0_F} \leqslant X_F^c \wedge y = 0_F$. Thus, $y/0_F \in \mathbb{T}$. Moreover, $y/0_F \in \mathbb{F}$ as $F \in \mathbb{F}$ and $\mathbb{F}$ is hereditary. Therefore, we have that $y/0_F \in \mathbb{T} \cap \mathbb{F} = \{0\}$, and hence, $y = 0_F$, which contradicts the hypothesis.

Thus, in all possible cases, $F \in \mathbb{C}$, and hence, $\mathbb{F} \subseteq \mathbb{C}$.

(ii) $\Longrightarrow$ *(iii)*. Clear.

(iii) $\Longrightarrow$ *(i)*. We show first that $\mathbb{C} \subseteq \mathbb{F}$. Let $C \in \mathbb{C}$. Write t for the idempotent radical associated to the torsion theory $(\mathbb{T}, \mathbb{F})$. Then, as by hypothesis $(\mathbb{T}, \mathbb{F})$ splits, there exists $y \in C$ such that $X_C^t \vee y = 1_C$ and $X_C^t \wedge y = 0_C$. We have the following cases:

(a) If $X_C^t = 0_C$, then $C \in \mathbb{F}$.

(b) If $y = 0_C$, then $X_C^t = X_C^t \vee y = 1_C$, and hence, $C \in \mathbb{C} \cap \mathbb{T} = \{0\}$. Therefore, $C = 0 \in \mathbb{F}$.

(c) If both of X_C^t and y are not 0_C, then X_C^t and y are independent elements as $X_C^t \wedge y = 0_C$. Let c stand for the idempotent radical associated to the torsion theory $(\mathbb{C}, \mathbb{T})$. Note that, as $X_C^t/0_C \in \mathbb{T}$, we have that $c(X_C^t/0_C) = 0$. Thus, when applying [3, Proposition 1.3] to the join $X_C^t \vee y = 1_C$, we get that

$$1_C = X_C^c = X^c_{X_C^t/0_C} \vee X^c_{y/0_C} = 0_C \vee X^c_{y/0_C}.$$

Hence, $X^c_{y/0_C} = 1_C$, so that $y = 1_C$. But then, $X_C^t = 0_C$, which is a contradiction.

Therefore, in all possible cases, $C \in \mathbb{F}$, and hence, $\mathbb{C} \subseteq \mathbb{F}$.

As the torsion theory $(\mathbb{C}, \mathbb{T})$ splits, we have that $\mathbb{F} \subseteq \mathbb{C}$, as it is shown in $(iv) \Longrightarrow (i)$. Therefore, $\mathbb{F} = \mathbb{C}$. □

5. Open classes.

Definition 5.1. Let L be a lattice with a least element and let $a \in L$.

We say that an element $b \in L$ is a *pseudocomplement* of a in L if b is maximal such that $a \wedge b = 0_L$, this is, if $a \wedge c = 0_L$ and $c \geqslant b$, then $c = b$.

We say that $b \in L$ is a *strong pseudocomplement* of a in L if b is the greatest element such that $a \wedge b = 0_L$, that is, if $a \wedge c = 0_L$, then $c \leqslant b$.

Definition 5.2. A lattice L with a least element is *(strongly) pseudocomplemented* if each element of L has a *(strong) pseudocomplement* in L.

We will denote by $\mathcal{L}_{\mathcal{M}_\mathcal{S}}$, $\mathcal{L}_{\mathcal{M}_\mathcal{I}}$ and $\mathcal{L}_{\mathcal{M}_\mathcal{C}}$ the subcategories of $\mathcal{L}_\mathcal{M}$ that are, respectively, upper continuous, lower continuous and continuous.

Now, if $r \in \mathcal{L}_{pr}$ and $\mathcal{C}_r$ denotes the class of all lattice preradicals $t \in \mathcal{L}_{pr}$ such that $t \wedge r = 0$, then, for any chain $\mathcal{C} \subseteq \mathcal{C}_r$ and any $L \in \mathcal{L}_{\mathcal{M}_\mathcal{S}}$ one has that

$$((\bigvee_{t\in\mathcal{C}} t) \wedge r)(L) = {}^{X_L^r \wedge (\bigvee_{t\in\mathcal{C}} X_L^t)}/_{0_L} = \bigvee_{t\in\mathcal{C}} {}^{(X_L^t \wedge X_L^r)}/_{0_L} = 0.$$

Thus, if we assume the global choice axiom, the big lattice of lattice preradicals on the category $\mathcal{L}_{\mathcal{M}_\mathcal{S}}$ is pseudocomplemented.

