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March 31, 1965

Fermion Decay Into Spin-3/2 Fermion Plus Spin-0 Boson

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ABSTRACT

The decay of a fermion of arbitrary spin into an unstable spin-3/2 fermion plus a spinless boson is treated with density-matrix techniques. The formalism described is an extension of that developed by Byers and Fenster for the decay of a fermion into spin-1/2 and spin-0 particles. Decay distributions are completely described for three successive decay processes. Various tests for spin and parity of the parent fermion are suggested.

I. INTRODUCTION

A formalism for treating strong or weak decay of a fermion into a spin-1/2 particle and a spinless boson was developed over a year ago by Byers and Fenster.[1] The purpose of this article is to extend the formalism to treat fermion decay into a spin-3/2 fermion plus a spinless boson. The angular distribution of the decay and also the angular dependence of the fermion's polarization components afford tests for spin and parity hypotheses concerning the parent fermion. In general, the analysis of second-rank tensor polarization of the spin-3/2 fermion is possible; in addition, vector and third-rank tensor polarizations may be analyzable.

II. THE DECAY MATRIX

The decay process

$$X \rightarrow Z + B \tag{1}$$
$$(\text{spin: } J \rightarrow 3/2 + 0)$$

may be described (in the rest frame of X) by expressing the spin-space density matrix of Z in terms of that for X:

$$\rho_Z = \mathcal{M}\,\rho_X \mathcal{M}^\dagger\,, \tag{2}$$

where $\mathcal{M}$ is the decay matrix.[2] We suppose ρ_X to be given in the usual J, M representation, with some convenient direction defined by X production (e. g., the production normal) as the quantization axis. Further, we wish to treat the decay of X into Z in the system which yields helicity states for the Z. Thus the decay matrix may be considered as having two parts: a rotation matrix which transforms ρ_X into the "helicity system" for Z (with quantization axis along $\hat{Z}$, the direction of "particle" Z in the X rest frame); and a diagonalized transition matrix (A) describing the decay $X \to Z + B$. That is, $\rho_Z = A(R\rho_X R^\dagger)A^\dagger$, where R represents a rotation operation.

The complete element of the decay matrix may be written

$$\mathcal{M}_{\lambda M} = A_\lambda \, [(2J+1)/4\pi]^{1/2} \, \mathcal{D}^{J*}_{M\lambda} \, (\phi, \theta, 0), \tag{3}$$

with $\lambda = -3/2, -1/2, 1/2$, and $3/2$ for the case under discussion here. The $\mathcal{D}^{J}_{M\lambda}$ is a matrix element for a rotation operator (and may also be referred to as a "symmetrical-top function").[3,4] The A_λ are the helicity amplitudes, the elements of the diagonalized transition matrix describing $X \to Z + B$. Their form depends on the spin of Z, the spin of X, and the relative X-Z parity.

The helicity amplitudes are obtained as follows. Each A_λ represents the probability amplitude for the breakup of a system of total spin J (with projection λ on the helicity axis) into a system which has spin 3/2 (and helicity component λ) and any allowed orbital angular momentum (with helicity component zero). The A_λ may have contributions from four orbital angular-momentum waves; $\ell = J - 3/2$ through $J + 3/2$; two of these ℓ waves have even parity and two have odd parity. The relative contributions from the different orbital states may be expressed in terms of the complex decay amplitude a_ℓ and the Clebsch-Gordan coefficient for combining ℓ and spin 3/2 to obtain spin J:[5]

$$\begin{aligned} A_\lambda &= \sum_\ell (-)^{\ell - J + 3/2} a_\ell [(2\ell+1)/(2J+1)]^{1/2} \, C(\ell\, 3/2\, J; 0, \lambda) \\ &= (-)^{\lambda - 3/2} \sum_\ell a_\ell \, C(J\, 3/2\, \ell; \lambda, -\lambda) \, . \end{aligned} \tag{4}$$

The second expression for A_λ given in Eq. (4) follows from the first by the use of symmetry properties of the Clebsch-Gordan coefficients; the form of the second expression is reasonable, in that a_ℓ multiplies the coefficient giving the probability amplitude for forming the angular momentum state ℓ. (For the familiar case of decay into spin 1/2, the helicity amplitudes $A_{1/2} = a + b$ and $A_{-1/2} = a - b$ may be found by evaluating $A_\lambda = (-)^{\lambda-1/2} \sum_\ell a_\ell C(J\,1/2\,\ell; \lambda, -\lambda)$ or by diagonalizing the transition matrix $a + b\,\vec{\sigma}\cdot\hat{p}$. See Appendix I for further discussion.)

It is perhaps more practical to discuss strong rather than weak decay of the X, and to keep the opposite-parity amplitudes separate. The A_λ amplitudes receive contributions from orbital angular momenta $\ell = J-3/2$ (not allowed for $J = 1/2$) and $J+1/2$ if the X has spin and parity, relative to the Z, of $1/2^+$, $3/2^-$, $5/2^+$, etc.; they have contributions from $\ell = J-1/2$ and $J+3/2$ if the X has $J^P = 1/2^-, 3/2^+, 5/2^-$, etc. The two sets of helicity amplitudes have the following forms:

$$A \propto \begin{bmatrix} a\alpha_J + c\sqrt{3}\,\beta_J \\ a\sqrt{3}\,\beta_J - c\alpha_J \\ a\sqrt{3}\,\beta_J - c\alpha_J \\ a\alpha_J + c\sqrt{3}\,\beta_J \end{bmatrix} \quad \text{and } A' \propto \begin{bmatrix} b\sqrt{3}\,\alpha_J + d\beta_J \\ b\beta_J - d\sqrt{3}\,\alpha_J \\ -b\beta_J + d\sqrt{3}\,\alpha_J \\ -b\sqrt{3}\,\alpha_J - d\beta_J \end{bmatrix} \tag{5}$$

