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### Scaling Laws and Zero Viscosity Limits for Wall-Bounded Shear Flows and for Local Structure in Developed Turbulence

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**Scaling laws and zero viscosity limits for wall-bounded shear flows  
and for local structure in developed turbulence**

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**ABSTRACT.** Scaling laws for wall-bounded turbulence are derived, and their properties are analyzed via zero viscosity asymptotics; a comparison of the results with recent experiments shows that the observed scaling law differs significantly from the customary logarithmic law of the wall. The Izakson-Millikan-von Mises derivation of turbulence structure, properly interpreted, confirms this analysis. Analogous relations for the local structure of turbulence are given, including results on the scaling of the higher-order structure functions; these results suggest that there are no Reynolds number independent corrections to the Kolmogorov exponent, and thus that the classical 1941 version of the Kolmogorov theory already gives the limiting behavior. The use of small viscosity asymptotics is explained, and the consequences of the theory and of the experimental evidence for the Navier-Stokes equations and for the statistical theory of turbulence are discussed.

## 1. Introduction

Turbulence remains the greatest challenge of classical physics, and though many researchers, including giants such as Kolmogorov, Heisenberg, Taylor, Prandtl and von Kármán have vastly enlarged our understanding of it, none of their results was derived from first principles. All existing results in turbulence theory depend on some additional assumptions, explicit or implicit, which must be reexamined as knowledge expands.

Turbulence at very large Reynolds numbers is generally considered to be one of the happier provinces of the entire turbulence realm, as it is widely thought that two of its results are well-established and have a chance to enter, basically untouched, into a future complete theory; these results are the logarithmic law in the wall region of wall-bounded turbulent shear flow, obtained in the early thirties by von Kármán and Prandtl [23],[33], and the Kolmogorov-Obukhov scaling laws for local structure [25],[31]. However, Kolmogorov and Obukhov themselves expressed doubts about the original version of their theory and proposed modifications. Doubts about the logarithmic law and alternative proposals have been presented by the first author starting with the monograph [1].

In the present work we shall reexamine these two laws. In the first section we shall show that the universal logarithmic law is apparently invalid and must be replaced by a different scaling law, as was already done with less detail in [3]. In particular, we reexamine here the well-known Izakson-Millikan-von Mises derivation of the logarithmic law and correct it. We then consider the local structure of turbulence, with particular attention to the higher-order structure functions. We conclude that, provided a certain constant differs from zero, the corrections to the Kolmogorov 1941 theory are Reynolds number dependent and vanish in the limit of vanishing viscosity; recent work on the statistical theory of turbulence shows that the constant is in fact not zero. Our tools include zero viscosity asymptotics, which we explain.

## 2. Reynolds number dependent scaling law vs. the universal logarithmic law in wall-bounded turbulence

Despite sixty years of active research, two conflicting laws for the distribution of the mean velocity  $u$  in the intermediate region of a turbulent shear flow, especially pipe flow (Figure 1), coexist in the literature ([35], see also [20]). The first is the scaling law:

$$(2.1) \quad \phi = A\eta^\alpha$$

where  $\phi = u/u_*$ ,  $\eta = u_*y/\nu$ ,  $u_*$  is the friction velocity  $u_* = \sqrt{\tau/\rho}$ ,  $\tau$  is the shear stress at the wall,  $\rho$  is the fluid density,  $\nu$  is the kinematic viscosity,  $y$  is the distance from the

wall, and  $A, \alpha$  are constants that are known to depend slightly on the Reynolds number  $Re$ . The second law is the "universal" von Kármán-Prandtl law of the wall,

$$(2.2) \quad \phi = \frac{1}{\kappa} \log \eta + B$$

where, following the logic of the derivation,  $\kappa$  and  $B$  are universal constants, independent of the Reynolds number, which cannot adjusted as the Reynolds number changes.

Both laws can be derived from the fundamental scaling relation:

$$(2.3) \quad \partial_y u = \frac{u_*}{y} \Phi(\eta, Re)$$

where  $Re$  is a properly defined Reynolds number, for instance, for pipe flow  $Re = \bar{u}d/\nu$ , where  $\bar{u}$  is the mean velocity (discharge rate divided by the cross-section area),  $d$  is the diameter of the pipe,  $\nu$  is again the kinematic viscosity of the fluid, and  $\Phi$  is an unknown dimensionless function of two dimensionless arguments.

If one assumes that the limit of the function  $\Phi$  in (2.3), when both arguments tend to infinity, exists and is finite and different from zero, ( in the language of [1], [21], this is an assumption of complete similarity), then one can define  $\kappa = 1/\Phi(\infty, \infty)$  and one is led to (2.2). In particular, the assumption of complete similarity causes the parameters  $d$  and  $\nu$  to drop from the expression for the velocity gradient at high enough  $Re$ . We have chosen for now, among the several derivations of (2.2), the one which emphasizes the role of an assumption about the limiting behavior of  $\Phi$ . The opposite self-similarity assumption, ("incomplete similarity"), leads to general scaling laws; a form suggested by the absence of a characteristic length scale is:

$$(2.4) \quad \frac{u}{u_*} = B(Re) \left( \frac{u_* y}{\nu} \right)^{\beta(Re)}$$

