Algebraic Modular Forms on $\mathbf{SO_5}(\mathbb{Q})$ and the Computation of Paramodular Forms

by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

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Summer 2018

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Abstract

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This dissertation describes a result that compares two level subgroups on different inner forms of $\mathbf{GSp}(4)$, and then uses this result and a conjecture of Ibukiyama's to compute paramodular forms for all prime levels below 400. In the process 78 generic forms were computed, of which 47 had not been previously computed.

To Laura

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Acknowledgments

Thanks to Professor Kenneth Ribet for giving me the encouragement and freedom to pursue this research. Professor John Voight was the one who originally got me interested in the questions that would become part of this research, and informed me of the opportunities I had to pursue it. Professor Nathan Ryan generously arranged a research visit in Uruguay, and Jeffery Hein and Professor Gonzalo Tornaría collaborated on the earliest form of the conjectures presented here.

In the fall 2015 semester I visited ICERM's semester-long program on "Computational Aspects of The Langlands Program". At this program Professor Ralf Schmidt helped explain some of the relevant representation theory, and Professor David P. Roberts gave valuable assistance with searching for hypergeometric motives. I also benefited from conversations with Professors Armand Brumer, John Brian Conrey, Lasina Dembélé, David Farmer, John Jones, Kimball Martin and Brooks Roberts. Many more visitors to ICERM expressed interest and encouragement, and sadly my memory has stopped me from listing them all here.

The Sage community assisted with at least one Sage bug discovered during this work.

This research used the Savio computational cluster resource provided by the Berkeley Research Computing program at the University of California, Berkeley (supported by the UC Berkeley Chancellor, Vice Chancellor for Research, and Chief Information Officer).

Lastly, thanks to Barb Waller and Victoria Lee for five years of smoothing bureaucratic obstacles.

During the course of the work I was supported by some of NSF RTG grant DMS-1646385. My visit to Uruguay that started this work was part of NSF grant OISE-1341090. This document was prepared using Professor Paul Vojta's Latex templates.

Chapter 1

Introduction

This dissertation represents the result of joint work with Gonzalo Tornaría and Jeffery Hein.

Once we leave the familiar world of modular forms behind, the computation of examples becomes much harder. Despite significant efforts, very few examples of paramodular forms (Siegel modular forms invariant under the paramodular subgroup) on $\mathbf{GSp}_4$ are known. While some forms can be computed by taking lifts from $\mathbf{GL}_2$, these lifts will not have interesting Galois representations attached, as the Galois representation "comes from" $\mathbf{GL}_2$. The paramodular forms most of interest are the generic ones, which are not lifts.

Our question then is to compute the L-function of such a paramodular form, in particular of weight 3. This question has no doubt arisen from the first studies of Siegel modular forms in some sense, although I do not know when the paramodular group first became of interest. Several previous works have attempted to compute these forms, either as functions or merely recover the Hecke action.

One of the more recent advances in this area is by Poor and Yuen [31] who work directly with multivariate power series representing the Fourier-Jacobi expansion of the forms, using known dimension formulas to determine if they spanned the space, and then applying the Hecke operators to the expansions.

Lansky and Pollack [26] earlier carried out a computation on a compact form of $\mathbf{PGSp_4}$, which is of course isomorphic to $\mathbf{SO_5}$ and hence the same as the group we are considering. However, they work directly with the representation in terms of matrices and brute force congruences to enumerate representatives. A direct comparison of efficiency is difficult: they do not seem to have gotten to the levels we did and mention their algorithm slows dramatically with level.

Chisholm [8] worked directly with quaternionic lattices to carry out computations on an inner form of $\mathbf{GSp}_4$, which may be an alternative route to compute some of these forms as well. However it is not immediately clear which level structure her algorithm applies to could be used to compute paramodular forms.

Ash, Gunnels and McConnell [1] used a modular symbols based approach to computing the space of modular forms on $\mathbf{SL_4}(\mathbb{Z})$, and find some of the same forms we do. However, their algorithms rapidly slow as the primes for which the Euler factor is desired increase, and so they could only manage to compute the first three Euler factors at levels 61 and 73.

One of the standard ways to compute classical modular forms of level N is to work instead on the compact group $D^\times$ for a well chosen quaternion algebra D. For a choice of level subgroup given by the units of a particular order $\mathcal{O}$, one obtains a finite space as domain for modular forms, as well as matrices that can be explicitly computed representing the actions of the Hecke operators on this space. This space is isomorphic to $S_2(\Gamma_0(N))$ if N is squarefree by Jacquet-Langlands [21].

Our work uses an analogous idea for the computation of paramodular forms. The group $\mathbf{GSp_4}$ modulo its center is $\mathbf{PGSp}_4$ which is exceptionally isomorphic to $\mathbf{SO}_{3,2}$. On the compact inner form $\mathbf{SO}_5$ we seek a level structure such that the space of modular forms with respect to that level structure will be easy to compute and that will transfer to paramodular forms. If we only require the forms are easy to compute one such class of level structures is given by integral lattices Λ by the work of Greenberg and Voight [15].

This dissertation demonstrates that a conjecture of Ibukiyama's implies that computing the modular forms of level $\mathbf{Aut}(\Lambda)$ for a well chosen lattice Λ recovers the Hecke eigenvalues of paramodular forms of prime level. Some of these lattices were previously used in the dissertation of Hein [20] to compute paramodular forms with prime levels up to 200, but without much basis for thinking they were paramodular forms. In addition to this theoretical contribution this dissertation extended those computations to prime levels less than 400, and implemented an algorithm of Dubey and Holenstein [11] for computing lattices from their genus symbols.

Our methods are able to compute many more Euler factors than other methods because they do not involve computing large numbers of Fourier coefficients. To compute the action of a Hecke operator T_p on a Siegel modular form from the Fourier expansion one needs Fourier coefficients corresponding to positive definite binary quadratic forms of discriminant p^2 times the lowest discriminant that appears in the series expansion of the form, and because the expressions for the Fourier series are given as products of Fourier series, this results in a very large number of coefficients being computed. By contrast we can compute T_p in time $O(p^{3+\epsilon})$ and T_{p^2} in time $O(p^{5+\epsilon})$ as described in Chapter 5 of Hein's thesis [20]. The implicit constants depend on the lattice and the size of the genus, but are independent of p.

Chapter 2 of this dissertation discusses some notation and basic properties of quadratic spaces and paramodular forms. Chapter 3 demonstrates some general theorems about algebraic modular forms we will employ to prove the main theorem. Chapter 4 conducts the comparison of level subgroups that connects our forms to those of Ibukiyama's. Chapter 5 addresses the question of what can be said with fewer hypothesis as well as finding lower bounds for the dimension of the space we compute. Then in Chapter 6 we discuss how the algorithms for computing these forms work, and how they are amenable to running on large clusters. Finally. Appendix A contains the table of all rational eigenforms we computed.

Beyond the work in this dissertation I will discuss just two interesting examples that we computed. These examples were computed with Hein's implementation in Magma [4], [20].

The first example is a nonholomorphic form. Yoshida lifts take a classical newform of weight 4 and one of weight 2 to one of several possible Siegel modular forms [35]. One particularly interesting one we find comes from a newform of weight 4 level 5 together with newform of weight 2 and level 42. The space of forms is the forms with level given by the genus of a maximal lattice of discriminant 105 with positive Hasse-Witt invariant at all primes. This is Yoshida lift with totally generic representation, and so is associated to a vector valued form, not a classically holomorphic one. There is no homomorphic form in the L-packet this lift lives in by the criterion of Saha and Schmidt [35].

Our evidence that this is a correct identification is that the Euler factors at $11, 13$ and 17 agree. Unfortunately there is no Sturm bound to assist us in possibly proving this example correct. However, this is enough to demonstrate that the associated representation is of Yoshida type, and that it is completely generic.

The second example is of a form with corresponding hypergeometric motive [33]. This example has conductor 182. The hypergeometric motive has $A = (3, 3)$, $B = (1, 1, 1, 1)$ and $t = 1/729$. The space of algebraic modular forms where the form appears has level given by the maximal lattice in a quadratic space with discriminant 91 and Hasse-Witt invariant -1 at 7 and 13. David Roberts assisted in discovering this example. Euler factors up to $p = 17$ agree.

We now discuss future directions, beginning with the closest and concluding with the further out. Ibukiyama and Kitayama [24] have generalized Ibukiyama's conjecture to square-free levels divisible by an odd number of odd primes (and not 2). It is plausible to assume that the calculations conducted in this thesis will generalize to this setting as well. In addition, there is a notion of weight that may be added to the algebraic modular forms we work with. Calculations with nontrivial weight have not yet been implemented or systematically performed by us, and some of the lemmas in this dissertation would need adaptation to such a setting. Such an addition would enable the calculation of paramodular forms in weights higher than 3.

Jeffery Hein proposed a precise conjecture about which forms appear in our calculations, along with a description of which forms we compute that lift to nonlifts in terms of theta series [20, Conjecture 3.5.6]. The work in this dissertation provides only a partial resolution, by demonstrating that the image is contained in the space of paramodular forms, and that it has dimension $1/2$ the dimension of the space of paramodular forms. Information about the nonvanishing of theta series of lattice of dimension 5 and the global epsilon factor of paramodular forms is needed to make progress on the remaining parts of Hein's conjecture.

While knowledge of the Euler factors determines the paramodular form [37] ,we do not know of an explicit formula for the Fourier Jacobi expansion given the Hecke eigenvalues. Therefore series expansion methods for paramodular forms currently produce more information about the Fourier expansions. It would be interesting to come up with a method to obtain the Fourier-Jacobi expansion from the data we compute here.

Finally weight 2 paramodular forms remain off-limits to the techniques described in this dissertation, similar to how weight 1 modular forms require different techniques to compute then other weights. Unfortunately this is the weight of interest in the paramodular conjecture [31], and so our methods cannot assist in investigating the paramodular conjecture.

Chapter 2

Notation and basic properties

We start by discussing quadratic forms. Most of this material is in Cassels [7], which we repeat to set the notation definitely. Let F be a field of characteristic 0. A *quadratic space* over F is a pair of a finite dimensional vector space V and function $q : V \to F$, where $q(x) = B(x, x)$ and B is a bilinear form. There is a choice of basis such that if $v = x_1e_1 + x_2e_2 + \ldots + x_ne_n$, $q(v) = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2$. The *discriminant* (a function of the quadratic form and the author, see [10, Chapter 15]) will be the product of the a_i, and is only defined up to multiplication by a square in F as the choice of basis may modify the discriminant by a square factor.

Now the more subtle set of definitions when we wish to consider $\mathbb{Z}$ or $\mathbb{Z}_p$ instead of a field. Let R be a domain (either $\mathbb{Z}$ or $\mathbb{Z}_p$ in this thesis) and F its field of fractions. An *integral quadratic form* for us is an expression q such that there is a matrix B with entries in R and diagonal entries divisible by 2 if 2 is not a unit in R such that $q(x) = (1/2)x^TBx$. By way of clarification xy and $x^2 + y^2$ are both integral quadratic forms.

Now let q be a quadratic form over $\mathbb{Z}$. We define two quadratic forms q, q' to be (globally) equivalent if $q(Tx) = q'(x)$ for some T in $\mathbf{GL}_n(\mathbb{Z})$. Two forms are locally equivalent at p if, treating the q and q' as forms with coefficients in $\mathbb{Z}_p$, there is a T in $\mathbf{GL}_n(\mathbb{Z}_p)$ such that $q(Tx) = q'(x)$.

A form is *positive definite* if all the eigenvalues of B over $\mathbb{R}$ are positive.

