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ON CROSS-PHASE AND THE QUENCHING OF THE TURBULENT DIFFUSION OF MAGNETIC FIELDS IN TWO DIMENSIONS

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ABSTRACT

Nonlinear closure models of the two-dimensional magnetohydrodynamic equations predict that the turbulent diffusivity of magnetic fields in high magnetic Reynolds number flows will be strongly suppressed below the value predicted by simple kinematic models. The consequences of such "resistivity quenching" for models of dissipation and transport in astrophysical plasmas are profound. However, to date there has been little examination of the underlying assumption implicitly made by such models—that the quenching is associated with a reduction in the cross-phase between the velocity and the magnetic potential, rather than a suppression of the turbulence itself. In this Letter, we revisit the two-dimensional problem in an attempt to address this issue. The object of our scrutiny is the normalized cross-phase and its dependence on the initial magnetic field strength. This parameter is a useful diagnostic of turbulent transport and is insensitive to the decay of magnetic field. We present the results of numerical simulations that are consistent with the current picture of resistivity quenching as primarily a suppression of transport of magnetic potential rather than turbulence intensity.

Subject headings: magnetic fields — MHD — turbulence

1. INTRODUCTION

Astrophysical flows are frequently both turbulent and magnetized, and modeling the processes of dissipation and transport in such flows represents a major challenge to our understanding of magnetic fields in astrophysical objects. In the magnetized case, the usual intuition about turbulent diffusion of a passively advected field in hydrodynamic flows is of limited utility, however, because the magnetic field can influence the turbulence itself via the Lorentz force. This magnetic "back-reaction" is a crucial feature of hydromagnetic turbulence, and can profoundly influence the nature of the turbulent diffusion of magnetic fields in astrophysical contexts.

In estimating the magnetic field required for a significant back-reaction, it is tempting to take the equipartition value, i.e., the value for which there is an equal amount of energy in the flow $\boldsymbol{u}$ and the field $\boldsymbol{B}$. Surprisingly, however, the presence of a large-scale, slowly varying mean field $\langle \boldsymbol{B} \rangle$ significantly lowers this threshold field. Using a combination of physical intuition and numerical simulation, Cattaneo & Vainshtein (1991) argued that in two-dimensional magnetohydrodynamic turbulence in a periodic domain the turbulent resistivity η_T will be suppressed below its kinematic value $\eta_{\rm kin}$ by a factor

$$\eta_T/\eta_{\rm kin} = (1 + {\rm Re}_m X^2)^{-1}, \qquad (1)$$

where ${\rm Re}_m = \eta_{\rm kin}/\eta$ is the usual magnetic Reynolds number, $\eta_{\rm kin} = \langle u^2 \rangle \tau_c$ is the kinematic diffusivity, $\langle \cdots \rangle$ represents an ensemble or spatial average, and τ_c is the velocity autocorrelation time (Monin & Yaglom 1971). The large-scale field $X = \langle B \rangle / \langle u^2 \rangle^{1/2}$ is measured in equipartition units. For a mean field magnitude larger than a fraction ${\rm Re}_m^{-1/2}$ of the equipartition value, the turbulent resistivity will be suppressed below the kinematic estimate.

Given the (literally) astronomical values of ${\rm Re}_m$ in astrophysical contexts (10^7 in the solar convection zone, for instance), this suppression can be significant indeed. It is not surprising, then, that the result of Cattaneo & Vainshtein (1991) engendered considerable debate in the community, particularly when it was extended to the α-effect in three dimensions (Kleeorin & Ruzmaikin 1982; Zel'dovich et al. 1983; Kulsrud & Anderson 1992; Gruzinov & Diamond 1994; Cattaneo & Hughes 1996). Notwithstanding the important implications of the α-quench for mean-field dynamo theories, it can be argued that the more constrained two-dimensional problem (concerned as it is with the suppression of the scalar η_T rather than the psuedotensor α_{ij}) represents the more telling phenomenon, and it is to this problem that we restrict our attention in the present Letter.