First note that  $Re$  can enter (2.4) only through a function  $\psi = \psi(Re)$  such that  $\psi(CRe)/\psi(Re) \rightarrow 1$  as  $Re \rightarrow \infty$  for any constant  $C$  (see [4]); indeed, a change in the definition of  $Re$ , for example through a change in the choice of reference length or velocity, multiplies  $Re$  by a constant  $C$ , and the property just mentioned is needed to make the scaling law invariant under this change of definition ("asymptotic covariance"). An example of a function  $\psi$  that satisfies these conditions is  $\psi(Re) = \ln Re$ . The experimental data suggest that indeed  $\psi(Re) = \ln Re$  is a good choice. An expansion of (2.4) around the state that corresponds to  $Re = \infty$  then leads to:

$$(2.5) \quad \partial_y u = \frac{u_*}{y} \left( B_0 + \frac{B_1}{\ln Re} + o\left(\frac{1}{\ln Re}\right) \right) \left( \frac{u_* y}{\nu} \right)^{\frac{\beta_1}{\ln Re} + o\left(\frac{1}{\ln Re}\right)}$$

We emphasize that neither the scaling law (2.5) nor the universal logarithmic law (2.2) can be considered as merely convenient representations of experimental data. They have precise and equally justified theoretical foundations; both are based on the assumption of self-similarity, but the logarithmic law is based on the assumption of complete similarity whereas the scaling law is based on the assumption of incomplete similarity. The question is, which of these assumptions, if any, is correct. This question can be answered in full only by further advances in the theory of Navier-Stokes equations and/or by further experimental studies, as is the case for the local structure discussed below.

The difference between the cases of complete and incomplete similarity is significant. In the first case the experimental data should cluster, in the  $\phi = u/u_*$ ,  $\ln \eta$  plane ( $\eta = u_*y/\nu$ ), on the universal straight line of the logarithmic law. Both the slope  $1/\kappa$  and the additive constant entering the logarithmic law (3.4) should, by the logic of the derivation, be universal, i.e. Reynolds number independent. In particular, it is not legitimate to say that in a certain range of Reynolds numbers there is one best fit for the constants in the logarithmic law but that in a different range the constants are different. In the second case the experimental points may occupy an area in the  $\phi$ ,  $\ln \eta$  plane bounded by the envelope of the family of scaling law curves having the Reynolds number as parameter. The envelope in turn can be approximated piecewise by various straight lines that depend on the range of Reynolds numbers under consideration. Scaling laws similar to (2.4) were popular among engineers, especially before the papers of von Kármán [23] and Prandtl [33]. Recently ([2],[5]) arguments in favor of the incomplete similarity (2.4)–(2.5) were proposed and the coefficients  $B_0$ ,  $B_1$  and  $\beta_1$  were determined from the Nikuradze data [30]:

$$(2.6) \quad B_0 = \frac{\sqrt{3}}{2}, \quad B_1 = \frac{15}{4}, \quad \beta_1 = \frac{3}{2}.$$

With the choice of parameters given in (2.6) the power law has the form

$$(2.7) \quad \phi = \left( \frac{1}{\sqrt{3}} \ln Re + \frac{5}{2} \right) \eta^{\frac{3}{2 \ln Re}}, \quad \phi = \frac{u}{u_*}, \quad \eta = \frac{u_*y}{\nu}$$

or, equivalently,

$$(2.8) \quad \phi = \left( \frac{1}{\sqrt{3}} \ln Re + \frac{5}{2} \right) \exp \left( \frac{3 \ln \eta}{2 \ln Re} \right).$$

It is important to note that the analyses below do not depend on the specific values of the constants that have been obtained by comparison with experiment; it is the form of the scaling law that matters, in particular, the fact the  $\alpha$  is inversely proportional to  $\ln Re$ .



The difficulty in distinguishing between complete and incomplete similarity on the basis of the experimental data was due until recently to a surprising reason: All the available experimental data were concentrated near the envelope of the family of curves (2.8) ( see the discussion in [5]). The envelope corresponding to the family of curves (2.8) is a smooth curve which can be approximated by piecewise linear functions of  $\ln \eta$ . Asymptotically, at very large  $Re$ , the envelope is the straight line [2]

$$(2.9) \quad \phi = \frac{\sqrt{3}}{2} e \ln \eta + 6.79 .$$

We now set out to extract from the scaling law (2.8) predictions for what happens at very large Reynolds number, beyond the range to which the constants were fitted. The success of this extrapolation is a validation of the scaling law (2.8). The main tool we use is *zero-viscosity asymptotics*; in [3], and again below, we justify zero-viscosity asymptotics by appealing to a statistical argument, but here a simpler explanation will suffice. We have already explained the importance of assumptions about the asymptotic behavior of  $\Phi$  in (2.3). Consider again equation (2.3), and its special case, equation (2.8). If one stands at a fixed distance from the wall, in a specific pipe with a given pressure gradient, one is not free to vary  $Re$  and  $\eta$  independently; the viscosity  $\nu$  appears in both, and if  $\nu$  is decreased, both arguments of  $\Phi$  will vary. The appropriate limit is the limit of vanishing viscosity, if it exists. The statistical theory described below asserts that this limit does exist, but we can check its existence in our specific case independently. When one takes this limit, one considers flows at ever larger  $\eta$  at ever larger  $Re$ ; the ratio  $\frac{3 \ln \eta}{2 \ln Re}$  tends to  $3/2$  because  $\nu$  appears in the same way in both numerator and denominator. To show this in more detail, using only physically meaningful quantities, proceed as follows: Note that in experimental measurements using any probe, including the Pitot tube used by Nikuradze [30], it is impossible to approach the wall closer than a certain distance  $\delta$ , say the diameter of the Pitot tube. Consider the experimental possibilities for a certain member of the family (2.8). It was shown in [5] that the experimental points presented by Nikuradze are close to the envelope. So we assume that up to some distance  $\Delta > \delta$  the experimental points are close to the envelope. What happens farther? Consider the combination  $3 \ln \eta / 2 \ln Re$ . It can be represented in the following form

$$(2.10) \quad \frac{3 \ln \eta}{2 \ln Re} = \frac{3 \left[ \ln \frac{u_* \Delta}{\nu} + \ln \frac{y}{\Delta} \right]}{2 \left[ \ln \frac{u_* \Delta}{\nu} + \ln \frac{\bar{u}}{u_*} + \ln \frac{d}{\Delta} \right]} .$$

