Aspects of Low Dimensional Quantum Gravity

by

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Abstract

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This thesis will be an exploration of various topics within low dimensional quantum gravity, with an emphasis on the gravitational path integral. We begin by studying two-dimensional Jackiw-Teitelboim (JT) gravity deformed by a gas of conical defects and we solve the model non-perturbatively. We then consider the problem of defining the Lorentzian gravity path integral through a contour rotation of the Euclidean path integral within the context of JT gravity. We demonstrate the agreement of integration domains and calculate the measure for the Lorentzian path integral. We then analyze ensemble averaging a family of two dimensional Conformal Field Theories and find a relation with an exotic three-dimensional bulk Chern-Simons theory.

Contents

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Chapter 1

Introduction

A complete understanding of quantum gravity would allow us to address a variety of interesting questions. A partial list of such questions includes understanding the origin of the universe, the mechanism behind the resolution of gravitational singularities, the structure and counting of black hole microstates, and the resolution of the black hole information paradox. We currently lack such a theory for our own universe.

Over the past two decades, the AdS/CFT correspondence has played a central role in quantum gravity research. The correspondence relates a conformal field theory in d spacetime dimensions to a theory of quantum gravity in $d+1$ dimensions [1]. Within the context of the AdS/CFT correspondence, there has been growing appreciation that lower dimensional models, such as two-dimensional Jackiw-Teitelboim (JT) gravity [2, 3], can give us valuable insight into difficult problems in quantum gravity. These models capture the relevant physics of interest while maintaining computational tractability. Moreover, recent advances in a wide variety of directions have highlighted the central role played by Euclidean gravitational path integral techniques [4–7]. The path integral is defined by

$$Z = \int_{\partial M = \Sigma} \mathcal{D}g \; e^{-S[g]}, \tag{1.1}$$

where we sum over all possible bulk geometries M with appropriate boundary conditions Σ, with an appropriate action for the bulk geometry $S[g]$. In two spacetime dimensions the path integral is well understood and we have good control over both perturbative and non-perturbative gravitational effects that are inaccessible in more complicated models. These effects are crucial in revealing quantum properties of black holes, including the information paradox [5, 6]. The broader goal of studying these simple models is to investigate the questions in the first paragraph within a simpler setting where we have full control, unlike higher dimensions.

In this thesis we will highlight a few selected works exploring both perturbative and non-perturbative aspects of lower dimensional theories of quantum gravity, with a special emphasis placed on gravitational path integral techniques. We now provide a brief overview of the thesis.

1.1 Jackiw-Teitelboim gravity and ensemble averaging

The standard form of the AdS/CFT correspondence states that a given quantum mechanical theory is dual to a theory of quantum gravity. This is encapsulated by the equation

$$Z_{\mathrm{QM}} = Z_{\mathrm{QG}}, \tag{1.2}$$

where on the left we evaluate the partition function of the quantum mechanical theory on the boundary while on the right we evaluate the quantum gravity partition function in the bulk. However, in recent years it has been appreciated that there exist examples of a modified version of this duality, where an *average* over quantum mechanical theories is dual to a theory of quantum gravity

$$\langle Z_{\mathrm{QM}} \rangle = Z_{\mathrm{QG}}. \tag{1.3}$$

On the left-hand side we average over a suitable space of quantum mechanical theories. The clearest example of this is the duality between two-dimensional Jackiw-Teitelboim (JT) gravity [2, 3, 8–11] and an ensemble of $d = 0 + 1$ dimensional quantum mechanical theories [4]. The duality is expressed by an equality between a matrix integral and a gravitational path integral

$$\int dH e^{-N \operatorname{Tr} V(H)} \operatorname{Tr}(e^{-\beta H}) = \int \mathcal{D}g \mathcal{D}\phi \, e^{-I_{\mathrm{JT}}[g,\phi]} \tag{1.4}$$

where H is a Hermitian matrix of size $N \times N$. To match the left-hand side to the right-hand side we formally take the double-scaling limit where we send $N \to \infty$ while tuning the potential $V(H)$ to match the JT gravity answer to leading order in the topological expansion. The matrix H is interpreted as the Hamiltonian of the $d = 0 + 1$ dimensional quantum mechanical system, and the boundary conditions imposed on the gravitational path integral are specified by the observable computed on the left-hand side.

The existence of theories of quantum gravity dual to averages over quantum mechanical systems is a radical departure from the standard statement of the AdS/CFT correspondence, and a natural question is whether there exist additional quantum gravitational theories with ensemble average duals. It was quickly realized realized that many two-dimensional quantum gravity theories had ensemble average duals [12–20]. In [15, 16] a deformation of JT gravity by a gas of conical defects was considered, with action (heuristically) given by

$$I[g, \phi] = I_{\mathrm{JT}}[g, \phi] + \lambda \int d^2x \sqrt{g} e^{-2\pi(1-\alpha)\phi}. \tag{1.5}$$

The above theories were shown to be dual to matrix integrals, as in equation (1.4) with modified potentials $V(H)$, in the special range of angles given by the parameter $\alpha \in [0, \frac{1}{2})$ [15, 16]. The connection of the above theory with conical defects comes from expanding the action perturbatively in the coupling λ, which enforces the localization constraint that the metric is of constant negative curvature with special cone points.

Chapter 2 is based on the work [21] where we study this theory in the full range $\alpha \in [0, 1]$ and show that for all choices of α the theory is dual to a matrix integral (1.4) with appropriately chosen matrix potential $V(H)$. This is accomplished by using the correspondence between JT gravity and the $(2, p)$ minimal string theory [4, 19, 20, 22] in the limit that $p \to \infty$. This approach circumvents various technical difficulties encountered for JT gravity with conical points in the regime of $\alpha \in [\frac{1}{2}, 1]$. We solve the theory non-perturbatively by providing an explicit form of the string equation which can be used to calculate observables to any order in the genus expansion.

1.2 Lorentzian path integrals and topology change

To define the gravitational path integral we must specify the types of metrics that are integrated over. Naturally, there are two common gravitational path integrals that are considered: Euclidean path integrals and Lorentzian path integrals, where we integrate over Euclidean or Lorentzian metrics respectively [23]. The Euclidean path integral is much better understood, and it is defined as a sum over all manifolds M with an integral over all Euclidean metrics g on M. The manifolds M are not required to have a foliation giving them fixed spatial topology, that is M can have topology change.

The clearest example of this is in two dimensions, where perturbative string theory is defined as an integral over two-dimensional Euclidean worldsheets. The string partition function for bosonic strings in Minkowski space is defined by the Polyakov path integral [24]

$$Z_{\text{string}} = \int \mathcal{D}g \mathcal{D}X e^{-S[g,X]}, \tag{1.6}$$

where we implicitly sum over all compact orientable surfaces M. The fact that we include manifolds M that change topology is crucial, since it is these contributions that implement perturbative interactions between strings. However, the fact that the above definition uses Euclidean metrics is in tension with the fact that the string is propagating in a Lorentzian signature target space, since the induced metric on the worldsheet have Lorentzian signature, and hence it seems more natural to consider strings on Lorentzian worldsheets. The trouble with this is that manifolds M that change spatial topology cannot be equipped with smooth Lorentzian metrics g. We will return to this issue momentarily.

Two-dimensional Euclidean gravity path integrals have also played a crucial role in recent developments related to quantum properties of black holes. A unitary page curve can be recovered by including non-perturbative, topology changing configurations when calculating the entanglement entropy of Hawking radiation emitted by a black hole [5, 6]. The number of microstates of certain higher dimensional supersymmetric black holes can be counted using two-dimensional Euclidean path integral techniques, obtaining an exact integer degeneracy that matches string theory computations [7]. The spectral form factor, which contains information on the spectrum of the Hamiltonian is also computed by Euclidean path integral techniques [4, 25–27]. A short list of other recent applications of path integrals techniques applied to black hole physics includes [28–42].

The above results highlight that topology changing contributions to the Euclidean path integral are crucial for recovering the expected quantum mechanical properties of black holes. Furthermore, two-dimensional Euclidean JT gravity naturally appears in the above calculations, either as a dimensional reduction of higher dimensional black holes, or as a toy model for a two-dimensional black hole.

However, black holes are inherently Lorentzian objects, and it is unclear how to interpret topology changing Euclidean configurations from a Lorentzian gravitational path integral perspective. While it is commonly believed that Euclidean and Lorentzian gravity path integrals should be equivalent after a suitable contour rotation, this equivalence has never been adequately demonstrated. Partial progress has been made within the framework of string perturbation theory, where strings propagating on Lorentzian signature worldsheets were considered by Mandelstam [43, 44]. To accommodate Lorentzian worldsheets with spatial topology change the metrics g were not entirely restricted to be Lorentzian but were allowed to become singular at points where strings interact and split apart or join together

$$\text{Interaction point:} \qquad \frac{1}{2}\sqrt{-g}R = 2\pi i\delta^{(2)}(x - x_I), \qquad \det g(x_I) = 0. \tag{1.7}$$

These singular Lorentzian worldsheets are now known as lightcone diagrams, and they provide a Lorentzian interpretation for the Euclidean perturbative string genus expansion.

Chapter 3 is based on the work [45] where we explore extending the formalism of Lorentzian lightcone diagrams to JT gravity. The aim is to establish a Lorentzian theory of JT gravity that incorporates topology changing configurations. In this case, the structure of the metric near the singular points takes the specific form

$$\frac{1}{2}\sqrt{-g}\,(R+2) = (2\pi i + \alpha)\,\delta^{(2)}(x - x_I), \qquad \det g(x_I) = 0, \tag{1.8}$$

where α is a positive real number. We define the Lorentzian path integral to be given by

$$Z_{\mathrm{JT}} \equiv \int \mathcal{D}g\mathcal{D}\phi e^{iI_{\mathrm{JT}}[g,\phi]}, \tag{1.9}$$

where we now include the constant negative curvature analogues of lightcone diagrams, with the metrics behaving as (1.8). We argue that by defining the Lorentzian theory on such lightcone diagrams the theory has the same domain of integration as the Euclidean JT gravity path integral, and we compute the Lorentzian integration measure. We are not able to prove that the Euclidean and Lorentzian measures agree, which is required to show that the Lorentzian path integral fully agrees with the Euclidean path integral. However, since the regions of integration agree it guarantees that any topology changing Euclidean configuration is mapped to a corresponding singular Lorentzian geometry, which provides a Lorentzian interpretation for Euclidean wormhole geometries.

1.3 Ensemble averaging in higher dimensions

In the previous sections we were primarily interested in two-dimensional gravitational theories. One surprising fact was that many two-dimensional models are actually dual to ensemble averages over quantum mechanical theories, and a natural question is whether this idea can be extended to higher dimensional versions of AdS/CFT. The basic idea is that while a single conformal field theory might have a very complicated AdS bulk dual, an average over a suitable moduli space of CFTs may have a very simple gravitational description that captures the statistics of the ensemble of theories. One challenge for this idea is that CFTs must satisfy additional consistency conditions compared to $0+1$ dimensional quantum mechanical theories, and so CFTs are much rarer in the space of quantum field theories. As a result, relatively few CFTs are explicitly known, and one challenge is to find a suitable moduli space of theories to average over.

Nevertheless, these ideas have been applied within the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$ in clever ways [46–59]. In [49] an ensemble of two-dimensional QFTs was defined by specifying that the density of states of the ensemble grows as Cardy's formula [60], and the OPE coefficients are taken to be Gaussian random variables with variance consistent with holographic CFT statistics. While each such QFT is not itself a CFT, the average over the ensemble produces data that is consistent with the theory being a holographic CFT to leading order in c. It was found that ensemble averaged quantities were correctly reproduced by semiclassical 3d gravity computations. One confusing aspect of this proposal is that the individual QFTs that are averaged over are not UV complete, yet the averaged quantities are reproduced by a semiclassical theory of gravity.

In [46, 47] the moduli space of two-dimensional $\mathbb{T}^D$ Narain CFTs was averaged over and it was found that many observables matched with an exotic theory of 3d gravity given by $U(1)^D \times U(1)^D$ Chern-Simons with action

$$S_{\mathrm{CS}} = \sum_{i=1}^{D} \int_M \left(A^{(i)} \wedge dA^{(i)} - B^{(i)} \wedge dB^{(i)}\right), \tag{1.10}$$

coupled to topological gravity, where we sum over all handlebody geometries in the case of a single asymptotic boundary. This provided an explicit example of how averaging over a complicated set of CFTs can give rise to a simple semiclassical gravity theory in one higher dimension. One feature of this duality is that the bulk dual for individual Narain theories is not known.[1] However, it turns out that a closely related family of CFTs known as symmetric product orbifolds of Narain theories, denoted by $\mathrm{Sym}^N(\mathbb{T}^4)$, are dual to tensionless string theory on $\mathrm{AdS}_3 \times S^3 \times \mathbb{T}^4$ as we take the number of copies N of the theory to be large [61–65]. In this case the bulk dual is known for each CFT in the ensemble, but it is very far from a semiclassical gravity theory.

Chapter 4 is based on [66] where we explore ensemble averaging the symmetric product orbifold of Narain theories, with the goal of finding a simple bulk theory that reproduces

[1] Although see [54] for some developments in identifying the bulk dual for an individual Narain theory.

the boundary average. One of our aims is to connect the standard story of holography with ensemble averaging. In particular, while each individual $\mathrm{Sym}^N(\mathbb{T}^4)$ theory at large N is a very complicated string theory that sensitively depends on the point in moduli space, the hope is that the average is reproduced by a very simple semiclassical theory. We now explain how this is partially realized.

The symmetric product orbifold CFTs are defined by taking a seed CFT X, taking some integer number of copies of the theory, and gauging by the S_N permutation symmetry exchanging the copies

$$\mathrm{Sym}^N(X) = X^{\otimes N}/S_N\,. \tag{1.11}$$

A natural expectation is that the averaged bulk dual should follow a similar structure which turns out to be correct. Many contributions to the ensemble average are reproduced by taking N copies of the bulk Chern-Simons theory (1.10) with total action

$$S = \sum_{I=1}^{N} \int_M \left(A_I \wedge dA_I - B_I \wedge dB_I\right), \tag{1.12}$$

and gauging the S_N permutation symmetry, where we have suppressed the summation over the D indices above. We will consider the theory on an asymptotic boundary torus, and the prescription we follow is to include bulk geometries to match the boundary ensemble average computation. We again couple the theory to topological gravity and define the gravitational path integral to be evaluated on handlebody geometries, which we denote by

$$Z_{\text{Bulk}} = \sum_{\substack{\text{handlebodies,}\\ \text{vortices}}} \int \mathcal{D}A\mathcal{D}B\; e^{-S[A,B]}\,\mathcal{V}\,. \tag{1.13}$$

In the above we also allow for certain singular gauge field configurations in the bulk which are implemented by inserting vortex operators $\mathcal{V}$ in the path integral that implement twisted boundary conditions on the gauge fields. We sum over all vortices in the theory.

We find the above bulk theory correctly captures some, but not all, of the terms in the boundary average of the partition function. We interpret the terms that are reproduced as "semiclassical" contributions that admit a simple gravitational description, while the other terms correspond to complicated wormhole configuration, see Chapter **??**. Nevertheless, at large N we are able to give an interpretation to certain geometries as arising from averaging over the amplitude of a single string propagating on an AdS_3 background. Furthermore, certain special quantities such as the averaged free energy in the grand canonical ensemble are fully reproduced by the semiclassical gravity calculation in the bulk. Therefore, the proposed bulk theory does not reproduce all contributions to every observable, and we leave a discussion of modifications of the gravity path integral to Chapter 4.6 that would allow us to reproduce additional terms of the boundary average from the bulk.

Chapter 2

2D Dilaton-gravity, deformations of the minimal string, and matrix models

2.1 Introduction

This chapter is based on [21] where we study two dimensional dilaton-gravity theories with ensemble average holographic duals. In recent years many insights regarding quantum gravity and black holes have been obtained by looking at simple models in two dimensions described by variants of Jackiw-Teitelboim (JT) dilaton-gravity [2, 3, 8–11] that can be solved exactly [4, 67–72]. Some examples are the study of shockwaves and their relation to quantum chaos [9], traversable wormholes [73–75], quantum effects for higher dimensional near extremal black holes [16, 29, 32, 36], and recently how unitarity of the black hole spectrum and dynamics emerges from the inclusion of spacetime wormholes in the gravity path integral [4–6, 76, 77].

Of particular importance is the study of non-perturbative effects and spacetime wormholes, which connects pure JT gravity with a matrix integral [4] in the double-scaling limit [78–80]. This gives a new twist on holography where a bulk gravitational theory is related to an ensemble average over boundary Hamiltonians. This has been generalized in various directions, for example [12, 28, 42, 81].

It was argued in [4] that pure JT gravity, including non-perturbative effects, is equal to the large p limit of the $(2, p)$ minimal string theory. This theory of 2D gravity has been known to be dual to a matrix integral for a long time [19, 20]. This correspondence has been further studied in [22, 82–89].

The connection between JT gravity and matrix integrals was generalized in [15, 16] to include a gas of defects, which in turn can be related to more general 2D dilaton-gravity building on some previous work [90]). The goal of the present paper is to identify deformations of the minimal string that in a certain limit are equal to these deformations of JT gravity. Developing this connection will also allow us to find exact solutions for deformations of JT gravity that are outside the reach of the methods used by [15, 16]. In the rest of this section we will give a brief summary of our results.

The $(2, p = 2m-1)$ minimal string theory consists of coupling the $(2, p)$ 2D minimal model to 2D gravity. After fixing the conformal gauge this theory can be recast as a combination of the minimal model, the Liouville gravity mode ϕ, and a set of bc ghosts. We will study deformations of this theory described by the action

$$I = I_{(2,p)} - \sum_{n=1}^{m-1} \tau_n \int \mathcal{O}_{1,n}\, e^{b(1+n)\phi}, \tag{2.1}$$

where $b = \sqrt{2/p}$ is the Liouville coupling. In this equation $I_{(2,p)}$ represents the undeformed minimal model coupled to Liouville gravity. The constants τ_n are the couplings of the deformations labeled by a gravitationally dressed minimal model primary $\mathcal{O}_{1,n}$. The cosmological constant μ is identified with the $n = 1$ deformation.

We will focus on the disk path integral with fixed length boundary conditions $\ell \equiv \oint_{\text{bdy}} e^{b\phi}$. This can be computed using 2D CFT techniques, although in general this is difficult to do. Alternatively we can use the fact that the theory is dual to a matrix integral, with the matrix interpreted as a random Hamiltonian. All information is then encoded in the leading order disk density of states $\rho(E)$, where E are the eigenvalues of the matrix (related to the boundary cosmological constant in the continuum description). Other observables and higher genus corrections are uniquely fixed by the loop equations [91]. In a remarkable work, Belavin and Zamolodchikov [92] proposed an exact expression for $\rho(E)$ valid to all orders in τ_n. Their only input is the fact that the theory is equivalent to a matrix integral and that correlators on the sphere satisfy the fusion rules of the minimal model, following the program started in [93].

Another theory that is dual to a matrix integral is JT gravity with a gas of defects. These conical singularities are characterized by two numbers, a weighting factor λ and a parameter α defined through the deficit angle $2\pi(1-\alpha)$. In general we have $0 < \alpha < 1$, with $\alpha = 0$ being a cusp and $\alpha = 1$ being basically no defect. This duality has been studied in [15, 16] for the case $0 < \alpha < 1/2$ which we will refer to as sharp defects. The first result of this paper is to show, using the Belavin-Zamolodchikov solution, that the large p limit of the deformed minimal string gives JT with a gas of defects. In this correspondence we identify each deformation with each defect species. The coupling τ_n is proportional to λ in the large p limit. We also scale the label of the minimal model operator as $n = \frac{p}{2}(1-\alpha)$, with fixed α identified as the other defect parameter. At finite p, α is a discrete parameter but becomes continuous at large p and bounded between zero and one.

Even though we can check this connection between deformations of the minimal string and JT with a gas of defects by comparing explicit solutions of the theories, having a more direct argument would be preferable. In order to do this we can write the minimal string as a minimal model coupled to Liouville gravity. Then we can write a Lagrangian representation of the minimal model as time-like Liouville (this connection is not completely understood;

see [94] for a recent discussion) and a field redefinition gives JT with defects (this is a simple generalization of [22, 86]). We summarize the relation between these theories in figure 2.1.

Taking the large p limit of the minimal string solution, we find the exact disk density of states for JT gravity coupled to general defects. The answer we obtain from the Belavin-Zamolodchikov solution is given in (2.67). Instead, we will present the result using a trick pointed out to us by T. Budd [95]. It is convenient to define a defect generating function $W(y)$ as

$$W(y) \equiv \sum_i \lambda_i e^{-2\pi(1-\alpha_i)y}, \tag{2.2}$$

where i is an index labeling defect species, which can be continuous. Providing a function $W(y)$ is equivalent to specifying the angle and weights of the defects. Since $0 < \alpha_i < 1$ the inverse Laplace transform of $W(y)$ should have support on $(-2\pi, 0)$.

Before presenting the solution to the disk density of states we need to specify the edge of the spectrum E_0 where $\rho(E < E_0) = 0$. In terms of the defect generating function, the large p limit of the Belavin-Zamolodchikov solution (2.67) gives E_0 as the largest solution of

$$\int_{\mathcal{C}} \frac{dy}{2\pi i} e^{2\pi y} \left(y - \sqrt{y^2 - 2W(y) - E_0} \right) = 0. \tag{2.3}$$

Using this result, the disk density of states for $E > E_0$, obtained in the same way, is given by

$$\rho(E) = \frac{e^{S_0}}{2\pi} \int_{\mathcal{C}} \frac{dy}{2\pi i} e^{2\pi y} \tanh^{-1} \left(\sqrt{\frac{E - E_0}{y^2 - 2W(y) - E_0}} \right). \tag{2.4}$$

The contour $\mathcal{C}$ is the one appropriate for an inverse Laplace transform, running along the imaginary direction with a real part such that all singularities are to the left. This solution matches the one found in [15, 16] when W involves only defects in the range $0 < \alpha \leq 1/2$. Since the connection between the minimal string deformations, defects, and Belavin-Zamolodchikov solution are valid for any value of α we claim that this solution is valid for JT gravity with a gas of general defects with $0 < \alpha < 1$. The solution we find for $\alpha > 1/2$ is very different from the $0 < \alpha \leq 1/2$ solution, analytically continued in α. This feature is most transparent in (2.67). The new terms we find have a nice geometrical interpretation as we explain later on[1]. It is an open problem to derive this result using the JT gravity path integral representation of the theory, since the methods of [15, 16] cannot be directly applied for reasons we review in next section.

Finally, we study the connection between JT gravity with defects and the quantization of 2D dilaton-gravity

$$I = -\frac{1}{2} \int \sqrt{g} \left[\Phi R + 2U(\Phi) \right]. \tag{2.5}$$

[1]When $\alpha > 1/2$ there is the possibility of defects merging and this produces new contributions to the density of states. This is reminiscent of the situation with conical defects in 3D gravity and 2D CFT associated to operators with $h < (c-1)/32$ [96]. We thank S. Collier for discussions on this.

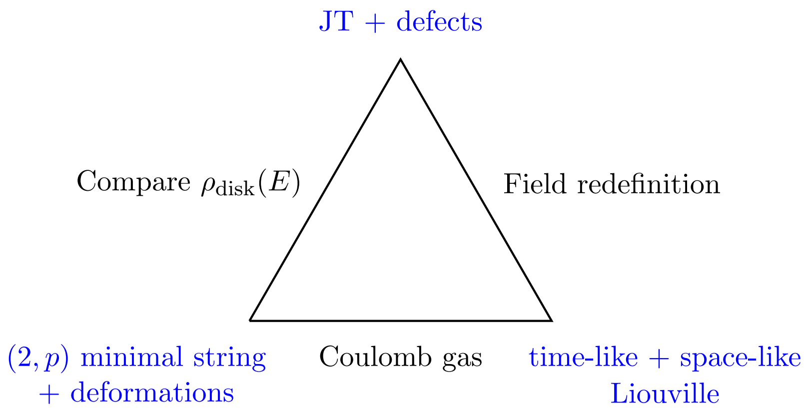


Figure 2.1: Relation between the three theories we are interested in and their respective deformations: the minimal string, time-like Liouville coupled to space-like Liouville and JT gravity with defects. The connection between the $(2, p)$ minimal string and the combination of Liouville theories is the least rigorous since it relies on a Coulomb gas representation of the minimal model.

In the simplest quantization scheme, as explained in [15], one can identify the dilaton potential with the defect generating function as $U(\Phi) = \Phi + W(\Phi)$. This can be justified by Taylor expanding the path integral in powers of W representing contributions from an arbitrary number of defects. The minimal string gives a new perspective on this connection. In order to do this we start from the action (2.1) and rewrite the minimal model in a Coulomb gas, or time-like Liouville, representation. This action can be combined with the gravitational Liouville mode into a 2D metric and a 2D dilaton which we identify with Φ [22, 86]. This approach to the quantization scheme gives a 2D dilaton-gravity theory with a slightly different identification between the parameters λ, α and the dilaton potential, see equation (2.89). We use this approach to propose a solution of dilaton-gravity with a polynomial potential. The advantage of this scheme is that it gives an answer consistent with a semiclassical limit. Some recent studies on other aspects of dilaton-gravity are [97–102].

The organization of the rest of the paper is as follows. In section 2.2 we review the solution for JT gravity with a gas of sharp defects and its connection to 2D dilaton-gravity. In section 2.3 we describe the minimal string theory and its deformations, and present the exact solution proposed by Belavin-Zamolodchikov. In section 2.4 we show the connection between these two theories. We use the minimal string to find a solution of JT gravity with a gas of defect with arbitrary angles. In section 2.5 we explain the connection with dilaton-gravity. We finish presenting conclusions and open directions in section 2.6. We leave technical details for Appendices.

Note: While this work was in progress, we became aware of a related work by T. Budd [95]. The author found the same results by computing the Weil-Petersson volumes via a geometric construction.

2.2 JT gravity with defects: Review

We are interested in analyzing the path integral of JT gravity with conical defects. We first review the genus expansion of pure JT gravity following [4], followed by a summary of JT gravity with a gas of "sharp" defects with deficit angles $\pi < \theta \equiv 2\pi(1-\alpha) < 2\pi$, or equivalently $0 < \alpha < 1/2$ [15, 16] [2]. These theories were shown to be dual to certain double-scaled matrix models. We will comment on this briefly but postpone the full discussion to sections 2.3 and 2.4, when we discuss defects with general deficit angles $0 < \theta < 2\pi$, or equivalently $0 < \alpha < 1$, and the connection to the deformed minimal string. We will refer to the range $\alpha > 1/2$ as "blunt" defects. The case $\alpha = 1/2$ is treated separately in section 2.4, where we show that it exhibits the same behavior as $\alpha < 1/2$.

Pure JT gravity

JT gravity in Euclidean signature is a 2D dilaton-gravity theory described by a metric g coupled to a scalar dilaton Φ with action

$$I_{\mathrm{JT}} = -\frac{S_0}{2\pi}\left(\frac{1}{2}\int_M \sqrt{g}R + \int_{\partial M}\sqrt{h}K\right) - \frac{1}{2}\int_M \sqrt{g}\Phi(R+2) - \int_{\partial M}\sqrt{h}\Phi(K-1), \qquad (2.6)$$

where we have set $\ell_{AdS} = 1$. The first term is the Einstein-Hilbert term, which by the Gauss-Bonnet theorem is equivalent to the Euler characteristic $\chi(M)$ of the manifold, so it is purely topological. For a surface with genus g and n boundaries, $\chi = 2 - 2g - n$. We will assume S_0 is some large parameter[3] so that e^{-S_0} plays the role of suppressing higher genus topologies in the path integral and the theory can be studied under an asymptotic "genus expansion." The dilaton Φ appears linearly in the action and merely acts as a Lagrange multiplier enforcing the constraint $R + 2 = 0$. This fixes the bulk manifold M to be a patch of hyperbolic surface, bounded by some boundary curve ∂M.

Thus, all of the non-trivial dynamics come from the boundary term. For geometries with circular asymptotic boundaries we choose boundary conditions such that the proper length of each boundary is fixed to be β/ϵ where ϵ is regarded as a "holographic" regulator which is eventually taken to zero. The dilaton is fixed to be a constant $\Phi_b = \gamma/\epsilon$ on each boundary. Taking $\epsilon \to 0$ while keeping the ratio β/γ fixed corresponds to sending ∂M to the asymptotic boundary. We set $\gamma = 1/2$ for simplicity. This gives the familiar Schwarzian action [9], which encodes fluctuations of the curve.

We are interested in path integrals over connected geometries with n asymptotic boundaries of regularized lengths β_i. We will denote this quantity by the n-point connected correlator $\langle Z(\beta_1)...Z(\beta_n)\rangle_C$. The topological term in the action naturally organizes the path

[2] We follow the conventions in [16], related to [15] via $\alpha_{\mathrm{W}} \to 1 - \frac{\alpha_{\mathrm{MT}}}{2\pi}$. As explained in those papers, taking $\alpha > 1$ generates a proliferation of defects near the boundary that spoils the AdS_2 asymptotics. For this reason, in this paper we will only focus on $0 \leq \alpha \leq 1$. This is also the natural range appearing from the minimal string, as explained in the next section.

[3] When the theory is regarded as an effective action for the near-horizon dynamics of near-extremal black holes, the prefactor S_0 has the interpretation as the Bekenstein-Hawking entropy.

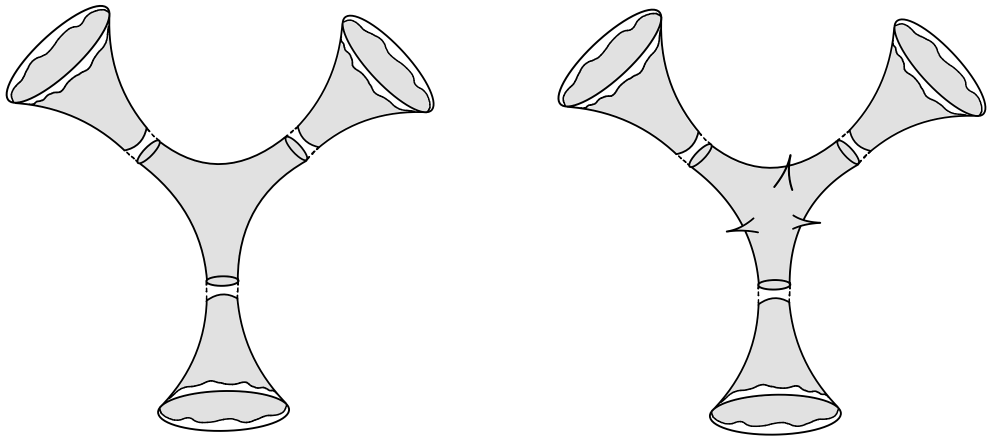


Figure 2.2: **Left:** The SSS construction of the correlator $\langle Z(\beta_1)Z(\beta_2)Z(\beta_3)\rangle_C$. The Weil-Petersson volume $V_{g=0,n=3}$ is glued to trumpets along each of the three geodesic boundaries. **Right:** Analogous construction of the correlator with sharp conical defects. The Weil-Petersson volume $V_{g=0,n=3,k=3}$ has three sharp defects.

integral into a topological expansion in the genus g of bulk manifolds. A manifold of genus g with n asymptotic boundaries has Euler characteristic $\chi = 2-2g-n$ so the genus expansion has the form

$$\langle Z(\beta_1)...Z(\beta_n)\rangle_C = \sum_{g=0}^{\infty} e^{-S_0(2g+n-2)} Z_{g,n}(\beta_1, ..., \beta_n). \tag{2.7}$$

In [4] an explicit expression for $Z_{g,n}$ was found in terms of volumes of moduli spaces of hyperbolic Riemann surfaces with geodesic boundaries. It will turn out that a similar construction generalizes to JT with sharp conical defects, so it will be useful to first understand the pure JT construction.

To compute $Z_{g,n}$ we must integrate over all constant negative curvature geometries of genus g with n asymptotic boundaries. All of these geometries can be constructed by gluing n trumpet geometries to an internal surface of genus g with n geodesic boundaries. See figure 2.2. We can now fix the lengths of the geodesic boundaries to be $b_1, ..., b_n$ and compute the contributions of the trumpets and internal geometry separately. Finally, we can integrate these contributions for all geodesic boundary lengths, with appropriate measure, to recover $Z_{g,n}$ as desired.

The path integral on each trumpet is

$$Z_{\text{trumpet}}(\beta, b) = \sqrt{\frac{1}{4\pi\beta}} e^{-\frac{b^2}{4\beta}}. \tag{2.8}$$

We must also integrate over all internal geometries of genus g with n geodesic boundaries of lengths b_i. These surfaces form the moduli space $\mathcal{M}_{g,n}(b_1, ..., b_n)$ of bordered Riemann surfaces. The proper measure for this moduli space is provided by the Weil-Petersson (WP) measure, and the integral over the moduli space reduces to the Weil-Petersson volume

$V_{g,n}(b_1, ..., b_n)$ [103]. Putting everything together, we integrate over all possible geodesic lengths b with the proper gluing measure bdb to get all stitchings of trumpets with internal moduli with the final result

$$Z_{g,n}(\beta_1, ..., \beta_n) = \int_0^\infty b_1 db_1 Z_{\text{trumpet}}(\beta_1, b_1) \cdots \int_0^\infty b_n db_n Z_{\text{trumpet}}(\beta_n, b_n) V_{g,n}(b_1, ..., b_n). \quad (2.9)$$

There are two special cases for which the above formula does not apply, the disk $Z_{0,1}(\beta)$ and the double trumpet $Z_{0,2}(\beta_1, \beta_2)$. It will be interesting to compare these amplitudes to the corresponding ones with defects so we quote the pure JT results here:

$$Z_{0,1}(\beta) = \sqrt{\frac{1}{16\pi\beta^3}} e^{\frac{\pi^2}{\beta}}, \quad Z_{0,2}(\beta_1, \beta_2) = \frac{\sqrt{\beta_1\beta_2}}{2\pi(\beta_1 + \beta_2)}. \quad (2.10)$$

The structure of the genus expansion implies that JT gravity is dual to a double-scaled Hermitian matrix model [4]. An integral transform of the JT amplitudes $Z_{g,n}$ satisfies the topological recursion property of matrix integrals, derived from the loop equations. This turns out to be a consequence of Mirzakhani's recursion relations [103] for $V_{g,n}$, which were shown to be equivalent to a topological recursion by Eynard and Orantin [104]. The gravitational path integral can now be interpreted as a matrix integral

$$\langle Z(\beta_1)...Z(\beta_n)\rangle = \int dH e^{-L \operatorname{Tr} V(H)} \operatorname{Tr}(e^{-\beta_1 H})... \operatorname{Tr}(e^{-\beta_n H}), \quad (2.11)$$

where H is a Hermitian matrix of size $L \times L$. Formally we take a double-scaling limit on the right-hand side where we send $L \to \infty$ while tuning the potential $V(H)$ such that the matrix model density of states matches the JT density of states at leading order. The matrix H is interpreted as the Hamiltonian of a dual quantum system and we interpret JT gravity as an ensemble average over independent quantum systems.

Sharp defects

We will now briefly review the work of [15, 16] on JT gravity with conical defects of deficit angle $0 < \alpha < 1/2$. The reason for this restriction will be apparent shortly.

We are again interested in path integrals over connected geometries with n asymptotic boundaries which we denote by $\langle Z(\beta_1)...Z(\beta_n)\rangle_C$, but allowing for the presence of a gas of defects. The path integral naturally organizes into a topological expansion in genus alongside an expansion in the number of defects inserted in the bulk

$$\langle Z(\beta_1)...Z(\beta_n)\rangle_C = \sum_{g=0}^{\infty} \sum_{k=0}^{\infty} e^{-S_0(2g+n-2)} \frac{\lambda^k}{k!} Z_{g,n,k}(\beta_1, ..., \beta_n). \quad (2.12)$$

The term proportional to λ^k inserts k conical defects in the bulk integrated over all possible insertion positions. The defects are indistinguishable so the symmetry factor $k!$ prevents overcounting identical configurations[4]. From this expression we see the weight λ acts as a fugacity with respect to the number of defects.

The factor $Z_{g,n,k}$ consists of an integral over all constant negative curvature geometries of genus g with k conical points and n asymptotic boundaries. It turns out that for conical defects with deficit angles $\pi < \theta < 2\pi$, or equivalently $0 < \alpha < 1/2$, all such geometries can be constructed by gluing trumpets to internal geometries of genus g with k conical points [105], the only exception being the disk with one defect, where a direct calculation [90] gives $Z_{0,1,k} = Z_{\text{trumpet}}(b = 2\pi i\alpha)$. Besides this exception, all of our formulas from the pure JT discussion carry over as long as we replace the moduli space that we are integrating over in

$$\begin{aligned} &Z_{g,n,k}\left(\beta_1,\ldots,\beta_n;\alpha_1,\ldots,\alpha_k\right) = \\ &\quad \int_0^\infty b_1 db_1\, Z_{\text{trumpet}}\left(\beta_1,b_1\right)\cdots\int_0^\infty b_n db_n\, Z_{\text{trumpet}}\left(\beta_n,b_n\right) V_{g,n,k}\left(b_1,\ldots,b_n;\alpha_1,\ldots,\alpha_k\right), \end{aligned} \tag{2.13}$$

with $V_{g,n,k}$ the Weil-Petersson volumes of the moduli space of surfaces of genus g with n geodesic boundaries and k conical points. For $\alpha < 1/2$ there is a simple relation between the WP volumes with conical points $V_{g,n,k}$ and volumes without conical points $V_{g,n+k}$ [105–107]. The WP volumes with k conical points can be found from the ordinary volumes by analytically continuing k of the $n+k$ geodesic boundary lengths to imaginary values

$$V_{g,n,k}\left(b_1,\ldots b_n;\alpha_1,\ldots,\alpha_k\right) = V_{g,n+k}\left(b_1,\ldots b_n, b_{n+1} = 2\pi i\alpha_1,\ldots,b_{n+k} = 2\pi i\alpha_k\right). \tag{2.14}$$

Using the above relation, and a formula for the genus zero WP volumes previously derived in [22], reference [16] re-summed the defect expansion at genus zero

$$\langle Z(\beta_1)...Z(\beta_n)\rangle_{C,g=0} = e^{-S_0(n-2)}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}Z_{0,n,k}(\beta_1,...,\beta_n). \tag{2.15}$$

We will give the answer for the case with multiplet species of sharp defects (λ_i,α_i). Performing the sum explicitly and doing an inverse Laplace transform gives the density of states

$$\langle\rho(E)\rangle_{g=0} = \frac{e^{S_0}}{2\pi}\int_{E_0}^{E}\frac{du}{\sqrt{E-u}}\left(I_0(2\pi\sqrt{u}) + \sum_i \lambda_i\frac{2\pi\alpha_i}{\sqrt{u}}I_1\left(2\pi\alpha_i\sqrt{u}\right)\right), \tag{2.16}$$

where I_n are modified Bessel functions of the first kind. Here, $E_0(\lambda)$ gives the spectral edge of the distribution and is the largest root of the string equation which we review in section 2.3. We will derive the same result from the minimal string in section 2.4.

JT gravity with defects is again dual to a matrix integral which can be shown by proving that integral transforms of $Z_{g,n,k}$ satisfy topological recursion. We omit the details, arriving at the same result through the minimal string in the next section.

[4]The moduli space and volumes $V_{g,n,k}$ are typically defined with distinguishable points, so $Z_{g,n,k}$ is defined here with distinguishable defects.

We finish by outlining the connection to dilaton-gravity. It was argued in [90] that an insertion of a defect with parameters (λ, α) in JT gravity is equivalent to the insertion in the path integral of

$$\lambda \int d^2x \sqrt{g} e^{-2\pi(1-\alpha)\Phi}. \tag{2.17}$$

By summing over an arbitrary number of defects and summing over possible defect species this insertion simply exponentiates [15, 16]. The end result is a modification of the action of JT gravity into a more general dilaton-gravity model

$$I = -\frac{1}{2} \int d^2x \sqrt{g} \left(\Phi R + 2U(\Phi) \right), \quad U(\Phi) = \Phi - \sum_i \lambda_i \, e^{-2\pi(1-\alpha_i)\Phi}, \tag{2.18}$$

where the sum i is over the defect species. As emphasized in [15] there is a choice of renormalization scheme behind this identification between a quantum theory (the sum over defects) and a classical dilaton potential. We will give a different scheme using the minimal string in section 2.5.

General defects

For blunt defects, corresponding to $\alpha > 1/2$, the previous methods cannot be used for the reasons we will emphasize now. The recipe of [4] involves first integrating out the dilaton and summing over hyperbolic surfaces with cone points. The second step involves finding geodesics homologous to the holographic boundaries and computing the path integral by gluing Weil-Petersson volumes with 'trumpet' partition functions. This step fails when $\alpha > 1/2$ since hyperbolic surfaces with these cone points do not necessarily always have geodesics that we could use to cut and glue.

There is a simple argument explaining why this is the case. Assume the opposite, namely that we can write a hyperbolic surface with k cone points labeled by α_i and for concreteness take a single geodesic boundary and no handles. We can use the Gauss-Bonnet formula on a two-dimensional surface with k cone points and with geodesic boundaries

$$\frac{1}{2} \int_M R + \sum_{i=1}^{k} 2\pi(1-\alpha_i) = 2\pi\chi(M). \tag{2.19}$$

For surfaces with $R = -2$ away from cone points and one geodesic boundaries this gives the inequality $\sum_{i=1}^{k}(1-\alpha_i) > 1$. When $\alpha_i < 1/2$, this inequality is always satisfied as long as we have more than a single defect. Instead, assume we have a single defect species with $\alpha > 1/2$. Then we need at least $k > 1/(1-\alpha)$ in order for the inequality to be satisfied. We show this in Figure 2.3.

Another manifestation of the fact that general defects are special is the fact that they can merge without having to pinch off the surface. Consider the case of a single species of

Figure 2.3: **Left:** Picture of a hyperbolic surface with a holographic boundary to the left and two sharp defects shown to the right with $\alpha < 1/2$. In dashed line we show a geodesic homologous to the holographic boundary. **Right:** Similar picture of a hyperbolic surface in the case $\alpha > 1/2$. As argued in the main text there is no geodesic to cut and glue.

defects with defect parameter α. In general, k of them can merge into a single "large" defect with cone angle

$$2\pi\alpha^{(k)} = 2\pi(1 - k(1-\alpha)) \tag{2.20}$$

as long as $\alpha^{(k)}$ is positive. This is not possible when $\alpha < 1/2$, so sharp defects do not merge in a smooth way. Instead, when $\alpha > 1/2$ we can merge no more than $k = \lfloor \frac{1}{1-\alpha} \rfloor$ defects.

For these reasons instead of following the approach of [4], we will exploit the connection between JT gravity with defects and the minimal string that we develop in section 2.3. The solution of the deformed minimal string given in [92] is insensitive to whether it corresponds to sharp defects or not. Therefore we will find the JT answer by taking a limit of the minimal string.

2.3 Deformations of the minimal string

In this section we review the relevant aspects of the $(2,p)$ minimal string and its deformations by adding "tachyon" operators in the action. These string theories are dual to a Hermitian one-matrix integral in the double-scaling limit. Therefore the tree-level string equation of the matrix integral, or equivalently the disk density of states, completely specifies the model in the double-scaling limit. The most general tree-level string equation corresponding to the deformed minimal string was proposed by Belavin and Zamolodchikov [92] building on the work [93].

The minimal string

We will begin with a brief review of the minimal string theory. We will interpret it as a theory of 2D gravity on the worldsheet coupled to a minimal model CFT. In the conformal gauge the physical metric g can be written in terms of a fiducial metric $\hat{g}$ and a scale factor as $g = e^{2b\phi}\hat{g}$, with b a parameter to be determined later. After gauge fixing, the minimal string reduces to a minimal model coupled to Liouville field theory associated to the scale factor ϕ, and bc ghosts. Below we review some useful facts of these building blocks that we will need later.

Minimal model: The minimal string can be defined for any minimal model labeled by coprime integers (p, p'). Since we will be interested in the theories dual to a one-matrix integral we focus on the Lee-Yang series $(2, p = 2m - 1)$, for integer m. The central charge of these theories is given by $c_M = 1 - 6q^2$, where we define $q = 1/b - b$ and $b = \sqrt{2/p}$. The interpretation of these parameters will become clear soon. We will consider theories with odd m since only those exist non-perturbatively; see for example [108].

The spectrum of these models includes a finite number of primaries labeled by a single integer with scaling dimension

$$\mathcal{O}_{1,n}, \qquad \text{with} \;\; \Delta_n = \frac{(nb^{-1} - b)^2 - (b^{-1} - b)^2}{4}, \qquad n = 1, \ldots, m - 1. \tag{2.21}$$

These primary operators satisfy very simple fusion rules. Using the Coulomb gas approach, some aspects of the minimal models can be reproduced by a time-like Liouville action, as discussed by Zamolodchikov [109] (see also [110]), in terms of a scalar field χ. The action is given by

$$I_{\rm MM}[\chi] = \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[-(\hat{\nabla}\chi)^2 - q\hat{R}\chi - 4\pi\mu e^{2b\chi} \right], \tag{2.22}$$

where μ is a parameter we will fix later. This gives a Lagrangian description of these CFT. This representation motivates the connection to dilaton-gravity following [86]. Within this time-like Liouville theory consider the following operators

$$\exp\left(2\hat{a}_n\chi\right), \qquad \hat{a}_n = \frac{b}{2}(1 - n), \qquad n = 1, \ldots, m - 1, \tag{2.23}$$

with scaling dimension $\Delta_{\hat{a}_n} = \hat{a}_n(q + \hat{a}_n)$. We associate these operators to the minimal model primaries $\mathcal{O}_{1,n} \leftrightarrow \exp\left(2\hat{a}_n\chi\right)$, although with a different normalization we present below. It was verified in [109] that some correlators of time-like Liouville reproduce the minimal model correlators, after imposing the fusion rules by hand. Some recent progress understanding this identification can be found in [94] (for a different approach see [111, 112]).

Liouville sector: The second building block of the minimal string is the Liouville gravity mode. This is an action for the scale factor ϕ in the conformal gauge, originating from the conformal anomaly, given by

$$I_L[\phi] = \frac{1}{4\pi} \int d^2z \sqrt{\hat{g}} \left[(\hat{\nabla}\phi)^2 + Q\hat{R}\phi + 4\pi\mu e^{2b\phi} \right], \tag{2.24}$$

where $Q = b + 1/b$. The central charge is given by $c_L = 1 + 6Q^2$. The cancellation of the conformal anomaly between the minimal model, Liouville and the ghosts fixes the parameter

$b = \sqrt{2/p}$. The primary operators are

$$V_a = \exp\big(2a\phi\big), \qquad a = \frac{Q}{2} + iP, \qquad \Delta_a = a(Q-a) \tag{2.25}$$

which we can write in terms of a or the Liouville momentum P, which is continuous. The normalizable states correspond to $P^2 > 0$ while non-normalizable local operator insertions have $P^2 < 0$. We will be interested in the case of geometries with boundaries. In this case we will define the FZZT boundary condition [113] with fixed boundary cosmological constant

$$\mu_B = \kappa \cosh\left(2\pi b s\right), \quad \kappa \equiv \frac{\sqrt{\mu}}{\sqrt{\sin \pi b^2}}, \tag{2.26}$$

in terms of the parameter s. Equivalently we will work with the fixed length boundary condition which is a Laplace transform of the FZZT brane. This can be thought of a Dirichlet boundary condition on the Liouville field that fixes the value of $\oint e^{b\phi} \to \ell$ along the boundary. For a detailed explanation of how to go between them see [22].

Minimal string: The minimal string theory is a combination of the minimal model CFT, the Liouville mode and a set of ghosts. These sectors are only coupled through the anomaly cancellation, and a possible integral over moduli space when computing observables. We will only consider in this paper states of ghost number one. These are "tachyon" operators that are given by gravitationally dressing minimal model primaries

$$\mathcal{T}_n \equiv \int \sqrt{\hat{g}}\, \mathcal{O}_{1,n}\, e^{2a_n \phi}, \qquad a_n = \frac{b}{2}(1+n), \qquad n = 1, \ldots, m-1. \tag{2.27}$$

For the operator $\mathcal{T}_n$ to be diffeomorphism invariant, the composite operator $\mathcal{O}_{1,n}\, e^{2a_n\phi}$ should have dimension $(1,1)$. The dressing parameter was determined from the condition

$$\Delta_n + \Delta_{a_n} = 1. \tag{2.28}$$

There is another solution for a_n to this equation, but in (2.27) we picked the one that is smooth in the $b \to 0$ limit.

In the rest of this section we will review a connection pointed out by Seiberg and Stanford [86] (see also Appendix F of [22]) between this theory and two-dimensional dilaton-gravity that we will exploit later on. The idea is to use the time-like Liouville representation of the minimal model. Then the action of the minimal string can be written as a sum of two Liouville modes $I_{\rm MS}[\chi, \phi] = I_{\rm MM}[\chi] + I_{\rm L}[\phi] + I_{\rm ghosts}$. The ghosts do not seem to be important so we will ignore them from now on, although this should be understood better. In conformal gauge we can perform the following field redefinition mixing the two modes

$$b\phi = \rho - \pi b^2 \Phi, \tag{2.29}$$

$$b\chi = \rho + \pi b^2 \Phi. \tag{2.30}$$

We rewrite the path integral now over ρ and Φ. We will call Φ the dilaton and define a new two-dimensional metric

$$g = e^{2\rho}\hat{g}. \tag{2.31}$$

Notice this metric is not the same as the worldsheet metric in the minimal string. In terms of this new metric and dilaton the minimal string can be rewritten as

$$I = -\frac{1}{2}\int \sqrt{g}\left[\Phi R + 4\mu \sinh\left(2\pi b^2 \Phi\right)\right]. \tag{2.32}$$

This has the most general two-derivative form of dilaton-gravity $-\frac{1}{2}\int \Phi R + 2U(\Phi)$, with dilaton potential given by $U(\Phi) = 2\mu \sinh(2\pi b^2\Phi)$ (this potential was studied for different reasons in [114–116]). This suggests that the minimal string is equal to a certain dilaton-gravity theory and some checks were performed in [22]. This theory simplifies in the limit $b \to 0$ where the potential is linear and the theory becomes Jackiw-Teitelboim gravity

$$I = -\frac{1}{2}\int \sqrt{g}\ \Phi(R + 2\Lambda), \tag{2.33}$$

where in the limit we keep $\Lambda \equiv 4\pi b^2\mu$, which becomes the absolute value of the two-dimensional cosmological constant, fixed and set to one. We will call this the pure JT gravity limit or *JT limit* for short. We can check that this identification is true by comparing the disk density of states [4]. We will extend this to the case of the minimal string deformed by tachyon operators in section 2.4.

String equation

From the perspective of the loop equations [117, 118], all the information of a matrix integral is encoded in the disk density of state $\rho_0(E)$, which is also equivalent to giving the matrix potential $V(H)$. Higher genus contributions are determined from the topological recursion [118, 119]; for an example on how this is done see [4]. Therefore, if two theories that are dual to a matrix integral share the same disk density of states it means the theories are equivalent to all orders in the genus expansion. Since we will work in the double-scaling limit we will 'label' the theory by $\rho_0(E)$ instead, since the precise matrix potential depends on how we regularize the theory away from the double-scaling limit.

It will be very useful when studying deformations to look at the matrix integral from a different perspective: the string equation [78–80] (building on previous work [19, 120–123]). To leading order in the genus expansion the string equation has the following form

$$\sum_k t_k u^k = x, \tag{2.34}$$

where x is a dummy variable we can use to compute certain observables and t_k are the KdV couplings. We include t_0 on the left-hand side so at the end of the day we will always fix

$x = 0$. This can be derived by taking the double-scaling limit of the orthogonal polynomial method; see for example [78]. Knowing $u(x)$ allows us to compute the main observable we are interested in. For example, the genus zero disk partition function is given by [124]

$$\left\langle \mathrm{Tr}\left[e^{-\beta H}\right]\right\rangle_{g=0} = e^{S_0}\frac{1}{\sqrt{2\pi\beta}}\int_0^\infty dx\, e^{-u(x)\beta}. \tag{2.35}$$

The string equation provides an independent way to compute higher genus corrections. The recipe is to replace in the string equation the powers of u by the Gelfand-Dickii differential operators $u^k \to R_k[u;\hbar]$ which to leading order in $\hbar \equiv e^{-S_0}$ coincide with $R_k[u] \sim u^k$ [125, 126]. Then one needs to solve again for $u(x;\hbar)$ and compute observables using the quantum mechanical free fermion perspective [124] which to leading order reduces to (2.35). This perspective is useful to derive analytical [22] and numerical results [84] which would be hard to obtain from the topological recursion.

To simplify the presentation, instead of giving the KdV couplings individually we will directly define and compute the function

$$\mathcal{F}(u) \equiv \sum_k t_k u^k. \tag{2.36}$$

The string equation to leading order becomes $\mathcal{F}(u) = x$. The KdV coupling can be easily extracted by a Taylor expansion. We allow for the possibility of the index k being unbounded, since this is needed for the JT gravity matrix integral.

The function $\mathcal{F}(u)$ fully specifies the double-scaling limit of a matrix integral. This is equivalent to giving the disk density of states which using (2.35) is related by the equation

$$\rho(E) = \frac{e^{S_0}}{2\pi}\int_{E_0\equiv u(0)}^{E}\frac{du}{\sqrt{E-u}}\frac{\partial\mathcal{F}}{\partial u}, \tag{2.37}$$

where E_0 is the largest root of the string equation with $x = 0$, given explicitly by $\mathcal{F}(E_0) = 0$. This is in general a complicated equation.

String equation for the $(2,p)$ minimal string: The minimal string with $p = 2m-1$ is dual to the mth-multicritical model of a one-matrix integral [19, 20]. In order to compare the matrix model with the results from the worldsheet CFT approach one has to turn on lower order couplings in a specific way [93]. We will now review the derivation of the 'string equation' that matches with the CFT continuum approach [92, 93]. This can be read off from the calculation of the disk partition function $Z(\ell)$ with fixed length boundary conditions [93, 113]. The final answer is given by

$$Z(\ell) \sim \int_0^\infty ds\, e^{-\mu_B(s)\ell}\sinh 2\pi b s \sinh 2\pi\frac{s}{b}. \tag{2.38}$$

From this expression we can extract the disk density of states as a function of p and also the cosmological constant through κ. This identifies $\mu_B(s)$ with the matrix eigenvalues E_{MS} (we leave E and u to refer to a different convention that we specify below). The answer picking a particular normalization is given by

$$\rho(E_{\rm MS}) = e^{S_0}\frac{p^2}{32\pi^4\kappa^2}\sinh\left(\frac{p}{2}\text{arccosh}\left(\frac{E_{\rm MS}}{\kappa}\right)\right). \tag{2.39}$$

Using (2.37) we can obtain the string 'string equation' associated to this theory. The result is given by

$$\mathcal{F}(u_{\rm MS}) = \frac{p^2}{32\pi^3\sqrt{2\kappa^3}}\left[P_m\left(\frac{u_{\rm MS}}{\kappa}\right) - P_{m-2}\left(\frac{u_{\rm MS}}{\kappa}\right)\right], \tag{2.40}$$

where $\kappa \sim \sqrt{\mu}$ and $P_n(x)$ are the Legendre polynomials. We review the definition and some useful properties in Appendix A.2. At large energies this precisely becomes the string equation of the m-th multicritical point of the matrix integral $u^m \sim x$, or equivalently $\rho_0(E_{\rm MS}) \sim (\sqrt{E_{\rm MS}})^{2m-1}$. As anticipated this answer is modified by turning on other lower couplings and near the edge this presents the universal $\rho(E_{\rm MS}) \sim \sqrt{E_{\rm MS}-\kappa}$ behavior.

JT gravity string equation: Now we will write down the JT gravity tree-level string equation and compare it with a limit of the minimal string. It is useful to rescale and shift the matrix, and therefore its eigenvalues E, and the variable u in the string equation correspondingly. Following [22] we define the variables E and u as

$$E_{\rm MS} = \kappa\Big(1 + \frac{8\pi^2}{p^2}E\Big), \qquad u_{\rm MS} = \kappa\Big(1+\frac{8\pi^2}{p^2}u\Big). \tag{2.41}$$

In terms of these variables the density of states and string equation of the minimal string are independent of μ and become

$$\rho(E) = \frac{e^{S_0}}{4\pi^2}\sinh\left(\frac{p}{2}\text{arccosh}\Big(1+\frac{8\pi^2}{p^2}E\Big)\right), \tag{2.42}$$

$$\mathcal{F}(u) = \frac{p}{16\pi^2}\left[P_m\Big(1+\frac{8\pi^2}{p^2}u\Big) - P_{m-2}\Big(1+\frac{8\pi^2}{p^2}u\Big)\right]. \tag{2.43}$$

We can now take the large p limit keeping E and u fixed. This gives

$$\rho_{\rm JT}(E) = \frac{e^{S_0}}{4\pi^2}\sinh\left(2\pi\sqrt{E}\right) \qquad \leftrightarrow \qquad \mathcal{F}_{\rm JT}(u) = \frac{\sqrt{u}}{2\pi}I_1\left(2\pi\sqrt{u}\right). \tag{2.44}$$

It is now evident that this coincides with the density of states (and therefore string equation) of the matrix integral dual to pure JT gravity.