We next define another rational invariant of quadratic forms. Let $q(x) = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2$ be a diagonal quadratic form over $\mathbb{Q}$. Recall the Hilbert symbol $(a, b)_p$ is 1 if $ax^2 + by^2 - z^2 = 0$ has a solution over $\mathbb{Q}_p$, and -1 otherwise. The *Hasse-Witt invariant* at p according to Cassels is $\prod_{i<j}(a_i, a_j)_p$. It is always 1 or -1, and is -1 at only finitely many primes. Other authors may include diagonal terms. It is a local invariant that only depends on the equivalence class of the quadratic form over $\mathbb{Q}_p$. We can extend this definition to quadratic forms over $\mathbb{Z}$ by treating the entries as though they are in $\mathbb{Q}$. The Hasse-Witt invariants at all primes (including the infinite prime) and the discriminant are the only invariants of quadratic forms over $\mathbb{Q}$: if two quadratic forms q and q' share these, they are isomorphic over the rationals.

Equivalence over $\mathbb{Z}$ implies equivalence over $\mathbb{Z}_v$ for all v including the real place. The converse is not true. The set of positive definite forms everywhere locally equivalent to a given positive definite form q modulo the relation of global equivalence is a finite set called the *genus*.

Quadratic forms over $\mathbb{Z}_q$ have canonical representatives in each equivalence class. For odd q these are simply diagonal forms, further reduced by converting $ap^r x^2 + bp^r y^2$ to $p^r x^2 + abp^r y^2$ for any units a, b. For even 2 the canonical form has a truly involved definition.

For a quadratic form $q(x) = 1/2x^\top Bx$, if we fix an appropriate (V, r) a quadratic space over $\mathbb{Q}$ (and associated bilinear form), then there is a matrix L such that $B = L^* L$ where L^* is the dual with respect to r. The rows of L span a lattice corresponding to q. Such a lattice is not unique as we can always rotate it. We define two lattices Λ and Π to be globally equivalent if there exist bases $\lambda_1, \lambda_2, \ldots \lambda_n \in \Lambda$ and $\pi_1, \pi_2, \ldots \pi_n \in \Pi$ such that $\langle \lambda_i, \lambda_j \rangle = \langle \pi_i, \pi_j \rangle$ for all i, j. This definition extends to lattices over $\mathbb{Z}_p$. It agrees with our definition of equivalence of a quadratic form.

For an integral lattice Λ, we let $\Lambda_q = \Lambda \otimes \mathbb{Z}_q$ for any prime q.

We will let $\mathbf{SO}_5(\Lambda)$ be the set of elements of $\mathbf{SO}_5$ of the underlying quadratic space that preserve Λ. The isomorphism class of the underlying quadratic space is determined by Λ. The level subgroup $\mathbf{Aut}(\Lambda)$ of $\mathbf{SO}_5(\mathbb{A})$ is the product $\otimes_p' \mathbf{SO}_5(\Lambda_p)$. Almost everywhere the lattice is unimodular, and therefore $\mathbf{SO}_5(\Lambda_p)$ is a hyperspecial subgroup of $\mathbf{SO}_5$. This will be proven later.

Let Λ be a lattice in a quadratic space (V, q). The *Clifford Algebra* $\mathbf{Cliff}(V)$ is the quotient of the tensor algebra by the relation $v \otimes v = q(v)$. It is $\mathbb{Z}/2\mathbb{Z}$ graded where the vectors are in the odd part. The *even Clifford algebra* $\mathbf{Cliff}^e(V)$ is the even part of the grading. Now, if Λ is integral then let $\mathrm{Cliff}^e(\Lambda)$ be the set of integral combinations of products of vectors in Λ and $\mathbb{Z}$. Since $v^2 = q(v)$ is always integral for an element in Λ, $\mathbf{Cliff}^e(\Lambda)$ is closed under multiplication. It is also discrete and therefore is an order in $\mathbf{Cliff}^e(\Lambda)$.

The operation of transposition reverses the order of vectors in the Clifford algebra and is an involution. We usually denote it σ. It respects the grading of the Clifford algebra. The *generalized spin group* is the group of elements u of $\mathbf{Cliff}^e(V)$ such that $uvu^{-1} \in V$ for all v in V and is denoted $\mathbf{GSpin}(V)$. It is a simply connected group. It has a subgroup $\mathbf{Spin}(V)$ consisting of elements u such that $u\sigma(u) = 1$ where σ is transposition. Conjugation by a single vector acts by reflection over that vector. One subgroup of $\mathbf{GSpin}$ is products of an even number of reflections, and in fact this subgroup with the center generates $\mathbf{GSpin}$.

We now discuss Siegel modular forms. Let

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Then $\mathbf{Sp}_4(\mathbb{Q})$ is the set of g in $\mathbf{M}_2(\mathbb{Q})$ such that $g^\top Jg = J$. The related group $\mathbf{GSp}_4(\mathbb{Q})$ is all g such that $g^\top Jg = n(g)J$ for some $n(g)$ in $\mathbb{Q}^\times$.

We define the paramodular group of level N as

$$K(N) = \mathbf{Sp}_4(\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

Let Ω be the space of symmetric complex 2 by 2 matrices whose imaginary parts are positive definite. We call Ω the Siegel upper half space. Elements of $\mathbf{Sp}_4(\mathbb{Q})$ and $\mathbf{GSp}_4(\mathbb{Q})$ act on Ω as follows: $g \cdot Z = (AZ+B)(CZ+D)^{-1}$ where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. This is analogous to the action of SL_2 on the upper half plane.

Now let F be a $\mathbb{C}$-valued function on the Siegel upper half space, and define the operator $|_{(3,0)}$ as $F|_{(3,0)}[g](Z) = \det(CZ+D)^{-3}F(g \cdot Z)$. A paramodular form of weight 3 and level p is a holomorphic function on the Siegel upper half space such that $F|_{(3,0)}[g] = F$ for all g in $K(p)$ [23]. It is a cusp form if it vanishes on the cusps of the associated symmetric space. We will denote the space of cuspforms of weight k for the paramodular group by $S_k(K(p))$.

Unexpectedly, the paramodular group has a local newforms theory due to Ralf Schmidt and Brook Roberts [32]. Furthermore the conductor of the spinor L function associated to a paramodular newform of level N is N. If we were to take $\Gamma_0(N)$ instead we might get L functions of conductor N^2, and so $\Gamma_0(N)$ is the "wrong" level in some sense.

Algebraic modular forms are a class of modular forms that do not involve analysis. Greenberg and Voight [15] following Gross [17] note the following: If G is a reductive group such that G_∞ is compact,and K is a product of compact open subgroups of G_p for each p, then $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K$ is finite. Let x_i be an enumeration of its elements. Given a G-vector space W of G, the space of algebraic modular forms of level K and weight W is the vector space $\sum H^0(x_i, W)$. If W is the trivial representation then this space is the space of functions on the x_i.

This space is acted on by double coset operators. Let $KtK = \bigsqcup t_iK$. Then we have $(H_tf)(x) = \sum f(t_ix)$, the usual decomposition of double cosets into disjoint single cosets. This is a Hecke operator, and distinct Hecke operators commute.

An f that is an eigenvector of all the Hecke operators an eigenform. If W is trivial one such function is the constant function. We call all the other eigenforms cuspidal, and call their span the space of cuspforms. The cuspforms and constant function are orthogonal under an inner product defined by Gross [17] as the Hecke operators are normal with respect to that inner product. This immediately implies that the space of modular forms has a basis of eigenforms.

If K' is a finite index subgroup of K, there is a Hecke equivariant embedding from $M(K, W)$ to $M(K', W)$. A function $f \in M(K, W)$ that is an eigenvector of the Hecke operators corresponds to an automorphic representation π of G with $\pi_\infty = W$, and a K invariant vector [15]. But this is a representation of G with a K' invariant vector.

We use $\hat{G}$ to denote the adelicization of an algebraic group G, and for a subgroup $U \subset \hat{G}$ and a set of places S we use U^S to denote the product of all components at places not in S, and U_S the product of all components at places in S.

Chapter 3

Generalities on algebraic modular forms

We start with the following simple result. When it comes to working with modular forms Greenberg and Voight take a product over finite places while Ibukiyama includes the infinite place and has the level structure include all of G_∞ [15, 23]. We will work with a group such that G_∞ is only compact mod center, not compact, so Greenberg and Voight's definition wouldn't apply. But if we consider a reductive group G which is compact, these two definitions agree, as shown by the following lemma.

Lemma 1. *The space of f on $\hat{G}$ such that $f(agu) = f(g)$ for a in $G(\mathbb{Q})$, u in U, and g in $\hat{G}$ is isomorphic as a Hecke module to the set of algebraic modular forms on G with level structure U^∞ when $U_\infty = G_\infty$ and G_∞ is compact.*

Proof. Let h be a function that satisfies Greenberg and Voight's definition. Let ρ be the projection from $\hat{G}$ to $G(\mathbb{A}_{\text{fin}})$.

Now construct a function $f = \rho \circ h$. It is immediately clear that f satisfies Ibukiyama's definition. Denote this by $\eta(f)$. It is immediate that η is linear.

Conversely if we have a function f satisfying Ibukiyama's definition, $f(xr) = f(x)$ for all $r \in G_\infty$. So f factors through ρ. Call the resulting function h. Now if we have a in G, u in U^∞ and g in $\hat{G}$, $h(agu) = f(agu) = f(g) = h(g)$. Therefore we have an isomorphism of vector spaces between the forms of Ibukiyama and Greenberg and Voight.

We now consider Hecke operators. We will consider only Hecke operators for sets of the form UtU where t is identity at the real place. Let $t_1, t_2, \ldots t_n$ be such that $UtU = \bigsqcup t_i U$. I claim that $K\rho(t)K = \bigsqcup \rho(t_i)K$, where K is $\rho(U)$. Suppose that x is a member of $K\rho(t)K$. Then we know that $x \times 1_\infty = t_i u$ for some t_i and u in U, but then $x = \rho(t_i)k$ for some k in K. Furthermore, if $\rho(t_i)k_1 = \rho(t_j)k_2$ we have $t_i u_1 = t_j u_2$ for some u_1, u_2 in U.

Now if the double coset operator acts on $f(x)$ we get $\sum f(xt_i)$. But then $\sum h(x\rho(t_i)) = \sum(h(\rho(xt_i)) = \sum \eta(f(xt_i)) = \eta(\sum f(xt_i))$. Therefore η is a Hecke equivariant linear isomorphism.

□

The second result is a general functoriality result. Suppose we have two algebraic groups G and H. If we have a central homomorphism ϕ from G to H, then we would like to descend algebraic modular forms in the sense of Ibukiyama, invariant under the kernel, from G to H as well as lift back up. This result lets us do that under some slight additional assumptions on ϕ. Note that for ϕ we know each factor is open [22], but this is not enough to show openness in the restricted direct product, which we must assume separately.

Lemma 2. *Suppose $\phi : G \to H$ is a homomorphism with kernel the center of reductive groups over $\mathbb{Q}$. If J is a level structure on $\hat{G}$ such that K_p contains the maximal open subgroup of $Z(G_p)$ for all p and all of G_∞, and $\phi(K)$ is open in $\hat{H}$, then there is a Hecke equivariant isomorphism between $M(G, K)$ and $M(H, \phi(K))$. Furthermore, this map preserves cuspidality.*

Proof. If g is an algebraic modular form on $\hat{G}$, then for any element z of $Z(\hat{G})$ we can apply strong approximation for $Z(G)$ to write $z = z'k$ where k is an element of U, and z' an element of $Z(G)(\mathbb{Q})$. Therefore $g(zx) = g(x)$ for all x and z. So we can descend g to a function $\tilde{g}$ on H. This function will be invariant under $\phi(K)$ and so is an algebraic modular form.

Conversely we can pull back a function on $\hat{H}$ to $\hat{G}$, and if the function invariant under $\phi(K)$ the pullback will be invariant under K. Therefore these spaces are isomorphic.