According to [2], at small  $\nu$ , i.e. large  $Re$ ,  $\bar{u}/u_* \sim ((1/\sqrt{3}) \ln Re + 5/2)$ , so that the term  $\ln \bar{u}/u_*$  in the denominator of the right-hand side of (2.10) is asymptotically small, of the

order of  $\ln \ln Re$ , and can be neglected at large  $Re$ . The crucial point is that due to the small value of the viscosity  $\nu$  the first term  $\ln(u_* \Delta / \nu)$  in both the numerator and denominator of (2.10) should be dominant, so that  $3 \ln \eta / 2 \ln Re$  is close to  $3/2$  ( $y$  is obviously less than  $d/2$ ). Therefore the quantity

$$1 - \ln \eta / \ln Re$$

can be considered in the intermediate region  $\Delta < y < d/2$  as a small parameter, so that the factor  $\exp(3 \ln \eta / 2 \ln Re)$  is approximately equal to

$$(2.11) \quad \begin{aligned} \exp \left[ \frac{3}{2} - \frac{3}{2} \left( 1 - \frac{\ln \eta}{\ln Re} \right) \right] &\approx e^{3/2} \left[ 1 - \frac{3}{2} \left( 1 - \frac{\ln \eta}{\ln Re} \right) \right] \\ &= e^{3/2} \left[ \frac{3}{2} \frac{\ln \eta}{\ln Re} - \frac{1}{2} \right]. \end{aligned}$$

According to (2.8) we have also

$$(2.12) \quad \eta \partial_\eta \phi = \partial_{\ln \eta} \phi = \left( \frac{\sqrt{3}}{2} + \frac{15}{4 \ln Re} \right) \exp \left( \frac{3 \ln \eta}{2 \ln Re} \right),$$

and the approximation (2.11) can also be used in (2.12). Thus in the intermediate asymptotic range of distances  $y$ :  $y > \Delta$ , but at the same time  $y$  slightly less than  $d/2$ , the following asymptotic relations should hold with accuracy  $o(1/\ln Re)$ :

$$(2.13) \quad \phi = e^{3/2} \left( \frac{\sqrt{3}}{2} + \frac{15}{4 \ln Re} \right) \ln \eta - \frac{e^{3/2}}{2\sqrt{3}} \ln Re - \frac{5}{4} e^{3/2},$$

and

$$(2.14) \quad \partial_{\ln \eta} \phi = \left( \frac{\sqrt{3}}{2} + \frac{15}{4 \ln Re} \right) e^{3/2}.$$

Note that this law has a finite limit independent of  $Re$ . At the same time it can be easily shown that for the envelope of the power-law curves the asymptotic relation is

$$(2.15) \quad \partial_{\ln \eta} \phi = \left( \frac{\sqrt{3}}{2} + O \left( \frac{1}{\ln^2 Re} \right) \right) e.$$

The difference in slopes between (2.14) and (2.15) is significant. It means that the individual members of the family (2.6) should have at large  $Re$  an intermediate part, represented in the plane  $\phi, \ln \eta$  by straight lines, with a slope different from the slope of the envelope by a factor  $\sqrt{e} \sim 1.65$ . Therefore the graph of the individual members of the family (2.6) should have the form presented schematically in Figure 2.

Recently an experimental paper by Zagarola et al. [39] presented new data obtained in a high-pressure pipe flow. High pressure creates a large density and therefore a low kinematic viscosity. The experimentalists were thus able to enlarge the range of Reynolds numbers by an order of magnitude in comparison with Nikuradze's [30]. The Reynolds number is varied in these experiments by changing the pressure and thus the kinematic viscosity, exactly as is done mathematically in our small viscosity asymptotics.

The experimental data are presented in Fig. 3, which is reproduced with permission from Fig. 4 of [39]; these data agree well with our small viscosity results: At small  $y$  the deviation from the envelope is too small to be noticed (part I of the curve in Figure 2); for larger  $y$ , up to a very close vicinity of the maximum of  $\phi$  achieved in the experiments, the data are split. To each Reynolds number corresponds its own curve with a pronounced linear part having a slope clearly larger than the slope of the envelope; the ratio of the slope of the curves of the family to that of the envelope is always larger than 1.5. Contrary to the opinion of Zagarola et al., we consider this graph to be a clear confirmation of the scaling law, and a strong argument against the universal logarithmic law according to which all the points up to a close vicinity of maxima should lie on the universal logarithmic straight line. The prediction of a difference of  $\sqrt{\epsilon}$  between the slopes of the individual velocity profiles and the slope of their envelope provides an easily verified criterion for assessing the agreement between the experimental data and the scaling law. Note that at high  $Re$  the difference between the proposed law and the universal logarithmic law is large enough to have a substantial impact on the outcome of engineering calculations.