While the seminal work of Cattaneo & Vainshtein (1991) convincingly demonstrated the existence of a quench of flux transport in two-dimensional magnetohydrodynamic turbulence in a periodic domain, the precise mechanism of this quench was not identified. In particular, it was not established whether the quench is due to a reduction in turbulence intensity or spatial transport. The distinction is an important one because theoretical models of resistivity quenching have, to date, implicitly assumed the latter, by calculating the turbulent resistivity directly via a closure of the turbulent flux of magnetic potential. In this Letter, we present the results of numerical simulations that indicate that it is primarily the normalized cross-phase in the flux of magnetic potential, rather than the turbulence intensity, that is reduced by the imposition of a large-scale mean field.

2. THE THEORY OF RESISTIVITY QUENCHING IN TWO DIMENSIONS

The usual equations of forced incompressible magnetohydrodynamics in two dimensions are

$$\partial_t \nabla^2 \psi = \{\psi, \nabla^2 \psi\} - \{A, \nabla^2 A\} + \nu \nabla^2 \nabla^2 \psi + \tilde{f},$$
$$\partial_t A = \{\psi, A\} + \eta \nabla^2 A, \qquad (2)$$

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where the magnetic field $\boldsymbol{B}$ and velocity field $\boldsymbol{u}$ are described, respectively, by the magnetic potential function $A(x, y, t)$ and the stream function $\psi(x, y, t)$ such that $\boldsymbol{B} = \nabla A \times \hat{z}$ and $\boldsymbol{u} = \nabla\psi \times \hat{z}$. The braces $\{U, V\}$ represent the Jacobian $\partial_x U \partial_y V - \partial_x V \partial_y U$, ν and η are the molecular viscosity and resistivity, and f is an imposed forcing.

We separate $\boldsymbol{B}$ into a mean component, $\langle \boldsymbol{B} \rangle = B_0 \hat{\boldsymbol{y}}$, say, and a fluctuating component $\boldsymbol{b}(x, y, t)$. Likewise, the magnetic potential $A(x, y, t)$ separates into a mean $\langle A \rangle = A_0(x)$ and fluctuating component $a(x, y, t)$, such that $B_0 = -\partial_x A_0$ and $\boldsymbol{b} = \nabla a \times \hat{z}$. We further assume that there is no mean flow so $\langle \boldsymbol{u} \rangle = 0$.

Averaging the equation for A in equation (2) over small scales and imposing periodic boundary conditions yields the equation of motion for the mean potential:

$$\partial_t A_0 + \partial_x \langle u_x a \rangle = \eta \partial_x^2 A_0. \tag{3}$$

We make the usual assumption that the effect of the turbulent small-scale fields $\boldsymbol{u}$ and $\boldsymbol{b}$ is to induce a turbulent down-gradient flux of the mean potential A_0 so that $\langle u_x a \rangle = -\eta_T \partial_x A_0$. Equation (3) then becomes

$$\partial_t A_0 = (\eta + \eta_T)\, \partial_x^2 A_0 = (1 + \mathrm{Nu}_m)\, \eta \partial_x^2 A_0, \tag{4}$$

where $\mathrm{Nu}_m = \eta_T/\eta$ is a magnetic analog of the Nusselt number. In general, we expect $\eta_T \leq \eta_{\mathrm{kin}}$, or, equivalently, $\mathrm{Nu}_m \leq \mathrm{Re}_m$, where equality corresponds to the kinematic case.

Multiplying both sides of the equation of motion for the fluctuating component a, averaging, and again imposing periodic boundary conditions, one obtains

$$\frac{1}{2}\, \partial_t \langle a^2 \rangle + B_0 \langle u_x a \rangle = \eta \langle b^2 \rangle. \tag{5}$$

Stationarity then implies that

$$\mathrm{Nu}_m = \frac{\eta_T}{\eta} = \frac{\langle b^2 \rangle}{B_0^2}\,. \tag{6}$$

This well-known relation, frequently referred to as the "Zel'dovich theorem," is ultimately a consequence of Alfvén's theorem. Equation (6) implies that most of the magnetic energy is contained in the small-scale fields. This is because, even when quenched, Nu_m is typically much larger than unity. This observation led Cattaneo & Vainshtein (1991) to postulate that it is the small-scale field $\boldsymbol{b}$, rather than the mean field B_0, that reduces the turbulent diffusion. Extrapolating from equation (6) (with $\langle u^2 \rangle = \langle b^2 \rangle$) and the kinematic estimate $\mathrm{Nu}_m \approx \mathrm{Re}_m$ then leads to the quenching result given in equation (1).