Deformations and Belavin-Zamolodchikov string equation

We now analyze deformations of the minimal string. We will review the exact string equation proposed by Belavin and Zamolodchikov [92], building on the work of Moore, Seiberg and

Staudacher [93]. In the next section we will make a connection between these deformations of the minimal string and the conical deformations of pure JT gravity introduced in [16] and [15].

We will focus on the following type of deformations by adding a combination of tachyon operators $\sum_n \tau_n \mathcal{T}_n$ to the minimal string action, where τ_n are the couplings of each deformation. Then the action of the deformed minimal string is, writing the deformation more explicitly,

$$I = I_{(2,p)} - \sum_{n=2}^{m-1} \tau_n \int \sqrt{\hat{g}}\, \mathcal{O}_{1,n}\, e^{b(1+n)\phi}, \qquad n = 1, \ldots, m-1, \tag{2.45}$$

where $a_n = b(n+1)/2$ is tuned such that the integrand is a marginal operator.

The outline of the derivation is the following. When computing tachyon correlation functions on the sphere, the structure is strongly constrained by the minimal model fusion rules and conformal invariance. For example the correlator $\langle \mathcal{T}_n \mathcal{T}_{n'} \rangle_{S^2} = 0$ unless $n = n'$. On the other hand, given a tree-level string equation one can derive these correlators by taking derivatives with respect to the couplings τ_n's. Strikingly, solely the conditions derived from the fusion rules completely fix the string equation. The final answer is given by

$$\begin{aligned} \mathcal{F}(u_{\rm MS}) \;\; &= \;\; \frac{p}{16\pi^2}\left(P_m\left(\frac{u_{\rm MS}}{\kappa}\right) - P_{m-2}\left(\frac{u_{\rm MS}}{\kappa}\right)\right) \\ &\quad + \sum_{L=1}^{\infty} \sum_{n_1,\ldots,n_L=1}^{m-1} \frac{1}{L!}\Pi_{i=1}^{L}\lambda_{n_i}\left(\frac{16\pi^2}{p^2}\right)^{L-1} P^{(L-1)}_{m-1-\sum_{i=1}^{L} n_i}\left(\frac{u_{\rm MS}}{\kappa}\right), \end{aligned} \tag{2.46}$$

where $\lambda_n \propto \tau_n$, with the prefactor determined below. We defined the Lth derivative of the nth Legendre polynomial as $P_n^{(L)}(x) \equiv \partial_x^L P_n(x)$ [5]. The sum is over a set of L integers $1 \leq n_i \leq m-1$ where $i = 1, \ldots, L$ and $L = 1, \ldots, \infty$. As explained in Appendix A.2, the Legendre polynomial is defined such that $P_n(x) = 0$ when $n < 0$. This implies that the sums in the second term in the right-hand-side only contribute as long as $m - 1 - \sum_i n_i \geq 0$. This constraint comes from analyzing resonance conditions between deformations.

The relation between the parameter λ_n associated to the deformation and the coupling in the action τ_n depends on the precise normalization of the minimal model operators. If we pick the conventional normalization we obtain

$$\lambda_n = \tau_n \;\mathrm{Leg}(n) \frac{p^2}{16\pi^2}, \tag{2.47}$$

where we defined the leg-factor

$$\mathrm{Leg}(n) \equiv \frac{i^{n-1}}{2\mu}\sqrt{\frac{\pi\mu\gamma(nb^2)}{(\pi\mu\gamma(b^2))^n}}\; \frac{\Gamma(\frac{1}{b^2}-1)}{\Gamma(\frac{1}{b^2}-n)}, \tag{2.48}$$

[5]Not to be confused with the associated Legendre polynomial which will not appear in this paper.

with $\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$. This relation comes from comparing the correction to the disk partition function to linear order in the deformation computed from the string equation or from the worldsheet CFT description. We do this in Appendix A.3. The leg-factor appears as well in computing sphere correlators. For example the normalized sphere two-point function is

$$\langle \mathcal{T}_n \mathcal{T}_{n'} \rangle_{S^2} = \delta_{n'n} \frac{(2m-3)(2m-1)(2m+1)}{2m-2n-1} \mathrm{Leg}(n)^2. \tag{2.49}$$

If we want to shift normalization for the minimal model operators by $\mathcal{O}_{1,n} \to N_n \mathcal{O}_{1,n}$ this can be achieved by shifting $\mathrm{Leg}(n) \to N_n \mathrm{Leg}(n)$. We will use this in section 2.5 to study these expressions in a normalization more natural for the time-like Liouville description of the minimal model.

Now we will analyze several special cases of these formulas in order to gain some intuition. To simplify we will consider the case of a single deformation with parameter n and look at various cases:

$\underline{n = m-1}$**:** This corresponds to the case of deforming the action by the operator with the lowest dimension in the minimal model. The level of the Legendre polynomial appearing in the string equation is $m - 1 - \sum_i n_i$. Therefore whenever an operator with $n = m-1$ is present, it will only contribute by itself to linear order exactly. Any other term vanishes. Then its contribution to the string equation is

$$\delta \mathcal{F} \sim \lambda_{m-1} P_0 \left(\frac{u_{\mathrm{MS}}}{\kappa} \right) = \lambda_{m-1}. \tag{2.50}$$

This is equivalent to shifting $x \to x - \lambda_{m-1}$ when computing observables. Therefore this gives an interpretation of the parameter x as a specific coupling in the minimal string.

$\underline{\frac{m-1}{2} \le n < m-1}$**:** In this case the string equation also simplifies as long as $n > n_\star$ with $n_\star \equiv (m-1)/2$. In this range, the string equation is linear in the λ_n's. This happens because the resonance conditions for these operators are very limited and there are no non-linear ambiguities due to contact terms [92]. The solution is given by

$$\mathcal{F}(u_{\mathrm{MS}}) = \frac{p}{16\pi^2} \left[P_m \left(\frac{u_{\mathrm{MS}}}{\kappa} \right) - P_{m-2} \left(\frac{u_{\mathrm{MS}}}{\kappa} \right) \right] + \sum_{n=n_\star}^{m-1} \lambda_n P_{m-n-1} \left(\frac{u_{\mathrm{MS}}}{\kappa} \right). \tag{2.51}$$

As n decreases from $n = m-1$ to $n_\star$, the order of the polynomial goes from order zero to order $(m-1)/2$.

$\underline{n < \frac{m-1}{2}}$**:** This is the most complicated range. As becomes clear from the string equation, deformations with lower n involve higher orders in λ_n. In this case there is no simplification and we need to consider the full expression.

<u>$n = 1$</u>**:** Finally, there is a simplification when we include operators with $n = 1$. This is the identity operator and we should reproduce a shift of the cosmological constant by $-\tau_1$. In this case the bound on the terms that can appear in the string equation is $m - 1 - \sum_i n_i = m - 1 - L > 0$, and therefore $L \leq m - 1$. The sum can be explicitly done using results in Appendix A.2 as

$$\begin{aligned} \mathcal{F}(u_{\mathrm{MS}}) &\sim \frac{P_m(u_{\mathrm{MS}}/\kappa) - P_{m-2}(u_{\mathrm{MS}}/\kappa)}{2m-1} + \sum_{n=1}^{m-1} \frac{\lambda^n}{n!} \left(\frac{16\pi^2}{p^2}\right)^n P^{(n-1)}_{m-n-1}(u_{\mathrm{MS}}/\kappa) && (2.52) \\ &\sim P_m\left(\frac{u_{\mathrm{MS}}}{\kappa\sqrt{1-2\tau_1 \mathrm{Leg}(1)}}\right) - P_{m-2}\left(\frac{u_{\mathrm{MS}}}{\kappa\sqrt{1-2\tau_1 \mathrm{Leg}(1)}}\right), && (2.53) \end{aligned}$$

where we omit a u independent prefactor that will not be important, and in the second line we have used $\lambda \frac{16\pi^2}{p^2} = \tau_1 \, \mathrm{Leg}(n = 1)$. We see that this gives back the undeformed minimal string equation with the replacement $\kappa \to \kappa\sqrt{1 - 2\tau_1 \, \mathrm{Leg}(1)}$. Using the minimal model normalization of operators we obtain $\mathrm{Leg}(n = 1) = \frac{1}{2\mu}$. Therefore we find that the shift of $\kappa \sim \sqrt{\mu}$ is equivalent to shifting the cosmological constant $\mu \to \mu - \tau_1$, consistent with equation (2.45).

After these clarifications we see the behavior of the deformed minimal string strongly depends on whether the parameter n is bigger or smaller than $n_\star$. This is reminiscent of the differences between sharp or blunt defect deformations with either $\alpha < 1/2$ and $\alpha > 1/2$. In the next section we will see this is not a coincidence.

2.4 JT gravity with defects from the minimal string

In the previous section we studied the exact solution for deformations of the $(2, p)$ minimal string. In this section we will show that the large p limit of these theories corresponds to JT gravity with a gas of defects. We will use this correspondence to find the disk density of states for JT gravity in the presence of general defects with an arbitrary defect angle.

Sharp defects from the minimal string

Having explained the exact solution of the deformed minimal string in section 2.3 we will now begin to take the JT gravity limit, namely $p \to \infty$. As a first clarification, we will take the energy and u to scale as

$$E_{\mathrm{MS}} = \kappa\Big(1 + \frac{8\pi^2}{p^2}E\Big), \qquad u_{\mathrm{MS}} = \kappa\Big(1 + \frac{8\pi^2}{p^2}u\Big), \tag{2.54}$$

where E and u are kept fixed in the $p \to \infty$ limit.

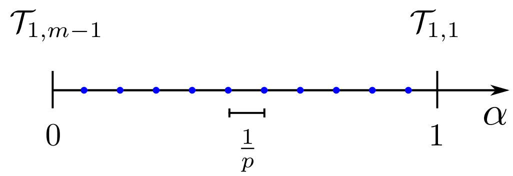


Figure 2.4: The relationship between the minimal string operators $\mathcal{T}_n$ and the JT defect parameter α. The spacing between the operators in α is discrete and of order $1/p$; as we take the JT limit by sending p to infinity, α becomes a continuous parameter.

As we take the large p limit, we need to choose a scaling for both the index n labeling the operator and the coupling τ_n. We choose the following scaling

$$n = \frac{p}{2}(1-\alpha), \quad \tau_n = \lambda \frac{16\pi^2}{p^2}\frac{1}{\mathrm{Leg}(n)}, \tag{2.55}$$

where we keep α and λ fixed in the JT limit. The rationale for this choice will be motivated and explained below.

At finite values of p, α is a discrete parameter since n is discrete and takes value in the set $\alpha \in \frac{2}{p}\cdot\mathbb{Z}$. As we take $p\to\infty$ this parameter becomes continuous. Since n in the minimal string is in the range $1 \leq n \leq m-1$, the parameter α lies between $0<\alpha<1$. The case $n=1$ corresponds to $\alpha = 1-\frac{2}{p} \to_{p\to\infty} 1$, while $n=m-1$ corresponds to $\alpha=\frac{1}{p}\to_{p\to\infty} 0$. We give a diagram showing the relation between n and α in figure 2.4. We saw the deformations in the minimal string changes drastically at the threshold $n_\star$. In terms of α this is given by $\alpha_\star = \frac{1}{2}+\frac{1}{2p} \to_{p\to\infty} \frac{1}{2}$. The string equation simplifies when $n_\star < n$ or equivalently $0<\alpha<1/2$. This is precisely the same range corresponding to sharp defects studied in [15, 16].

We begin by considering the large p limit restricted to $n_\star < n$ or $\alpha < 1/2$. We write the string equation (2.51) in terms of variables that we keep fixed in the JT limit. Deformations of the string equation in this range are exactly linear so we can consider one defect without loss of generality. The finite p string equation perturbed by one defect is

$$\mathcal{F}(u) = \frac{p}{16\pi^2}\left[P_m\left(\frac{u_{\mathrm{MS}}}{\kappa}\right) - P_{m-2}\left(\frac{u_{\mathrm{MS}}}{\kappa}\right)\right] + \lambda\, P_{m-\frac{p}{2}(1-\alpha)-1}\left(\frac{u_{\mathrm{MS}}}{\kappa}\right). \tag{2.56}$$

Using identities from Appendix A.2 the large p limit with α and λ fixed gives the following answer

$$\mathcal{F}(u) \to \frac{\sqrt{u}}{2\pi} I_1\left(2\pi\sqrt{u}\right) + \lambda\, I_0\left(2\pi\alpha\sqrt{u}\right), \tag{2.57}$$

which precisely coincides with the string equation of JT gravity with a gas of sharp defects [15, 16]! In that context the gas of defects are characterized by a deficit angle $2\pi(1-\alpha)$ (with α related to the minimal model operator) and a weighing factor λ (related to the coupling τ). It is easy to extend this to the case of multiple defect species from (2.51)

$$\mathcal{F}(u) \to \frac{\sqrt{u}}{2\pi} I_1\left(2\pi\sqrt{u}\right) + \sum_i \lambda_i\, I_0\left(2\pi\alpha_i\sqrt{u}\right), \tag{2.58}$$

obtaining again a match with JT gravity.

We have shown that the minimal string deformed by operators $n_\star < n$ at large p becomes JT gravity with sharp defects $\alpha < 1/2$. We have only compared tree-level string equations, but we would like to stress that this is enough to argue that both theories are the same to all orders in perturbation theory. This is because both theories are dual to a matrix integral, and all observables are determined by the matrix potential which can be determined from the tree-level disk density of states. Of course, this is true up to corrections not captured by the double-scaling limit.

Another check

The upshot of the previous calculation is that the insertion of a defect from the dilaton-gravity perspective is equivalent to a tachyon vertex operator insertion in the minimal string. It is instructive to check this more explicitly in a simple example, taken from [22, 90]. We compute the expectation value of the disk partition function with a single tachyon insertion and fixed boundary length ℓ. This is easy to do using Liouville CFT techniques and gives the answer

$$\langle \mathcal{T}_n \rangle \sim \int_0^\infty ds\ e^{-\mu_B(s)\ell} \cos\left(4\pi P_n s\right), \tag{2.59}$$

where $P_n = \pm i(1 - nb^2)/(2b)$ is the Liouville momentum of the gravitational dressing associated to $\mathcal{O}_{1,n}$ with $n = (1-\alpha)/b^2$. Replacing this value and writing the answer in terms of α gives

$$\langle \mathcal{T}_n \rangle \sim \int_0^\infty ds e^{-\ell\kappa\cosh(2\pi b s)} \cosh\left(\frac{2\pi\alpha s}{b}\right). \tag{2.60}$$

To take the JT limit of the result written in this form we set $s = bk$ and $\ell = \beta/(2\pi^2\kappa b^4)$ and take $b \to 0$ while keeping β and k fixed

$$\langle \mathcal{T}_n \rangle \sim \int_0^\infty dk e^{-\beta k^2} \cosh\left(2\pi\alpha k\right) \sim \frac{1}{\sqrt{\beta}} e^{\frac{\pi^2\alpha^2}{\beta}}. \tag{2.61}$$

This is precisely, for a specific choice of units, the path integral of JT gravity on the disk with fixed renormalized length β. This check is valid for any $0 < \alpha < 1$.

In the small b limit these are heavy operators. This match can then be understood from the perspective of the semiclassical evaluation of the Liouville path integral with one heavy insertion and fixed boundary length. The classical solution is precisely a hyperbolic metric with one conical defect at the operator insertion. For a review in the present context see Appendix B.1 of [22].

Cases with $E_0 = 0$

In analogy with an analysis in [15], we can study deformations of the minimal string with $n_\star < n$ that give $E_0 = 0$. This requires at least two different defects and in general $\sum_n \lambda_n = 0$. In this case we can analytically compute the density of states.

Using the equation for the density of states (2.37) with the string equation (2.56) in terms of u variables, the integral identity in (A.6) immediately gives

$$\begin{aligned} e^{-S_0}\rho(E) &= \frac{1}{4\pi^2}\sinh\left(\frac{p}{2}\text{arccosh}\left(\frac{E_{\text{MS}}}{\kappa}\right)\right) \\ &\quad + \sum_{n=n_\star}^{m-1}\lambda_n\frac{2\kappa}{p\sqrt{E_{\text{MS}}^2-\kappa^2}}\cosh\left(\frac{p-2n}{2}\text{arccosh}\left(\frac{E_{\text{MS}}}{\kappa}\right)\right), \end{aligned} \tag{2.62}$$

where we have written the final result in terms of E_{MS} for simplification purposes. It is surprising from the worldsheet CFT perspective of the minimal string that all higher order corrections in λ_n vanish! This is a nontrivial consequence of the Belavin-Zamolodchikov analysis that deserves further study. A similar observation was made in [15] in the context of JT gravity. To make the connection we take the large p limit of this expression giving

$$e^{-S_0}\rho(E) \to \frac{1}{4\pi^2}\sinh\left(2\pi\sqrt{E}\right) + \sum_i \lambda_i \frac{\cosh\left(2\pi\alpha_i\sqrt{E}\right)}{2\pi\sqrt{E}}. \tag{2.63}$$

This is precisely the density of states of JT gravity with sharp defects for $E_0 = 0$.

Solution with general defects

One defect species

The most general string equation for an arbitrary deformation is complicated, so we will begin by analyzing a simpler case where only one deformation τ_n is turned on with $n < n_\star$. Then (2.46) simplifies into the following equation

$$\mathcal{F}(u) = \frac{p}{16\pi^2}\left[P_m\left(\frac{u_{\text{MS}}}{\kappa}\right) - P_{m-2}\left(\frac{u_{\text{MS}}}{\kappa}\right)\right] + \sum_{L=1}^{\lfloor\frac{m-1}{n}\rfloor}\frac{\lambda_n^L}{L!}\left(\frac{16\pi^2}{p^2}\right)^{L-1} P_{m-Ln-1}^{(L-1)}\left(\frac{u_{\text{MS}}}{\kappa}\right), \tag{2.64}$$

where we remind the reader that $P_n^{(L)}(x) \equiv \partial_x^L P_n(x)$ is the derivative of the Legendre polynomial. The new feature compared to (2.51) is the fact that now the string equation is non-linear as a function of the deformation. This modification will survive the large p limit.

Now we take the JT limit. In order to do this we take $p \to \infty$, or equivalently $m \to \infty$, while scaling the index of the minimal model operator as $n = p(1-\alpha)/2$ and the coupling as in (2.55). This scaling gives JT gravity with a gas of defects with weight λ and defect angle α. The final answer for the string equation obtained from (2.64), using identities in Appendix A.2, is given by

$$\begin{aligned} \mathcal{F}(u) &= \frac{\sqrt{u}}{2\pi}I_1\left(2\pi\sqrt{u}\right) + \lambda I_0\left(2\pi\alpha\sqrt{u}\right) \\ &\quad + \sum_{L=2}^{\lfloor\frac{1}{1-\alpha}\rfloor}\frac{\lambda^L}{L!}\left(\frac{2\pi(1-L(1-\alpha))}{\sqrt{u}}\right)^{L-1} I_{L-1}\left(2\pi(1-L(1-\alpha))\sqrt{u}\right). \end{aligned} \tag{2.65}$$

The first line contains the two terms that already appear in the string equation for sharp defects. In the second line we show the new terms that appear when $\alpha > 1/2$. The nonlinear terms of order λ^L appear only if $L \leq \lfloor \frac{1}{1-\alpha} \rfloor$. This has a geometric interpretation since there is a bound on the number of individual defects that can be merged into a single defect. Moreover, the combination appearing in each term, $2\pi(1 - L(1-\alpha))$ is precisely the deficit angle after merging L defects of angle $2\pi(1-\alpha)$. It would be nice to understand how to derive this formula directly from the perspective of JT gravity but we leave this for future work.

Finally, using this tree-level string equation we can find the density of states to leading order in the genus expansion. Using (2.37) this is given by

$$\rho(E) = \frac{e^{S_0}}{2\pi} \sum_{L=0}^{\lfloor \frac{1}{1-\alpha} \rfloor} \frac{\lambda^L}{2L!} \int_{E_0}^{E} \frac{du}{\sqrt{E-u}} \left(\frac{2\pi(1-L(1-\alpha))}{\sqrt{u}} \right)^L I_L \left(2\pi(1-L(1-\alpha))\sqrt{u} \right). \quad (2.66)$$

The edge of the spectrum E_0 appearing in this expression is defined as the largest root of (2.65) solving $\mathcal{F}(u_0 = 1 + \frac{8\pi^2}{p^2} E_0) = 0$. For $\alpha \leq 1/2$ this coincides with the answer found in [15, 16] and generalizes it to $\alpha > 1/2$. Contrary to previous expectations this is a non-analytic function of α and the result changes drastically as α approaches one.

General case

We now generalize the previous discussion to an arbitrary number of defect species put together. In order to do this we take the JT gravity limit of the most general deformed minimal string equation (2.46). The calculation is very similar to what we have already done and therefore we give the final answer

$$\mathcal{F}(u) = {\sum_{\{\ell_i\}}}' \frac{\prod_i \lambda_i^{\ell_i}}{L!} \left(\frac{2\pi(1 - \sum_i \ell_i(1-\alpha_i))}{\sqrt{u}} \right)^{L-1} I_{L-1}\Big(2\pi\Big(1 - \sum_i \ell_i(1-\alpha_i)\Big)\sqrt{u}\Big), \quad (2.67)$$

where $L = \sum_i \ell_i$ and the sum includes all permutations of λ_i for a particular configuration $\{\ell_i\}$. The case with all $\ell_i = 0$ and therefore $L = 0$ gives back the pure JT gravity contribution. We see this is a simple generalization of the one species case (2.65). The prime in the summation means that we only include configurations $\{\ell_i\}$ such that the following identity is satisfied

$$1 - \sum_i \ell_i(1-\alpha_i) > 0. \quad (2.68)$$

This bound has a simple geometric interpretation. It specifies the maximum number of defects α_i that can merge together, and it is a straightforward generalization of the bound on L appearing in (2.65).

Structure of the perturbative expansion

We now explain general features of the perturbative expansion in λ of the partition function, and we relate it to some geometrical intuition developed in section 2.2. For simplicity we focus on the case of one defect species.

To begin consider the case of sharp defects. For $0 < \alpha \leq 1/2$ the first orders in the expansion are

$$\alpha \leq \frac{1}{2}: \qquad Z_{\rm JT} = e^{S_0}\frac{e^{\frac{\pi^2}{\beta}}}{4\sqrt{\pi}\beta^{3/2}}, \quad Z_{1-\rm def} = e^{S_0}\frac{e^{\frac{\pi^2\alpha^2}{\beta}}}{2\sqrt{\pi\beta}}, \quad Z_{2-\rm def} = e^{S_0}\frac{\sqrt{\beta}}{2\sqrt{\pi}}, \quad \ldots, \tag{2.69}$$

where the dots indicate contribution with larger number of defects. The exponential appearing in $Z_{\rm JT}$ and $Z_{1-\rm def}$ come from classical solutions, while the prefactor comes from perturbative quantum corrections. On the other hand for more than one defect there is no classical solution. This manifests itself in the fact that for $Z_{k-\rm def}$ with $k > 1$ the answer is always given by a finite polynomial in $\sqrt{\beta}$ (it is easy to show this property using the string equation).

Consider now the case $1/2 < \alpha < 2/3$ for which the maximum order in λ in the string equation is $\lfloor \frac{1}{1-\alpha} \rfloor = 2$. The perturbative expansion now has new terms

$$\begin{aligned}\frac{1}{2} < \alpha < \frac{2}{3}: \qquad & Z_{\rm JT} = e^{S_0}\frac{e^{\frac{\pi^2}{\beta}}}{4\sqrt{\pi}\beta^{3/2}}, \quad Z_{1-\rm def} = e^{S_0}\frac{e^{\frac{\pi^2\alpha^2}{\beta}}}{2\sqrt{\pi\beta}}, \quad Z_{2-\rm def} = e^{S_0}\frac{\sqrt{\beta}}{2\sqrt{\pi}}e^{\frac{\pi^2(1-2\alpha)^2}{\beta}}, \\ & Z_{3-\rm def} = e^{S_0}\frac{\pi^{3/2}}{6\sqrt{\beta}}(\beta(12-6\alpha(4-3\alpha))-\pi^2), \quad \ldots.\end{aligned} \tag{2.70}$$

The first two terms corresponding to zero or one defect are unchanged. This is true for defects in the whole range $0 < \alpha < 1$. But now for $1/2 < \alpha < 2/3$ we see the two-defect term gets modified and a new exponential term appears. This is consistent with the analysis in section 2.2, in particular the exponent of $Z_{2-\rm def}$ is precisely the same as the classical contribution of two defects of angle α merging into a single one of angle $2\pi(2\alpha - 1) > 0$. The contribution of $Z_{k-\rm def}$ for all $k > 2$ can be shown to be given by a finite sum of powers, consistent with the fact that for $1/2 < \alpha < 2/3$ there is no classical solution with three or more merged defects[6].

A similar structure is valid in the whole range $0 < \alpha < 1$ consistent with section 2.2. If we look at the high temperature limit of small β, then the first terms have the behavior

$$Z_{L-\rm def} \sim e^{S_0}\beta^{L-\frac{3}{2}}e^{\frac{\pi^2(1-L(1-\alpha))^2}{\beta}}, \quad \text{for} \quad L \leq \left\lfloor \frac{1}{1-\alpha} \right\rfloor, \tag{2.71}$$

[6]For $1/2 < \alpha < 2/3$ and $k > 2$ there is a geodesic in the geometry we can use to glue with trumpets. Nevertheless in this case a different calculation of the WP volume is required [105]. Therefore, more generally contributions with $Z_{k-\rm def}$ and $k > \lfloor \frac{1}{1-\alpha} \rfloor$ do not need to match with the ones from sharp defects continued to values of $\alpha > 1/2$, and we find they are different.

while for $Z_{k-\text{def}}$ with $k > L$ the answer is always a finite sum of powers. It would be interesting to understand these results directly from a JT gravity with defects path integral calculation, but we leave this for future work. Some progress in this direction was done in [95]. We collect more results regarding the perturbative expansion in Appendix A.5.

Summary: Types of defects

In this section we summarize the different types of defects and their correspondence to the minimal string deformations outlined at the end of section 2.3.

$\underline{\alpha = 0}$**:** This gives a gas of cusps, and is obtained in the large p limit of the $\mathcal{T}_{m-1}$ deformation. The string equation in this case is simply

$$\mathcal{F}(u) = \frac{\sqrt{u}}{2\pi} I_1\left(2\pi\sqrt{u}\right) + \lambda. \tag{2.72}$$

Just like the $\mathcal{T}_{m-1}$ deformation, this is equivalent to shifting the variable $x \to x - \lambda$ in the string equation and gives a geometric interpretation for this dummy variable used to compute expectation values. From the dilaton-gravity perspective this amounts to adding a term to the dilaton potential proportional to $e^{-2\pi\Phi}$.

$\underline{0 < \alpha \leq 1/2}$**:** These are the sharp defects studied in [15, 16] and they are obtained from deformations between $\mathcal{T}_{n_\star}, \dots, \mathcal{T}_{m-2}$. The string equation is exactly linear in the deformation parameter λ

$$\mathcal{F}(u) \to \frac{\sqrt{u}}{2\pi} I_1\left(2\pi\sqrt{u}\right) + \lambda\, I_0\left(2\pi\alpha\sqrt{u}\right). \tag{2.73}$$

For $\alpha = 1/2$, the contribution with $L = 2$ is exactly zero[7]. This was implicitly assumed in [16] since this case is relevant to the application to 3D, and here we can verify this.

$\underline{1/2 < \alpha < 1}$**:** They correspond to blunt defects, for which the SSS recipe cannot be applied. The string equation is non-linear in the deformation

$$\mathcal{F}(u) \to \sum_{L=0}^{\lfloor \frac{1}{1-\alpha} \rfloor} \frac{\lambda^L}{L!} \left(\frac{2\pi(1 - L(1-\alpha))}{\sqrt{u}}\right)^{L-1} I_{L-1}\left(2\pi(1 - L(1-\alpha))\sqrt{u}\right), \tag{2.74}$$

and we propose the new terms are related to the possibility of defects merging and new classical solutions compared with sharp defects.

[7] This happens whenever $(1-\alpha)^{-1}$ is an integer since then the contribution from $L_{\max} = \lfloor \frac{1}{1-\alpha} \rfloor$ is proportional to $(1 - L_{\max}(1-\alpha))^{L_{\max}-1} = 0$.

$\underline{\alpha = 1}$**:** The deformation with $\alpha = 1$ corresponds to the large p limit of the deformation $\mathcal{T}_1$, which in the minimal string simply shifts the cosmological constant. From the JT perspective, when $\alpha = 1$ we have a gas of insertions with vanishing deficit angle and therefore we expect to recover pure JT gravity. Indeed this is the case:

$$\mathcal{F}(u) \;\to\; \frac{\sqrt{u}}{2\pi} I_1\left(2\pi\sqrt{u}\right) + \sum_{L=1}^{\infty} \frac{\lambda^L}{L!}\left(\frac{2\pi}{\sqrt{u}}\right)^{L-1} I_{L-1}\left(2\pi\sqrt{u}\right) \tag{2.75}$$

$$= \frac{\sqrt{u+2\lambda}}{2\pi} I_1\left(2\pi\sqrt{u+2\lambda}\right). \tag{2.76}$$

This is precisely the JT gravity string equation up to a simple shift of $u \to u + 2\lambda$. The density of states associated to this is simply

$$\rho(E) = \frac{e^{S_0}}{4\pi^2}\sinh\left(2\pi\sqrt{E-E_0}\right), \quad E_0 = -2\lambda. \tag{2.77}$$

In section 2.5, we give a possible interpretation of this shift from the dilaton-gravity perspective. It is interesting that the $\alpha \to 1$ limit of the partition function in the disk with a single defect does not give back pure JT gravity. Instead we need to sum over a gas of points with zero deficit angle in order to recover the undeformed theory.

Defect generating function

We have presented the solution to JT gravity with a gas of generic defect species parameterized by their weight λ_i and angle α_i. It will be convenient to recast this information about the theory in the following defect generating function $W(y)$. This is defined as

$$W(y) \equiv \sum_i \lambda_i e^{-2\pi(1-\alpha_i)y}. \tag{2.78}$$

Characterizing the species present is equivalent to giving the function $W(y)$, with some restriction on W coming from $0 < \alpha_i < 1$.

The solution found through the string equation (2.67) is not very transparent when describing the spectrum of defects in terms of $W(y)$. As we show in Appendix A.4, using some integral identities for Bessel functions, the string equation (2.67) can be exactly rewritten as an inverse Laplace transform[8]

$$\mathcal{F}(u) = \int_{\mathcal{C}} \frac{dy}{2\pi i} e^{2\pi y}\left(y - \sqrt{y^2 - u - 2W(y)}\right), \tag{2.79}$$

which now depends on the function $W(y)$ in a very simple way. The contour is along the imaginary axis with all singularities to the left.

[8]We thank T. Budd for pointing this out [95].

Moreover, this expression can be inserted in (2.37) to obtain a general formula for the disk density of states as a function of the defect generating function $W(y)$

$$\rho(E) = \frac{e^{S_0}}{2\pi} \int_{\mathcal{C}} \frac{dy}{2\pi i}\, e^{2\pi y}\, \tanh^{-1}\left(\sqrt{\frac{E-E_0}{y^2-2W(y)-E_0}}\right), \tag{2.80}$$

where the edge of the spectrum E_0 can be found by solving $\mathcal{F}(E_0)=0$. From these expressions it becomes evident that adding a gas of defects with angle $\alpha \to 1$ has the effect of shifting the generating function by a y-independent constant. From (2.79) we see such a shift $W(y) \to W(y)+c$, for some constant c, can be absorbed by a shift $u \to u-2c$, or equivalently a shift in the energy $E \to E-2c$. We have seen this explicitly in a simpler case at the end of the previous section.

These expressions will be extremely useful in the next section when we reinterpret this theory as a solution of 2D dilaton-gravity.

2.5 Dilaton-gravity

We will argue that there is a connection between deformations of the minimal string and dilaton-gravity theories, and we study the large p limit of these theories. The minimal string formulation implies a precise relation between the defect parameters and the dilaton potential which differs from the one proposed in [15]. We then use the Belavin-Zamolodchikov string equation to propose an exact solution of these dilaton-gravity theories.

The minimal string as 2D dilaton-gravity

To explain the first point we again use the argument of [86] (see also [22]) to rewrite the minimal string action in terms of a time-like Liouville field. We apply the same field redefinition (2.29)-(2.30) to the tachyon insertions present in the deformation of the minimal string action. Ignoring changes in normalization, this gives

$$\int \sqrt{\hat{g}}\ \mathcal{O}_{1,n}\ e^{2\alpha_L \phi} \to \int \sqrt{\hat{g}}\ e^{2\alpha_M \chi}\ e^{2\alpha_L \phi} \to \int \sqrt{g}\ e^{-2\pi b^2 n \Phi}, \tag{2.81}$$

where g is the JT gravity metric and Φ the JT dilaton. The final proposal is that the deformed minimal string is equivalent to a two-dimensional dilaton-gravity theory

$$I = -\frac{1}{2}\int \sqrt{g}\left[\Phi R + 2U(\Phi)\right], \tag{2.82}$$

with the following dilaton potential

$$U(\Phi) = 2\mu \sinh\left(2\pi b^2 \Phi\right) + \sum_{n=1}^{m-1} \tau_n\, e^{-2\pi b^2 n \Phi}, \tag{2.83}$$

where $n = 1, \ldots, m-1$ [9]. The first term is the undeformed minimal string potential derived in section 2.3. We can take the JT limit of this dilaton-gravity action using the scaling introduced in (2.55) $n = (1-\alpha)/b^2$. Each deformation term becomes

$$\tau_n \int \sqrt{g}\; e^{-2\pi b^2 n \Phi} \to \tau_n \int \sqrt{g}\; e^{-2\pi(1-\alpha)\Phi}, \tag{2.84}$$

which is the same dilaton potential associated to one defect species. This gives yet another perspective on why deformations of the minimal string matched with JT gravity with defects in the previous section.

The conventional normalization of the minimal model operator does not match with the time-like Liouville exponential required in this derivation. Therefore the parameter τ_n here is rescaled with respect to the one used in the previous section. We analyze this in detail in the next section.

Minimal string normalization

The Belavin-Zamolodchikov string equation, which gives the exact solution of the theory, is written in (2.46) in terms of λ_n, related to the coupling in the dilaton potential τ_n by

$$\lambda_n = \tau_n \, \frac{p^2}{16\pi^2} \, \mathrm{Leg}(n). \tag{2.85}$$

The leg-factor is defined in (2.49) and depends on the precise normalization of the minimal model operator. In order to compute the dilaton potential in (2.83) as a function of the λ_n's we need to compute the leg-factor corresponding to the exponential time-like Liouville normalization,

$$e^{2\hat{a}_n \chi} = N_n^{(E)} \mathcal{O}_{1,n}, \tag{2.86}$$

where $\mathcal{O}_{1,n}$ denotes the minimal model with the standard normalization used in equation (2.48). We exclude the definition of the prefactor $N_n^{(E)}$ for convenience; its precise expression can be found in Appendix C of [109]. Instead we quote directly the result for the leg-factor corresponding to the exponential normalization[10]

$$\mathrm{Leg}(n) = \frac{\gamma(nb^2)}{2\mu_M \gamma(-b^2)} \left(\frac{-\gamma(-b^2)}{\gamma(b^2)} \right)^{\frac{1+n}{2}}, \tag{2.87}$$

where μ_M is the cosmological constant of the time-like Liouville field which we take to be $-\mu$, consistent with (2.22), and $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. If we take the JT limit by sending

[9] It would be interesting to understand from the dilaton-gravity perspective whether the exponent in the dilaton potential has to be quantized at finite p. A more complete understanding of the minimal model as time-like Liouville would very likely answer this question (see [94] for some progress in this direction).

[10] Our definition has an extra factor of $-1/2$ relative to C.18 in [109]; this is to normalize the correlation functions to be consistent with the convention of [92].

$b \to 0$ for a deformation with $n = (1-\alpha)/b^2$ we find that

$$\tau_n = \frac{2\pi\lambda_n}{\gamma(1-\alpha)} e^{-2(1-\alpha)c}, \tag{2.88}$$

where c is the Euler–Mascheroni constant and we have again set $4\pi b^2\mu = 1$ to recast the JT action in the standard form. The exponential term can be removed by a simple shift of the dilaton.

The final result of this section is that, following the minimal string quantization, the string equation (2.79) provides an exact solution for the following 2D dilaton-gravity

$$U(\Phi) = \Phi + \sum_i \frac{2\pi\lambda_i}{\gamma(1-\alpha_i)} e^{-2\pi(1-\alpha_i)\Phi}, \tag{2.89}$$

where the sum is over species of defects with parameters α_i and λ_i. The advantage of this choice is the existence of a good semiclassical limit, which we can probe in the limit $\alpha \to 1$ where the backreaction from the conical defect is small. Using that $\gamma(1-\alpha) \approx (1-\alpha)^{-1}$, the dilaton potential becomes approximately $U(\Phi) = \Phi + \sum_i 2\pi(1-\alpha_i)\lambda_i e^{-2\pi(1-\alpha_i)\Phi}$. It was checked in Appendix D of [16] that this extra factor of $2\pi(1-\alpha)$ guarantees a match with the semiclassical dilaton-gravity calculation. We will come back to this in the next section where we analyze polynomial dilaton potentials.

Polynomial potentials

In this section we explain how to generate nearly polynomial dilaton potentials

$$U(\Phi) = \Phi + \sum_{n=2}^{N} \eta_n \Phi^n + \cdots \tag{2.90}$$

by choosing certain combinations of defect parameters in (2.89). By nearly polynomial we mean that for some large interior region in the geometry the potential has the desired form, but near the asymptotic boundary where the dilaton is very large it reduces to JT.

One way to construct a potential of lowest degree m is to introduce $m+1$ defects with $\theta_i = 2\pi(1-\alpha_i) \approx 0$. After expanding the exponentials in $\theta_i\Phi$ the terms up to Φ^m can be cancelled by tuning the couplings τ_i, with one coupling leftover to specify the free parameter η_m. The behavior of the potential is such that there is an arbitrarily large polynomial region defined by $\theta_i\Phi \ll 1$ which smoothly connects to a JT region near the asymptotic boundary where the dilaton becomes large. This ensures that the boundary is asymptotically AdS_2 and these theories are unambiguously defined through the string equation (2.67).

We demonstrate the above procedure with a simple example of the quadratic potential $U(\Phi) = \Phi + \frac{\eta}{2}\Phi^2 + \cdots$, for which the semiclassical limit was studied in [127]. We begin by turning on three general defects with corresponding potential

$$U(\Phi) = \Phi + \tau_1 e^{-\theta_1\Phi} + \tau_2 e^{-\theta_2\Phi} + \tau_3 e^{-\theta_3\Phi}, \tag{2.91}$$

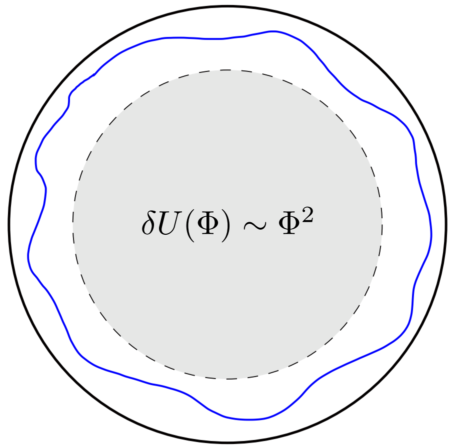


Figure 2.5: We illustrate the geometry for the quadratic deformation constructed in section 2.5. The blue curve corresponds to the Schwarzian boundary, while the shaded region corresponds to the portion of the disk where the deformation to the dilaton potential is approximately quadratic. The region in between corresponds to the transition from the nearly polynomial behavior to AdS_2 asymptotics.

where the linear term is the undeformed JT potential. For $\theta_i \Phi \ll 1$ we can expand the exponentials to find

$$U(\Phi) = \Phi + \tau_1 + \tau_2 + \tau_3 - (\tau_1\theta_1 + \tau_2\theta_2 + \tau_3\theta_3)\Phi + \frac{1}{2}(\tau_1\theta_1^2 + \tau_2\theta_2^2 + \tau_3\theta_3^2)\Phi^2 + \cdots \tag{2.92}$$

To get the leading order behavior of Φ^2 we can choose the following parameters $\theta_1 = \theta_2/2 = \theta_3/3 \equiv \theta$ and $\tau_1 = -\tau_2/2 = \tau_3 \equiv \tau$, with θ close to zero. If we now formally take $\theta \to 0$ while keeping the combination $\eta \equiv 2\theta^2\tau$ fixed at a small but finite value, we arrive at the following potential

$$U(\Phi) = \Phi + \frac{\eta}{2}\Phi^2. \tag{2.93}$$

The quantity that will be relevant to us will be the prepotential, which is defined by

$$\widehat{U}(\Phi) = 2\int_0^\Phi U(\Phi')d\Phi' = \Phi^2 + \frac{\eta}{3}\Phi^3. \tag{2.94}$$

Higher order polynomial potentials can be constructed by tuning defect parameters in a similar way.

Note that since we have constructed this potential by turning on defects this approximation is only valid when $\theta\Phi \ll 1$. At any finite value of θ, this condition only holds for a region near the center of the disk. The deformation can be turned on in a large portion of the bulk by taking θ small, but there is a transition to JT asymptotics near the boundary. See figure 2.5 for an illustration of this. As we will see in the next section, as long as we assume η is fixed to be small but finite and only work perturbatively in η, we can consistently take the $\theta \to 0$ limit.

Solution from string equation

We now solve for the density of states from the string equation for the quadratic potential and compare to the answer from the semiclassical gravity calculation. The string equation for this theory is given by choosing the same combination of defect parameters as above in the defect generating function (2.78) which gives

$$W(y) = -\frac{\eta}{6}y^3 \tag{2.95}$$

where we have taken the $\theta \to 0$ limit and added a defect with zero deficit angle to cancel the constant term. Examining (2.79) and (2.80), we see that the combination $y^2 - 2W(y)$ appearing in the string equation can be identified with the prepotential $\widehat{U}(\Phi \to y)$, with the y^2 term corresponding to the JT term. To make this identification, it is important to use the quantization we obtained from the minimal string in equation (2.89). Now using either (2.37) or (2.80) we can obtain the exact density of states

$$\rho(E) = \frac{e^{S_0}}{2\pi}\int \frac{dy}{2\pi i}e^{2\pi y}\ \tanh^{-1}\left(\sqrt{\frac{E}{\widehat{U}(y)}}\right), \quad \widehat{U}(y) = y^2 + \frac{\eta}{3}y^3 \tag{2.96}$$

where we have used $E_0 = 0$ to all orders in η. This can now be evaluated perturbatively in η and we find

$$\rho(E) = \frac{e^{S_0}}{2\pi}\left(\frac{\sinh\left(2\pi\sqrt{E}\right)}{2\pi} - \frac{\eta}{6}E\sinh\left(2\pi\sqrt{E}\right) + \cdots\right). \tag{2.97}$$

For each term in perturbation theory in η we do the y integral along a contour in the imaginary direction with all singularities to the left, appropriate to an inverse Laplace transform. This is the prediction for the density of states from the string equation for the quadratic potential. It would be interesting to connect this solution to the quantization of the non-local boundary action derived by Kitaev and Suh [127].

We now comment on the validity of the string equation (2.79) and density of states (2.80) for more general choices of $W(y)$ beyond sums of exponentials. The expression for the string equation is poorly behaved when we formally send $\theta \to 0$ to achieve (2.95). The integrand is unbounded as y becomes large and the branch cut structure becomes complicated. These issues are ameliorated if one works directly with the density of states (2.80) where the integrand is better behaved. Still working non-perturbatively in η, an issue remains regarding the choice of contour of the y integral. This is not a problem in the case of defects producing exponential terms so to solve this we can go back to consider finite θ. In this case the contour is uniquely defined and we expect this to indicate unambiguously which contour to use in (2.80). We leave a more thorough investigation of this issue for future work.

Check: Semiclassical limit

We will see now that the large E limit of the density of states we propose matches the semiclassical gravity calculations obtained in [127]; see also [128] and more recently [129]. In the semiclassical limit, the energy in terms of the prepotential is $E = \widehat{U}(\Phi_0)$ where Φ_0 is the value of the dilaton at the horizon. Since the entropy is proportional to the dilaton at the horizon, the density of states is immediately given by

$$\rho_{\text{classical}}(E) \approx \frac{e^{S_0}}{8\pi^2} \exp\left(2\pi\, \widehat{U}^{-1}(E)\right). \tag{2.98}$$

We now show that this formula matches (2.97) for the case of quadratic potential. Taking the large energy limit of (2.97) and expanding for small η we find

$$\log \rho(E) = S_0 + 2\pi\sqrt{E} - \frac{\eta}{3}\pi E + \mathcal{O}(\eta^2, 1/\sqrt{E}) \tag{2.99}$$

This expression matches with the semiclassical answer above since expanding (2.98) for small η gives $\log \rho_{\text{classical}} = S_0 + 2\pi \widehat{U}^{-1}(E) \approx S_0 + 2\pi\sqrt{E} - \frac{\eta}{3}\pi E + \mathcal{O}(\eta^2)$.

Moreover, at each order in η we can keep the terms in the exact density of states that grow fastest at large energies. We have checked these terms match with (2.98) to leading order in large E up to $\mathcal{O}(\eta^5)$. This suggests that our string equation correctly captures the semiclassical behavior of general dilaton potentials that can be built by turning on defect deformations. We note that this result was obtained directly from a polynomial $W(y)$ and is therefore independent of how it is regulated by a sum of exponentials in the asymptotically AdS_2 region.

2.6 Discussion

An important result of [4] was the realization that after the inclusion of spacetime wormholes in the gravity path integral, pure JT gravity in asymptotically AdS_2 is dual to an ensemble of quantum mechanical systems, realizing a new paradigm of holography. In this paper we have generalized this story by showing that a large class of pure dilaton-gravity theories in two dimensions are dual to an ensemble of quantum systems. We expect this duality to hold for pure gravity in higher dimensions since wormholes play a very similar role, although computing their precise contribution is harder (for some progress see [130]).

More specifically, we have pointed out a connection between deformations of the $(2, p)$ minimal string by tachyon-like operators in the large p limit and JT gravity with a gas of defects, and studied the connection between these theories formulated as 2D dilaton-gravities with general potentials. Using the minimal string we found an exact solution for JT gravity with defects when the defect angle is blunt, or in our notation $\alpha > 1/2$. The structure of the solution is more complicated than the case of sharp defects studied in [15, 16], and we gave a geometrical interpretation of these new features. Finally we have used this solution to propose an exact disk density of states for a general class of dilaton potentials.

We finish with some open questions.

Path integral with general defects

In the presence of general defects we can still integrate out the dilaton first, producing an integral over the moduli space of hyperbolic surfaces with cone points. When $\alpha < 1/2$, the geometry has geodesics that can be used to cut and glue by decomposing the surface into trumpets and pairs of pants, following the recipe of [4]. For the case $\alpha \geq 1/2$ we have used the minimal string to solve the theory. In this case there might not exist geodesics we can use to cut and glue and it would be very interesting to understand how to directly evaluate the JT path integral. Moreover, after integrating out the dilaton, we are left with an integral over the moduli space of hyperbolic surfaces with cone points, for which the corresponding Weil-Petersson volumes have only been calculated for $\alpha < 1/2$ [105, 106]. Therefore, our results can be seen as a physicist derivation of what these volumes are for more general defects. It would be interesting to understand this from first principles [95].

A possible approach to solve this problem is to rewrite the JT path integral with defects as a sum over discretized surfaces with constant negative curvature. This sum over discrete surfaces cannot be done with a matrix integral, which does not fix the curvature locally. Instead, a related problem was actually studied some time ago using the model of dually weighted graphs [131] [11]. Matching these discrete results to the continuum approach is an open problem being pursued in [132].

Leg-factor from JT gravity

When relating the exact solution of JT gravity with defects to the 2D dilaton-gravity formulation through the dilaton potential, some ambiguities appear. This was explained in [15]. In general different renormalization procedures can lead to different quantizations of the same classical theory and 2D dilaton-gravity is no exception.

The natural choice for JT gravity made in [15] associates the classical dilaton potential with the defect generating function (2.78). In this paper we have studied a different quantization provided by the minimal string, recast as a time-like Liouville coupled to a standard gravitational Liouville theory. This gives a different identification between the defect parameters and the dilaton potential (2.89). The advantage of this choice is that the quantum theory has a good semiclassical limit.

It would be nice to understand the origin of this mismatch in (2.89) from a detailed evaluation of the dilaton-gravity path integral. In order to do this it might be important to fill in the gaps regarding how the minimal model appearing in the minimal string is equivalent to a time-like Liouville theory (for some progress in this direction see [94]).

Negativities in spectral density

It was noticed in [16] and [15] that the disk density of states for JT gravity with defects can become negative if the defect weight λ is too large, above some angle-dependent critical

[11] We thank V. Kazakov for pointing this out and for several discussions.

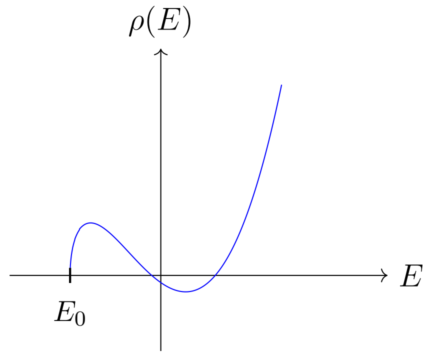


Figure 2.6: Disk density of states for a matrix model with string equation $u(u^2-1)+\lambda = x$, dual to the only deformation of the $(2,5)$ minimal string. For concreteness we used $\lambda = 0.5$, giving $E_0 = -1.2$.

value. It is an open question to understand what resolves this problem and the matrix integral formulation of the theory might help answer this.

One complication is that the matrix integral associated to JT gravity with defects is difficult and requires turning on an infinite number of operators in the matrix potential. In this section we would like to conclude with the observation that similar negativities already appears in the $(2,p)$ minimal string at finite p, and is thus inherited by the general class of dilaton-gravity theories derived from the $p \to \infty$ limit.

We will present the simplest case we found, which is the $(2,5)$ minimal string, corresponding to the Lee-Yang CFT coupled to 2D gravity. Besides the identity this CFT contains only one non-trivial operator $\mathcal{O}_{1,2}$, and this gives rise to the only possible deformation. We study the minimal string with the following deformation term in the action $\delta I \sim \lambda \int \mathcal{O}_{1,2}\, e^{2a_2\phi}$. After some rescalings, the string equation associated to this theory is $\mathcal{F}(u) = u(u^2-1)+\lambda$. From this string equation we can compute the disk density of states using (2.37) and we find that it becomes negative at some energies for negative enough λ. See figure 2.6 for an example. This model corresponds to a simple matrix potential which can be easily studied to see whether there is a phase transition to a two-cut model as suggested in [15] (another approach to this issue has been taken in [133])[12]. In the case of JT gravity, understanding this regime might be necessary to study theories with a dS_2 region inside the bulk [134, 135].

[12]The negativity seems to appear for values of λ where the interpretation of the matrix integral as a sum over random surfaces of finite size fails [19]. We thank V. Kazakov for pointing this out.

Chapter 3

Lorentzian Topology Change in JT Gravity

3.1 Introduction

This Chapter is based on [45] where we explore defining a Lorentzian theory of JT gravity with topology changing configurations. Euclidean wormholes have played an important role in recent developments in quantum gravity. The study of two dimensional Jackiw-Teitelboim (JT) gravity [4] has been central to these developments. Topology changing Euclidean configurations have been found to be important in the calculation of a unitary page curve [5, 6], the spectral form factor[4, 25, 27], and a variety of other interesting effects, see [28, 31, 33] for a few selected examples.

However, so far a satisfactory Lorentzian explanation for Euclidean wormhole calculations has been lacking. The standard procedure is to calculate a Euclidean amplitude and then analytically continue it to obtain a Lorentzian result. The role that Euclidean wormholes play in the Lorentzian theory is not apparent in this analytic continuation. Similarly, in the standard canonical quantization treatment of Lorentzian JT gravity[136] the topology of the Lorentzian spacetime is fixed, and it's unclear how the Euclidean path integral formulation of the theory[4] can be related to the Lorentzian formulation.

In this work we formulate a Lorentzian theory of JT gravity that includes Lorentzian topology changing configurations. The theory is defined through a special analytic continuation of the standard Euclidean path integral. Euclidean wormholes are turned into topology changing Lorentzian configurations with degenerate points in the metric, see figure 3.1. Our proposal is inspired by a formulation of bosonic string theory on degenerate Lorentzian worldsheets known as the interacting string picture[43, 44]. In [137, 138] it was argued that the interacting string picture is equivalent to the standard Euclidean path integral formulation to all orders in the genus expansion. This is accomplished by gauge-fixing the Euclidean path integral to lightcone diagrams[137], which are a special class of metrics that can be analytically continued to Lorentzian signature where they become the topology

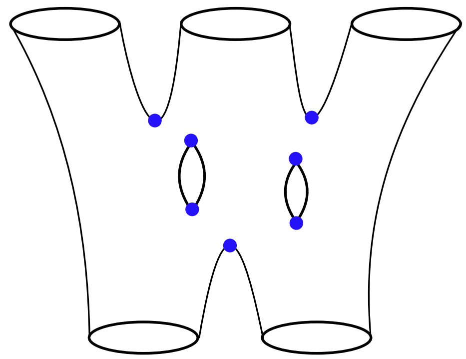


Figure 3.1: Lorentzian topology changing transition where two spatial circles evolves into three circles. The metric is Lorentzian everywhere except the splitting points (blue circles) where it is degenerate. At the splitting points the topology of spatial slices changes. The spacetime is an analytic continuation of a genus two Euclidean geometry with five circular boundaries.

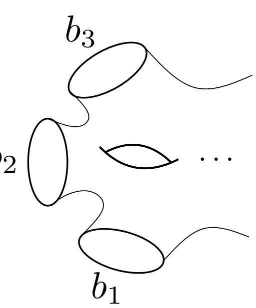
changing geometries of the interacting string picture.

To define the Lorentzian JT path integral we closely follow the construction of the bosonic string genus expansion with singular Euclidean/Lorentzian worldsheets [137–139]. We start with the standard Euclidean JT path integral and with a suitable gauge choice and analytic continuation we end up with a Lorentzian path integral over degenerate metrics. We now briefly explain this construction, leaving technical details to the main sections.

Summary of results

We begin with the two dimensional Euclidean gravity path integral with specified boundary conditions. In this paper we consider boundary conditions given by n geodesic circles of given lengths $\vec{b} = (b_1, \cdots, b_n)$. For simplicity we assume the bulk geometry has fixed genus g and is fully connected. The path integral is computed by

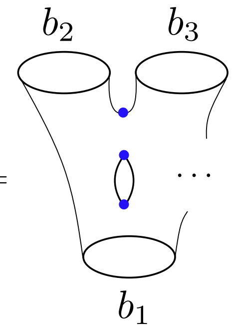
$$Z = \int \frac{\mathcal{D}g_{\mu\nu}\mathcal{D}\Phi}{\mathrm{Vol}} e^{-I_{\mathrm{JT}}[g,\Phi]} = b_2 \tag{3.1}$$



The integral over Euclidean metrics can be split into an integral over the Weyl factor ω and over the moduli $\hat{g}$, where all metrics can be represented as $g = e^{2\omega}\hat{g}$. The moduli space of $\hat{g}$ is the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$ of genus g with n punctures. It was shown by Giddings and Wolpert [137] that this Moduli space has a representative metric

$\hat{g}$ that is flat with curvature singularities at isolated points, this is known as a Euclidean lightcone metric. Choosing the Euclidean lightcone metric as our representative metric $\hat{g}$ gives us a path integral over singular Euclidean geometries

$$Z = \int_{\text{moduli}} \mathrm{d}(\text{measure}) \int \mathcal{D}\omega \mathcal{D}\Phi e^{-I_{\text{JT}}[e^{2\omega}\hat{g},\Phi]} = \quad \cdots \tag{3.2}$$



In the above figure, the circles (blue) corresponds to points where the metric $\hat{g}$ becomes degenerate $\det \hat{g} = 0$. It is at these points that the topology of spatial slices changes. The integral over the moduli is partially over all possible locations of the degenerate points. The Euclidean lightcone diagram has a globally defined euclidean time τ in terms of which the metric is flat everywhere except the degenerate points. The boundary conditions are now specified at $\tau = \pm\infty$, and we must make a choice to send boundary conditions either to the future or the past. In the above figure the boundary of length b_1 has been sent to the past while the boundaries of length b_2, b_3 have been sent to the future.