**3. The Izakson-Millikan-von Mises (IMM) overlap argument.** We now examine in detail a well-known and widely used argument that appears to back the logarithmic law of the wall (see e.g [14],[29]). In this argument, it is assumed that outside the wall sublayer (see Figure 1) one has a generalized law of the wall,

$$(3.1) \quad \phi = u/u_* = f(u_*y/\nu),$$

where  $f$  is a dimensionless function; the influence of the Reynolds number  $Re$ , which contains the external length scale (for pipe flow, the diameter  $d$  of the pipe) is neglected. In the external region, adjacent to the axis of the pipe in pipe flow, one assumes a "defect law",

$$(3.2) \quad u_{CL} - u = u_*g(2y/d),$$

where  $u_{CL}$  is the average velocity at the centerline and  $g$  is another dimensionless function. Here the neglect of the effect of  $Re$  means that the effect of viscosity is neglected. It is

assumed furthermore that for some interval in  $y$  the laws (3.1) and (3.2) overlap, so that

$$(3.3) \quad u_{CL} - u = u_{CL} - u_* f(u_* y / \nu) = u_* g(2y/d).$$

After differentiation of (3.3) with respect to  $y$  followed by multiplication by  $y$  one obtains

$$(3.4) \quad \eta f'(\eta) = \xi g'(\xi) = \frac{1}{\kappa},$$

where  $\eta = u_* y / \nu$ ,  $\xi = 2y/d$ , and  $\kappa$  is a constant; integration then yields the law of the wall

$$(3.5) \quad f(\eta) = \frac{1}{\kappa} \ln \eta + B,$$

as well as the defect law

$$(3.6) \quad g(\xi) = -\frac{1}{\kappa} \ln \xi + B_*,$$

with

$$B_* = \frac{u_{CL}}{u_*} - \frac{1}{\kappa} \ln \frac{u_* d}{2\nu} - B.$$

We will now show that this elegant derivation survives, and indeed is compatible with our conclusions, provided the effects of the Reynolds number are taken into account in a manner consistent with the data.

We begin by noting that in the nearly linear portion II of the graph of figure 1 the flow can be described by a local logarithmic law with a Reynolds number dependent effective von Kármán constant  $\kappa_{eff} = \kappa(Re)$ :

$$(3.7) \quad \kappa_{eff} = \frac{2}{\sqrt{3}e^{3/2} + \frac{15}{2}(\ln Re)e^{3/2}};$$

as  $Re \rightarrow \infty$ ,  $\kappa(Re)$  tends to the limit  $\kappa_\infty = \frac{2}{\sqrt{3}e^{3/2}} \sim 0.2776\dots$ , smaller than the usual von Kármán constant  $\kappa = \frac{2}{\sqrt{3}e} \sim .425\dots$  by a factor  $\sqrt{e} \sim 1.65\dots$ . With this in mind, the IMM procedure can be modified as follows: The law of the wall, equation (3.1), becomes

$$(3.8) \quad \phi = u/u_* = f(u_* y / \nu, Re),$$

so that the influence of  $Re$ , which contains the external scale, is included. The defect law (3.2) is replaced by the Reynolds number dependent defect law

$$(3.9) \quad u_{CL} - u = u_* g(2y/d, Re),$$

so that the influence of the molecular viscosity  $\nu$  is preserved. Now assume that the laws (3.7) and (3.8) overlap on some  $y$  interval:

$$u_{CL} - u = u_{CL} - u_* f(u_* y / \nu, Re) = u_* g(2y/d, Re).$$

Replacing  $f$  by its expression (2.7) yields:

(3.10)

$$g(2y/d, Re) = \phi_{CL} - \left(\frac{1}{\sqrt{3}} \ln Re + \frac{5}{2}\right) e^{3/2} - \left(\frac{\sqrt{3}}{2} + \frac{15}{4 \ln Re}\right) e^{3/2} \ln(2y/d) + e^{3/2} \left(\frac{\sqrt{3}}{2} + \frac{15}{4 \ln Re}\right) \ln \frac{u_*}{2\bar{u}},$$

where  $\phi_{CL} = u_{CL}/u_*$ . This calculation is self-consistent, and differs from the original IMM procedure by matching a Reynolds number dependent defect law to the actual curves of the scaling law (2.7) rather than to their envelope.

Another way of looking at the calculation we have just performed is to note that if one requires an overlap between a law of the wall that does not depend on  $d$  and a defect law that does not depend on  $\nu$ , one obtains an overlap that depends on neither  $d$  nor  $\nu$ ; this enforces complete similarity and results in the von-Kármán-Prandtl law, which can be obtained by simply removing the quantities  $d$  and  $\nu$  from the list of arguments in equation (2.3). On the other hand, more realistic requirements on the laws being matched leave room for incomplete similarity and are consistent with the scaling law (2.1). Note that the experimental results, for example Figure 7 in [39], exhibit clearly the dependence of the profile in the neighborhood of the centerline on  $\nu$ . The matching was successfully carried out because the scaling law has an intermediate range that is approximately linear in  $\ln \eta$ ; the success of the matching does not depend on the specific values of the constants  $B_0$ ,  $B_1$  and  $\beta_1$  in (2.6). Landau's derivation of the log-law [29] can be repaired in an analogous way.