It is worthwhile noting that any scheme intended to circumvent the resistivity quench must grapple with this very robust result. Ultimately, this means a relaxation of the assumptions made in the derivation of equation (6): for instance, by the imposition of a boundary flux of magnetic potential (Blackman & Field 2000; Silvers 2006), or another source of microscopic irreversibility, such as nonlinear wave-wave interactions (Keating & Diamond 2008). We will not pursue these topics here.

Numerical simulations have, for the most part, firmly established the existence of resistivity quenching in two-dimensional magnetohydrodynamic turbulence (Cattaneo 1994; Silvers 2005, 2006). Less certain is the nature of the physical mechanism underlying this phenomenon. Cattaneo (1994) identified a subtle modification of the Lagrangian energy spectrum and attributed this to the emergence of nondiffusive behavior associated with the development of long-term memory in the system.

To date, theories of resistivity quenching have implicitly assumed a suppression of the turbulent flux of magnetic potential $\langle u_x a \rangle$, rather than the turbulence intensity. In these theories, a direct calculation of the flux is made via closure calculations, such as the eddy-damped quasi-normal Markovian approximation (Pouquet et al. 1976; Pouquet 1978; Gruzinov & Diamond 1994, 1996). These yield expressions of the form

$$\langle u_x a \rangle = B_0 \sum_k \tau_k \left(\langle u^2 \rangle_k - \langle b^2 \rangle_k \right), \tag{7}$$

where τ_k is the decorrelation time of the fluid and the field. For simplicity we replace τ_k by its spectral average τ, dropping the spectral summations. Substituting for $\langle b^2 \rangle$ in equation (7) and using the Zel'dovich theorem given in equation (6) yields the quenching formula of Cattaneo & Vainshtein (1991) written here in terms of the magnetic Nusselt number:

$$\mathrm{Nu}_m = \frac{\mathrm{Re}_m}{1 + \mathrm{Re}_m X^2}\,. \tag{8}$$

Note that equation (7), which is a generic result independent of the particular closure scheme used, implies that, at least for high Re_m and roughly unit magnetic Prandtl number, the turbulent resistivity $\eta_T = \langle u_x a \rangle / B_0 \approx \eta_{\mathrm{kin}} + \eta_{\mathrm{mag}}$, where $\eta_{\mathrm{kin}} \approx \langle u^2 \rangle \tau$ is the contribution to the turbulent resistivity from the velocity field, and $\eta_{\mathrm{mag}} \approx -\langle b^2 \rangle \tau$ is the magnetic contribution, strictly negative. The form of equation (7) therefore suggests an appealing physical picture of resistivity quenching as a struggle between two competing cascades, or, equivalently, two competing couplings. In scale space, the velocity field, which tends to strain apart isocontours of a, is characterized by a forward cascade of a^2 to smaller scales. On the other hand, the Lorentz force causes like-signed current filaments to be attracted to one another, leading to the coagulation of blobs of magnetic potential. This tendency leads to an inverse cascade of a^2 to larger scales. These two cascades are parameterized by turbulent diffusivities of opposite sign, as in equation (7). In real space, the competing couplings exactly cancel for fully "Alfvénized" turbulence, for which there is an equipartition of kinetic and magnetic energy. This paints a picture, complementary to the one described above, of resistivity quenching as a consequence of the conversion of eddy energy into Alfvén wave energy.

The scenarios just described are, of course, strongly dependent on the assumption that the physical content of resistivity quenching is a reduction in the flux of magnetic potential. To test this hypothesis, we examine the normalized cross-phase, defined as

$$\mathcal{P} = \langle u_x a \rangle \Big/ \sqrt{\langle u_x^2 \rangle \, \langle a^2 \rangle}. \tag{9}$$

As is clear from this definition, the cross-phase and the turbulent flux are closely related. One important difference is that, unlike the flux, which inevitably decays as the magnetic field dissipates, $\mathcal{P}$ remains normalized during the time evolution of the system. This permits one to directly compare the cross-phase for a variety of large-scale fields. By contrast, such a

TABLE 1
Results from Numerical Simulations for Different Initial Fields

X^2	$\mathcal{P}_{av}$	$\mathcal{P}_{max}$	$\mathcal{P}_{min}$	$\Delta\mathcal{P}$
1.0	0.0001	0.007	−0.009	0.0048
0.1	0.0005	0.025	−0.023	0.0129
0.03	0.0008	0.045	−0.022	0.0163
0.015	0.0012	0.048	−0.043	0.0200
0.01	0.0035	0.064	−0.033	0.0240
0.002	0.0144	0.036	−0.039	0.0236
0.0	0.0290	0.146	−0.078	0.0337

survey is not possible for the turbulence intensity, which will also decay.