To turn the above Euclidean path integral into a Lorentzian path integral, we will analytically continue the Euclidean lightcone geometries. Since the time τ is globally defined, we can analytically continue $\tau \to i\tau$ to get a Lorentzian signature metric $\hat{g}$, known as a Lorentzian lightcone diagram. We now have a path integral over degenerate Lorentzian metrics that incorporate topology changing transitions

$$Z_{\text{L}} = \int_{\text{moduli}} \mathrm{d}(\text{measure}) \int \mathcal{D}\omega \mathcal{D}\Phi e^{iI_{\text{JT}}[e^{2\omega}\hat{g},\Phi]}. \tag{3.3}$$

Our ***definition*** for the Lorentzian JT path integral will be the above integral over Lorentzian lightcone diagrams. The role of the Weyl factor ω will be to give the geometry constant negative curvature away from the degenerate points.

In the above procedure we started with the usual Euclidean path integral and through a gauge choice and a suitable analytic continuation we ended up with a Lorentzian path integral. It might then be expected that the amplitudes computed with this Lorentzian path integral should agree with the corresponding Euclidean amplitudes. Indeed, this is the argument of D'Hoker and Giddings[138] that the interacting string picture[44] is equal to the Euclidean path integral to all orders in the genus expansion[1]. In the case of JT gravity there are some additional subtleties that arise due to the degenerate points, and we return

[1] Giddings and D'Hoker [138] stopped short of analytically continuing the Euclidean lightcone diagrams to Lorentzian signature. Thus they argued that the analytically continued interacting string picture was equivalent to the usual Euclidean path integral.

to this question in section 3.3 and in the discussion. We now briefly summarize the rest of the paper.

In Section 3.2 we review the basic aspects of Euclidean and Lorentzian lightcone diagrams. We also review the work of Giddings and Wolpert [137] where it was shown that Euclidean lightcone diagrams give a single cover of the moduli space of punctured Riemann surfaces. Lastly, we discuss the problem of finding a Weyl factor to turn a Euclidean/Lorentzian lightcone metric into a constant negative curvature geometry with degenerate points.

In Section 3.3 we fill in the technical details of the path integral over lightcone diagrams. We explain how boundary conditions are implemented, and we discuss the integration measure over the moduli space of lightcone diagrams. The inclusion of degenerate points introduces certain ambiguities into the path integral, and we discuss how these ambiguities modify the relation of the Euclidean amplitudes to the Lorentzian amplitudes.

In the Appendices we construct the Lorentzian pair of pants with constant negative curvature, and we include additional details on punctured Riemann surfaces and the integration measure.

3.2 Lightcone diagrams

In this section we review aspects of Euclidean and Lorentzian lightcone diagrams, their connection to punctured Riemann surfaces, and how to construct constant negative curvature analogues of lightcone diagrams.

Euclidean and Lorentzian lightcone diagrams

A Euclidean/Lorentzian lightcone diagram is a two dimensional geometry with degenerate metric g built out of flat Euclidean/Lorentzian cylinders joined together at singular points, see figure 3.2. The two basic properties of a lightcone diagram are the number of asymptotic cylinders n and the genus g. The number of cylinders running off to infinity is given by $n \geq 2$ with a fixed number extending to past or future infinity[2]. In the next section we will see that in/out states are specified by introducing boundary conditions on the asymptotic cylinders. The genus g and number of boundaries n determine the number of times the diagram splits apart and joins together in the interior of the geometry. At the splitting and joining points the metric is degenerate, and there are $2g-2+n$ such degenerate points. We will also call these interaction/singular points, and denote where they occur by the subscript z_I. The curvature on a Euclidean lightcone diagram is given by

$$\frac{1}{2}\sqrt{g}R = -2\pi \sum_{I=1}^{2g-2+n} \delta^2(z - z_I), \qquad \det g(z_I) = 0. \tag{3.4}$$

[2]At least one cylinder must run to the future and one to the past.

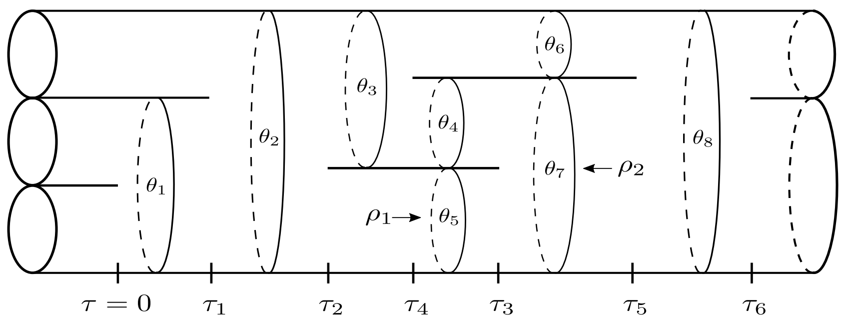


Figure 3.2: Euclidean/Lorentzian lightcone diagram of genus two with five boundaries. The geometry is flat except at interactions points τ_i with delta functions of curvature. A lightcone diagram is built by gluing pairs of pants together. The interaction times are labelled by τ, the twist angles by θ, and the free internal radius by ρ.

Note that since the determinant of the metric is zero at the interaction points the action on a lightcone diagram can be badly behaved at these points. Even though the metric is singular, the singularity is sufficiently mild. The Gauss-Bonnet theorem continues to hold

$$\frac{1}{2}\int \sqrt{g}R = 2\pi\chi, \tag{3.5}$$

where $\chi = 2 - 2g - n$. All the curvature required for Gauss-Bonnet to be satisfied is localized at the interaction points. In Euclidean signature we introduce coordinates τ, σ on the diagrams where τ is a time coordinate and σ is a periodic coordinate around the cylinder. We can represent the diagrams by making identifications in the complex plane $w = \tau + i\sigma$, see Fig. 3.3 for an example of how to build a pair of pants. To travel around a singular point we must go through angle 4π in w coordinates, so the singularity is a double cover of the Euclidean plane.

We can easily find the metric near the singular points using complex coordinates

$$ds^2 = |z|^{2\alpha} dz d\bar{z} = e^{2\omega} dz d\bar{z}, \qquad \frac{1}{2}\sqrt{g}R = -2\pi\alpha\delta^{(2)}(z). \tag{3.6}$$

which is a cone of opening angle $2\pi(1+\alpha)$ with conformal mode $\omega = \frac{1}{2}\log|z|^{2\alpha}$. The singularities of the Euclidean lightcone diagram correspond to $\alpha = 1$. We can see this by the coordinate $w = z^2$, going through angle 2π in z takes us through an angle of 4π in w, which is a double cover of the Euclidean plane as required.

Lorentzian lightcone diagrams

The above discussion was largely restricted to Euclidean lightcone diagrams. However, by an analytic continuation of the time coordinate $\tau \to i\tau$ we obtain a metric that is Lorentzian everywhere except at the degenerate points. These geometries are Lorentzian lightcone

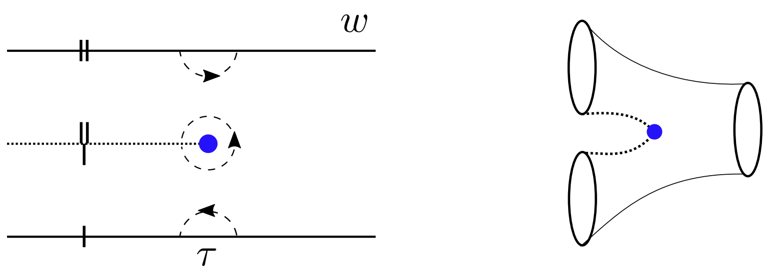


Figure 3.3: Euclidean Pair of pants constructed by identification in the complex $w = \tau + i\sigma$ plane. The middle line (dashed) is infinitesimally split up to the interaction time τ and identified with the top and bottom segments as indicated. The singular point is indicated by the circle (blue). A closed loop around the singular point goes through angle 4π.

diagrams, and the metrics on them are known as almost Lorentzian metrics. The curvature is given by

$$\frac{1}{2}\sqrt{-g}R = 2\pi i \sum_{I=1}^{2g-2+n} \delta^2(x - x_I), \qquad \det g(x_I) = 0. \tag{3.7}$$

These geometries are constructed by gluing Lorentzian pants along geodesic boundaries. The pair of pants cannot be given an everywhere Lorentzian metric, but with the inclusion of the above singularity at the splitting point it becomes possible to equip it with an almost Lorentzian metric [140, 141]. We can explicitly write down a regularized complex metric near the splitting point of the pants. In the conventions of [141] this metric is given by

$$ds^2 = \left(x^2 + y^2 + \gamma\right)\left(dx^2 + dy^2\right) - (2 \pm i\epsilon)(xdx - ydy)^2. \tag{3.8}$$

The splitting point is located at $x = y = 0$, and we have introduced regulators $\gamma, \epsilon > 0$. The almost Lorentzian pants are obtained from the above metric as we take the regulators to zero, with appropriate identifications of the x, y plane[140]. Analogous to the Euclidean case, the almost Lorentzian pants are a double cover of Minkowski space. The γ regulator temporarily removes the degenerate point and the ϵ regulator deforms the metric into what is known as an allowable complex metric[3] [141]. This is important for us because allowable complex metrics satisfy an analytically continued version of the Gauss-Bonnet theorem [140, 141].

Ultimately we will be integrating over all lightcone diagrams, and our Euclidean action will contain a topological term suppressing each geometry by it's Euler characteristic $e^{S_0 \chi}$, where S_0 is some large constant. Upon analytic continuation of the time coordinate the topological term will be proportional to

$$\frac{i}{2}\int d^2x\sqrt{-g}R. \tag{3.9}$$

[3] A complex metric is defined as allowable if the following condition is satisfied[141, 142]. Consider a basis where the metric is diagonal $g_{ij} = \lambda_i \delta_{ij}$, if $\sum_i |\operatorname{Arg}\lambda_i| < \pi$ at every point on the manifold then the metric is allowable. Allowable metrics satisfy certain nice properties. For example, this condition ensures that the path integral of a p-form gauge field converges on the background g.

For allowable complex metrics the above integral is equal to an analytically continued version of the Gauss-Bonnet theorem [141]

$$\frac{1}{2}\int d^2x\sqrt{-g}R = -2\pi i\chi. \tag{3.10}$$

All metrics considered in this paper can be deformed to be allowable, and so we maintain the same topological expansion in Lorentzian as in Euclidean signature. We can directly see that the above formula holds for the Lorentzian lightcone metric (3.7). An important property is that given an allowable complex metric g, $e^{2\omega}g$ is also allowable if ω is real. Since the Lorentzian pants are allowable so is any rescaling of the pants, which will be useful when we need to consider the constant negative curvature pants geometry.

Moduli space of lightcone diagrams

We now explain the moduli space of inequivalent Euclidean/Lorentzian lightcone diagrams of genus g with n boundaries. The starting point is to specify the geodesic radius r_i of each of the n boundaries and whether they run off to past or future infinity. A positive radius indicates that the boundary originates from past infinity, a negative radius indicates the boundary goes to future infinity. We demand that the sum of the lengths vanishes $\sum_i r_i = 0$. So that the total length of incoming boundaries is equal to the total length of outgoing boundaries. We denote the total length of incoming/outgoing boundaries by $r_{\max}$.

A general diagram is constructed by gluing Euclidean/Lorentzian pairs of pants together along geodesic boundaries[4]. The different ways to glue pants together gives the moduli space of lightcone diagrams. From figure 3.2, the moduli space is parameterized by the following coordinates

Interaction Times	$\tau_i \in [0, \infty)$	$i = 1, \dots, 2g + n - 3$
Twist Angles	$\theta_j \in [0, 2\pi)$	$j = 1, \dots, 3g + n - 3$
Internal Radius	$\rho_k \in [0, r_{\max}]$	$k = 1, \dots, g$

The moduli space is of real dimension $6g - 6 + 2n$. The interaction times τ define the points at which the cylinders split apart at degenerate points. The diagram can be translated in time so the first interaction time can always be set at $\tau = 0$. When gluing two cylinders together we can twist them by a relative angle $\theta \in [0, 2\pi)$, this gives the twist angles. Since the pair of pants conserves total geodesic boundary length, the sum of radii of the diagram at any intermediate time must be identical to the sum of beginning or ending radii. We have a freedom in how the total geodesic length is distributed at intermediate times between cylinders, which gives the last parameter ρ_k where we integrate over lengths of intermediate cylinders subject to the constraint that the total length is fixed to $r_{\max}$.

[4]The pair of pants used in the construction have boundaries with waist of length b_1 and legs of lengths b_2, b_3 with the constraint that $b_1 = b_2 + b_3$. There are other Lorentzian pair of pants geometries without this constraint, but they are not used in this construction.

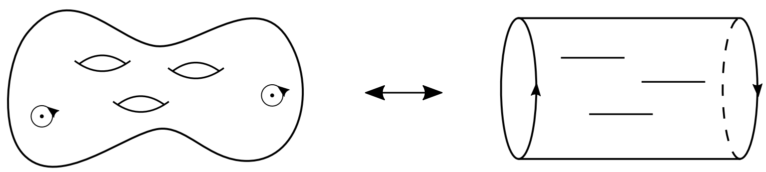


Figure 3.4: A genus three surface Σ with two punctures. We can put a metric on Σ given by (3.13) to turn it into a Euclidean lightcone diagram with two asymptotic cylinders.

The end result is that the integral over all lightcone diagrams is given by integrating over interaction times τ, twist angles θ, and intermediate radii ρ

$$\int [d\tau][d\theta][\rho d\rho] \equiv \frac{1}{S}\left(\prod_{i=1}^{2g+n-3}\int_0^\infty d\tau_i\right)\left(\prod_{j=1}^{3g+n-3}\int_0^{2\pi} d\theta_j\right)\left(\prod_{k=1}^{g}\int_0^{r_{\max}} \rho_k d\rho_k\right). \tag{3.11}$$

The above integration range actually overcounts identical lightcone diagrams, and we must divide by a symmetry factor S to integrate over one copy of the moduli space[137]. This symmetry factor has never been explicitly computed, but see [143] for a recent discussion.

Punctured Riemann surfaces and lightcone diagrams

In this section we'd like to explain the connection between punctured Riemann surfaces and lightcone diagrams. Giddings and Wolpert[137] showed that the moduli space of Euclidean lightcone diagrams gives a single cover of the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$. This is accomplished by showing that every Riemann surface with $n \geq 2$ punctures can be equipped with a unique Euclidean lightcone metric and vice versa. The integral over the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$ can therefore be represented as an integral over the moduli space of Euclidean lightcone diagrams.

We now briefly explain how a given Riemann surface can be equipped with a Euclidean lightcone metric. Consider a genus g Riemann surface Σ with local coordinates z and n punctures located at points z_i. The goal will be to find a unique meromorphic one-form $\omega = \omega(z)dz$ on Σ with simple poles at the punctures z_i with real residues r_i such that their sum is zero $\sum_{i=1}^n r_i = 0$. The specific choice of the r_i does not matter. We also require that the integral of the form ω is imaginary along any closed cycle[5] on the surface Σ

$$\oint \omega(z)dz \in i\mathbb{R}. \tag{3.12}$$

Under these conditions, Giddings and Wolpert [137] showed that the one-form ω exists, is

[5]Without this condition there are many one-forms ω with the appropriate simple poles, however they fail to give a global lightcone diagram. One way to understand this condition is that on a lightcone diagram a closed loop starts and ends at the same time τ, so around any closed loop $\Delta\tau + i\Delta\sigma = \oint \omega(z)dz$ must be purely imaginary.

unique, and that it defines a unique Euclidean lightcone metric[6] on Σ by

$$ds^2 = \omega\overline{\omega} = |\omega(z)|^2 dz d\bar{z} = dw d\bar{w}, \tag{3.13}$$

see figure 3.4. In the last equality we have defined the lightcone coordinates $w = \tau + i\sigma = \int_{z_0}^{z} \omega(z) dz$, where z_0 is an arbitrary point on Σ. In these coordinates the metric is explicitly flat with the only breakdown occurring near the poles or zeroes of the form. Near a puncture $z \sim z_i$ the form behaves as $\omega = \frac{r_i}{z-z_i} dz + \ldots$, so the metric will behave as

$$ds^2 = \frac{r_i^2}{|z - z_i|^2} dz d\bar{z} + \ldots \tag{3.14}$$

The form will also have $2g - 2 + n$ zeros at certain points z_I on Σ, near such points the form behaves as $\omega = (z - z_I)\, dz + \ldots$, and the metric will behave as

$$ds^2 = |z - z_I|^2 dz d\bar{z} + \ldots \tag{3.15}$$

Note that near z_I the metric is that of the interaction point of a Euclidean lightcone diagram in equation (3.6). Thus, the zeros of ω correspond to degenerate points where the topology of spatial slices of Σ change, while the simple poles are located at the punctures.

We now explain how to extract the moduli of the lightcone diagram (τ, θ, ρ) from the metric (3.13), see figure 3.5. One useful property is that the time coordinate $\tau = \mathrm{Re} \int_{z_0}^{z} \omega(z) dz$ is globally defined and path independent, since it is defined as the integral of a globally defined one-form[7]. The interaction time differences can be determined by performing an integral between zeros of the form $\tau_I - \tau_J = \mathrm{Re} \int_{z_J}^{z_I} \omega dz$. The radius of an asymptotic cylinder is given by integrating around a puncture $\oint dw = \oint_{z_i} \omega(z) dz = 2\pi i r_i$. Near the punctures we have $w = r_i \log(z - z_i)$, and so the puncture is mapped to $\tau = \pm\infty$ if the radius r_i is negative/positive respectively.

The lengths of intermediate cylinders are given by integrating along cycles at constant time τ, which gives $\oint dw = \oint \omega(z) dz = 2\pi i \rho_i$. Integration over cycles with non-constant τ allows us to reconstruct the twist angles θ, see [137] for additional details. The above procedure gives us the moduli (τ, θ, ρ) which uniquely fixes a lightcone diagram for a given Riemann surface Σ, but the reverse direction also holds in mapping a lightcone diagram to a Riemann surface[137]. To summarize, given a punctured Riemann surface Σ we can equip it with a unique Euclidean lightcone metric. Integrating over the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$ is thus equivalent to integrating over the moduli space of Euclidean lightcone diagrams.

$$\int_{\mathcal{M}_{g,n}} \ldots = \int [d\tau][d\theta][\rho d\rho] \ldots \tag{3.16}$$

In the above we have left out the integration measure to which we return to in section 3.3.

[6]The construction of ω involves the period matrix of the Riemann surface, which is why each Riemann surface is mapped to a unique lightcone diagram.

[7]The integral along two paths can differ by at most an imaginary constant if they enclose a puncture, but the time τ is the real part of the integral so it does not matter.

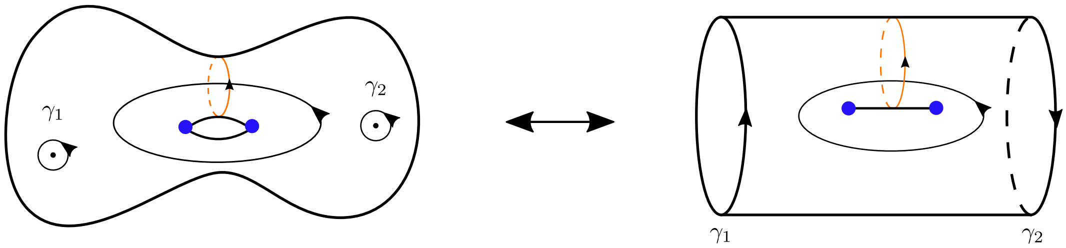


Figure 3.5: A torus with two punctures mapped to a lightcone diagram with two interaction points and two asymptotic cylinders. The circles (blue) on the torus denote where the one-form ω has zeros. The two cycles on the torus are illustrated with one at constant time τ (orange). The loops γ_i enclose the punctures which run off to infinity in the lightcone metric.

Negative curvature lightcone diagrams

We are ultimately interested in JT gravity, so we want to construct the constant negative curvature analogues of lightcone diagrams. We would like to start with a Euclidean/Lorentzian lightcone metric $\hat{g}$ and turn it into a constant negative curvature geometry $g = e^{2\omega}\hat{g}$ with a Weyl factor ω, with suitable singularities at the degenerate points. The reason for this is that when considering JT gravity we will find we have a constraint $R = -2$ away from the degenerate points, but there is no condition on the behavior of ω at such points. We find that the Weyl factor is not unique, and we have a large class of geometries that differ by the strength of the conical singularity at the degenerate points. The only constraint is that the resulting geometries satisfy Gauss-Bonnet.

Negative curvature Euclidean lightcone diagrams

Since $\hat{g}$ already satisfies Gauss-Bonnet, we will choose Weyl factor to weaken the delta functions at the splittings points, and redistribute the curvature uniformly so that Gauss-Bonnet remains satisfied. Consider a Riemann surface Σ where we prescribe conical singularities of opening angles $2\pi\alpha_I$ at points z_I, where the metric locally behaves as $ds^2 = |z|^{2(1-\alpha_I)}dzd\bar{z}$ near z_I. Does there exist a constant negative curvature metric g on Σ with the prescribed singularity structure? The answer turns out to be yes, assuming the specification of α_I does not violate Gauss-Bonnet

$$\frac{1}{4\pi}\int_{\Sigma\backslash\{z_i\}}\sqrt{g}R = \chi(\Sigma) + \sum_I (1-\alpha_I) < 0. \tag{3.17}$$

Under the above condition, it was shown[8][145, 146] that the metric g exists, is unique, and is in the conformal equivalence class of metrics on Σ. We conclude that starting from a Euclidean lightcone metric $\hat{g}$, we can find a unique ω such that $g = e^{2\omega}\hat{g}$ has constant

[8]For a review of various existence and uniqueness theorems on metrics with conical singularities see [144].

negative curvature and any desired singularity structure consistent with Gauss-Bonnet

$$\frac{1}{2}\sqrt{g}\,(R+2) = -2\pi \sum_{I=1}^{2g-2+n} (1-\alpha_I)\delta^2(z-z_I). \tag{3.18}$$

We have written the curvature in this way to emphasize that the effect of α_I is to take curvature from the interaction points and redistribute it uniformly across the rest of the surface $\Sigma\backslash\{z_i\}$. Note that this amounts to studying JT gravity with conical excesses, which appears to be the closest relative to Euclidean lightcone diagrams. However, the utility of lightcone diagrams is they can be easily analytically continued to singular Lorentzian geometries, which we now discuss.

Negative curvature Lorentzian lightcone diagrams

We would like to construct negative curvature Lorentzian lightcone diagrams in a similar way by rescaling the metric $g = e^{2\omega}\hat{g}$, where $g, \hat{g}$ are almost Lorentzian geometries. As far as we are aware, not much is known about almost Lorentzian metrics and the existence of such Weyl factors. However, we would like to argue that it is plausible that there exists an appropriate ω such that the curvature of $g = e^{2\omega}\hat{g}$ takes the form

$$\frac{1}{2}\sqrt{-g}\,(R+2) = 2\pi \sum_{I=1}^{2g-2+n} (i+\alpha_I)\,\delta^2(z-z_I), \tag{3.19}$$

under the condition that α_I is chosen so that the complex Gauss-Bonnet theorem is not violated, which requires that $\alpha_I > 0$. The purpose of the α_I can can be seen from the form of Gauss-Bonnet for complex metrics[140, 141]

$$\frac{1}{2}\int_\Sigma \sqrt{-g}R = -2\pi i\chi(\Sigma). \tag{3.20}$$

The effect of the α_I is to cancel out the contribution of the bulk volume

$$\frac{1}{4\pi}\int_{\Sigma\backslash\{z_I\}} \sqrt{-g}R + \sum_{I=1}^{2g-2+n} \alpha_I = 0, \tag{3.21}$$

so that the remaining terms in (3.19) give the Euler characteristic. While we cannot prove that such Weyl factor exists, we think it is reasonable that it does for the following reasons. It's existence would be the natural extension of the Euclidean results of [145, 146] which gives the unique metric (3.18). In both cases the role of the α_I is to cancel out the bulk volume term to satisfy Gauss-Bonnet. Furthermore, since the Lorentzian lightcone metric $\hat{g}$ is complex allowable, so is $e^{2\omega}\hat{g}$, which implies that Gauss-Bonnet (3.20) holds for both metrics. If $e^{2\omega}\hat{g}$ is to have constant negative curvature, it's singular points must contribute in a way to cancel out the bulk volume, otherwise Gauss-Bonnet would be violated. Finally, in appendix A.6 we construct the Lorentzian constant negative curvature pairs of pants with the same singularity structure as advocated above.

To summarize, we believe it is reasonable that there exists a Weyl factor that turns a Lorentzian lightcone diagram into a constant negative curvature analogue given by (3.19), with the only constraint on α_I being that Gauss-Bonnet (3.20) is satisfied. One important point is that while flat lightcone diagrams are unique, the constant negative curvature analogues are not. When integrating over the Weyl factor we must make a choice of which geometries to include, which amounts to deciding which α_I to include. We are not able to give a conclusive answer to this question, but we return to it in the discussion.

3.3 JT gravity on lightcone diagrams

JT gravity is a two dimensional dilaton gravity theory with metric $g_{\mu\nu}$ and dilaton Φ. We will start with the Euclidean JT gravity path integral with boundary conditions given by n geodesic circles of given lengths $\vec{b} = (b_1, \cdots, b_n)$. We assume the bulk geometry has genus g and is connected, with the extension to the full topological expansion easily following. This amplitude can be computed by performing the path integral over all compact geometries Σ of genus g with n vertex operators V inserted[9]

$$Z = \int \frac{\mathcal{D}g_{\mu\nu}\mathcal{D}\Phi}{\text{Vol}} e^{-I_{\text{JT}}[g,\Phi]} V_1 \dots V_n, \tag{3.22}$$

where Vol is the volume of the diffeomorphism group. The Euclidean JT action on a compact surface Σ is given by [2, 3, 10]

$$I_{\text{JT}}[g,\Phi] = -\frac{S_0}{4\pi}\int_\Sigma \sqrt{g}R - \frac{1}{2}\int_\Sigma \sqrt{g}\Phi(R+2). \tag{3.23}$$

The first term is purely topological. Taking into account the vertex operators, the total topological contribution will be given by $e^{S_0\chi}$ where $\chi = 2-2g-n$. We ignore this term from now on, although at certain points various constants will be absorbed into the definition of S_0. The vertex operator that inserts a geodesic boundary of length b is given by [147]

$$V = e^{-S_0}\int_\Sigma d^2z\sqrt{g}e^{-2\pi\Phi}\cos\left(b\Phi\right). \tag{3.24}$$

The path integral over all two dimensional Euclidean metrics can be decomposed into an integral over the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces, and over the Weyl factor ω of the metric [148, 149]

$$\int \frac{\mathcal{D}g_{\mu\nu}\mathcal{D}\Phi}{\text{Vol}} = \int_{\mathcal{M}_g} \mathrm{d}\mu \int \mathcal{D}\omega\mathcal{D}\Phi e^{-25S_L[\omega,\hat{g}]}. \tag{3.25}$$

[9]The other way to compute this amplitude is to perform the path integral over surfaces with n boundaries with appropriate boundary conditions. However, the formalism of lightcone diagrams most easily applies to the situation where boundary conditions are specified by vertex operator insertions.

The above path integral is performed over metrics $g = e^{2\omega}\hat{g}$, where $\hat{g}$ is a choice of representative metric[10] for each point in the moduli space $\mathcal{M}_g$. In the above integral $\mathrm{d}\mu$ is the standard integration measure on the moduli space $\mathcal{M}_g$ [148, 149], since it is quite complicated we write it schematically and go into additional detail in appendix A.7. The Liouville action S_L will turn out to be proportional to the Euler characteristic after we integrate out the dilaton, and we absorb it into the definition of the topological term. This integral treats the moduli of the Riemann surface on a different footing from the positions of the vertex operators. Giddings and D'Hoker [138] gave an argument that the integral over vertex operator positions $\int d^2 z_i \sqrt{g}$ can be absorbed into the integral over $\mathcal{M}_g$ to give an integral over the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$ of genus g with n punctures[138]

$$\int_{\mathcal{M}_g} \mathrm{d}\mu \prod_{i=1}^{n} \left(e^{-S_0} \int d^2 z_i \sqrt{g} e^{-2\pi\Phi} \cos\left(b_i \Phi\right) \right) = \int_{\mathcal{M}_{g,n}} \mathrm{d}\mu \prod_{i=1}^{n} W_i. \tag{3.26}$$

The measure $\mathrm{d}\mu$ on the right is now for the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$, and various factors have been absorbed into the definition of the wavefunctions W_i. The integral over $\mathcal{M}_{g,n}$ automatically takes into account the integral over all distinct points z_i at which the wavefunctions can be inserted, and it could have been our starting point for the Euclidean path integral[11] (3.22). On the right hand side we have defined the wavefunctions

$$W_i = e^{-2\pi\Phi(z_i)} \cos\left(b_i \Phi(z_i)\right), \tag{3.27}$$

located at the punctures at points z_i on the surface. The role that the wavefunctions W_i play is that they impose boundary conditions at the punctures in the following way. Consider performing the path integral over the dilaton near a particular puncture z_i with wavefunction W_i. When we integrate over the dilaton Φ near the puncture we get the following constraint on the behavior of the metric

$$\frac{1}{2}\sqrt{g}(R+2) = (2\pi \pm i b_i)\delta^2(z - z_i). \tag{3.28}$$

Each wavefunction comes with two branches denoted by $\pm ib$, which arise from the expansion of $\cos(b\Phi)$, and so the constraint is a linear combination of the two branches[12]. Since each wavefunction comes with two branches, performing the dilaton integral over the entire surface gives a sum over all possible branches of the constraints.

[10]The standard choice is to pick $\hat{g}$ such that it has constant negative curvature for $g \geq 2$.

[11]The two are equal up to overall normalization factors that can be absorbed, see [138, 149].

[12]It might at first be surprising that a geodesic boundary is given by an imaginary conical defect since a geodesic boundary does not give a delta function of curvature. This was explained in Appendix A of [147]. A geodesic boundary is given by the end of the trumpet geometry, after reaching the geodesic on the trumpet the radial coordinate can be analytically continued into the imaginary axis to reach an imaginary defect. We should thus imagine that the geometry is continued into the complex radial direction near the insertion points.

The end result is we have re-written the path integral over two dimensional Euclidean metrics with vertex operator insertions as a path integral over the moduli space of punctured Riemann surfaces with wavefunctions inserted at the punctures

$$Z = \int_{\mathcal{M}_{g,n}} \mathrm{d}\mu \int \mathcal{D}\omega \mathcal{D}\Phi e^{-I_{\mathrm{JT}}[e^{2\omega}\hat{g},\Phi]} W_1 \dots W_n. \tag{3.29}$$

In the above integral $\mathrm{d}\mu$ is the standard measure on the moduli space $\mathcal{M}_{g,n}$ [138, 149]. The path integral measure for $\mathcal{M}_{g,n}$ is quite complicated and is written out in appendix A.7 alongside additional details on the moduli space.

Lightcone diagrams

To perform the above integral we must choose a representative metric $\hat{g}$ for each point in $\mathcal{M}_{g,n}$, and then perform the integral over the Dilaton Φ and Weyl factor ω. At this point we must make a choice to separate the n wavefunctions W_i into those that run to the Euclidean past or future[13]. We must have at least one wavefunction in the future and one in the past for the Euclidean lightcone metric $\hat{g}$ to be a good gauge choice. Using the results summarized in section 3.2, we choose the representative metric $\hat{g}$ to be given by the unique Euclidean lightcone diagram for each point in $\mathcal{M}_{g,n}$, which we write below for convenience

$$\frac{1}{2}\sqrt{\hat{g}}\hat{R} = -2\pi \sum_{I=1}^{2g-2+n} \delta^2(z - z_I), \qquad \det \hat{g}(z_I) = 0. \tag{3.30}$$

As explained in section 3.2, the Euclidean lightcone metric $\hat{g}$ has a globally defined Euclidean time τ. In terms of the τ coordinate on Σ, the wavefunctions W_i we choose to define the in state are located at $\tau = -\infty$, while those that define the out state are at $\tau = \infty$. In lightcone diagram gauge the measure $\mathrm{d}\mu$ takes a particularly simple form[138, 149]

$$\int_{\mathcal{M}_{g,n}} \mathrm{d}\mu = \int [d\tau][d\theta][\rho d\rho] \frac{2\pi \det'(-\hat{\nabla}^2)}{\int_\Sigma d^2z\sqrt{\hat{g}}}, \tag{3.31}$$

up to a term proportional to the Euler characteristic that we absorbed into S_0. The prime on the determinant indicates that we exclude zero modes. We include more details on the derivation of the above measure in appendix A.7. The determinant of the Laplacian is defined to be with respect to the Euclidean lightcone metric $\hat{g}$ on Σ, and we discuss the determinants in more detail slightly later. We end up with the following path integral

$$Z = \int_{W_i} [d\tau][d\theta][\rho d\rho] \frac{2\pi \det'(-\hat{\nabla}^2)}{\int_\Sigma d^2z\sqrt{\hat{g}}} \int \mathcal{D}\omega \mathcal{D}\Phi e^{-I_{\mathrm{JT}}[e^{2\omega}\hat{g},\Phi]}. \tag{3.32}$$

[13] In bosonic string theory we would send the vertex operators defining the in state to the Euclidean past, while vertex operators that define the out state would be sent to the future.

We have denoted the integral over lightcone diagrams as $\int_{W_i}$ to emphasize that the wavefunctions W_i impose boundary conditions at the ends of the cylinders at $\tau = \pm\infty$. Note that so far all we have done is rewritten (3.22) in a particular gauge choice. This is essentially the argument for the equivalence of the Polyakov formulation of bosonic string theory and the interacting string picture [43, 44] on Euclidean lightcone diagrams [138, 150]. The only new ingredients we have encountered in considering this formalism for JT gravity is that we have to integrate over the Weyl factor, and there is no canonical choice for separating the boundaries into those that are sent to the past or future.

Lorentzian JT path integral

Since Euclidean lightcone diagrams come with a global notion of Euclidean time τ, we can analytically continue $\tau \to i\tau$ to get Lorentzian lightcone diagrams with metric

$$\frac{1}{2}\sqrt{-\hat{g}}\hat{R} = 2\pi i \sum_{I=1}^{2g-2+n} \delta^2(z - z_I), \qquad \det \hat{g}(z_I) = 0. \tag{3.33}$$

Our ***definition*** for the Lorentzian JT path integral will be given by the above analytic continuation applied to (3.32), which gives us

$$Z_{\mathrm{L}} = \int_{W_i} [d\tau][d\theta][\rho d\rho] \frac{2\pi \det'(-\hat{\nabla}^2)}{\int_\Sigma d^2 z\sqrt{\hat{g}}} \int \mathcal{D}\omega\mathcal{D}\Phi e^{iI_{\mathrm{JT}}[e^{2\omega}\hat{g},\Phi]}. \tag{3.34}$$

The Lorentzian JT action is now given by

$$I_{\mathrm{JT}}[g,\Phi] = \frac{S_0}{4\pi}\int_\Sigma \sqrt{-g}R + \frac{1}{2}\int_\Sigma \sqrt{-g}\Phi(R+2). \tag{3.35}$$

where we use the notation $g = e^{2\omega}\hat{g}$. We will ignore the first term which is topological by complex Gauss-Bonnet (3.20) and focus on the second term. On a lightcone diagram we use (3.33) to find that the second term is given by

$$\frac{1}{2}\int_\Sigma \sqrt{-g}\Phi(R+2) = 2\pi i \sum_{I=1}^{2g-2+n} \Phi(z_I) + \frac{1}{2}\int_{\Sigma\setminus\{z_I\}} \sqrt{-g}\Phi\left(R+2\right). \tag{3.36}$$

We see that the JT action on a lightcone diagram picks up point terms at the degenerate points where the geometry undergoes a topology changing transition. The usual JT gravity constraint of constant negative curvature $R = -2$ no longer holds at such points. If the constraint did hold on all of Σ the path integral would be zero since a suitable Lorentzian topology changing metric does not exist. There is one issue, if we integrate over the dilaton $\Phi(z_I)$ at the interaction point the path integral will diverge. However, since there are $2g-2+n$ interaction points we can absorb this infinite divergence into the definition of S_0 in the topological term.

We can now perform the path integral over the dilaton Φ. Since we are in Lorentzian signature, we choose our contour for Φ to be along the real axis. Rewriting the action in terms of the Weyl factor we find

$$\int \mathcal{D}\Phi \exp\left(i\int_{\Sigma\backslash\{z_I\}}\sqrt{-\hat{g}}\Phi\left(-\hat{\nabla}^2\omega+e^{2\omega}\right)\right)=\delta\left(-\hat{\nabla}^2\omega+e^{2\omega}\right)=\frac{\delta\left(\omega-\omega_0\right)}{\det\left(-\hat{\nabla}^2+2e^{2\omega_0}\right)}. \tag{3.37}$$

Here ω_0 is a configuration that satisfies the delta function constraint. This constraint enforces that the metric $g=e^{2\omega}\hat{g}$ has constant negative curvature $R=-2$ away from the interaction points. The end result is the following path integral

$$Z_{\mathrm{L}}=\int_{W_i}[d\tau][d\theta][\rho d\rho]\frac{2\pi\det'(-\hat{\nabla}^2)}{\int_\Sigma d^2z\sqrt{\hat{g}}}\int\mathcal{D}\omega\frac{\delta\left(\omega-\omega_0\right)}{\det\left(-\hat{\nabla}^2+2e^{2\omega_0}\right)}. \tag{3.38}$$

The reason we have not carried out the integral over ω is the following. As explained in section 3.2, there exist multiple Weyl factors that satisfy the constraint (3.37) while having different behavior at the degenerate points given by (3.19), which we restate here for convenience

$$\frac{1}{2}\sqrt{-g}\left(R+2\right)=2\pi\sum_{I=1}^{2g-2+n}\left(i+\alpha_I\right)\delta^2(z-z_I). \tag{3.39}$$

We must choose our contour of integration for ω to decide which configurations should be included. A configuration is specified by picking out a preferred choice of α_I. There appear to be two natural choices. We can choose a certain α_I as special and the Weyl factor ω_0 can be forced to localize onto such a geometry for all lightcone diagrams. The other obvious option is to integrate over all possible ω (i.e. all α_I). The benefit of choosing a particular α_I as special would be that we would get the closest analogue to Euclidean JT gravity where only one ω contributes for each point in moduli space $\mathcal{M}_{g,n}$. For simplicity we will assume that the Weyl factor localizes to a single configuration, after which we are left with an integral over the moduli space $\mathcal{M}_{g,n}$ with given measure

$$Z_{\mathrm{L}}=\int_{W_i}[d\tau][d\theta][\rho d\rho]\frac{2\pi\det'(-\hat{\nabla}^2)}{\int_\Sigma d^2z\sqrt{\hat{g}}}\frac{1}{\det\left(-\hat{\nabla}^2+2e^{2\omega_0}\right)}. \tag{3.40}$$

The integration range is defined in (3.11), and the wavefunctions W_i implement boundary conditions as discussed around (3.28). The end result is that we must choose a set of boundary conditions and then evaluate the above integral.

We now discuss the definition of the functional determinants appearing in the above integral. Even though the integral is over Lorentzian lightcone geometries, we will choose the functional determinants to be analytically continued and defined on the corresponding Euclidean lightcone geometries to make them well defined. The Laplacian $\hat{\nabla}^2$ is defined with respect to the degenerate lightcone metric (3.4). The corresponding determinant $\det(-\hat{\nabla}^2)$ has been studied by [139, 151–153], and in principle it is known in terms of basic objects on the underlying punctured Riemann surface. However, the second determinant is more complicated. Using the conformal anomaly we can relate it to $\det(-\nabla^2+2)$ where ∇^2 is now defined with respect to a constant negative curvature metric $g = e^{2\omega}\hat{g}$ with conical singularities at the splitting points. There has been some recent progress on computing closely related determinants[14] [154], but as of now determinants on singular Riemann surfaces are not fully understood. Regardless, evaluating the above determinants will introduce some dependence on the moduli into the above integral.

Relation to Euclidean amplitudes

We would like to discuss the relation of our proposed Lorentzian path integral (3.40) to the standard Euclidean amplitudes. We first briefly review why the Euclidean JT path integral gives Weil-Petersson volumes, originally explained in [4]. For simplicity, consider a compact surface of genus g, the Euclidean path integral over all smooth metrics on this surface reduces to [4]

$$Z = \int_{\mathcal{M}_g} \mathrm{d(WP)}(\det \hat{P}_1^\dagger \hat{P}_1)^{1/2} \int \mathcal{D}\omega \frac{\delta(\omega)}{\det(-\hat{\nabla}^2+2)}, \tag{3.41}$$

where we have gauge fixed to a smooth metric $\hat{g}$ of constant negative curvature, and as a result the Weyl factor localizes to $\omega = 0$. We have written the integral in this way to compare to (3.38). The measure d(WP) is known as the Weil-Petersson measure. In the above we have two determinants: $(\det \hat{P}_1^\dagger \hat{P}_1)^{1/2}$ which originates from gauge fixing and is theory independent, and $\det(-\hat{\nabla}^2+2)$ which is special to JT gravity and arises from the integral over the dilaton Φ. In [4] it was pointed out that the ratio of these determinants is unity, up to a factor that can be absorbed into the coupling constant S_0. We are left with an integral over the Weil-Petersson measure, which gives us the Weil-Petersson volume

$$Z = \int_{\mathcal{M}_g} \mathrm{d(WP)} = V_g. \tag{3.42}$$

We can similarly ask whether our Lorentzian path integral (3.40) localizes to the Weil-Petersson measure. Possible issues might arise both from the theory independent measure (3.31), and from the theory dependent contribution (3.37). However, choosing lightcone diagrams as the representative metric for the moduli space $\mathcal{M}_{g,n}$ is simply a gauge choice, so the theory independent measure (3.31) cannot depend on this choice. Indeed, it was pointed out in [138] that (3.31) secretly contains the Weil-Petersson measure d(WP) in lightcone coordinates, and that the determinant of the Laplacian $\det(-\hat{\nabla}^2)$ roughly comes from the punctured Riemann surface analogue of the gauge fixing determinant $(\det \hat{P}_1^\dagger \hat{P}_1)^{1/2}$.

[14] In [154] various determinants were studied on punctured Riemann surfaces with conical singularities of opening angles $\frac{2\pi}{n+1}$ with $n \in \mathbb{Z}^+$. However, in our case we require conical excesses instead of defects so the results do not appear immediately applicable.

The only possible issue with recovering the Euclidean amplitudes can therefore be from the determinant arising from the integral over the Weyl factor in (3.37). However, this is precisely where we ran into an ambiguity since ω is not constrained to behave in any particular way at the degenerate points. Therefore the determinant in (3.37) should somehow depend on the choice of singularity structure at the degenerate points, and it's unclear if we get an exact cancellation as in (3.41) for Euclidean JT. Without a better understanding of determinants on singular surfaces it is difficult to say anything more concrete. However, since the only deviation comes from a finite set of points we believe it's quite likely that the resulting amplitudes computed by (3.40) share similar features with Euclidean JT gravity, we return to this point in the discussion.

3.4 Discussion

In this work we have proposed a definition for the Lorentzian JT gravity path integral that includes topology changing configurations. This is accomplished by integrating over metrics which are Lorentzian signature everywhere except at special points where the metric becomes degenerate, and where the spatial topology changing transitions occur.

Our proposal is inspired by a formulation of bosonic string theory on singular Lorentzian worldsheets [43, 44, 137–139], and we followed similar logic to define the Lorentzian JT path integral. Using lightcone diagrams we analytically continued the Euclidean path integral to define a Lorentzian path integral over degenerate Lorentzian metrics. This analytic continuation gives a Lorentzian interpretation for the Euclidean path integral genus expansion. We end with a few comments and potential future directions.

Relation to Euclidean JT and degenerate points

We found that the most serious ambiguity in the definition of our Lorentzian theory is how to properly treat the degenerate points. There appears to be no standard prescription for dealing with such points in Lorentzian signature, although see [40] for a recent discussion. In Euclidean signature the degenerate points can always be removed with a singular Weyl factor to give a smooth Euclidean metric. However, in Lorentzian signature this cannot be done since the degenerate points are crucial for the existence of an almost Lorentzian metric with spatial topology change. This gives us some freedom to choose the behavior of the metric at such points[15] (3.19), and this choice propagates into the final path integral in (3.40).

As discussed in section 3.3, to understand the relation between Lorentzian and Euclidean amplitudes we must understand how the ambiguity at the singular points modifies the path integral measure. Indeed, in both the Lorentzian and Euclidean cases the domain of integration is given by the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$, so the only

[15] This ambiguity does not appear when considering the bosonic string on lightcone diagrams[138] since we don't integrate over the Weyl factor.

difference can arise from the respective measures on this space. Since the determinants on singular Riemann surfaces appearing in (3.40) are not fully understood it is difficult to give a conclusive answer. However, it is likely that this ambiguity modifies the path integral measure, and so we don't expect the Lorentzian amplitudes to exactly match the Euclidean amplitudes.

Regardless, we expect the Lorentzian amplitudes to have qualitatively similar behavior to Euclidean JT gravity amplitudes for reasons of random matrix universality. In [27] it was shown that wormhole contributions in a wide range of dilaton-gravity models, with integration measures deviating from pure JT gravity, have universal behavior independent of the specific details of the theory. Indeed, we do not expect the specific details of the integration measure to seriously modify universal behavior, such as the existence of a page curve[5, 6] or the spectral form factor [4, 25, 27].

Towards non-perturbative Lorentzian physics

Non-perturbative Euclidean wormholes have played an important role in a wide variety of recent calculations, see [4–6, 25, 27, 28, 31, 33] for some selected examples. It would be interesting to understand the Lorentzian origin of these calculations using the Lorentzian path integral developed in this work. As an example, it would be interesting to understand the Replica wormhole computations [5, 6] or the swap entropy [155] using topology changing Lorentzian wormholes, see also [38, 39]. Additionally, it would be interesting to understand how the inclusion of topology changing Lorentzian metrics modifies the structure of the JT gravity Hilbert space[136], and whether there is a canonical quantization interpretation for Lorentzian topology change.

To address the above questions it will be necessary to introduce asymptotically AdS boundaries, which amounts to studying lightcone diagrams with boundaries. These diagrams were studied in the context of open strings[43, 44], but the proof of Giddings and Wolpert [137] was never extended to them. Thus we cannot claim that these diagrams give a single cover of the desired moduli space, but it seems quite likely that they do. Regardless, we can directly define the Lorentzian JT theory with asymptotic AdS boundaries to live on such lightcone diagrams, but for now we leave an examination of the above questions to future work.

Simpler representation of moduli space

One of the features of lightcone diagrams is that they provide a particularly simple representation for the integral over the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$. The standard procedure is to introduce Fenchel-Nielsen coordinates on the moduli space in terms of which the the domain of integration is incredibly complicated, see [4] for a discussion.

However, lightcone diagrams have a very simple region of integration (3.11)

$$\int_{\mathcal{M}_{g,n}} \ldots = \frac{1}{S} \prod_{i,j,k} \int_0^\infty d\tau_i \int_0^{2\pi} d\theta_j \int_0^{r_{\max}} \rho_k d\rho_k \ldots \tag{3.43}$$

while the integration measure can be quite complicated, see (3.40). It would be interesting to understand the integration measure in (3.40) in terms of the lightcone coordinates on the moduli space. It might be possible that in corners of moduli space the measure takes a simplified form in terms of these coordinates. More generally, it would be interesting to understand if the ratio of determinants in (3.40) simplifies as in the case of Euclidean JT gravity.

Chapter 4

Averaging the symmetric product orbifold

This Chapter is based on [66] where we study ensemble averaging over a family of two dimensional conformal field theories and it's holographic gravitational dual. The AdS/CFT correspondence states that a given conformal field theory is dual to a theory of quantum gravity. However, it has recently been appreciated that some simple theories of quantum gravity, defined by a sum over geometries and weighed by a semiclassical action, are dual to an average over a suitable ensemble of boundary theories. The clearest example of this is JT gravity which is a two-dimensional gravity theory dual to an ensemble of one dimensional quantum mechanical theories [4].

This idea has been extended to a wide class of two-dimensional gravitational theories [12, 15, 16, 21, 156], and progress has also been made in extending these concepts to higher-dimensional theories of gravity [46–50]. In more than two bulk dimensions it is a priori unclear how to construct ensemble averages over dual microscopic theories, so many approaches have focused on two-dimensional CFTs with a large number of symmetries such as Narain CFTs [51–56], and WZW models [57–59]. We will consider the former. In [46, 47] it was demonstrated that averaging over the family of $\mathbb{T}^D$ Narain CFTs is dual to a bulk theory given by Chern-Simons coupled to topological gravity. Narain CFTs can be defined by the action

$$I = \int \mathrm{d}^2 z \left(G_{mn} \delta^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n + i B_{mn} \varepsilon^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n \right) , \tag{4.1}$$

where the choice of target metric G_{mn} and B_{mn} is a choice of moduli and defines the theory. We denote the Narain CFT partition function by $Z_{\mathbb{T}^D}(m, \Sigma)$ where Σ is a two dimensional Riemann surface on which the theory is defined and m is a particular choice of the moduli defining the theory. Averaging over the moduli with an appropriate measure, it was found that the resulting averaged partition function $\langle Z_{\mathbb{T}^D}(m, \Sigma) \rangle$ could be reproduced by a bulk Chern-Simons calculation. The bulk theory takes the form of $2D$ copies of abelian Chern-

Simons with total gauge group $G = \mathrm{U}(1)^D \times \mathrm{U}(1)^D$ and action given by

$$S_{\mathrm{CS}} = \sum_{i=1}^{D} \int_M \left(A_i \wedge dA_i - B_i \wedge dB_i\right), \tag{4.2}$$

where the $2D$ gauge fields A_i, B_i transform under independent copies of U(1). It was found that summing over a class of bulk three-manifolds, bulk handlebodies M with asymptotic boundary $\partial M = \Sigma$, precisely reproduces the average over Narain CFTs

$$\langle Z_{\mathbb{T}^D}(m, \Sigma)\rangle = \sum_{\text{handlebodies } M} Z_G(M), \tag{4.3}$$

where on the right we evaluate the Chern-Simons path integral with action (4.2) on each handlebody. The subscript G on the partition function denotes the gauge group of the Chern-Simons theory. We go into additional details on the proposed duality between the Narain average and a bulk Chern-Simons theory in Section 4.1.

In this paper we will extend this duality by ensemble averaging over a related family of two-dimensional CFTs, symmetric product orbifolds of Narain CFTs. The process to construct a symmetric product orbifold is to take N tensor copies of a seed CFT X, and gauge the S_N permutation symmetry exchanging the copies of the theory

$$\mathrm{Sym}^N(X) = X^{\otimes N}/S_N\,. \tag{4.4}$$

We go into additional details on constructing such theories in Section 4.1. Applying this procedure to the Narain theories we can construct a family of CFTs, denoted by $\mathrm{Sym}^N(\mathbb{T}^D)$, labelled by a choice of integer N and a point in moduli space m. We denote the partition function of such theories by $Z_{\mathbb{T}^D \wr S_N}(m, \Sigma)$. Since this family of theories has the same moduli space as the Narain theories we can again perform the ensemble average over such theories. We now summarize our main results.

Summary of main results

Ensemble averaging $\mathrm{Sym}^N(\mathbb{T}^D)$: The goal of this paper is to provide a bulk dual for the ensemble average of the symmetric product orbifold of Narain CFTs. Following the standard holographic prescription, the bulk dual should be given by a sum over a suitable set of bulk geometries. The philosophy we will take in this work is that the rules for the bulk path integral should be dictated by the boundary ensemble average. In particular, the choice of which bulk geometries to include is determined by consistency with the boundary answer [46].

We now restrict our attention to $\mathrm{Sym}^N(\mathbb{T}^D)$ CFTs defined on a boundary torus with modular parameter τ. In Section 4.2 we explain how to ensemble average the partition function of this class of theories $\langle Z_{\mathbb{T}^D \wr S_N}(m, \tau)\rangle$. The final result is given in equation (4.91), and is a formal expression in terms of the Siegel-Weil Formula (4.20), which we introduce and explain in Section 4.1. We expect this average to be holographically dual to a sum over bulk geometries with an asymptotic boundary torus. We find this is partially realized. The averaged partition function can be schematically expressed as follows[1]

$$\langle Z_{\mathbb{T}^D \wr S_N}(m,\tau)\rangle = \sum_{\substack{\text{handlebodies } M, \\ \text{vortices}}} Z_{\text{Bulk}}(M) + \text{non-semiclassical geometries.} \tag{4.5}$$

In the above we have split the boundary average into two terms. We will refer to the first term as a "semiclassical" contribution while the second term is a "non-semiclassical" contribution. We define contributions as semiclassical or not based on whether they can be reproduced by the standard rules of the gravitational path integral. Let us first explain the bulk origin of the semiclassical contribution.

Semiclassical Contributions: The semiclassical contribution is reproduced by a standard gravitational path integral where we sum over handlebody geometries bounding the asymptotic torus with the inclusion of "vortices" (analogous to 't Hooft loops for discrete gauge groups) running along the non-contractible cycle of the geometry. We have schematically represented the contribution of each such geometry by $Z_{\text{Bulk}}(M)$, which is given by a one-loop exact Chern-Simons calculation. In the case of a boundary torus, handlebody geometries are three manifolds of the form $D^2 \times S^1$. On each handlebody geometry we evaluate the partition function of a Chern-Simons theory with gauge group $\text{U}(1)^D \times \text{U}(1)^D \wr S_N$.[2] The bulk action of the Chern-Simons theory with this group is given by N copies of the previous action in equation (4.2)

$$S_{\text{CS}} = \sum_{i=1}^{N} \int_{\mathcal{M}} \left(A_{(i)} \wedge \mathrm{d}A_{(i)} - B_{(i)} \wedge \mathrm{d}B_{(i)}\right) , \tag{4.6}$$

where we have suppressed the summation over the D indices present in (4.2) for simplicity. In total the theory has $2DN$ gauge fields. Evaluating the partition function of this theory is slightly non-trivial since the structure of the gauge group is a wreath product, and in Section 4.3 we explain how to accomplish this for bulk handlebody geometries.

The final aspect of equation (4.5) that we must explain is the summation over vortices. A vortex is a gauge theory line operator that we choose to place along the non-contractible cycle of the handlebody. This operator implements twisted boundary conditions on the

[1] As we will discuss later in Section 4.1, both the average over the Narain moduli space and sum over handlebodies can diverge, assuming the degree N of the orbifold group and the boundary genus g are sufficiently large (in a way that will be made precise below), see also [46]. We will largely ignore these divergence issues since, even when the sum over handlebodies diverges, the individual summands still make sense as on-shell bulk partition functions. Of course, if one considers the full microscopic theory without averaging, no such divergence should appear.

[2] The notation $\wr$ indicates that the gauge group takes the form of a wreath product group, which is defined by taking N copies of $\text{U}(1)^D \times \text{U}(1)^D$ and gauging the symmetry permuting them. We explain this structure in Section 4.3.

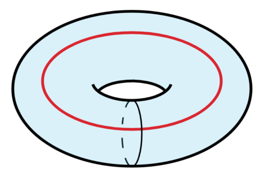
gauge fields $A_{(i)}, B_{(i)}$ as they travel around the vortex

$$A_{(i)} \to A_{\pi(i)}, \qquad B_{(i)} \to B_{\pi(i)}, \tag{4.7}$$

where $\pi \in S_N$ are permutations. The inclusion of vortices amounts to including gauge field configurations that are singular in the interior of the handlebody, and we explain how to evaluate the path integral on a handlebody with a vortex insertion in Section 4.3. The summation over vortices amounts to a summation over all twisted boundary conditions implemented by permutations π on the gauge fields.

Putting everything together, in sections 4.3 and 4.3 we evaluate the Chern-Simons partition function on handlebody geometries with vortex operator insertions and show that we precisely reproduce what we denote as the semiclassical contribution to the averaged partition function

$$\langle Z_{\mathbb{T}^D \wr S_N}(m,\tau)\rangle \supset \sum_{\substack{\text{handlebodies } M,\\ \text{vortices}}} Z_{G\wr S_N}(M) =$$

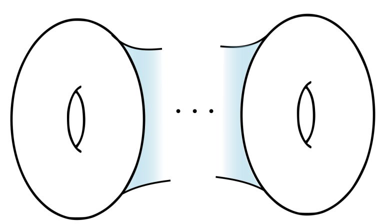


$$. \tag{4.8}$$

In the above figure the vortex is the red line running in the interior of the handlebody. The Chern-Simons partition function $Z_{G\wr S_N}$ implicitly depends on the twisted boundary conditions the vortex implements. Thus, at least a portion of the ensemble averaged partition function can be reproduced by a standard bulk theory given by $U(1)^D \times U(1)^D \wr S_N$ Chern-Simons with the inclusion of bulk vortices.