Definition 5.3. We say that a class $C \subseteq \mathcal{L}_\mathcal{M}$ is *open* if it is closed under isomorphisms, initial intervals, and quotient intervals.

Theorem 5.4. *The big lattice of open classes in $\mathcal{L}_\mathcal{M}$ is strongly pseudocomplemented.*

Proof. Let $C \subseteq \mathcal{L}_\mathcal{M}$ be an open class. We will show that the class

$$C^{\perp_{\leqslant,/}} = \{L \in \mathcal{L}_\mathcal{M} | \ L \text{ contains no nonzero subinterval in } C\}$$

is an open class, and further, is a strong pseudocomplement of C in $\mathcal{L}_\mathcal{M}$. First, it is straightforward that $C^{\perp_{\leqslant,/}}$ is closed under isomorphisms and under initial and quotient intervals. Now, if $L \in C \cap C^{\perp_{\leqslant,/}}$, as $1_L/0_L$ is an interval of L, one has that $L = 0$. Lastly, suppose that $D \subseteq \mathcal{L}_\mathcal{M}$ is an open class satisfying $C \cap D = \{0\}$. Then, for $L \in D$, since D is open, any subinterval of L belongs to D. Thus, L cannot contain nonzero subintervals in C. Therefore, $L \in C^{\perp_{\leqslant,/}}$. □

Lemma 5.5. *Let C be an open class in $\mathcal{L}_{\mathcal{M}_\mathcal{S}}$ and let $L \in \mathcal{L}_{\mathcal{M}_\mathcal{S}}$. If $\{x_i\}_{i\in I}$ is an upper directed subset of L such that $x_i/0_L \in C^{\perp_{\leqslant,/}}$, for all $i \in I$, then ${}^{(\bigvee_{i\in I} x_i)}/_{0_L} \in C^{\perp_{\leqslant,/}}$.*

Proof. We proceed by contradiction. Assume there exists a non-trivial subinterval $b/a \in C$ contained in ${}^{(\bigvee_{i\in I} x_i)}/_{0_L}$. Thus, the hypotheses give that

$$b = b \wedge (\bigvee_{i\in I} x_i) = \bigvee_{i\in I} (x_i \wedge b).$$

Further, as $C^{\perp_{\leqslant,/}}$ is hereditary, ${}^{x_i \wedge b}/_{0_L} \in C^{\perp_{\leqslant,/}}$ for all $i \in I$.

Note that, if $a \geqslant x_i \wedge b$ for all $i \in I$, it would follow that $a = b$, which is not possible since we are assuming that $b/a \neq 0$. Hence, there exists $i_0 \in I$ such that $x_{i_0} \wedge b \nleqslant a$. This way, by the modularity of L, it follows that

$$0 \neq {}^{x_{i_0} \wedge b}/_{(x_{i_0} \wedge b) \wedge a} \cong {}^{(x_{i_0} \wedge b) \vee a}/_a = {}^{(x_{i_0} \vee a) \wedge b}/_a \in C.$$

The latter leads us to a contradiction, as ${}^{x_{i_0} \wedge b}/_{(x_{i_0} \wedge b) \wedge a} = {}^{x_{i_0} \wedge b}/_{x_{i_0} \wedge a}$ would be a non-zero subinterval in C of ${}^{x_{i_0} \wedge b}/_{0_L} \in C^{\perp_{\leqslant,/}}$. Therefore, ${}^{(\bigvee_{i\in I} x_i)}/_{0_L} \in C^{\perp_{\leqslant,/}}$. □

Remark 5.6. Following a similar argument, one shows the dual result to Lemma 5.5. This states that, if C is an open class in $\mathcal{L}_{\mathcal{M}_\mathcal{I}}$, and if $L \in \mathcal{L}_{\mathcal{M}_\mathcal{I}}$ has a lower directed subset $\{x_i\}_{i\in I}$ such that the intervals $1_l/x_i \in C^{\perp_{\leqslant,/}}$ for all $i \in I$, then ${}^{1_L}/_{\bigwedge_{i\in I} x_i} \in C^{\perp_{\leqslant,/}}$.