#### 4. The scaling laws in the inertial range and their consequences

The analogy between the inertial range in the local structure of developed turbulence and the intermediate range in turbulent shear flow near a wall has been noted long ago (see e.g. [8],[37]), and we appeal to it to justify the extension of the scaling analysis above to the case of local structure, where the experimental data are much poorer. In the inertial range of local structure the general scaling law that corresponds to (2.1) is:

$$(4.1) \quad D_{LL} = (\langle \epsilon \rangle r)^{\frac{2}{3}} \Phi\left(\frac{r}{\Lambda}, Re\right),$$

where  $D_{LL} = \langle [u_L(\mathbf{x} + \mathbf{r}) - u_L(\mathbf{x})]^2 \rangle$  is the basic component of the second order structure function tensor which determines all the other components for incompressible flow,  $u_L$  is

the velocity component along the vector  $\mathbf{r}$  joining two observation points  $\mathbf{x}$  and  $\mathbf{x} + \mathbf{r}$ ,  $\varepsilon$  is the total rate of energy dissipation,  $r = |\mathbf{r}|$  is the length of the vector  $\mathbf{r}$ ,  $\Lambda$  is an external length scale, e.g. the Taylor scale, and  $Re$  is a properly defined Reynolds number, for example one based on the Taylor scale. The brackets  $\langle \dots \rangle$  denote an ensemble average. By the logic of the derivation of (4.1) the function  $\Phi$  should be a universal function of its arguments, identical for all flows. Formula (4.1) is assumed to hold only at very high Reynolds numbers  $Re$  and very small  $r/\Lambda$ . It should be equally valid for different definitions of  $Re$  and  $\Lambda$ . Therefore, as above,  $Re$  can enter (4.1) only through an appropriate function  $\psi(Re)$ . The classical “K-41” Kolmogorov theory [25] results from the assumption of complete similarity, in which, for  $r/\Lambda$  small enough and  $Re$  large enough,  $\Phi$  can be taken as a constant different from zero, and one obtains  $D_{LL}$  proportional to  $(\langle \varepsilon \rangle r)^{\frac{2}{3}}$ .

Various corrections to that law have been proposed (for recent reviews, see e.g. [19],[26],[27]). Many of them involve the addition of an extra length scale to the problem, an addition that is hard to justify. We now explore what can be deduced from the much more plausible assumption of incomplete similarity. An expansion of  $\Phi$  for flows in which  $Re$  is large gives, in analogy with (2.5) [4]:

$$(4.2) \quad D_{LL} = (\langle \varepsilon \rangle r)^{\frac{2}{3}} \left[ A_0 + \frac{A_1}{\ln Re} + o\left(\frac{1}{\ln Re}\right) \right] \left(\frac{r}{\Lambda}\right)^{\frac{\alpha_1}{\ln Re} + o\left(\frac{1}{\ln Re}\right)},$$

where we have written, by analogy with the wall-bounded case,  $\ln Re$  for the more general  $\psi(Re)$ . In the present problem, the molecular viscosity  $\nu$  appears only in the variable  $Re$ , so that the limit of vanishing viscosity and the limit of infinite  $Re$  coincide. Note the asymptotic covariance of (4.2) in the external length scale  $\Lambda$ : Indeed, replacing  $\Lambda$  by a different length scale  $\Lambda_1$ , we find

$$\begin{aligned} \left(\frac{r}{\Lambda}\right)^{\alpha_1/\ln Re} &= \left[\left(\frac{r}{\Lambda_1}\right)\left(\frac{\Lambda_1}{\Lambda}\right)\right]^{\alpha_1/\ln Re} \\ &= \left(\frac{r}{\Lambda_1}\right)^{\alpha_1/\ln Re} \exp\left[\alpha_1 \ln\left(\frac{\Lambda_1}{\Lambda}\right)/\ln Re\right] \sim \left(\frac{r}{\Lambda_1}\right)^{\alpha_1/\ln Re} \end{aligned}$$

for large  $Re$ .

The classical “K-41” Kolmogorov theory now corresponds to  $A_0 \neq 0$  in (4.2); then, for large  $Re$ , the famous Kolmogorov 2/3 law is obtained

$$(4.3) \quad D_{LL} = A_0 (\langle \varepsilon \rangle r)^{\frac{2}{3}}.$$

In real measurements for finite but accessibly large  $Re$ ,  $\alpha_1/\ln Re$  is small in comparison with 2/3, and the deviation in the power of  $r$  in (4.2) should be unnoticeable. On the

other hand, the variations in the “Kolmogorov constant” have been repeatedly noticed (see [28],[34],[36]). Complete similarity is possible only if  $A_0 \neq 0$ . If  $A_0 \neq 0$  one has a well-defined turbulent state with a 2/3 law in the limit of vanishing viscosity, and finite  $Re$  effects can presumably be obtained by expansion about that limiting state. In the limit of vanishing viscosity, there are no corrections to the “K-41” scaling, as was also deduced in [10] by a statistical mechanics argument.

Kolmogorov [25] proposed similarity relations also for the higher order structure functions:

$$D_{LL\dots L}(r) = \langle [u_L(\mathbf{x} + \mathbf{r}) - u_L(\mathbf{x})]^p \rangle,$$

where  $LL\dots L$  denotes  $L$  repeated  $p$  times; the scaling gives  $D_{LL\dots L} = C_p(\langle \varepsilon \rangle r)^{p/3}$ .

Experiments, mainly by Benzi et al, see [19], apparently show some self-similarity, obviously incomplete, so that  $D_{LL\dots L}$  is proportional to  $r^{\zeta_p}$ , with exponents  $\zeta_p$  always smaller than  $p/3$  for  $p \geq 3$ , so that  $\zeta_4 = 1.28$  instead of 1.33,  $\zeta_5 = 1.53$  instead of 1.67,  $\zeta_6 = 1.77$  instead of 2.00,  $\zeta_7 = 2.01$  instead of 2.33, and  $\zeta_8 = 2.23$  instead of 2.67. A possible explanation can be of the same kind as for  $p = 2$ :

$$(4.4) \quad D_{LL\dots L} = (C_p^0 + \frac{C_p^1}{\ln Re} + o(\frac{1}{\ln Re})) (\langle \varepsilon r \rangle^{p/3} (r/\Lambda)^{\gamma_p / \ln Re + \dots}).$$

In other words, at  $Re = \infty$  the classic “K41” theory is valid, but the experiments were performed at Reynolds numbers too small to reveal the approach to complete similarity. If this explanation is correct, the coefficients  $\gamma_p$  are negative starting with  $p = 4$ , and therefore the influence of the external scale could be very strong.