for the $1/2^+$ and the $1/2^-$ parity sequences, respectively. All four A_λ elements are actually applicable only to the decay of an X with $J \geq 3/2$, since the $\lambda = 3/2$ and $-3/2$ spin states are not accessible to an initial particle with spin 1/2. The coefficients a, b, c, and d in these matrices represent the complex amplitudes a_ℓ for decay through the channels with $\ell = J-3/2, -1/2, +1/2$, and $+3/2$, respectively; the remaining symbols are dependent on the original X particle's spin, with $\alpha_J = \sqrt{J+3/2}$ and with $\beta_J = \sqrt{J-1/2}$. These A_λ amplitudes are subject to the constraint of normalization (total decay probability being equal to 1):

$$\mathrm{Tr}\, A A^\dagger \ \text{ or } \ \mathrm{Tr}\, A' A'^\dagger = 1. \tag{6}$$

Equation (6) is consistent with the usual condition $\sum_\ell a_\ell^2 = 1$.

III. FINAL DENSITY MATRIX

Expressions for the angular distribution and for the polarization components of the Z "particle" are obtained by expressing ρ_X in matrix form and carrying out the transformations of Eqs. (2) and (3). We take

$$\rho_X = (2J_X+1)^{-1} \sum_{L=0}^{2J_X} \sum_M (2L+1)\, t_{LM}^* T_{LM} \tag{7}$$

in the manner of Byers and Fenster.[6] It is convenient to use the irreducible tensors T_{LM} as basis operators in spin space.[3,7] These are traceless tensors, except that $T_{00} = I$; their use simplifies the satisfaction of normalization and hermiticity requirements for the density matrix. Further, they combine naturally with the orthogonal $Y_{LM}(\theta,\phi)$ and $\mathcal{D}^L_{MM'}(\phi,\theta,0)$ in decay distributions derived from the density matrix. These tensors have forms in spin space which correspond to those of the Y_{LM} in coordinate space; e.g., $Y_{11} \propto (x+iy)/r$ and $T_{11} \propto (S_x+iS_y)/|S|$, where the S_x and S_y are spin operators and $|S|$ is $\sqrt{J(J+1)}$. The T_{LM} tensors obey the symmetry relation $T_{L,-M} = (-)^M T^\dagger_{LM}$.

In Eq. (7), the T_{LM} operators have a maximum rank (L) equal to $2J_X$. The t_{LM} represent the expectation values $\langle T_{LM}\rangle$ which describe the X initial spin state. The expression for an element of the final Z density matrix becomes, by the use of Eqs. (2) and (3),

$$\rho_Z\Big]_{\lambda\lambda'} = \sum_{M'',M'} A_\lambda \mathcal{D}^{J*}_{M''\lambda} \left[\sum_{L,M} (2L+1)\, t^*_{LM} T_{LM} \right]_{M''M'} \mathcal{D}^J_{M'\lambda'} A^*_{\lambda'}/4\pi. \tag{8}$$

Then, as shown by Byers and Fenster, Clebsch-Gordan coefficients may be substituted for the matrix elements of the T_{LM}:

$$T_{LM}\Big]_{M''M'} = C(J\,L\,J;\, M'M) \text{ with } M'' = M' + M \tag{9}$$

for the representation where T_{L0} is diagonal. This substitution yields

$$\rho_Z\Big]_{\lambda\lambda'} = (A_\lambda A^*_{\lambda'}/4\pi) \sum_{M'',M'} \sum_{L,M} (2L+1)\, t^*_{LM}\, C(J\,L\,J; M'M)\, \mathcal{D}^{J^*}_{M''\lambda}\, \mathcal{D}^{J}_{M'\lambda'} \,. \tag{10}$$

With the use of various properties of the $\mathcal{D}$ functions and of Clebsch-Gordan coefficients, this reduces to[6, 8]

$$\begin{aligned}\rho_Z\Big]_{\lambda\lambda'} = {} & (-)^{J-\lambda'}\, (A_\lambda A^*_{\lambda'}/4\pi) \sqrt{2J+1} \\ & \times \sum_{L,M} \Big[\sqrt{2L+1}\; C(J\,J\,L; \lambda, -\lambda') \\ & \times t^*_{LM}\, \mathcal{D}^{L^*}_{M,\lambda-\lambda'}(\phi, \theta, 0)\Big] .\end{aligned} \tag{11}$$

Evidently the derivation of Eq. (11) is a general one, which is valid for any spin of X or of Z (integer as well as half-integer).

The elements of the ρ_Z density matrix may be used to derive theoretical expressions for decay distributions after simplification of terms. It is convenient to define the symbol[9]

$$n^{(2\lambda)}_{L\mathfrak{m}} \equiv (-)^{J+\mathfrak{m}-\lambda} [(2J+1)/4\pi]^{1/2}\, C(J\,J\,L; \lambda, \mathfrak{m}-\lambda), \tag{12}$$

where λ assumes the usual values from +3/2 to -3/2 and where $\mathfrak{m}$ has a value of 0, 1, 2, or 3 ($\mathfrak{m}$ being $\lambda-\lambda'$). The diagonal elements of the density matrix then may be expressed as

$$\begin{aligned}\rho_{\lambda\lambda} &= |A_\lambda|^2 \sum_{L,M} n^{(2\lambda)}_{L0}\, t^*_{LM}\, Y_{LM}(\theta, \phi) \\ &= |A_\lambda|^2 \sum_{L,M} n^{(2\lambda)}_{L0}\, t_{LM}\, Y^*_{LM}(\theta, \phi)\end{aligned} \tag{13}$$