Next these operations are Hecke invariant. Let KtK be a double coset operating on g. Let t_iK be a decomposition of the double coset. Now consider $\phi(KtK)$. I claim that $\phi(t_iK)$ is a decomposition of that double coset. If we have that $\phi(t_i)\phi(K) = \phi(t_j)\phi(K)$, that means $\phi(t_it_j^{-1})$ is in $\phi(K)$, and hence $t_it_j^{-1}$ is in $K\ker(\phi)$.

But now we have that $k_1tk_2^{-1}t^{-1} = k_3z$ for some $k \in K$ and z in $Z(G)$. From there we have $tk_2^{-1}t^{-1} = k_3k_1^{-1}z$, followed by $k_2^{-1}t^{-1} = t^{-1}k_3k_1^{-1}z$, and lastly $t^{-1} = t^{-1}k_3k_1^{-1}k_2z$. But now we obtain z is an element of K, ergo t_i and t_j give the same coset of K. Armed with this fact the proof that the action of the double coset of K on g followed by projection is identical with the action on the projection of g of the double coset of $\phi(K)$ is immediate.

The elements of the Hecke algebra that become the trivial element under this map are those for KtK with t central. But their action on $M(G, K)$ was sending $g(x)$ to $g(tx)$, which by above is just $g(x)$.

Finally cuspidality. As the noncuspidal form is simply the constant function 1 it is clear what happens to the eigenforms that are not cuspidal, and hence to the ones that are. □

These two lemmas will be applied at the end of the next chapter.

Chapter 4

Level structure and Ibukiyama's conjecture

Ibukiyama proposed the following generalization of the Eichler correspondence [23]: Let B be a quaternion algebra ramified only at p and ∞ and $\mathcal{O}$ a maximal order of D.

Then let $G = \{g \in \mathbf{M}_2(D) : g\bar{g}^\top = n(g)\}$. Now let $U_q = \mathbf{M}_2(\mathcal{O}_q)^\times \cap G_q$ and let

$$S = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

and $G_p^* = \{g \in \mathbf{M}_2(D) : gS\bar{g}^\top = n(g)S\}$ and

$$U_p^* = \left(\begin{bmatrix} \mathcal{O}_p & \pi^{-1}\mathcal{O}_p \\ \pi\mathcal{O}_p & \mathcal{O}_p \end{bmatrix}\right) \cap G_p^*$$

G_p^* and G_p are isomorphic, and after applying this (unspecified) isomorphism Ibukiyama wrote a level as $U = G_\infty \prod U_p$ for all p.

We will now specialize to the case of trivial weight. Unwrapping the definition from Ibukiyama and reversing the order of arguments to agree with Greenberg and Voight, we consider functions f on $\hat{G}$ such that $f(agu) = f(g)$ for a in $G(\mathbb{Q})$, u in U, and g in $\hat{G}$. This space, which will be denoted $S(U)$ is finite. Ibukiyama conjectures that there is a Hecke equivariant map from $S(U)$ to $S_3(K(p))$, the space of paramodular forms of weight 3.

Our goal is to demonstrate the existence of a Hecke equivariant injection from the algebraic modular forms on $\mathbf{SO}_5(\mathbb{Q})$ with level $\mathbf{Aut}(\Lambda)$ for a particular lattice Λ to be specified momentarily into $S(U)$, with U as above.

We review the definition of D locally. As stated in the book of Voight [42, Theorem 13.3.10], D_p has generators i, j and k such that $ij = k$, all of these anticommute, and $i^2 = r$, $j^2 = p$ where r is a quadratic nonresidue modulo p. For all other primes q, D_q is isomorphic to $\mathbf{M}_2(\mathbb{Q}_q)$. At the real place D is isomorphic to the Hamiltonians.

The maximal order in $\mathbf{M}_2(\mathbb{Q}_q)$ is conjugate to $\mathbf{M}_2(\mathbb{Z}_q)$. For D_p it is $\mathbb{Z}_p \oplus \mathbb{Z}_p i \oplus \mathbb{Z}_p j \oplus \mathbb{Z}_p k$.

Let (V, q) be a 5-dimensional quadratic space with Hasse-Witt invariant -1 at p and 2, and denote by $\mathbf{SO}_5$ the algebraic group of its isometries. This is the quadratic space that will underlie our lattice Λ that we will consider. The even Clifford algebra of this space is isomorphic to $\mathbf{M}_2(D)$ as an algebra with involution as observed in Hashimoto and Ibukiyama [19]. Later we will demonstrate that G is **GSpin**.

Our functoriality result will enable us to transfer forms back and forth from G to $\mathbf{SO}_5$ if we show the image under f is open. If we show the image of U is contained within the stabilizer of Λ, then level raising will give an injection from $S(\Lambda)$ to $S(U)$ and then Ibukiyama's conjecture will let us get to $S_3(K(p))$. This analysis is the bulk of our proof.

We will specify the genus of Λ as follows: if we let $q(x) = 1/2x^\top Bx$ be the associated quadratic form to Λ then B is locally isomorphic to the unique form with discriminant $2p$ at odd primes not p. At the remaining primes 2 and p we can take the following local components:

$$B_p = \begin{cases} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & p \end{pmatrix} & p \equiv 3, 5 \mod 8 \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2r & 0 \\ 0 & 0 & 0 & 0 & pr \end{pmatrix} & p \equiv 1, 7 \mod 8 \end{cases}$$

$$B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2p \end{pmatrix}$$

where r is an arbitrary quadratic nonresidue modulo p. Furthermore, we require B to be positive definite. This is a valid genus because it satisfies the compatibility conditions, and $q(x)$ has integral coefficients as the diagonal of B has all even entries. We will let Λ be any lattice in this genus. As was shown by Greenberg and Voight the space of algebraic modular forms of trivial weight and level $\mathbf{Aut}(\Lambda)$ is equal to the space of functions on the genus of Λ, and so the choice of a particular representative that we have made do not change the space.

It is alternatively the maximal lattice in a quadratic space over $\mathbb{Q}$ with Hasse-Witt invariant -1 at p and 2 and discriminant $2p$. This can be seen by computing the Hasse-Witt invariant, and then observing that the lattice is a maximal integral lattice over $\mathbb{Z}_p$ for each p. We know that $\mathbf{Aut}(\Lambda)$ is a level subgroup of $\mathbf{SO}_5$.

In order to say things about the conductor of forms in $S(\Lambda)$ we will prove the following lemma

Lemma 3. *For odd q not dividing $\det(B)$, $\mathbf{Aut}(\Lambda_q)$ is a hyperspecial subgroup of $\mathbf{SO}(\Lambda_q \otimes \mathbb{Q}_q)$.*

Proof. Note that as q does not divide $\det(B)$, Λ_q has dual Λ_q. Therefore the group scheme $\underline{G}(S) = \{m \in M_2(S) \text{s.t.} m^\top Bm = B\}$ is smooth and clearly represents $\mathbf{Aut}(\Lambda_q)$ by Gan and Yu's construction. The special fiber is a finite group of Lie type and is reductive, therefore $\mathbf{Aut}(\Lambda_q)$ is hyperspecial [41, Section 3.8]. □

This then implies the eigenforms in $S(\Lambda)$ will be in representations unramified away from $2p$. This implies the local Hecke algebra will determine the Euler factors for the corresponding automorphic representation, and the Satake isomorphism will give the formula to turn Hecke eigenvalues into an Euler factor.

Furthermore, for our choice of lattice $\mathbf{Aut}(\Lambda_2)$ is a hyperspecial maximal subgroup as Λ is semiregular at 2 due to having Murphy's discriminant p [28, Section 3]. This implies that the automorphic representations are unramified at 2 as well.

For this lattice the even Clifford algebra of the underlying quadratic space is isomorphic as an algebra with involution to $\mathbf{M}_2(D)$ with conjugate transpose as the involution where D is some quaternion algebra ramified at p and ∞. This can be shown by taking an orthogonal basis of V and then examining the subalgebras e_1e_2, e_1e_3 and $e_1e_2e_3e_4$ and $e_1e_2e_3e_5$.

Lemma 4. *If V is a five dimensional quadratic vector space $\mathbf{GSpin}(V)$ is equal to the group G of g in $\mathbf{Cliff}^e(V)$ such that $g\sigma(g) = n(g)$ for some $n(g)$ in $\mathbb{Q}$.*

Proof. The condition $g\bar{g}^\top = n(g)$ implies that when g acts by conjugation its action restricts to the subspace $m = \bar{m}^\top$. This space has an inner product $\mathrm{Tr}(xy)$ and is 6 dimensional. The inner product is preserved by the action by conjugation. If we exclude the scalars, we obtain a 5 dimensional vector space which is the dual of V, on which G acts by conjugation. The kernel of this action is the scalars, as can be seen by examining commutation in $\mathbf{M}_2(D)$.

This space T is the space spanned by products of four elements of a chosen orthogonal basis of V. It has a pairing with V, $p(t, v) = (tv + \sigma(tv))/2$ which takes values in multiples of the product of five elements of that same fixed orthogonal basis. We will now investigate the action of conjugation by a vector v. We know that this conjugation in particular preserves T and V.

Let τ_v denote the negative conjugation by v, and let v^* be the element of T such that $p(v^*, v) = 1$ and $p(v, w) = 0$ for all w orthogonal to v. Note that w^* anticommutes with v as we can express it as $vxyz$ where x, y and z are mutually orthogonal and form a basis with v and w. This is in the space fixed by transposition, and has the right properties to be w^*.

Then

$$
\begin{aligned}
p(\tau_v v^*, v) &= -\frac{1}{-2q(v)}(vv^* - vv + \sigma(vv^* - vv)) \\
&= -\frac{1}{2q(v)}(q(v)vv^* + \sigma(q(v)vv^*)) \\
&= -\frac{1}{2}(vv^* + \sigma(vv^*)) \\
&= -\frac{1}{2}(v^*v + \sigma(vv^*) \\
&= -p(v^*, v) \\
p(\tau_v v^*, w) &= -\frac{1}{-2q(v)}(vv^* - vw + \sigma(vv^* - vw)) \\
&= -\frac{1}{2q(v)}(vv^*wv + \sigma(vv^*wv)) \\
&= -\frac{1}{2q(v)}v(v^*w + \sigma(v^*w))v \\
&= 0 \\
p(\tau_v w^*, w) &= -\frac{1}{-2q(v)}(vw^* - vw + \sigma(vw^* - vw)) \\
&= \frac{1}{2q(v)}(vw^*vw + \sigma(vw^*vw)) \\
&= \frac{1}{2q(v)}(vvw^*w + \sigma(vvw^*w)) \\
&= p(w^*, w) \\
p(\tau_v w^*, v) &= \frac{1}{-2q(v)}(vw^* - vv + \sigma(vw^* - vv)) \\
&= \frac{1}{2q(v)}v(w^*v + \sigma(w^*v)v \\
&= 0 \\
p(\tau_v w^*, z) &= \frac{1}{-2q(v)}(vw^* - vz + \sigma(vw^* - vz)) \\
&= \frac{1}{-2q(v)}(vw^*zv + \sigma(vw^*zv)) \\
&= \frac{1}{-2q(v)}v(w^*z + \sigma(w^*z)v \\
&= 0
\end{aligned}
$$

The above equations demonstrate that t_v acts by reflection over v^*, and so every element of G can be written as a product of vectors that will act by reflection on T and a scalar. Therefore the elements of G preserve V when acting by conjugation, and are therefore elements in $\mathbf{GSpin}(V)$. But every product of t_v is an element of G, and so we see that G is exactly $\mathbf{GSpin}(V)$. □

We now embark on our study of $f(U_q)$ for all primes q. There are four distinct cases to handle.