Recall that the *skeleton* of a strongly pseudocomplented lattice L is the set of elements of L that are strong pseudocomplements.

Theorem 5.7. *The skeleton of the big lattice of open classes in $\mathcal{L}_{\mathcal{M}_S}$ is contained in the big lattice of hereditary torsion classes.*

Proof. As previously noted, the pseudocomplement $C^{\perp_{\leqslant,/}}$ of an open class C is closed under initial and quotient intervals. We will then show that it is closed under extensions and "suprema of initial intervals".

(Extensions) Let $0 \to a/0_L \hookrightarrow L \overset{-\vee a}{\to} 1_L/a \to 0$ be an exact sequence such that $a/0_L$ and $1_L/a$ belong to $C^{\perp_{\leqslant,/}}$. In case L is not an element of $C^{\perp_{\leqslant,/}}$, we would have a nonzero interval d/c of L belonging to C. Now, L being a modular lattice, ${}^{a\vee c}/_{c} \cong {}^{a}/_{a\wedge c} \in C^{\perp_{\leqslant,/}}$ as $a/0_L \in C^{\perp_{\leqslant,/}}$. Also, ${}^{(a\vee c)\wedge d}/_{c}$ lies in C since it is a subinterval of d/c; however, it is also a subinterval of ${}^{a\vee c}/_{c} \in C^{\perp_{\leqslant,/}}$. Therefore $(a \vee c) \wedge d = c$.

Consider now the composite $d/c \overset{i}{\hookrightarrow} 1_L/c \overset{-\vee a}{\to} {}^{1_L}/_{a\vee c}$. We claim that $(\_\vee a)\circ i$ is a linear monomorphism. If $x \in d/c$ satisfies that $x \vee a = c \vee a$, then, as $x \leq a \vee x$ and $x \leq d$, it holds that $x \leq (a \vee x) \wedge d = (a \vee c) \wedge d = c$. As $x \in d/c$ we conclude that $x = c$.
Lastly, we have that ${}^{1_L}/_{a\vee c} \in C^{\perp_{\leqslant,/}}$ as $1_L/a \in C^{\perp_{\leqslant,/}}$. Thus, $d/c \in C \cap C^{\perp_{\leqslant,/}}$, which is a contradiction coming from initially assuming that L is not an element of $C^{\perp_{\leqslant,/}}$.

(Suprema of initial intervals) Let $L \in \mathcal{L}_{\mathcal{M}_S}$ and let $\{x_i\}_{i\in I} \subseteq L$ such that $x_i/0_L \in C^{\perp_{\leqslant,/}}$ for all $i \in I$. Our task is to show that ${}^{(\bigvee_{i\in I} x_i)}/_{0_L} \in C^{\perp_{\leqslant,/}}$. Indeed, assume that I is well ordered. Let us denote $y_j = \bigvee_{i\leqslant j} x_i$. We will prove, relying on the well-ordering of I, that $y_j/0_L \in C^{\perp_{\leqslant,/}}$ for all $j \in I$. Let us then assume that $y_i/0_L \in C^{\perp_{\leqslant,/}}$ for all $i < k$, but $y_k/0_L \notin C^{\perp_{\leqslant,/}}$. Then, there is an interval $0 \neq b/a \in C$ contained in $y_k/0_L$. Also, note that

$$y_k = \bigvee_{i\leqslant k} x_i = (\bigvee_{i<k} x_i) \vee x_k = (\bigvee_{i<k} y_i) \vee x_k.$$

As $\{y_i\}_{i<k}$ is an upper directed subset of L, with $y_i/0_L \in C^{\perp_{\leqslant,/}}$ for all $i < k$, it follows by Lemma 5.5 that ${}^{(\bigvee_{i<k} y_i)}/_{0_L} \in C^{\perp_{\leqslant,/}}$.