with $Y_{LM}(\theta, \phi)$ replacing $\sqrt{(2L+1)/4\pi}\; \mathcal{D}^L_{M0}(\phi, \theta, 0)^*$. The three elements just above the diagonal of ρ_Z are similar, but contain $\mathcal{D}^{L^*}_{M1}$; the two elements above these contain $\mathcal{D}^{L^*}_{M2}$; etc. The density matrix thus has the following form [with $D^L_{M\mathfrak{m}}$ replacing the orthonormal function $\sqrt{(2L+1)/4\pi}\, \mathcal{D}^L_{M\mathfrak{m}}(\phi, \theta, 0)$] :

$$\rho_Z = \sum_{L,M} \begin{bmatrix} t^*_{LM}\left|A_{\frac{3}{2}}\right|^2 n^{(3)}_{L0} Y_{LM} & t^*_{LM} A_{\frac{3}{2}} A^*_{\frac{1}{2}} n^{(3)}_{L1} D^{L*}_{M1} & t^*_{LM} A_{\frac{3}{2}} A^*_{-\frac{1}{2}} n^{(3)}_{L2} D^{L*}_{M2} & t^*_{LM} A_{\frac{3}{2}} A^*_{-\frac{3}{2}} n^{(3)}_{L3} D^{L*}_{M3} \\ \text{———} & t^*_{LM}\left|A_{\frac{1}{2}}\right|^2 n^{(1)}_{L0} Y_{LM} & t^*_{LM} A_{\frac{1}{2}} A^*_{-\frac{1}{2}} n^{(1)}_{L1} D^{L*}_{M1} & t^*_{LM} A_{\frac{1}{2}} A^*_{-\frac{3}{2}} n^{(1)}_{L2} D^{L*}_{M2} \\ \text{———} & \text{———} & t^*_{LM}\left|A_{-\frac{1}{2}}\right|^2 n^{(-1)}_{L0} Y_{LM} & t^*_{LM} A_{-\frac{1}{2}} A^*_{-\frac{3}{2}} n^{(-1)}_{L1} D^{L*}_{M1} \\ t_{LM} A_{-\frac{3}{2}} A^*_{\frac{3}{2}} n^{(3)}_{L3} D^{L}_{M3} & \text{———} & \text{———} & t^*_{LM}\left|A_{-\frac{3}{2}}\right|^2 n^{(-3)}_{L0} Y_{LM} \end{bmatrix} . \quad (14)$$

As the density matrix is self-adjoint, all terms of ρ_Z below the diagonal are easily obtained from the terms above.

The $n_{L\mathcal{M}}^{(2\lambda)}$ coefficients may be related by the use of symmetry properties and recursion relations for the Clebsch-Gordan coefficients; e. g.,

$$n_{L0}^{(-2\lambda)} = (-)^L \, n_{L0}^{(2\lambda)} \text{ for any } L, \lambda \tag{15}$$

and $$n_{L0}^{(3)} = \left\{1 - L(L+1)[(J+3/2)(J-1/2)]^{-1}\right\} n_{L0}^{(1)} \text{ for even } L.$$

See Appendix II for other $n_{L\mathcal{M}}^{(2\lambda)}$ expressions.

IV. DECAY DISTRIBUTIONS

The angular distribution and all possible polarization distributions for the Z may be found by taking the expectation values of all spin operators required to describe the Z spin state: T_{00} (the identity), T_{10}, T_{20}, T_{21}, T_{22}, T_{30}, T_{31}, and T_{32}. These are the same tensor operators as those described above; but here they have a dimensionality of 4 (are represented by 4 by 4 matrices) rather than $2J_X+1$, as above. The theoretical expressions for the expectation values are derived (in terms of the t_{LM} describing the original X spin state) by taking the trace of $\rho_Z T_{LM}$ (and normalizing through division by $\mathrm{Tr}\,\rho_Z$).[10]

The angular distribution of the Z in the X rest frame is found, with the use of Eq. (15), by evaluating[11, 12]

$$\mathrm{Tr}(\rho_Z T_{00}) = \mathrm{Tr}\,\rho_Z \equiv I(\theta,\phi)$$

$$= \sum_{L_e}^{2J-1} \sum_M [(A_{3/2}^2 + A_{-3/2}^2)\, n_{L0}^{(3)} + (A_{1/2}^2 + A_{-1/2}^2)\, n_{L0}^{(1)}]\, t_{LM} Y_{LM}^*(\theta,\phi). \tag{16}$$

The index L_e takes on only even values because the combination of

λ and $-\lambda$ elements of ρ_Z causes odd-L contributions to cancel (for strong decay). The appropriate forms of the A_λ amplitudes and of the n_{L0} coefficients may be substituted to predict the angular distribution for any spin and parity of the initial X system. As an example, the distribution for $J^P = 3/2^-$ is [if the production normal is the polar axis, Fig.(1)]

$$I(\theta,\phi) = 4(a^2+c^2)\,n_{00}t_{00}Y_{00} - 8\,\mathrm{Re}\,a^* c\, n^{(1)}_{20}[t_{20}Y_{20} + 2\,\mathrm{Re}(t_{22}Y^*_{22})] \;; \qquad (17)$$

whereas that for $J^P = 3/2^+$ is

$$I(\theta,\phi) = 20(b^2+d^2)\,n_{00}t_{00}Y_{00} - 8(2b^2-2d^2+3\,\mathrm{Re}\,b^* d)\,n^{(1)}_{20}[t_{20}Y_{20} + 2\,\mathrm{Re}(t_{22}Y^*_{22})]\,. \qquad (18)$$

Because of the normalization requirement of Eq. (6), the first terms (or average cross sections) are identical in these two cases. (Each is equal to $n_{00}t_{00}Y_{00} = 1/4\pi$.) The complexity of the $I(\theta,\phi)$ distribution demanded by experimental data of course gives information on J, the X spin. Polarization determinations are necessary to establish the X parity, as well as to obtain more information on the spin.