4.1 The archimedean prime

We have that U_∞ is all of G_∞. Here there is very little to say beyond the surjectivity of the map over fields which we have already proven, so $f(U_\infty)$ is all of $\mathbf{Aut}(\Lambda_\infty)$. See Garrett's notesheet on sporadic isogenies for details [14].

4.2 The case of good odd primes

Let q be an odd prime not p. We will use D to refer to $D \otimes \mathbb{Q}_q$ by abuse of notation. First we have to write down the isomorphism between the Clifford algebra and $\mathbf{M}_2(D)$ in a way that is q-integrality preserving. While D is also the matrix algebra, it is perhaps less confusing to forget that temporarily. First note that Λ_q is isomorphic to the $\mathbb{Z}_q$ span of e_1, e_2, e_3, e_4, e_5 where the e_i are an orthogonal basis of V, and e_5 has length p.

Then note that e_1e_2, e_2e_3, e_1e_3 are a set of elements that generate a quaternion algebra, and $e_1e_2e_3e_4$, $e_1e_2e_3e_5$, e_4e_5 is another set that generates a quaternion algebra. These sets of elements commute.

Therefore we have an algebra we can write as $\left(\frac{-1,-1}{\mathbb{Q}_q}\right) \otimes \left(\frac{-1,-p}{\mathbb{Q}_q}\right)$. Transposition acts by conjugation on the $\left(\frac{-1,-1}{\mathbb{Q}_q}\right)$ component and akin to matrix transpose on $\left(\frac{-1,-2p}{\mathbb{Q}_q}\right)$. To give an isomorphism as algebras with involution we send $\left(\frac{-1,-p}{\mathbb{Q}_p}\right)$ to the space of matrices, and then send $\left(\frac{-1,-1}{\mathbb{Q}_q}\right)$ to D.

We first examine the isomorphism between $\left(\frac{-1,-p}{\mathbb{Q}_q}\right)$ and $\left(\frac{1,1}{\mathbb{Q}_q}\right)$. If we write a linear map that preserves the norm form on the traceless part this map is an isomorphism of quaternion algebras by Proposition 3.14 in Chapter 3 of Voight [42]. But the norm form of $\left(\frac{-1,-p}{\mathbb{Q}_q}\right)$ is $-x^2 - py^2 - pz^2$ which is isomorphic over $\mathbb{Z}_q$ to $x^2 + y^2 - z^2$. Therefore there this map preserves the order as the linear map and its inverse can be picked to have integral entries.

Now there is a well-known isomorphism (example 1.4 of [42]) between $\left(\frac{1,1}{\mathbb{Q}_q}\right)$ and a matrix algebra where we send i to $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and j to $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. As 2 is a q-adic unit this sends our order to matrices with $\mathbb{Z}_q$ entries.

When we apply both of these isomorphisms to the two factors of our tensor product, we get an isomorphism between $\mathbf{Cliff}^e(\Lambda_q)$ and $\mathbf{M}_2(D)$ where $D = \left(\frac{-1,-1}{\mathbb{Q}_q}\right)$. Furthermore, this isomorphism preserves the order. Now we examine the order we have in $\left(\frac{-1,-1}{\mathbb{Q}_q}\right)$. This is a split quaternion algebra by computing the Hilbert symbol and again we will have an order preserving isomorphism to $\mathbf{M}_2(\mathbb{Z}_q)$ (where we remember what D is).

We have now demonstrated that the algebra isomorphism between $\mathbf{M}_2(D_q)$ and $\mathbf{Cliff}^e(V_q)$ in fact sends the order $\mathbf{Cliff}^e(\Lambda_q)$ to the order $\mathbf{M}_2(\mathcal{O})$. The next paragraphs will use this fact to prove that U_q has image equal to $\mathbf{Aut}(\Lambda_q)$.

As Λ_q is unimodular, every element g of $\mathbf{SO}(\Lambda_q)$ has a representative as a product of reflections by vectors in Λ_q whose norm is a q-adic unit [7, Corollary 2, Lemma 3.3, Chapter 8]. This will lift to an element of **GSpin** which will have integral coefficients and hence be in U_q.

Conversely, if we have an element of U_q, g, it has $\mathbb{Z}_q$ integral coefficients and hence is an element of $\mathbf{Cliff}^e(\Lambda_q)$ and therefore preserves Λ_q when acting by conjugation [7, Chapter 8]. Therefore $f(U_q)$ is isomorphic to $\mathbf{SO}(\Lambda_q)$.

4.3 The case of p

For the prime p we have more work to do and must resort to use of Bruhat-Tits theory. We will denote the special form used to define G_p by $\mathbf{GU}_2(D)$, and let $\mathbf{SO}_5$ be the algebraic group of isometries of a 5 dimensional quadratic space with discriminant $2p$ and Hasse-Witt invariant -1.

Our analysis starts by noting that $\mathbf{SO_5}$ is a connected lie group. We have a morphism f from $\mathbf{GU}_2(D)$ to $\mathbf{SO_5}$, and we would like to investigate its integral structure.

Now recall some basic properties of the enlarged Bruhat-Tits building. To a reductive group G over $\mathbb{Q}_p$, $\mathcal{B}^e(G, \mathbb{Q}_p)$ is a metric space on which G acts, which is an extension of $\mathcal{B}(G, \mathbb{Q}_p)$. To a point x in $\mathcal{B}^e(G, \mathbb{Q}_p)$ there is a smooth scheme P_x whose o_L points are the stabilizer of x in G_L [18].

Here $D(H)$ will be the derived subgroup of H, and $\tilde{D}(H)$ its simply connected cover as an algebraic group. Confusion with D for a quaternion algebra should be minimal.

Lemma 5. *Let $f : G \to H$ be a central surjection of algebraic groups over a local field K. Let $\hat{f}$ denote the $G(K)$ invariant of buildings described by Langvogt[25] Then if f is either an isogeny or has kernel including the center of G, for any x a point in $\mathcal{B}^e(G)$, there exists a O_K morphism $\tilde{f} : P_x \to P_{\hat{f}(x)}$ with image including all O_K points, where P_y denotes the stabilizer scheme associated to y.*

Proof. We also have that for finite Galois extensions L of K, or unramified (and not necessarily finite) extensions L , the stabilizer of x in $\mathcal{B}^e(G, L)$ is the o_L points of P_x the parahoric in $\mathcal{B}^e(G, K)$ by Galois descent.

Therefore we have a scheme map from P_x to $P_{\hat{f}(x)}$ as explained by Paul Broussous and nfdc23[5], see also Bruhat and Tits [6, paragraph 1.7.2].

First we consider the case of f an isogeny.

Then the map on buildings is an isometry, and therefore $f^{-1}(\mathbf{Stab}(\hat{f}(x))$ is exactly $\mathbf{Stab}(x)$. Now, for each element y of $\mathbf{Stab}(x)$ there is a finite Galois extension L_y such that there is a $z \in G(L_y)$ such that $f(z) = x$. Therefore there is a O_{L_y} point of P_x that maps onto any point of $P_{\hat{f}(x)}$. and $\tilde{f}$ is hence dominant as its image is all O_K points of $P_{\hat{f}(x)}$.

We now consider the case of full kernel. Here we examine Langvogt's construction in more detail. The map $G \to H$ is factored into $G \to Z(G) \times \tilde{H} \to \tilde{H} \to H$ where the arrows on the ends are isogenies dual to the multiplication map, and the one in the middle is a projection.

We only need to look at the middle arrow as the outer two are handled by our argument for isogenies. For every element h in the stabilizer of $\hat{f}(x)$ there is an element (g_1, g_2) that maps to it, and g_2 stabilizes x in $\mathcal{B}^e(G_i)$. But then $(1, g_2)$ also stabilizes x, and maps to h.

The composition of all three arrows has image all O_K points of $P_{\hat{f}(x)}$. $\square$

We now consider the decomposition $Z(G) \times \tilde{D}(G)$ in the case of $G = \mathbf{GU_2}(\mathrm{D})$. $Z(G)$ is the diagonal matrices with values in K Clearly $D(G)$ is a subgroup of $\mathbf{U_2}(D)$, and as $\mathbf{U_2}(D)$ is a simply connected and simple algebraic group we have equality. Because $\mathbf{U_2}(D)$ is simply connected, $\tilde{D}(G)$ is $D(G)$.

Now U_p is a compact connected subgroup of $\mathbf{GU_2}(D)$, which is connected.

It therefore stabilizes a point x in the building $\mathcal{B}^e(\mathbf{GU_2}(D), \mathbb{Q}_p)$, and is contained in the stabilizer. Since we are interested in the image of U_p under a map f whose kernel is the center, it is enough to look at the image of $U'_p = U_p \cap \mathbf{U_2}(D)$ since we can draw a commutative square as in Langvogt's construction.

Since U'_p is the stabilizer of a lattice $\mathcal{O} \oplus \varpi\mathcal{O}$ in $\mathbf{U_2}(D)$, there is a smooth group scheme $\mathcal{P}$ that represents the stabilizer of the point U'_p fixes and has generic fiber $\mathbf{U_2}(D)$ [13, Section 5.8] This is a maximal stabilizer.

If we look in the extended building of $\mathbf{GU_2}(D)$ we see the stabilizer of x must be $\mathcal{G}_m \times \mathcal{P}$, as $\mathbf{GU_2}(D)$ is isogenous to $\mathbf{GL}_1 \times \mathbf{U_2}(D)$.

Let $\mathcal{M}$ be the stabilizer scheme in $\mathbf{SO}_5$ corresponding to our point $\hat{f}(x)$. This is a maximal stabilizer as we can pull back any bigger stabilizers to x, and the stabilizer of x was maximal.

By the above we have a map of group schemes $\mathbf{G}_m \times \mathcal{P} \to \mathcal{M}$ which surjects onto O_K points of $\mathcal{M}$ and is trivial on $\mathbf{G}_m$. This doesn't mean we get all O_K points of M from O_K points of P: there may be some extension required to hit a particular point.

Since these schemes are smooth we can use Hensel's lemma, so we have a surjection of special fibers $\mathbf{G}_m \times \mathcal{P}_{\mathbb{F}_p} \to \mathcal{M}_{\mathbb{F}_p}$. This implies we have a surjection of the maximal reductive quotient of $\mathbf{G}_{m,\mathbb{F}_p} \times \mathcal{P}_{\mathbb{F}_p}$ onto the maximal reductive quotient of $\mathcal{M}_{\mathbb{F}_p}$ since the maximal reductive quotient of a group surjects onto all reductive quotients of that group. However, if G surjects onto H, then $D(G)$ surjects onto $D(H)$. Now, the maximal reductive quotient of the special fiber of $\mathcal{P}_{\mathbb{F}_p}$ is $\mathbf{Res}_{\mathbb{F}_p^2/\mathbb{F}_p}\mathbf{Sp}_2$.

There are two classes of maximal compacts in our $\mathbf{SO}_5(\mathbb{Q}_p)$. The first corresponds to the stabilizer of a nonmaximal lattice, the second to the stabilizer of a maximal one by Proposition 6.3.9 in Gan-Yu [13]. These exist because we can rewrite our quadratic form as a hyperbolic plane plus a three dimensional anisotropic space. If u, v are vectors in the hyperbolic plane such that $\langle u, v\rangle = 1$, then we can take either u, v as our vectors, or u, pv. The first choice gives our chosen lattice, the second a nonmaximal one, and both satisfy the condition in Gan and Yu for its automorphism group to be a maximal compact subgroup.

The two possibilities for the maximal reductive quotient of special fiber of $\mathcal{M}$ are $\mathbf{O_3} \times \mathbf{O_2}$ and $\mathbf{^2O_4}$ corresponding to the nonmaximal and the maximal one respectively.