We assume that $\langle a^2\rangle^{1/2} \approx \langle B_{\max}^2\rangle^{1/2} l$, where $\langle B_{\max}^2\rangle \approx B_0^2 \mathrm{Re}_m$ is the maximum value of the magnetic field immediately prior to decay (Zel'dovich 1957) and l is the gradient length-scale of the magnetic potential, here assumed to be the scale on which the system is forced. In addition, we assume isotropy, so that $\langle u_x^2\rangle \approx \frac{1}{2}\langle u^2\rangle$. The cross-phase can then be expressed in terms of the magnetic Nusselt number with the use of Fick's law $\langle u_x a\rangle = \eta_T B_0$, which yields

$$\mathcal{P} \approx \mathrm{Nu}_m \mathrm{Re}_m^{-3/2}, \qquad (10)$$

where we have ignored numerical factors of order unity. Equation (8) then implies that

$$\mathcal{P} \approx \frac{\mathrm{Re}_m^{-1/2}}{1 + \mathrm{Re}_m X^2}. \qquad (11)$$

3. RESULTS

Equations (2) were solved using a second-order pseudospectral scheme in space and an integrating factor method in time. In nondimensional variables, $1/\nu$ and $1/\eta$ are of the same order as the Reynolds number Re and magnetic Reynolds number Re_m: as in Cattaneo & Vainshtein (1991) we set these to be $1/\nu = 500$ and $1/\eta = 1000$, respectively. The system was driven by random excitation of wavenumbers in the range $5 < k < 6$ via two overlapping functions (see Silvers 2006 for more details).

Owing to the difficulty of obtaining a stationary state in two-dimensional magnetohydrodynamic turbulence, numerical simulations were carried out for a variety of initial (zero mean) field strengths B_0, rather than a large-scale mean field, and the subsequent decay was investigated. This is the same approach taken by Cattaneo & Vainshtein (1991). Initial field strengths were chosen so that the initial magnetic energy per unit volume,

$$E_0^M = \frac{1}{V}\int_V \frac{B_0^2}{2}\, dV, \qquad (12)$$

took the values EX^2, where $X^2 = 1$, 0.1, 0.03, 0.015, 0.010, and 0.002. The equipartition energy of the flow, E, was obtained by averaging the kinetic energy per volume over 100 time units prior to the introduction of the magnetic field. The initial magnetic field was introduced via the magnetic potential

$$A_0 = \sqrt{4E_0^M} \cos y. \qquad (13)$$

For the purposes of comparison, the kinematic case was also examined.

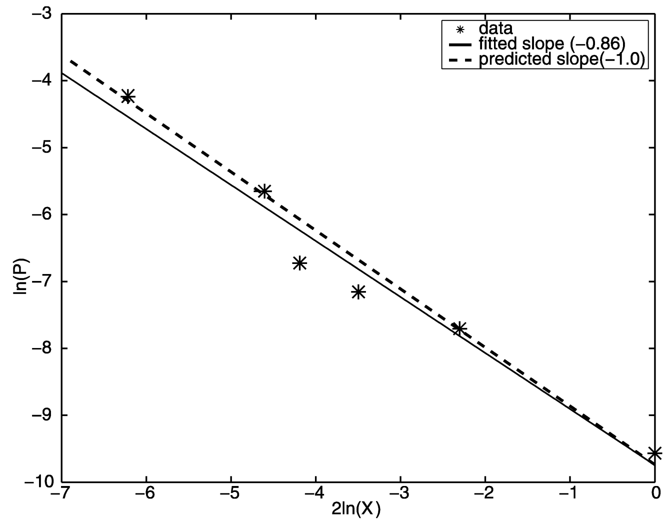

Fig. 1.—Log-log plot of the time-averaged normalized cross-phase $\mathcal{P}$ vs. the square of the initial field strength measured in equipartition units $X^2 = B_0^2/\langle u^2\rangle$. The measured slope of −0.86 is in close agreement with the predicted value of −1.0.