Non-semiclassical Contributions: Let us now explain the "non-semiclassical" contribution to equation (4.5). The ensemble average of the partition function $\langle Z_{\mathbb{T}^D \wr S_N}\rangle$ contains averages over multiple disconnected products of partition functions of the seed theory. A useful example is to consider the case of $N = 2$, where the boundary average contains a contribution

$$\langle Z_{\mathbb{T}^D \wr S_2}(m,\tau)\rangle \supset \frac{1}{2}\langle Z_{\mathbb{T}^D}(m,\tau) Z_{\mathbb{T}^D}(m,\tau)\rangle = \sum_{\text{geometries}}$$



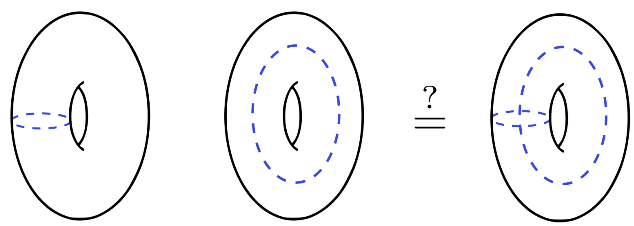
$$. \tag{4.9}$$

Where in the above figure we have used the results summarized in Section 4.1 to represent the average $\langle Z^2_{\mathbb{T}^D}(m,\tau)\rangle$ as a gravitational path integral with two asymptotic boundary tori [46]. Generic non-semiclassical contributions arise from geometries of a similar nature, where for general N the boundary average instructs us to include terms with up to N asymptotic boundary tori.

While such terms have a geometric interpretation as bulk configurations with multiple asymptotic boundaries, typically, they do not have a holographic interpretation as a single geometry with a single asymptotic boundary. To assign a holographic interpretation to a bulk wormhole geometry M with n asymptotic boundaries we require that M is suitably "symmetric". The precise notion of this symmetry is subtle and we elaborate on it in Section 4.3, but it is reminiscent of the $\mathbb{Z}_n$ replica symmetry in the context of the replica trick [157]. In the case that M is completely disconnected, the requirement is that the same boundary cycle is contractible in the interior of each disconnected bulk geometry. Such a symmetry is highly non-generic, and most wormhole configurations contributing to the average do not have a simple interpretation. However, a subset of such geometries do have such a symmetry, and have already been implicitly included in the "semi-classical" Chern-Simons computation, see Section 4.3.

It's useful to give a simple example of a geometry that does not have a semiclassical interpretation. Consider a contribution to equation (4.9) where one torus is filled in with a handlebody with contractible spatial cycle, while the other is filled in with a handlebody with contractible time cycle

$$\langle Z_{\mathbb{T}^D \wr S_2}(m,\tau)\rangle \supset$$



, (4.10)

where the contractible cycles are identified by the dashed lines in the figure. A holographic interpretation would require a single bulk geometry with both spatial and time cycles contractible in the interior. A bulk manifold M cannot have both of these cycles contract, and so such a contribution has no hope of being reproduced by a standard sum over geometries. To include such contributions we would need to seriously modify the standard rules for the bulk path integral, and allow different gauge fields A_I to live on "independent" manifolds with different contractible cycles. See Section 4.3 and the discussion in Section 4.6.

Averaging correlators: In Section 4.4 we consider ensemble averaging correlation functions of twist operators in the symmetric orbifold, focusing specifically on the case of the $\mathrm{Sym}^2(\mathbb{T}^D)$ orbifold. These are non-local operators implementing twisted boundary conditions for the fundamental fields in the orbifolded theory. Hence, these objects are naturally identified as being the dual to the vortices of the the bulk Chern-Simons theory mentioned around (4.7). Following the elegant approach of references [158, 159], the monodromy implemented by twist fields trivialises on the covering space and it can be shown that the correlation functions reduce to a product of covering map data and the seed partition function on the (branched) covering space. For example for the case of the sphere we get (4.179). Therefore the Siegel-Weil formula may be used in performing the average of the latter resulting in a modular sum. How is this interpreted from the bulk perspective? Using the identification of vortices and twist operators we will show that summing over all inequivalent configurations of vortices ending on pairs of equivalent twist operators reduces to the aforementioned modular sum.

More specifically, as in [160] we consider vortex configurations, which are rational tangles in the language of knot theory. Such tangles exhibit handlebodies as branched covering spaces such that the modular sum can be understood as a sum over hyperbolic metrics.

Averaging Supersymmetric Narain CFTs: In Section 4.5, we consider the ensemble average over the supersymmetric version of Narain CFTs. We propose that the ensemble average is holographically dual to a supersymmetric version of $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ Chern-Simons, and we reproduce the boundary ensemble averaged torus partition function from a bulk supersymmetric Chern-Simons sum over handlebody geometries. Furthermore, we consider the symmetric product orbifold $\mathrm{Sym}^N(\mathbb{T}^D)$ of supersymmetric Narain theories and show that, similar to the non-supersymmetric case, a supersymmetric Chern-Simons theory with gauge group $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_N$ reproduces many "semiclassical" contributions to the averaged partition function.

Averaging the Tensionless String: The symmetric orbifold $\mathrm{Sym}^N(\mathbb{T}^4)$ at large N is dual to type IIB string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ with one unit of pure NS-NS flux (the so-called 'tensionless string') [61–65, 161–166]. Part of our motivation for considering the average of $\mathrm{Sym}^N(\mathbb{T}^D)$ theories is to understand whether the tensionless string can be ensemble averaged to produce a semiclassical sum over geometries. We are partially successful, averaging a single string propagating on an AdS_3 background gives a "semiclassical" geometry as defined above. Averaging over multiple strings on a single background gives rise to the "non-semiclassical" geometries. We leave a more complete discussion of this to Section 4.6.

In Section 4.1 we review some preliminary results necessary for the rest of the work. We explain the Narain average/Chern-Simons duality of [46, 47], and we review the basic construction of symmetric product orbifold CFTs. In Section 4.2 we begin by explaining how to ensemble average over a symmetric product CFT. We then apply this to the class of Narain CFTs $\mathrm{Sym}^N(\mathbb{T}^D)$ to obtain a boundary answer for the average. In Section 4.3 we interpret the boundary average as a holographic Chern-Simons theory with the inclusion of bulk vortices. In Section 4.4 we consider correlation functions of twist operators in the CFT, and we show that a bulk dual is given by a sum over vortex configurations equivalent to a specific sum over hyperbolic three-manifolds. In Section 4.5 we consider the supersymmetric extension of the Narain theories, and we provide a bulk dual given by supersymmetric Chern-Simons theory, with additional details left to appendix A.8. In Section 4.6 we end with a discussion of our results.

4.1 Preliminaries

In this section we review the necessary technology used throughout the main parts of the paper. In particular, we review the Narain-ensemble/U(1) gravity duality proposed by [46, 47], as well as the basics of permutation orbifolds. Readers familiar with Narain averaging and permutation orbifolds should feel free to skip this section.

Narain averaging and the sum over geometries

Consider a sigma-model with target space a D-dimensional torus $\mathbb{T}^D$. The action reads

$$I = \int \mathrm{d}^2 z \left(G_{mn} \delta^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n + i B_{mn} \varepsilon^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n \right) \, . \tag{4.11}$$

where G_{mn} is the metric on the $\mathbb{T}^D$ target space and B_{mn} is a two-form field, and the target coordinates are compact $X^m \sim X^m + 2\pi$. We take the theory to be defined on a Riemann surface with locally flat metric $\delta^{\alpha\beta}$, with $\varepsilon^{\alpha\beta}$ being the Levi-Civita symbol. This CFT belongs to a family of two-dimensional CFT's with left and right-moving current algebras of type $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ with central charges $(c_L, c_R) = (D, D)$, namely the *Narain* family of CFTs. The moduli space of Narain CFTs is parameterised by D^2 parameters encoded in choosing a target metric G_{mn} and the two form field B_{mn}. Equivalently the moduli space is given by the double quotient space (for more details, see e.g. [167]):

$$\mathcal{M}_D = \mathrm{O}\left(D, D; \mathbb{Z}\right) \backslash \mathrm{O}\left(D, D\right) / \mathrm{O}\left(D\right) \times \mathrm{O}\left(D\right) \, . \tag{4.12}$$

The partition function on a torus with modular parameter τ is given by

$$Z_{\mathbb{T}^D}(m, \tau) = \frac{\Theta(m,\tau)}{|\eta(\tau)|^{2D}} \, , \tag{4.13}$$

where $m \in \mathcal{M}_D$ labels a particular point in the moduli space of Narain CFTs, and $\eta(\tau)$ is the Dedekind eta function. The only moduli dependence enters through $\Theta(m, \tau)$ which is the Siegel-Narain theta function.[3] As this moduli space carries a natural measure via the Zamolodchikov metric,[4] one can ensemble-average over the space of Narain CFTs. Since the only dependence on the moduli enters through $\Theta(m, \tau)$ we must consider the formal expression:

$$\langle \Theta(m, \tau) \rangle := \int_{\mathcal{M}_D} \mathrm{d}\mu(m) \Theta(m, \tau) \, = \frac{E_{\frac{D}{2}}(\tau)}{(\mathrm{Im}\, \tau)^{\frac{D}{2}}} \, , \tag{4.14}$$

with $\mathrm{d}\mu(m)$ being the normalized Zamolodchikov measure. The expression for $\langle \Theta(m, \tau) \rangle$ is found by the use of the *Siegel-Weil* [168–171] formula, and it is given by the real analytic Eisenstein series $E_s(\tau)$ which is defined as

$$E_s(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} (\mathrm{Im}\, \gamma \cdot \tau)^s \, . \tag{4.15}$$

[3] The partition function $Z(m, \tau)$, the theta function $\Theta(m, \tau)$, and the Eisenstein series $E_s(\tau)$ which we introduce later, also depend on $\bar{\tau}$. To avoid clutter, we omit this dependence in the notation and keep it implicit.

[4] This metric is calculated by computing the two-point function of the exactly marginal operator $\mathcal{O} \sim \delta G_{mn} \delta^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n + i \delta B_{mn} \varepsilon^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n$. This metric is equivalently the Haar measure of $\mathrm{O}(D, D; \mathbb{R})$ descendend to the quotient.

Here Γ_∞ is the subset of the modular group which leaves invariant the imaginary part $\operatorname{Im}\tau$, and $\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})$ is the left quotient [46].[5] The sum over modular images can be represented by matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ with coprime $(c,d)=1$. Using the above we can write the final expression for the ensemble-averaged torus partition function as a sum over modular images

$$\langle Z_{\mathbb{T}^D}(m,\tau)\rangle = \frac{E_{\frac{D}{2}}(\tau)}{(\operatorname{Im}\tau)^{\frac{D}{2}}\,|\eta(\tau)|^{2D}} = \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}\,. \tag{4.16}$$

The above averaging procedure can be generalized to partition functions on arbitrary Riemann surfaces by using the higher genus analogue of the Siegel-Weil formula. The averaged partition function of the sigma-model on a genus g surface Σ_g with period matrix Ω takes on the form

$$\langle Z_{\mathbb{T}^D}(m,\Omega)\rangle = \frac{E_{\frac{D}{2}}(\Omega)}{(\det\,\operatorname{Im}\Omega)^{\frac{D}{2}}\,\left|\det{}'\,\overline{\partial}\right|^{D}}\,. \tag{4.17}$$

The determinant $\det{}'\,\overline{\partial}$ appearing in (4.17) is of the operator $\overline{\partial}$ on Σ_g omitting the zero-modes. We have introduced the higher genus generalization of the Eisenstein series

$$E_s(\Omega) = \sum_{\Gamma_0} \left(\det\,\operatorname{Im}\Omega_{\Gamma_0}\right)^s\,, \tag{4.18}$$

with Ω_{Γ_0} being the period matrix defined with respect to what is known as a Lagrangian sublattice Γ_0. We defer the discussion of Lagrangian sublattices to slightly later in this section, but for now a particular Γ_0 should be thought of as specifying a distinguished set of asymptotic boundary cycles which will be contractible in the bulk manifold when we interpret the ensemble average holographically, see Figure 4.1.

The average can also be defined over products of partition functions on disconnected Riemann surfaces. Suppose we have a product of n partition functions on associated Riemann surfaces of genus g_i with period matrices Ω_i. We can form a matrix Ω which is the direct sum of the period matrices of the respective Riemann surfaces

$$\Omega = \bigoplus_{i=1}^{n}\Omega_i. \tag{4.19}$$

The ensemble average over disconnected Riemann surfaces is then given by the following generalization of the Siegel-Weil formula

$$\langle Z_{\mathbb{T}^D}(m,\Omega_1)\ldots Z_{\mathbb{T}^D}(m,\Omega_n)\rangle = \frac{E_{\frac{D}{2}}(\Omega)}{\prod_{i=1}^{n}(\det\,\operatorname{Im}\Omega_i)^{\frac{D}{2}}\left|\det{}'\,\overline{\partial}_{\Sigma_{g_i}}\right|^{D}}\,, \tag{4.20}$$

[5] The subgroup Γ_∞ consists of all $\mathrm{SL}(2,\mathbb{Z})$ matrices of the form $\begin{pmatrix} \pm1 & n \\ 0 & \pm1 \end{pmatrix}$. Two matrices $\gamma,\gamma'\in\mathrm{SL}(2,\mathbb{Z})$ are considered equivalent in $\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})$ if $\gamma = h\cdot\gamma'$ for some $h\in\Gamma_\infty$. The sum in (4.15) includes one matrix from each equivalence class in $\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})$.

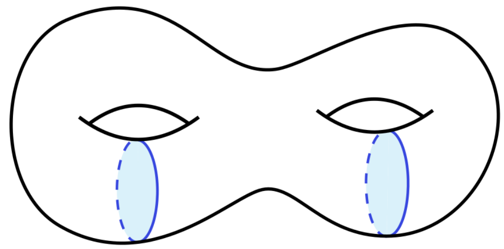


Figure 4.1: A given Lagrangian sublattice Γ_0 appearing in the sums in (4.16), (4.17), and (4.20) will be holographically associated with a choice of asymptotic boundary cycles that becomes contractible in the interior of the geometry. In this figure we have genus two handlebody with the drawn cycles contractible in the bulk.

where in the above Ω is no longer the period matrix of a single Riemann surface, but a direct sum of period matrices of disconnected Riemann surfaces.

Double Torus Average

We will primarily be interested in the Narain average over products of partition functions on disconnected tori boundaries. In this case the period matrices are just the modular parameters of the tori $\Omega_i = \tau_i$, and $\Omega = \text{diag}(\tau_1, \dots, \tau_n)$ is a diagonal square matrix. The averaged partition function for products of disconnected torus boundaries is then given by (4.20) with the appropriate diagonal matrix Ω. We explicitly work out the case of two tori with identical modular parameters τ since it will be used later. Applying (4.18) and (4.20) to the case $\Omega = \text{diag}(\tau, \tau)$ we obtain

$$\langle Z_{\mathbb{T}^D}(m,\tau) Z_{\mathbb{T}^D}(m,\tau)\rangle = \frac{1}{\text{Im}(\tau)^D |\eta(\tau)|^{4D}} \sum_{\Gamma_0 \subset H_1(\Sigma \sqcup \Sigma, \mathbb{Z})} (\det \text{Im}(\Omega_{\Gamma_0}))^{D/2} . \tag{4.21}$$

Where we have used that $\det' \overline{\partial} = |\eta(\tau)|^2$ on the torus. In the above $\Gamma_0 \subset H_1(\Sigma \sqcup \Sigma, \mathbb{Z})$ is a sum over possible contractible cycles on the two tori. This sum contains a set of contribution that give the disconnected average $\langle Z(\tau)\rangle^2$, in addition to wormhole contributions.

We now explain how to see the contribution of the disconnected average $\langle Z(\tau)\rangle^2$ in the sum. Let $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}$ be the A-cycles of the two tori, while $\mathcal{B}^{(1)}, \mathcal{B}^{(2)}$ are their B-cycles. Take the contractible cycles specified by Γ_0 to be given by independent modular transformations γ_i of the $\mathcal{A}^{(i)}$ cycles on the two respective torii. This corresponds to a choice of Γ_0 and Ω_{Γ_0} given by

$$\Gamma_0 = \text{Span}_{\mathbb{Z}}\left(\gamma_1(\mathcal{A}^{(1)}), \gamma_2(\mathcal{A}^{(2)})\right) , \qquad \Omega_{\Gamma_0} = \begin{pmatrix} \gamma_1 \cdot \tau & 0 \\ 0 & \gamma_2 \cdot \tau \end{pmatrix} . \tag{4.22}$$

The above choice of Γ_0 is *decomposable*[6] and amounts to picking all possible choices of contractible cycles on the two tori independently, and we postpone an explanation of how to

[6] Intuitively, a decomposable Γ_0 amounts to picking independent contractible cycles on all surfaces[46].

obtain Ω_{Γ_0} to the next subsection. This choice immediately gives the following contribution to the average

$$\langle Z_{\mathbb{T}^D}(m,\tau) Z_{\mathbb{T}^D}(m,\tau)\rangle \supset \frac{1}{\mathrm{Im}(\tau)^D |\eta(\tau)|^{4D}} \sum_{\gamma_1,\gamma_2 \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma_1 \cdot \tau)^{\frac{D}{2}}\, \mathrm{Im}(\gamma_2 \cdot \tau)^{\frac{D}{2}} . \quad (4.23)$$

By comparing to equation (4.16) we notice that this is the disconnected contribution squared $\langle Z_{\mathbb{T}^D}(m,\tau)\rangle^2$. Wormhole contributions arise from other choices for Γ_0, an example of which is given by

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}} \left(\mathcal{A}^{(1)} + \mathcal{A}^{(2)}, \mathcal{B}^{(1)} - \mathcal{B}^{(2)}\right) . \quad (4.24)$$

The above choice corresponds to a bulk wormhole geometry of the form $\Sigma \times [0,1]$, where Σ is a torus. We examine this case in greater detail in Section 4.3. To summarize, the average over products of partition functions contains disconnected contributions which can be identified with special choices of Γ_0, alongside wormhole contributions which correspond to more non-trivial choices of contractible cycles.

Lagrangian Sublattices

We now briefly explain Lagrangian sublattices since they appear in the Eisenstein series (4.18). Consider a Riemann surface Σ_g of genus g. The surface has $2g$ canonical cycles which are labelled by $\mathcal{A}_i, \mathcal{B}_i$ with $i = 1, \dots, g$. The first homology group of the surface $H_1(\Sigma_g)$ is generated by these $2g$ cycles, and we have

$$H_1(\Sigma_g) \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{2g}. \quad (4.25)$$

Once we have made a choice of cycles $\mathcal{A}_i, \mathcal{B}_i$ we can choose a basis of g holomorphic one-forms ω_j by imposing the condition that

$$\oint_{\mathcal{A}_i} \omega_j = \delta_{ij}. \quad (4.26)$$

The period matrix of Σ_g is then defined to be given by

$$\oint_{\mathcal{B}_i} \omega_j = \Omega_{ij}. \quad (4.27)$$

To all cycles $\gamma, \gamma' \in H_1(\Sigma_g)$ we can associate an intersection number $\langle \gamma, \gamma' \rangle$ that counts the number of times that γ and γ' cross. A Lagrangian sublattice is defined to be a primitive[7] subgroup $\Gamma_0 \subset H_1(\Sigma_g)$ generated by g cycles $\tilde{\mathcal{A}}_i$ such that their mutual intersection numbers

[7] A primitive sublattice/subgroup Γ_0 of $H_1(\Sigma)$ is defined such that given $v \in \Gamma_0$ there does not exist an integer n such that $v = nu$ for some $u \in H_1(\Sigma)$. This is to exclude situations where the lattice Γ_0 is generated by $2[\mathcal{A}]$ in the case of the torus. Such a lattice would be generated by the cycle that winds twice around the $[a]$ cycle of the torus. Holographically, such a choice would require demanding the twice wound $[a]$ cycle be contractible in the bulk.

vanish $\langle \tilde{\mathcal{A}}_i, \tilde{\mathcal{A}}_j \rangle = 0$. That is, for a genus g surface a Lagrangian sublattice is a choice of g non-intersecting cycles. Once we have picked g cycles to define the Lagrangian sublattice we must choose a dual pair of cycles $\tilde{\mathcal{B}}_i$ that do not mutually intersect, but intersect the original cycles once $\langle \tilde{\mathcal{A}}_i, \tilde{\mathcal{B}}_j \rangle = \delta_{ij}$. We now define holomorphic differentials $\tilde{\omega}_j$, constructed out of the original differentials ω_j, such that we have

$$\oint_{\tilde{\mathcal{A}}_i} \tilde{\omega}_j = \delta_{ij}. \tag{4.28}$$

The period matrix Ω_{Γ_0} associated to the Lagrangian sublattice is then defined to be

$$\oint_{\tilde{\mathcal{B}}_i} \tilde{\omega}_j = (\Omega_{\Gamma_0})_{ij}\,. \tag{4.29}$$

In the case of multiple disconnected Riemann surfaces the first homology group is given by a direct sum of the homology groups. For two surfaces of genera g_1 and g_2 we have $H_1(\Sigma_{g_1} \sqcup \Sigma_{g_2}, \mathbb{Z}) \cong H_1(\Sigma_{g_1}, \mathbb{Z}) \oplus H_1(\Sigma_{g_2}, \mathbb{Z})$. A Lagrangian sublattice is then a group $\Gamma_0 \subset H_1(\Sigma_{g_1} \sqcup \Sigma_{g_2}, \mathbb{Z})$ generated by $g_1 + g_2$ cycles that have zero mutual intersection numbers. The period matrix associated to Γ_0 is defined in an identical way to the case of a single surface, and generalizes to any number of disconnected surfaces. The new ingredient with disconnected surfaces is that the cycles $\tilde{\mathcal{A}}_i$ that define Γ_0 can now be linear combinations of cycles on disconnected surfaces, as explained for the average over two disconnected tori earlier. To summarize, the sum over Lagrangian sublattices appearing in the Eisenstein series (4.18) is a sum over all possible choices of non-intersecting boundary cycles.

Holographic Dual: U(1) Gravity

In references [46, 47] a three dimensional bulk dual was proposed for the average over $\mathbb{T}^D$ Narain CFTs. It takes the form of a $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ Chern-Simons theory with $2D$ independent U(1) gauge fields A^i, B^i and action

$$S_{\mathrm{CS}} = i \sum_{i=1}^{D} \int_M \left(A^i \wedge \mathrm{d}A^i - B^i \wedge \mathrm{d}B^i\right) - \frac{1}{2} \int_{\partial M} d^2 z \sqrt{g} g^{ab} \left(A^i_a A^i_b + B^i_a B^i_b\right). \tag{4.30}$$

In the above we have included the proper boundary term with boundary metric g_{ab}, which corresponds to a choice of boundary Riemann surface. A choice of boundary conditions that make the variational problem well defined are given by asymptotically fixing $A_{\bar{z}} = 0$ and $B_z = 0$, see [51, 172, 173].[8]

In principle to compute the bulk partition function we should specify asymptotic boundary conditions and evaluate the Chern-Simons path integral over all bulk manifolds consistent

[8] It turns out that the only bulk configurations that contribute also have $A_z = B_{\bar{z}} = 0$ on the boundary. This can be seen by noticing that the holonomy of A, B around the contractible cycle must vanish since the connection is flat [173].

with those boundary conditions. However, it was shown in reference [46] that in the case of a single torus boundary, the bulk partition function defined by summing over only bulk handlebodies exactly reproduced the Narain average.[9] A torus handlebody is a manifold of the form $M \cong D_2 \times S^1$. There is an entire family of distinct handlebodies labelled by elements of $\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})$, with the distinction being which asymptotic cycle of the boundary torus is contractible in the interior of the handlebody. Summing over the contribution of each handlebody we find [46]

$$\sum_{\text{handlebodies } M} Z_{\mathrm{CS}}(M) = \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}, \tag{4.31}$$

where each term in the sum corresponds to a one-loop partition function of Chern-Simons on the handlebody specified by γ. We note that each choice of γ picks out a boundary cycle that is contractible in the interior of the handlebody.[10] In terms of the representation of γ given above equation (4.16), the contractible cycle is given by $c\tau + d$ with $(c,d) = 1$. This precisely reproduces the Narain average partition function (4.16) on the torus. Similarly, it was shown in [46] that summing over handlebodies with a single higher genus asymptotic boundary correctly reproduces the higher genus Narain average (4.17), where the contribution of each handlebody is again given by the one-loop Chern-Simons partition function.

From the above discussion there are two key points:

- To reproduce the Narain average, the sum over bulk geometries should not include every manifold with appropriate asymptotic boundary conditions.
- The sum over Lagrangian sublattices in the Siegel-Weil formula identifies which asymptotic boundary cycles are contractible in the bulk.

The first point is surprising since naively every bulk geometry should be included, but we take the perspective that the boundary ensemble average will dictate what bulk geometries we ultimately include in the sum. For the second point, there are infinitely many three-manifolds with the same contractible bulk cycles, but U(1) gravity seems to pick out a distinguished bulk manifold with given contractible cycles. We follow the interpretation of [46] where it was proposed that this simple theory of gravity cannot resolve finer topological features of the bulk manifold other than which boundary cycles are contractible in the interior.

In the case of averaging over disconnected partition functions the bulk picture is more complicated. The average given by (4.20), which we rewrite for convenience, can still be

[9] The statement that the bulk partition function is given by $\mathrm{U}(1)^{2D}$ Chern-Simons theory is rather subtle. See [46] for a discussion on subtleties related to whether the gauge group of the Chern-Simons theory should be U(1) or $\mathbb{R}$.

[10] The sum over $\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})$ also appears in pure AdS_3 gravity [174], where it is interpreted as the sum over the family of $\mathrm{SL}(2,\mathbb{Z})$ black holes. As an example, the term with $\gamma\cdot\tau = \tau$ is the contribution from the handlebody with the spatial circle contractible in the interior, while the term with $\gamma\cdot\tau = -1/\tau$ corresponds to the handlebody with contractible time circle. In pure AdS_3 gravity these handlebodies would correspond to Thermal AdS_3 and the BTZ black hole respectively.

given a bulk interpretation

$$\langle Z_{\mathbb{T}^D}(m,\Omega_1)\ldots Z_{\mathbb{T}^D}(m,\Omega_n)\rangle = \frac{\sum_{\Gamma_0}\left(\det\,\mathrm{Im}\,\Omega_{\Gamma_0}\right)^{\frac{D}{2}}}{\prod_{i=1}^{n}\left(\det\,\mathrm{Im}\,\Omega_i\right)^{\frac{D}{2}}\left|\det{}'\,\overline{\partial}_{\Sigma_{g_i}}\right|^{D}}\,. \tag{4.32}$$

Each Lagrangian sublattice Γ_0 in the sum again corresponds to a bulk manifold with certain asymptotic cycles contractible in the interior. This sum includes both disconnected handlebody contributions whose each connected component appeared when averaging a single partition function, as well as new wormhole contributions where the bulk manifold connects multiple disconnected boundaries.

For an independent bulk computation of (4.32) we should evaluate the Chern-Simons path integral on the bulk manifold specified by Γ_0. For wormhole geometries there are again infinitely many bulk manifolds with the same contractible cycles specified by a particular Γ_0, and it's unclear which one is picked out by the Narain average. Furthermore, we are left with the problem of evaluating the Chern-Simons partition function on the given wormhole geometry, for which we know of no general results, but see comments in [46]. We will forgo these issues and assume the bulk theory is directly defined by (4.32).

Divergence of the Ensemble Average

The average of the $\mathbb{T}^D$ partition function only converges when $D-1>g$, where g is the genus of the boundary. In the case of multiple disconnected boundaries this generalizes to

$$\langle Z_{\mathbb{T}^D}(m,\Omega_1)\ldots Z_{\mathbb{T}^D}(m,\Omega_n)\rangle < \infty, \qquad D-1>\sum_{i=1}^{n} g_i. \tag{4.33}$$

We note that the Eisenstein series (4.18) precisely diverges when $D-1\leq g$. When the average over moduli space such as (4.14) diverges we cannot, strictly speaking, claim that the average is given by an Eisenstein series since the average is not well defined. Nevertheless, we will define the divergent ensemble average for $D-1\leq g$ to be given by the standard Eisenstein series (4.18), where each term in the sum gives a finite contribution but the full sum does not converge. From a bulk perspective each term in the sum may be associated with a finite contribution from a given bulk geometry, but each geometry is not sufficiently suppressed to make the sum convergent.

From a boundary perspective this is slightly puzzling since for every member of the ensemble the partition function is well defined, but when the average is performed a new divergence appears. This can be understood as follows, the Narain CFTs are defined by a target T^D torus. There are points in moduli space where the target torus decompactifies and we get infinitely many light states that give a divergence to the thermal partition function. The simplest example of this is $D=1$ where the target is a circle S^1. As the radius of the circle goes to infinity we decompactify to target $\mathbb{R}$, and the momentum modes with zero

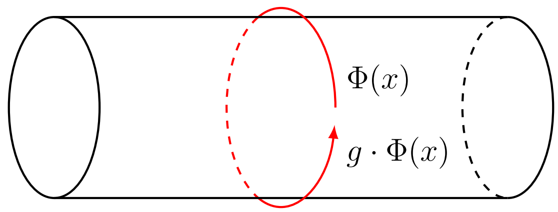


Figure 4.2: An orbifold CFT M/G allows for twisted boundary conditions of fundamental fields around non-contractible loops.

winding become light. The measure for the ensemble average suppresses these dangerous corners of moduli space, but when averaging sufficiently many products of partition functions the growth of light states near decompactification points eventually wins out over the measure suppression and gives a divergent answer.

Symmetric orbifold CFTs

Orbifolds in general

Before discussing the symmetric orbifold, let us recall some basic facts of orbifold conformal field theories in general. Let M be a smooth manifold with a discrete symmetry group G. We can define the quotient space M/G by identifying points

$$p \sim g \cdot p\,, \tag{4.34}$$

where p is a point in M and $g \in G$. If the action of G has fixed points, then a generic point of M/G will not locally look like $\mathbb{R}^n$, but will have conical singularities. Such a space M/G is known as an *orbifold*.

Now, consider a conformal field theory which is a sigma-model with target space M, i.e. a CFT whose fundamental fields are maps $\Phi : \Sigma \to M$ from some two-dimensional surface Σ into M. By abuse of notation, we will also refer to this CFT as M. The CFT inherits the symmetry group G of the target space M, and we can define a new conformal field theory with target space M/G by starting with the CFT M and 'gauging' the discrete symmetry G of the original theory. Specifically, we demand that the field configurations Φ and $g \cdot \Phi$ are physically equivalent, and in the path integral only integrate over equivalence classes $[\Phi]$ of field configurations under the action of G.

Naively, one can avoid the over-counting by picking a representative of $[\Phi]$ in the path integral and only integrating over those representatives. This would effectively result in dividing the path integral of the CFT M by an overall factor of $1/|G|$. However, picking a unique representative of $[\Phi]$ is only possible locally, and globally one needs to be more careful. If Σ has a non-contractible loop based at a reference point $x \in \Sigma$, then it is possible that $\Phi(x)$ obtains a monodromy g upon being transported around this loop (see Figure 4.2). This is a perfectly allowed field configuration, since $\Phi(x)$ and $g \cdot \Phi(x)$ represent the same point in the target space orbifold M/G, yet such a configuration does not allow one to smoothly choose a unique representative of the equivalence class $[\Phi]$ at every point.

Twisted boundary conditions like those in Figure 4.2 are characterized by assigning a monodromy g to each loop γ based at a point x, such that transporting the field Φ around γ returns $g(\gamma)\cdot\Phi(x)$. The assignment of an element $g\in G$ to each loop satisfies the following properties:

- If γ is the trivial loop, then $g(\gamma)=\mathrm{id}$.
- The monodromy $g(\gamma)$ depends only on the homotopy class of the loop γ, i.e. $g(\gamma)$ is invariant under smooth deformations of the loop γ.
- Given two loops γ_1,γ_2 based at the same point, $g(\gamma_1\circ\gamma_2)=g(\gamma_1)g(\gamma_2)$, where $\gamma_1\circ\gamma_2$ is the composition of the loops γ_1 and γ_2.

The above three properties are equivalent to specifying a group homomorphism $g:\pi_1(\Sigma)\to G$.[11] Each such twisted boundary condition should in principle appear in the path integral of M/G, and so the path integral should sum over them. The result is that the path integral of M/G on Σ can be expressed as

$$Z=\frac{1}{|G|}\sum_{g:\pi_1(\Sigma)\to G}\int_g \mathcal{D}\Phi\, e^{-S[\Phi]}\,, \tag{4.35}$$

where the subscript g in the path integral instructs us to integrate over field configurations which obey the twisted boundary conditions. The factor of $1/|G|$ is again included so that physically equivalent fields are not overcounted.

Permutation orbifolds

Consider a 2D CFT X whose fundamental fields are labeled collectively by Φ. If X has central charge c, we can construct a CFT with arbitrarily large central charge Nc by considering the N^{th} tensor power of X

$$X^{\otimes N}:=\underbrace{X\otimes\cdots\otimes X}_{N\text{ times}}\,. \tag{4.36}$$

The theory $X^{\otimes N}$ contains, as its fundamental fields, N-tuples $\Phi=(\Phi_1,\ldots,\Phi_N)$ of fundamental fields of X. Since $X^{\otimes N}$ is constructed from N copies of an identical seed theory the Hilbert space is an N times tensor product of the original Hilbert space, and there is an obvious symmetry of permuting the individual copies. Let $\Omega\subset S_N$ be a permutation group acting on the letters $\{1,\ldots,N\}$. Then for any permutation $\pi\in\Omega$, there is a natural action on the fundamental fields of $X^{\otimes N}$ given by

$$\pi\cdot\mathbf{\Phi}=\left(\Phi_{\pi(1)},\ldots,\Phi_{\pi(N)}\right). \tag{4.37}$$

[11]We assume that Σ is connected, so that $\pi_1(\Sigma)$ is independent of the chosen basepoint.

We can define a new CFT, known as the permutation orbifold or $X \wr \Omega$, by gauging the action of Ω on $X^{\otimes N}$. That is, we define the orbifold theory

$$X \wr \Omega := X^{\otimes N}/\Omega\,. \tag{4.38}$$

The use of the wreath product symbol $\wr$ to denote permutation orbifolds will be clarified later when we discuss Chern-Simons theories with permutation symmetry.

A special case of a permutation orbifold comes from taking the permutation group Ω to be the full symmetric group S_N. In this case, the permutation orbifold $X \wr S_N$ is called the *symmetric orbifold* and is often denoted by $\mathrm{Sym}^N(X)$. We will mostly focus on symmetric orbifold theories in this paper, but most statements we make generalize in a straightfoward manner to generic permutation orbifolds.

Partition functions

Let Σ be a Riemann surface of genus one. Its cycles are denoted by A and B, and given a point $z \in \Sigma$, we let $A \cdot z$ denote the operation of transporting z along a cycle homotopic to A. Within the permutation orbifold, as with any orbifold, we impose the gauging of the discrete group Ω by allowing the fundamental fields $\mathbf{\Phi}$ to have non-trivial monodromies when transported around non-contractible loops on Σ. That is, given permutations π_A and π_B, we allow the twisted boundary conditions

$$\begin{aligned} \mathbf{\Phi}(A \cdot z) &= \pi_A \cdot \mathbf{\Phi}(z)\,, \\ \mathbf{\Phi}(B \cdot z) &= \pi_B \cdot \mathbf{\Phi}(z)\,. \end{aligned} \tag{4.39}$$

Now, given that $A \cdot B \cdot A^{-1} \cdot B^{-1}$ is a contractible cycle on the torus (see Figure 4.3), we should not pick up a monodromy when traversing it. Thus, we have

$$\mathbf{\Phi}(z) = \mathbf{\Phi}(A \cdot B \cdot A^{-1} \cdot B^{-1} \cdot z) = (\pi_A \pi_B \pi_A^{-1} \pi_B^{-1}) \cdot \mathbf{\Phi}(z)\,, \tag{4.40}$$

which is only consistent if

$$\pi_A \pi_B = \pi_B \pi_A\,, \tag{4.41}$$

i.e. if the permutations π_A and π_B commute. This is precisely the requirement that the permutations π_A, π_B define a group homomorphism $\pi_1(\Sigma) \cong \mathbb{Z} \oplus \mathbb{Z} \to S_N$. Now, from the general theory of orbifolds, we know that in the path integral of $X \wr \Omega$ on Σ we are required to sum over all twisted boundary conditions π_A, π_B which commute. That is, the partition function is given by

$$Z_\Omega(\Sigma) = \frac{1}{|\Omega|} \sum_{[\pi_A, \pi_B]=0} Z_{\pi_A,\pi_B}(\Sigma)\,, \tag{4.42}$$

where $Z_{\pi_A,\pi_B}(\Sigma)$ is the path integral of $X^{\otimes N}$ on Σ with the twisted boundary conditions imposed by π_A, π_B. For a general choice of π_A, π_B the fields are not single valued, they permute amongst themselves as we travel around different cycles of the torus.

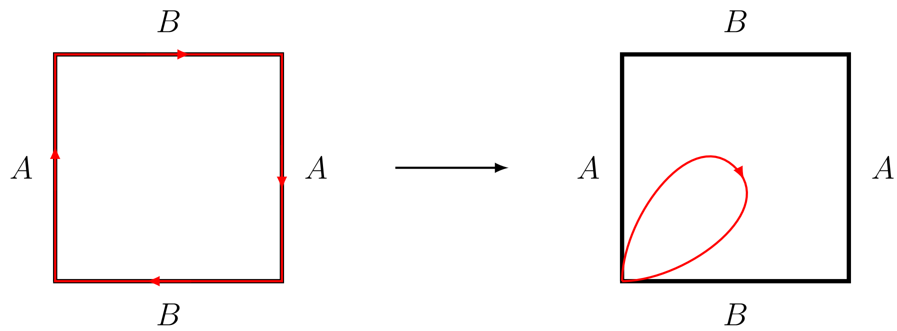


Figure 4.3: The loop $A \cdot B \cdot A^{-1} \cdot B^{-1}$ on a torus is contractible. Thus, twisted boundary conditions π_A, π_B must satisfy $\pi_A \pi_B = \pi_B \pi_A$.

Let us now specialize to the case $\Omega = S_N$. In order to evaluate the individual summands in equation (4.42), we use a standard trick in the theory of orbifolds by considering a covering space $\widetilde{\Sigma}$ on which the fields become single-valued [175]. The space $\widetilde{\Sigma}$ is constructed by taking N copies of Σ, letting the field Φ_i live on the i^{th} copy of Σ, and 'stitching' together the copies of Σ via the twisted boundary conditions π_A, π_B.

The resulting surface $\widetilde{\Sigma}$ is an N-fold covering space of Σ in the topological sense (this process is easy to visualize in the case of a one-dimensional theory on a circle, see Figure 4.4). The partition function with twisted boundary conditions reduces to a partition function of the seed theory X on $\widetilde{\Sigma}$, i.e.

$$Z_{\pi_A,\pi_B}(\Sigma) = Z(\widetilde{\Sigma}) \,, \tag{4.43}$$

where Z denotes the partition function of X. Therefore, we naively write

$$Z_{S_N}(\Sigma) \stackrel{?}{=} \frac{1}{N!} \sum_{\widetilde{\Sigma} \to \Sigma} Z(\widetilde{\Sigma}) \,, \tag{4.44}$$

where $Z(\widetilde{\Sigma})$ is the partition function of X on $\widetilde{\Sigma}$.

The above equation is actually not quite right. This is because not all pairs of boundary conditions (π_A, π_B) give topologically inequivalent covering spaces. Indeed, if we define $(\pi'_A, \pi'_B) = (\pi\pi_A\pi^{-1}, \pi\pi_B\pi^{-1})$, the resulting covering space is the same, since the effect of conjugating by π it simply permutes the sheets of $\widetilde{\Sigma} \to \Sigma$ (the copies of Σ), which is a homeomorphism. If we let

$$\mathcal{O}_{\pi_A,\pi_B} = \left\{ (\pi\pi_A\pi^{-1}, \pi\pi_B\pi^{-1}) | \pi \in S_N \right\} \,, \tag{4.45}$$

where we do not double-count equal pairs of permutations, then each covering space $\widetilde{\Sigma} \to \Sigma$ occurs precisely $|\mathcal{O}_{\pi_A,\pi_B}|$ times in the sum (4.42).[12] We define the 'symmetry factor' of a

[12] The set $\mathcal{O}_{\pi_A,\pi_B}$ is the orbit set of the element $(\pi_A, \pi_B) \in \mathrm{Hom}(\pi_1(\Sigma), S_N)$ under the S_N action which acts as conjugation. Covering spaces are in one-to-one correspondence with the coset $\mathrm{Hom}(\pi_1(\Sigma), S_N)/S_N$ of this action. Readers familiar with mathematical aspects of gauge theory will recognize this coset as the space of principal S_N bundles over Σ, which is just a fancy word for a covering space.

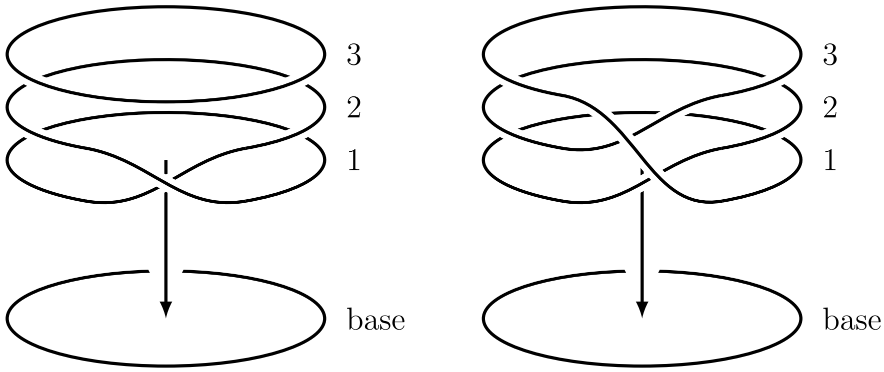


Figure 4.4: Twisted boundary conditions on the circle as 3-fold covering spaces. Left: the fields $\{\Phi_1, \Phi_2, \Phi_3\}$ satisfy twisted boundary conditions $\Phi_1(2\pi) = \Phi_2(0)$, while Φ_3 is single-valued. Right: The fields satisfy boundary conditions $\Phi_1(2\pi) = \Phi_2(0)$, $\Phi_2(2\pi) = \Phi_3(0)$, and $\Phi_3(2\pi) = \Phi_1(0)$.

covering space $\widetilde{\Sigma} \to \Sigma$ to be the quotient $|\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)| = N!/|\mathcal{O}_{\pi_A,\pi_B}|$. Thus,

$$Z_{S_N}(\Sigma) = \sum_{\widetilde{\Sigma} \to \Sigma} \frac{Z(\widetilde{\Sigma})}{|\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)|} \,. \tag{4.46}$$

The factor $|\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)|$ is precisely the degree of the group of *deck transformations*: homeomorphisms of the covering space $\widetilde{\Sigma}$ which preserve the projection $\widetilde{\Sigma} \to \Sigma$.[13]

For a base space which is a torus, it is a topological fact that the covering spaces $\widetilde{\Sigma} \to \Sigma$ considered above are always given by disjoint unions of tori. That is, each covering space we want to consider is given by

$$\widetilde{\Sigma} = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n \to \Sigma \,, \tag{4.47}$$

where Σ_i is a torus with modular parameter τ_i that is not necessarily equal to the initial modular parameter. Since the partition function of a CFT on a disjoint union of spaces is the product of the partition functions, we have, for each covering space $\widetilde{\Sigma} \to \Sigma$,

$$Z(\widetilde{\Sigma}) = \prod_{i=1}^{n} Z(\Sigma_i) \,. \tag{4.48}$$

Thus, in order to calculate the partition function $Z_{S_N}(\Sigma)$ of the symmetric orbifold theory $X \wr S_N$ on a torus Σ, one only needs to know the generic torus partition function $Z(\Sigma)$ for

[13] Since a covering space can be considered a homomorphism $\phi : \pi_1(\Sigma) \to S_N$, we can equivalently define a deck transformation to be an automorphism $\psi : S_N \to S_N$ which leaves ϕ invariant, i.e. for which $\psi \circ \phi = \phi$. Group theoretically, conjugation by elements of S_N defines an action on $\mathrm{Hom}(\pi_1(\Sigma), S_N)$. The group of deck transformations of a covering space $\phi \in \mathrm{Hom}(\pi_1(\Sigma), S_N)$ is the stabilizer $\mathrm{Stab}(\phi)$ under the S_N action. By the orbit-stabilizer theorem, we have $|\mathrm{Stab}(\phi)||\mathcal{O}(\phi)| = |S_N| = N!$, or $N!/|\mathcal{O}(\phi)| = |\mathrm{Stab}(\phi)| = |\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)|$. Note that we do not require Σ to be a torus, and this statement works for any topological space Σ.

the seed theory X – all of the other data is contained in the combinatorics of the covering spaces. This simplification does not occur for partition functions of $X \wr S_N$ on higher-genus surfaces: as we will see later, if Σ has genus g, calculating the partition function $Z_{S_N}(\Sigma)$ requires knowing the partition functions of the seed theory X on surfaces of many different genera.

Example: $N = 2$

For $N = 2$, the above discussion can be made very concrete. The only two permutations in S_2 are the identity e and the two-cycle π. Since S_2 is abelian, all permutations commute among each other, and we can immediately write down the sum (4.42) as

$$Z_2(\Sigma) = \frac{1}{2}\left(Z_{e,e}(\Sigma) + Z_{\pi,e}(\Sigma) + Z_{e,\pi}(\Sigma) + Z_{\pi,\pi}(\Sigma)\right) . \tag{4.49}$$

If we realize the torus Σ as a parallelogram in the complex plane $\mathbb{C}/\{m+n\tau\}$, we can choose the A-cycle to act as $A \cdot z = z + \tau$ and the B-cycle to act as $B \cdot z = z + 1$. Then $Z_{\pi,e}$ is the partition function of fields (Φ_1, Φ_2) with $\Phi_1(z + \tau) = \Phi_2(z)$. This is single valued on the torus obtained by making the A-cycle twice as long, i.e.

$$\mathbb{C}/\{m + 2n\tau\} . \tag{4.50}$$

This is a torus with modular parameter 2τ, and so

$$Z_{\pi,e}(\tau) = Z(2\tau) . \tag{4.51}$$

Similarly,

$$Z_{e,\pi}(\tau) = Z(\tfrac{\tau}{2}) , \quad Z_{\pi,\pi}(\tau) = Z(\tfrac{\tau+1}{2}) . \tag{4.52}$$

All of the above covering tori are depicted in Figure 4.5. The partition function $Z_{e,e}(\tau)$ is just the partition function of $X^{\otimes 2}$ with no twisted boundary conditions, and so

$$Z_{e,e}(\tau) = Z(\tau)^2 . \tag{4.53}$$

Putting the above discussion together, we can write the full $X \wr S_2$ symmetric orbifold partition function (4.49) as

$$Z_2(\tau) = \frac{1}{2}Z(\tau)^2 + \frac{1}{2}Z(\tfrac{\tau}{2}) + \frac{1}{2}Z(2\tau) + \frac{1}{2}Z(\tfrac{\tau+1}{2}) . \tag{4.54}$$

Each term in this sum can be seen as the partition function of the seed theory X evaluated on a covering space $\widetilde{\Sigma} \to \Sigma$, see Figure 4.5. A similar, more complicated discussion can be done for $N = 3$, which requires 18 pairs of commuting permutations in S_3. The end result is

$$\begin{aligned} Z_3(\tau) =& \frac{1}{6}Z(\tau)^3 + \frac{1}{2}Z(\tau)Z(2\tau) + \frac{1}{2}Z(\tau)Z(\tfrac{\tau}{2}) + \frac{1}{2}Z(\tau)Z(\tfrac{\tau+1}{2}) \\ &+ \frac{1}{3}Z(3\tau) + \frac{1}{3}Z(\tfrac{\tau}{3}) + \frac{1}{3}Z(\tfrac{\tau+1}{3}) + \frac{1}{3}Z(\tfrac{\tau+2}{3}) . \end{aligned} \tag{4.55}$$

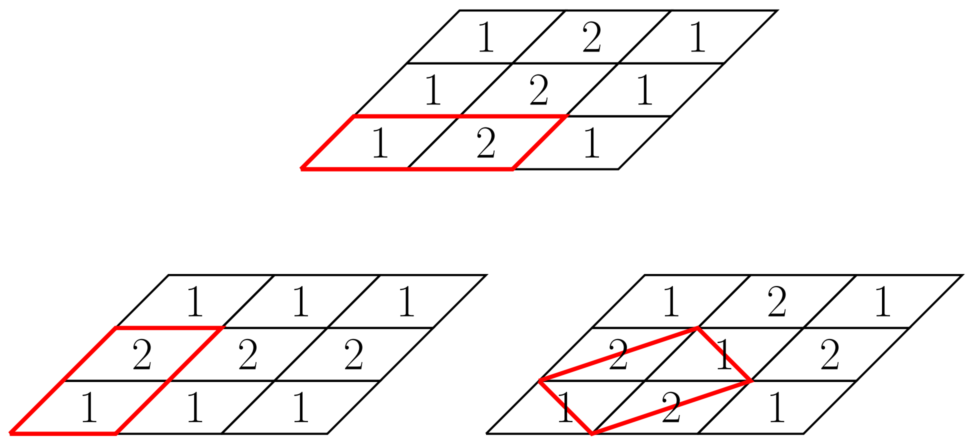


Figure 4.5: The connected covering spaces for the torus $N = 2$ symmetric orbifold. Individual cells represent the base torus (with modular parameter τ), and the numbers label which copy of the seed theory lives on which sheet of the base torus. The permutations π_A, π_B prescribe how to stitch together the copies of the seed theory onto the covering space. The covering space itself is the fundamental domain for which the arrangement of labels $1, 2$ is periodic (shown in red). The fundamental domains in the above examples have modular parameter $\tau/2$, 2τ, and $(\tau + 1)/(1 - \tau) \sim (\tau + 1)/2$, respectively.

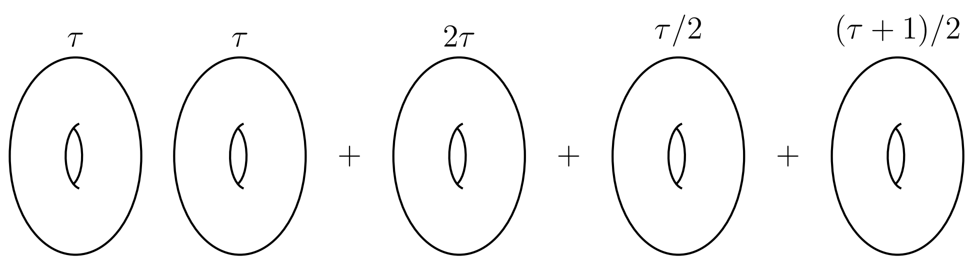


Figure 4.6: All covering space geometries contributing to the S_2 partition function. The Riemann-Hurwitz formula guarantees that these covering spaces are also tori.

General N

A general expression for the S_N symmetric orbifold partition function can be found by working in the grand canonical ensemble

$$\mathrm{Sym}(X) := \bigoplus_{N=0}^{\infty} (X \wr S_N) \,. \tag{4.56}$$

Working with $\mathrm{Sym}(X)$ allows us to work with all symmetric orbifold theories at once. We can keep track of the specific orbifold $X \wr S_N$ by introducing a chemical potential p which keeps track of N. In analogy to second-quantized statistical mechanics, we define the 'grand canonical' partition function $\mathfrak{Z}(p,\tau)$ by

$$\mathfrak{Z}(p,\tau) = \sum_{N=0}^{\infty} p^N Z_N(\tau)\,. \tag{4.57}$$

In the case of the torus, it can be shown [176, 177] that the grand canonical partition function has a simple expression in terms of Hecke operators, namely

$$\mathfrak{Z}(p,\tau) = \exp\left(\sum_{k=1}^{\infty} p^k T_k Z(\tau)\right)\,, \tag{4.58}$$

where the k^{th} Hecke operator is given by

$$T_k Z(\tau) = \frac{1}{k}\sum_{ad=k}\sum_{b=0}^{d-1} Z\left(\frac{a\tau+b}{d}\right)\,. \tag{4.59}$$

The relationship between Hecke operators and permutation orbifolds is that (4.59) sums over all *connected* covering spaces of the original torus with degree k. The integers a, d in the sum indicate how many times the covering space wraps around the A and B cycles, respectively. The integer b then indicates a Dehn twist on the base torus, which is a Dehn twist around the B cycle of angle $2\pi b/d$ on the covering torus. The exponential (4.58) then can be expanded to include all disconnected covering spaces. Taking the degree N coefficient then produces the partition function of the symmetric orbifold $X \wr S_N$, which can be written as

$$Z_N(\tau) = \sum_{\text{paritions of } N}\prod_{k=1}^{N} \frac{1}{N_k!}\left(T_k Z(\tau)\right)^{N_k}\,, \tag{4.60}$$

where partitions of N sums over N_k such that $\sum_{k=1}^{N} kN_k = N$.

Finally, we list an equivalent definition of the Hecke operators in terms of modular transformations. Let M_k denote the space of 2×2 integer matrices with determinant k, and let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ denote the modular group. The coset $\Gamma\backslash M_k$ is given by

$$\Gamma\backslash M_k = \left\{ \begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \middle|\, ad = k\,,\quad b = 0,\ldots,d-1 \right\}\,. \tag{4.61}$$

Thus, we can write

$$T_k Z(\tau) = \frac{1}{k}\sum_{\gamma\in \mathrm{SL}(2,\mathbb{Z})\backslash M_k} Z\left(\gamma\cdot\tau\right)\,, \tag{4.62}$$

where 2×2 matrices are taken to act on τ in the usual way, i.e.

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot\tau = \frac{a\tau+b}{c\tau+d}\,. \tag{4.63}$$

Higher genus

Finally, we mention the situation for permutation CFTs formulated on surfaces Σ with genus $g > 1$. We will focus on the case $\Omega = S_N$.

Recall that the uniformization theorem states that every surface Σ_g with genus $g > 1$ can be expressed as a quotient $\mathbb{H}^2/G$ of the upper-half-plane by a Fuchsian group G.[14] Consider a (connected) covering space $\Sigma_{g'} \to \Sigma_g$ of order N. By the Riemann-Hurwitz formula, the genus g' of $\Sigma_{g'}$ is related to the genus g of Σ by

$$g' = N(g-1) + 1 \,. \tag{4.64}$$

One can classify such covering spaces in the following way: let H be a subgroup of G with index $[H : G] = N$.[15] Then the quotient $\mathbb{H}^2/H$ is a covering space of $\Sigma = \mathbb{H}^2/G$. The covering map is given by mapping the equivalence class $[z]_H = \{h \cdot z | h \in H\}$ to $[z]_G = \{g \cdot z | g \in G\}$, and this map is N-to-1. It can be shown that all connected covering spaces of Σ can be found in the following way, and the covering spaces are determined uniquely by the subgroups H up to conjugation by elements of G.

Using the above construction, it has been shown that the grand canonical partition function $\mathfrak{Z}(p, \Sigma)$ of the grand canonical ensemble (4.56) on a higher-genus Riemann surface $\Sigma_g = \mathbb{H}^2/G$ can be written as [178]

$$\mathfrak{Z}(p, \mathbb{H}^2/G) = \exp\left(\sum_{\substack{H \subset G \\ \text{up to conjugation}}} \frac{p^{[H:G]}}{[H : G]} Z(\mathbb{H}^2/H) \right) , \tag{4.65}$$

where $Z(\mathbb{H}^2/H)$ is the partition function of the seed theory X on the covering surface $\Sigma_{g'} = \mathbb{H}^2/H$. The sum in the exponential with fixed value of $[H : G]$ can be thought of as a higher-genus generalization of the Hecke operators (4.59).

Expanding out (4.65) will generally give terms of the form

$$\prod_{i=1}^{n} Z(\mathbb{H}^2/H_1) \cdots Z(\mathbb{H}^2/H_n) \,, \tag{4.66}$$

where $H_1, \ldots, H_n \subset G$ have degrees $N_i = [H_i : G]$, with some combinatorial factors we don't care about. Isolating the p^N coefficient (i.e. the terms contributing to the S_N partition function) requires

$$\sum_{i=1}^{n} [H_i : G] = \sum_{i=1}^{n} N_i = N \,. \tag{4.67}$$

[14] A Fuchsian group is a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, which acts on the upper-half-plane in the usual way, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \,.$$

[15] The index of a subgroup H of G is the number of left cosets or equivalently the number $|G/H|$ of right cosets of H in the group G.