As $\mathbf{O}_2$ is commutative it vanishes when take derived subgroups, and so the two choices for the derived subgroups are $D(\mathbf{O}_3) = \Omega_3$ and $D(\mathbf{^2O_4})$. But there is no finite extension L of $\mathbb{F}_p$ such that the L points of $\mathbf{Res}_{\mathbb{F}_p^2/\mathbb{F}_p}\mathbf{Sp_2}$ surject onto $\Omega_3(\mathbb{F}_p)$. If L is of odd degree the points are just $\mathbf{Sp}_2(L \otimes \mathbb{F}_{p^2})$, and if L is of even degree we get $\mathbf{Sp}_2(L) \times \mathbf{Sp}_2(L)$. Once we mod out by the center we either have a simple group or a product of simple groups, and the simple groups are not the same as $\Omega_3(\mathbb{F}_p)$ but are the same as $\mathrm{D}(\mathbf{^2O_4}(\mathbb{F}_p))$ by Steinberg's classification [40]. But this is exactly the special fiber of the parahoric associated to the maximal lattice, and therefore we have a map to the parahoric associated to the maximal lattice and not the other one.

Therefore U_p has image contained in the stabilizer of the lattice we are considering.

4.4 The oddest prime

We are left with the prime 2. Unlike the case of q odd we cannot necessarily write a diagonalization of a lattice with the right invariants (in fact we cannot for the lattice we are considering) and many of the divisions by 2 in the q case must be treated more gently. The calculations in this subsection have been checked with Sage for all 4 possible $\mathbb{Z}_2$ isomorphism classes of lattices and so are likely correct.

We begin by recalling our lattice. It is $H \oplus H \oplus \langle p\rangle$ where H is the hyperbolic plane, and $\langle p\rangle$ a 1-dimensional subspace with norm p. Note that there are only 4 such equivalence classes of lattices, as there are 4 square classes of p in $\mathbb{Z}_2$.

Now, if we denote the natural basis with e_1, e_2, e_3, e_4, e_5, and take $a_1 = e_1+e_2$, $a_2 = e_1-e_2$, $a_3 = e_3 + e_4$, $a_4 = e_3 - e_4$, and $a_5 = p$ then the a_i are orthogonal, and we have exactly the same commuting elements as in the good prime case.

Unlike the good prime case our algebra is now naturally $\left(\frac{1,-1}{\mathbb{Q}_2}\right) \otimes \left(\frac{1,p}{\mathbb{Q}_2}\right)$, with the first factor being generated by a_1a_2, a_2a_3, a_1a_3. We send $\left(\frac{1,p}{\mathbb{Q}_2}\right)$ to the matrix algebra, and $\left(\frac{1,-1}{\mathbb{Q}_2}\right)$ to the quaternion algebra (which confusingly is also the matrix algebra). This is to ensure the Clifford transposition acts by conjugate transpose.

We start with $a_1a_2a_3a_4, a_1a_2a_3a_5, a_4a_5$. There is an integral isomorphism given by a splitting of the norm form to $\left(\frac{1,1}{\mathbb{Q}_2}\right)$, but this is not enough as 2 is not a 2-adic unit. Now, only the square class of p matters, so we consider 4 cases: $p = 1, 3, 5, 7$.

If $p = 1$, then we already have the algebra $\left(\frac{1,1}{\mathbb{Q}_2}\right)$. Calculation confirms that each element of $\mathbf{M}_2(\mathbb{Z}_2)$ is a linear combination of products of the e_i: we apply the isomorphism and notice that $(i+1)/2$ and $(j+k)/2$, $(j-k)/2$ are all integral combination of e_i. These form a basis for $\mathbf{M}_2(\mathbb{Z}_2)$. In the other cases the reduced norm forms are all integrally equivalent to that of $\left(\frac{1,p}{\mathbb{Q}_2}\right)$ and the generated order is thus preserved. Now, the isomorphism between the norm forms of $\left(\frac{1,p}{\mathbb{Q}_2}\right)$ and $\left(\frac{1,1}{\mathbb{Q}_2}\right)$ is also integral, and in each case the order is generated by a set of integral combinations of the e_i.

We now investigate the maximal order in the quaternion algebra generated by a_1a_2, a_1a_3, a_2a_3. If we take the right generators we get $\left(\frac{1,1}{\mathbb{Q}_2}\right)$ by using a_1a_2 and a_2a_3 as generators. This quaternion algebra is isomorphic to $\mathbf{M}_2(\mathbb{Q}_2)$ and has the maximal order of matrices with integer coefficients. Working through the same calculation we see all of these matrices are linear combinations of products of the e_i with 2-adic integral coefficients. Therefore the elements of U_2 are elements of the order generated by the lattice we started with. As before this demonstrates that U_2 maps into $\mathbf{SO}(\Lambda_2)$ as conjugation by these elements preserves Λ_2. Unlike the odd q case, we do not necessarily have surjectivity from this arguments as the reflections by vectors in Λ with unit norm do not necessarily generate all of $\mathbf{SO}(\Lambda)$ but only a subgroup.

4.5 Conclusion

We know that each U_q has image contained in $\mathbf{SO}(\Lambda_q)$ for Λ_q a localization of the specified lattice Λ. Almost everywhere the image of U_q is equal to $\mathbf{SO}(\Lambda_q)$, therefore we the product of $f(U_q)$ is an open subgroup in $\mathbf{SO}(\hat{V})$. Forms on $S(\Lambda)$ will descend to forms on $f(U_q)$ by level raising, and by the functoriality lemma we get a map to $S(U_q)$. Applying Ibukiyama's conjecture gives us the following result:

Theorem 1. *Let Λ be the maximal lattice in a quadratic space (V, q) with discriminant p and Hasse-Witt invariant -1 at p and 2. Denote the space of cuspidal algebraic modular forms as $S(\Lambda)$. Conditional on the Ibukiyama conjecture there is a Hecke equivariant injection from $S(\Lambda)$ to $S_3(K(p))$.*

Chapter 5

Further results

5.1 Unconditional comparisons

While Ibukiyama's conjecture plays a key role in the above theorem, we can get quite far without it.

Let π' be an cuspidal automorphic representation of $\mathbf{SO}_5(V)$ that we compute. Our $\mathbf{SO}_5$ is an inner form of the split $\mathbf{SO}_5$, so they have the same L-group. Langlands functoriality suggests there exists conjectural transfer of π' to forms π on the split $\mathbf{SO}_5$.

Now the split $\mathbf{SO}_5$ is exceptionally isomorphic to $\mathbf{PGSp}_4$ which implies there is corresponding a π with trivial central character on $\mathbf{GSp}_4(\mathbb{Q})$. These π are characterized by having the same Euler factors as π' at all places [3].

By the classification of the cuspidal spectrum due to Ralf Schmidt [37], we can look at a single Euler factor for a good prime q, and if that Euler factor satisfies Ramanujan bounds we know the π lies in a generic L-packet. Such an L-packet has a classically paramodular representative, and as the conductor exponent is 0 away from p (since the π'_q, and hence π_q are unramified for those q), that representative must have level p^k for some k. Such a representative is uniquely determined by the strong multiplicity one theorem for holomorphic paramodular forms.

We now examine the parameters at the infinite place in more detail. I am grateful to Ralf Schmidt for explaining some of the facts used here to me. The representation associated to the π' we consider is trivial. Since $\mathbf{SO}_5$ is compact the Harish-Chandra parameter of a representation is 1/2 the sum of all the roots plus the Blattner parameter of the representation. For the trivial representation this gives us $(2, 1)$. On $\mathbf{GSp}_4$ for trivial central characters the minimal K-type in region 1 is $(1, 2)$ plus the Blattner parameter [36], and so is $(3, 3)$, exactly the one corresponding to holomorphic Siegel modular forms of weight 3.

Therefore, if we assume that the Langlands functoriality between inner forms of $\mathbf{SO}_5$ and $\mathbf{GSp}_5$ holds, the image of π' is an L-packet containing an automorphic representation π generated by a vector related to a classical paramodular form of level p^k and weight 3.

While a weak lift is known [38, Introduction] this is not enough to conclude anything definite about π. If we look at a finite number of places all the places examined could be ones where the weak lift fails to provide information about π. Therefore any interpretation of the computational results in the next section depends at least partially on conjecture.

We limited the above discussion to generic forms because these are the ones whose L-function is not a product of L-functions of automorphic forms on lower-rank groups such as $\mathbf{GL}_1$ or $\mathbf{GL}_2$ which can be computed easily. All of the interesting L-functions of forms to compute on $\mathbf{GSp}_4$ are the generic ones.

5.2 Lower bounds on dimension

We can compare the dimension of our space to the dimension of Ibukiyama's space of forms via the mass formula. While we don't have an equation for the dimension of our space we can use a lower bound.

Our source for the mass formula is Conway and Sloan [9]. Its conventions differ slightly from common presentations of the mass formula as it uses a scale-invariant local mass instead of a local density. Let our dimension be $D(p)$ and let the dimension of Ibukiyama's space be $C(p)$ We will put a lower bound on $D(p)/C(p)$ as p goes to ∞. Denote our quadratic form for level p by f_p.

Note that each lattice has at least 2 automorphisms and so $2M(f_p)/C(p) < D(p)/C(p)$ for all p. There is a formula for $C(p)$ [24], $C(p) = \sum_{i=0}^{12} H_i(3,0,p) + 1$ where the H_i are large formulas. We have $C(p) = p^2 2^{-6} 3^{-2} 5^{-1} + O(p)$ as H_0 contains the first time and all the other H_i are $O(1)$ or $O(p)$.

The mass formula expresses $M(p)$ as a product of local factors modifying a local standard density. Each local factor is a product of cross terms and type terms. The global standard density for low dimensions is tabulated, and for dimension 5 it is 1/720. The local factor is determined by writing the form as an infinite orthogonal of scaled unimodular forms, then determining the type and cross terms from that expression. At primes not dividing the discriminant the local density is the local standard density.

The p-adic constituents of f_p are a 4 dimensional form with a nonresidue discriminant and a 1 dimensional form which may or may not have a nonresidue discriminant. Luckily in both cases we have $M_p(f_p) = p^4/(2(p^4-1)) \cdot 1/2p^2$. Further $M_p(f_p)/\mathrm{std}_p(f_p) = \frac{p^6(1-p^{-2})(1-p^{-4})}{2(p^4-1)}$.

We have that f_p has the 2-adic constituents a 4 dimensional type II form, a 1 dimensional type I form, and a bound love form in the language of Conway and Sloan. This tells us that $M_2(f_p) = 4/45$, and $M_2(f_p)/\mathrm{std}_2(f_p) = 1/8$. Therefore $M(f_p) = 1/720 \cdot 1/8 \cdot \frac{p^6(1-p^{-2})(1-p^{-4})}{2(p^4-1)}$ as the standard density for dimension 5 is 1/720.

A simple calculation demonstrates that

$$\lim_{p\to\infty} \frac{2M(p)}{C(p)} = \frac{1}{2}$$

and so

$$\lim_{p\to\infty} \frac{D(p)}{C(p)} > \frac{1}{2}$$

Therefore as the level increases we expect to compute at least 1/2 of all the forms in Ibukiyama's space.

In dimension 3 this sort of analysis with the mass formula gives exactly 1/2 using the computation of Birch [2]. In that case we know that we obtain exactly the minus space of Aitkin-Lehner by the results in Hein's dissertation [20], and so the correct asymptotic fraction of forms obtained is 1/2 [12].

Chapter 6

Computational applications

6.1 Greenberg-Voight algorithm and the computation of paramodular forms

The above would not be very interesting if it was not for the fact that we can comparatively easily compute $S(\Lambda)$. The reader interested in much of this can consult the dissertation of Jeffery Hein [20], as well as the article by Greenberg and Voight [15]. Voight and Greenberg described how to efficiently compute the space of modular forms of $\mathbf{SO}(V)$ if the level structure used was that of isometries of a lattice, and Jeffery Hein [20] includes more details of the algorithm.

