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F. N. Spiess

In using the data available during the conduct of a bearings only approach two rather different procedures are usually followed. One is to attempt the complete solution for range, course and speed by taking usually six bearings while maneuvering the submarine in a rather specialized way. The other methods involve the use usually of three bearings and an assumed range or speed to calculate the course and the speed or range (whichever was not assumed). The most elaborate example of the use of this method is the analyzer in the Mark 101 fire control system. The purpose of this paper is to show that using four bearings it is first actually possible to solve the problem completely (range, course and speed solution) in a very general way, and second to give in detail several simple, direct methods for making such a solution without restricting the motion of the attacking submarine.

Once the general method of attack on the situation has been understood, it becomes obvious that the problem of obtaining satisfactory solutions in the case of a bearings only approach has been reduced to simple terms, and uncertainties and restrictions presently accompanying this type of attack have been completely eliminated. Graphical methods capable of giving quick reliable solutions are available now, and it may be that in the near future simple computers can be devised to give accurate solutions rapidly enough so that the final bearing used can be inserted almost at the firing point to give both a target speed and range check.

The actual mathematical proof that in general four bearings will determine the range, course and speed of the target is given at the end of this paper in Appendix 1. The proof is made analytically by writing the four equations of the known bearing lines and of a fifth straight line (the target course) whose position and slope are initially unknown, and along which the target moves with uniform speed. The coordinates of the points of intersection of this fifth line with each of the other four are determined and the requirement is made that the distances between successive points be proportional to the time between successive observations. In this way enough conditions are obtained to allow the unique determination of the two components of target velocity and the range. Some consideration of these equations shows, perhaps, why this method has not been recognized previously. That is that in some special cases there is no one unique solution. One of these is the very simple one in which the submarine maintains exactly constant course and speed or remains at one point throughout the entire series of observations. It is proved in Appendix 1 that a sufficient condition to avoid this degenerate situation is to have the component of the submarine's motion perpendicular to the first bearing be not equivalent to uniform motion in that direction. Any other arbitrary motion does not affect the uniqueness of the solution of the problem. Consideration of several examples will lead one to realize that the accuracy will increase with increasing difference between the observed bearing at one part of this sequence and that at another part. This will lead one to favor approach tactics which produce this result. These may, however, fit in with the situation rather appropriately, for it is likely that one might, for instance, favor stopping to obtain two bearings, then running hard in a direction to close the track and target, and then slowing to take two more bearings—this procedure would usually lead to good results in the application of the method of solution proposed here.

Three completely graphical methods of constructing the solution have already been devised and undoubtedly others exist. After studying these methods and the equations in the appendices it will become obvious that the principles set forth here make it possible to construct an electrical or mechanical computer which could do the complete job. The two simplest graphical methods are described below. The third graphical method, which is a little more elegant, but also somewhat more cumbersome than the other two, is not included here. A paper giving this third method and some forms of the equations more suitable for computer design has been prepared, and is available on request.

The first method is the least accurate, but is the simplest and clearly illustrates the fundamental principles involved. The necessary equipment is the standard submarine DRT and attached drafting machine, and the set of rulers giving the distance run in successive time intervals at some particular speed. Plot in the four bearings from the actual successive positions of the submarines. Select rulers for two different speeds, preferably bracketing the target's estimated speed, although this is not necessary. Lay the first ruler across the bearings in such a way that the first time interval is intercepted by the first two bearings and the second time interval between second and third bearings. Make a mark at the time corresponding to the fourth bearing--this will be the position the target would have had if making the speed corresponding to the ruler used. With the second speed scale carry out the same procedure. Connect the two points so located--the target will be on this line at the time of the fourth bearing, thus the intersection of the fourth bearing with this line is the actual location of the target. It is proved in Appendix 1 that the extrapolated line of position is actually a straight line, thus justifying the use of only two points in determining its location. If by this time a fifth bearing is available, repeat the procedure using the second third and fourth bearings to find the extrapolated line of position at the time of the fifth bearing. The intersection of the fifth bearing with the extrapolated line of position is the location of the target at the time of the fifth observation. If a fifth bearing is not yet available invert the original procedure, using the second, third and fourth bearings to find the extrapolated line of position at the time of the first observation, and its intersection with the first bearing. In either event two positions of the target have been determined, with a known time between them, and the problem has been completely solved without any assumptions.