Since $\mathbb{H}^2/H_i$ defines a Riemann surface $\Sigma_{g_i'}$ of genus $g_i'-1=N_i(g-1)$, we can equivalently write the above product as

$$Z(\Sigma_{g_1'})\cdots Z(\Sigma_{g_n'})\,, \tag{4.68}$$

where the genera g_i' are constrained by the requirements

$$\sum_{i=1}^{n}(g_i'-1)=N(g-1) \quad \text{and} \quad (g_i'-1)|(g-1)\,. \tag{4.69}$$

Thus, calculating the S_N orbifold partition function on a surface of genus g requires knowledge of the seed theory partition functions of surfaces of various genera.

Spin structures

We now generalize the above discussion to CFTs with fermionic degrees of freedom. In this, case, in addition to a surface Σ, one must make a choice of spin structure. A surface of genus g has $2g$ non-contractible cycles, and a choice of spin structure on Σ is a choice of periodic or antiperiodic boundary conditions for fermions on Σ around each cycle.[16] There are 2^{2g} such choices of spin structure, and the path integral of a fermionic CFT on Σ is highly dependent on the choice of spin structure.

Specializing to genus $g=1$, the spin structure is labeled by two half integers α,β, such that a fermion ψ satisfies

$$\psi(A\cdot z)=e^{2\pi i\alpha}\psi(z)\,,\quad \psi(B\cdot z)=e^{2\pi i\beta}\psi(z)\,. \tag{4.70}$$

Explicitly, if we choose A to be the cycle along the τ-direction and B to be the cycle along the 1-direction, we write

$$\psi(z+\tau)=e^{2\pi i\alpha}\psi(z)\,,\quad \psi(z+1)=e^{2\pi i\beta}\psi(z)\,. \tag{4.71}$$

We will denote the partition function of a (fermionic) CFT X with spin structure $(\alpha\,,\beta)$ and conformal structure τ as

$$Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) \quad \text{or} \quad Z_{\vec{e}}\,(\tau)\,, \tag{4.72}$$

where $\vec{e}=(\alpha\,\beta)^T$ is a column vector.

If we consider the symmetric product of X, a formula similar to (4.58) can be given which takes into account spin structures. We now must sum over permutations of the N fermions such that $\psi(A\cdot z)=(-1)^\alpha\pi_A\cdot\psi(z)$ and $\psi(B\cdot z)=(-1)^\beta\pi_B\cdot\psi(z)$. The effect of this is that the spin structure on the covering space can differ from the base space. The end result is given by

$$\mathfrak{Z}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(p,\tau)=\exp\left(\sum_{k=1}^{\infty}p^k\mathcal{T}_kZ\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\right)\,, \tag{4.73}$$

[16] A spin structure can also be defined as a homomorphism $\phi:\pi_1(\Sigma)\to\mathbb{Z}_2$. Since $\mathbb{Z}_2$ is abelian, if Σ is connected this is equivalently a homomorphism $\phi:H_1(\Sigma,\mathbb{Z})\to\mathbb{Z}_2$.

where the fermionic Hecke operator $\mathcal{T}_k$ acts as

$$\mathcal{T}_k Z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau) = \frac{1}{k} \sum_{ad=k} \sum_{b=0}^{d-1} Z \begin{bmatrix} a\alpha + b\beta \\ d\beta \end{bmatrix} \left(\frac{a\tau + b}{d} \right) . \tag{4.74}$$

Here, we understand the parameters of the spin structure to be added modulo 2. The spin structure $(a\alpha + b\beta, d\beta)$ is the spin structure $(\alpha\,, \beta)$ pulled back to the covering torus. Note that if we take $\gamma \in \Gamma\backslash M_k$, then the spin structure on the covering space is written as

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \gamma \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} , \tag{4.75}$$

and so, writing the spin structure as a column vector $\vec{e} = (\alpha\, \beta)^T$, we can write the Hecke operator compactly as

$$\mathcal{T}_k Z_{\vec{e}} = \frac{1}{k} \sum_{\gamma \in \Gamma\backslash M_k} Z_{\gamma\cdot\vec{e}}\,(\gamma \cdot \tau)\,. \tag{4.76}$$

4.2 Averaging the symmetric orbifold

As mentioned in the introduction, the fundamental objects of study in this paper are the permutation orbifolds $\mathbb{T}^D \wr \Omega$ of Narain CFTs. In this section, we compute the averaged torus partition functions of these orbifolds over the moduli space $\mathcal{M}_D$ of $\mathbb{T}^D$ targets. We focus primarily on the symmetric orbifold case $\Omega = S_N$. We begin by computing the averaged partition function of the $\mathbb{T}^D \wr S_2$ orbifold, which is simple enough to write down explicitly, but still complicated enough to exhibit general features which persist at larger N. We then turn to the case of $N > 2$. Finally, we briefly comment on the generalization to higher-genus surfaces.

Generalities

Before jumping straight into averaging symmetric orbifold partition functions, let us briefly comment on the general structure one might expect to see. Given a surface Σ, we know that the symmetric orbifold partition function of some theory X on Σ is expressed by summing the partition function of the seed theory X on all possible N-fold covering spaces $\widetilde{\Sigma} \to \Sigma$, weighted by an appropriate automorphism factor. We again quote the result:

$$Z_N(\Sigma) = \sum_{\widetilde{\Sigma}\to\Sigma} \frac{Z(\widetilde{\Sigma})}{|\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)|} , \tag{4.77}$$

where the sum is restricted to N-sheeted covering surfaces.

Now, let us assume that X is not a single theory, but an element of some moduli space $m \in \mathcal{M}$. Furthermore, we assume that, as in the case of U(1)-gravity, there is a sense in which averaging the partition function $Z(\Sigma, m)$ over the moduli space $\mathcal{M}$ admits a holographic interpretation in terms of 'filling in' the manifold Σ. That is, we assume

$$\int_{\mathcal{M}} \mathrm{d}\mu(m)\, Z(\Sigma, m) = \sum_{\partial M = \Sigma} Z_{\text{grav}}(M)\,, \tag{4.78}$$

where we sum over three-dimensional 'bulk manifolds' M with boundary Σ, weighted by some sort of gravitational path integral Z_{grav} evaluated on M.[17] With this assumption in mind, we can automatically compute the average of the symmetric orbifold partition function $Z_N(\Sigma, m)$ over the moduli space $\mathcal{M}$, and the result will take the schematic form

$$\int_{\mathcal{M}} \mathrm{d}\mu(m)\, Z_N(\Sigma, m) = \int_{\mathcal{M}} \mathrm{d}\mu(m) \sum_{\widetilde{\Sigma}\to\Sigma} \frac{Z(\widetilde{\Sigma}, m)}{|\mathrm{Aut}(\widetilde{\Sigma}\to\Sigma)|} = \sum_{\widetilde{\Sigma}\to\Sigma} \sum_{\partial\widetilde{M}=\widetilde{\Sigma}} \frac{Z_{\text{grav}}(\widetilde{M})}{|\mathrm{Aut}(\widetilde{\Sigma}\to\Sigma)|}\,. \tag{4.79}$$

That is, to average the symmetric orbifold partition function, we first sum over covering spaces $\widetilde{\Sigma}\to\Sigma$, and then sum over bulk manifolds $\widetilde{M}$ with boundary $\widetilde{\Sigma}$.

In principle, the above procedure is straightforward (although most likely analytically intractable). However, it leaves much to be desired in terms of the standard holographic dictionary. Specifically, each term in equation (4.79) is calculated on a bulk manifold which has boundary $\widetilde{\Sigma}$, while holographically one expects gravitational quantities dual to CFT data to be computed on manifolds with boundary Σ. Moreover, the automorphism factors $|\mathrm{Aut}(\widetilde{\Sigma}\to\Sigma)|$ don't have an immediate interpretation in terms of a gravitational path integral on $\widetilde{M}$.

A natural solution to both of the above problems arises if we think of $\widetilde{M}$ not as a true gravitational bulk manifold, but as a covering space of a gravitational bulk manifold M with boundary Σ. As we will discuss in more detail below, if the bulk is described by a gauge theory which has a discrete S_N factor, then the path integral of the gravitational theory on the bulk M is naturally calculated by passing to a covering space $\widetilde{M}$. In direct analogy to the symmetric orbifold, we would expect the three-dimensional partition function on M of a theory with S_N gauge symmetry to take the schematic form

$$\sum_{\widetilde{M}\to M} \frac{Z_{\text{grav}}(\widetilde{M})}{|\mathrm{Aut}(\widetilde{M}\to M)|}\,, \tag{4.80}$$

where the sum is over all covering spaces $\widetilde{M}\to M$ of degree N. The full gravitational path integral would then be given by summing over all appropriate bulk geometries M with boundary Σ. In order for this procedure to reproduce the averaged symmetric orbifold partition function (4.79), we would demand

$$\sum_{\widetilde{M}\to M} \sum_{\partial M=\Sigma} \frac{Z_{\text{grav}}(\widetilde{M})}{|\mathrm{Aut}(\widetilde{M}\to M)|} \overset{!}{=} \sum_{\widetilde{\Sigma}\to\Sigma} \sum_{\partial\widetilde{M}=\widetilde{\Sigma}} \frac{Z_{\text{grav}}(\widetilde{M})}{|\mathrm{Aut}(\widetilde{\Sigma}\to\Sigma)|}\,. \tag{4.81}$$

[17] We allow the notion of a 'bulk manifold' to be rather vague. In the case of U(1) gravity, this role is played by Lagrangian sublattices $\Gamma_0 \subset H_1(\Sigma, \mathbb{Z})$.

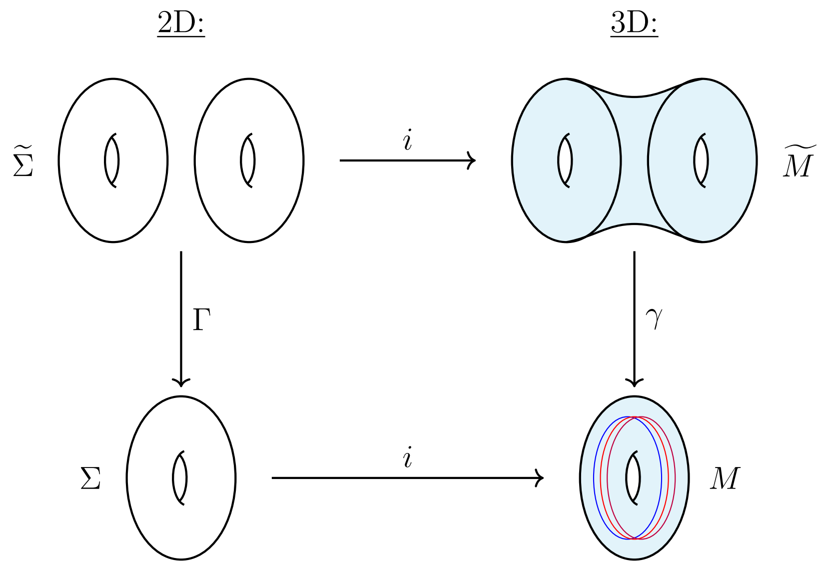


Figure 4.7: We construct a bulk manifold $\widetilde{M}$ by 'filling in' the covering space $\widetilde{\Sigma}$. If $\widetilde{M}$ is a (branched) covering space of another bulk manifold M with boundary Σ, then M is interpreted as a bulk geometry in our theory of quantum gravity, and the branch loci of the covering $\gamma : \widetilde{M} \to M$ are interpreted as 'vortices' for the bulk gauge theory. For bulks $\widetilde{M}$ which do not cover a bulk M with boundary Σ, a semiclassical interpretation of that contribution to the gravitational path integral is less clear.

Unfortunately, such a strict prescription has little chance of working. This is because there are many ways to fill a covering surface $\widetilde{\Sigma}$ with a three-manifold $\widetilde{M}$ which itself does not cover any three manifold M whose boundary is Σ. In other words, given a surface Σ, a covering $\widetilde{\Sigma}$, and a bulk manifold $\widetilde{M}$ bounded by $\widetilde{\Sigma}$, the bottom-right corner of the diagram in Figure 4.7 does not always exist. However, the opposite statement is always true: given a surface Σ, a three-manifold M with $\partial M = \Sigma$, and a covering space $\widetilde{M}$ of M, the boundary of $\widetilde{M}$ is always a covering space of $\widetilde{\Sigma}$. This means that the left-hand side of (4.81) is strictly contained in the right-hand side, and so at the very least some information of the averaged symmetric orbifold CFT can be recovered from a bulk theory with gauge group S_N.

We will return to the bulk/boundary duality later in Section 4.3, where we explicitly calculate the two sides of (4.81). We will find that, although the two sides do not match, one is contained within the other and we find a match between the *connected parts* of both sides in the case of a single torus boundary. We will refer to the terms that match as "semiclassical" contributions, since they can be obtained through the usual rules of the bulk gravity path integral. We also postulate that the "extra" terms appearing in the right-hand-side of (4.81) represent quantum gravity corrections that go beyond the usual semiclassical picture, since it seems difficult to interpret these extra terms as standard bulk geometries.

Example: $N = 2$

Let us make some of the general statements made above more explicit by considering the case of the $\mathbb{T}^D \wr S_2$ orbifold. As discussed in Section 4.1, the partition function of the orbifold $X \wr S_2$ on a torus of modular parameter τ is given by

$$Z_{\mathbb{T}^D \wr S_2}(\tau) = \frac{1}{2} Z_{\mathbb{T}^D}(\tau)^2 + \frac{1}{2} Z_{\mathbb{T}^D}(2\tau) + \frac{1}{2} Z_{\mathbb{T}^D}(\tfrac{\tau}{2}) + \frac{1}{2} Z_{\mathbb{T}^D}(\tfrac{\tau+1}{2}), \tag{4.82}$$

where $Z(\tau)$ is the partition function of X. We will split up the partition function into a "connected" and a "disconnected" part

$$Z_{\mathbb{T}^D \wr S_2}(\tau) = Z_{\mathbb{T}^D \wr S_2,\, \text{conn.}}(\tau) + Z_{\mathbb{T}^D \wr S_2,\, \text{dis.}}(\tau), \tag{4.83}$$

where connectedness refers to whether the covering space is connected or not. That is, the connected part only contains terms with a single partition function Z. If X is a Narain theory $X = \mathbb{T}^D$, then we can write the $\mathbb{T}^D \wr S_2$ partition function as

$$Z_{\mathbb{T}^D \wr S_2}(m, \tau) = \frac{1}{2} \frac{\Theta(m, 2\tau)}{|\eta(2\tau)|^{2D}} + \frac{1}{2} \frac{\Theta(m, \frac{\tau}{2})}{|\eta(\frac{\tau}{2})|^{2D}} + \frac{1}{2} \frac{\Theta(m, \frac{\tau+1}{2})}{|\eta(\frac{\tau+1}{2})|^{2D}} + \frac{1}{2} \left(\frac{\Theta(m, \tau)}{|\eta(\tau)|^{2D}} \right)^2, \tag{4.84}$$

where $\Theta(m, \tau)$ is the Narain theta function evaluated at the point $m \in \mathcal{M}_D$ of the Narain moduli space. The "connected" and "disconnected" parts are given by

$$\begin{aligned} Z_{\mathbb{T}^D \wr S_2,\, \text{conn.}}(m, \tau) &= \frac{1}{2} \frac{\Theta(m, 2\tau)}{|\eta(2\tau)|^{2D}} + \frac{1}{2} \frac{\Theta(m, \frac{\tau}{2})}{|\eta(\frac{\tau}{2})|^{2D}} + \frac{1}{2} \frac{\Theta(m, \frac{\tau+1}{2})}{|\eta(\frac{\tau+1}{2})|^{2D}}, \\ Z_{\mathbb{T}^D \wr S_2,\, \text{dis.}}(m, \tau) &= \frac{1}{2} \left(\frac{\Theta(m, \tau)}{|\eta(\tau)|^{2D}} \right)^2. \end{aligned} \tag{4.85}$$

These correspond to contributions to the symmetric orbifold partition function from double-covers of the torus which are connected and disconnected, respectively. Using the Siegel-Weil formula (4.14) we have

$$\int_{\mathcal{M}_D} \mathrm{d}\mu \, \frac{\Theta(m, \tau)}{|\eta(\tau)|^{2D}} = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \frac{1}{|\eta(\gamma \cdot \tau)|^{2D}}, \tag{4.86}$$

and we can immediately write down the average of the connected piece:

$$\langle Z_{\mathbb{T}^D \wr S_2,\, \text{conn.}}(m, \tau) \rangle = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \left(\frac{1}{|\eta(\gamma \cdot 2\tau)|^{2D}} + \frac{1}{|\eta(\gamma \cdot (\frac{\tau}{2}))|^{2D}} + \frac{1}{|\eta(\gamma \cdot (\frac{\tau+1}{2}))\,|^{2D}} \right). \tag{4.87}$$

The average for the disconnected part was worked out in Section 4.1, and requires the Siegel-Weil formula for disconnected surfaces (4.20). The period matrix Ω of the disjoint union $\Sigma \sqcup \Sigma$ is given by $\Omega = \mathrm{diag}(\tau, \tau)$, and following the rules of [46], we can write the average of

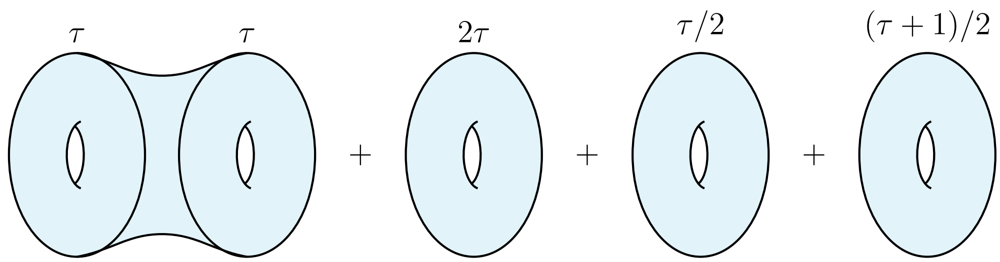


Figure 4.8: The "bulk geometries" contributing to the averaged $\mathbb{T}^D \wr S_2$ partition function. The geometry on the left is understood to be a connected three-manifold whose boundary is the disjoint union of two tori.

the disconnected component as a sum over Lagrangian sublattices Γ_0 of the total homology lattice $H_1(\Sigma \sqcup \Sigma, \mathbb{Z}) \cong H_1(\Sigma, \mathbb{Z}) \oplus H_1(\Sigma, \mathbb{Z})$, namely

$$\langle Z_{\mathbb{T}^D \wr S_2,\, \text{dis.}}(m,\tau)\rangle = \frac{1}{2\,\mathrm{Im}(\tau)^{2D}|\eta(\tau)|^{4D}} \sum_{\Gamma_0 \subset H_1(\Sigma\sqcup\Sigma,\mathbb{Z})} \left(\det \mathrm{Im}(\Omega_{\Gamma_0})\right)^{D/2}, \tag{4.88}$$

where Ω_{Γ_0} is the period matrix Ω evaluated on the sublattice Γ_0 as explained in Section 4.1. Each Lagrangian sublattice Γ_0 is to be associated with a bulk geometry with boundary $\Sigma \sqcup \Sigma$ such that the boundary cycles Γ_0 are contractible in the bulk. Choices of Γ_0 correspond to either "Wormhole" geometries, or completely disconnected bulk geometries.

In both the connected and disconnected case, the averaged partition function has an interpretation as a sum over geometries shown in Figure 4.8. For the connected piece, $\langle Z_{\text{conn}}\rangle$ is a sum over $\mathrm{U}(1)^{2D} \times \mathrm{U}(1)^{2D}$ Chern-Simons partition functions on geometries with boundary tori of modular parameter 2τ, $\frac{\tau}{2}$, or $\frac{\tau+1}{2}$. The disconnected piece $\langle Z_{\text{dis}}\rangle$ is written as a sum over both disconnected handlebody geometries and "wormhole" geometries whose boundary is the disjoint union $\Sigma \sqcup \Sigma$. From a holographic perspective this sum over geometries is unsatisfactory for two reasons:

- Each geometry comes with a symmetry factor of $\frac{1}{2}$, which, from the point of view of a bulk Chern-Simons theory has no reason to be there.

- Holographically, we expect the bulk dual of a CFT on a surface Σ to consist of a gravitational theory on bulk manifold $\mathcal{M}$ whose boundary is Σ. However, none of the geometries mentioned above have boundary Σ, but rather their boundaries are double covers of Σ.

Thus, we need a way to interpret the contributions of (4.87) and (4.88) in terms of geometries whose boundary is given by Σ. We now explain under what conditions this can be done.

We will first make some formal statements before giving a more intuitive picture towards the end of the paragraph. Let us denote one of the bulk geometries contributing to (4.87) and (4.88) by $\widetilde{\mathcal{M}}$ and its boundary by $\widetilde{\Sigma}$. Suppose that $\widetilde{\mathcal{M}}$ has symmetry group G such that if we consider the quotient geometry $\widetilde{\mathcal{M}}/G$ we obtain a manifold with desired asymptotic boundary $\partial\widetilde{\mathcal{M}}/G = \Sigma$. Then we can holographically interpret the geometry $\widetilde{\mathcal{M}}/G$ as contributing to the averaged partition function through the usual rules of AdS/CFT. It turns out that we can give such an interpretation to certain terms in the sum, as we now explain for the case of $N = 2$. This is achieved by recalling that the boundaries of all of the above geometries are covering spaces of the original torus Σ. There exists a 2-to-1 map

$$\widetilde{\Sigma} \to \Sigma \tag{4.89}$$

which is the covering map of Σ by $\widetilde{\Sigma}$. This map can be thought of as the quotient map of a $\mathbb{Z}_2$ automorphism (deck transformation) $\iota : \widetilde{\Sigma} \to \widetilde{\Sigma}$ which projects $\widetilde{\Sigma}$ to the quotient space $\widetilde{\Sigma}/\iota \cong \Sigma$.[18] One might hope that the bulk geometries $\widetilde{\mathcal{M}}$ with boundary $\widetilde{\Sigma}$ also inherit a $\mathbb{Z}_2$ automorphism that restricts to ι on the boundary. For many of the bulk geometries this is indeed the case, such as with the solid tori which fill the connected covering spaces, shown in Figure 4.8. For these geometries we can define a (possibly singular) bulk manifold $\mathcal{M} = \widetilde{\mathcal{M}}/\mathbb{Z}_2$ which has boundary Σ, and thus provides a promising candidate for the bulk manifold which should appear in the sum over geometries contributing to the averaged $\mathbb{T}^D \wr S_2$ partition function. However, as we will explore more in detail later, many of the bulk geometries appearing in the averaged $\mathbb{T}^D \wr S_2$ do not inherit a $\mathbb{Z}_2$ automorphism compatible with the automorphism of the boundary $\widetilde{\Sigma}$. Specifically, the geometries in Figure 4.8 with connected boundary all inherit the $\mathbb{Z}_2$ automorphism of the boundary, and so they can indeed be thought of as covering spaces of well-defined three-manifolds. However, for the geometries whose boundaries are disconnected, the $\mathbb{Z}_2$ symmetry of the boundary can be 'broken' by specific details of the bulk.

Let us also explain this in a more pictorial way. Consider the concrete case of the potential wormhole contribution depicted in the leftmost panel of Figure 4.8. The corresponding boundary contribution, the leftmost panel of figure 4.6, is invariant under swapping of the two tori. This symmetry of the two coverings leads to the factor $\frac{1}{2}$ as explained in Section 4.1. Analogously, we can also consider the second to left panel in Figure 4.6. The upper and lower half of the torus correspond to the two covering sheets. Again, we observe a symmetry by exhanging these two sheets. Similar statements can be made about the remaining two covering surfaces pictured in 4.6. In order to define a theory on a potential quotient three manifold, we must be able to extend these symmetries of the covering surfaces into the bulk. It is then clear that for the 3 connected contributions pictured in Figure 4.8 this can be done. However, for the wormhole contribution it is also clear that this may only be possible if the two boundary tori are imbued with the same complex structure.

Geometries for which this is not possible can be avoided by either only considering the average of the connected contribution to the $\mathbb{T}^D \wr S_2$ partition function, for which such a $\mathbb{Z}_2$ automorphism always exists, or by trying to find a suitable generalization of what is meant

[18]The S_2 orbifold on the torus is a special case since all covering spaces $\widetilde{\Sigma} \to \Sigma$ satisfy $\Sigma \cong \widetilde{\Sigma}/\mathbb{Z}_2$. In general the relationship between covering and base spaces is not a simple quotient.

by a 'bulk geometry with boundary Σ'. We will return to this point in Section 4.3 in the context of the bulk gauge theory description.

General N

For general N the theory is given by the $\mathbb{T}^D \wr S_N$ orbifold. The partition function for fixed N can be extracted from the grand canonical partition function in equation (4.58) by extracting the term proportional to p^N in the series expansion. For general N the form of the partition function is quite complicated and is given by (4.60),

$$Z_{\mathbb{T}^D \wr S_N}(m,\tau) = \sum_{\text{parititons of } N} \prod_{k=1}^{N} \frac{1}{N_k!} \left(T_k Z_{\mathbb{T}^D}(m,\tau)\right)^{N_k}, \tag{4.90}$$

where again partitions of N sum over N_k such that $\sum_{k=1}^{N} kN_k = N$. We can immediately obtain the ensemble average by applying the Siegel-Weil formula (4.20) to the above expression

$$\langle Z_{\mathbb{T}^D \wr S_N}(m,\tau)\rangle = \sum_{\text{parititons of } N} \left\langle \prod_{k=1}^{N} \frac{1}{N_k!} \left(T_k Z_{\mathbb{T}^D}(m,\tau)\right)^{N_k} \right\rangle. \tag{4.91}$$

However, this expression is quite formal since it is a complicated sum over various wormhole and non-wormhole geometries with up to N asymptotic boundaries. It is useful to split the partition function into a sum over "connected" and "disconnected" contributions

$$Z_{\mathbb{T}^D \wr S_N}(m,\tau) = Z_{\text{conn}}(m,\tau) + Z_{\text{dis}}(m,\tau). \tag{4.92}$$

The "connected" part of the partition function only includes contributions from connected covering spaces, and it has a simple expression since it can be extracted from (4.58) by keeping contributions proportional to p^N and a single copy of Z, giving

$$Z_{\text{conn}}(m,\tau) = T_N Z_{\mathbb{T}^D}(m,\tau) = \frac{1}{N} \sum_{ad=k} \sum_{b=0}^{d-1} Z_{\mathbb{T}^D}\left(m, \frac{a\tau+b}{d}\right). \tag{4.93}$$

The ensemble average over the connected part is simple since there are no wormhole contributions. We find

$$\langle Z_{\text{conn}}(m,\tau)\rangle = \frac{1}{N} \sum_{ad=k} \sum_{b=0}^{d-1} \left\langle Z_{\mathbb{T}^D}\left(m, \frac{a\tau+b}{d}\right)\right\rangle, \tag{4.94}$$

$$= \frac{1}{N} \sum_{ad=k} \sum_{b=0}^{d-1} \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\left(\frac{a\tau+b}{d}\right))|^{2D}}. \tag{4.95}$$

where the quantity on the right is simply the torus average given in (4.16). We will later see that we can give this term a holographic interpretation as a Chern-Simons path integral with vortices, similar to the case of $N = 2$.

The "disconnected" contribution $Z_{\text{dis}}(m,\tau)$ is more difficult to write out explicitly, but it would contain sums over multiple copies of partition functions with different modular parameters. As an explicit example, from (4.55) we see that for $N=3$ the "disconnected" piece would be given by

$$Z_{\text{dis}}(\tau) = \frac{1}{6}Z_{\mathbb{T}^D}(\tau)^3 + \frac{1}{2}Z_{\mathbb{T}^D}(\tau)Z_{\mathbb{T}^D}(2\tau) + \frac{1}{2}Z_{\mathbb{T}^D}(\tau)Z_{\mathbb{T}^D}(\tfrac{\tau}{2}) + \frac{1}{2}Z_{\mathbb{T}^D}(\tau)Z_{\mathbb{T}^D}(\tfrac{\tau+1}{2}). \tag{4.96}$$

We can again apply the Siegel-Weil formula for disconnected surfaces (4.20) to perform the average over the "disconnected" partition function. This provides a bulk interpretation for the disconnected piece as a sum over wormhole geometries with up to N asymptotic boundary tori. Similar to the discussion for $N=2$, we are unable to give a holographic interpretation to the disconnected piece as a sum over bulk geometries with a single asymptotic boundary torus of modular parameter τ.

Averaging at higher genus

Finally, we make some brief comments on how to generalize the above discussion to symmetric orbifold partition functions on higher genus surfaces, only focusing on the connected component of the partition function. In Section 4.1 we mentioned that the symmetric orbifold partition function on higher genus surface $\Sigma_g = \mathbb{H}^2/G$ can be neatly packaged in the grand canonical ensemble as a sum over subgroups $H \subset G$ where G is the Fuchsian group acting on the upper-half-plane. Specifically,

$$\mathfrak{Z}(p, \mathbb{H}^2/G) = \exp\left(\sum_{\substack{H\subset G \\ \text{up to conjugation}}} \frac{p^{[H:G]}}{[H:G]} Z(\mathbb{H}^2/H) \right). \tag{4.97}$$

The connected component of the $X \wr S_N$ orbifold can then be easily extracted by isolating the p^N coefficient in the exponential, namely

$$Z_{N,\text{conn}}(\mathbb{H}^2/G) = \frac{1}{N} \sum_{\substack{H\subset G,\ [H:G]=N \\ \text{up to conjugation}}} Z(\mathbb{H}^2/H)\,. \tag{4.98}$$

This sum is precisely what one would expect: a sum over all connected covering spaces $\mathbb{H}^2/H$ of the surface $\Sigma \cong \mathbb{H}^2/G$. An equivalent yet slightly less algebraic notation would be to write the connected contribution as

$$Z_{N,\text{conn}}(\Sigma_g) = \frac{1}{N} \sum_{\widetilde{\Sigma}\to\Sigma_g} Z(\widetilde{\Sigma})\,, \tag{4.99}$$

where the sum is over all genus $g' = N(g-1)+1$ surfaces which cover Σ_g.

We can now consider the ensemble average of this quantity by specifying $X = \mathbb{T}^D$. By the dictionary of [46], we know that the average of the partition function $Z(\widetilde{\Sigma})$ is given as a sum over all handlebodies bounded by $\widetilde{\Sigma}$ (or equivalently as a sum over all Lagrangian sublattices $\Gamma_0 \subset H_1(\widetilde{\Sigma}, \mathbb{Z})$) weighted by the Chern-Simons action on that handlebody. Explicitly,

$$\langle Z_{N,\text{conn}}(m, \Sigma_g) \rangle = \frac{1}{N} \sum_{\widetilde{\Sigma} \to \Sigma_g} \sum_{\partial \widetilde{M} = \widetilde{\Sigma}} Z_{\text{CS}}(\widetilde{M}) \,. \tag{4.100}$$

While this expression is short and conceptually simple, its practical computation is very difficult. Specifically, computing the Chern-Simons path integral on the handlebody $\widetilde{M}$ requires knowledge of the period matrix $\widetilde{\Omega}$ of $\widetilde{\Sigma}$, which is difficult to generically compute [179]. That said, the averaged partition function (4.100) is in principle well-defined, up to divergence issues in the resulting Eisenstein series.[19]

4.3 The bulk theory

The torus theory $\mathbb{T}^D$ possesses affine $\mathrm{U}(1)^D_L \times \mathrm{U}(1)^D_R$ currents, whose zero-modes generate a global symmetry algebra. These currents are given by the left- and right-moving derivatives of the worldsheet scalars, namely

$$J^m = \partial X^m \,, \quad \overline{J}^m = \overline{\partial} X^m \,, \quad m = 1, \ldots, D \,. \tag{4.102}$$

By the standard holographic dictionary, these currents are dual to gauge fields in the bulk,

$$J^m \to A^m \,, \quad \overline{J}^m \to B^m \,, \tag{4.103}$$

where A transforms under the left $\mathrm{U}(1)^D$ and B transforms under the right $\mathrm{U}(1)^D$. In the Narain ensemble, this identification of global symmetries on the boundary with gauge symmetries in the bulk is all that needs to be done, and we can write the bulk action as a $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ Chern-Simons theory, namely

$$S = \int_{\mathcal{M}} (A \wedge \mathrm{d}A - B \wedge \mathrm{d}B) \,. \tag{4.104}$$

Here, we implicitly perform a sum over the $\mathrm{U}(1)^D$ indices m, see (4.30). The rules of Narain averaging then also tell us that the path integral must be summed over bulk geometries $\mathcal{M}$.

[19] The $\mathbb{T}^D$ averaged partition function on a surface of genus g' diverges when $D \leq g' + 1$. Since each term in (4.100) is evaluated on a surface of genus $g' = N(g-1)+1$, where g is the genus of Σ_g, we require

$$D > N(g-1) + 2 \tag{4.101}$$

for convergence.

For the permutation orbifold $\mathbb{T}^D \wr \Omega$, things are a bit different. We now have N copies of each current, and so in the bulk we expect N copies of the gauge fields A, B. Thus, we postulate the following action:

$$S = \sum_{i=1}^{N} \int_{\mathcal{M}} \left(A_{(i)} \wedge \mathrm{d}A_{(i)} - B_{(i)} \wedge \mathrm{d}B_{(i)} \right) . \tag{4.105}$$

Where again the $U(1)^D$ indices are implicit. Since the boundary theory contains N copies of the original $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ symmetry the bulk theory has N copies of the $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ gauge symmetry

$$A_{(i)} \to A_{(i)} + \mathrm{d}\lambda^{A}_{(i)} , \quad B_{(i)} \to B_{(i)} + \mathrm{d}\lambda^{B}_{(i)} . \tag{4.106}$$

We must now understand how quotienting by Ω modifies the bulk theory. Before we quotient the boundary theory by Ω we have a permutation symmetry given by $J^m \to J^{\pi(m)}$, $\overline{J}^m \to \overline{J}^{\pi(m)}$. This immediately maps into a symmetry of the bulk Chern-Simons theory $A_{(i)} \to A_{(\pi(i))}$, $B_{(i)} \to B_{(\pi(i))}$. After gauging the permutation symmetry Ω, it is natural to expect the bulk theory to also carry this gauge symmetry. This promotes the permutation symmetry of Chern-Simons fields to a gauge symmetry

$$A_{(i)} \to A_{(\pi(i))} , \quad B_{(i)} \to B_{(\pi(i))} . \tag{4.107}$$

The full gauge group of the bulk dual of the $\mathbb{T}^D \wr \Omega$ permutation orbifold should then be the group generated by combinations of the $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ transformations (4.106) and the permutations (4.107). The group generated by these two transformations is a semidirect product of $(\mathrm{U}(1)^D \times \mathrm{U}(1)^D)^N$ with the permutation group Ω, and in the mathematical literature is denoted by the *wreath product*[20]

$$\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr \Omega . \tag{4.108}$$

Armed with the above discussion, a natural duality to propose would be:

$$\begin{gathered}\text{Narain-ensemble of } \mathbb{T}^D \wr \Omega \text{ orbifolds} \\ \Longleftrightarrow \\ \mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr \Omega \text{ Chern-Simons coupled to topological gravity}\end{gathered} \tag{4.109}$$

Although this gauge group is not discrete, it contains a discrete factor[21] of Ω. Thus, the above theory will exhibit behaviors universal to all discrete gauge theories. One of these is the

[20] The wreath product is just mathematical notation meaning the gauge group which is generated by composing (4.106) and (4.107). The relationship between the wreath product symbol $\wr$ and permutation orbifolds $X \wr \Omega$ is that if a CFT X has symmetry group G, then the permutation orbifold $X \wr \Omega$ has symmetry group $G \wr \Omega$.

[21] Indeed, $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr \Omega$ is equivalent to $(\mathrm{U}(1)^D \times \mathrm{U}(1)^D)^N \times \Omega$ as a topological space (but not as a group).

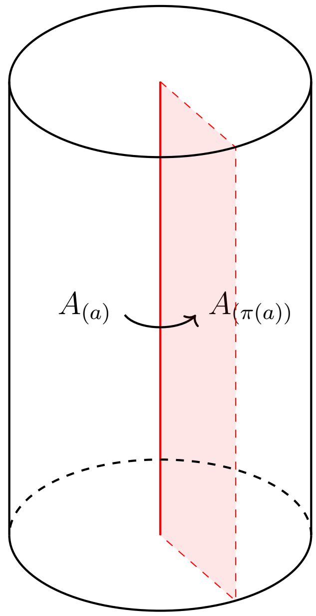


Figure 4.9: The effect of including a vortex associated to a permutation π in the bulk dual of $\mathbb{T}^D \wr \Omega$.

existence of *twist operators* in the bulk, which in three-dimensional discrete gauge theories take the form of one-dimensional vortices [180, 181]. We now explain the bulk partition function of this theory and how gauging by Ω introduces new features such as vortices.

The Chern-Simons path integral with action (4.105) and gauge group $G \wr S_N$, where G is some Lie group, on a bulk manifold M is given by

$$Z_{G \wr S_N} = \frac{1}{N!} \sum_{\text{bundles}} \int \mathcal{D}A\mathcal{D}B \; e^{-S_{\text{CS}}[A,B]} \,. \tag{4.110}$$

The factor of $N!$ comes from taking permutations of the fields to be gauge equivalent. We take this into account by integrating over all possible fields A, B but divide by $N!$ not to overcount. In the path integral we are to integrate over gauge connections on all $G \wr S_N$ bundles over M. Typically, the total gauge group is a Lie group and so there exists only a single trivial bundle. However, if the gauge group contains a discrete factor, such as S_N, then there are additional bundles that must be included in the path integral [182, 183]. This is one of the new features when dealing with gauge theories with discrete groups. As we will see slightly later, the effect of including the sum over bundles is that we must include gauge field configurations where the fields A, B are twisted as we travel around the non-contractible cycles in M.

We will find that to match to the boundary ensemble average we will need to include gauge field configurations that also have non-trivial monodromies around the contractible cycles in M, but these are not reproduced by the sum over bundles above. However, there is a way to include such configurations by including codimension two "vortices" in the path integral. We thus claim that the correct bulk path integral of interest is given by

$$Z_{\text{Bulk}} \equiv \frac{1}{N!} \sum_{M} \sum_{\substack{\text{bundles,} \\ \text{vortices}}} \int \mathcal{D}A\mathcal{D}B \; e^{-S_{\text{CS}}[A,B]} \mathcal{V} \,, \tag{4.111}$$

where we have modified the path integral by inserting an additional vortex operator $\mathcal{V}$. We specify what the summation over vortices means slightly later. A vortex is a line operator embedded into the manifold M which enforces that the gauge fields A, B pick up certain monodromies as they travel around the vortex. The sum over vortices needs to be put in by hand, and we take the philosophy that the summation over vortex configurations should be chosen to match the boundary answer. In the bulk path integral we also sum over a set of bulk manifolds M with given asymptotic boundary structure, and the specific choice of bulk manifolds M will be clarified later.

In the remainder of this section we go into additional details regarding the summation over bundles and vortices. We then explain the bulk path integral calculation in the case of $N = 2$ with an asymptotic torus boundary, and we clarify which terms in the boundary ensemble average are reproduced by the bulk calculation. We discuss the case of larger N and higher genus boundaries, and we comment on bulk non-handlebody contributions.

Topological theories with finite gauge group

Before fully exploring our proposed bulk theory, let us first make some general remarks about topological field theories with finite gauge group [183], see [184] for an expository introduction.

A field theory with a discrete gauge group Ω is formally very similar to 2D orbifold CFTs. Indeed, an orbifold CFT can be formulated as a 2D theory with discrete gauge group. In three-dimensions, we can similarly consider field theories whose gauge groups are discrete. Let M be some three-manifold. If the field content of the theory is Φ, then gauging the group Ω amounts to identifying

$$\Phi(p) \sim g \cdot \Phi(p) \tag{4.112}$$

at all points p. Again, just as in the orbifold CFT case, this leads to interesting behavior when M is not simply-connected. In this case, we can imagine transporting Φ around some non-contractible loop γ based at p. It is perfectly fine if Φ itself is not single-valued, but rather picks up a monodromy $\phi(\gamma) \in \Omega$ upon being transported around γ. That is,

$$\Phi(\gamma \cdot p) = \phi(\gamma) \cdot \Phi(p) \,, \tag{4.113}$$

where $\Phi(\gamma \cdot p)$ is shorthand for the value of Φ after being transported along γ. Since Φ and $\phi(\gamma) \cdot \Phi$ are physically equivalent, the above should be a perfectly allowed configuration in the path integral.

Similarly to the discussion in Section 4.1, the assignment of a twisted boundary condition to each loop γ must be consistent with the process of concatenation of loops. That is, the twisted boundary conditions must form a homomorphism $\phi : \pi_1(M) \to \Omega$. Moreover, for any $g \in \Omega$, the homomorphisms $\phi(\gamma)$ and $g^{-1}\phi(\gamma)g$ represent the same physical field configuration, since it is related to ϕ simply by a global field redefinition $\Phi \to g \cdot \Phi$. That is, contributions to the path integral are defined only up to conjugation by elements of Ω (this is the Ω group action on the map ϕ). Mathematically, the set of such twisted boundary conditions are in one-to-one correspondence with the representation variety

$$\mathrm{Hom}(\pi_1(M), \Omega)/\Omega\,, \tag{4.114}$$

which is equivalently the moduli space of flat Ω-bundles on M.[22]

We can also consider a fully topological theory whose only content is the discrete gauge group Ω. The field content of the theory is trivial, and the only nontrivial observables are the partition functions on M. It is given as a sum over all homomorphisms $\phi : \pi_1(M) \to \Omega$ (i.e. twisted boundary conditions) weighted by the automorphism group $\mathrm{Aut}(\phi)$ of that homomorphism. That is, the partition function of the TQFT with gauge group Ω is simply:

$$Z_\Omega(M) = \sum_{\substack{\phi:\pi_1(M)\to\Omega \\ \text{up to conjugation}}} \frac{1}{|\mathrm{Aut}(\phi)|}\,. \tag{4.115}$$

As a trivial example, if $M \cong \mathrm{S}^3$ is the three-sphere, then $\pi_1(\mathrm{S}^3)$ is trivial, hence the only choice for ϕ is $\phi(\mathrm{id}_{\pi_1(\mathrm{S}^3)}) = \mathrm{id}_\Omega$ and we simply have

$$Z_\Omega(\mathrm{S}^3) = \frac{1}{|\Omega|}\,. \tag{4.116}$$

Now, we are specifically interested in the case of $\Omega = S_N$. As we discussed in Section 4.1, a set of twisted boundary conditions $\phi : \pi_1(M) \to S_N$ can be identified with a covering space $\widetilde{M} \to M$, and the group of automorphisms becomes the group of deck transformations $\mathrm{Aut}(\widetilde{M} \to M)$. Thus, the S_N topological gauge theory on M computes a weighted sum over all covering spaces of M of degree N, weighted by the order of the group of deck transformations. Explicitly,

$$Z_{S_N}(M) = \sum_{\substack{\widetilde{M}\to M \\ \text{degree } N}} \frac{1}{|\mathrm{Aut}(\widetilde{M} \to M)|}\,. \tag{4.117}$$

The above construction can be rather nicely extended to the case of not a finite gauge group, but a gauge group $G \wr S_N$ where G is some Lie group and $\wr$ is the wreath product defined above. In this case, the path integral is over all gauge connections on $G \wr S_N$ bundles over M. Now, topologically, a $G \wr S_N$ bundle $E \to M$ is equivalent to a G bundle $E \to \widetilde{M}$ over the S_N bundle $\widetilde{M} \to M$ (i.e. a degree N covering space of M).[23] Thus we can trade

[22]Since Ω is discrete, all Ω-bundles are flat.
[23]This statement is already implicit in [185].

the path integral of a $G \wr S_N$ gauge theory for a sum over path integrals of a G gauge theory on the covering spaces of M. The final result is

$$Z_{G\wr S_N}(M) = \sum_{\substack{\widetilde{M}\to M \\ \text{degree } N}} \frac{Z_G(\widetilde{M})}{|\mathrm{Aut}(\widetilde{M}\to M)|} \,. \tag{4.118}$$

Another way to view the above construction is to take N copies of a gauge theory on G, and then to gauge the S_N permutation symmetry of the resulting product theory. This is equivalent to defining a gauge theory with gauge group $G \wr S_N$, and the arguments for computing path integrals in symmetric orbifold theories from Section 4.1 carries over, and we arrive at (4.118), in direct analogy to the logic that allowed us to derive equation (4.46).[24]

To summarize the above discussion, the summation over non-trivial bundles appearing in equation (4.111) induces a summation over non-trivial boundary conditions on the gauge fields around the non-contractible cycles in M. This can be rewritten as a sum over covering spaces of M given by equation (4.118). Note that this does not induce twisted boundary conditions around the contractible cycle. To include such configurations we need to include bulk vortices, which we now discuss.

Vortices

Another property of gauge theories with discrete gauge groups is the presence of so-called vortices. As explained above, vortices are codimension-2 objects which impose twisted boundary conditions on gauge fields transported around them. These are the 3-dimensional analogues of twist fields in orbifold CFTs, and as we will see in Section 4.4 are holographically dual to twist fields.

Informally, a vortex (often also called a *monodromy defect* [188] or a *Gukov-Witten operator* [189]) is a codimension-2 extended object in a gauge theory which has the property that being transported around it induces a monodromy of the gauge group Ω. This is analogous to the Aharonov-Bohm effect, in which the wavefunction of a charged particle picks up a phase upon being transported around a solenoid. In the case of a gauge theory with a discrete gauge group, a field Φ picks up a monodromy $\Phi \to g \cdot \Phi$ upon being transported around the vortex.

Formally, a vortex is a codimension-2 sublocus L of a three-manifold M which carries charge $[g]$, where $[g]$ is some conjugacy class of elements of the discrete gauge group Ω.[25] We

[24] In fact, given any TQFT Z, not just gauge theories, one can construct a 'symmetric product' theory by formally averaging Z over covering spaces, see [186]. Such TQFTs are useful, for example, in computing generalizations of Hurwitz numbers, see also [187].

[25] We assign a charge to $\mathcal{V}$ in terms of conjugacy classes of Ω because otherwise the vortex $\mathcal{V}$ would not be a gauge-invariant object. This is because gauge transformations would act on a charge g as hgh^{-1}. This generically changes the group element g, but leaves it in the conjugacy class. This is spelled out more concretely in Section 4.4 in the context of orbifold twist fields. For abelian groups like those considered in [160] this problem does not arise.

can formally write a vortex operator $\mathcal{V}_{[g],L}$ of charge $[g]$ associated to a sublocus L by the operator

$$\mathcal{V}_{[g],L} = \sum_{\sigma\in[g]} \mathcal{V}_{\sigma,L}\,, \tag{4.119}$$

where we sum over permutations σ in the conjugacy class $[g]$, which makes the operator gauge invariant. In discrete gauge theories the vortex operator cannot typically be represented in terms of fundamental fields of the theory. It's action is implemented by imposing twisted boundary conditions on the fields as they are transported around the vortex: transporting the fields A_I, B_I around $\mathcal{V}_{\sigma,L}$ enforces the twisted boundary condition $A_I \to A_{\sigma(I)}$ and $B_I \to B_{\sigma(I)}$.

We can combine vortex operators with the non-trivial sum over bundles to implement twisted boundary conditions around multiple cycles. Suppose our bulk manifold is a torus with boundary cycles a, b, and that a vortex operator imposes twisted boundary conditions around the a cycle given by π_a, while a non-trivial bundle imposes twisted boundary conditions π_b around the b cycle. There will only exist non-trivial gauge field configurations if $[\pi_a, \pi_b] = 0$. That is, going around the same set of cycles in different orders gives the same boundary conditions for the fields. The end result is that when summing over bundles and vortices the only contributions come from combinations with consistent boundary conditions. For a general bulk manifold M this condition is formalized as follows.

The partition function of a discrete gauge theory on M in the presence of a vortex L of charge $[g]$ is defined similarly to the partition function on M alone. Let $\ell \in \pi_1(M \setminus L)$ be a generator of the fundamental group of $M \setminus L$ which winds once around L. Then the path integral in the presence of the vortex is defined by summing over all homomorphisms $\phi : \pi_1(M \setminus L) \to \Omega$ (up to conjugation) such that $\phi(\ell)$ lies in the conjugacy class $[g]$. Specifically,

$$Z_\Omega(M; L, [g]) = \sum_{\substack{\phi:\pi_1(M\setminus L)\to\Omega \\ \text{up to conjugation} \\ \phi(\ell)\in[g]}} \frac{1}{|\mathrm{Aut}(\phi)|}\,. \tag{4.120}$$

Again, we can specialize to the case where Ω is the permutation group S_N. Conjugacy classes of S_N are labeled uniquely by cycle-types of permutations. Let us denote the charge associated to L with $[\pi]$ and let us assume that $[\pi]$ is the conjugacy class of permutations with cycle-type $w_1, \ldots, w_k$, where

$$w_1 + \cdots + w_k = N\,. \tag{4.121}$$

Then a homomorphism $\phi : \pi_1(M \setminus L) \to S_N$ with $\phi(\ell) \in [\pi]$ defines a covering space of M which is *branched* over L with branching structure given by the cycle type $w_1, \ldots, w_k$. In the language of Figure 4.9, this branched covering space $\widetilde{M} \to M$ is the N-fold cover such that the gauge fields are single-valued. Thus,

$$Z_{S_N}(M; L, [\pi]) = \sum_{\substack{\widetilde{M}\to M \\ \text{branched over } L}} \frac{1}{|\mathrm{Aut}(\widetilde{M} \to M)|}\,, \tag{4.122}$$

where the branching over L has branching structure $w_1, \ldots, w_k$.

Just as in the case without vortices, we can also enrich the above partition function by considering a topological gauge theory with group $G \wr S_N$, where G is any Lie group. The partition function then becomes a sum over G partition functions on branched covers of M over L, i.e.

$$Z_{G\wr S_N}(M; L, [\pi]) = \sum_{\substack{\widetilde{M}\to M \\ \text{branched over } L}} \frac{Z_G(\widetilde{M})}{|\mathrm{Aut}(\widetilde{M} \to M)|} \,. \tag{4.123}$$

Note that this is essentially the three-dimensional generalization of the covering space construction for calculating correlation functions of twist fields in two-dimensional CFTs [158, 175].

The upshot of the above construction is that the introduction of a vortex into a topological $G \wr S_N$ gauge theory amounts to computing partition functions on the branched cover $\widetilde{M}$ of M over the branching locus L. This is completely analogous to the case in 2D orbifold CFTs, where correlation functions of twist fields are computed by passing to the branched cover, branched at the points where the twist fields are inserted [175]. In fact, as we will see, this analogy is made precise in the holographic setting, and we will find that vortices in $G \wr S_N$ gauge theory which intersect the boundary ∂M are dual to twist fields in the symmetric orbifold CFT. We delay this discussion to Section 4.4.

Below we will perform some calculations of the bulk $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_N$ Chern-Simons theory. We will start by taking the simple case of a torus boundary and $N = 2$, where we can be quite explicit. We then move on to the case of a torus boundary but for generic N. Finally, we will make comments about the calculation of bulk partition functions on 3-manifolds with higher-genus boundaries.

Example: $N = 2$

Let us now turn to calculating the $G \wr S_N$ Chern-Simons partition functions on bulk manifolds where $G = \mathrm{U}(1)^D \times \mathrm{U}(1)^D$. For now, let us work with the simple case $N = 2$. We fix an asymptotic boundary Σ which is a torus of modular parameter τ. As discussed above, we can reduce the problem of calculating $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_2$ partition functions to the task of computing $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ partition functions on degree 2 covering spaces $\widetilde{M}$ of M.

Let us start by letting M be a handlebody, i.e. a solid torus. Then $\pi_1(M) \cong \mathbb{Z}$, and there are precisely two covering spaces for a given M: the covering space which is simply two disconnected copies of M, and the handlebody whose asymptotic boundary is a torus of modular parameter 2τ. We know that the Chern-Simons partition function on a handlebody with contractible spatial cycle and modular parameter τ is simply

$$Z_G(M) = \frac{1}{|\eta(\tau)|^{2D}} \,, \tag{4.124}$$

and so we have

$$Z_{G\wr S_2}(M) = \frac{1}{2}\frac{1}{|\eta(\tau)|^{4D}} + \frac{1}{2}\frac{1}{|\eta(2\tau)|^{2D}}\,, \tag{4.125}$$

where the factors of two come from the automorphism factors of the covering spaces.

Now, we can also introduce a nontrivial vortex. Let L run along the non-contractible cycle of M. The only nontrivial conjugacy class of S_2 is $[(1\,2)]$, and there are precisely two covering spaces of M branched over L with that structure: a handlebody with modular parameter $\tau/2$ and a handlebody with modular parameter $(\tau+1)/2$. Thus, the the partition function of the $G \wr S_2$ Chern-Simons theory on M with vortex L is given by

$$Z_{G\wr S_2}(M;L) = \frac{1}{2}\frac{1}{\left|\eta(\frac{\tau}{2})\right|^{2D}} + \frac{1}{2}\frac{1}{\left|\eta(\frac{\tau+1}{2})\right|^{2D}}\,. \tag{4.126}$$

In a theory of quantum gravity it is natural to sum over bulk manifolds with fixed asymptotic boundary, which in the case of the handlebody M is given by a sum over modular images of the boundary torus. The natural partition function after coupling $G \wr S_2$ Chern-Simons theory to topological gravity is then given by

$$Z_{\text{Bulk}} = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})}\left(\underbrace{\frac{1}{|\eta(\gamma\cdot\tau)|^{4D}}}_{\text{disconnected}} + \frac{1}{|\eta(2\,\gamma\cdot\tau)|^{2D}} + \underbrace{\frac{1}{\left|\eta(\frac{\gamma\cdot\tau}{2})\right|^{2D}} + \frac{1}{\left|\eta(\frac{\gamma\cdot\tau+1}{2})\right|^{2D}}}_{\text{vortex}}\right)\,. \tag{4.127}$$

In the above we have identified the term associated with a disconnected covering space of the torus, as well as the contributions arising from including the vortex. The last three terms represent covering spaces $\widetilde{M}$ which are connected, the last two of which are branched over the vortex L. All of the covering spaces, both with and without vortex, are shown in Figure 4.10.

Comparing to the symmetric orbifold

We would like to compare the above bulk calculation to the Narain-averaged symmetric orbifold result, which we recall takes the form

$$\begin{aligned}\langle Z_{\mathbb{T}^D\wr S_2}(m,\tau)\rangle = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})}&\left(\frac{1}{|\eta(\gamma\cdot(2\tau))|^{2D}} + \frac{1}{\left|\eta(\gamma\cdot(\frac{\tau}{2}))\right|^{2D}} + \frac{1}{\left|\eta(\gamma\cdot(\frac{\tau+1}{2}))\right|^{2D}}\right)\\ &+ \frac{1}{2}\left\langle Z_{\mathbb{T}^D}(\tau,m)^2\right\rangle\,,\end{aligned} \tag{4.128}$$

where the second line is the disconnected part of the partition function. Comparing the connected parts of (4.127) and (4.128), we see that the modular parameters in the sum do

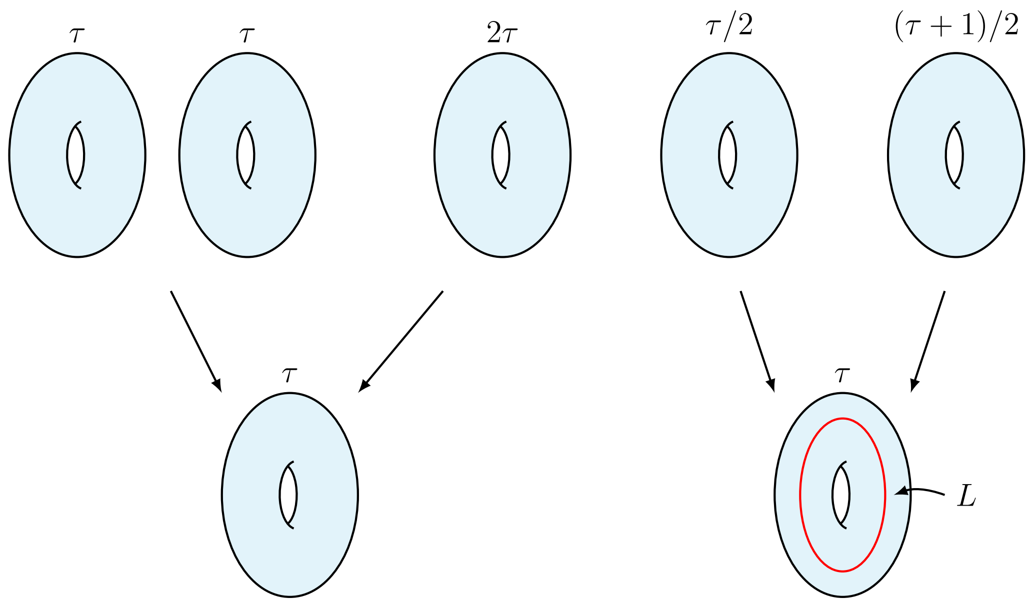


Figure 4.10: The four double covering spaces of a solid torus M without a vortex (on the left) and with a vortex (on the right). The connected covering spaces are also solid tori with modified modular parameters, while the disconnected covering space is simply $M \sqcup M$.

not quite match. However, it is an algebraic fact that these two sums actually coincide, i.e.

$$\begin{aligned}\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})}&\left(\frac{1}{|\eta(2\,\gamma\cdot\tau)|^{2D}}+\frac{1}{\left|\eta(\frac{\gamma\cdot\tau}{2})\right|^{2D}}+\frac{1}{\left|\eta(\frac{\gamma\cdot\tau+1}{2})\right|^{2D}}\right)\\ &=\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})}\left(\frac{1}{|\eta(\gamma\cdot(2\tau))|^{2D}}+\frac{1}{\left|\eta(\gamma\cdot(\frac{\tau}{2}))\right|^{2D}}+\frac{1}{\left|\eta(\gamma\cdot(\frac{\tau+1}{2}))\right|^{2D}}\right),\end{aligned}\tag{4.129}$$

see, for example, Theorem 6.9 and 6.10 of [190]. Thus, the connected part of the $\mathrm{U}(1)^D\times\mathrm{U}(1)^D\wr S_2$ Chern-Simons theory coupled to 3D gravity precisely reproduces the Narain average of the connected part of the symmetric orbifold theory $\mathbb{T}^D\wr S_2$.

