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Author
Thierry-Mieg, J.

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J. Thierry-Mieg

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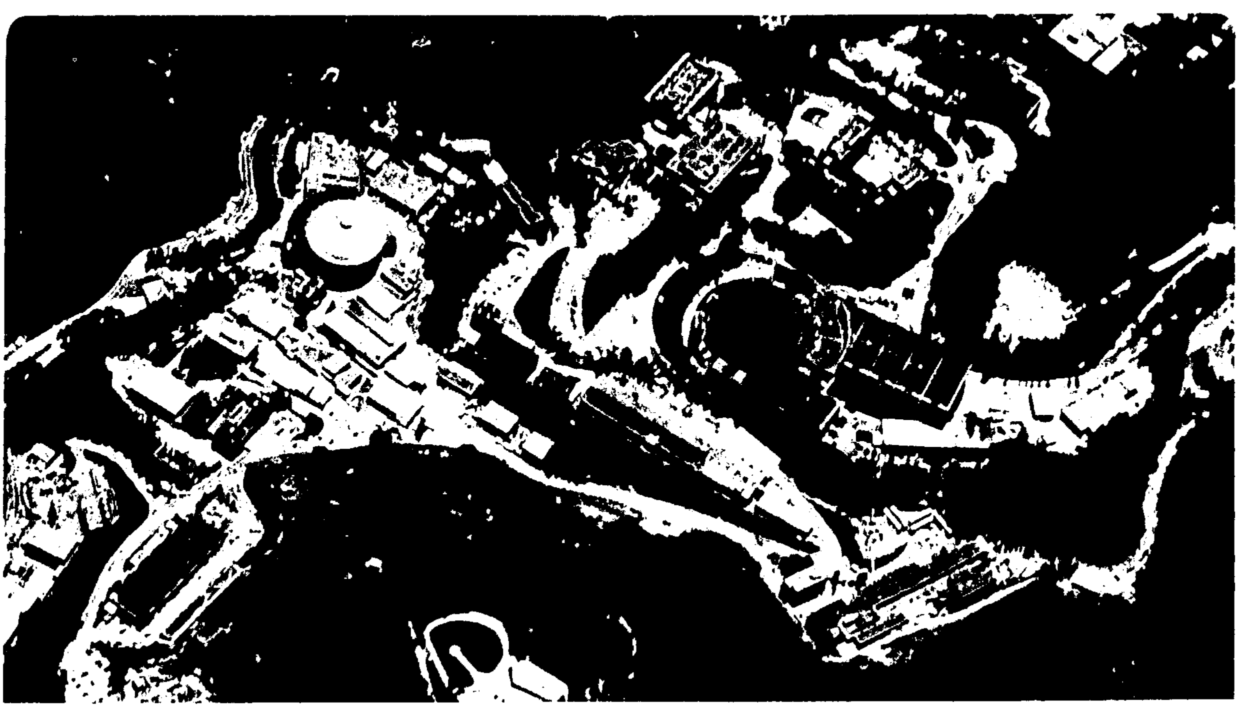

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REMARKS CONCERNING THE $E_8 \times E_8$ AND D_{16} STRING THEORIES[1]

by

Jean Thierry-Mieg[2]

CNRS - Observatoire de Meudon, France

and

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720, U.S.A.

Abstract

Completing the results of Green and Schwarz, we prove that the $E_8 \times E_8$ and D_{16} supersymmetric theories are both anomaly-free in 10 dimensions and discuss their relation with the even unimodular lattices.

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[2] Participating Guest at Lawrence Berkeley Laboratory.

INTRODUCTION

Green and Schwarz [1] have recently established that in dimension 10, N = 1 supergravity coupled to a Yang-Mills supermultiplet is anomaly free if the gauge group is D_{16} = SO(32). It was remarked by the present author that $E_8 \times E_8$ is a second solution. This later solution was overlooked by Green and Schwarz in their original analysis because it cannot be readily extended from supergravity to superstring. Indeed, a string may carry à la Chan Paton only an orthogonal or symplectic group [2].

The aim of this letter is to present the two solutions in a unified notation, to analyse the relation between the group D_{16} and $E_8 \times E_8$ and to propose a reformulation of string theory.

Our analysis will bring together four seldom related subjects:

a) In section 1 we discuss the cohomology of Lie groups in order to exhibit the 2 solutions.

b) In section 2, we define the root lattices and their generalization, the even unimodular lattices, which have been used over the last decades in arithmetic [3] and coding theory [4]. They share with strings the critical dimensions 2, 10, 18, 26. They single out the same groups $E_8 \times E_8$ and D_{16} [3], and we note that they are constructed over the super-orthogonal groups and their Majorana Weyl spinors.

c) In section 3, we recall the occurence in supergravity of a peculiar phenomenon nicknamed Cartan crystallization: when the dimension in which the theory is expressed is varied, the sum of the dimension of the space plus the rank of the internal symmetry group is preserved [5,6].

d) In Section 4, we describe the Frenkel Kac strings [7] . These dual models allow for a fusion of the three concepts that we have just described.