In all cases, an initial rapidly growing phase was observed: this is due to the generation of small-scale gradients in a and the corresponding stretching of field lines (Zel'dovich 1957). Decay then follows: for sufficiently strong initial field the decay is initially ohmic followed by decay at the kinematic rate. For weak fields the decay is purely kinematic. This qualitative behavior closely matches that observed by Cattaneo & Vainshtein (1991).

Also observed in each case was a temporary suppression of the kinetic energy after the introduction of the magnetic field. A similar suppression was observed by Cattaneo (1994): it is important to note, however, that this suppression can be attributed to coupling with Alfvén waves generated when the magnetic field is first switched on and not to any quenching of the velocity field directly.

As the magnetic field decays to zero, so too does the cross-phase. However, *the magnitude of the normalized cross-phase remains of roughly constant amplitude, even as the field decays*. For this reason, the cross-phase is a more useful diagnostic than the turbulence intensity itself.

Table 1 contains the results of our numerical simulations. For each X^2, the normalized cross-phase $\mathcal{P}$ was observed to fluctuate greatly and, indeed, even changed sign, as can be seen from the maximum and minimum measured values $\mathcal{P}_{max}$ and $\mathcal{P}_{min}$. The time-averaged value $\mathcal{P}_{av}$, however, indicates a systematic down-gradient diffusion of magnetic potential. Also shown in Table 1 is the standard deviation $\Delta\mathcal{P}$ for each case. Owing to the turbulent nature of the simulations, these values are larger than the time-averaged value of $\mathcal{P}$ by as much as an order of magnitude. This is unfortunate but unavoidable, and it is important that the results of all investigations of this kind be considered in this light.

Figure 1 depicts a log-log plot of the time-averaged normalized cross-phase against the square of the initial field strength expressed in equipartition units, measured during the initial part of the decay. A fit is obtained with slope −0.86, in good agreement with the Cattaneo & Vainshtein (1991) prediction of −1.0 in the limit $\mathrm{Re}_m X^2 \gg 1$ (see eq. [11] above). Although it is difficult to accurately determine Re_m in decay problems such as this one, it is likely of the order of $1/\eta =$

1000. Using this estimate, the mean fields considered here are at most only weakly supercritical, and this fact may be responsible for the relatively minor discrepancy between the predicted and fitted values for the slope.

Not shown in Figure 1 is the kinematic case $X^2 = 0$, which might be considered the extreme weak field case. For very weak fields such that $\mathrm{Re}_m X^2 \ll 1$, equation (8) implies that $\mathrm{Nu}_m \approx \mathrm{Re}_m$. Substituting this value into equation (10) yields $\mathcal{P} \approx \mathrm{Re}_m^{-1/2}$. If we estimate $\mathrm{Re}_m \approx 1000$, we find that the cross-phase should level off at about $\ln(\mathcal{P}) \approx -3.45$. We found close agreement between this value and the numerically obtained value of $\ln(\mathcal{P}) = -3.54$.

4. CONCLUSIONS

We have shown that the cross-phase, or, equivalently, the transport of magnetic potential at fixed amplitude, is reduced by a mean field, as demonstrated by the dependence of the normalized cross-phase on B_0 (Fig. 1) which is the principle result of this Letter. The reduction in cross-phase is *sufficient* to explain the suppression of η_T predicted by Cattaneo & Vainshtein (1991), indirectly implying that resistivity quenching is *not* due to a reduction in the turbulence intensity. Note that this study does not discount the existence of a quench of the velocity, only that such a quench, if it exists at all, is not responsible for resistivity quenching in two-dimensional MHD turbulence. This result adds weight to existing theoretical models of resistivity quenching (Gruzinov & Diamond 1994, 1996).

Finally, we note that the subtle, but crucial, distinction between suppression of turbulence and suppression of transport is not confined to magnetohydrodynamics. An analogy can be made with polymer hydrodynamics (Lumley 1973; Tabor & DeGennes 1986; Gruzinov & Diamond 1996; Groisman & Steinberg 2001), where the introduction of polymer additives to turbulent pipe flow can suppress cross-pipe momentum transport. Indeed, the analogy is a particularly suggestive one, as the reduction in drag is associated with an equipartition of mechanical and elastic energy, just as resistivity quenching seems to be associated with the conversion of eddy energy to Alfvén wave energy, for which there is an equipartition of mechanical and magnetic energy.

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