We briefly review the mathematics involved. For Λ a lattice, Π is a p-neighbor if $|\Lambda/(\Lambda \cap \Pi)| = |\Pi/(\Pi \cap \Lambda)| = p$. The genus of Λ is the transitive closure of the p-neighbor relation for any p that does not divide the discriminant (this is not true in general but is for our class of Λ [15]. The set of p-neighbors of a given lattice can be computed in time $O(p^n)$ where n is the dimension. This definition can be extended to p^k neighbors, where we require $\Lambda/(\Lambda \cap \Pi) \cong \Pi/(\Lambda \cap \Pi) \cong (\mathbb{Z}/p\mathbb{Z})^k$ as abelian groups.

Now, if we number the representatives of the genus $\Lambda_1, \Lambda_2, \dots \Lambda_n$. we can define a matrix T_{p^k} as follows: the ij-th entry of t_{p^k} is the number of p^k neighbors of Λ_i that are isomorphic to Λ_j. These T_{p^k} are matrices that act as the Hecke operators of $\mathbf{SO}(V)$ on the space of algebraic modular forms of trivial weight and level $\mathbf{Aut}(\Lambda)$ for p not dividing the discriminant of Λ.

Therefore to compute a set of commuting matrices representing the various T_{p^k}, we can iterate over each element of the genus, compute the p-neighbors, and determine which elements of the genus they are isomorphic to. This can be done for any lattice and applies by the prior section to the computation of $S(K(p))$.

Jeffery Hein computed the Euler factors for primes less then 100 for all forms with prime levels up to 197. We extend this here to computing the Euler factors up to 31 for all prime levels less than 400, with an implementation written in Sage [34]. These computations were carried out on the Savio cluster at the University of California, Berkeley.

In order to take better advantage of all the computing resources available, we divide the calculations into three parts. First we compute the lattice and the genus by taking the transitive closure of the 3 neighbor relation. Then each of the T_{p^k} we desire to compute can be independently computed. Finally we accumulate all the matrices and find a basis that simultaneously diagonalizes them and compute the eigenvalues for each eigenvector, then transform that into Euler factors. The bulk of the time taken is in the second step, where millions of lattices must be computed and compared for isomorphism to a list of given ones.

The first step is to write down a lattice in a genus given the invariants of the genus. I implemented the method of Dubey and Holenstein [11], although only for positive definite quadratic forms, for doing this. The input to this algorithm is a finite list of primes where the lattice is not locally unimodular along with a matrix at each prime. The output is a matrix whose genus symbol is the list that was input.

In the second step we want to compute a set of representatives for the genus. This can be done by computing the transitive closure of the p-neighbor relation for enough p. In theory we could use the mass formula and compute the automorphism group of each lattice to determine if we are done. However, because the lattices we work with have squarefree discriminant it is enough to take the transitive closure of the 3-neighbor relation [15, Corollary 5.10].

Once the genus representatives are computed, we can then compute each T_{p^k} that is wanted independently. For each lattice in the list of representatives we enumerate all the p^k neighbors as described by Hein [20, Algorithm 5.2.7]. This new lattice is LLL reduced, an operation that preserves the isomorphism class but reduces the size of the entries of the Gram matrix [27]. Then we search in the list of representatives for the representative equivalent to the neighbor, and increment the appropriate entry of the matrix. In practice, we in fact have to go further in exposing parallelism and compute each row of the matrix independently to make best use of the available computing power. Isometry testing is farmed out by Sage out to PARI [29], which in turn implements an algorithm due to Plesken and Souvignier [30]. Once this process is complete we write out the matrix.

At the end of the previous step of the computation we have a set of mutually commuting matrices. We split the $\mathbb{Z}$-module on which they act into irreducible submodule, and then compute the eigenvalues for the single Galois orbit of a common eigenvector in each submodule, as William Stein did in the classical case [39].

Then we compute the Euler factor for the spinor L function at p, or, if only T_p and not T_{p^2} is computed, the truncated Euler factor with only a linear term. Using some of the good primes we determine if the form is a Saito-Kurokawa lift,a Yoshida lift, or neither and hence generic. The Euler factors and classification are the output of the program. Since the Euler factors are defined via the Satake parameters, our choice of generators of the Hecke algebra doesn't change them.

6.2 Results

On each space with prime odd level less than 400 and not 3 we compute the T_p and T_{p^2} for odd $p < 31$. We skip primes equal to the level as well as the theory does not tell us the operator to use. The restriction to odd p only is a result of laziness of the present author: it is considerably more involved to compute the p-neighbors for 2, and this was not implemented in the current code. Jeffery Hein previously computed T_p and T_{p^2} for all $p < 100$ on spaces with level less than 200.

The resulting code may be found at `github.com/wbl/paramod_comp`. All of the generic forms with rational coefficients discovered appear in the appendix of this thesis, and all the data is available on request. For levels up to 400 we find 78 Galois orbits of generic forms, of which 21 are rational. We have found one form that potentially has the same L-function as a hypergeometric motive.

In each case we have been able to compare with prior computations we find agreement with the Euler factors that are predicted. We also examine the found Saito-Kurokawa lifts for low levels and found corresponding classical modular forms that give rise to them. This gives some confidence that our algorithm, implementation, and the underlying conjectures are correct.

The conductor is for our example of a correspondence with geometry is 257. The hypergeometric motive as in Magma [33] has $A = (2,2,2,2)$, $B = (1,1,1,1)$ and $t = -1/256$. All Euler factors for primes less then 31 agree.

In this example David Yuen has been able to compute the Euler factor for 2 independently and it agrees with what we have here. David Roberts assisted with finding this example. Note that we can easily compute a large number of Euler factors on both the automorphic and algebraic sides, and so it would seem plausible that we might be able to apply some variation of Faltings-Serre to prove isomorphism.

Unfortunately the reduction of this Galois representation modulo 2 is probably trivial. While variants of Faltings-Serre due to Grenié [16] could in principle applied to prove isomorphism of the 2-adic Galois representations, the tower of fields that is required is prohibitively painful to construct and the class of extensions that must be examined to determine this triviality is also large.

Appendix A

Tables of rational forms

The following tables include the Euler factors for all rational generic eigenforms I computed. Some of the data was previously published in Hein [20]. Each table contains the primes on the left and the associated Euler factor on the right. The name of the form is p followed by a letter which bears no connection to anything about the space.

p	$L_p(x)$
3	$729 \cdot x^4 + 81 \cdot x^3 + 3 \cdot x^2 + 3 \cdot x + 1$
5	$15625 \cdot x^4 - 375 \cdot x^3 + 85 \cdot x^2 - 3 \cdot x + 1$
7	$117649 \cdot x^4 + 3087 \cdot x^3 + 56 \cdot x^2 + 9 \cdot x + 1$
11	$1771561 \cdot x^4 + 5324 \cdot x^3 + 1738 \cdot x^2 + 4 \cdot x + 1$
13	$4826809 \cdot x^4 + 6591 \cdot x^3 + 1469 \cdot x^2 + 3 \cdot x + 1$
17	$24137569 \cdot x^4 - 181781 \cdot x^3 + 7922 \cdot x^2 - 37 \cdot x + 1$
19	$47045881 \cdot x^4 + 514425 \cdot x^3 + 7486 \cdot x^2 + 75 \cdot x + 1$
23	$148035889 \cdot x^4 - 121670 \cdot x^3 + 13938 \cdot x^2 - 10 \cdot x + 1$
29	$594823321 \cdot x^4 - 5170468 \cdot x^3 + 22504 \cdot x^2 - 212 \cdot x + 1$
31	$887503681 \cdot x^4 + 178746 \cdot x^3 - 34658 \cdot x^2 + 6 \cdot x + 1$

Table A.1: 61A

p	$L_p(x)$
3	$729 \cdot x^4 + 54 \cdot x^3 + 3 \cdot x^2 + 2 \cdot x + 1$
5	$15625 \cdot x^4 + 130 \cdot x^2 + 1$
7	$117649 \cdot x^4 - 2401 \cdot x^3 - 420 \cdot x^2 - 7 \cdot x + 1$
11	$1771561 \cdot x^4 + 87846 \cdot x^3 + 3124 \cdot x^2 + 66 \cdot x + 1$
13	$4826809 \cdot x^4 - 35152 \cdot x^3 - 1482 \cdot x^2 - 16 \cdot x + 1$
17	$24137569 \cdot x^4 - 250563 \cdot x^3 + 3400 \cdot x^2 - 51 \cdot x + 1$
19	$47045881 \cdot x^4 - 109744 \cdot x^3 + 2109 \cdot x^2 - 16 \cdot x + 1$
23	$148035889 \cdot x^4 - 1861551 \cdot x^3 + 22954 \cdot x^2 - 153 \cdot x + 1$
29	$594823321 \cdot x^4 + 439002 \cdot x^3 + 6844 \cdot x^2 + 18 \cdot x + 1$
31	$887503681 \cdot x^4 + 5511335 \cdot x^3 + 23622 \cdot x^2 + 185 \cdot x + 1$

Table A.2: 73A

p	$L_p(x)$
3	$729 \cdot x^4 + 135 \cdot x^3 + 42 \cdot x^2 + 5 \cdot x + 1$
5	$15625 \cdot x^4 - 375 \cdot x^3 + 80 \cdot x^2 - 3 \cdot x + 1$
7	$117649 \cdot x^4 - 5145 \cdot x^3 + 182 \cdot x^2 - 15 \cdot x + 1$
11	$1771561 \cdot x^4 - 34606 \cdot x^3 + 1342 \cdot x^2 - 26 \cdot x + 1$
13	$4826809 \cdot x^4 + 32955 \cdot x^3 + 156 \cdot x^2 + 15 \cdot x + 1$
17	$24137569 \cdot x^4 + 294780 \cdot x^3 + 2278 \cdot x^2 + 60 \cdot x + 1$
19	$47045881 \cdot x^4 - 219488 \cdot x^3 - 6650 \cdot x^2 - 32 \cdot x + 1$
23	$148035889 \cdot x^4 - 608350 \cdot x^3 + 6302 \cdot x^2 - 50 \cdot x + 1$
29	$594823321 \cdot x^4 - 585336 \cdot x^3 - 4234 \cdot x^2 - 24 \cdot x + 1$
31	$887503681 \cdot x^4 - 4230322 \cdot x^3 + 9982 \cdot x^2 - 142 \cdot x + 1$

Table A.3: 79A

p	$L_p(x)$
3	$729 \cdot x^4 + 162 \cdot x^3 + 12 \cdot x^2 + 6 \cdot x + 1$
5	$15625 \cdot x^4 - 2000 \cdot x^3 + 265 \cdot x^2 - 16 \cdot x + 1$
7	$117649 \cdot x^4 + 5831 \cdot x^3 + 252 \cdot x^2 + 17 \cdot x + 1$
11	$1771561 \cdot x^4 + 2662 \cdot x^3 - 671 \cdot x^2 + 2 \cdot x + 1$
13	$4826809 \cdot x^4 + 101062 \cdot x^3 + 1352 \cdot x^2 + 46 \cdot x + 1$
17	$24137569 \cdot x^4 - 324258 \cdot x^3 + 5899 \cdot x^2 - 66 \cdot x + 1$
19	$47045881 \cdot x^4 - 288078 \cdot x^3 + 6498 \cdot x^2 - 42 \cdot x + 1$
23	$148035889 \cdot x^4 - 596183 \cdot x^3 + 1472 \cdot x^2 - 49 \cdot x + 1$
29	$594823321 \cdot x^4 + 1560896 \cdot x^3 + 18328 \cdot x^2 + 64 \cdot x + 1$
31	$887503681 \cdot x^4 - 6494438 \cdot x^3 + 46934 \cdot x^2 - 218 \cdot x + 1$