- they may carry an arbitrary Lie group.
- they generalize to even unimodular lattices [8].
- they naturally undergo Cartan crystallization.. We therefore suggest that a Frenkel Kac string based on the 26 dimensional even unimodular lattice $E_{25/1}$ generalizes the Venezziano model and yields, through Cartan crystallization, the 10 dimensional string with symmetry group $E_8 \times E_8$ or D_{16}.

We shall not discuss the supersymmetry of the string model but we stress again that the classification of the even unimodular lattices involves supersymmetry.

1. THE ANOMALY CONDITION

Green and Schwarz [1] have studied the anomalies of D = 10, N = 1 supergravity coupled to a Yang Mills supermultiplet. They embellish the usual model with an elegant generalization of the Chapline Manton [9] field strength for the 2-form B:

$$H = dB - \omega_3 - \lambda\psi_3 \qquad (1.1)$$

ω_3 amd ψ_3 denote the Chern Simons 3-forms of the Lorentz and Yang-Mills gauge groups:

$$d\omega_3 = Tr(RR)$$
$$d\psi_3 = Tr(FF) \qquad (1.2)$$

Using the results of Alvarez Gaumé and Witten[10], they have selected the acceptable gauge groups in 2 steps:

a) The intrinsic $Tr(R^6)$ gravitational anomaly is absent if and only if the dimension of the group is 496 ([10], eq.119 p.327)

b) The pure gauge and the mixed anomalies can be removed if and only if $\lambda = 1/30$ and:

$$Tr(F^6) = 1/48\, Tr(F^2)Tr(F^4) - 1/14400\, Tr(F^2)^3 \qquad (1.3)$$

Miraculously, D_{16} meets these requirements. Let us show that $E_8 \times E_8$, which has dimension 496, is the unique alternative solution.

Let $H^i(G)$ denote the i^{th} cohomology space of a real Lie group G. The Betti numbers $b_i = \dim (H^i)$ are given by the Poincaré polynomial [11]:

$$P(G, E) = \sum_{i=0}^{\dim G} b_i t^i = \prod_{j=1}^{\text{rank } G} (1 + t^{s_j}) \quad (1.4)$$

The s_j are the degree of the Chern Simons forms ω_{s_j} which generate the exterior algebra $H^*(G)$. To each of them corresponds a generator P_{m_j} of degree $m_j = 1/2\ (s_j + 1)$ of the Weyl invariant polynomials over the adjoint representation:

$$P_{m_j}(F) = d\, \omega_{s_j} (A, F) \quad (1.5)$$

Over a Lie group, A is the Cartan left invariant form, F vanishes and $\omega(A, 0)$ is closed. Over a principal fiber bundle, by the Maurer Cartan BRS condition, $A = A + c$ is the sum of the Yang Mills field and Faddeev Popov ghost form [12] $F = F$ is horizontal and if $2m_j$ exceeds the dimension of the base manifold the Chern Simons form $\omega(A + c, F)$ is closed and generates all solutions [13] of the Wess-Zumino equation [14], therefore, all possible anomalies [15].

In the case of E_8 [11], the s_j read: (3, 15, 23, 27, 35, 39, 47, 59) and one way verify that they add up to 248. Therefore in E_8 the first 2 Weyl generators are Tr F^2 and Tr F^8 and we have:

$$\begin{aligned} \text{Tr } F^4 &= \delta\, (\text{Tr } F^2)^2 \\ \text{Tr } F^6 &= \varepsilon\, (\text{Tr } F^2)^3 \end{aligned} \quad (1.6)$$

To evaluate the coefficients δ and ε it is sufficient to choose a single generator in the algebra. Consider the 2 groups $G_1 = SO(32)$ and $G_2 = E_8 \times E_8$ and their regular subalgebra $H_1 = SU(2) \times SU(2) \times SO(28)$ and $H_2 = SU(2) \times E_7 \times E_8$ respectively.

With respect to SU(2), the adjoint representation splits in both cases as a triplet, 56 doublets and 381 singlets. The corresponding diagonal Cartan generator of SU(2) thus reads, up to a normalization:

$$h = \text{diag}(2, 0, -2, (1, -1)^{56}, 0^{381}) \quad (1.7)$$

where the superscript denote repetitions [4]. Substituting in equation (1.6), we read that in $E_8 \times E_8$ $\delta = 1/100$ and $\varepsilon = 1/7200$ and the second condition (1.3) of Green and Schwarz is satisfied.