It is of great interest to relate relations such as (4.2) and (4.4) to the properties of the Navier-Stokes or Euler equations. In fact, in turbulence one deals not with single solutions of the Navier-Stokes equations but with ensembles of solutions, i.e. with time and viscosity dependent probability measures  $p_t^v$  on a space  $S$  of acceptable solutions of these equations, where it is assumed that for  $t' > t$ , the measure  $p_{t'}^v$  is carried by a set that has evolved from the set that carried  $p_t^v$  by the action of the Navier-Stokes equations (see [38]). A general technical scaffolding that supports these constructions cannot be exhibited at present because of the lack of sufficient understanding of the Navier-Stokes equations.

In [9],[10], it was concluded that expected values with respect to  $p_t^v$  of quadratic functionals of the velocity field  $\mathbf{u}$  converged to averages with respect to  $p_t^0$ . Indeed, an equilibrium probability density was constructed for velocity or vorticity fields as a limiting case of the probability densities one encounters in vortex systems near the  $\lambda$  point or the percolation

threshold, and a probability density for the Euler equations was constructed by perturbation near that equilibrium. This probability density  $p_t^0$  had finite mean energy densities at every point and already exhibited the Kolmogorov spectrum (the importance of a finite energy density is explained in [38]). The conclusion from this analysis is that as the viscosity is reduced one approaches a zero-viscosity limit for turbulence which is not very different from what one finds at a non-zero but very small viscosity, and that in particular  $A_0 \neq 0$  in equation (4.2). These considerations led to the zero-viscosity asymptotics used above, and indeed the success of zero-viscosity asymptotics in turn provides experimental support to the construction of [9], [10]. The scaling law (4.2) is fully compatible with the analysis of [9],[10]. It shows that as the Reynolds number tends to infinity second order moments of the velocity field converge to an inviscid limit.

The task that remains is to relate equation (4.2) and the statistical argument of [9],[10] to rigorous analytical properties of the Navier-Stokes and Euler equations. Far from boundaries we shall not distinguish explicitly between the zero viscosity limit of the Navier-Stokes equations and the Euler equations. The first question to consider is whether estimates such as (4.2) hold strongly “path-wise”, i.e., whether, for each individual solution of the Navier-Stokes equations one has in some appropriate norm  $\| \cdot \|$

$$(4.5) \quad \| \mathbf{u}^\nu - \mathbf{u}^0 \| = O\left(\frac{1}{\log Re}\right),$$

where  $\mathbf{u}^\nu$  is a solution of the Navier-Stokes equations with viscosity  $\nu$  (Reynolds number  $Re$ ),  $\mathbf{u}^0$  is a the solution of the Euler equations with the same data, and solutions that occur with zero probability are excluded. It is interesting to note that a specific result about average energy conservation by ensembles of functions that are Hölder continuous on the average was conjectured by Onsager to hold “path-wise”, and that this turned out to be essentially true [12],[16],[32].

If the solutions of the Euler equations were smooth, then an analog of equation (4.2) would hold with a faster rate of convergence, specifically, with  $Re$  replacing  $\ln Re$  [6]. However, the solutions of the Euler equations in three space dimensions are not smooth ( for numerical evidence, see e.g. [9],[22]; discussions can be found in [7],[15]; specific conjectures about the loss of smoothness are presented e.g. in [19], [32]). The question that must be resolved is whether these solutions are smooth enough to support estimates such as (4.2) or (4.4). In [17],[18] Constantin and Wu examined the decrease in the rate of convergence of solutions of the Navier-Stokes equations to the solutions of the Euler equations for vortex patches in two dimensions, as the boundaries of the patches lost smoothness. They extrapolated from their analysis that the loss of convergence is too severe for estimates such as (4.5).



However, one should note the following: (i) Though the measures  $p_i$  are unknown, it is unlikely that vorticity fields whose range is discrete carry a finite measure; they are too exceptional. Numerical experiment [28] is indeed consistent with [16], [17] for the patch problem, but there are no numerical results showing a loss of convergence to the Euler equations in any other two-dimensional problem. (ii) The whole discussion in the present paper relates to turbulence in three space dimensions and it is very risky to extrapolate from two to three dimensions; indeed, the construction of the "near-equilibrium" ensemble of [9],[10] is intrinsically three dimensional, and cannot be specialized to two dimensions [13].

In addition, strong path-wise convergence is much more than is necessary to explain equation (4.2). It would be enough, for example, if the the measures  $p_i^\nu$  converged to  $p_i^0$  at the rate given by (4.2). Weak convergence results on the convergence of  $p_i^\nu$  to  $p_i^0$  are known [38], but they do not yet yield a rate of convergence. Most important, all one needs for equation (4.2) is that for quadratic functionals  $F = F[\mathbf{u}(\cdot)]$  of the velocity  $\mathbf{u}$  (but not of its derivatives),

$$(4.6) \quad \int F dp_i^\nu - \int F dp_i^0 = O\left(\frac{1}{\ln Re}\right).$$

This is a much weaker requirement; how to prove such estimates, in an appropriate technical setting, is an interesting open question. For the higher order structure functions of (4.4) one needs convergence for functionals  $F$  of higher degree; the argument of [9],[10] does not preclude the possibility that for large enough degree  $p$  this convergence would fail.