The disconnected part

Now that we have shown that the connected parts of the Narain-averaged symmetric orbifold and topological gravity partition functions agree, let us move on to the disconnected part. The disconnected part of the symmetric orbifold partition function is given by

$$\left\langle Z_{\mathbb{T}^D\wr S_2,\,\mathrm{dis.}}(\tau)\right\rangle=\frac{1}{2}\left\langle Z_{\mathbb{T}^D}(m,\tau)^2\right\rangle=\frac{1}{2|\eta(\tau)|^{4D}\,\mathrm{Im}(\tau)^D}\sum_{\Gamma_0}\left(\det\mathrm{Im}\,\Omega_{\Gamma_0}\right)^{D/2},\tag{4.130}$$

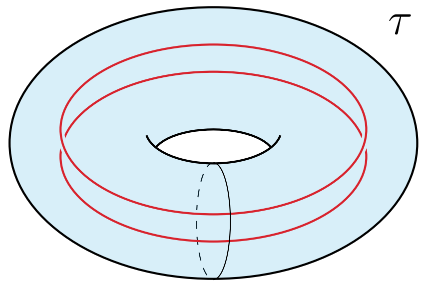


Figure 4.11: A vortex configuration for $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_2$ Chern-Simons theory on a solid torus. The double-cover of this geometry is topologically $\Sigma \times I$.

where

$$\Omega = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} \tag{4.131}$$

is the period matrix of the double cover $\Sigma \sqcup \Sigma$ of the boundary torus Σ, and the sum is over Lagrangian sublattices of $H_1(\Sigma \sqcup \Sigma, \mathbb{Z})$ (see Section 4.1). Intuitively, the sum is over bulk manifolds $\widetilde{M}$ with boundary $\Sigma \sqcup \Sigma$ such that the sublattice Γ_0 is contractible in $\widetilde{M}$. On the other hand, the disconnected piece of the bulk $G \wr S_2$ partition function (4.127) is

$$Z_{\text{Bulk, dis.}} = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \frac{1}{|\eta(\gamma \cdot \tau)|^{4D}} = \frac{1}{2|\eta(\tau)|^{4D} \, \mathrm{Im}(\tau)^D} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma \cdot \tau)^D \,, \tag{4.132}$$

where we have used the fact that $|\eta(\tau)|^4 \, \mathrm{Im}(\tau)$ is modular invariant.

Clearly, (4.130) and (4.132) are not equal. However, the sum in (4.130) actually contains the full sum (4.132). Let $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ be the A-cycles of the two boundaries, while $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ are their B-cycles. Then the Lagrangian sublattices of the form

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}} \left(\gamma(\mathcal{A}^{(1)}), \gamma(\mathcal{A}^{(2)}) \right) \,, \tag{4.133}$$

where $\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})$ is a modular transformation which acts on the homology cycles of Σ in the usual way, contribute

$$(\det \mathrm{Im} \, \Omega_{\Gamma_0})^{D/2} = \mathrm{Im}(\gamma \cdot \tau)^D \,, \tag{4.134}$$

and thus reproduce the elements in the sum (4.132). This makes sense, given that sublattices (4.133) correspond to manifolds $\widetilde{M}$ which are the disjoint union $M \sqcup M$ of two handlebodies of modular parameter $\gamma \cdot \tau$, which are precisely the covering spaces that appeared in the disconnected part of the Chern-Simons partition function. Thus, although (4.130) is not precisely reproduced by (4.132), we have the inclusion

$$\left\langle Z_{\mathbb{T}^D \wr S_2, \text{dis.}}(\tau) \right\rangle \supset Z_{\text{Bulk, dis.}} \,, \tag{4.135}$$

where by $\supset$ we mean that the sum on the left-hand-side contains all elements of the sum on the right-hand-side.

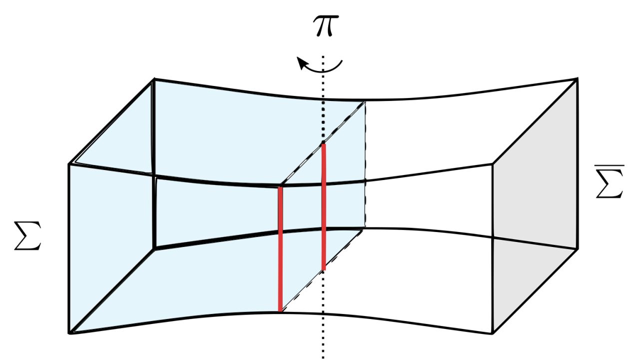


Figure 4.12: The three-manifold $\widetilde{M} = \Sigma \times I$ has an orientation-preserving involution given by rotating the geometry around it's center axis by angle π. The fixed points of this involution are the two circles shown in red, which run through the bulk. The quotient $\widetilde{M}/\mathbb{Z}_2$ (light blue) is a solid torus with two vortices running through the non-contractible cycle.

So what about the other geometries contributing to (4.130) which aren't reproduced by the Chern-Simons calculation? It turns out that at least the $T^2 \times I$ wormhole can be recovered from the Chern-Simons theory if we include more complicated vortex configurations.[26] For example, (4.130) contains the Lagrangian sublattice

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}}\left(\mathcal{A}^{(1)} + \mathcal{A}^{(2)}, \mathcal{B}^{(1)} - \mathcal{B}^{(2)}\right) , \tag{4.136}$$

which geometrically corresponds to a bulk manifold $\widetilde{M} \cong \Sigma \times I$. This manifold can actually be included in the Chern-Simons calculation if we include a two vortex configuration as shown in Figure 4.11. The two vortices in the figure individually act on the fields by the swap $[(1\,2)]$. The branched covering space of this vortex configuration has two boundaries, since one does not pick up a monodromy upon being transported along a cycle at the boundary, and it is not difficult to see that the topology of the covering space is indeed $\Sigma \times I$ (see Figure 4.12). One can also come up with stranger covering spaces which are not topologically $\Sigma \times I$ by, for example, applying Dehn twists to the vortex in Figure 4.11.

So now the natural question is: are there contributions to (4.130) which cannot be recovered by a sufficiently complicated vortex configuration on the bulk Chern-Simons theory? As we will show, the answer is actually yes. Both the averaged symmetric orbifold and the Chern-Simons theory are summing over geometries in two different ways. The symmetric orbifold theory is summing over covering spaces $\widetilde{\Sigma}$ of the boundary manifold Σ, and then summing over fillings (Lagrangian sublattices) of $\widetilde{M}$. The Chern-Simons theory is summing fillings (Lagrangian sublattices) of M and (branched) covering spaces $\widetilde{M}$ of M. Every contribution to the Chern-Simons theory, so long as $\widetilde{M}$ is specified by a Lagrangian sublattice of its boundary, computes a contribution to the symmetric orbifold partition function, since $\widetilde{\Sigma} := \partial\widetilde{M}$ is always a covering space of Σ. However, given a covering space $\widetilde{\Sigma}$ of Σ and a

[26]Strictly speaking, the Chern-Simons calculation on the $T^2 \times I$ wormhole has not yet been carried out and matched to the expected boundary answer.

bulk three-manifold $\widetilde{M}$ specified by a Lagrangian sublattice of $\widetilde{\Sigma}$, it is not always true that $\widetilde{M}$ is a covering space (branched or otherwise) of a three-manifold M with boundary Σ. Put pictorially, the diagram

$$\begin{tikzcd} \widetilde{\Sigma} \arrow[r, hook, "\widetilde{i}"] \arrow[d, "\Gamma"] & \widetilde{M} \\ \Sigma & \end{tikzcd} \tag{4.137}$$

does not always have a completion of the form

$$\begin{tikzcd} \widetilde{\Sigma} \arrow[r, hook, "\widetilde{i}"] \arrow[d, "\Gamma"] & \widetilde{M} \arrow[d, "\Gamma"] \\ \Sigma \arrow[r, hook, "i"] & M \end{tikzcd} \tag{4.138}$$

where $\Gamma : \widetilde{M} \to M$ is a (branched) covering map.

In order for $\widetilde{M}$ to be a covering space of a three-manifold M whose boundary is Σ, we need the Lagrangian sublattice Γ_0 to be 'compatible' with the covering space structure of $\widetilde{\Sigma}$. The covering space $\Gamma : \widetilde{\Sigma} \to \Sigma$ comes equipped with a group of automorphisms (deck transformations), which are self-homeomorphisms ϕ of $\widetilde{\Sigma}$ such that $\Gamma(\phi(p)) = \phi(p)$ for all $p \in \widetilde{\Sigma}$. This group $\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)$ can be thought of as the set of symmetries of the covering space $\widetilde{\Sigma}$. Now, we specify a bulk manifold $\widetilde{M}$ by picking a Lagrangian sublattice $\Gamma_0 \subset H_1(\widetilde{\Sigma}, \mathbb{Z})$, and it is clear that $\widetilde{M}$ can only inherit the symmetries of $\widetilde{\Sigma}$ if the group $\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)$ leaves the Lagrangian sublattice Γ_0 invariant.[27]

A natural condition for $\widetilde{M} \to M$ to be a covering space respecting the structure of $\widetilde{\Sigma} \to \Sigma$ is for the Lagrangian sublattice Γ_0 to be invariant under the action of the deck transformations $\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)$.[28] We emphasize, however, that this is not a sufficient condition, and that there could be Lagrangian sublattices Γ_0 which are invariant under the group $\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)$, but for which the desired covering space $\widetilde{M} \to M$ does not exist.

Let us now return to the S_2 symmetric orbifold example. The sum in equation (4.130) is over all Lagrangian sublattices Γ_0 of $H_1(\Sigma \sqcup \Sigma, \mathbb{Z})$. The group of deck transformations of the covering map $\Sigma \sqcup \Sigma \to \Sigma$ is simply $\mathbb{Z}_2$, generated by swapping the two copies of the tori.

[27] An element $\phi \in \mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)$ has a natural action $\phi_* : H_1(\widetilde{\Sigma}, \mathbb{Z}) \to H_1(\widetilde{\Sigma}, \mathbb{Z})$ given by simply pushing-forward one-cycles with ϕ.

[28] In the case that $\widetilde{\Sigma}$ is a regular covering of Σ and $\widetilde{M} \to M$ is not branched, this is straightfoward. A regular covering $\widetilde{\Sigma} \to \Sigma$ has precisely the structure of a principal $G = \mathrm{Aut}(\widetilde{\Sigma} \to \Sigma)$ bundle. The diagram 4.138 requires that $\widetilde{M} \to M$ is also a principal G bundle, i.e. the deck transformations of $\widetilde{M}$ should be those of $\widetilde{\Sigma}$. Put another way, if $\widetilde{M}$ does not admit an action of the deck transformations G, then there is an obstruction for a base space M to exist. We suspect that a similar logic exists if $\widetilde{\Sigma}$ is not regular and $\widetilde{M} \to M$ is branched. We thank Ivano Basile for pointing this out to us.

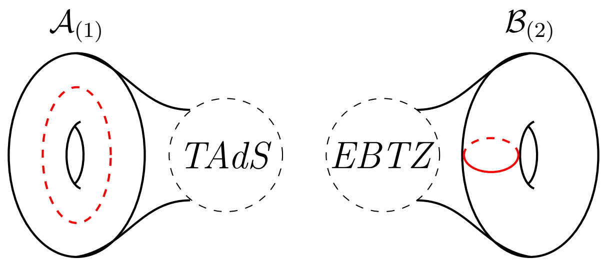


Figure 4.13: The geometry corresponding to the choice of Lagrangian sublattice (4.140) is a disjoint union of thermal AdS_3 and the BTZ black hole. Such a geometry does not have an interpretation when we have a single bulk manifold M, but it can be included in the gravitational path integral if we allow different gauge fields A_I to live on independent bulk manifolds M_I with the "same" asymptotic boundary.

In terms of the homology, this acts as:

$$\phi_*(\mathcal{A}^{(1)}) = \mathcal{A}^{(2)}\,, \quad \phi_*(\mathcal{B}^{(1)}) = \mathcal{B}^{(2)}\,. \tag{4.139}$$

According to the discussion above, only Lagrangian sublattices which are left invariant under the action of ϕ_* have a chance of being computed by a Chern-Simons calculation on a bulk manifold M. As an example of a Lagrangian sublattice which does not work, take

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}}\left(\mathcal{A}^{(1)}, \mathcal{B}^{(2)}\right)\,. \tag{4.140}$$

This sublattice corresponds to a three-manifold which is a disjoint union of thermal AdS_3 and the Euclidean BTZ black hole. Of course, there is no three-manifold with single torus boundary whose 2-fold cover is a disjoint union of thermal AdS_3 and Euclidean BTZ, and so this Lagrangian sublattice has no chance of being reproduced by a Chern-Simons calculation. Indeed, Γ_0 is not invariant under the deck transformation ϕ_*, as

$$\phi_*(\Gamma_0) = \mathrm{Span}_{\mathbb{Z}}\left(\mathcal{A}^{(2)}, \mathcal{B}^{(1)}\right) \neq \Gamma_0\,. \tag{4.141}$$

As another example, we can take the sublattice

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}}\left(\mathcal{A}^{(1)} + \gamma(\mathcal{A}^{(2)}), \mathcal{B}^{(1)} - \gamma(\mathcal{B}^{(2)})\right)\,, \tag{4.142}$$

for some modular transformation γ. This sublattice corresponds to a wormhole with topology $\Sigma \times I$ for which one boundary has modular parameter τ and the other has modular parameter $\gamma(\tau)$. The deck transformation ϕ_* only leaves Γ_0 invariant if the modular parameter is trivial, i.e. $\gamma = \mathrm{id}$, for which we simply recover the double-vortex configuration in Figure 4.11.

While the average of the disconnected component of the symmetric orbifold partition function contains bulk geometries which do not inherit the $\mathbb{Z}_2$ automorphism present in the

2D theory, we note that such bulk geometries come in pairs. For example, the two Lagrangian sublattices

$$\Gamma_0^{(1)} = \mathrm{Span}_{\mathbb{Z}}\left(\gamma(\mathcal{A}^{(1)}), \rho(\mathcal{A}^{(2)})\right) , \quad \Gamma_0^{(2)} = \mathrm{Span}_{\mathbb{Z}}\left(\rho(\mathcal{A}^{(1)}), \gamma(\mathcal{A}^{(2)})\right) \tag{4.143}$$

transform into one another under the $\mathbb{Z}_2$ automorphism swapping the two boundary tori. More generally, geometries contributing to (4.130) either transform as singlets under the $\mathbb{Z}_2$ automorphism, or as doublets. Sublattices Γ_0 which transform as singlets correspond to bulk geometries M which inherit the boundary automorphism, and thus can be realized as the double cover of a manifold $M/\mathbb{Z}_2$, which we think of as a bulk geometry with bulk vortices.

It is tempting to also think of pairs $(\Gamma_0^{(1)}, \Gamma_0^{(2)})$ of sublattices which transform as $\mathbb{Z}_2$ doublets as also corresponding, in some abstract sense, to the 'double cover' of a generalized bulk geometry (or 'microgeometry', borrowing the terminology of [65]). Such a generalized notion of a bulk geometry is not entirely far-fetched, and has been considered before in the context of pure 3D gravity (see the discussion in Section 4.2 of [174], and also Section 3.1 of [191]). In that case, holomorphic factorization of the CFT dual to pure gravity required the introduction of geometries for which the left- and right-moving boundary gravitons lived on separate bulk geometries with the same boundary. Analogously, it is possible that the theory of quantum gravity dual to the Narain-averaged symmetric orbifold includes, in its sum over 'geometries', contributions for which separate copies of the Chern-Simons gauge fields probe different classical bulks with the same boundary.

General N

Now that we have seen the details of how the Narain-averaged symmetric orbifold and $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_N$ Chern-Simons theory are related for $N = 2$, let us turn our attention to generic N. We find it convenient to work in the grand canonical ensemble. We define the grand canonical partition function of the symmetric orbifold to be

$$\mathfrak{Z}(\tau, p) = \sum_{N=0}^{\infty} p^N Z_{\mathbb{T}^D \wr S_N}(\tau) , \tag{4.144}$$

which, as we have seen, admits a nice expression in terms of Hecke operators

$$\mathfrak{Z}(\tau, p) = \exp\left(\sum_{n=1}^{\infty} p^n T_n Z_{\mathbb{T}^D}(\tau)\right) . \tag{4.145}$$

We can also define the grand canonical ensemble of the bulk Chern-Simons theory by specifying a solid torus M with boundary modular parameter τ, as well as a vortex locus L, which we keep implicit, which runs along the non-contractible cycle of M

$$\mathfrak{Z}_{\mathrm{CS}}(M, p) = \sum_{N=0}^{\infty} p^N Z_{G \wr S_N}(M). \tag{4.146}$$

As in the case of $N = 2$, we propose that the correct thing to do is to include the vortex L in the definition of the bulk partition function and to allow it to take any charge. That is, the $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_N$ partition function is computed by summing over all degree N covering spaces of M branched along L with any allowed branching structure. Since $\pi_1(M \backslash L) \cong \pi_1(\partial M)$ (i.e. $M \backslash L$ retracts onto ∂M) the covering spaces of M branched over L are in one-to-one correspondence with the (unbranched) covering spaces of the boundary torus. This means the combinatorial counting of bulk and boundary covering spaces matches. The grand canonical partition function includes both connected and disconnected covering spaces, and through standard combinatorial arguments it is given by the exponential of the connected covering spaces

$$\mathfrak{Z}_{\mathrm{CS}}(M,p) = \exp\left(\sum_{n=1}^{\infty} p^n T_n Z_G\left(\tau\right)\right) . \tag{4.147}$$

Indeed, the connected covering spaces are simply handlebodies whose boundaries are N-fold covering spaces of the boundary torus of M. From the above we can extract a formula for the partition function of the $G \wr S_N$ Chern-Simons theory with a vortex along the non-contractible cycle by keeping all terms with a power of p^N which is

$$Z_{G \wr S_N}(M) = \sum_{\text{parititons of } N} \prod_{k=1}^{N} \frac{1}{N_k!} \left(T_k Z_G(\tau)\right)^{N_k} , \tag{4.148}$$

where the partitions of N are $\sum_{k=1}^{N} k N_k = N$. This can be compared to the boundary partition function given in equation (4.60). The connected part of the partition function is given by

$$Z_{G \wr S_N,\mathrm{conn.}}(M) = T_N Z_G(\tau) . \tag{4.149}$$

Which can also be obtained by recalling that the Hecke operator T_N sums over all connected covering spaces of the original torus. To obtain the bulk partition function we additionally need to sum over all bulk handlebodies M, which is implemented by the sum over modular images.

We claim that the averaged free energy of the grand canonical symmetric orbifold exactly equals the free energy of the grand canonical Chern-Simons theory, summed over all solid tori M, i.e.

$$\int_{\mathcal{M}_D} \mathrm{d}\mu \, \log \mathfrak{Z}(\tau, p) = \sum_M \log \mathfrak{Z}_{\mathrm{CS}}(M,p) . \tag{4.150}$$

This is equivalent to stating that the connected covering space contribution to the symmetric orbifold is exactly reproduced by the bulk Chern-Simons theory. To check this claim, note that the averaged symmetric orbifold free energy takes the form

$$\begin{aligned}\int_{\mathcal{M}_D} \mathrm{d}\mu\, \log \mathfrak{Z}(\tau,p) &= \sum_{n=1}^{\infty} p^n \left\langle T_n Z(\tau)\right\rangle \\ &= \sum_{n=1}^{\infty} \frac{p^n}{n} \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\gamma'\in \mathrm{SL}(2,\mathbb{Z})\backslash M_n} \frac{1}{|\eta(\gamma\cdot\gamma'\cdot\tau)|^{2D}}\,,\end{aligned} \tag{4.151}$$

where M_n is the set of all 2×2 integer matrices and we have used the definition (4.62) of the n^{th} Hecke operator. Now, the sum over geometries in the Chern-Simons free energy is implemented by a sum over modular images of the boundary torus. We have

$$\begin{aligned}\sum_M \log \mathfrak{Z}_{\mathrm{CS}}(M,p) &= \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}(2,\mathbb{Z})} \sum_{n=1}^{\infty} p^n T_n Z_G(\gamma\cdot\tau) \\ &= \sum_{n=1}^{\infty} \frac{p^n}{n} \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\gamma'\in \mathrm{SL}(2,\mathbb{Z})\backslash M_n} \frac{1}{|\eta(\gamma'\cdot\gamma\cdot\tau)|^{2D}}\,,\end{aligned} \tag{4.152}$$

where we have again used the definition of the Hecke operator T_n. Equations (4.151) and (4.152) appear to yield different results, since the summand of one includes the modular parameter $\gamma\cdot\gamma'\cdot\tau$, while the other includes $\gamma'\cdot\gamma\cdot\tau$. However, it turns out[29] (see Theorem 6.9 and 6.10 of [190]) that the sums (4.151) and (4.152) are composed of all the same terms, simply shuffled around. That is,

$$\sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\gamma'\in \mathrm{SL}(2,\mathbb{Z})\backslash M_n} \frac{1}{|\eta(\gamma\cdot\gamma'\cdot\tau)|^{2D}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\gamma'\in \mathrm{SL}(2,\mathbb{Z})\backslash M_n} \frac{1}{|\eta(\gamma'\cdot\gamma\cdot\tau)|^{2D}}\,. \tag{4.153}$$

This proves the claim of (4.150).

Since the free energy of the grand canonical partition function (either in the case of the symmetric orbifold or of Chern-Simons) computes the connected contribution, we have the following result:

> *The ensemble average of the connected part of the $\mathbb{T}^D \wr S_N$ orbifold torus partition function is equal to the connected part of the $U(1)^D\times U(1)^D \wr S_N$ Chern-Simons partition function, summed over all handlebodies bounded by the CFT torus for all N.*

$$\left\langle Z_{\mathbb{T}^D\wr S_N,\,\mathrm{conn.}}(\tau)\right\rangle = Z_{\text{Bulk, conn.}}\,. \tag{4.154}$$

As we have seen in the $N=2$ example, however, the disconnected parts of the two theories cannot so easily be matched.

[29] It turns out that given γ_1,γ_1' there exists a γ_2,γ_2' with $\gamma_i\in\Gamma_\infty\backslash SL(2,\mathbb{Z})$ and $\gamma_i'\in M_n\backslash SL(2,\mathbb{Z})$ such that $\gamma_1\cdot\gamma_1'=\gamma_2'\cdot\gamma_2$. This decomposition is unique, and summing over all possible γ_1,γ_1' is equivalent to summing over all possible γ_2,γ_2' [190]. Hence, the two sums are equivalent.

Disconnected Part

We now briefly comment on what portion of the disconnected partition function the Chern-Simons calculation reproduces. We restrict to a torus handlebody M with a single vortex operator running along the non-contractible cycle L. As explained earlier, since $\pi_1(M\backslash L) \cong \pi_1(\partial M)$ the branched covering spaces of M exactly match the covering spaces of the boundary torus. The disconnected part of the bulk Chern-Simons partition function is given by discarding the connected covering spaces from the full answer in equation (4.148) and summing over modular images

$$Z_{\text{Bulk, dis.}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \sum_{\text{parititons of } N} \prod_{k=1}^{N-1} \frac{1}{N_k!} \left(T_k Z_G(\gamma\cdot\tau)\right)^{N_k} , \tag{4.155}$$

where now we sum over disconnected partitions of N by imposing $\sum_{k=1}^{N-1} kN_k = N$. Similarly, the contribution of disconnected covering spaces to the boundary ensemble average is given by (4.60)

$$\left\langle Z_{\mathbb{T}^D\wr S_N,\,\text{dis.}}(\tau)\right\rangle = \sum_{\text{parititons of } N} \left\langle \prod_{k=1}^{N-1} \frac{1}{N_k!} \left(T_k Z(\tau)\right)^{N_k} \right\rangle , \tag{4.156}$$

where we must use the disconnected Siegel-Weil formula (4.20) to evaluate the average. One of the contributions to the above average will be given by filling in the "same cycle", specified by a modular parameter γ, on each disconnected covering torus in (4.156). More precisely, these are the configurations where the preimage of the cycles on the covering tori map to the same cycle on the base torus.

This precisely matches the bulk computation in (4.155) after using Theorem 6.9 of [190] to commute the sum over modular images with the Hecke operator sum. However, the other terms appearing in the boundary average will not be reproduced by the bulk computation. We therefore have that the bulk Chern-Simons computation is strictly contained within the boundary average

$$\left\langle Z_{\mathbb{T}^D\wr S_N,\,\text{dis.}}(\tau)\right\rangle \supset Z_{\text{Bulk, dis.}} . \tag{4.157}$$

Higher-genus boundaries

Let us now make a few comments about the case of geometries whose boundaries have genus $g \geq 2$. We only comment on the connected components.

In the symmetric orbifold theory on a surface Σ_g of genus g, the connected component of the partition function can be expressed as

$$Z_{\mathbb{T}^D\wr S_N,\text{conn.}}(m,\Sigma_g) = \frac{1}{N} \sum_{\widetilde{\Sigma}\to\Sigma_g} Z_{\mathbb{T}^D}(m,\widetilde{\Sigma}) , \tag{4.158}$$

where the sum is over all connected unramified covering surfaces $\widetilde{\Sigma} \to \Sigma_g$ of degree N. As in the case of the torus, these covering spaces are constructed by summing over all twisted

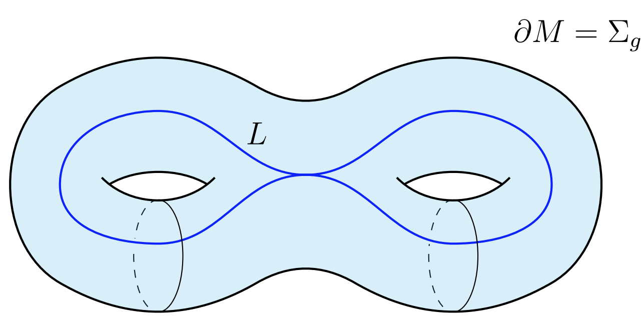


Figure 4.14: A handlebody M is a singular foliation of its boundary Σ_g. The leaves of the foliation become singular at the 'center' L of M, which roughly resembles Σ_g with the contractible cycles collapsed to points. In the Chern-Simons theory dual to the Narain-averaged symmetric orbifold, we treat L as the locus of permutation gauge vortices.

boundary conditions around the cycles of Σ_g, with the requirement that around contractible cycles the resulting boundary conditions are trivial. As noted above, such surfaces have constrained topology, and specifically have genus

$$g' = N(g-1)+1\,, \tag{4.159}$$

We argued in Section 4.2 that the average of this connected component over the Narain moduli space is given as a sum of the $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ Chern-Simons partition function on handlebodies whose boundaries are the covering surfaces $\widetilde{\Sigma}_{g'}$:

$$\Big\langle Z_{\mathbb{T}^D \wr S_N,\mathrm{conn.}}(m,\Sigma_g)\Big\rangle = \frac{1}{N}\sum_{\widetilde{\Sigma}_{g'}\to\Sigma_g}\ \sum_{\partial\widetilde{M}=\widetilde{\Sigma}} Z_G(\widetilde{M})\,. \tag{4.160}$$

We can now try to interpret the above sum in terms of Chern-Simons theory. Let M be a handlebody bounded by Σ_g. Next, let L be the codimension 2 locus shown in Figure 4.14. The fundamental group $\pi_1(M\backslash L)$ is isomorphic to the fundamental group of the boundary of M, i.e.

$$\pi_1(M\setminus L)\cong\pi_1(\Sigma_g)\,. \tag{4.161}$$

Since covering spaces are classified by choices of consistent twisted boundary conditions around different cycles, that is homomorphisms $\phi:\pi_1(\Sigma_g)\to S_N$, we have that covering spaces of M branched over L have the same structure as covering spaces of Σ_g.[30] Furthermore, the covering spaces $\widetilde{M}$ will also be handlebodies. There is one subtlety regarding the singular locus L. Earlier we demanded that vortex operators were defined along a dimension one submanifold of M, but L is not a manifold so it is not obvious in what sense a vortex operator can be associated to L. However, it remains perfectly consistent to impose

[30]The isomorphism $\pi_1(M\setminus L)\cong\pi_1(\Sigma_g)$ is due to the fact that a handlebody can be foliated by copies of its boundary, up to a singular locus given by L. That is, $M\setminus L$ is homeomorphic to $\Sigma_g\times[0,1)$, see [172].

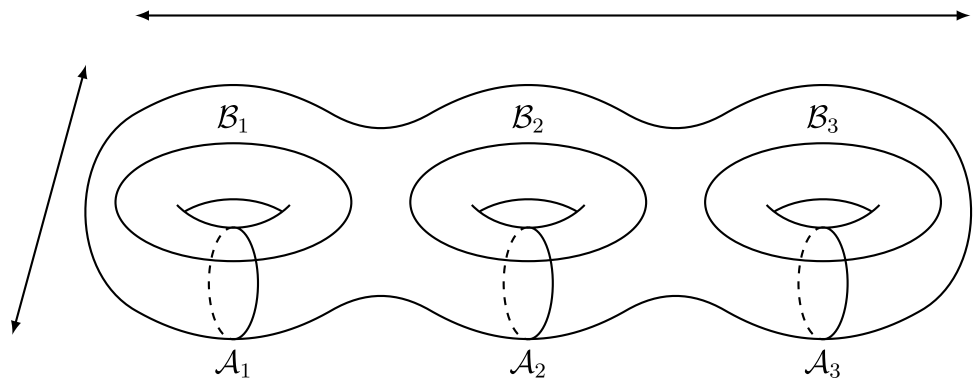


Figure 4.15: A genus 3 surface which double covers a genus 2 surface. The deck transformation acts as a product of two reflections (shown), and maps the homology cycles as $\mathcal{A}_1, \mathcal{B}_1 \to \mathcal{A}_3, \mathcal{B}_3$ and fixes $\mathcal{A}_2, \mathcal{B}_2$.

twisted boundary conditions on the bulk Chern-Simons fields as they travel around L. Since $\pi_1(\Sigma_g) \cong \pi_1(M\backslash L)$ a choice of twisted boundary conditions on Σ_g descends to a consistent set of monodromies for the gauge fields as they travel around different cycles of L, and we define our bulk theory by demanding such monodromies.

Coupling our Chern-Simons theory to topological gravity results in summing over all handlebodies M with boundary Σ_g. Summarizing, we have the bulk contribution

$$\sum_{\partial M = \Sigma_g} Z_{\text{CS,conn.}}(M) = \frac{1}{N} \sum_{\partial M = \Sigma_g} \sum_{\substack{\widetilde{M} \to M \\ \text{branched over } L}} Z_G(\widetilde{M}) \,. \tag{4.162}$$

This formula, however, will not reproduce the full symmetric orbifold answer (4.160), even at the level of the connected parts. Algebraically, this is rather straightfoward to see: the sum over handlebodies $\widetilde{M}$ in (4.160) amounts to summing over Lagrangian sublattices of $H_1(\widetilde{\Sigma}, \mathbb{Z})$, which can be repackaged into a sum over the quotient space $P_{g'}\backslash \mathrm{Sp}(2g', \mathbb{Z})$, where $P_{g'}$ is the parabolic subgroup of $\mathrm{Sp}(2g', \mathbb{Z})$. On the other hand, the sum over handlebodies in (4.162) is over Lagrangian sublattices of $H_1(\Sigma_g, \mathbb{Z})$, which in turn is a sum over images of a fixed sublattice under the action of $P_g\backslash \mathrm{Sp}(2g, \mathbb{Z})$. Since $g' > g$ when $g \geq 2$ (and $N \geq 2$), these sums can't possibly be equal. On the other hand, when $g = 1$ (i.e. for the torus partition function), we always have $g' = g$, and the sums contain all of the same terms.

The more geometrical reason for the fact that not all geometries appearing in the average of the symmetric orbifold partition function can be recovered from the Chern-Simons theory was already discussed in Section 4.3. We can think of the symmetric orbifold calculation as finding a (connected) branched covering space $\widetilde{\Sigma} \to \Sigma_g$, and then choosing a Lagrangian sublattice Γ_0 of $H_1(\widetilde{\Sigma}, \mathbb{Z})$, which in turn defines a handlebody $\widetilde{M}$. However, if Γ_0 does not fill in the cycles of $\widetilde{\Sigma}$ in a symmetric way (i.e. such that Γ_0 is invariant under the deck transformations $\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma_g)$), then $\widetilde{M}$ will not be the branched cover of some 3-manifold M whose boundary is Σ_g. As a simple example, let us take $g = 2$ and take the two-fold covering space shown in Figure 4.15. The covering surface has genus $g' = 2g - 1 = 3$, and its homology cycles are labeled by $\mathcal{A}_i, \mathcal{B}_i$ for $i = 1, \ldots, 3$ (also shown in Figure 4.15). The deck transformation group is $\mathrm{Aut}(\widetilde{\Sigma} \to \Sigma_2) \cong \mathbb{Z}_2$, and simply consists of the operation of swapping the two sheets of the cover. On the homology elements, the nontrivial element $\phi \in \mathrm{Aut}(\widetilde{\Sigma} \to \Sigma_2)$ acts as

$$\phi_*(\mathcal{A}_1) = \mathcal{A}_3\,, \quad \phi_*(\mathcal{A}_2) = \mathcal{A}_2\,, \quad \phi_*(\mathcal{B}_1) = \mathcal{B}_3\,, \quad \phi_*(\mathcal{B}_2) = \mathcal{B}_2\,. \tag{4.163}$$

Any Lagrangian sublattice Γ_0 which is not invariant under this group of deck transformations has no hope of describing a bulk manifold $\widetilde{M}$ which is a covering space of a manifold M with boundary $\partial M = \Sigma_2$. For example, the sublattice

$$\Gamma_0 = \mathrm{Span}_{\mathbb{Z}}\left(\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_3\right) \tag{4.164}$$

is Lagrangian but is clearly not invariant under the deck transformations of $\widetilde{\Sigma}$.

In the case of the torus, this simply doesn't happen for connected covering spaces. This is because the set of deck transformations of a connected covering space acts trivially on homology, i.e. all Lagrangian sublattices of the covering space are invariant under deck transformations. This explains, at least qualitatively, why we were able to get a match between the connected parts of the symmetric orbifold calculation and the bulk Chern-Simons theory in the case of a genus one boundary.

Non-handlebody contributions

Among the gravitational contributions considered so far, all have arisen either from handlebodies with genus-g boundaries, potentially with vortices in the bulk. We now show that for surfaces of genus $g \geq 2$, the Narain-averaged partition function of $\mathbb{T}^D \wr S_N$ will also include *smooth* bulk geometries which are not handlebodies. We work again with $N = 2$ for simplicity.

Consider a surface Σ_g of genus g, and consider the $\mathbb{T}^D \wr S_2$ partition function. It will take the form

$$Z_{\mathbb{T}^D \wr S_2}(m, \Sigma_g) = \frac{1}{2} Z_{\mathbb{T}^D}(m, \Sigma_g) Z_{\mathbb{T}^D}(m, \Sigma_g) + \cdots\,, \tag{4.165}$$

where we are concentrating only on the contribution from the double cover $\Sigma_g \sqcup \Sigma_g \to \Sigma_g$ (of course, there will be other contributions, but we will not need them for our purposes). Upon averaging over the Narain moduli space, we have

$$\left\langle Z_{\mathbb{T}^D \wr S_2}(m, \Sigma_g) \right\rangle = \frac{1}{2} \left\langle Z_{\mathbb{T}^D}(m, \Sigma_g) Z_{\mathbb{T}^D}(m, \Sigma_g) \right\rangle + \cdots\,. \tag{4.166}$$

As we know, the average of $Z(\Sigma_g)Z(\Sigma_g)$ is computed by summing over all Lagrangian sublattices $\Gamma \subset H_1(\Sigma_g \sqcup \Sigma_g, \mathbb{Z})$, weighted by an appropriate one-loop determinant. A special class of sublattices Γ are constructed by picking a basis $(\mathcal{A}_i, \mathcal{B}_i)$ for the homology group of Σ_g, and letting

$$\Gamma = \mathrm{Span}_{\mathbb{Z}}\left(\mathcal{A}_i^{(1)} + \mathcal{A}_i^{(2)}, \mathcal{B}_i^{(1)} - \mathcal{B}_i^{(2)}\right)\,, \tag{4.167}$$

where the (1) and (2) superscripts differentiate between the two copies of Σ_g. This is the Lagrangian sublattice associated to a manifold which is topologically of the form $\widetilde{M} \cong \Sigma_g \times I$. For $g \geq 2$, such manifolds are hyperbolic with metric

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \cosh^2 \rho \, \mathrm{d}s^2_{\Sigma_g} \,, \tag{4.168}$$

where the coordinate $\rho \in \mathbb{R}$ is the coordinate along the interval. Furthermore, for special choices of complex structure on Σ_g the boundary admits a fixed point free $\mathbb{Z}_2$ involution which reverses the orientation of Σ_g. Combining this boundary involution with the bulk reversal $\rho \to -\rho$ we obtain a bulk involution ι [191, 192]. This involution acts without fixed points, and we can consider the quotient manifold

$$M := \widetilde{M}/\iota \,. \tag{4.169}$$

As noted in [191, 192], this geometry is smooth and hyperbolic with boundary Σ_g, but is not a handlebody. Specifically, it has the following properties. Given the inclusion map $i : \Sigma_g \hookrightarrow \mathcal{M}$, the induced map $i_* : \pi_1(\Sigma_g) \hookrightarrow \pi_1(\mathcal{M})$:

- is injective. This means that the non-contractible cycles of Σ_g do not become contractible when viewed as cycles of $\mathcal{M}$.
- is not surjective. This means that there are non-contractible cycles of $\mathcal{M}$ which are not visible from the boundary Σ_g.

The first property means that $\mathcal{M}$ is not associated to *any* Lagrangian sublattice $\Gamma_0 \subset H_1(\Sigma_g, \mathbb{Z})$. The second property means that there are generators of the fundamental group which do not exist on the boundary. This is why $\mathcal{M}$ is able to be both smooth and have a connected double cover $\widetilde{M}$ which has a disconnected boundary.

By the standard U(1)-gravity dictionary, we should be able to associate the contribution of the sublattice (4.167) in the averaged partition function (4.166) to the path integral of $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ Chern-Simons theory on the wormhole geometry $\widetilde{M}$. This, in turn, should be reproduced by the $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_2$ partition function on M with a nontrivial monodromy around the 'internal' generator(s) of $\pi_1(M)$ (i.e. the generator(s) of $\pi_1(M)$ which are not inherited from $\pi_1(\Sigma_g)$).

We emphasize that the gravitational instanton associated to (4.167) appears to be a *smooth* bulk manifold M with a connected boundary, but which is not a handlebody. This is worth emphasizing, since U(1) gravity with a connected boundary (i.e. the bulk dual of the non-orbifolded Narain ensemble) includes only handlebodies in its sum over geometries.[31] It would be thus be interesting to explore further in what sense the ensemble average of $\mathbb{T}^D \wr S_N$ CFTs includes more generic gravitational instantons which are not visible in U(1) gravity, but which are generally expected to be included in a more general theory of three-dimensional quantum gravity (for example, semiclassical gravity).

[31] Strictly speaking, U(1) gravity only classifies bulk geometries by their associated Lagrangian sublattice, which for connected boundaries are in one-to-one correspondence with handlebodies.

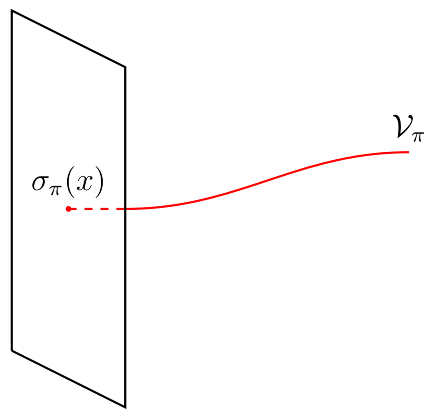


Figure 4.16: A twist field $\sigma_\pi(x)$ is holographically dual to a vortex $\mathcal{V}_\pi$ in the bulk which ends at the point x.

4.4 Averaging correlation functions

In the previous sections, we have considered the ensemble average of partition functions of permutation orbifolds $\mathbb{T}^D \wr \Omega$ (mostly for $\Omega = S_N$) and showed that a large family of contributions can be recovered from a Chern-Simons theory with gauge group $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr \Omega$ coupled to topological gravity. In this section, we consider correlation functions of permutation orbifolds and explore their bulk interpretations.

Before jumping into calculations, let us briefly summarize the main idea. In Section 4.3, we found that the averaged partition function of the symmetric orbifold CFT is partially reproduced by a sum over bulk geometries which include nontrivial vortices. Roughly, these vortices are holographically dual to the twisted-sector states of the orbifold theory on the boundary, see Figure 4.16. Now, let us consider a correlation function of twist fields[32]

$$\langle \sigma_{\pi_1}(x_1) \cdots \sigma_{\pi_n}(x_n) \rangle \ ,$$

where $\pi_i \in S_N$ are permutations specifying the monodromy of fundamental fields around the point x_i. Just as in the case of the partition function of the symmetric orbifold, such correlators are determined by passing to a covering space $\Sigma \to \mathbb{CP}^1$ which is branched over the points $x_1, \ldots, x_n$, such that the branching structure is induced by the monodromies $\pi_1, \ldots, \pi_n$. Upon averaging, one is then instructed to sum over bulk geometries $\widetilde{M}$ filling in Σ, specified by a choice of Lagrangian sublattice of $H_1(\Sigma, \mathbb{Z})$.

Holographically, one would expect this correlator to be dual to a bulk computation involving vortices $\mathcal{V}_{\pi_i}$ ending at the points x_i on the boundary. Such vortices will need to end somewhere in the bulk, and in principle one should sum over all bulk configurations of the vortices. In terms of the bulk $G \wr S_N$ Chern-Simons theory, the calculation of the path integral in the presence of a nontrivial vortex configuration corresponds to passing to the branched covering of $\mathbb{H}^3$ over the locus of the vortices, with the branching structure

[32]The exact definition of twist fields is discussed in detail below.

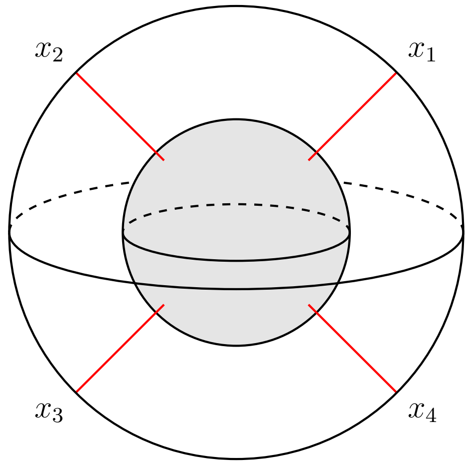


Figure 4.17: The bulk geometry dual to the four-point function of twist fields in the symmetric orbifold. In principle, the vortices can branch and tangle in complicated ways in the bulk.

determined by the monodromy specified on the components of the branching locus. Given that the vortices end on the boundary at the points $x_1, \ldots, x_n$, a branched cover of $\mathbb{H}^3$ will induce a branched cover of the boundary. However, the precise organization of the vortices in the bulk can drastically effect the topology of the 3-dimensional branched covering. Thus, each topological choice of configuration of vortices in the bulk will specify a 3-manifold $\widetilde{M}$ which is a branched covering of $\mathbb{H}^3$, and whose boundary is a branched covering Σ of $\mathbb{CP}^1$ branched at $x_1, \ldots, x_n$ with monodromies $\pi_1, \ldots, \pi_n$.

The above discussion suggests that it is possible to reproduce the Narain average of symmetric orbifold correlators by summing over topologically distinct vortex configurations in $G \wr S_N$ Chern-Simons theory on $\mathbb{H}^3$. However, this is not always possible. For the same reasons discussed in Section 4.3, while it is always true that a branched cover $\widetilde{M}$ over $\mathbb{H}^3$ always has boundary Σ which is a branched cover of $\mathbb{CP}^1$, the converse is much more difficult to satisfy. Given a branched cover $\Sigma \to \mathbb{CP}^1$, it is not always the case that a given 3-manifold $\widetilde{M}$ with boundary Σ admits the structure of a branched cover of $\mathbb{H}^3$ (or any other 3-manifold with boundary $\mathbb{CP}^1$ for that matter).

For the examples we explicitly consider in this section, however, this turns out to not be an issue. Specifically, in this section we consider the symmetric orbifold $\mathbb{T}^D \wr S_2$, for which there is a unique twist operator $\sigma_{(1\,2)}$.[33] We consider the correlation functions of the form

$$\left\langle \sigma_{(1\,2)}(x_1) \cdots \sigma_{(1\,2)}(x_n) \right\rangle \, . \tag{4.170}$$

Just as in [160], we find that the Narain averages of these correlators are indeed reproduced by a sum over bulk vortex configurations in $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_2$ Chern-Simons theory, for which

[33] It is possible to consider $\mathbb{T}^D \wr S_N$ theories with $N > 2$. We leave this to future work.

the vortices are constrained to lie in *rational tangles*.[34] As such, we are able to reproduce the averaged symmetric orbifold correlation function via a bulk Chern-Simons calculation.[35]

Correlators in permutation orbifolds

Before considering the ensemble average of correlation functions, let us first review the calculation of correlation functions in permutation orbifolds. For the rest of this section, we only concern ourselves with correlators of fields on the sphere ($\mathbb{CP}^1$) for simplicity.

As an orbifold theory, a permutation orbifold $X \wr \Omega$ contains non-local operators known as twist-fields which implement a monodromy on the fundamental fields of the theory. Denoting by $\mathbf{\Phi} = (\Phi^{(1)}, \ldots, \Phi^{(N)})$ the collective fundamental fields of the tensor theory $X^{\otimes N}$, a twist operator σ_π associated to a permutation $\pi \in \Omega$ is defined by the monodromy relation

$$\mathbf{\Phi}(e^{2\pi i} z + \zeta)\, \sigma_\pi(\zeta) = (\pi \cdot \mathbf{\Phi})(z + \zeta)\, \sigma_\pi(\zeta)\,, \tag{4.171}$$

where

$$\pi \cdot \mathbf{\Phi} = \left(\Phi^{(\pi(1))}, \ldots, \Phi^{(\pi(N))}\right). \tag{4.172}$$

On their own, twist fields σ_π are not gauge-invariant.We can see this as follows. Consider the monodromy of an element $(\rho \cdot \mathbf{\Phi})\,(z)$ around the twist field $\sigma_\pi(\zeta)$. The twist field acts on $\mathbf{\Phi}(z)$:

$$(\rho \cdot \mathbf{\Phi})\,(e^{2\pi i} z{+}\zeta)\sigma_\pi(\zeta) = (\rho \cdot \pi \cdot \mathbf{\Phi})\,(z{+}\zeta)\sigma_\pi(\zeta) = \left(\rho \cdot \pi \cdot \rho^{-1} \cdot (\rho \cdot \mathbf{\Phi})\right)(z{+}\zeta)\sigma_\pi(\zeta)\,. \tag{4.173}$$

From this we can infer the action of the twist fields on the field $(\rho \cdot \mathbf{\Phi})\,(z)$, which lies on the same gauge slice as $\mathbf{\Phi}(z)$:

$$(\rho \cdot \mathbf{\Phi})\,(e^{2\pi i} z + \zeta)\sigma_\pi(\zeta) = (\rho \cdot \mathbf{\Phi})\,(e^{2\pi i} z + \zeta)\sigma_{\rho\pi\rho^{-1}}(\zeta)\ . \tag{4.174}$$

Where on the RHS the twist field acts on $(\rho \cdot \mathbf{\Phi})$ and on the LHS on $\mathbf{\Phi}(z)$. Hence under an overall permutation $\rho \in \Omega$, twist fields transform as

$$\sigma_\pi \to \sigma_{\rho\pi\rho^{-1}}\ . \tag{4.175}$$

From a twist field σ_π, we can construct a gauge-invariant twist field $\sigma_{[\pi]}$ by

$$\sigma_{[\pi]} = \mathcal{N}_{[\pi]} \sum_{\rho \in [\pi]} \sigma_\rho\,, \tag{4.176}$$

[34]Our analysis differs slightly from that of [160], in that they consider orbifolds of the form $\mathbb{T}^D/\mathbb{Z}_n$, which are qualitatively different from symmetric orbifolds. As such, their bulk theory is of the form $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \rtimes \mathbb{Z}_n$.

[35]We emphasize that for $N > 2$, the analysis does not work out so cleanly, and not every term in the averaged correlation functions will be reproducible by a bulk Chern-Simons calculation on a classical background.

where $\mathcal{N}_{[\pi]} = 1/\sqrt{|[\pi]|}$ is a normalization factor so that $\sigma_{[\pi]}$ is canonically normalized.[36] Note that $\sigma_{[\pi]}$ only depends on the conjugacy class of π in Ω. A generic correlator of gauge-invariant twist fields thus takes the form

$$\left\langle \prod_{i=1}^{n} \sigma_{[\pi_i]}(x_i) \right\rangle = \prod_{i=1}^{n} \left(\mathcal{N}_{[\pi_i]}\right) \sum_{\rho_1,\ldots,\rho_n \in \Omega} \left\langle \prod_{i=1}^{n} \sigma_{\rho_i \pi_i \rho_i^{-1}}(x_i) \right\rangle . \tag{4.177}$$

That is, the correlators of gauge-invariant twist fields can be expressed purely in terms of an appropriate sum over correlators of the 'pure' twist fields σ_π. From now on, we calculate only the correlators of pure twist fields σ_π, keeping in mind that we should sum over conjugacy classes to obtain a gauge-invariant result. Expression (4.177) will in general have disconnected contributions, meaning contributions for which some terms of the right hand side factorize. Here we focus on the connected part of the correlators. This can be done by looking at correlators for which the group elements that appear in the twist fields generate a transitive subgroup of the permutation group acting on the elements of $\{1, 2, ..., N\}$ that appear in the correlator (see e.g. [193]).

Given a set of twist fields σ_{π_i}, we can compute the (sphere) correlation function

$$\langle \sigma_{\pi_1}(x_1) \cdots \sigma_{\pi_n}(x_n) \rangle \tag{4.178}$$

in the following way [158, 159]. Within the path integral, the twist fields define twisted boundary conditions of the fundamental field on the punctured sphere $\mathbb{CP}^1 \setminus \{x_1, \ldots, x_n\}$, and so we can pass to a covering space Σ of $\mathbb{CP}^1$ which is ramified over the points x_i such that the fundamental fields are single-valued on Σ.[37] The covering space Σ is related to the base sphere via a holomorphic map $\Gamma : \Sigma \to \mathbb{CP}^1$, and so we can exploit the conformal symmetry to pull back the fields of the seed theory X to the covering space Σ, at the expense of a conformal anomaly in the path integral measure when X has non-zero central charge. The result is that the correlation function of twist fields is given by

$$\langle \sigma_{\pi_1}(x_1) \cdots \sigma_{\pi_n}(x_n) \rangle = e^{-S_L[\Phi_\Gamma]} Z_X(\Sigma) \,, \tag{4.179}$$

where $Z_X(\Sigma)$ is the partition function of the seed theory on the surface Σ. The genus of the covering surface Σ is fixed by the Riemann - Hurwitz formula. Specifically, it is given in terms of the genus g of the base surface, the number M of distinct elements of $\{1, 2, ..., N\}$ that appear in the correlator and the lengths of the cycles w_j, $j = 1, 2, ..., n$. Concretely, we have

$$g' - 1 = M(g-1) + \frac{1}{2} \sum_{j=1}^{n} (w_j - 1) \ . \tag{4.180}$$

[36] Here, by 'canonically normalized' we mean that the two-point function satisfies $\langle \sigma_{[\pi]}(x_1) \sigma_{[\pi]}(x_2) \rangle = 1/(x_1 - x_2)^{2h(\pi))}$, where $h(\pi)$ is the conformal weight of the twist field.

[37] Note that, unlike in previous sections, we are now using Σ as opposed to $\widetilde{\Sigma}$ to refer to the covering space, since the base space is already specified to be $\mathbb{CP}^1$.

Since we focus on sphere correlators, $g = 0$ and thus

$$g' = 1 - M + \frac{1}{2}\sum_{j=1}^{n}(w_j - 1)\,. \tag{4.181}$$

Here, the Liouville action S_L is given by

$$S_L[\Phi] = \frac{c}{48\pi}\int_\Sigma \mathrm{d}^2 z\sqrt{g}\left(-\Phi\Delta\Phi + R\Phi\right)\,, \tag{4.182}$$

where c is the central charge of the seed CFT X, R is the scalar curvature on Σ, and Δ is the Laplacian on Σ.[38] The conformal anomaly is found by evaluating this action on the scalar

$$\Phi_\Gamma = \log|\partial\Gamma|^2\,. \tag{4.183}$$

This corresponds to a metric on the covering space

$$ds^2 = e^{\Phi_\Gamma}\mathrm{d}z\,\mathrm{d}\overline{z} \tag{4.184}$$

where $z, \overline{z}$ are local coordinates on the covering space.

Narain-averaging correlators

Let us now specify our CFT to be a Narain theory $X = \mathbb{T}^D$. We write the correlators of twist fields in $\mathbb{T}^D \wr \Omega$ as

$$\langle\sigma_{\pi_1}(x_1)\ldots\sigma_{\pi_n}(x_n)\rangle = e^{-S_L[\Phi_\Gamma]}Z(m,\Sigma)\,, \tag{4.185}$$

where $Z(m,\Sigma)$ is the partition function of the $\mathbb{T}^D$ theory on the surface Σ. We can now average this result over the moduli m and we find

$$\int_{\mathcal{M}_D}\mathrm{d}\mu(m)\ \langle\sigma_{\pi_1}(x_1)\ldots\sigma_{\pi_n}(x_n)\rangle = e^{-S_L[\Phi_\Gamma]}\int_{\mathcal{M}_D}\mathrm{d}\mu(m)\,Z(m,\Sigma)\,. \tag{4.186}$$

However, the average of the partition function Z is readily written down in terms of a sum over Lagrangian sublattices of $H_1(\Sigma,\mathbb{Z})$, i.e.

$$\int_{\mathcal{M}_D}\mathrm{d}\mu(m)\ \langle\sigma_{\pi_1}(x_1)\ldots\sigma_{\pi_n}(x_n)\rangle = e^{-S_L[\Phi_\Gamma]}\sum_{\Gamma_0\subset H_1(\Sigma,\mathbb{Z})} Z_{\mathrm{CS}}(\Gamma_0)\,, \tag{4.187}$$

where $Z_{\mathrm{CS}}(\Gamma_0)$ is shorthand for the $\mathrm{U}(1)^D\times\mathrm{U}(1)^D$ Chern-Simons partition function on the bulk manifold defined by the Lagrangian sublattice Γ_0. In the case that Σ is a connected

[38] In our conventions, we have $\sqrt{g}\,\Delta\Phi = \partial(\sqrt{g}\,\overline{\partial}\Phi) + \overline{\partial}(\sqrt{g}\,\partial\Phi)$.

surface, we can instead express the averaged correlation function in terms of a sum over handlebodies bounded by Σ, namely

$$\int_{\mathcal{M}_D} \mathrm{d}\mu(m)\ \langle \sigma_{\pi_1}(x_1)\dots\sigma_{\pi_n}(x_n)\rangle = e^{-S_L[\Phi_\Gamma]} \sum_{\substack{\text{handlebodies}\\ \partial M=\Sigma}} Z_{\rm CS}(\mathcal{M})\,. \tag{4.188}$$

In the rest of this section, we will consider examples of such correlators and interpret them in terms of correlators of vortices of $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr \Omega$ Chern-Simons theory on the hyperbolic ball. Completely analogously to [160], we find that the sum over geometries on the covering space can be interpreted as a sum over configurations of bulk vortices.