There is no other solution. Any other group where Tr(F^6) factorizes is a product of [11] A_n $n < 5$, G_2, F_4, E_6, E_7 and in every of these components there are too few SU(2) doublets to fulfill condition (1.3).

2. Even unimodular lattices

In this section, we shall recover D_{16} and $E_8 \times E_8$ as the common solution of a purely geometric and a priori unrelated problem.

In Minkowski space $R^{p,q}$, consider $p + q$ linearly independent vectors α_i. The set of all points with integral contravariant coordinates $\Gamma = \{x, x = m^i\alpha_i, m^i \in Z\}$ is called a lattice. The dual [16] or reciprocal [17] lattice is the set of all points with integral covariant coordinates $\Gamma^* = \{k, k_i = k\cdot\alpha_i, k_i \in Z\}$. If $\Gamma \subset \Gamma^*$, the lattice is called integral. Furthermore, if the square length of all the base vectors α_i is even, then all square distances are even and the lattice is called even. A root lattice,

$\Gamma(G)$, is associated to every Lie algebra G. It is generated by a system of simple roots and its geometry is specified by the Dynkin diagram of the Lie group, reading the number of links between pair of nodes modulo 2. Therefore it is sufficient to consider the algebras of type A, D, E. The others are redundant, for example:

$$\Gamma(G_2) = \Gamma(A_2), \quad \Gamma(B_2) = \Gamma(D_2) \qquad (2.1)$$

If the common square length of the roots is scaled to 2, the dual lattice $\Gamma(G)^*$ carries all finite dimensional representations of G and includes $\Gamma(G)$.

This construction can be applied to a basic classical superalgebra [18] with one modification. The Cartan space of a superalgebra is not Euclidean but Lorentzian and the nilpotent Fermi roots are on the light cone.

The even unimodular, or self dual, lattices are particularly interesting: they generalize the E Lie algebras, they are highly symmetric, very densely packed [4] and related to the sporadic finite groups [19]. Their occurence in the present problem results from their relation with the Majorana Weyl condition and superalgebras:

a) Even unimodular lattices [16] and Majorana Weyl fermions [20] exist in the same dimensions 8m + n, n. We are grateful to Yuval Ne'eman for this observation.

b) The lattice $E_{8m+n/n}$, generated by the root lattice of the superalgebra $D_{8m+n/n}$ = OSp (16m + 2n/2n) augmented by its Majorana (self dual) Weyl (chiral) superspinor representation [21] is even unimodular.

In fact, all even unimodular lattices in Minkowski space are of this super-E type [3].

i) Euclidean space

In dimensions 8, the unique even unimodular lattice is $\Gamma_8 = \Gamma(E_8)$. Also, E_8 is the unique Lie algebra such that the fundamental representation is the adjoint. In dimension 16, there are 2 even unimodular lattices: $\Gamma_{16} = \Gamma(E_{16})$ and $\Gamma_8 \times \Gamma_8$. These two lattices have a remarkable property: although they are not isomorphic, they have the same number of nodes at any distance from the origin (same θ function)[3] and we expect the equivalence of the S matrix of their respective string models. This is the promised relation between $E_8 \times E_8$ and D_{16} which are precisely the maximal root diagrams of $\Gamma_8 \times \Gamma_8$ and Γ_{16}. Remark, however, that Γ_{16} has twice as many points as $\Gamma(D_{16})$.

In dimension 24 occurs an intriguing lattice discovered by Leech in 1966 [22]. Each mode has 196560 nearest neighbors, all sitting at square distance 4, but there are 23 inequivalent types of empty deep holes of square radius 2. Each of these holes corresponds to one of the other 23 even unimodular Niemeier lattices [16, 23] built around a particular Lie algebra of rank 24, for examples D_{24}, $D_{16}\,E_8$, $E_8{}^3$, $A_2{}^{11}$. Remarkedly, the Leech and the 23 Niemeier lattices have only 2 linearly independent θ functions[3].

In dimension 32, there are over 80 million even unimodular lattices [3] and the situation seems out of control!

ii) Minkowski space

The situation is Minkowski space is simpler. The lattice $E_{8n+1/1}$, denoted $II_{8n+1,1}$ by Conway and Sloane [24], is the unique even unimodular lattice [3]. This lattice can be explicitly constructed as follows. In an orthonormal frame, the "super" sum of the coordinates of its points (space minus time) is even and all coordinates, are either integral, generating the D sublattice, or half integral, corresponding to the superspin representation [21].

The richness of the Euclidean case is recovered by projection. If W is a null vector, then $W^{\perp}/W$ is Euclidean even unimodular. By choosing W carefully, it is possible to project $E_{25/1}$ onto any of the 23 Niemeier, lattices or the Leech [24]. It is interesting to note that in the case of the root lattice of a Lie superalgebra $W^{\perp}/W$ carries the weights of all W-atypical representations of finite superdimension.

iii) Superlattices.