The estimate in equation (4.2) has an important bearing on the perturbative treatments of turbulence theory (see e.g. [26],[27]). In most of these treatments the ground state is a solution of the Stokes equation, and one climbs to high  $Re$  by renormalizing a perturbation series in  $Re$ . This difficult expansion makes some sense if the limit of vanishing  $\nu$  (or infinite  $Re$ ) is singular, but not if that limit is well-behaved; if one can find an approximation to Euler turbulence one can go to finite  $Re$  by an expansion in powers of  $1/\ln Re$ .

The mathematical situation for wall- bounded flow is at present so uncertain (see e.g [24]) that we are unable to offer an appropriate conjecture that would relate scaling laws in wall-bounded turbulence to properties of the Navier-Stokes equations.

## 5. Conclusions.

The following conclusions have been reached above: (i) The customary universal logarithmic law of the wall must be jettisoned and replaced by a power law; (ii) it is very likely

that the corrections to the classical "K-41" scaling of the inertial range of local structure in fully developed turbulence are Reynolds-number dependent and disappear in the limit of infinite Reynolds number; (iii) zero viscosity asymptotics, based on a statistical description of fully-developed turbulence in which the zero viscosity limit is well-behaved, constitutes a powerful tool for the analysis of turbulence at high Reynolds numbers.

We also wish to point out that our combination of similarity theory and of asymptotics based on a statistical theory represents a step forward in the effort to derive the properties of turbulent flow from first principles.

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## Figure Captions

Figure 1. The intermediate region of wall-bounded shear flow (e.g., flow in a cylindrical pipe).

Figure 2. The individual members of the family of scaling laws (2.7)–(2.8) near the envelope in the plane  $\phi, \ln \eta$  have a straight intermediate interval with a slope essentially larger than that of the envelope. The horizontal scale is logarithmic.

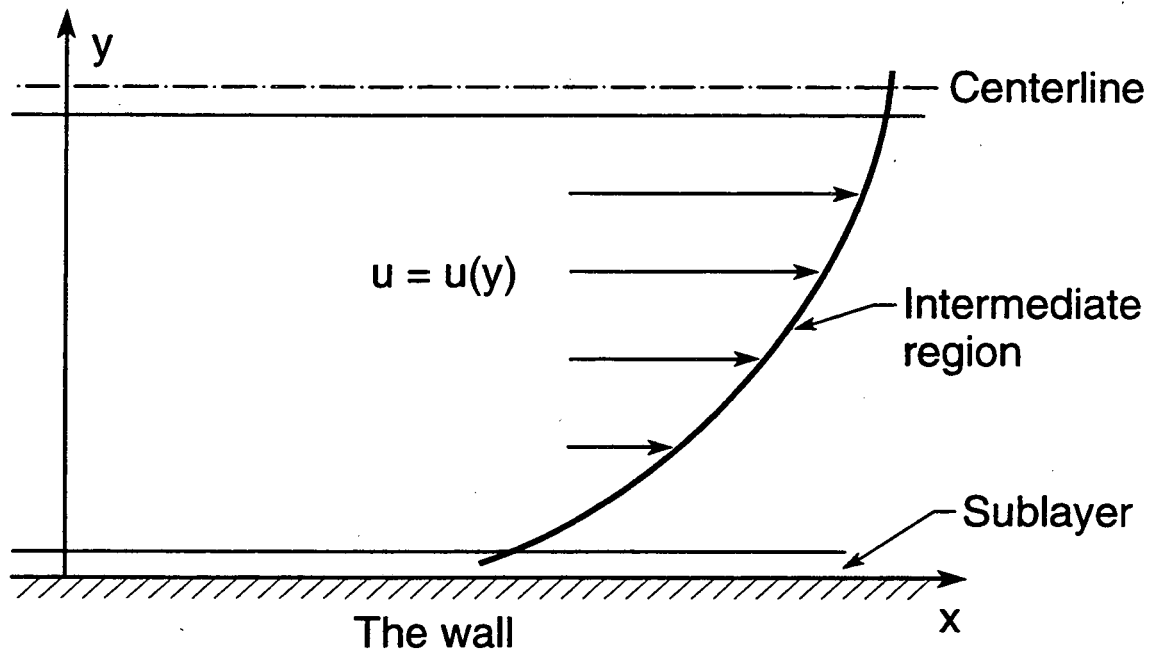
I: A part close to the envelope;

II: straight intermediate part;

III: the fast growing ultimate part having no physical meaning;