Although $\langle T_{10}\rangle_Z$, $\langle T_{11}\rangle_Z$, and $\langle T_{1,-1}\rangle_Z$, the components of "vector polarization" of the Z, are produced by the $X \to Z$ decay process, strong decay of the Z cannot serve for analysis of this polarization. A tensor component of Z polarization which will be found in the angular distribution of Z decay is $\langle T_{20}\rangle_Z \propto \langle 3\,S_z^2 - \bar{S}^2\rangle$. (This is the Z spin alignment along its direction of flight, as the density matrix ρ_Z used to derive $\langle T_{20}\rangle = \mathrm{Tr}(\rho\, T_{20})$ is in the helicity representation.) Further contributors to the Z decay distribution are $\langle T_{2,\pm1}\rangle_Z$ and $\langle T_{2,\pm2}\rangle_Z$; however, these are observable only if azimuthal as well as polar decay angles (relative to $\hat{Z}$) are investigated. The expressions for these tensor polarization components are given by the following: [$I\langle T_{\ell,-m}\rangle = (-)^m I\langle T_{\ell m}\rangle^*$]

$$I\,\langle T_{20}\rangle_Z = \mathrm{Tr}(\rho_Z T_{20}) = (1/5)^{1/2}\sum_{L,M}$$

$$\times\left[(A^2_{3/2}+A^2_{-3/2})\,n^{(3)}_{L0} - (A^2_{1/2}+A^2_{-1/2})\,n^{(1)}_{L0}\right] t_{LM} Y^*_{LM}(\theta,\phi)$$

$$I\,\langle T_{21}\rangle_Z = (2/5)^{1/2}\sum_{L,M}\left[-A_{1/2}A^*_{3/2}+A_{-3/2}A^*_{-1/2}(-)^{2J+L}\right]$$

$$\times\, n^{(3)}_{L1}\, t_{LM}\,[(2L+1)/4\pi]^{1/2}\,\mathcal{D}^{L}_{M1}(\phi,\theta,0) \qquad (19)$$

$$I\,\langle T_{22}\rangle_Z = (2/5)^{1/2}\sum_{L,M}\left[A_{-1/2}A^*_{3/2}-A_{-3/2}A^*_{1/2}(-)^{2J+L}\right]$$

$$\times\, n^{(3)}_{L2}\, t_{LM}\,[(2L+1)/4\pi)]^{1/2}\,\mathcal{D}^{L}_{M2}(\phi,\theta,0)\ .$$

As these are unnormalized, they represent $I(\theta,\phi)$ times $\langle T_{\ell m}\rangle_Z(\theta,\phi)$. [All of the relations in Eq. (19) may be readily derived with the use of the $T_{\ell m}$ matrices for spin 3/2, which can be calculated from Eq. (9). These are presented in matrix form in Appendix III.]

In order for the polarization components of Z to be analyzed, the nature of Z decay must be examined. The simplest possibility is the strong decay

$$Z \rightarrow F + b \qquad (20)$$

$$(\text{spin: } 3/2 \rightarrow 1/2 + 0),$$

where F may be an unstable fermion (Ξ or Λ) or a stable one (p or n). The original Byers-Fenster formalism may be applied to obtain angular and polarization distributions for F (in the Z rest frame) in terms of $\langle T_{\ell m}\rangle_Z$ parameters described above. (If the fermion F has spin 3/2 rather than 1/2, the expressions developed above for ρ_Z should be reapplied to determine ρ_F and hence the various $I\langle T_{\ell m}\rangle_F$.)

The formalism predicts that the angular distribution of F (spin 1/2) in the Z rest frame is (with azimuthal angle ignored)

$$\mathcal{J}(\psi) = \frac{1}{4\pi} I(\theta,\phi)\,[1 - \langle T_{20}\rangle_Z \sqrt{5}(3\cos^2\psi-1)/2] \; . \qquad (21)$$

Here the angle ψ must refer to the angle between $\hat{F}$ and $\hat{Z}$ ($\hat{Z}$ being now defined as the direction of transformation into the Z rest frame), a "correlation" angle; this is required by the interpretation given $\langle T_{20}\rangle_Z$ in deriving Eq. (19). Equation (21) has a particularly simple form if the direction of Z (specified by angles θ and ϕ) is averaged over, as all terms then vanish in $I(\theta,\phi)$ and $I\langle T_{20}\rangle$ except for the L, M = 0, 0 terms of Eqs. (16) and (19); thus $I(\theta,\phi)$ becomes equal to $\mathrm{Tr}\,A A^+/4\pi = 1/4\pi$ and $I\langle T_{20}\rangle_Z$ becomes a constant dependent on the helicity amplitudes for $X \to Z$. As the helicity amplitudes are functions of J_X and these functions depend on the X parity, some spin-parity information may be extracted from a simple $\hat{F} - \hat{Z}$ correlation analysis. If only the lower ℓ wave [amplitude a or b of Eq. (5)] is included, the expected distribution is[13]

$$\mathcal{J}(\psi) \propto a^2[4J-(1/2)(-2J+3)(3\cos^2\psi-1)] \propto [1+\left(\frac{2J-3}{2J+1}\right)\cos^2\psi] \qquad (22)$$

for the $3/2^-$, $5/2^+$, $7/2^-$, etc. parity sequence ($\ell = J-3/2$); and it is

$$\mathcal{J}(\psi) \propto b^2[4J+4-(J+5/2)(3\cos^2\psi-1)] \propto [1-\left(\frac{6J+15}{10J+13}\right)\cos^2\psi] \qquad (23)$$