The Bose Fermi graduation of the $D_{m/n}$ roots is given by the parity of the sum of the time coordinates. This graduation extends to the spin representation only if n is even. We are grateful to Neil Marcus for discussing this point. In such a case, one may speak of a superlattice. The first example is $E_{10/2}$ whose existence was conjectured by Julia in connection with D = 11 supergravity [25].

4. Cartan crystallization

Let us now address a completely different question: dimensional reduction in supergravity. Morel and I [5] and Julia [6], have observed that the Bose and Fermi degrees of freedom of maximal (N = 8) supergravity can be inbedded in the adjoint representation of $E_{8(8)}$ in every dimension, from 3 to 11, in which the theory can be written. These nine realizations, including chiral and nonchiral D = 10, correspond to the 9 splittings of $E_{8(8)}$ associated with the removal of a root of the extended Dynkin diagram. The regular subalgebra exhibited in this way is of the form $H_D = S\ell(D-2) \times K_{D(D)}$, or SO(8,8) in the particular nonchiral D = 10 case, and one interpret $S\ell(D-2)/SO(D-2)$ as the graviton and $K_{D(D)}$ quotiented by its maximal compact subgroup as the scalar sector. The remaining tensors and vectors span the non compact generators of $E_{8(8)}/H_D$ and the Fermions are obtained by conjugation. Replacing $E_{8(8)}$ by $D_{8(8)}$ yields "N = 4" supergravity [26].

This observation, difficult to interpret in field theory and not explained by the Kaluza-Klein formalism, shows that in supergravity the dimension of space can be transmuted into the rank of the internal symmetry group. This gives a strong significance to the fact that the rank of the only anomaly free symmetry groups of the superstring 16, added to its critical dimension 10, yields the critical dimension 26 of the Veneziano model.

We called this phenomenon Cartan crystallization [5] because the easiest visualization is to consider a discrete approximation of space time where the distances between points are given by the dynamical metric field and fluctuate. This lattice is a liquid. If d directions crystalize into a root lattice, one recovers a lower dimensional gravity with internal symmetry, a sort of discrete principal fiber bundle. In the reciprocal space, d continuous directions of space are transfered into the maximal torus of the internal group [25] . We shall see how to implement this construction in the next section.

5. Frenkel Kac string

In their work on representations of affine Lie algebras, Frenkel and Kac [7] have constructed Vertex operators, analogous to those of the Veneziano model [27], which satisfy the duality condition. The particularity of their scheme is that the impulsion carried by the vertex is quantized, and takes value on the root lattice of the Lie algebra. In string language, the transverse excitations occur in a Cartan space and the Weyl group plays the role of the discretized transverse Lorentz group.

The construction holds for an arbitrary Lie group, i.e. $E_8 \times E_8$, and has been extended by Goddard and Olive [8] to even unimodular lattices. Moreover, starting from a lattice or rank d + n and dequantizing d dimensions, one obtains a string in d-space carrying in a nontrivial way a rank n internal symmetry group. The spectrum of the model is very different from Chan Paton. Here, the Regge trajectories relate the square mass to the sum of the squared spin and isospin and arbitrarily large representations of the internal symmetry group occur among massive states.

5. Conclusion

Consider a lattice approximation $E_{25/1}$ to the Minkowski space time in dimension 25 + 1. The space transverse to a light cone lattice vector is an even unimodular lattice (Section 2). Splitting this lattice into two sectors $\Lambda_d \times \Lambda_{26-d}$, one obtains a string in d-dimension with internal Frenkal Kac symmetry of rank 26-d (Section 3-4). The splitting is consistent only if it chooses 2 regular sublattices. If we require that Λ_d it self be unimodular, this singlularizes a naked string in d = 26, a d = 18 model with E_8 internal symmetry, two d = 10 model, with $E_8 \times E_8$ or D_{16} symmetry and 24 models in d = 2 corresponding to the Leech or any Niemeier lattice. We expect that the known string theories can all be reconstructed in this way, including a new d = 18 model. The relation between the Veneziano model and the Leech lattice had already been evoked [8, 25] but the concept of Cartan crystallization and the observation that the only acceptable symmetry group of the d = 10 superstring correspond to the rank 16 even unimodular lattices bridges for the first time the dimensional gap separating the different dual models [27] and one may hope to explain dynamically the emergence of supersymmetry.

In dimension 4, the model should be quite different, but by looking for a rank 2 root sublattice of a Neimeier lattice [16] we expect that the Veneziano model, properly reduced to dimension 4, should exhibit as internal symmetry group a real form of the group $SU(3)^{11}$, eleven copies of SU(3)!

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TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720