Table A.4: 89A

p	$L_p(x)$
3	$729 \cdot x^4 + 108 \cdot x^3 + 18 \cdot x^2 + 4 \cdot x + 1$
5	$15625 \cdot x^4 - 1000 \cdot x^3 + 110 \cdot x^2 - 8 \cdot x + 1$
7	$117649 \cdot x^4 - 1372 \cdot x^3 + 322 \cdot x^2 - 4 \cdot x + 1$
11	$1771561 \cdot x^4 + 5324 \cdot x^3 + 154 \cdot x^2 + 4 \cdot x + 1$
13	$4826809 \cdot x^4 + 87880 \cdot x^3 + 2522 \cdot x^2 + 40 \cdot x + 1$
17	$24137569 \cdot x^4 - 334084 \cdot x^3 + 4182 \cdot x^2 - 68 \cdot x + 1$
19	$47045881 \cdot x^4 + 109744 \cdot x^3 - 3382 \cdot x^2 + 16 \cdot x + 1$
23	$148035889 \cdot x^4 + 462346 \cdot x^3 - 4738 \cdot x^2 + 38 \cdot x + 1$
29	$594823321 \cdot x^4 - 7024032 \cdot x^3 + 58174 \cdot x^2 - 288 \cdot x + 1$
31	$887503681 \cdot x^4 + 5719872 \cdot x^3 + 27962 \cdot x^2 + 192 \cdot x + 1$

Table A.5: 113A

p	$L_p(x)$
3	$729 \cdot x^4 - 21 \cdot x^2 + 1$
5	$15625 \cdot x^4 + 250 \cdot x^3 + 210 \cdot x^2 + 2 \cdot x + 1$
7	$117649 \cdot x^4 - 686 \cdot x^3 + 399 \cdot x^2 - 2 \cdot x + 1$
11	$1771561 \cdot x^4 + 18634 \cdot x^3 + 2365 \cdot x^2 + 14 \cdot x + 1$
13	$4826809 \cdot x^4 + 74698 \cdot x^3 + 1560 \cdot x^2 + 34 \cdot x + 1$
17	$24137569 \cdot x^4 + 73695 \cdot x^3 - 6290 \cdot x^2 + 15 \cdot x + 1$
19	$47045881 \cdot x^4 - 109744 \cdot x^3 + 4313 \cdot x^2 - 16 \cdot x + 1$
23	$148035889 \cdot x^4 - 1885885 \cdot x^3 + 13616 \cdot x^2 - 155 \cdot x + 1$
29	$594823321 \cdot x^4 - 975560 \cdot x^3 - 25665 \cdot x^2 - 40 \cdot x + 1$
31	$887503681 \cdot x^4 + 4528232 \cdot x^3 - 5766 \cdot x^2 + 152 \cdot x + 1$

Table A.6: 167A

p	$L_p(x)$
3	$729 \cdot x^4 + 27 \cdot x^3 - 39 \cdot x^2 + x + 1$
5	$15625 \cdot x^4 + 85 \cdot x^2 + 1$
7	$117649 \cdot x^4 + 5488 \cdot x^3 + 623 \cdot x^2 + 16 \cdot x + 1$
11	$1771561 \cdot x^4 + 31944 \cdot x^3 + 1474 \cdot x^2 + 24 \cdot x + 1$
13	$4826809 \cdot x^4 - 4394 \cdot x^3 + 2756 \cdot x^2 - 2 \cdot x + 1$
17	$24137569 \cdot x^4 - 284954 \cdot x^3 + 7123 \cdot x^2 - 58 \cdot x + 1$
19	$47045881 \cdot x^4 + 20577 \cdot x^3 - 10925 \cdot x^2 + 3 \cdot x + 1$
23	$148035889 \cdot x^4 - 559682 \cdot x^3 + 13087 \cdot x^2 - 46 \cdot x + 1$
29	$594823321 \cdot x^4 + 2682790 \cdot x^3 + 44515 \cdot x^2 + 110 \cdot x + 1$
31	$887503681 \cdot x^4 + 6285901 \cdot x^3 + 32147 \cdot x^2 + 211 \cdot x + 1$

Table A.7: 173A

p	$L_p(x)$
3	$729 \cdot x^4 + 216 \cdot x^3 + 48 \cdot x^2 + 8 \cdot x + 1$
5	$15625 \cdot x^4 - 1125 \cdot x^3 + 10 \cdot x^2 - 9 \cdot x + 1$
7	$117649 \cdot x^4 - 7889 \cdot x^3 + 385 \cdot x^2 - 23 \cdot x + 1$
11	$1771561 \cdot x^4 + 15972 \cdot x^3 + 1672 \cdot x^2 + 12 \cdot x + 1$
13	$4826809 \cdot x^4 + 83486 \cdot x^3 + 2548 \cdot x^2 + 38 \cdot x + 1$
17	$24137569 \cdot x^4 - 456909 \cdot x^3 + 5644 \cdot x^2 - 93 \cdot x + 1$
19	$47045881 \cdot x^4 - 102885 \cdot x^3 + 3211 \cdot x^2 - 15 \cdot x + 1$
23	$148035889 \cdot x^4 + 2348231 \cdot x^3 + 20631 \cdot x^2 + 193 \cdot x + 1$
29	$594823321 \cdot x^4 - 2560845 \cdot x^3 + 4901 \cdot x^2 - 105 \cdot x + 1$
31	$887503681 \cdot x^4 - 59582 \cdot x^3 - 16554 \cdot x^2 - 2 \cdot x + 1$

Table A.8: 197A

p	$L_p(x)$
3	$729 \cdot x^4 - 27 \cdot x^3 - 21 \cdot x^2 - x + 1$
5	$15625 \cdot x^4 + 1000 \cdot x^3 + 155 \cdot x^2 + 8 \cdot x + 1$
7	$117649 \cdot x^4 + 2058 \cdot x^3 + 273 \cdot x^2 + 6 \cdot x + 1$
11	$1771561 \cdot x^4 + 39930 \cdot x^3 + 2387 \cdot x^2 + 30 \cdot x + 1$
13	$4826809 \cdot x^4 - 79092 \cdot x^3 + 3263 \cdot x^2 - 36 \cdot x + 1$
17	$24137569 \cdot x^4 + 49130 \cdot x^3 - 4828 \cdot x^2 + 10 \cdot x + 1$
19	$47045881 \cdot x^4 - 912247 \cdot x^3 + 13718 \cdot x^2 - 133 \cdot x + 1$
23	$148035889 \cdot x^4 - 730020 \cdot x^3 + 9706 \cdot x^2 - 60 \cdot x + 1$
29	$594823321 \cdot x^4 + 3633961 \cdot x^3 + 32277 \cdot x^2 + 149 \cdot x + 1$
31	$887503681 \cdot x^4 - 506447 \cdot x^3 - 13516 \cdot x^2 - 17 \cdot x + 1$

Table A.9: 223A

p	$L_p(x)$
3	$729 \cdot x^4 + 135 \cdot x^3 + 18 \cdot x^2 + 5 \cdot x + 1$
5	$15625 \cdot x^4 - 2125 \cdot x^3 + 280 \cdot x^2 - 17 \cdot x + 1$
7	$117649 \cdot x^4 + 13720 \cdot x^3 + 994 \cdot x^2 + 40 \cdot x + 1$
11	$1771561 \cdot x^4 - 75867 \cdot x^3 + 2002 \cdot x^2 - 57 \cdot x + 1$
13	$4826809 \cdot x^4 - 21970 \cdot x^3 - 1430 \cdot x^2 - 10 \cdot x + 1$
17	$24137569 \cdot x^4 - 221085 \cdot x^3 + 2788 \cdot x^2 - 45 \cdot x + 1$
19	$47045881 \cdot x^4 - 294937 \cdot x^3 - 2622 \cdot x^2 - 43 \cdot x + 1$
23	$148035889 \cdot x^4 - 365010 \cdot x^3 + 10534 \cdot x^2 - 30 \cdot x + 1$
29	$594823321 \cdot x^4 + 926782 \cdot x^3 - 13630 \cdot x^2 + 38 \cdot x + 1$
31	$887503681 \cdot x^4 + 8401062 \cdot x^3 + 55242 \cdot x^2 + 282 \cdot x + 1$

Table A.10: 229A

p	$L_p(x)$
3	$729 \cdot x^4 + 27 \cdot x^3 - 6 \cdot x^2 + x + 1$
5	$15625 \cdot x^4 + 1125 \cdot x^3 + 40 \cdot x^2 + 9 \cdot x + 1$
7	$117649 \cdot x^4 + 686 \cdot x^3 - 98 \cdot x^2 + 2 \cdot x + 1$
11	$1771561 \cdot x^4 + 17303 \cdot x^3 - 1694 \cdot x^2 + 13 \cdot x + 1$
13	$4826809 \cdot x^4 - 52728 \cdot x^3 - 234 \cdot x^2 - 24 \cdot x + 1$
17	$24137569 \cdot x^4 - 230911 \cdot x^3 + 8908 \cdot x^2 - 47 \cdot x + 1$
19	$47045881 \cdot x^4 - 720195 \cdot x^3 + 8930 \cdot x^2 - 105 \cdot x + 1$
23	$148035889 \cdot x^4 + 24334 \cdot x^3 - 6210 \cdot x^2 + 2 \cdot x + 1$
29	$594823321 \cdot x^4 - 536558 \cdot x^3 + 27434 \cdot x^2 - 22 \cdot x + 1$
31	$887503681 \cdot x^4 + 4885724 \cdot x^3 + 14942 \cdot x^2 + 164 \cdot x + 1$

Table A.11: 229B

p	$L_p(x)$
3	$729 \cdot x^4 + 54 \cdot x^3 - 12 \cdot x^2 + 2 \cdot x + 1$
5	$15625 \cdot x^4 - 1000 \cdot x^3 + 100 \cdot x^2 - 8 \cdot x + 1$
7	$117649 \cdot x^4 + 2058 \cdot x^3 + 546 \cdot x^2 + 6 \cdot x + 1$
11	$1771561 \cdot x^4 + 50578 \cdot x^3 + 1540 \cdot x^2 + 38 \cdot x + 1$
13	$4826809 \cdot x^4 - 70304 \cdot x^3 + 741 \cdot x^2 - 32 \cdot x + 1$
17	$24137569 \cdot x^4 - 230911 \cdot x^3 + 2822 \cdot x^2 - 47 \cdot x + 1$
19	$47045881 \cdot x^4 - 68590 \cdot x^3 + 5225 \cdot x^2 - 10 \cdot x + 1$
23	$148035889 \cdot x^4 + 1581710 \cdot x^3 + 20102 \cdot x^2 + 130 \cdot x + 1$
29	$594823321 \cdot x^4 - 731670 \cdot x^3 + 14587 \cdot x^2 - 30 \cdot x + 1$
31	$887503681 \cdot x^4 + 893730 \cdot x^3 + 23343 \cdot x^2 + 30 \cdot x + 1$