Four-point functions

We begin with the simplest nontrivial example: the four-point function of the twist field $\sigma_{(1\,2)}$ in the orbifold $\mathbb{T}^D \wr S_2$. Since $S_2 \cong \mathbb{Z}_2$, this analysis is very similar to that of [160].

Without loss of generality, we can use the $\mathrm{SL}(2,\mathbb{C})$ isometry on the sphere to place three points at $0, 1, \infty$ and leave the fourth point arbitrary. That is, we consider the correlator

$$\langle \sigma_{(1\,2)}(0)\sigma_{(1\,2)}(1)\sigma_{(1\,2)}(u)\sigma_{(1\,2)}(\infty)\rangle\ . \tag{4.189}$$

Since S_2 is abelian, this is actually a fully gauge-invariant correlator. This correlator is also connected as the permutation (12) acts in a transitive way on the elements $\{1,2\}$. The covering space on which the fundamental fields are single-valued is given by the torus whose modular parameter is related to the cross-ratio u via the modular λ function [175]. That is,

$$u(\tau) = 1-\lambda(\tau) = 1-\frac{\vartheta_2(\tau)^4}{\vartheta_3(\tau)^4}\,. \tag{4.190}$$

The explicit form of the covering map Γ is

$$\Gamma(z) = \frac{\mathfrak{p}(z;\tau)-\mathfrak{p}(\frac{1}{2};\tau)}{\mathfrak{p}(\frac{\tau}{2};\tau)-\mathfrak{p}(\frac{1}{2};\tau)}\,, \tag{4.191}$$

where $\mathfrak{p}$ is the Weierstrass function. Given the form of the covering map, the conformal anomaly $e^{-S_L[\Phi_\Gamma]}$ can be explicitly worked out and, after regularization, takes the form [175]

$$e^{-S_L[\Phi_\Gamma]} = \frac{1}{2^{2c/3}|u(1-u)|^{c/12}}\,. \tag{4.192}$$

Therefore overall :

$$\langle \sigma_{(1\,2)}(0)\sigma_{(1\,2)}(1)\sigma_{(1\,2)}(u)\sigma_{(1\,2)}(\infty)\rangle = \left(\frac{1}{2^{2c/3}|u(1-u)|^{c/12}}\right)\left(\frac{\Theta(m,\tau)}{|\eta(\tau)|^{2D}}\right)\,. \tag{4.193}$$

Putting everything together, the average of the four-point function of $\sigma_{(1\,2)}$ is computed using the Siegel-Weil formula, and we find

$$\begin{aligned}\int_{\mathcal{M}_D} \mathrm{d}\mu(m)\ &\langle \sigma_{(1\,2)}(0)\sigma_{(1\,2)}(1)\sigma_{(1\,2)}(u)\sigma_{(1\,2)}(\infty)\rangle \\ &= \frac{1}{2^{2c/3}|u(1-u)|^{c/12}} \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}\,,\end{aligned} \tag{4.194}$$

where τ is obtained from u by inverting (4.190) and the central charge c is just the dimension D of the torus target.[39] Specifically, τ admits a closed-form expression in terms of hypergeometric functions

$$\tau = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2},1:u)}{{}_2F_1(\frac{1}{2},\frac{1}{2},1:1-u)}\,. \tag{4.195}$$

The bulk interpretation

As in [160], we can interpret the sum over modular images in (4.194) in terms of $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_2$ Chern-Simons theory in the following way. Consider the CFT sphere on which we calculate the correlation functions to be the boundary of the ball $\mathbb{H}^3$. We extend each twist field $\sigma_{(1\,2)}$ as a vortex in the bulk which meets the boundary at the point x_i. Each vortex in the bulk implements a monodromy $A_{(1)} \to A_{(2)}$ in the bulk gauge field. Since a vortex cannot just end at one point, we need to join pairs of boundary points by vortices. Let us choose for the moment a vortex which joins the point at $x = 0$ with the point at $x = 1$ and another which joins $x = u$ with $x = \infty$. The two strands are in principle allowed to cross and link in the bulk in an arbitrary fashion, so long as they do not cross. For example, the 'trivial' configuration $\mathcal{T}_0$ on the left of Figure 4.19 connects 0 to 1 and u to ∞ in the simplest way possible – with no crossings in the bulk. The bulk geometry found by taking the double branched cover of the ball over the vortex $\mathcal{T}_0$ has the property that the cycle generated by a loop encircling u and ∞ is contractible in the bulk. For an illustration, see Figure 4.18. Indeed, as was noted in [160], the branched cover of the geometry in Figure 4.18 is a handlebody with torus boundary.[40]

In [160], the sum over modular images in equation (4.194) was argued to arise from a sum over vortex configurations in the bulk which are topologically 'rational tangles'. A rational tangle is a vortex configuration which is obtained by applying successive exchanges (braidings) of the points $0, 1, u, \infty$ on the trivial tangle $\mathcal{T}_0$. For example, the three tangles shown in Figure 4.19 are the trivial tangle $\mathcal{T}_0$, the tangle obtained by starting with $\mathcal{T}_0$ and braiding the ends at $1, u$ around each other twice, and the tangle obtained from $\mathcal{T}_0$ by braiding the ends at u, ∞ three times. Note that the direction of swapping matters.

[39] We keep the central charge c arbitrary since, for example, for Narain theories with supersymmetry, the central charge is instead $c = 3D/2$.

[40] This can be seen by 'cutting open' the hyperbolic ball along branch cuts associated to the vortices and gluing a second copy along the same branch cuts, see Figure 2 of [160].

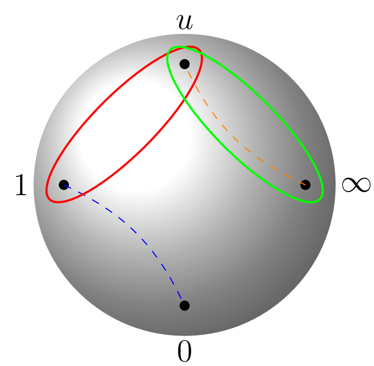


Figure 4.18: An illustration of the (non)contractibility of loops encircling points on the sphere. The green loop is contractible when continued in the bulk whereas the red loop is not (it has to cross the orange and blue vortices that live inside the sphere).

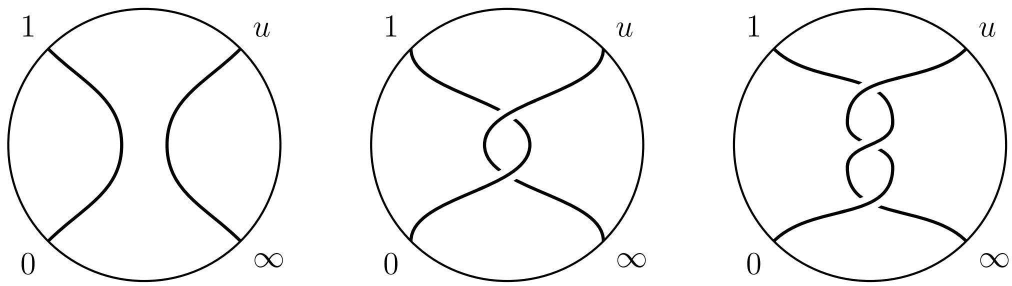


Figure 4.19: Some rational tangles contributing to the four-point function of twist fields in the averaged $\mathbb{T}^D \wr S_2$ orbifold.

Mathematically, a rational tangle is obtained from the trivial tangle $\mathcal{T}_0$ in the following way: Let $B_4(\mathrm{S}^2)$ be the braid group of points on the sphere. There is a natural action of $B_4(\mathrm{S}^2)$ on the ends of the vortices at $x = 0$, $x = 1$, $x = u$, and $x = \infty$. A basis of generators for B_4 are the 'standard braids' $\sigma_1, \sigma_2, \sigma_3$ which swap the pairs $(0, 1)$, $(1, u)$ and (u, ∞) respectively (with specified orientation) as in Figure 4.20. All rational tangles can be seen as the action of an element of $B_4(\mathrm{S}^2)$ on the trivial tangle $\mathcal{T}_0$.

The braid group $B_4(\mathrm{S}^2)$ is generated by $\sigma_1, \sigma_2, \sigma_3$, which satisfy the following relations:

$$
\begin{gathered}
\sigma_1\sigma_3 = \sigma_3\sigma_1\,, \quad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\,, \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \\
\sigma_1\sigma_2\sigma_3\sigma_3\sigma_2\sigma_1 = 1\,.
\end{gathered} \tag{4.196}
$$

The last of these relations is specific to the braid group on the sphere. Note that the braid group does not act on rational tangles faithfully. In particular, the combination $\sigma_1\sigma_3^{-1}$ acts trivially on any rational tangle since it corresponds to a reflection about the East-West axis as in Figure 4.21, see [194].

Let $\mathcal{N}\langle\sigma_1\sigma_3^{-1}\rangle$ be the normal subgroup of $B_4(\mathrm{S}^2)$ generated by $\sigma_1\sigma_3^{-1}$.[41] Since this is

[41] That is, $\mathcal{N}\langle\sigma_1\sigma_3^{-1}\rangle$ is the smallest normal subgroup of $B_4(\mathrm{S}^2)$ which contains $\sigma_1\sigma_3^{-1}$.

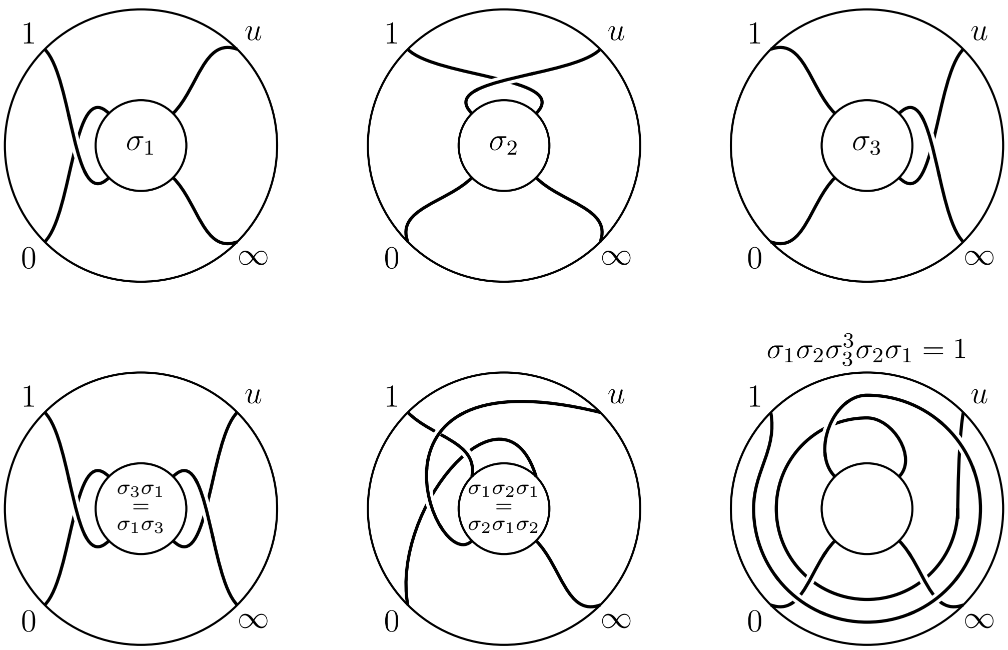


Figure 4.20: The action of the braid of points on the sphere $B_4(\mathrm{S}^2)$. The two circles depict cross sections of two-spheres. Inside the smaller two-sphere the strands could be arbitrarily tangled. The first row shows the action of the generators, the second depicts some of the relations that these generators satisfy. In the last drawing, the strands can be "untangled": the one that connects to 1 from the front of the inner S^2 and the one connected to u behind the inner S^2.

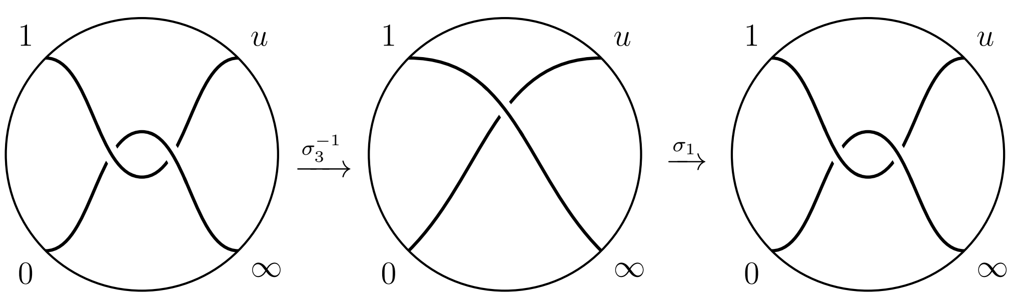


Figure 4.21: The action of $\sigma_1\sigma_3^{-1}$ on the tangle obtained by acting with σ_3^2. We see that this corresponds to the same tangle. One can see this for example by drawing the first tangle as seen from "behind the page".

a normal subgroup, the quotient $B_4(\mathrm{S}^2)/\mathcal{N}\left\langle\sigma_1\sigma_3^{-1}\right\rangle$ is a group which acts in a well-defined manner on the space of tangles. This quotient effectively imposes $\sigma_1 \sim \sigma_3$ in the defining relations (4.196) of the braid group. It turns out that this quotient is nothing more than the modular group, i.e.

$$B_4(\mathrm{S}^2)/\mathcal{N}\left\langle\sigma_1\sigma_3^{-1}\right\rangle \cong \mathrm{PSL}(2,\mathbb{Z})\,. \tag{4.197}$$

The isomorphism is seen by the direct matrix identification

$$\sigma_1, \sigma_3 \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\,. \tag{4.198}$$

Finally, note that not even $B_4(\mathrm{S}^2)/\mathcal{N}\left\langle\sigma_1\sigma_3^{-1}\right\rangle$ acts faithfully on the space of rational tangles. This is because $\sigma_1 \cdot \mathcal{T}_0 = \mathcal{T}_0$. Thus, the set of operations which acts faithfully on the set of rational tangles is

$$(B_r(\mathrm{S}^2)/\mathcal{N}\left\langle\sigma_1\sigma_3^{-1}\right\rangle)/\left\langle\sigma_1\right\rangle \cong \Gamma_\infty\backslash\mathrm{PSL}(2,\mathbb{Z})\,. \tag{4.199}$$

Thus, the sum over rational tangles produces a sum over the coset $\Gamma_\infty\backslash\mathrm{SL}(2,\mathbb{Z})$. This sum also precisely reproduces the sum over Lagrangian sublattices, since each rational tangle admits a combination of the cycles shown in Figure 4.18 which becomes contractible in the bulk. Furthermore, the branched double cover of each rational 2-tangle is a handlebody with torus boundary.[42]

Higher-point functions

The above analysis tells us how to interpret the four-point function of twist fields in the S_2 permutation orbifolds in terms of rational tangles of vortices in the bulk. Let us now consider the case of higher-point functions. In order to have a non-vanishing correlation function, we require that the number of twist field insertions is even, and so we consider the correlator

$$\left\langle\sigma_{(1\,2)}(x_1)\cdots\sigma_{(1\,2)}(x_{2g+2})\right\rangle\,, \tag{4.200}$$

where g is some non-negative positive integer. The covering space associated to this correlation function is a genus-g hyperelliptic curve defined by the equation

$$\Sigma_g = \left\{(x,y)\in\mathbb{CP}^1\times\mathbb{CP}^1\,\middle|\, y^2 = \prod_{i=1}^{2g+2}(x-x_i)\right\}\,. \tag{4.201}$$

Indeed, around the point $x = x_i$, the solutions $y(x)$ have a square-root branch cut, as required by the form of the correlator. The correlation function can be expressed as

$$\left\langle\sigma_{(1\,2)}(x_1)\cdots\sigma_{(1\,2)}(x_{2g+2})\right\rangle = e^{-S_L[\Phi_\Gamma]}Z(\Sigma_g)\,, \tag{4.202}$$

[42] This follows either from the fact that the double cover of $\mathcal{T}_0$ is a torus handlebody and the fact that all rational 2-tangles are homeomorphic to $\mathcal{T}_0$. Intuitively, the action of the braid group on $\mathcal{T}_0$ can be thought of as implementing Dehn surgery on the double cover of $\mathcal{T}_0$. In the mathematics literature, this is often referred to as the 'Montesinos Trick' [195].

where Γ is the covering map

$$\begin{aligned}\Gamma :&\Sigma_g \to \mathbb{CP}^1\\ &(x,y) \mapsto x\,.\end{aligned} \tag{4.203}$$

This gives a double cover of $\mathbb{CP}^1$ by Σ_g since each value of x has two values of y satisfying (4.201). Here and in what follows we will not be explicit about the form of the conformal anomaly.

Upon ensemble averaging, we have

$$\int_{\mathcal{M}_D} \mathrm{d}\mu(m)\ \langle \sigma_{(1\,2)}(x_1)\cdots\sigma_{(1\,2)}(x_{2g+2})\rangle = e^{-S_L[\Phi_\Gamma]} \sum_{\gamma\in P\backslash \mathrm{Sp}(2g,\mathbb{Z})} Z_{\mathrm{CS}}(\gamma\cdot\Omega)\,, \tag{4.204}$$

where Ω is the period matrix of Σ_g, and $Z_{\mathrm{CS}}(\gamma\cdot\Omega)$ is shorthand for the expression

$$Z_{\mathrm{CS}}(\gamma\cdot\Omega) = \frac{(\det \mathrm{Im}(\gamma\cdot\Omega))^{D/2}}{(\det\mathrm{Im}(\Omega))^{D/2}|\det{}'\,\bar\partial_{\Sigma_g}|^D}\,, \tag{4.205}$$

which is the $\mathrm{U}(1)^D\times\mathrm{U}(1)^D$ Chern-Simons partition function on a handlebody bounded by Σ_g [46]. The sum over modular images of Ω under the action of $P\backslash\mathrm{Sp}(2g,\mathbb{Z})$ defines a sum over different inequivalent handlebodies with boundary Σ_g. For later convenience, we introduce a basis for the homology group of Σ_g by choosing the branch cuts in equation (4.201) to be between neighboring points x_{2i-1} and x_{2i} for $i = 1,\ldots,g+1$, and choosing the $\mathcal{A}$ and $\mathcal{B}$ cycles of Σ_g as in Figure 4.22. Just as in the case of the four-point function, we can interpret the sum over handlebodies as a sum over rational tangles of vortices in the bulk. We do this in the following way: let $\mathcal{T}_0$ be the 'trivial' tangle defined by connecting x_{2i-1} to x_{2i} as shown in Figure 4.23. The double cover of the ball branched over this surface will be a genus g handlebody such that the $\mathcal{A}$ cycles of Σ_g (those surrounding the branch cuts between x_{2i-1} and x_{2i}) are contractible in the bulk. Now, let us pick out the point x_{2g+2} to be special. We can generate all rational tangles connecting the points $x_1,\ldots,x_{2g+2}$ via actions of the braid group B_{2g+1} (where we treat the points $x_1,\ldots,x_{2g+1}$ as individual strands, keeping x_{2g+2} fixed) on the trivial tangle $\mathcal{T}_0$. Via the double cover $\Gamma:\Sigma_g\to\mathbb{CP}^1$ associated to this configuration of points, the action of the braid group on the configuration X can be shown to induce an action on the fundamental group $\pi_1(\Sigma_g)$ or, similarly, its abelianization $H_1(\Sigma_g,\mathbb{Z})$. Specifically, if we let $\sigma_k\in B_{2g+1}$ be the element of the braid group which swaps the points x_k and x_{k+1} (with specified orientation), and we pick a homology basis $\mathcal{A}_i,\mathcal{B}_i$ on Σ_g, then we can take the action of the braid group on $H_1(\Sigma_g,\mathbb{Z})$ to be (breaking up cases for k even and odd)

$$\begin{aligned}
\sigma_{2i}\cdot\mathcal{A}_i &= \mathcal{A}_i+\mathcal{B}_i\,,\\
\sigma_{2i}\cdot\mathcal{B}_i &= \mathcal{B}_i\,,\\
\sigma_{2i}\cdot\mathcal{A}_{i+1} &= \mathcal{A}_{i+1}-\mathcal{B}_i\,,\\
\sigma_{2i-1}\cdot\mathcal{B}_{i-1} &= \mathcal{B}_{i-1}+\mathcal{A}_i\,,\\
\sigma_{2i-1}\cdot\mathcal{A}_i &= \mathcal{A}_i\,,\\
\sigma_{2i-1}\cdot\mathcal{B}_i &= \mathcal{B}_i-\mathcal{A}_i\,.
\end{aligned} \tag{4.206}$$

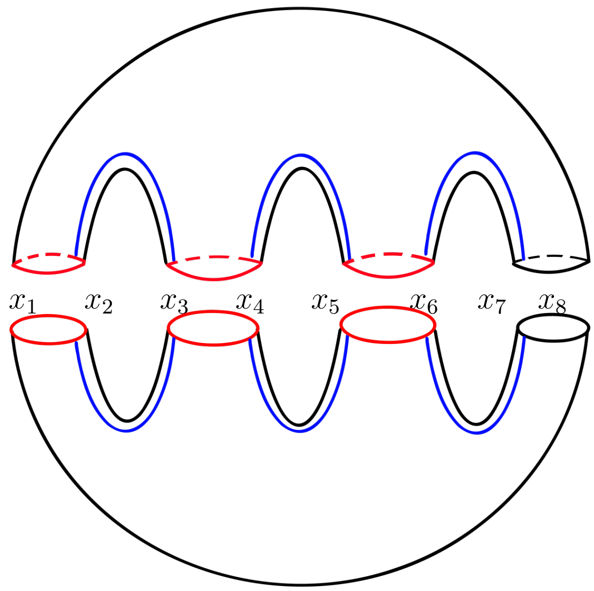


Figure 4.22: The homology basis of the genus g surface Σ_g. The red and blue curves represent the $\mathcal{A}$ and $\mathcal{B}$ cycles respectively. In the preimage, the red cycles are given by the branch cuts between x_i and x_{i+1}.

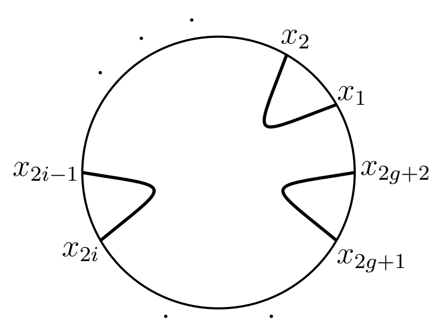


Figure 4.23: The trivial tangle $\mathcal{T}_0$.

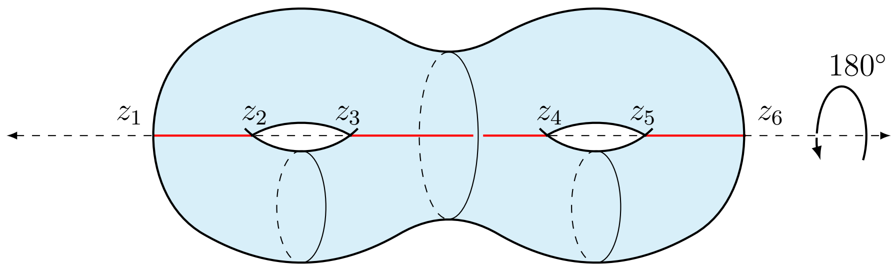


Figure 4.24: A handlebody filling in the hyperelliptic curve Σ_g ($g = 2$ shown). The hyperelliptic involution induces a 180° rotation around the symmetry axis, and the fixed points of this rotation are $g + 1$ intervals. The $\mathbb{Z}_2$ quotient of the handlebody under the hyperelliptic involution is homeomorphic to a ball with $g + 1$ vortices. The points z_i are the preimages of the points x_i in the correlator (4.202). Since a tangle is homeomorphic to the trivial tangle if and only if it is rational, the double-cover of a rational $(g+1)$-tangle is always a handlebody.

Furthermore, all other actions are trivial. This action defines a representation $\rho : B_{2g+1} \to \mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z})) \simeq \mathrm{GL}(2g, \mathbb{Z})$ of dimension $2g$ of the braid group by integer matrices. Furthermore, it is not hard to see that the above action preserves the intersection form $\langle \mathcal{A}_i, \mathcal{B}_j \rangle = \delta_{ij}$, and thus the representation of the braid group is actually via symplectic matrices. That is, there is a homomorphism

$$\rho : B_{2g+1} \to \mathrm{Sp}(2g, \mathbb{Z}) \,. \tag{4.207}$$

This is known as the *symplectic* representation of the braid group [196]. Given an element $\sigma \in B_{2g+1}$, the symplectic representation $\rho(\sigma)$ acts on the period matrix of Σ_g as

$$\Omega \to \sigma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1} \,, \tag{4.208}$$

where A, B, C, D are the block entries of $\rho(\sigma)$ as a symplectic matrix.

Now, while the action of the braid group on the $2g + 2$ endpoints of the $g + 1$ vortices induces $\mathrm{Sp}(2g, \mathbb{Z})$ transformations on the period matrix Ω of the hyperelliptic curve Σ, it is also possible to show that the resulting rational tangle (obtained by acting on the trivial tangle with the braid group) indeed has a double cover which is a handlebody of genus g, whose boundary is Σ_g.[43] Each rational tangle in the $G \wr S_2$ Chern-Simons theory, therefore, reproduces a handlebody in the sum (4.204). However, it is not clear that every element of the sum (4.204) over handlebodies is reproduced by a rational tangle of vortices in the Chern-Simons theory. Indeed, this is not the case. To see this, consider the image of the symplectic representation ρ. For $g = 1, 2$ this image is indeed the full modular group $\mathrm{Sp}(2g, \mathbb{Z})$, but for $g \geq 3$, $\rho(B_{2g+1})$ is only a finite-index subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ [196]. Thus, for $g \geq 3$, i.e. for n-point functions of twist fields with $n \geq 8$, the sum over rational tangles cannot reproduce the correct sum over the symplectic group.

[43] This follows again by the Montesinos trick – braid group actions on the trivial tangle lift to Dehn twists on the branched double cover, and so the double cover of any rational $(g + 1)$-tangle is related to a genus g handlebody by a Dehn twist, and is therefore also a genus g handlebody.

The reason why is because many handlebodies in the sum (4.204) do not respect the covering map structure of the boundary. Specifically, the covering map $\Gamma : \Sigma_g \to \mathbb{CP}^1$ has the structure of a $\mathbb{Z}_2$ quotient by the so-called *hyperelliptic involution* ι, which sends $(x, y) \to (x, -y)$ in equation (4.201). Given a choice of Lagrangian sublattice $\Gamma \subset H_1(\Sigma_g, \mathbb{Z})$ it is not guaranteed that the hyperelliptic involution extends to a symmetry of the bulk (when ι does extend to a symmetry of the bulk, it is represented by a 180° rotation around a fixed axis [197, 198], see Figure 4.24).[44] Those handlebodies M which admit an extension of the hyperelliptic involution are then branched covers $M \to M/\iota$ of 3-balls, for which the branching loci are rational tangles. Those handlebodies M for which the hyperelliptic involution does not extend into the bulk cannot, as far as we know, be thought of as 2-fold branched covers of a 3-manifold with spherical boundary, and thus represent another example of terms in the averaged symmetric orbifold theory which cannot be recovered by a semiclassical Chern-Simons calculation in the bulk.

The conformal anomaly

In the above discussion, we demonstrated how, in certain special cases, the ensemble average of correlators of twist fields in the symmetric orbifold can be reproduced by an appropriate sum over vortex configurations in a bulk $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_N$ Chern-Simons theory. In doing so, we only discussed the origin of the Eisenstein series appearing in the averaged correlation function from the Chern-Simons perspective, and we ignored the conformal anomaly which appears in the symmetric orbifold correlators. Let us now briefly comment on the origin of this prefactor from the Chern-Simons side.

First, let us recall the origin of the conformal anomaly in the symmetric orbifold setting. The 2-sphere $\mathbb{CP}^1$ can be locally taken to have fiducial metric $\mathrm{d}x\,\mathrm{d}\overline{x}$. In calculating the correlator of twist-fields on $\mathbb{CP}^1$, each contribution is given by an appropriate covering map $\Gamma : \Sigma \to \mathbb{CP}^1$. The associated contribution to the symmetric orbifold path integral arises from calculating the correlation functions of the pullback of the twist fields under Γ on the surface Σ. If Γ has the correct branching properties, then the pullbacks of these fields are untwisted, and the calculation reduces to the computation of a $\mathbb{T}^D$ correlator on Σ, see [158].

Importantly, in this construction, the path integral on Σ is evaluated with respect to the pullback metric

$$\Gamma^*(\mathrm{d}x\,\mathrm{d}\overline{x}) = |\partial\Gamma(z)|^2\,\mathrm{d}z\,\mathrm{d}\overline{z}\,, \tag{4.209}$$

where z is a local coordinate on Σ. This metric can be locally put back into the form of the fiducial metric $\mathrm{d}z\,\mathrm{d}\overline{z}$ via a Weyl rescaling $e^{-\Phi}$ with

$$\Phi = \log|\partial\Gamma|^2\,. \tag{4.210}$$

This Weyl rescaling comes at the cost of the conformal anomaly

$$\exp\left(-\frac{c}{6}S_L[\Phi]\right) = \exp\left(-\frac{c}{48\pi}\int_\Sigma \mathrm{d}^2z\,\sqrt{g}\left(-\frac{1}{2}\Phi\Delta\Phi + R\Phi\right)\right)\,, \tag{4.211}$$

[44] For low genera ($g = 1, 2$), however, the hyperelliptic involution always extends into the bulk.

where the metric g on Σ is the fudicial metric $\mathrm{d}z\,\mathrm{d}\overline{z}$, or more accuraltely the Weyl-transformed metric $e^{-\Phi}\Gamma^*\left(\mathrm{d}s^2_{\mathbb{CP}^1}\right)$.

On the Chern-Simons theory side, the conformal anomaly arises in a nearly identical way. Let us for the moment consider $G \wr S_N$ Chern-Simons theory for some generic group G. Given a bulk manifold M with spherical boundary $\partial M = \mathbb{CP}^1$ (in our case $M = \mathbb{H}^3$ is the 3-ball), we can consider the Chern-Simons path integral with vortex locus L. To compute this path integral, we must consider the branched cover $\widetilde{M}$ of M branched along the vortex locus L. Let $\gamma : \widetilde{M} \to M$ be the appropriate branched covering. We define $\Gamma = \gamma|_{\partial M}$ to be the induced branched covering $\Gamma : \Sigma \to \mathbb{CP}^1$ on the boundaries, where $\Sigma = \partial\widetilde{M}$. Now, we can gauge fix the Chern-Simons path integral by explicitly choosing a metric g on M. Then the $G \wr S_N$ Chern-Simons path integral is given by the G Chern-Simons path integral evaluated on $\widetilde{M}$ with metric $\gamma^* g$.

The Chern-Simons path integral is not completely independent of the choice of metric, and in particular depends on the boundary value of the metric on a specific bulk manifold. It is convenient to locally pick $g|_{\partial M} = \mathrm{d}x\,\mathrm{d}\overline{x}$. Then the induced boundary metric on $\widetilde{M}$ is simply the pullback of the boundary metric of M by γ, which is exactly the metric (4.209). We can perform a Weyl transformation by $e^{-\Phi}$ on the metric on $\widetilde{M}$[45] in order to bring the boundary metric of $\widetilde{M}$ into canonical form. However, the path integral of Chern-Simons theory is also not invariant under such transformations, and picks up a conformal anomaly [199]

$$\exp\left(-\frac{\dim(G)}{12}S_L[\Phi]\right) = \exp\left(-\frac{\dim(G)}{96\pi}\int_\Sigma \mathrm{d}^2 z\,\sqrt{g}\left(-\frac{1}{2}\Phi\Delta\Phi + R\Phi\right)\right)\,, \tag{4.212}$$

where the integral is over the boundary $\Sigma = \partial\widetilde{M}$ evaluated with the Weyl-transformed boundary metric. For the case of $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_N$ Chern-Simons theory, this reproduces the correct conformal anomaly of the symmetric orbifold theory since

$$\dim(G) = 2D = 2c\,. \tag{4.213}$$

The fact that the Chern-Simons theory picks up the same conformal anomaly as a boundary $\mathbb{T}^D$ CFT is, of course, to be expected. By the CS/WZW correspondence [200, 201], the partition function of $\mathrm{U}(1)^D \times \mathrm{U}(1)^D$ Chern-Simons theory on a smooth handlebody with boundary conditions specified in Section 4.1 should reproduce the vacuum character $|\chi_{\text{vac}}|^2$ of a (non-chiral) $\mathrm{U}(1)^D$ WZW model on the boundary. The vacuum character of a WZW model, however, picks up the appropriate conformal anomaly under Weyl transformations of the boundary metric.

[45] Here, Φ is understood to be a scalar field on $\widetilde{M}$ which reduces to $\log|\partial\Gamma|^2$ on the boundary.

4.5 Adding supersymmetry to the Narain ensemble

Narain averaging with supersymmetry

In this section we consider ensemble averaging the supersymmetric variants of the Narain theories. The supersymmetric $\mathbb{T}^D$ theories are obtained by simply adding D free fermions to the bosonic theory. Each fermion is a partner to one of the D free scalars. As discussed in 4.1, we must make a choice of boundary conditions for the fermions as they traverse nontrivial cycles. For a CFT on a torus we label the spin structure by two half integers α, β such that

$$\psi(A \cdot z) = e^{2\pi i \alpha}\psi(z)\,, \quad \psi(B \cdot z) = e^{2\pi i \beta}\psi(z)\,, \tag{4.214}$$

where again we take A to be the τ-cycle while B is the 1-cycle. The supersymmetric $\mathbb{T}^D$ partition function on the torus with a choice of spin structure is then given by

$$Z_{\mathbb{T}^D}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = Z_{\text{ferm}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)^D Z_{\mathbb{T}^D}(\tau) = |\det \overline{\partial}_{\alpha,\beta}|^D \frac{\Theta_D(m,\tau)}{|\eta(\tau)|^{2D}}\,, \tag{4.215}$$

where we have split the partition function into the bosonic and fermionic parts. Here, $\overline{\partial}_{\alpha,\beta}$ is the Dirac operator acting on spin $(\frac{1}{2}, 0)$ fermions with spin structure α, β. The determinant of these operators are given by theta functions. Specifically,

$$\left|\det \overline{\partial}_{\alpha,\beta}\right| = \frac{1}{|\eta(\tau)|} \times \begin{cases} |\vartheta_1(\tau)|\,, & \alpha = \beta = 0\,,\\ |\vartheta_2(\tau)|\,, & \alpha = \frac{1}{2}, \quad \beta = 0\,,\\ |\vartheta_3(\tau)|\,, & \alpha = \beta = \frac{1}{2}\,,\\ |\vartheta_4(\tau)|\,, & \alpha = 0\,, \quad \beta = \frac{1}{2}\,. \end{cases} \tag{4.216}$$

The exact form of the theta functions will not matter much in what follows. It suffices to remark on their modular properties. The partition function Z_{ferm} of a single free fermion, given by the above determinant, satisfies the modularity property

$$Z_{\text{ferm}}\begin{bmatrix}a\alpha + b\beta\\ c\alpha + d\beta\end{bmatrix}\left(\frac{a\tau + b}{c\tau + d}\right) = Z_{\text{ferm}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\,, \tag{4.217}$$

when acted on by an element of $\mathrm{SL}(2,\mathbb{Z})$. The partition function of the supersymmetric Narain theory can now be readily averaged, since only the theta function Θ_D depends on the Narain moduli. The result is

$$\int_{\mathcal{M}_D} \mathrm{d}\mu\, Z_{\mathbb{T}^D}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = Z_{\text{ferm}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)^D \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \frac{1}{|\eta(\gamma \cdot \tau)|^{2D}}\,. \tag{4.218}$$

The above partition function is recovered straightforwardly from the partition function of $\mathcal{N} = (1,1)$ Chern-Simons theory, summed over geometries. The boundary fermions in the

Chern-Simons theory have specified spin structure on the boundary, namely the same spin structure specified for the Narain CFT.

As explained around equation (4.16), each element of the sum over geometries is specified by a choice (c,d) of coprime integers such that the cycle $cA+dB$ becomes contractible in the bulk. Because the fermion picks up a sign of $(-1)^{2\alpha}$ around the A-cycle and $(-1)^{2\beta}$ around the B-cycle, the monodromy around the contractible cycle is given by $(-1)^{2(c\alpha+d\beta)}$. This flies in the face of the usual intuition that the boundary fermions must satisfy antiperiodic boundary conditions around a contractible bulk cycle. This only occurs if $c\alpha+d\beta$ is half-integer.

To be concrete, let us take the boundary theory to be in the NS sector, i.e. $\alpha=\beta=1/2$. Then in the bulk geometry associated to (c,d), the boundary fermion is periodic when $c+d$ is even, and antiperiodic when $c+d$ is odd. We can break up the sum over geometries into two sums:

$$\underbrace{Z_{\text{ferm}}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\tau)^D\sum_{\substack{(c,d)=1\\ c+d\text{ odd}}}\frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}}_{\text{`good' boundary conditions}}+\underbrace{Z_{\text{ferm}}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\tau)^D\sum_{\substack{(c,d)=1\\ c+d\text{ even}}}\frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}}_{\text{`bad' boundary conditions}}\,. \tag{4.219}$$

The 'good' boundary conditions correspond to the fermions being anti-periodic around the contractible cycle, while the bad correspond to the opposite. We give an explanation of this from the bulk perspective in the next subsection.

We can also consider the symmetric product orbifold of this supersymmetric theory, the only modification being the inclusion of the fermion partition function. It is easier to directly consider the grand canonical partition function in equation (4.73) which we showed can be written in terms of supersymmetric Hecke operators $\mathcal{T}_k$

$$\mathfrak{Z}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(p,\tau)=\exp\left(\sum_{k=1}^{\infty}p^k\mathcal{T}_kZ_{\mathbb{T}^D}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\right)\,. \tag{4.220}$$

We can average the above partition function in the same way we carried out the bosonic average. Since the fermions do not depend on the moduli they factor out of the average. It is convenient to consider the average of the connected part of the partition function

$$\begin{aligned}\int_{\mathcal{M}_D}\mathrm{d}\mu\,\log\mathfrak{Z}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(p,\tau)&=\sum_{k=1}^{\infty}p^k\left\langle\mathcal{T}_kZ\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\right\rangle\\ &=\sum_{k=1}^{\infty}\frac{p^k}{k}\sum_{ad=k}\sum_{b=0}^{d-1}Z_{\text{ferm}}\begin{bmatrix}a\alpha+b\beta\\ d\beta\end{bmatrix}\left(\frac{a\tau+b}{d}\right)^D\left\langle Z_{\mathbb{T}^D}\left(\frac{a\tau+b}{d}\right)\right\rangle,\end{aligned} \tag{4.221}$$

where we have split it up into the bosonic part which the average acts on, and the fermionic part. We will again reproduce the above average by performing a summation over bundles and vortices, now including fermions.

The bulk dual

The natural candidate bulk theory is a supersymmetric version of Chern-Simons. We will now briefly explain this supersymmetric theory, leaving our conventions and additional details to appendix A.8. In [202] supersymmetric Chern-Simons in flat Minkowski space was considered in the presence of a boundary. The boundary broke half the supersymmetry down to $\mathcal{N} = (1,0)$. In the conventions of [202] the flat space action is given by

$$S_{\text{CS}}^{\mathcal{N}=(1,0)} = \int_M d^3x(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \overline{\lambda}\lambda) - \frac{1}{2}\int_{\partial M} d^2x\sqrt{h}(h^{mn}A_mA_n + \overline{\chi}_-\gamma^m\partial_m\chi_-). \quad (4.222)$$

In the above λ, χ are Majorana fermions and the notation $\chi_\pm$ means we project the fermion onto it's top/bottom component respectively. The boundary is located at $x^3 = 0$ and the coordinates on the boundary are given by (x^1, x^2) indexed by the label m, the boundary metric is h and γ are the gamma matrices.

Due to the presence of the boundary, the action is not invariant under the most general supersymmetry transformation. However, it is invariant under half of the supersymmetry transformations $\mathcal{N} = (1,0)$.[46] These transformations are given by

$$\begin{aligned} \delta A_\mu &= (\overline{\lambda}\gamma_\mu\epsilon_+) + (\overline{\epsilon}_+\partial_\mu\chi_-), \\ \delta\lambda_a &= -\epsilon^{\mu\nu\rho}(\gamma_\rho\epsilon_+)_a\partial_\mu A_\nu, \\ \delta\chi_- &= (\gamma^\mu\epsilon_+)A_\mu = (\gamma^m\epsilon_+)A_m. \end{aligned} \quad (4.223)$$

Where ϵ_+ is a two component spinor projected onto only it's top component. In the path integral we must choose boundary conditions that satisfy the variational principle and that are left invariant under the above supersymmetry transformations. It was found in [202, 203] that one such choice of boundary conditions is given by fixing given by $A_- = 0$ and $2\gamma^2\lambda_+ + \partial_-\chi_- = 0$, where we have defined $A_\pm = A_1 \pm A_2$ and $\partial_\pm = \partial_1 \pm \partial_2$. The boundary action in terms of these fields is given by[47]

$$S_{\text{bdy}}^{\mathcal{N}=(1,0)} = \frac{1}{2}\int_{\partial M} d^2x(A_+A_- + \overline{\chi}_-\gamma^2\partial_-\chi_-). \quad (4.224)$$

The second boundary condition we fixed, relating χ_- and λ_+, guarantees that our boundary condition for the gauge field is invariant under the ϵ_+ supersymmetry transformation $\delta A_- = 0$. Similarly, the other boundary condition $2\gamma^2\lambda_+ + \partial_-\chi_- = 0$ is also invariant under ϵ_+ transformations. We don't need to set any additional boundary conditions because the second term in the boundary action (4.224) varies into an equation of motion for χ_- on the boundary. One of the interesting features of this theory is that there is a dynamical boundary fermion χ_-, decoupled from the other fields, without any corresponding bulk action.

[46] The action is invariant under these transformations without having to impose any boundary conditions on the fields. This approach was advocated as "supersymmetry without boundary conditions" in [202].

[47] For this one uses the identity $\gamma^1\chi_- = -\gamma^2\chi_-$.

Similarly, there is a $\mathcal{N}=(0,1)$ supersymmetric Chern-Simons theory invariant under ϵ_- transformations, given by the action

$$S_{\text{CS}}^{\mathcal{N}=(0,1)} = \int_M d^3x(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \overline{\lambda}\lambda) + \frac{1}{2}\int_{\partial M} d^2x\sqrt{g}(g^{mn}A_mA_n + \overline{\chi}_+\gamma^m\partial_m\chi_+). \tag{4.225}$$

In the above λ, χ_+ are again Majorana fermions, except now χ has been projected onto the top component. The boundary action can be re-written as

$$S_{\text{bdy}}^{\mathcal{N}=(0,1)} = \frac{1}{2}\int_{\partial M} d^2x(-A_+A_- + \overline{\chi}_+\gamma^2\partial_+\chi_+). \tag{4.226}$$

A consistent choice of boundary conditions is given by $A_+ = 0$ and $(-2\gamma^2\lambda_- + \partial_+\chi_+) = 0$. We can consider the combined action

$$S = S_{\text{CS}}^{\mathcal{N}=(1,0)} - S_{\text{CS}}^{\mathcal{N}=(0,1)}, \tag{4.227}$$

where in total the bulk theory has $\mathcal{N}=(1,1)$ supersymmetry with each half realized independently by one of two theories. The total action will depend on two independent Chern-Simons fields A, B, two independent auxiliary fields λ_1, λ_2 and two, fermions $\chi_\pm$ which have been projected onto the top/bottom component and effectively function as single component fermions. When considering the partition function of the total theory it will factorize into a contribution from the independent Chern-Simons fields, the two copies of the auxiliary fields λ_i, and the boundary fermions χ_-, χ_+. Since we are integrating over the auxiliary fields λ_i there is no restriction on the field configurations the boundary fermions take.

After analytically continuing to Euclidean signature and defining the theory on a bulk handlebody with an asymptotic boundary torus the full partition function is given by the product of contributions of: $U(1)\times U(1)$ Chern-Simons, a holomorphic and anti-holomorphic 2d free fermion, and an overall normalization given by integrating over the auxiliary λ_i. Dropping the normalization given by the auxiliary fields gives the partition function

$$Z_{\text{ferm}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)Z_{\text{CS}}(\tau) = \left|\det\overline{\partial}_{\alpha,\beta}\right|\frac{1}{|\eta(\tau)|^2}\,. \tag{4.228}$$

where the first factor comes from the Chern-Simons contribution and the second comes from the free fermions.

We take D copies of the above theory and perform a summation over all bulk handlebodies. The choice of asymptotic boundary conditions fixes the spin structures α, β around two particular boundary cycles A, B. When summing over bulk handlebodies the standard prescription is to only include handlebodies which can inherit the spin structure specified at the asymptotic boundary [174]. That is, if the cycle $cA + dB$ is contractible then it must be true that the fermions are anti-periodic around that cycle. However, in our case the fermions reduce to a boundary term, and so we do not have such a constraint since the spin structure does not need to be extended to the entire handlebody. Summing over all handlebodies, taking into account that the fermions give identical contributions due to (4.217), we find

$$Z_{\text{Bulk}}(\tau) = Z_{\text{ferm}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau)^D \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \left| \frac{1}{\eta(\gamma \cdot \tau)^2} \right|^D . \tag{4.229}$$

We see that the bulk supersymmetric Chern-Simons theory precisely reproduce the boundary ensemble average.

Let us now explain how the above is modified when we consider the symmetric product orbifold of the supersymmetric Narain theories. We are again interested in implementing twisted boundary conditions along the contractible and non-contractible cycles for both the gauge fields and the fermions.

Let's first consider the non-contractible cycle. In the case of the gauge fields we found that our gauge group was given by $(\mathrm{U}(1)^D \times \mathrm{U}(1)^D) \wr S_N$, and the summation over non-trivial S_N bundles gave us twisted boundary conditions for the gauge fields along the non-contractible cycle. The story for the fermions is similar. Fermions are sections of a spinor bundle with fiber denoted by S, and before quotienting the S_N symmetry the N fermions takes values in this spinor bundle. After quotienting we take the fiber of the spinor bundle to be given by $S \otimes \{1, \ldots, N\}$, and we must sum over all non-trivial bundles. The group S_N acts on the fiber by permutations acting on the the set $\{1, \ldots, N\}$. This implements a sum over all possible twisted boundary conditions for the fermions $\psi_I \to e^{2\pi i \alpha} \psi_{\pi(I)}$ when the S_N portion of the fiber is non-trivial, identical to the case of the gauge fields in Section 4.3.

For the contractible cycle, we can again use a vortex operator $\mathcal{V}$ to implement twisted boundary conditions for the fermions. We implement the action of the operator by specifying the monodromies it implements on the fermions, namely $\psi_I \to e^{2\pi i \beta} \psi_{\pi(I)}$ as we travel around the contractible cycle. Summing over all possible choices of vortices, similar to the gauge theory case in section 4.3, gives a summation over all twisted boundary conditions along the contractible cycle.

Combining these two ingredients together we find that a summation over bundles and vortices again implements the twisted boundary conditions necessary for the symmetric product orbifold. Since the fermions and gauge fields both transform in the adjoint of the discrete gauge group they acquire the same monodromy $A_I \to A_{\pi(I)}$ and $\psi_I \to e^{2\pi i \alpha} \psi_{\pi(I)}$ as they travel around a vortex. Since the fields acquire the same monodromies it immediately follows that that, identical to the bosonic case, performing a summation over bundles and vortices implements a summation over all degree N covering spaces of the base torus.

If we consider the grand canonical partition function of the supersymmetric Chern-Simons theory on handlebody M, since we are summing over covering spaces, we find that it is again given by the exponential of connected covers

$$\mathfrak{Z}_{\text{SCS}}(M, p) = \exp\left(\sum_{k=1}^{\infty} p^k \mathcal{T}_k Z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau) \right) . \tag{4.230}$$

Comparing to the boundary answer in equation (4.221), we immediately have that the connected covering space contributions match between the bulk and the boundary theories after summing over handlebodies M

$$\int_{\mathcal{M}_D} \mathrm{d}\mu \, \log \mathfrak{Z} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau, p) = \sum_M \log \mathfrak{Z}_{\mathrm{SCS}}(M, p) \, . \tag{4.231}$$

4.6 Conclusions and discussion

In this work our goal was to provide a bulk dual for the ensemble average over $\mathrm{Sym}^N(\mathbb{T}^D)$ CFTs. In the case of a torus boundary, we showed that a Chern-Simons theory with gauge group $\left(\mathrm{U}(1)^D \times \mathrm{U}(1)^D\right) \wr S_N$, with the inclusion of bulk vortices, reproduces what we denoted as the "semiclassical" portion of the boundary average. However, there remain many terms that do not have a clear semiclassical interpretation.

We argued that some of these terms can be included by appropriately modifying the rules of the gravity path integral. For instance, disconnected handlebodies that break replica symmetry can be reproduced by letting independent gauge fields A_I live on "independent" bulk manifolds M_I with distinct contractible cycles but the "same" asymptotic boundary, see the discussion around equation (4.143). Furthermore, we argued that the simplest wormhole geometry in equation (4.136) can be reproduced by having two vortices in the bulk. It's possible that other wormholes can also be reproduced by more complicated vortex configurations.

We will briefly contrast our results with the expectations for pure AdS_3 gravity [174, 204]. With a boundary torus, the only classical saddle-points for the Einstein-Hilbert action are handlebody geometries. However, in [160] it was argued that another reasonable class of geometries to include are conical defect geometries which are singular orbifolds. We note this is quite similar to what we have found. We also sum over handlebody geometries and the gauge theory analogue of conical defect geometries, since the bulk vortices make the field strength singular at the vortex. In addition, in AdS_3 gravity it is expected that additional off-shell geometries should be included [16, 48] in the path integral. While off-shell geometries seem to differ from the non-semiclassical geometries we have found, it is interesting that we also find contributions beyond simple handlebodies.

Ensemble Averaging Strings

As mentioned in the introduction, one motivation to consider the ensemble averaged symmetric orbifold comes from string theory. It has recently been shown that the tensionless string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ is precisely dual to the (super) symmetric orbifold $\mathrm{Sym}^N \mathbb{T}^4$ CFT at large N [61–65]. The tensionless string has string length equal to the AdS length, and so the theory is far from any sensible semiclassical limit. While all observables of the theory can be calculated through worldsheet path integrals, there is no known sense in which the string partition function can be approximated by a gravitational path integral with a sum over smooth bulk geometries. One hope is that a suitable sum over geometries may

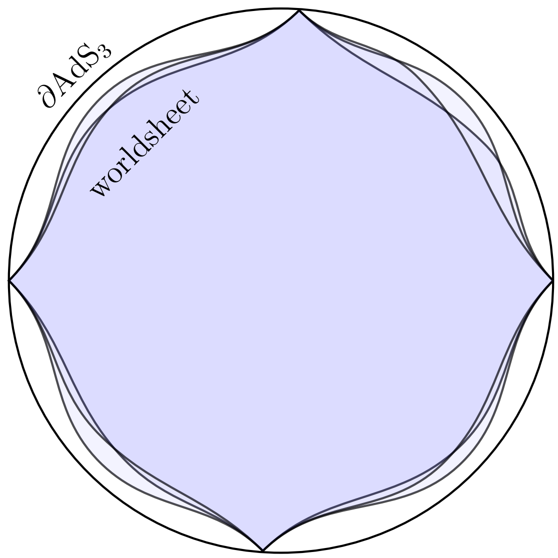


Figure 4.25: A cartoon representation of a worldsheet propagating within the bulk in the tensionless limit. The dynamics of the string are constrained to the boundary of spacetime.

emerge if we average over the moduli of the tensionless string. We now explain how this is partially realized.

The natural partition function for the tensionless string is given by the grand canonical ensemble of $\mathrm{Sym}(\mathbb{T}^4)$ [65]. The genus-one partition function of the grand canonical ensemble is given by[48]

$$\mathfrak{Z}(p,\tau) = \exp\left(\sum_{k=1}^{\infty} \underbrace{p^k T_k Z(\tau)}_{\text{worldsheets}}\right) . \tag{4.232}$$

The Hecke operators T_k enumerate over connected covering spaces that cover the boundary of AdS_3. Holographically, the connected covering spaces are to be interpreted as worldsheets of strings propagating in the target AdS_3 which wind the boundary k times. The argument of the exponential only includes contributions with a single string on the AdS_3 background. The grand canonical partition function $\mathfrak{Z}$, after expanding the exponential, then includes any number of disconnected strings propagating on the AdS_3 background. The free energy $F = \log \mathfrak{Z}$ is then, in this sense, a 'first quantized' partition function of the string theory, while the partition function $\mathfrak{Z}$ can be thought of as a 'second quantized' quantity.

Within the main text, we considered averaging both the connected and disconnected contributions of the symmetric orbifold, or equivalently we considered the averages of both the first and second quantized partition functions. From the point of string theory, these averages have the following interpretation:

- **First quantized average:** Averaging the string theory at the level of the worldsheet

[48]Here and in what follows we ignore spin structures.

sigma model on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$. This results in considering only an average of the free energy $F = \log \mathfrak{Z}$, which in turn is given by

$$\langle F(p,\tau)\rangle = \sum_{k=1}^{\infty} p^k \langle T_k Z(\tau)\rangle \ . \tag{4.233}$$

As discussed in Section 4.3, this average is reproduced by a sum over torus handlebodies of the free energy of the grand-canonical $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_N$ Chern-Simons theory with vortices in the bulk.

- **Second quantized average:** Averaging the string theory at the level of the second quantized theory of strings, i.e. at the level of string field theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$. The resulting partition function is

$$\langle \mathfrak{Z}(p,\tau)\rangle = \left\langle \exp\left(\sum_{k=1}^{\infty} p^k T_k Z(\tau)\right)\right\rangle \ . \tag{4.234}$$

As we have seen throughout this paper, the resulting expansion can be reproduced partially by bulk $\mathrm{U}(1)^D \times \mathrm{U}(1)^D \wr S_N$ Chern-Simons quantities on semiclassical bulk geometries, but many of the terms in the average cannot be recovered this way, and are thought of as arising from 'generalized' geometries.

Of course, from the point of view of string theory, these are both completely valid course-grainings of the $\mathbb{T}^4$ compactification. It's interesting to note that a single string (4.233) propagating on AdS_3 has a simple ensemble averaged interpretation in terms of a sum over handlebodies with vortices. When we average over multiple strings (4.234) on one background we get additional "non-semiclassical" geometries connecting the strings, for which we have no standard holographic interpretation.

One subtlety with the above is the issue of convergence. As explained in Section 4.1, the average of $\mathbb{T}^D$ and therefore of the $\mathrm{Sym}(\mathbb{T}^D)$ partition functions only converges when $D > g+1$. For the specific case of the tensionless string, $D = 4$, this means that most contributions to the grand canonical average diverge. However, the first quantized average (4.233) converges. Nevertheless it is still tempting to consider each individual term in the average as arising from a bulk geometry. A bulk interpretation of the divergence is that the sum over wormhole geometries diverges when we have too many asymptotic boundaries.

Semiclassical Limit

It would be interesting if there was an appropriate semiclassical limit of $\langle Z_{\mathbb{T}^D \wr S_N}(\tau)\rangle$ where the dominant geometry is given by the $\left(U(1)^D \times U(1)^D\right) \wr S_N$ Chern-Simons theory, as opposed to one of the more mysterious contributions. We do not have a coupling constant we can tune, but there does appear to be a natural notion of a semiclassical limit for the Narain average which is given by sending $D \to \infty$ [46, 205].