Let us make $z_k = \bigvee_{i<k} y_i$, so that $y_k = z_k \vee x_k$. Due to the modularity of L, we have that

$$\begin{aligned} C^{\perp_{\leqslant,/}} \ni z_k/0_L \supseteq {}^{z_k\wedge b}/_{z_k\wedge a} &= {}^{z_k\wedge b}/_{(z_k\wedge b)\wedge a} \\ &\cong {}^{(z_k\wedge b)\vee a}/_{a} = {}^{(z_k\vee a)\wedge b}/_{a} \in C. \end{aligned}$$

Thus, $z_k \wedge b = z_k \wedge a$. It follows that

$$\begin{aligned} b/a = {}^{b}/_{(z_k\wedge a)\vee a} &= {}^{b}/_{(z_k\wedge b)\vee a} = {}^{b}/_{(z_k\vee a)\wedge b} \\ &\cong {}^{(z_k\vee a)\vee b}/_{z_k\vee a} = {}^{z_k\vee b}/_{z_k\vee a}. \end{aligned}$$

By making $b' = z_k \vee b$ and $a' = z_k \vee a$, we get that $b/a \cong b'/a' \subseteq y_k/0_L$ is such that $b' \geqslant z_k$ and $b' \geqslant b$.

Analogously, setting $b'' = x_k \vee b'$ and $a'' = x_k \vee a'$ yields that

$$b'/a' \cong b''/a'' \subseteq y_k/0$$

with $b'' \geqslant x_k$ and $b'' \geqslant b'$. Further, as $y_k = (z_k \vee x_k) \leqslant b'' \leqslant y_k$, it follows that $b'' = y_k$. Note that, as $a'' < b'' = (z_k \vee x_k)$, a'' cannot be greater or equal to both of z_k and x_k. Hence, we can assume without loss of generality that $x_k \nleqslant a''$. Then, $(x_k \wedge a'') < x_k$, and thus,

$$0 \neq {}^{x_k}/_{x_k \wedge a''} \cong {}^{x_k \vee a''}/_{a''} \in C,$$

which is a contradiction since $x_k/0_L \in C^{\perp_{\leqslant,/}}$. Because the contradiction arises from assuming the existence of b/a, we conclude that $y_k/0_L \in C^{\perp_{\leqslant,/}}$.

Finally, since $y_i/0_L \in C^{\perp_{\leqslant,/}}$ for all $i \in I$, and $\{y_i\}_{i\in I}$ is upper directed, by Lemma 5.5 it follows that ${}^{(\bigvee_{i\in I} y_i)}/_{0_L} = {}^{(\bigvee_{i\in I} x_i)}/_{0_L} \in C^{\perp_{\leqslant,/}}$. □

Remark 5.8. By duality, from the proof of Theorem 5.7 we obtain that the skeleton of open classes in $\mathcal{L}_{\mathcal{M}_\mathcal{I}}$ is contained in the big lattice of cohereditary torsion-free classes. Moreover, the skeleton of the big lattice of open classes in $\mathcal{L}_{\mathcal{M}_\mathcal{C}}$ is contained in the big lattice of TTF classes.

Corollary 5.9. *The big lattice of left exact preradicals in $\mathcal{L}_{\mathcal{M}_\mathcal{S}}$ is strongly pseudocomplemented.*

Proof. We know that the big lattice of left exact preradicals in $\mathcal{L}_{\mathcal{M}_\mathcal{S}}$ is isomorphic to the big lattice of hereditary pretorsion classes in $\mathcal{L}_{\mathcal{M}_\mathcal{S}}$. A hereditary pretorsion class is an open class. Due to Theorem 5.7, the pseudocomplement as an open class of a hereditary pretorsion class turns out to be a hereditary torsion class, to which corresponds a left exact radical. □

Acknowledgement

The authors thank the referee for his precise and accurate suggestions and comments, which allowed them to enhance this work.

References

[1] Pardo-Guerra, S., Rincón-Mejía, H. A., Zorrilla-Noriega, M. G. (2019). Some isomorphic big lattices and some properties of lattice preradicals. *J. Algebra App.* 19:2050140. World Scientific, Journal of Algebra and Its Applications, 2019.

[2] Albu, T., Iosif, M. (2012). The category of linear modular lattices. *Societatea de Stiinte Matematice din Romania. Bull. Mathématique.*

[3] Albu, T., Iosif, M. (2015). Lattice preradicals with applications to Grothendieck categories and torsion theories. *J. Algebra* 44:339-366

[4] Stenström, B. (1975). *Rings of quotients, An introduction to methods of ring theory*. Die Grundlehren der Mathematischen Wissenschaften, Band 217. Springer-Verlag, New York-Heidelberg, 1975.

[5] Alejandro Alvarado-García, Hugo Alberto Rincón-Mejía, y José Rios-Montes. *On big lattices of classes of R-modules defined by closure properties*. Advances in ring theory. Trends in Mathematics. Birkhauser Basel. 2010.