Table A.12: 233A

p	$L_p(x)$
3	$729 \cdot x^4 + 54 \cdot x^3 + 39 \cdot x^2 + 2 \cdot x + 1$
5	$15625 \cdot x^4 - 1125 \cdot x^3 + 215 \cdot x^2 - 9 \cdot x + 1$
7	$117649 \cdot x^4 + 6860 \cdot x^3 + 119 \cdot x^2 + 20 \cdot x + 1$
11	$1771561 \cdot x^4 - 51909 \cdot x^3 + 671 \cdot x^2 - 39 \cdot x + 1$
13	$4826809 \cdot x^4 - 39546 \cdot x^3 + 1287 \cdot x^2 - 18 \cdot x + 1$
17	$24137569 \cdot x^4 + 39304 \cdot x^3 - 3604 \cdot x^2 + 8 \cdot x + 1$
19	$47045881 \cdot x^4 - 1344364 \cdot x^3 + 20026 \cdot x^2 - 196 \cdot x + 1$
23	$148035889 \cdot x^4 - 413678 \cdot x^3 - 1196 \cdot x^2 - 34 \cdot x + 1$
29	$594823321 \cdot x^4 - 2195010 \cdot x^3 + 26361 \cdot x^2 - 90 \cdot x + 1$
31	$887503681 \cdot x^4 + 6375274 \cdot x^3 + 48453 \cdot x^2 + 214 \cdot x + 1$

Table A.13: 251A

p	$L_p(x)$
3	$729 \cdot x^4 + 6 \cdot x^2 + 1$
5	$15625 \cdot x^4 + 500 \cdot x^3 - 130 \cdot x^2 + 4 \cdot x + 1$
7	$117649 \cdot x^4 + 2744 \cdot x^3 + 238 \cdot x^2 + 8 \cdot x + 1$
11	$1771561 \cdot x^4 - 31944 \cdot x^3 + 1078 \cdot x^2 - 24 \cdot x + 1$
13	$4826809 \cdot x^4 - 26364 \cdot x^3 + 3406 \cdot x^2 - 12 \cdot x + 1$
17	$24137569 \cdot x^4 - 176868 \cdot x^3 - 3162 \cdot x^2 - 36 \cdot x + 1$
19	$47045881 \cdot x^4 + 1254 \cdot x^2 + 1$
23	$148035889 \cdot x^4 - 194672 \cdot x^3 + 11086 \cdot x^2 - 16 \cdot x + 1$
29	$594823321 \cdot x^4 - 1073116 \cdot x^3 + 28014 \cdot x^2 - 44 \cdot x + 1$
31	$887503681 \cdot x^4 + 7149840 \cdot x^3 + 59582 \cdot x^2 + 240 \cdot x + 1$

Table A.14: 257A

p	$L_p(x)$
3	$729 \cdot x^4 - 54 \cdot x^3 + 48 \cdot x^2 - 2 \cdot x + 1$
5	$15625 \cdot x^4 + 125 \cdot x^3 + 80 \cdot x^2 + x + 1$
7	$117649 \cdot x^4 - 2744 \cdot x^3 + 658 \cdot x^2 - 8 \cdot x + 1$
11	$1771561 \cdot x^4 - 27951 \cdot x^3 - 286 \cdot x^2 - 21 \cdot x + 1$
13	$4826809 \cdot x^4 - 65910 \cdot x^3 + 442 \cdot x^2 - 30 \cdot x + 1$
17	$24137569 \cdot x^4 - 108086 \cdot x^3 - 6018 \cdot x^2 - 22 \cdot x + 1$
19	$47045881 \cdot x^4 - 137180 \cdot x^3 + 1102 \cdot x^2 - 20 \cdot x + 1$
23	$148035889 \cdot x^4 - 717853 \cdot x^3 + 7406 \cdot x^2 - 59 \cdot x + 1$
29	$594823321 \cdot x^4 + 48778 \cdot x^3 - 10092 \cdot x^2 + 2 \cdot x + 1$
31	$887503681 \cdot x^4 + 3515338 \cdot x^3 + 37510 \cdot x^2 + 118 \cdot x + 1$

Table A.15: 269A

p	$L_p(x)$
3	$729 \cdot x^4 + 54 \cdot x^3 + 30 \cdot x^2 + 2 \cdot x + 1$
5	$15625 \cdot x^4 - 750 \cdot x^3 + 10 \cdot x^2 - 6 \cdot x + 1$
7	$117649 \cdot x^4 + 2401 \cdot x^3 + 70 \cdot x^2 + 7 \cdot x + 1$
11	$1771561 \cdot x^4 + 14641 \cdot x^3 - 1078 \cdot x^2 + 11 \cdot x + 1$
13	$4826809 \cdot x^4 - 72501 \cdot x^3 + 2236 \cdot x^2 - 33 \cdot x + 1$
17	$24137569 \cdot x^4 + 39304 \cdot x^3 + 782 \cdot x^2 + 8 \cdot x + 1$
19	$47045881 \cdot x^4 - 823080 \cdot x^3 + 16758 \cdot x^2 - 120 \cdot x + 1$
23	$148035889 \cdot x^4 - 1301869 \cdot x^3 + 13478 \cdot x^2 - 107 \cdot x + 1$
29	$594823321 \cdot x^4 + 3390071 \cdot x^3 + 15428 \cdot x^2 + 139 \cdot x + 1$
31	$887503681 \cdot x^4 + 2025788 \cdot x^3 + 42222 \cdot x^2 + 68 \cdot x + 1$

Table A.16: 283A

p	$L_p(x)$
3	$729 \cdot x^4 - 54 \cdot x^3 - 15 \cdot x^2 - 2 \cdot x + 1$
5	$15625 \cdot x^4 - 500 \cdot x^3 + 65 \cdot x^2 - 4 \cdot x + 1$
7	$117649 \cdot x^4 + 7203 \cdot x^3 + 539 \cdot x^2 + 21 \cdot x + 1$
11	$1771561 \cdot x^4 - 6655 \cdot x^3 + 1419 \cdot x^2 - 5 \cdot x + 1$
13	$4826809 \cdot x^4 - 101062 \cdot x^3 + 3510 \cdot x^2 - 46 \cdot x + 1$
17	$24137569 \cdot x^4 + 619038 \cdot x^3 + 9707 \cdot x^2 + 126 \cdot x + 1$
19	$47045881 \cdot x^4 - 226347 \cdot x^3 + 6536 \cdot x^2 - 33 \cdot x + 1$
23	$148035889 \cdot x^4 - 705686 \cdot x^3 + 16744 \cdot x^2 - 58 \cdot x + 1$
29	$594823321 \cdot x^4 + 1438951 \cdot x^3 - 20938 \cdot x^2 + 59 \cdot x + 1$
31	$887503681 \cdot x^4 + 3306801 \cdot x^3 + 34689 \cdot x^2 + 111 \cdot x + 1$

Table A.17: 317A

p	$L_p(x)$
3	$729 \cdot x^4 - 162 \cdot x^3 + 42 \cdot x^2 - 6 \cdot x + 1$
5	$15625 \cdot x^4 + 1625 \cdot x^3 + 160 \cdot x^2 + 13 \cdot x + 1$
7	$117649 \cdot x^4 - 1372 \cdot x^3 + 154 \cdot x^2 - 4 \cdot x + 1$
11	$1771561 \cdot x^4 + 7986 \cdot x^3 + 946 \cdot x^2 + 6 \cdot x + 1$
13	$4826809 \cdot x^4 - 219700 \cdot x^3 + 6110 \cdot x^2 - 100 \cdot x + 1$
17	$24137569 \cdot x^4 + 299693 \cdot x^3 + 4964 \cdot x^2 + 61 \cdot x + 1$
19	$47045881 \cdot x^4 - 34295 \cdot x^3 + 1178 \cdot x^2 - 5 \cdot x + 1$
23	$148035889 \cdot x^4 - 876024 \cdot x^3 + 16974 \cdot x^2 - 72 \cdot x + 1$
29	$594823321 \cdot x^4 - 829226 \cdot x^3 + 2378 \cdot x^2 - 34 \cdot x + 1$
31	$887503681 \cdot x^4 - 566029 \cdot x^3 + 24366 \cdot x^2 - 19 \cdot x + 1$

Table A.18: 331A

p	$L_p(x)$
3	$729 \cdot x^4 - 189 \cdot x^3 + 33 \cdot x^2 - 7 \cdot x + 1$
5	$15625 \cdot x^4 + 1625 \cdot x^3 + 195 \cdot x^2 + 13 \cdot x + 1$
7	$117649 \cdot x^4 - 5488 \cdot x^3 + 448 \cdot x^2 - 16 \cdot x + 1$
11	$1771561 \cdot x^4 - 14641 \cdot x^3 + 1562 \cdot x^2 - 11 \cdot x + 1$
13	$4826809 \cdot x^4 + 70304 \cdot x^3 + 1274 \cdot x^2 + 32 \cdot x + 1$
17	$24137569 \cdot x^4 + 112999 \cdot x^3 + 1751 \cdot x^2 + 23 \cdot x + 1$
19	$47045881 \cdot x^4 + 438976 \cdot x^3 + 3078 \cdot x^2 + 64 \cdot x + 1$
23	$148035889 \cdot x^4 + 547515 \cdot x^3 + 3358 \cdot x^2 + 45 \cdot x + 1$
29	$594823321 \cdot x^4 - 6219195 \cdot x^3 + 61132 \cdot x^2 - 255 \cdot x + 1$
31	$887503681 \cdot x^4 - 7239213 \cdot x^3 + 41571 \cdot x^2 - 243 \cdot x + 1$

Table A.19: 349A

p	$L_p(x)$
3	$729 \cdot x^4 - 54 \cdot x^3 + 30 \cdot x^2 - 2 \cdot x + 1$
5	$15625 \cdot x^4 + 250 \cdot x^3 - 70 \cdot x^2 + 2 \cdot x + 1$
7	$117649 \cdot x^4 - 210 \cdot x^2 + 1$
11	$1771561 \cdot x^4 - 10648 \cdot x^3 - 1034 \cdot x^2 - 8 \cdot x + 1$
13	$4826809 \cdot x^4 - 4394 \cdot x^3 + 3250 \cdot x^2 - 2 \cdot x + 1$
17	$24137569 \cdot x^4 - 235824 \cdot x^3 + 510 \cdot x^2 - 48 \cdot x + 1$
19	$47045881 \cdot x^4 - 164616 \cdot x^3 + 10982 \cdot x^2 - 24 \cdot x + 1$
23	$148035889 \cdot x^4 - 486680 \cdot x^3 + 21390 \cdot x^2 - 40 \cdot x + 1$
29	$594823321 \cdot x^4 - 3316904 \cdot x^3 + 26390 \cdot x^2 - 136 \cdot x + 1$
31	$887503681 \cdot x^4 + 3694084 \cdot x^3 + 31806 \cdot x^2 + 124 \cdot x + 1$

Table A.20: 353A

p	$L_p(x)$
3	$729 \cdot x^4 + 54 \cdot x^3 + 30 \cdot x^2 + 2 \cdot x + 1$
5	$15625 \cdot x^4 - 875 \cdot x^3 + 40 \cdot x^2 - 7 \cdot x + 1$
7	$117649 \cdot x^4 + 1029 \cdot x^3 + 406 \cdot x^2 + 3 \cdot x + 1$
11	$1771561 \cdot x^4 + 18634 \cdot x^3 + 1342 \cdot x^2 + 14 \cdot x + 1$
13	$4826809 \cdot x^4 - 96668 \cdot x^3 + 1742 \cdot x^2 - 44 \cdot x + 1$
17	$24137569 \cdot x^4 - 73695 \cdot x^3 + 8908 \cdot x^2 - 15 \cdot x + 1$
19	$47045881 \cdot x^4 - 212629 \cdot x^3 + 4218 \cdot x^2 - 31 \cdot x + 1$
23	$148035889 \cdot x^4 - 2482068 \cdot x^3 + 27646 \cdot x^2 - 204 \cdot x + 1$
29	$594823321 \cdot x^4 + 2585234 \cdot x^3 + 37642 \cdot x^2 + 106 \cdot x + 1$
31	$887503681 \cdot x^4 + 1578923 \cdot x^3 - 12834 \cdot x^2 + 53 \cdot x + 1$

Table A.21: 359A

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