For $N = 2$ this can be argued very explicitly. The proper limit is to take $D \to \infty$ while holding the temperature $\tau = i\beta$ sufficiently large or small. In this limit, the contribution of wormholes to $\langle Z^2 \rangle$ is exponentially suppressed relative to the handlebodies [205]. Furthermore, it's straightforward to see that the disconnected handlebodies that dominate $\langle Z^2 \rangle$ will be replica symmetric in the large D limit.[49] The symmetric disconnected covers are captured by the Chern-Simons calculation, and therefore we find in this limit

$$\lim_{D\to\infty} \lim_{\beta\to\infty} \langle Z_{\mathbb{T}^D \wr S_2}(i\beta) \rangle = \sum_{\substack{\text{handlebodies } M, \\ \text{vortices}}} Z_{G\wr S_2}(M) + \mathcal{O}(\beta^{-cD}), \tag{4.235}$$

where the subleading terms are exponentially suppressed in D, with c some constant [205], relative to the dominant term in the semiclassical contribution. An obvious generalization of this equation is true when $\beta \to 0$. Thus, at least in this limit a semiclassical geometry dominates the average. It would be interesting to know if this holds for general N, which would require understanding whether disconnected symmetric handlebodies dominate $\langle Z^N \rangle$ in the large D limit.

Open Questions and future directions

We will end with a discussion of some open questions and possible future directions.

OPE Randomness Hypothesis: In [206] the OPE randomness hypothesis was proposed. This amounts to a generalization of ETH to an ansatz for the OPE coefficients of chaotic CFTs. The proposal conjectures that OPE coefficients involving heavy operators are random variables obeying an approximately Gaussian distribution. Both the $\mathbb{T}^D$ and $\mathrm{Sym}(\mathbb{T}^D)$ theories should evade these proposals since the theories are not chaotic. The symmetry currents furnish selection rules which should lead to deviations away from the aforementioned subleading non-Gaussianities. It would be interesting to determine the statistics of the OPE coefficients in the Narain ensemble.

Deforming Towards Supergravity: There is a general expectation that the tensionless regime can be deformed to the semiclassical supergravity regime [207] by turning on an exactly marginal operator in a twisted sector of the orbifold theory, see for example [208–213]. It would be interesting to understand if it is possible to ensemble average over the Narain moduli space with this marginal deformation turned on. One potential approach would be to use conformal perturbation theory, and our discussion of correlators in Section 4.4 might be useful for such an approach. It would be very interesting to see how the sum over bulk geometries is modified as we move away from the tensionless point.

[49] This is because non-replica symmetric disconnected covers are subleading due to the Hawking-Page transition at large/small temperatures.

Averaging the D1/D5 moduli space: As mentioned above, symmetric product orbifold theories $\mathrm{Sym}^N(\mathbb{T}^D)$ live in a larger moduli space than that of their Narain seed theory, and specifically for seed theories of central charge $c \leq 6$, symmetric product orbifolds admit marginal deformations by twist fields, which break the orbifold structure. One special such example is the D1/D5 system compactified on $\mathbb{T}^4$ [207], whose moduli space contains the 'orbifold point' $\mathrm{Sym}^N(\mathbb{T}^4)$ along with its Narain moduli, as well as four exactly marginal operators in the twist-2 sector.[50] It would be interesting to consider an average over the full 20-dimensional moduli space of the D1/D5 CFT. Such a computation, however, would require knowledge of unprotected quantities on the full D1/D5 moduli space, which are currently poorly understood beyond low-orders in conformal perturbation theory.

Large N and Phase Transitions: The grand canonical ensemble of $\mathrm{Sym}(\mathbb{T}^4)$ is dual to the tensionless string with any number of strings propagating on the background geometry. It was argued in [64] that at large N the grand canonical partition function is dominated by a configuration of strings that wind $\sim N$ times around the AdS_3 boundary. This corresponds to the average of the grand canonical ensemble being dominated by Chern-Simons with gauge group $G \wr S_N$ with large N. In the case of a single winding string this would correspond to a geometry with a single vortex, whereas multiple winding strings would partially be reproduced by multiple disconnected covering spaces with multiple vortices. It was shown that there are phase transitions between different stringy configurations as the chemical potential was tuned [63], it would be interesting to better understand these transitions from the perspective of the Chern-Simons geometries considered in this work.

Stringy Ensemble-Averaging: In this work we have attempted to construct an effective theory of a course-grained string theory. In view of the universal complexity of full-fledged UV compactifications there may be some logic in considering some form of course-graining as a starting point. It would be highly interesting to seek out other examples of string compactifications amenable to some form of ensemble averaging. See [160] for work in this direction.

[50]These operators roughly correspond to introducing a non-zero 't Hooft coupling λ.

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Appendix A

Appendix

A.1 JT Gravity With Conical Defects

A.2 Useful Formulas

In this Appendix we will collect useful identities that will be relevant in the main text.

Legendre polynomials

The Legendre polynomials $P_n(x)$ play a crucial role in the minimal string theory string equation. $P_n(x)$ is a polynomial of order n defined by

$$P_n(x) \equiv \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n , \tag{A.1}$$

where n is an integer. The normalization is chosen such that $P_n(1) = 1$. These polynomials can also be written as a particular case of hypergeometric function

$$P_n(x) = {}_2F_1 \left(-n, n+1, 1, \frac{1-x}{2} \right) . \tag{A.2}$$

The following relation is also useful

$$\frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1} = (x-1)\, {}_2F_1 \left(-n, n+1, 2, \frac{1-x}{2} \right) \tag{A.3}$$

and the derivative of this combination of polynomials is given by

$$\frac{d}{dx} \left(P_{n+1}(x) - P_{n-1}(x)\right) = (2n+1) P_n(x). \tag{A.4}$$

This is precisely the combinations appearing in the string equation of the undeformed minimal string.

Integrals: The following integrals are particularly relevant since they appear in the calculation of the minimal string density of states from the string equation. Having in mind this application, we call $p = 2m - 1$ and introduce an arbitrary parameter κ. Then the following identities hold

$$\int_\kappa^E \frac{du}{\sqrt{E-u}} P_{m-1}\left(\frac{u}{\kappa}\right) = \frac{2\sqrt{2\kappa}}{p} \sinh\left(\frac{p}{2}\text{arccosh}\left(\frac{E}{\kappa}\right)\right), \tag{A.5}$$

and, taking L to be an integer,

$$\int_\kappa^E \frac{du}{\sqrt{E-u}} \partial_u P_L\left(\frac{u}{\kappa}\right) = \frac{\sqrt{2\kappa}}{\sqrt{E^2-\kappa^2}} \cosh\left(\frac{2L+1}{2}\text{arccosh}\left(\frac{E}{\kappa}\right)\right) - \frac{1}{\sqrt{E-\kappa}}. \tag{A.6}$$

When computing disk partition functions the two integral representation of Bessel functions are very useful

$$\int_1^\infty dt\, e^{-st}\, \frac{2}{p} \sinh\left(\frac{p}{2}\, \text{arccosh}(t)\right) = \frac{1}{s} K_{p/2}(s) \tag{A.7}$$

and

$$\int_1^\infty dt\, e^{-st}\, \frac{\cosh\left(\frac{p}{2}\, \text{arccosh}(t)\right)}{\sqrt{t^2-1}} = K_{p/2}(s) \tag{A.8}$$

Finally another useful identity

$$\frac{P_m(x) - P_{m-2}(x)}{2m-1} + \sum_{n=1}^{m-1} \frac{\lambda^n}{n!} \partial_x^{n-1} P_{m-n-1}(x) = (1-2\lambda)^{m/2} \frac{P_m(\frac{x}{\sqrt{1-2\lambda}}) - P_{m-2}(\frac{x}{\sqrt{1-2\lambda}})}{2m-1}. \tag{A.9}$$

This identity can be proven using the representation of the Legendre polynomial as a contour integral used in [92].

Large order limit: It will be relevant in order to take the JT limit to consider the large order limit of these Legendre polynomials. In particular we will use the following relation

$$P_{m-n-1}\Big(1 + \frac{8\pi^2}{(2m-1)^2} u\Big) \to I_0(2\pi\alpha\sqrt{u}), \qquad n = \frac{2m-1}{2}(1-\alpha) \tag{A.10}$$

where we take $m \to \infty$ with $0 < \alpha < 1$ and u fixed. To prove this we can rewrite the Legendre polynomial as a hypergeometric function, take the $m \to \infty$ limit of the Taylor expansion in u, and then recognize precisely the Taylor coefficient of the Bessel function on the right hand side.

A related useful limit will be

$$\frac{2m-1}{8\pi} \left[P_m\Big(1 + \frac{8\pi^2}{(2m-1)^2} u\Big) - P_{m-2}\Big(1 + \frac{8\pi^2}{(2m-1)^2} u\Big)\right] \to \sqrt{u}\, I_1\left(2\pi\sqrt{u}\right), \tag{A.11}$$

where we take $m \to \infty$. For a proof of this limit see [22].

Bessel function identities

The modified Bessel functions of the first kind $I_n(x)$ appear from taking the JT limit of the minimal string theory string equation. They satisfy the symmetry property $I_{-n}(x) = I_n(x)$ and the following recurrence relation

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m \Big(x^{-n} I_n(x)\Big) = x^{-(n+m)} I_{n+m}(x) \tag{A.12}$$

which occur in the derivation of the JT with defects string equation.

Integrals: The following integral of the modified Bessel functions appears in the calculation of the JT with defects density of states from the string equation

$$\int_0^E \frac{du}{2\sqrt{E-u}} \left(\frac{a}{\sqrt{u}}\right)^{n+1} I_{n+1}\Big(a\sqrt{u}\Big) = \sqrt{\frac{\pi}{2}} \left(\frac{a}{\sqrt{E}}\right)^{n+1/2} L_{n+1/2}\Big(a\sqrt{E}\Big). \tag{A.13}$$

where L_ν is the modified Struve function, evaluated at half-integer order. These functions are related to the more familiar modified Bessel functions by

$$L_{n+1/2}(x) = I_{-n-1/2}(x) - \frac{1}{2^n}\sqrt{\frac{2}{\pi}} \sum_{m=0}^{n} \frac{(-1)^m (2m)!}{n!(n-m)!} x^{n-2m-1/2}. \tag{A.14}$$

The Bessel functions at half-integer order actually have elementary form [93]

$$I_{-n-1/2}(x) = \frac{1}{\sqrt{2\pi x}} \sum_{m=0}^{n} \frac{(n+m)!}{m!(n-m)!} (2x)^{-m} \Big((-1)^m e^x + (-1)^n e^{-x}\Big). \tag{A.15}$$

A useful way of writing the above summation is as follows

$$I_{-n-1/2}(x) = (-1)^n \sqrt{\frac{2}{\pi x}} \Big(p_n\left(1/x\right) \cosh x + q_{n-1}\left(1/x\right) \sinh x\Big) \tag{A.16}$$

where we define the finite polynomials

$$p_n\left(x\right) = \sum_{\substack{0\le m\le n:\\ m=n \bmod 2}} \frac{(n+m)!}{m!(n-m)!} (x/2)^m, \quad q_n\left(x\right) = \sum_{\substack{0\le m\le n+1:\\ m=n+1 \bmod 2}} \frac{(n+1+m)!}{m!(n+1-m)!} (x/2)^m. \tag{A.17}$$

It is easy to see that p_n and q_n have degree n, are vanishing for $n < 0$ and have definite parity.

Laplace transforms: The Bessel functions are related to elementary functions via the following useful Laplace transforms. When computing the partition function from the JT with defects density of states we have the following integral: for any $\nu > -1$,

$$\int_0^\infty dE\, e^{-\beta E}\left(\frac{a}{\sqrt{E}}\right)^\nu I_\nu\left(a\sqrt{E}\right) = 2^{-\nu}\beta^{-\nu-1}e^{\frac{a^2}{4\beta}}. \tag{A.18}$$

On the other hand, for any $\nu > -1/2$,

$$\int_0^\infty d\varphi\, e^{-\varphi y}\left(\frac{\varphi}{\sqrt{u}}\right)^\nu I_\nu(\sqrt{u}\varphi) = \frac{2^\nu\Gamma(\nu+1/2)}{\sqrt{\pi}}\left(y^2-u\right)^{-\nu-1/2}. \tag{A.19}$$

The case $\nu = -1$ is special and evaluates to

$$\int_0^\infty d\varphi\, e^{-\varphi y}\,\frac{\sqrt{u}}{\varphi}I_{-1}(\sqrt{u}\varphi) = y - \sqrt{y^2-u}. \tag{A.20}$$

These expressions are used in simplifying the JT with defects string equation.

A.3 Disk one-point function: Normalization

In this Appendix we compare the minimal string bulk one point function calculation to the matrix model prediction originally computed by [214], finding agreement as expected. We leave out the technical details, referring the interested reader to [22, 214, 215]. We mostly follow the conventions of [22] for the continuum calculations.

Liouville approach: First we will compute the continuum partition function to leading order in the deformation to the action $I_{\rm MS} \to I_{\rm MS} - \tau_n \mathcal{T}_n$. Then we will call $Z_0^{\rm MS}(\ell)$ the disk partition function in the undeformed minimal string and $Z_1^{\rm MS}(\ell) = \tau_n\langle\mathcal{T}_n\rangle_\ell$ the linear order correction in τ_n. In both cases the continuum calculation factorizes into a Liouville part and a minimal model matter part.

The undeformed minimal string disk partition function with boundary length ℓ, following the conventions of [22], is given by

$$\begin{aligned} Z_0^{\rm MS}(\ell) &= \frac{8\pi}{b}\left(\pi\mu\gamma(b^2)\right)^{\frac{1}{2b^2}}\frac{(1-b^2)}{\Gamma(b^{-2})}\int_\kappa^\infty d\mu_B\; e^{-\ell\mu_B}\sinh\left(\frac{1}{b^2}\text{arccosh}\frac{\mu_B}{\kappa}\right) && \text{(A.21)}\\ &= \frac{8\pi}{b}\left(\pi\mu\gamma(b^2)\right)^{\frac{1}{2b^2}}\frac{1}{\Gamma(\frac{1}{b^2}-1)}\frac{1}{\ell}K_{\frac{1}{b^2}}(\kappa\ell). && \text{(A.22)}\end{aligned}$$

The linear order contribution is given by the fixed length tachyon one point function. Following the conventions of [22] it is given by

$$Z_1^{\rm MS}(\ell) = \tau_n\langle\mathcal{T}_n\rangle_\ell = \tau_n\frac{4\pi}{b}\left(\pi\mu\gamma(b^2)\right)^{-iP/b}\frac{\Gamma\left(1+2iPb\right)}{\Gamma\left(-2iP/b\right)}K_{\frac{2iP}{b}}(\kappa\ell)\times\langle\mathcal{O}_{1,n}\rangle_{1,1}, \tag{A.23}$$

where $P = i(1-nb^2)/(2b)$, K_ν is the modified Bessel function of the second kind, and the matter one point function $\mathcal{O}_{1,n}$ is evaluated with identity brane boundary conditions and the identity operator normalized as $\langle \mathcal{O}_{1,1}\rangle_{1,1} = 1$. The matter one point function can be found from the modular S-matrix of the minimal model and is equal to [216]

$$\langle \mathcal{O}_{1,n}\rangle_{1,1} = \sqrt{\frac{S^{1,n}_{1,1}}{S^{1,1}_{1,1}}} = i^{n-1}\sqrt{\frac{\sin(\pi n b^2)}{\sin(\pi b^2)}}. \tag{A.24}$$

It is convenient to write the final answer in terms of the leg-factor

$$\mathrm{Leg}(n) = \frac{i^{n-1}}{2}\sqrt{\frac{\pi\gamma(nb^2)}{\mu(\pi\mu\gamma(b^2))^n}}\,\frac{\Gamma(\frac{1}{b^2}-1)}{\Gamma(\frac{1}{b^2}-n)}. \tag{A.25}$$

The normalized tachyon one point function is now given by

$$\frac{Z_1^{\mathrm{MS}}(\ell)}{Z_0^{\mathrm{MS}}(\ell)} = \tau_n\,\mathrm{Leg}(n)\frac{\kappa\ell K_{\frac{p}{2}-n}(\kappa\ell)}{K_{\frac{p}{2}}(\kappa\ell)}, \tag{A.26}$$

where we used that $b^2 = 2/p$ and rewrote the Liouville momentum P in terms of the minimal model operator n. This is the continuum prediction of the normalized disk one-point function. Now we will compare it with the matrix model approach.

Matrix Model: From the matrix model we will compute the correction to the disk partition function to linear order in λ_n. We expand both the density of states $\rho(E) = \rho_0(E) + \rho_1(E) + \mathcal{O}(\lambda_n^2)$ and the partition function $Z^{\mathrm{MM}}(\ell) = Z_0^{\mathrm{MM}}(\ell) + Z_1^{\mathrm{MM}}(\ell) + \mathcal{O}(\lambda_n^2)$. To this order the string equations for any deformation n is given by

$$\mathcal{F}(u) = \frac{p}{16\pi^2}\left[P_m\left(\frac{u}{\kappa}\right) - P_{m-2}\left(\frac{u}{\kappa}\right)\right] + \lambda_n P_{m-n-1}\left(\frac{u}{\kappa}\right) + \mathcal{O}(\lambda_n^2) \tag{A.27}$$

The undeformed disk density of states is

$$\rho_0(E) = \frac{p}{2\pi\kappa}\int_\kappa^E \frac{du}{\sqrt{E-u}}P_{m-1}\left(\frac{u}{\kappa}\right) = \frac{2\sqrt{2\kappa}}{2\pi p}\frac{p}{16\pi^2}\frac{p}{\kappa}\sinh\left(\frac{p}{2}\mathrm{arccosh}\left(\frac{E}{\kappa}\right)\right), \tag{A.28}$$

and the partition function is

$$Z_0^{\mathrm{MM}}(\ell) = \int_\kappa^\infty dE\,\rho_0(E)e^{-\ell E} = \frac{\sqrt{2\kappa}}{2\pi}\frac{p^2}{16\pi^2}\frac{1}{\kappa\ell}K_{p/2}(\kappa\ell). \tag{A.29}$$

To order $\sim \lambda_n$ we get a contribution from the linear order correction to the string equation and also from the linear order correction to E_0 applied to the density of states coming from the zeroth order string equation.

Lets consider first the latter. The contribution from the order λ_n^0 term in the string equation is

$$\frac{1}{2\pi}\int_{E_0}^{E}\frac{du}{\sqrt{E-u}}\partial_u\mathcal{F}_0(u)=\rho_0(E)-\frac{1}{2\pi}\int_{\kappa}^{E_0}\frac{du}{\sqrt{E-u}}\partial_u\mathcal{F}_0(u). \tag{A.30}$$

Now define $E_0=\kappa+\delta E_0$, where δE_0 is of order λ_n, and expand to linear order in δE_0. Therefore the second integral above is over a small range. We can approximate

$$\frac{1}{2\pi}\int_{E_0}^{E}\frac{du}{\sqrt{E-u}}\partial_u\mathcal{F}_0(u)=\rho_0(E)-\frac{\delta E_0\partial_u\mathcal{F}_0(\kappa)}{2\pi\sqrt{E-\kappa}}+\mathcal{O}(\lambda_n^2). \tag{A.31}$$

Finally, if we expand the equation for E_0, the string equation $\mathcal{F}_0(E_0)+\lambda_n\mathcal{F}_1(E_0)+\mathcal{O}(\lambda^2)=0$, to linear order in λ_n and use the fact that $\mathcal{F}_0(\kappa)=0$, we obtain

$$\mathcal{F}_0(\kappa)+\delta E_0\partial_u\mathcal{F}_0(\kappa)+\lambda_n\mathcal{F}_1(\kappa)+\mathcal{O}(\lambda_n^2)=0\quad\Rightarrow\quad\delta E_0=-\frac{\lambda_n\mathcal{F}_1(\kappa)}{\partial_u\mathcal{F}_0(\kappa)}. \tag{A.32}$$

In our case, the string equation is given by (A.28). Therefore $\mathcal{F}_1(u)=P_{m-n-1}(u/\kappa)$ and $\mathcal{F}_1(\kappa)=1$, giving $\delta E_0\partial_u\mathcal{F}_0(\kappa)=-\lambda_n$. Inserting this in equation (A.31) we get the linear order correction from the shift in the edge of the spectrum as

$$\frac{1}{2\pi}\int_{E_0}^{E}\frac{du}{\sqrt{E-u}}\partial_u\mathcal{F}_0(u)=\rho_0(E)+\frac{\lambda_n}{2\pi\sqrt{E-\kappa}}+\mathcal{O}(\lambda_n^2). \tag{A.33}$$

Adding all terms, the final answer for the linear order density of states is

$$\begin{aligned}\rho_1(E) &= \frac{\lambda_n}{2\pi}\int_{\kappa}^{E}\frac{du}{\sqrt{E-u}}\partial_u P_{m-n-1}\left(\frac{u}{\kappa}\right)+\frac{\lambda_n}{2\pi\sqrt{E-\kappa}} &\text{(A.34)}\\ &= \lambda_n\frac{\sqrt{2\kappa}}{2\pi\sqrt{E^2-\kappa^2}}\cosh\left(\frac{p-2n}{2}\operatorname{arccosh}\left(\frac{E}{\kappa}\right)\right). &\text{(A.35)}\end{aligned}$$

Notice that after a change of variables $E(s)=\kappa\cosh(2\pi bs)$, this equation becomes precisely the density of states (2.60) obtained from the Liouville one-point function. The partition function can be easily computed using the identity (A.8), obtaining

$$Z_1^{\rm MM}(\ell)=\int_\kappa^\infty dE\ \rho_1(E)e^{-\ell E}=\frac{\sqrt{2\kappa}}{2\pi}\lambda_n K_{\frac{p}{2}-n}(\kappa\ell). \tag{A.36}$$

Putting the two results together we can compute the ratio between the zeroth and linear order partition function

$$\frac{Z_1^{\rm MM}(\ell)}{Z_0^{\rm MM}(\ell)}=\lambda_n\frac{16\pi^2}{p^2}\frac{\kappa\ell K_{\frac{p}{2}-n}(\kappa\ell)}{K_{\frac{p}{2}}(\kappa\ell)}. \tag{A.37}$$

This is the prediction from the matrix model.

Comparison: Now we can compare the continuum result (A.26) with the matrix model calculation (A.37). Demanding that both results agree gives the following identification between the parameter in the string equation λ_n and the parameter in the action τ_n given by

$$\lambda_n = \tau_n \,\mathrm{Leg}(n) \frac{p^2}{16\pi^2}, \tag{A.38}$$

with $\mathrm{Leg}(n)$ given by (A.25). This dictionary between the string equation and the coupling of the deformation in the action can also be obtained by comparing the matrix model sphere correlation functions with the one computed by Liouville. This is done for example in [217] and also [92] and it matches with the identification (A.38) obtained from the fixed length disk correlator.

A.4 String equation simplification

In this Appendix, we will show that the string equation for JT gravity with general defects (2.67) can be written as in (2.79)

$$\mathcal{F}(u) = \int_{\mathcal{C}} \frac{dy}{2\pi i}\, e^{\varphi y}\left(y - \sqrt{y^2 - u - 2W(y)}\right)\Big|_{\varphi=2\pi}, \quad W(y) = \sum_i \lambda_i e^{-2\pi(1-\alpha_i)y}, \tag{A.39}$$

which is an inverse Laplace transform evaluated at $\varphi = 2\pi$. The contour $\mathcal{C}$ is taken to be along to the imaginary direction with all singularities to the left. For simplicity, we will restrict to a single defect species α and evaluate the formula by Taylor expanding in the coupling λ

$$y - \sqrt{y^2 - u - 2W(y)} \;=\; y - \sqrt{y^2 - u} + \sum_{L=1}^{\infty} \frac{\lambda^L}{L!} \frac{2^{L-1}\Gamma(L-1/2)}{\sqrt{\pi}} \frac{e^{-2\pi L(1-\alpha)y}}{(y^2-u)^{L-1/2}} \tag{A.40}$$

Using (A.20) the first term evaluates to the string equation for pure JT gravity

$$\int_{\mathcal{C}} \frac{dy}{2\pi i}\, e^{\varphi y}\left(y - \sqrt{y^2 - u}\right)\Big|_{\varphi=2\pi} = \frac{\sqrt{u}}{2\pi} I_1(2\pi\sqrt{u}). \tag{A.41}$$

For $L \geq 1$, we can make use of the formula (A.19) to write

$$\left(\frac{2\sqrt{u}}{\varphi}\right)^{L-1} \int_{\mathcal{C}} \frac{dy}{2\pi i}\, e^{\varphi y}\, \frac{2^{L-1}\Gamma(L-1/2)}{\sqrt{\pi}} (y^2-u)^{-L+1/2} = \left(\frac{\varphi}{\sqrt{u}}\right)^{L-1} I_{L-1}(\sqrt{u}\varphi) \tag{A.42}$$

and that for $a \geq 0$

$$f(\varphi - a)\Theta(\varphi - a) = \int_{\mathcal{C}} \frac{dy}{2\pi i}\, e^{\varphi y} \tilde{f}(y) e^{-ay}. \tag{A.43}$$

The answer at $\mathcal{O}(\lambda^L)$ is then

$$\left(\frac{\varphi - 2\pi L(1-\alpha)}{\sqrt{u}}\right)^{L-1} I_{L-1}\Big((\varphi - 2\pi L(1-\alpha))\sqrt{u}\Big)\,\Theta(\varphi - 2\pi L(1-\alpha)), \quad L \geq 0. \tag{A.44}$$

Substituting this back into the sum and setting $\varphi = 2\pi$, we find

$$\mathcal{F}(u) = \frac{\sqrt{u}}{2\pi} I_1(2\pi\sqrt{u}) + \sum_{L=1}^{\lfloor \frac{1}{1-\alpha} \rfloor} \frac{\lambda^L}{L!}\left(\frac{2\pi(1-L(1-\alpha)}{\sqrt{u}}\right)^{L-1} I_{L-1}\Big(2\pi(1-L(1-\alpha))\sqrt{u}\Big) \tag{A.45}$$

which is precisely the string equation for a single species (2.65). The string equation for an arbitrary number of species follows from an analogous calculation, resulting in the replacements $\lambda^L \to \Pi_i \lambda_i^{\ell_i}$ and $L(1-\alpha) \to \sum_i \ell_i(1-\alpha_i)$ with $\ell_i = 0, 1, \ldots, L$ and $L = \sum_i \ell_i$. This is precisely (2.67), as claimed.

Although we performed a Taylor expansion in λ, we note that the truncation of the series due to the step functions implies that this derivation holds for arbitrary large values of the couplings.

A.5 Density of states with general defects

In this Appendix, we will explicitly evaluate the disk density of states for JT gravity with general defects (2.66), which we reproduce here for convenience

$$\rho(E) = \frac{e^{S_0}}{2\pi} \sum_{L=0}^{\lfloor \frac{1}{1-\alpha} \rfloor} \frac{\lambda^L}{L!} \int_{E_0}^{E} \frac{du}{2\sqrt{E-u}} \left(\frac{2\pi(1-L(1-\alpha))}{\sqrt{u}}\right)^{L} I_L\Big(2\pi\big(1-L(1-\alpha)\big)\sqrt{u}\Big), \tag{A.46}$$

from which we can compute the partition function

$$Z(\beta) = \frac{1}{2\pi}\int_{E_0}^{\infty} dE\, \rho(E) e^{-\beta E}. \tag{A.47}$$

For the case of $E_0 = 0$, we will evaluate these expressions exactly to all orders in λ. For $E_0 \neq 0$, we will only write down the perturbative expansion.

Case $E_0 = 0$

Let us first consider the case with $E_0 = 0$. A single defect species can never satisfy this condition, so $E_0 \neq 0$ always in that case. Nevertheless, for the sake of notational convenience, we will perform the calculations with a single defect species and make the replacements $\lambda^L \to \Pi_i \lambda_i^{\ell_i}$ and $L(1-\alpha) \to \sum_i \ell_i(1-\alpha_i)$, with $\ell_i = 0, 1, \ldots, L$ and $L = \sum_i \ell_i$, at the end.

In this case, the integral in the density of states formula can be evaluated exactly using (A.13) and (A.14) and the result is

$$\begin{aligned}\rho(E) &= \frac{e^{S_0}}{2\pi}\sum_{L=0}^{\lfloor\frac{1}{1-\alpha}\rfloor}\frac{\lambda^L}{L!}\sqrt{\frac{\pi}{2}}\left(\frac{2\pi\big(1-L(1-\alpha)\big)}{\sqrt{E}}\right)^{L-1/2}\Bigg[I_{-L+1/2}\Big(2\pi\big(1-L(1-\alpha)\big)\sqrt{E}\Big) \\ &\quad -\sum_{m=0}^{L-1}\frac{(-1)^m(2m)!}{(L-1)!(L-1-m)!}\Big(2\pi\big(1-L(1-\alpha)\big)\sqrt{E}\Big)^{L-2m-3/2}\Bigg]. \qquad (A.48)\end{aligned}$$

This form of the density of states will be useful when we discuss the $E_0 \neq 0$ case in the perturbative coupling regime in the next section. From (A.16) and (A.17), we see that the low energy behavior at $\mathcal{O}(\lambda^L)$ is such that a divergent contribution of $\cosh\big(\#\sqrt{E}\big)/E^{L-1/2} \sim 1/E^{L-1/2}$ in the first term is precisely cancelled by a contribution of $-1/E^{L-1/2}$ in the second term, leading to the expected square-root behavior

$$\rho(E) = e^{S_0}\sum_{L=0}^{\lfloor\frac{1}{1-\alpha}\rfloor}\frac{\lambda^L}{(L!)^2}2^L\pi^{2L}(1-L(1-\alpha))^{2L}\sqrt{E}+\mathcal{O}(E^{3/2}). \qquad (A.49)$$

These expressions generalize those found in [15, 16].

The partition function can be computed from the Laplace transform of (A.48) given in (A.18) and the exact answer is

$$\begin{aligned}Z(\beta) &= \frac{e^{S_0}}{4\sqrt{\pi}}\sum_{L=0}^{\lfloor\frac{1}{1-\alpha}\rfloor}\frac{\lambda^L}{L!}\frac{2^L}{\beta^{3/2-L}}\left(e^{\frac{\pi^2(1-L(1-\alpha))^2}{\beta}}-\sum_{m=0}^{L-1}\frac{1}{m!}\left(\frac{\pi^2\left(1-L(1-\alpha)\right)^2}{\beta}\right)^m\right) \qquad (A.50)\\ &= \frac{e^{S_0}}{4\sqrt{\pi}}\sum_{L=0}^{\lfloor\frac{1}{1-\alpha}\rfloor}\frac{\lambda^L}{L!}\frac{2^L}{\beta^{3/2-L}}e^{\frac{\pi^2(1-L(1-\alpha))^2}{\beta}}\left(1-\frac{\Gamma\Big(L,\frac{\pi^2(1-L(1-\alpha))^2}{\beta}\Big)}{\Gamma(L)}\right), \qquad (A.51)\end{aligned}$$

where we have used the series representation of the incomplete Gamma function at integer order

$$\Gamma(n,x) = (n-1)!\,e^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}. \qquad (A.52)$$

It is interesting to note that in the large temperature regime $\beta \to 0$, the exponential terms dominate and we have

$$\begin{aligned}e^{-S_0}Z(\beta) &\approx \frac{1}{4\sqrt{\pi}}\sum_{L=0}^{\lfloor\frac{1}{1-\alpha}\rfloor}\frac{\lambda^L}{L!}\frac{2^L}{\beta^{3/2-L}}\,e^{\frac{\pi^2(1-L(1-\alpha))^2}{\beta}} \qquad (A.53)\\ &= \frac{e^{\frac{\pi^2}{\beta}}}{4\sqrt{\pi}\beta^{3/2}}+\lambda\frac{e^{\frac{\pi^2\alpha^2}{\beta}}}{2\sqrt{\pi}\beta^{1/2}}+\lambda^2\frac{\beta^{1/2}}{2\sqrt{\pi}}\,e^{\frac{\pi^2(2\alpha-1)^2}{\beta}}\Theta(2\pi(2\alpha-1))+\mathcal{O}(\lambda^3) \qquad (A.54)\end{aligned}$$

In fact, we will show below for the case $E_0 \neq 0$ that the exponential contributions are exact in the perturbative regime at each order of the coupling where the defect merging condition is satisfied.

Case $E_0 \neq 0$

Let us now move on to the general case with $E_0 \neq 0$. We will again focus on a single defect species, though the results generalize easily to any number of species.

In this case, we need to perform the integral in the density of states from E_0 to E. Since we have already found the exact answer for the integral from 0 to E, all that remains is to compute the integral from 0 to E_0 and subtract it from the previous answer. In practice, this is difficult as the integral in general does not have a closed form. In addition, one needs to solve for E_0, defined as the largest root to the equation $\mathcal{F}(E_0) = 0$, which in this case is a highly nonlinear equation involving sums of modified Bessel functions.

To avoid these issues, we will focus only on the perturbative coupling regime where things simplify greatly. Since we know that $E_0 = 0$ for pure JT gravity, i.e. $\lambda = 0$, the leading contribution to E_0 must be $\mathcal{O}(\lambda)$. The integral from 0 to E_0 is thus over a small range, and we can Taylor expand the integrand and evaluate the integral perturbatively in E_0. To extract the contribution at $\mathcal{O}(\lambda^L)$, one needs to solve for E_0 perturbatively up to $\mathcal{O}(\lambda^{L-1})$ from the string equation. In general, this can be done iteratively as the solution at a given order depends only on the solution at lower orders. While all of this can be done explicitly, it is rather tedius in practice and we will not do it here.

Fortunately, there is a quick way of extracting the answer up to $\mathcal{O}\big(\lambda^{\lfloor \frac{1}{1-\alpha} \rfloor}\big)$ by demanding (A.48) match the exact answer (2.77) at $\alpha = 1$. In that case, we found that the density of state reduces to that of pure JT gravity with the spectral edge shifted to $E_0 = -2\lambda$. The Taylor expansion in λ is

$$\rho(E) = \frac{e^{S_0}}{4\pi^2} \sinh\left(2\pi\sqrt{E + 2\lambda}\right) = \frac{e^{S_0}}{2\pi} \sum_{L=0}^{\infty} \frac{\lambda^L}{L!} \sqrt{\frac{\pi}{2}} \left(\frac{2\pi}{\sqrt{E}}\right)^{L-1/2} I_{-L+1/2}\left(2\pi\sqrt{E}\right). \quad \text{(A.55)}$$

Comparing this to (A.48) at $\alpha = 1$, we see that the polynomials in $1/\sqrt{E}$ must be precisely the contributions of the integral from 0 to E_0. We conclude that up to $\mathcal{O}\big(\lambda^{\lfloor \frac{1}{1-\alpha} \rfloor}\big)$, the density of states is

$$\rho(E) \approx \frac{e^{S_0}}{2\pi} \sum_{L=0}^{\lfloor \frac{1}{1-\alpha} \rfloor} \frac{\lambda^L}{L!} \sqrt{\frac{\pi}{2}} \left(\frac{2\pi\big(1 - L(1-\alpha)\big)}{\sqrt{E}}\right)^{L-\frac{1}{2}} I_{-L+\frac{1}{2}}\Big(2\pi\big(1 - L(1-\alpha)\big)\sqrt{E}\Big). \quad \text{(A.56)}$$

Finally, the partition function is, up to the same order,

$$Z(\beta) = \frac{e^{S_0}}{4\sqrt{\pi}} \sum_{L=0}^{\lfloor \frac{1}{1-\alpha} \rfloor} \frac{\lambda^L}{L!} \frac{2^L}{\beta^{3/2-L}} e^{\frac{\pi^2(1-L(1-\alpha))^2}{\beta}} + \mathcal{O}\left(\lambda^{\lfloor \frac{1}{1-\alpha} \rfloor + 1}\right). \quad \text{(A.57)}$$

It is easy to show using the string equation that the higher order terms take the form of finite polynomials in $\beta^{-1/2}$.

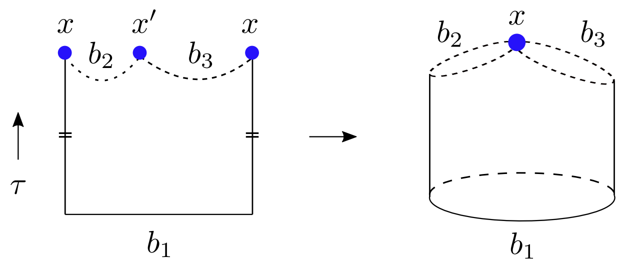

Figure A.1: Building the Lorentzian constant negative curvature pair of pants with geodesic boundaries by identifying points x, x'. The dashed lines are geodesics that form a figure eight with a singular point between them. The geometry on the right is smooth everywhere except at the point $x \sim x'$.

A.6 Lorentzian pair of pants

In this section we explain how to construct the constant negative curvature Lorentzian pair of pants[1]. This can be accomplished with an identification similar to the one used for the flat pants in figure 3.3. We start with the Lorentzian $R = -2$ cylinder which is the analytic continuation of the double trumpet

$$ds^2 = \frac{-d\tau^2 + d\sigma^2}{\cosh^2 \tau}, \qquad \tau \in (-\infty, \infty)\,, \qquad \sigma \in [0, b_1]. \tag{A.58}$$

From the identification $\sigma \sim \sigma + b_1$ we have a spatial geodesic of length b_1 at $\tau = 0$. We now pick two points x, x' at constant τ but at different values of σ, and we connect these points with two geodesics traversing both sides of the cylinder, see figure A.1. The geodesics will have lengths b_2, b_3 such that $b_1 \geq b_2 + b_3$. Solving for the geodesic lengths in terms of x, x', it's possible to show that we can find x, x' to give any geodesic lengths b_2, b_3 satisfying this constraint. We now identify the points $x \sim x'$ and discard the portion of the geometry above the geodesics to get a Lorentzian pair of pants with a singular point at $x \sim x'$.

When the geodesics meet at x, x' there is a jump in the tangent vector, so there is a delta function in the Extrinsic curvature of the boundary curve at those points. This is taken into account in the Gauss-Bonnet theorem through the jump angles α_i, of which there are two in the above construction

$$\frac{1}{2}\int \sqrt{g}R + \sum_i \alpha_i = 2\pi\chi. \tag{A.59}$$

We'll absorb the jump angles into a delta function contribution to the Ricci scalar at the point $x \sim x'$. The above is the Euclidean version of the Gauss-Bonnet theorem, the analytically continued version for almost Lorentzian metrics is given by (3.10) [140, 141]

$$\frac{1}{2}\int \sqrt{-g}R = -2\pi i\chi. \tag{A.60}$$

[1]We thank Don Marolf for suggesting this construction.

and we see that the effect of the jump angles is to cancel out the bulk volume contribution as explained in section 3.2. The scalar curvature of the pair of pants is then given by

$$\frac{1}{2}\sqrt{-g}\,(R+2) = (2\pi i + \alpha)\,\delta^2(x). \tag{A.61}$$

where we have relabeled the sum of jump angles as just α. One way to see that the imaginary delta function contribution appears after the identification of $x \sim x'$ is to note that without the Weyl factor in (A.58) this is the construction of the flat Lorentzian pants, where this contribution can be explicitly evaluated[140]. We can also attach pant legs to the boundaries b_2 or b_3 as follows. Consider a Lorentzian cylinder with two geodesic boundaries where one boundary has a corner so that the geometry exists. We can glue the boundary with the corner to the pants at b_2 or b_3 making sure the corner coincides with the point $x \sim x'$. This will give additional contributions to the α term, and gives Lorentzian pants with any desired geodesic boundary lengths.

A.7 Details on punctured Riemann surfaces and measures

In this section we include some additional details on punctured Riemann surfaces and on the integration measure. A punctured surface Σ is defined by taking a compact surface $\overline{\Sigma}$ of genus g and removing n distinct points $\Sigma = \overline{\Sigma}\backslash\{z_1, \ldots, z_n\}$, after which Σ is no longer compact and its Euler characteristic is given by $\chi(\Sigma) = 2 - 2g - n$. When performing an integral over a punctured surface the punctures do not contribute to the integral. The moduli space of punctured Riemann surfaces can be constructed by considering equivalence classes of singular Euclidean metrics on Σ [139]. We consider metrics that behave as

$$ds^2 = \frac{c_i}{|z - z_i|^2} dz d\bar{z} + \ldots \tag{A.62}$$

near the punctures at z_i, where c_i is some constant. We consider all such metrics related by smooth Weyl transformations to be in the same conformal equivalence class. The set of all such equivalence classes gives us the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$. In the literature it is also common to include additional singular metrics in the conformal equivalence class[137, 149, 151], such as the lightcone diagram metrics described in section 3.2. These metrics are related to other metrics in the equivalence class through Weyl transformations that are singular at isolated points, see [44] for an explicit example at genus zero.

The path integral over metrics on a genus g surface with n punctures was analyzed in [138] and is given by

$$\int \frac{\mathcal{D}g}{\mathrm{Vol}} = \int_{\mathcal{M}_{g,n}} [dm] \frac{\det\langle \mu_\alpha, \phi_\beta\rangle}{\det(\phi_\alpha, \phi_\beta)^{\frac{1}{2}}} (\det \hat{P}_1^\dagger \hat{P}_1)^{1/2} \int \mathcal{D}\omega e^{-26 S_L[\omega,\hat{g}]}. \tag{A.63}$$

Where on the left side Vol is the volume of the diffeomorphism group of the punctured surface, and we have gauge fixed to a metric $\hat{g}$ on the right. Compare this to equation (3.29) in the main text[2]. In the above integral m is a coordinate on the moduli space and $(\det \hat{P}_1^\dagger \hat{P}_1)^{1/2}$ comes from gauge fixing to the representative metric $\hat{g}$, and in total the above is the integration measure for the moduli space $\mathcal{M}_{g,n}$. For a detailed explanation of the various determinants in the above measure see [138, 148, 149]. In the main text we introduced the shorthand notation $\mathrm{d}\mu$ for the integration measure for simplicity, we now define it in terms of the above objects for easier comparison to the literature[138, 148, 149]

$$\int_{\mathcal{M}_{g,n}} \mathrm{d}\mu \equiv \int_{\mathcal{M}_{g,n}} [dm] \frac{\det \langle \mu_\alpha, \phi_\beta \rangle}{\det (\phi_\alpha, \phi_\beta)^{\frac{1}{2}}} (\det \hat{P}_1^\dagger \hat{P}_1)^{1/2}. \tag{A.64}$$

Gauge fixing to the lightcone metric $\hat{g}$, the various determinants in the above integral were worked out in [138] (see equations (2.22) and (4.7-4.8) in [138], see also section V.G in [149]). The product of determinants significantly simplifies, and we find the final result

$$\int \frac{\mathcal{D}g}{\mathrm{Vol}} = \int [d\tau][d\theta][\rho d\rho] \frac{2\pi \det'(-\hat{\nabla}^2)}{\int_\Sigma d^2 z \sqrt{\hat{g}}} \int \mathcal{D}\omega e^{-26 S_L[\omega, \hat{g}]}. \tag{A.65}$$

Compare this to equations (3.31) and (3.32). The prime on the determinant indicates that we remove zero modes.

A.8 Supersymmetric Chern-Simons

In this section we include additional details on our conventions and on $\mathcal{N}=1$ supersymmetric $U(1)$ Chern Simons in the presence of a boundary, closely following [202, 203]. We first discuss our conventions for Lorentzian signature supersymmetry. The gamma matrices satisfy the standard algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \gamma^{\mu\nu} = \eta^{\mu\nu} - \epsilon^{\mu\nu\rho}\gamma_\rho, \tag{A.66}$$

(the first of the right equations is a definition for $\gamma^{\mu\nu}$ and they are explicitly given by

$$\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.67}$$

The metric takes the form $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$, and the spacetime coordinates are $x^\mu = (x^1, x^2, x^3)$ where x^1 is the time component, while the boundary is located at $x^3 = 0$. We will use indices m, n to indicate components restricted to the boundary $x^m = (x^1, x^2)$. All spinors considered are Majorana $\overline{\lambda}^a \equiv C^{ab}\lambda_b$ where $C = -C^\intercal$ is the charge conjugation matrix.

[2]Note that in (3.25) we are integrating over $\mathcal{M}_g$ instead of punctured surfaces $\mathcal{M}_{g,n}$. The integration measure slightly differs between the two cases, see section 3 of [138].

Spinor indices are contracted top right to bottom left: $\overline{\lambda}\chi = \lambda^a\chi_a$, $\overline{\lambda}\gamma^\mu\chi = \lambda^a(\gamma^\mu)_a{}^b\chi_b$, where the gamma matrices implicitly have the index structure $(\gamma^\mu)_a{}^b$. Spinor indices a, b are raised and lowered with the antisymmetric charge conjugation matrix C: $\lambda^a = C^{ab}\lambda_b$, which is defined through $C\gamma^\mu C^{-1} = -(\gamma^\mu)^\intercal$. We introduce projection operators $P_\pm = \frac{1}{2}(1 \pm \gamma^3)$ and define fields $\chi_\pm$ with $P_\pm\chi = \chi_\pm$, which projects onto the top or bottom component of the spinor respectively.

$\mathcal{N} = (1, 0)$ Lorentzian Chern Simons

We first discuss the case of $\mathcal{N} = (1, 0)$ supersymmetric $U(1)$ Chern Simons in the presence of a boundary in Minkowski space. The action can be constructed through the use of spinor superfields,[3] including appropriate boundary terms it is given by [202, 203]

$$S_{\text{CS}}^{\mathcal{N}=(1,0)} = \int_M d^3x(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \overline{\lambda}\lambda) - \frac{1}{2}\int_{\partial M} d^2x\sqrt{h}(h^{mn}A_mA_n + \overline{\chi}_-\gamma^m\partial_m\chi_-). \quad (A.68)$$

In the above λ, χ are Majorana fermions, with $\chi_- = P_-\chi$ being a purely boundary fermion. The dynamical boundary fermion χ_- is unusual, and is required to cancel the boundary terms produced by the susy variation. We also have the boundary metric h_{mn}. The full susy variations under ϵ are given by [202]

$$\begin{aligned} \delta A_\mu &= -(\bar{\epsilon}\gamma_\mu\lambda) + (\bar{\epsilon}\partial_\mu\chi), \\ \delta\lambda_a &= -\epsilon^{\mu\nu\rho}(\gamma_\rho\epsilon)_a\partial_\mu A_\nu, \\ \delta\chi_a &= (\gamma^\mu\epsilon)_a A_\mu. \end{aligned} \quad (A.69)$$

However, the action (A.68) is only invariant up to a boundary term under a general variation (see (A.86)). For the full action to be invariant we must restrict to variations $\epsilon_+ = P_+\epsilon$, which means the supersymmetry is broken down to $\mathcal{N} = (1, 0)$ in the presence of the boundary. This is true even without imposing any boundary conditions on the fields. The explicit susy variations under ϵ_+ are

$$\begin{aligned} \delta A_\mu &= -(\bar{\epsilon}_+\gamma_\mu\lambda) + (\bar{\epsilon}_+\partial_\mu\chi_-), \\ \delta\lambda_a &= -\epsilon^{\mu\nu\rho}(\gamma_\rho\epsilon_+)_a\partial_\mu A_\nu, \\ \delta\chi_- &= (\gamma^m\epsilon_+)A_m, \end{aligned} \quad (A.70)$$

and the action is invariant under this subset of transformations. Using the above, we can obtain the following variations

$$\delta A_- = \bar{\epsilon}_+\left(2\gamma^2\lambda_+ + \partial_-\chi_-\right), \quad (A.71)$$

$$\delta\left(2\gamma^2\lambda_+ + \partial_-\chi_-\right) = \gamma^2\epsilon_+\partial_+A_-. \quad (A.72)$$

[3]The spinor superfield formalism includes an additional complex scalar that is the superpartner of χ, but it does not end up contributing to the action. We exclude it below, but see [202].

where we have defined the notation $A_\pm = A_1 \pm A_2$, $\partial_\pm = \partial_1 \pm \partial_2$ which is distinct from the spinor $\pm$ projection notation and used $\gamma^1\lambda_+ = \gamma^2\lambda_+$. For a good variational principle, the boundary conditions we impose are $A_- = 0$ and $2\gamma^2\lambda_+ + \partial_-\chi_- = 0$ [202, 203]. We see that the ϵ_+ transformations leave these boundary conditions invariant. The boundary kinetic term for χ_- does not need to be cancelled for a good variational principle since it gives the equations of motion. In the path integral with these boundary conditions the integral over χ_- is effectively not constrained since we also integrate over λ_+, thus χ_- is essentially a free dynamical one-component boundary fermion. The integral over λ will give some overall normalization constant since it is non-dynamical.

Similarly, we can construct an action with $\mathcal{N} = (0, 1)$ invariant under ϵ_- transformations by modifying the boundary term

$$S_{\mathrm{CS}}^{\mathcal{N}=(0,1)} = \int_M d^3x(\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \overline{\lambda}\lambda) + \frac{1}{2}\int_{\partial M} d^2x\sqrt{g}(g^{mn}A_mA_n + \overline{\chi}_+\gamma^m\partial_m\chi_+), \tag{A.73}$$

where λ, χ_+ are again Majorana fermions, and χ_+ has been projected onto it's top component. In this case the variations under ϵ_- are given by

$$\begin{aligned}
\delta A_\mu &= -(\overline{\epsilon}_-\gamma_\mu\lambda) + (\overline{\epsilon}_-\partial_\mu\chi_+),\\
\delta\lambda_a &= -\epsilon^{\mu\nu\rho}(\gamma_\rho\epsilon_-)_a\partial_\mu A_\nu,\\
\delta\chi_+ &= (\gamma^m\epsilon_-)A_m.
\end{aligned} \tag{A.74}$$

Which immediately gives us the variations

$$\delta A_+ = \overline{\epsilon}_-\left(-2\gamma^2\lambda_- + \partial_+\chi_+\right), \tag{A.75}$$

$$\delta\left(-2\gamma^2\lambda_- + \partial_+\chi_+\right) = -\gamma^2\epsilon_-\partial_-A_+. \tag{A.76}$$

Proper boundary conditions in this case correspond to $A_+ = 0$ and $-2\gamma^2\lambda_- + \partial_+\chi_+ = 0$ which are both preserved under ϵ_- transformations. In the main text we defined a total theory given by the difference of the above actions

$$S = S_{\mathrm{CS}}^{\mathcal{N}=(1,0)} - S_{\mathrm{CS}}^{\mathcal{N}=(0,1)}, \tag{A.77}$$

so that the bulk theory has the full $\mathcal{N} = (1, 1)$ supersymmetry realized by different sectors. Since $\overline{\chi}_-\gamma^m\partial_m\chi_+ = \overline{\chi}_+\gamma^m\partial_m\chi_- = 0$ the boundary fermion term for the above action can be rewritten in a simple form

$$\int_{\partial M} \mathrm{d}^2x\overline{\chi}\gamma^m\partial_m\chi, \tag{A.78}$$

which makes it clear that we have a dynamical 2d free fermion on the boundary. The gauge fields do not interact with the fermions so the full path integral will simply be a product of the Chern-Simons contribution and the fermion contribution. In each case, the boundary fermions $\chi_\pm$ are projected onto the top/bottom component and function as single component spinors, so individually their partition functions will contribute a determinant of either a holomorphic or anti-holomorphic derivative after analytic continuation to Euclidean signature.

There is one additional subtlety to address, the above theories were defined on a flat background. However, for our purposes we are interested in supersymmetric Chern-Simons on a manifold with an asymptotic boundary torus. In general such manifolds do not admit flat metrics, so we need to consider the theory on a curved background [218]. While Chern-Simons is itself background independent, the supersymmetric version depends on a choice of background metric g through the quadratic bulk fermion term. Ignoring the boundary term, the action is of the form

$$S = \int_M d^3x(\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \sqrt{g}\, \overline{\lambda}\lambda). \tag{A.79}$$

However, the metric dependence is quite mild since the fermion is non-dynamical, and it produces an overall normalization for the partition function. Nevertheless, to preserve supersymmetry we need to choose a bulk metric g that satisfies the killing spinor equations. The end result is that the supersymmetry transformations will mildly depend on the background metric [218]. In the case of bulk handlebodies we can choose g to be given by the corresponding AdS_3 metric [219] and supersymmetry will be preserved, but for more general three-manifolds that appear when considering wormhole geometries little is known.

Details of the variations.

In this subsection we include additional details details[4] on the variation of the supersymmetic Chern-Simons action in equation (A.68). For convenience we split the action into a bulk and boundary piece

$$S = \int_M \mathrm{d}^3x \left(\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \overline{\lambda}\lambda\right), \tag{A.80}$$

$$S_\partial = -\frac{1}{2}\int_{\partial M} \mathrm{d}^2x \left(A^m A_m + \overline{\chi}_- \gamma^m \partial_m \chi_-\right). \tag{A.81}$$

Where we used $h_{mn} = \mathrm{diag}\,(-1,1)$, where Latin indices m,n are again boundary indices and take values in $\{1,2\}$. Varying with the full supersymmetric variations (A.69) we find

$$\delta S = 2\int_M \mathrm{d}^3x \epsilon^{\mu\nu\rho}\partial_\nu A_\rho \left(-\bar{\epsilon}\gamma_\mu\lambda + \bar{\epsilon}\partial_\mu\chi\right) + \int_{\partial M} \mathrm{d}^2x \epsilon^{3mn} A_m \left(-\bar{\epsilon}\gamma_n\lambda + \bar{\epsilon}\partial_n\chi\right) \tag{A.82}$$

$$+ 2\int_M \mathrm{d}^3x \epsilon^{\mu\nu\rho}\partial_\nu A_\rho \left(\bar{\epsilon}\gamma_\mu\lambda\right). \tag{A.83}$$

[4]Some useful identities include $\overline{\lambda}\chi = \overline{\chi}\lambda$, $\overline{\lambda}\gamma_\mu\epsilon = -\bar{\epsilon}\gamma_\mu\lambda$, and since the spinors are Majorana we have $\overline{\lambda}^a\chi_a = C^{ab}\lambda_b\chi_a$.

The second line cancels the first term, and we can rewrite the remaining bulk term as a boundary term to obtain

$$\delta S = -\int_{\partial M} \mathrm{d}^2 x \epsilon^{3nm} A_n \left(\bar{\epsilon}\gamma_m \lambda + \bar{\epsilon}\partial_m \chi\right) \ . \tag{A.84}$$

The variation of the boundary term gives

$$\delta S_\partial = \int_{\partial M} \mathrm{d}^2 x \left(A^m \left(\bar{\epsilon}\gamma_m \lambda - \bar{\epsilon}\partial_m \chi\right) + \left(\partial_m \overline{\chi}_- \epsilon A^m - \partial_m \overline{\chi}_- \epsilon^{m\mu\rho}\gamma_\rho \epsilon A_\mu\right)\right) . \tag{A.85}$$

The variation of the full action (A.68) is thus a total boundary term

$$\begin{aligned} \delta S_{\mathrm{CS}}^{\mathcal{N}=(1,0)} = &\int_{\partial M} \mathrm{d}^2 x \left(-\epsilon^{3nm} A_n \left(\bar{\epsilon}\partial_m \chi + \bar{\epsilon}\gamma_m \lambda\right) + A^m \left(\partial_m \overline{\chi}_- \epsilon\right) - \epsilon^{m\mu\nu} A_\mu \left(\partial_m \overline{\chi}_- \gamma_\nu \epsilon\right)\right) \\ &+ \int_{\partial M} \mathrm{d}^2 x A^m \left(\bar{\epsilon}\gamma_m \lambda - \bar{\epsilon}\partial_m \chi\right) \ . \end{aligned} \tag{A.86}$$

Now we specialize to $P_+\epsilon = \epsilon_+$ variations. We immediately have the following cancellations. The third term cancels the last term since $\overline{\chi}_- \epsilon_+ = \bar{\epsilon}_+ \chi$, while the first term is cancelled by fourth term since in the fourth term only $\overline{\chi}_- \gamma_3 \epsilon_+ = \overline{\chi}_- \epsilon_+$ survives. Finally, the second term cancels the fifth term since we have $\epsilon^{3nm}\gamma_m = \gamma^3 \gamma^n$ and $\bar{\epsilon}_\pm \gamma^3 = \mp \bar{\epsilon}_\pm$ due to the factor of the charge conjugation matrix C in $\bar{\epsilon}$. Note again that no special choice of boundary conditions was needed to make the action invariant under ϵ_+ variations.

The variation of the $\mathcal{N} = (0,1)$ supersymmetric Chern-Simons action under ϵ_- works similarly, and we obtain

$$\begin{aligned} \delta S_{\mathrm{CS}}^{\mathcal{N}=(0,1)} = &\int_{\partial M} \mathrm{d}^2 x \left(-\epsilon^{3nm} A_n \left(\bar{\epsilon}\partial_m \chi + \bar{\epsilon}\gamma_m \lambda\right) - A^m \left(\partial_m \overline{\chi}_+ \epsilon\right) + \epsilon^{m\mu\nu} A_\mu \left(\partial_m \overline{\chi}_+ \gamma_\nu \epsilon\right)\right) \\ &+ \int_{\partial M} \mathrm{d}^2 x A^m \left(-\bar{\epsilon}\gamma_m \lambda + \bar{\epsilon}\partial_m \chi\right) . \end{aligned} \tag{A.87}$$

Specializing to $P_-\epsilon = \epsilon_-$ variations we again find that all the boundary terms cancel in a similar way as in (A.86).