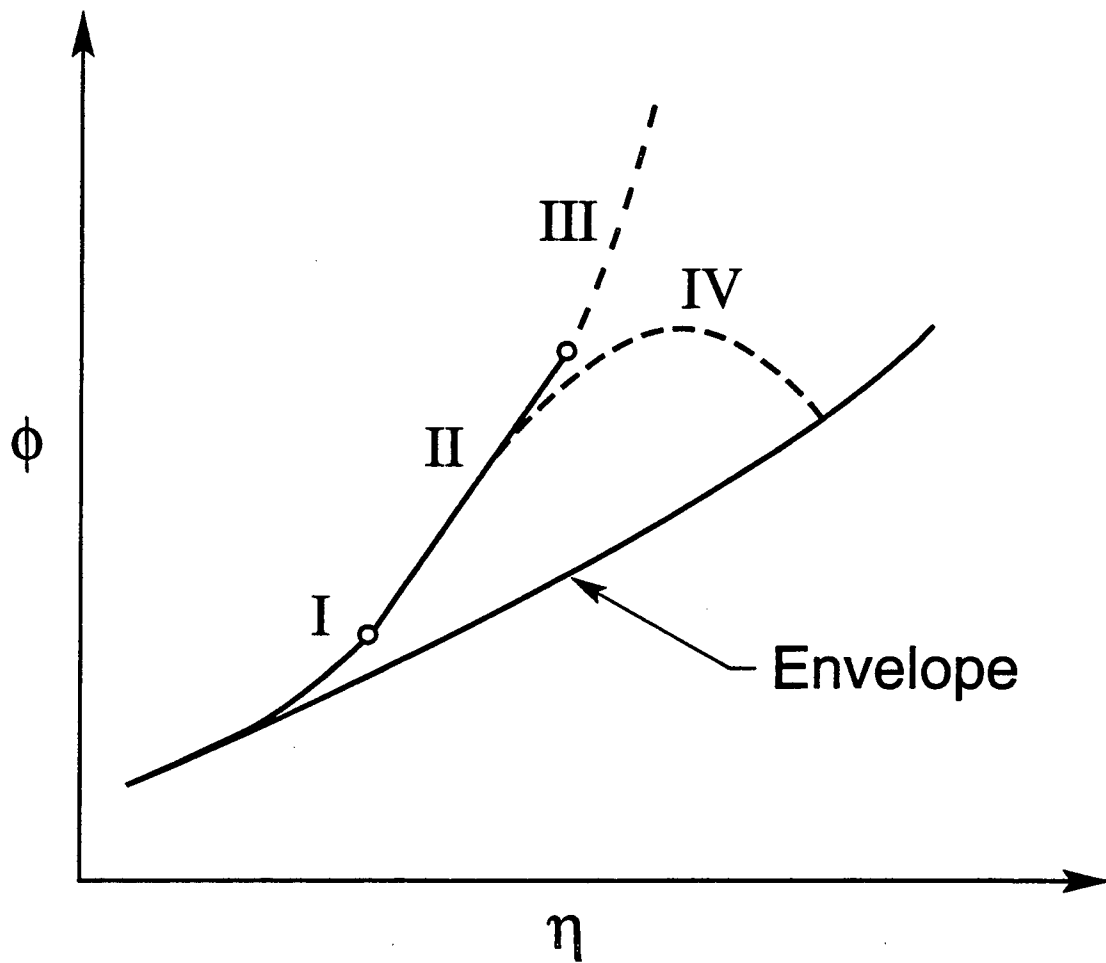
IV: the region near the maximum where the scaling law is not valid.

Figure 3. A graph of the velocity profiles normalized using inner scaling variables for 13 different Reynolds numbers between  $32 \times 10^3$  and  $35 \times 10^6$ . Also shown is a log-law with  $\kappa$  and  $B$  equal to 0.44 and 6.3, respectively. (Reproduced with permission from Zagarola et al.[39])



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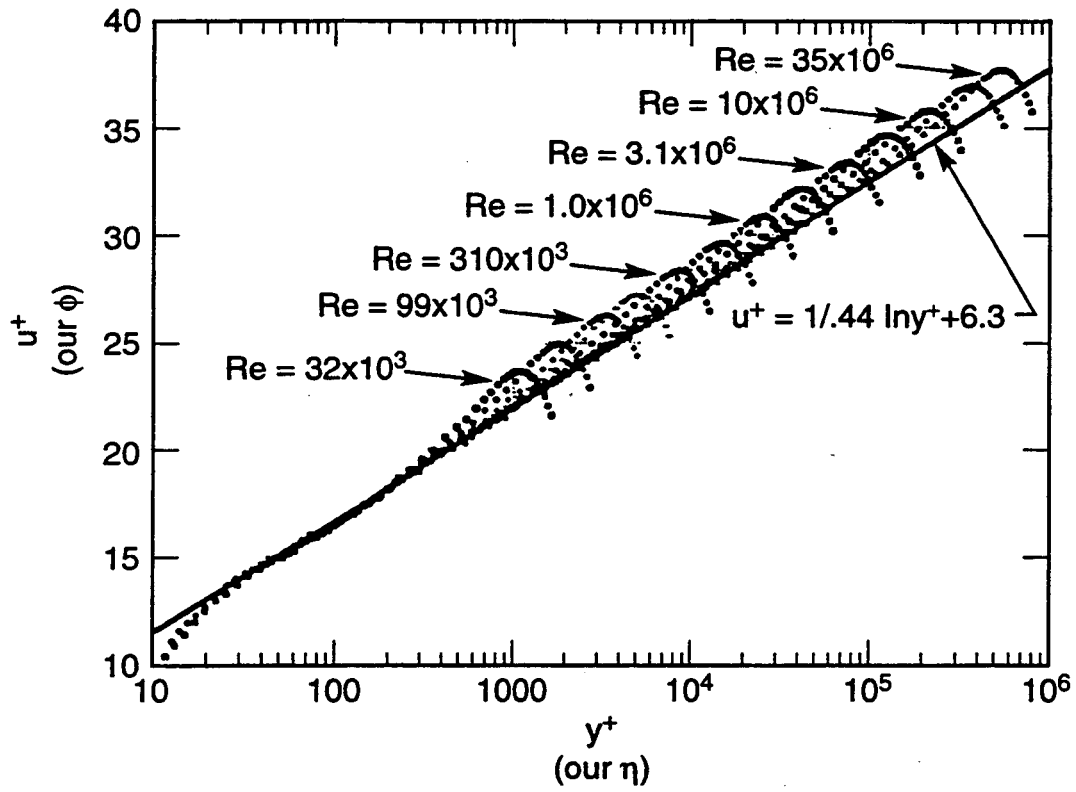
Figure 1



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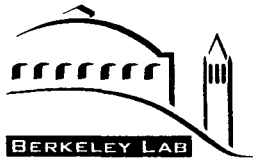
Figure 2





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Figure 3



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