for the $3/2^+$, $5/2^-$, $7/2^+$, etc. parity sequence ($\ell = J-1/2$). For the case of $J_X = 1/2$, there is no parity discrimination; the correlation distribution for $1/2^+$ or $1/2^-$ is

$$\mathcal{J}(\psi) \propto [1 + 3\cos^2\psi]. \qquad (24)$$

If all angles are observed in the $X \to Z \to F$ decay chain, the angular distribution of the spin-1/2 F may be expressed as follows:

$$\mathcal{J}'(\theta,\phi;\psi,\zeta) = \frac{1}{4\pi}\, I(\theta,\phi)\,\{1 - \langle T_{20}\rangle(\theta,\phi)\sqrt{5}\,(3\cos^2\psi - 1)/2$$
$$+2\,(15/2)^{1/2}\,[\,\mathrm{Re}\,\langle T_{21}\rangle(\theta,\phi)\cos\zeta + \mathrm{Im}\,\langle T_{21}\rangle(\theta,\phi)\sin\zeta]\sin\psi\cos\psi$$
$$-(15/2)^{1/2}\,[\mathrm{Re}\,\langle T_{22}\rangle(\theta,\phi)\cos 2\zeta + \mathrm{Im}\,\langle T_{22}\rangle(\theta,\phi)\sin 2\zeta]\,\sin^2\psi\}, \tag{25}$$

where θ, ϕ angles give the direction of $\hat{Z}$, and ψ, ζ angles give the direction of $\hat{F}$ (in the X and Z rest frames, respectively). The experimental evaluation of these angles should make use of "direct Lorentz transformations" to move reference axes from one rest frame to the next.[14] Further, the aximuthal angle ζ of $\hat{F}$ should be referred to the x axis used for constructing ρ_Z; this is found by taking $\hat{x} = \widehat{Z \times (Z \times \mathfrak{z})}$ where $\mathfrak{z}$ is the polar axis for θ (probably the production normal). See Fig. (1).

If the fermion F is unstable, its decay provides an analyzer for the (vector) polarization components of the F. It is only in these polarizations that the odd-ℓ $\langle T_{\ell m}\rangle_Z$ appear, if the F has been produced by strong decay. The expressions for these $\langle T_{\ell m}\rangle_Z$ in terms of the t_{LM} parameters describing the original X spin state are (with L_0 taking only odd values)[15]

$$I\,\langle T_{10}\rangle_Z \equiv \mathrm{Tr}(\rho_Z T_{10}) = (1/15)^{1/2} \sum_{L_0, M}$$
$$\times\, [3(A^2_{3/2} + A^2_{-3/2})\,n^{(3)}_{L0} + (A^2_{1/2} + A^2_{-1/2})\,n^{(1)}_{L0}]\, t_{LM}\, Y^*_{LM}(\theta,\phi)$$

$$I\,\langle T_{11}\rangle_Z = -(2/15)^{1/2} \sum_{L, M} \{[A_{1/2}A^*_{3/2} + A_{-3/2}A^*_{-1/2}(-)^{2J+L}]\sqrt{3}\, n^{(3)}_{L1} \tag{26}$$

$$+2A_{-1/2}A^*_{1/2}n^{(1)}_{L1}\Big\}\; t_{LM}[(2L+1)/4\pi]^{1/2}\mathcal{D}^L_{M1}(\phi,\theta,0) \qquad \text{cont. (26)}$$

$$I\langle T_{30}\rangle_Z = (1/35)^{1/2}\sum_{L,M}\Big\{[A^2_{3/2}+A^2_{-3/2}]$$

$$\times\, n^{(3)}_{L0} - 3[A^2_{1/2}+A^2_{-1/2}]\,n^{(1)}_{L0}\Big\}\, t_{LM}Y^*_{LM}(\theta,\phi)$$

$$I\langle T_{31}\rangle_Z = 2(1/35)^{1/2}\sum_{L,M}$$

$$\times\Big[-A_{1/2}A^*_{3/2}n^{(3)}_{L1}+\sqrt{3}\,A_{-1/2}A^*_{1/2}n^{(1)}_{L1}-A_{-3/2}A^*_{-1/2}n^{(-1)}_{L1}\Big]t_{LM}\mathcal{D}^L_{M1}(\phi,\theta,0) \times[(2L+1)/4\pi]^{1/2}$$

$$I\langle T_{32}\rangle_Z = (2/7)^{1/2}\sum_{L,M}\Big[A_{-1/2}A^*_{3/2}n^{(3)}_{L2}-A_{-3/2}A^*_{1/2}n^{(1)}_{L2}\Big]t_{LM}\mathcal{D}^L_{M2}(\phi,\theta,0) \times[(2L+1)/4\pi]^{1/2}$$

$$I\langle T_{33}\rangle_Z = -2(1/7)^{1/2}A_{-3/2}A^*_{3/2}\sum_{L,M}n^{(3)}_{L3}t_{LM}\mathcal{D}^L_{M3}(\phi,\theta,0)[(2L+1)/4\pi]^{1/2}$$