Again in the above paragraph it is apparent that the method only fails to give a unique answer if the fourth bearing coincides with the line of position developed from the other three bearings as discussed above and in Appendix 1.

The second method goes through roughly the same steps, but uses an accurate geometric construction (see Appendix 2) to locate the extrapolated bearing line. Although this construction can be made by using a slide rule and a pair of dividers in addition to the drafting machine and DRT, it is faster and simpler to use a transparent plate which can be laid over the DRT plot. This plate should have ruled on it a set of closely spaced parallel lines, and perpendicular to these lines another line on which at least two different scales (for time intervals) are marked off, all with their zeros coinciding with one line of the set of parallel lines. To illustrate the procedure used here let us assume the four bearings were taken at 0, 2, 5 and 7 minutes and have been plotted on the DRT. Lay the plate over the plot with 0 line along the first bearing. Using a grease pencil mark on the plate the point of intersection of the 2 minute line (using scale 1) with bearing number 2 and the 5 minute line with bearing number 3. Extend the line through these two points to intersect the 7 minute line on scale 1 of the plate. Repeat using scale 2. Find the intersection of the line determined by the two 7 minute points (one by scale 1 and one by scale 2) with the fourth bearing. Mark this point directly on the chart--the target was there at 7 minutes. Turn the plate around to put the zero line on bearing 4 and repeat using bearings 2, 3, and 4 to find the line which intersects bearing number 1. This intersection point was occupied by the target at 0 time. Two points actually occupied by the target at an interval of 7 minutes have been found, and the problem is solved. At the moment this method appears to give the best balance between speed and accuracy.

APPENDIX 1

Formulation Of The Equations And Development Of Several Relationships Of Interest

The purpose of this appendix is to develop the necessary equations which allow us to prove that there is a unique answer to this problem, and to put them in several different forms which allow convenient solution, or illustrate particular points.

As a beginning we call the four points from which the bearings were actually taken P_1, P_2, P_3 and P_4 and their intersections with the course line P_1^1, P_2^1, P_3^1, and P_4^1 respectively. We will say the bearings were taken at times t_1, t_2, t_3 and t_4. If we represent these in a rectangular coordinate system we can place the coordinate system with its origin at P_1 and its y axis lying along the first bearing line. The slopes of the other bearing lines are then m_2, m_3, and m_4. The components of the target velocity expressed in this coordinate system will be v_x and v_y. At this point we have ten unknowns: x_1^1, y_1^1, x_2^1, y_2^1, x_3^1, y_3^1, x_4^1, y_4^1, v_x and v_y. We have the equations of the four bearing lines:

first bearing, l_1, $x = 0$

second bearing, l_2, $y - y_2 = m_2 (x - x_2)$

third bearing, l_3, $y - y_3 = m_2 (x - x_3)$

fourth bearing, l_4, $y - y_4 = m_4 (x - x_4)$.

And the equations of motion of the target in parametric form, taking $t_1 = 0$:

$$y - y_1^1 = v_y t \text{ and } x = v_x t.$$

If we write these in terms of their points of intersection we find exactly ten linear equations relating our ten unknown:

$$x_1^1 = 0 \tag{1}$$

$$y_2^1 - y_2 = m_2 \; (x_2^1 - x_2) \tag{2}$$

$$y_3^1 - y_3 = m_3 \; (x_3^1 - x_3) \tag{3}$$

$$y_4^1 - y_4 = m_4 \; (x_4^1 - x_4) \tag{4}$$

$$(y_2^1 - y_1^1) = v_y \; t_2 \tag{5}$$

$$(y_3^1 - y_1^1) = v_y \; t_3 \tag{6}$$

$$(y_4^1 - y_1^1) = v_y \; t_4 \tag{7}$$

$$x_2^1 = v_x \; t_2 \tag{8}$$

$$x_3^1 = v_x \; t_3 \tag{9}$$

$$x_4^1 = v_x \; t_y \tag{10}$$

These can be put into various forms for easy solution. The first which will be used corresponds with the graphical methods developed in the text by setting up the extrapolated line of position and finding its intersection with the fourth bearing. This is done by solving (10) for v_x and (7) for v_y. Then substituting these values into (8) and (9), and (5) and (6) respectively. Then substituting the resulting expressions for x_2^1, y_2^1, x_3^1, and y_3^1 into (2) and (3). The resulting equations are