The contributions of these expectation values to the longitudinal and transverse polarization components of the F are given by the following:

$$\mathscr{I}\;\mathbf{P}\cdot\hat{\mathbf{F}} = (4\pi)^{-1/2}\Big\{0.448[\langle T_{10}\rangle Y_{10}+2\,\mathrm{Re}(\langle T_{11}\rangle Y^*_{11})] -1.34[\langle T_{30}\rangle Y_{30}+\sum_\nu 2\,\mathrm{Re}(\langle T_{3\nu}\rangle Y^*_{3\nu})]\Big\} \qquad (27)$$

$$\mathscr{I}\;(\mathbf{P}\cdot\hat{\mathbf{x}}'+i\mathbf{P}\cdot\hat{\mathbf{y}}') = -\gamma(4\pi)^{-1/2}\Big\{1.27\sqrt{3/4\pi}[\langle T_{10}\rangle\mathcal{D}^1_{01}+\langle T_{11}\rangle\mathcal{D}^1_{11} -\langle T_{11}\rangle^*\mathcal{D}^1_{-1,1}] -1.55\sqrt{7/4\pi}[\langle T_{30}\rangle\mathcal{D}^3_{01}+\sum_\nu(\langle T_{3\nu}\rangle\mathcal{D}^3_{\nu 1}-\langle T_{3\nu}\rangle^*\mathcal{D}^3_{-\nu,1})]\Big\}$$

where $\hat{x}' = \widehat{F\times(F\times Z)}$ and $\hat{y}' = \widehat{Z\times F}$. The summation index ν runs from 1 to 3. The $Y_{\ell m}$ and $\mathcal{D}^{\ell}_{mm'}$ symbols represent the functions $Y_{\ell m}(\psi,\zeta)$ and $\mathcal{D}^{\ell}_{mm'}(\zeta,\psi,0)$, respectively; also, in both equations, the substitution of $(-)^m\langle T_{\ell m}\rangle^*$ for $\langle T_{\ell,-m}\rangle$ has been made. In the second of these equations, the symbol γ is to be taken as +1 or -1 if the relative Z-F parity is such that the angular momentum in Z decay is J-1/2 or J+1/2, respectively. The polarization components of Eq. (27) are determined experimentally by taking $(3/\alpha)\sum \hat{p}\cdot\hat{F}$, $(3/\alpha)\sum \hat{p}\cdot\hat{x}'$, and $(3/\alpha)\sum \hat{p}\cdot\hat{y}'$, where α is the usual asymmetry parameter for F decay and $\overline{p}$ is the decay momentum in the F rest frame; the sums are taken over all events with F at some particular ψ, ζ orientation.

V. EXPERIMENTAL TESTS FOR SPIN AND PARITY

Prescriptions for Tests

One possible test for parity and spin of the X is to be found in the sign and magnitude of the $\cos^2\psi$ coefficient of Eqs. (22) through (24), which are valid under the assumption that only the lower angular-momentum wave contributes to X decay. Equation (23) is not very sensitive to spin assumption, and Eq. (24) yields no information on parity. Other possible tests, some of them more general than the above, are presented in the following paragraphs.

The $\langle T_{2m}\rangle_Z$ values describing the Z spin state may be determined from the angular distribution observed for the process $Z \to F$. [See Eq. (25).] If F undergoes weak decay, the $\langle T_{\ell m}\rangle_Z$ with $\ell = 1$ and 3 may also be determined [Eq. (27)]. (In principle, a scattering of F with a known analyzing target would also yield the $\langle T_{1m}\rangle_Z$ and $\langle T_{3m}\rangle_Z$.) The experimental evaluations of $I(\theta,\phi)$ for $X \to Z$ and the three $\langle T_{2m}\rangle_Z(\theta,\phi)$ from $X \to Z \to F$ yield a total of four evaluations of each even-L t_{LM} describing the initial X state; further, the two $\langle T_{1m}\rangle_Z$ and the four $\langle T_{3m}\rangle_Z$ yield six evaluations of each odd-L t_{LM} describing the X state. Odd-L (even-L) t_{LM} may also be obtained from $\langle T_{2m}\rangle$ ($\langle T_{1m}\rangle$ and $\langle T_{3m}\rangle$) for $m \neq 0$; but these arise from interference of the two orbital amplitudes permitted for a given X parity [the a and c or b and d amplitudes of Eq. (5)] and thus are probably small.

Every $n^{(2\lambda)}_{L\mathcal{m}}$ coefficient appearing with a t_{LM} in the $I(\theta,\phi)$ or $\langle T_{\ell m}\rangle$ distributions may be expressed as $n^{(1)}_{L0}$ times some factor containing J_X and L. (See Appendix II.) The A_λ helicity amplitudes also depend explicitly on J for $J \geqslant 3/2$. Thus, by comparison of the $A_\lambda A^*_{\lambda'} n^{(2\lambda)}_{L\mathcal{m}}$ coefficients of $Y^*_{\ell m}$ or $\mathcal{D}^{\ell}_{mm'}$ from one distribution with those in another distribution, various tests of J may be made. One way of estimating J is to construct a function similar to a χ^2 which compares values of $n^{(1)}_{L0}\, t_{LM}$ obtained in two or more distributions and to treat J as a variable parameter in this function;[16] another possible approach is construction of a general likelihood function treating all stages of decay and maximizing of this function for various J assumptions. A final and possibly very useful method is evaluation of a J-dependent function multiplying some $n^{(1)}_{L0}\, t_{LM}$ by taking ratios of terms in various distributions.[17]

A possible approach in setting up a general spin test function might be the following. Let the definition of "moment" be the coefficient of Y^*_{LM} or $[(2L+1)/4\pi]^{1/2}\mathcal{D}^{L}_{MM'}$ projected out of a distribution, Eq. (16), (19), or (26) (by weighting that distribution with Y_{LM} or $[(2L+1)/4\pi]^{1/2}\mathcal{D}^{L*}_{MM'}$ and summing over all events).[18] If $t_{LM}(1)$ stands for the L, M moment obtained from one distribution, and if $f_J(1)$ is the function of J which must be divided into this moment to obtain $n^{(1)}_{L0}\, t_{LM}$, and if $t_{LM}(2)$ represents a similar term from a second distribution, etc., then a comparison can be made of the four evaluations of even-L t_{LM}'s by calculating the following for various J values. (A minimum "χ^2" yields the best J estimate.)[16]

$$"\chi^2_J" \equiv \sum_{\substack{L,M,\\ L'M'}} \sum_{i,j} \{[t_{LM}(i)/f_J(i)] - \langle t_{LM}(i)/f_J(i)\rangle\}\, G^{-1}_{LM(i),\, L'M'(j)} \times \{[t_{L'M'}(j)/f_J(j)] - \langle t_{L'M'}(j)/f_J(j)\rangle\}. \tag{28}$$

The indices i and j designate the four evaluations; and $\langle\ \rangle$ represents an average of these. The symbol G stands for a variance matrix. {That is, $G_{LM(i),\ L'M'(j)}$ is the second-moment matrix, the average value of $[t_{LM}(i)-\langle t_{LM}(i)\rangle]\ [t_{L'M'}(j)-\langle t_{L'M'}(j)\rangle]$.} A "$\chi^2$" for the six evaluations of odd-L moments may be developed by analogy with Eq. (28).

Construction of a likelihood function is not difficult; the proper distribution function is $\mathcal{J}(\psi,\zeta)$ if the fermion F does not decay, and is $\mathcal{J}\times[1+\alpha\,\vec{P}_F\cdot\hat{p}]$ if it does decay. Either of these distribution functions would be most useful if expressed in terms of $n_{L0}^{(1)}\,t_{LM}$ times the $f_J(i)$ function of J discussed above. A high-spin form of the likelihood function, one appropriate for the maximum J assumed, might be used; then a maximum could be sought as a function of J, $n_{L0}t_{LM}$, and the ℓ-wave amplitudes (without changing the form of the likelihood function).

Finally, known functions of spin may be evaluated by taking ratios of corresponding L, M moments found in two different experimental distributions. For example, after substitution of expressions for helicity amplitudes and for $n_{Lm}^{(2\lambda)}$ coefficients (see Appendix II), the ratio of an even-L moment in $I(\theta,\phi)$ to the same moment of $I\langle T_{20}\rangle(\theta,\phi)$ yields [from Eqs. (16) and (19)][19]

$$\frac{\text{I moment}}{I\langle T_{20}\rangle\ \text{moment}} \equiv \langle Y_{LM}\rangle \Big/ \big\langle\langle T_{20}\rangle\, Y_{LM}\big\rangle \approx \sqrt{5}\left[\frac{4J(2J-1)-2L(L+1)}{(3-2J)(2J-1)-2L(L+1)}\right] \tag{29a}$$

for one parity of decay (orbital wave $\ell = J-3/2$); and the same ratio yields for the other parity ($\ell = J-1/2$),

$$\langle Y_{LM}\rangle \Big/ \big\langle\langle T_{20}\rangle\, Y_{LM}\big\rangle \approx \sqrt{5}\left[\frac{4(2J-1)(J+1)-6L(L+1)}{(2J+5)(2J-1)-6L(L+1)}\right]. \tag{29b}$$

(In terms of explicit functions of angles, the moment ratio of Eq. (29) is evaluated for a particular L, M by dividing $\sum_i Y_{LM}(\theta_i,\phi_i)$ by the quantity

$\sum_i Y_{20}(\psi_i)\, Y_{LM}(\theta_i, \phi_i)/n_{20}^{(1)}$. The sums are to be taken separately over real and imaginary parts; the index i runs over all events; and the $n_{20}^{(1)}$ constant is that of Eq. (12) with J set equal to 3/2.) Equation (29) provides a separate test for every non-zero even-L, M moment. Also, the ratio of moments from $I\langle T_{22}\rangle$ and $I\langle T_{21}\rangle$ yields, for even $L \geq 2$ and $J \geq 3/2$,

$$\left| \left\langle \langle T_{22}\rangle \mathcal{D}_{M2}^{L*} \right\rangle \Big/ \left\langle \langle T_{21}\rangle \mathcal{D}_{M1}^{L*} \right\rangle \right| = (J+1/2)/[(L+2)(L-1)]^{1/2} \quad (30)$$

(In terms of explicit functions, the moment ratio of Eq. (30) is evaluated by dividing $\sum_i^N Y_{22}(\psi_i \zeta_i)\, \mathcal{D}_{M2}^{L*}(\phi_i, \theta_i, 0)$ by $\sum_i^N Y_{21}(\psi_i, \zeta_i)\, \mathcal{D}_{M1}^{L*}(\phi_i, \theta_i, 0)$. Here the $n_{20}^{(1)}$ constant is common to both terms and can be ignored.)

Care must be taken in the interpretation of these J estimations; the ratio of two normally distributed quantities is itself not normally distributed.[20] Ratio calculations may be made separately for the real and imaginary parts of each moment. (Only M values greater than zero need be used, as any moment with -M is the charge conjugate of that with +M.)

A simple test may be made for the X-Z relative parity for any J assumption, the test being the determination of the relative sign of a moment in $I\langle T_{22}\rangle$ with respect to the corresponding moment in $I\langle T_{21}\rangle$. [The helicity amplitudes for these moments are the same except for sign, which depends on parity, as shown in Eq. (5).] Thus if $\Gamma = +1$ or -1 for $J^P = 3/2^-, 5/2^+$, etc., or $3/2^+, 5/2^-$, etc., respectively,

$$\left\langle \langle T_{22}\rangle \mathcal{D}_{M2}^{L*} \right\rangle \Big/ \left\langle \langle T_{21}\rangle \mathcal{D}_{M1}^{L*} \right\rangle = \Gamma(J+1/2)/[(L+2)(L-1)]^{1/2} \quad (31)$$

for even $L \geq 2$ and $J \geq 3/2$. [This is of course the same moment ratio as in Eq. (30).] A χ^2 which tests the equality of corresponding moments with $\Gamma = +1$ or -1 is easily constructed. The parity test of Eq. (31), and a similar one (below) for odd-ℓ $I\langle T_{\ell m}\rangle$ are analogous to the test that may be made for the parity of decay into spin 1/2: the determination of relative sign of any two corresponding moments in transverse $(I \langle T_{11} \rangle)$ and longitudinal $(I \langle T_{10} \rangle)$ polarizations.[1]