$$y_4^1\ (t_2/t_4) + y_1^1\left[1 - (t_2/t_4)\right] - y_2 = m_2\left[x_4^1\ (t_2/t_4) - x_2\right] \qquad (11)$$

$$y_4^1\ (t_3/t_4) + y_1^1\left[1 - (t_3/t_4)\right] - y_3 = m_2\left[x_4^1\ (t_3/t_4) - x_3\right] \qquad (12)$$

The resulting single equation after elimination of y_1^1 between (11) and (12) is:

$$y_4^1\ (t_3 - t_2) - (y_3 - m_3x_3)\ (t_4 - t_2) + (y_2 - m_2x_2)\ (t_4 - t_3)$$

$$= x_4^1\left[m_3\ (t_3/t_4)\ (t_4 - t_2) - m_2\ (t_4/t_4)\ (t_4 - t_3)\right]\ . \qquad (13)$$

This above equation (13) is a linear one relating x_4^1 and y_4^1 obtained without use of Eq. (4) and is the actual equation of the extrapolated line of position, based on the time of the fourth bearing and the information contained in the other three bearings. An important point here is that the linearity of this equation shows that the extrapolated line of position is actually a straight line, and not some more complicated curve. This point is important since it shows clearly that one is justified in locating this line by the determination of two points on it, as was done in the text above. The final step of simultaneous solution of Eqs. (4) and (13) is equivalent to the graphical determination of the intersection of the extrapolated line of position and the fourth bearing.

We will not bother to carry out the last step of this solution, which will result in general in the unique determination of all the quantities of interest. Further use can be made of these two, however, to study the case in which a unique solution is not obtainable. This is the case in which the two equations represent lines which coincide. That case can only occur when rather special relationships exist among the points of observation and the slopes of the bearing lines:

$$(t_3 - t_2)\,(y_4 - m_4x_4) = (y_3 - m_3x_3)\,(t_4 - t_2) = (y_2 - m_2x_2)\,(t_4 - t_3) \quad (14)$$

$$\text{with } m_4t_4(t_3 - t_2) = m_3t_3(t_4 - t_2) - m_2t_2(t_4 - t_3). \quad (15)$$

These can be solved to eliminate m_4, giving

$$y_2(t_4 - t_3) + y_3(t_2 - t_4) + y_4(t_3 - t_2) + m_3\left[x_3 - x_4(t_3/t_4)\right](t_4 - t_2)$$

$$- m_2\left[x_2 - x_4(t_2/t_4)\right](t_4 - t_3) = 0. \quad (15a)$$

This condition may be satisfied through some fortuitous relationship between the bearing slopes m_2 and m_3, over which the attacker does not have complete control, or by some relationship among the points of observation, which are at the attacker's disposal. If m_2 and m_3 are completely arbitrary the only relationship which will cause occurrence of this degeneracy is the simultaneous satisfaction of:

$$y_2 (t_4 - t_3) + y_3 (t_2 - t_4) + y_4 (t_3 - t_2) = 0 \qquad (16a)$$

$$x_3 = x_4 (t_3/t_4) \qquad (16b)$$

$$x_2 = x_4 (t_2/t_4). \qquad (16c)$$

Equations (16b) and (16c) say that the motion perpendicular to the first bearing is equivalent to uniform motion. Thus if the component of the submarine's motion along a line perpendicular to the first bearing line is not equivalent to uniform motion then this special case cannot occur without the chance cooperation of the target. It can be seen that all conditions (16) will be satisfied if the attacking submarine is stopped (for then $x_2 = x_3 = x_4 = 0$ and $y_2 = y_3 = y_4 = 0$) for the entire sequence, or if it is moving in a way equivalent to uniform motion

$$(y_2/t_2) = (y_3/t_3) = (y_4/t_4) \text{ and } (x_2/t_2) = (x_3/t_3) = (x_4/t_4) .$$

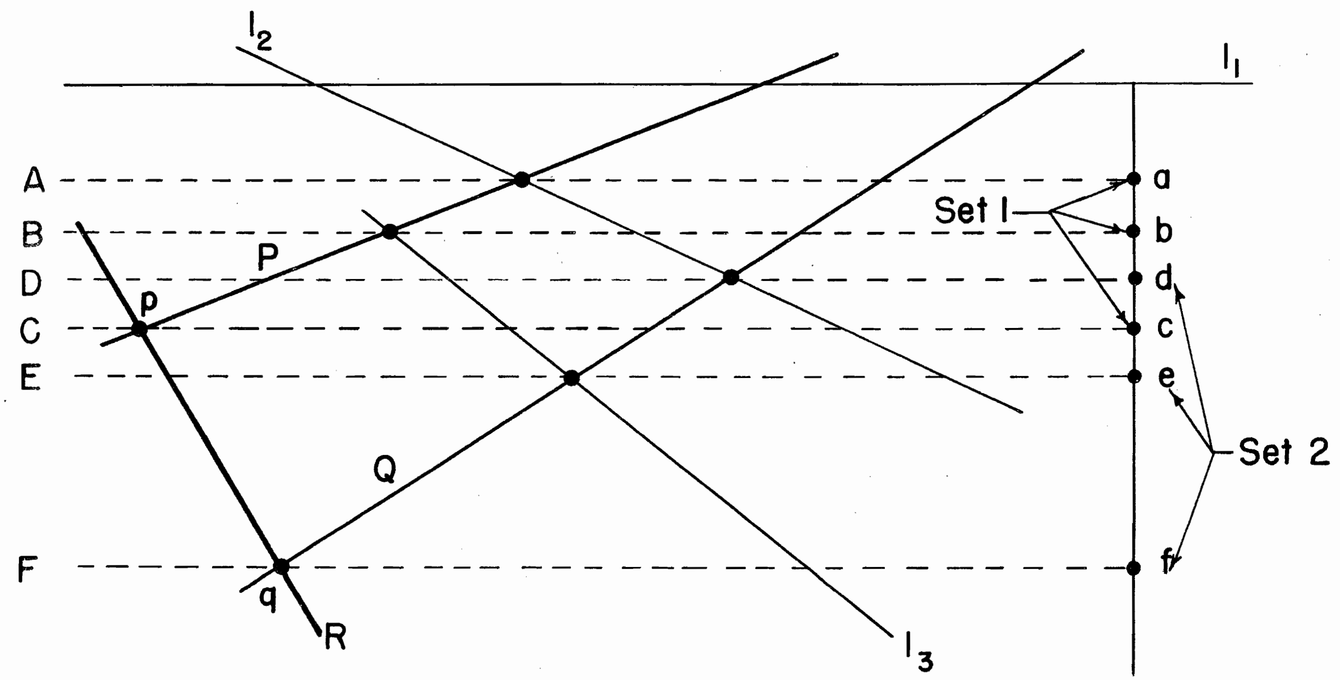
APPENDIX 2

Geometric Construction of the Extrapolated Line of Position

Given three bearings, observed at times t_1, t_2, and t_3, to find the line of position of the target at some future time t_4.

Construction (See Figure)

On a line perpendicular to l_1 lay off two sets of three points each. Space the points such that the distance of each from l_1 is proportional to $t_2 - t_1$, $t_3 - t_1$, and $t_4 - t_1$. Use a different constant of proportionality for each of the two sets. Call these points a, b, c, and d, e, f. Through each of these draw a line parallel to l_1. Call these lines A, B, C and D, E, F respectively. Draw a line through the intersection of A and l_2 and the intersection B and l_3. Extend the line to intersect C at p. Draw a line through the intersection of D and l_2 and the intersection of E and l_3. Extend this line to intersect F at q. Call the first of these lines P and second Q. Call R the line through p and q. Since P is divided by the parallels l_1, A, B, and C into intervals of the proper length, it is also so divided by l_1, l_2, l_3 and R, and similarly Q is divided appropriately by l_1, l_2, l_3 and R. Thus since it is known that the extrapolated line of position is a straight line (Appendix 1) then R is the desired line. The intersection of R with the fourth bearing line will then occur at the position occupied by the target at the time t_4.



l_2
l_1
A
B
D
C
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