If the F polarization can be analyzed, more tests for spin and parity may be found in the odd-ℓ $\langle T_{\ell m} \rangle$ distributions. Some obvious tests [from Eq. (26)] are the following:[18] With the higher orbital wave neglected,[19]

$$I \langle T_{10} \rangle \text{ moment} / I \langle T_{30} \rangle \text{ moment} \equiv \left\langle \langle T_{10} \rangle \, Y_{LM} \right\rangle \Big/ \left\langle \langle T_{30} \rangle \, Y_{LM} \right\rangle$$

$$\approx (7/3)^{1/2} (3/2) \left[\frac{(2J-1)(2J+1) - 2L(L+1)}{(3-4J)(2J-1) - L(L+1)} \right] \quad (32a)$$

for one parity of decay (orbital wave $\ell = J - 3/2$); and the same ratio yields for the other parity $(\ell = J - 1/2)$

$$\left\langle \langle T_{10} \rangle \, Y_{LM} \right\rangle \Big/ \left\langle \langle T_{30} \rangle \, Y_{LM} \right\rangle \approx (7/3)^{1/2} (1/6) \left[\frac{(2J-1)(10J+13) - 18L(L+1)}{2J-1 - L(L+1)} \right] \quad (32b)$$

These equations are valid for any odd-L moments. (In terms of explicit functions, this ratio may be found experimentally from

$$\sum_i (\hat{p} \cdot \hat{F})_i \, Y_{10}(\psi_i) \, Y_{LM}(\theta_i, \phi_i) / n_{10}^{(1)} \text{ divided by } \sum_i (\hat{p} \cdot \hat{F})_i \, Y_{30}(\psi_i) Y_{LM}(\theta_i, \phi_i) / n_{30}^{(1)}$$

or from $\sum_i (\hat{p} \cdot \hat{x}' + i\hat{p} \cdot \hat{y}') \mathcal{D}_{01}^{1*}(\zeta_i, \psi_i, 0) \, Y_{LM}(\theta_i, \phi_i) / n_{11}^{(1)}$ divided by

$$\sum_i (\hat{p} \cdot \hat{x}' + i\hat{p} \cdot \hat{y}') \mathcal{D}_{01}^{3*}(i) \, Y_{LM}(i) / n_{31}^{(1)}, \text{ with } n_{1m}^{(1)} \text{ and } n_{3m}^{(1)}$$

evaluated for $J = 3/2$.)

Further tests may be made by combining $I \langle T_{10} \rangle$ and $I \langle T_{30} \rangle$ and also $I \langle T_{11} \rangle$ and $I \langle T_{31} \rangle$. From Eq. (26) it is apparent that

$$(15)^{1/2} I \langle T_{10} \rangle - 3(35)^{1/2} I \langle T_{30} \rangle = 20 A_{1/2}^2 \sum_{L_o, M} n_{L0}^{(1)} t_{LM} Y_{LM}^*(\theta, \phi) \quad (33)$$

and that

$$(5/2)^{1/2} I \left\langle T_{11} \right\rangle - \frac{1}{2}(35)^{1/2} \; I \left\langle T_{31} \right\rangle =$$

$$-(5/\sqrt{3}) \; A_{-1/2} A^{*}_{1/2} \sum_{L,M} n^{(1)}_{L1} t_{LM} \mathcal{D}^{L}_{M1}(\phi, \theta, 0) \left[(2L+1)/4\pi\right]^{1/2}. \tag{34}$$

Dividing Eq. (33) into Eq. (34) yields the relative sign of $A_{1/2}$ and $A_{-1/2}$ and hence the relative X-Z parity. Also, the ratio of n_{L1} to n_{L0} gives a function of spin J. Thus, with the parity factor Γ defined as above, and with R and S representing the left-hand sides of Eqs. (33) and (34), respectively,

$$\text{S moment/R moment} \equiv \left\langle S \mathcal{D}^{L*}_{M1} \right\rangle \; [2L+1)/4\pi]^{1/2} / \left\langle R Y_{LM} \right\rangle$$

$$= \Gamma \; (2J+1) \; / \; 4[3L(L+1)]^{1/2} \tag{35}$$

for any odd L. This relation is rigorously true, without any approximation for amplitudes; it is meaningful for odd-L values. (Explicit functions for moments of $I\left\langle T_{\ell m} \right\rangle$ distributions in Eqs. (33) and (34) are easily written out, as derived from the longitudinal or transverse polarization distributions of the E, Eq. (27); they have the same form as the functions given for the evaluation of Eq. (32).)

Another combination of the above distributions gives

$$3(15)^{1/2} I \left\langle T_{10} \right\rangle + (35)^{1/2} \; I \left\langle T_{30} \right\rangle = 20 \, A^{2}_{3/2} \sum_{L,M} n^{(1)}_{L0} t_{LM} Y^{*}_{LM}(\theta, \phi). \tag{36}$$

We may compare this to the $I \left\langle T_{33} \right\rangle$ distribution to evaluate both spin and parity. With the left side of Eq. (36) designated as U, and with the appropriate $\mathcal{D}^{L*}_{M3}$ and Y_{LM} moments ($L \geq 3$),

$$I \left\langle T_{33} \right\rangle \text{ moment / U moment} = \Gamma(2J+1)\left[1/10\sqrt{7}\right]\left[2-L(L+1)\right]$$

$$\times \left[(L+3)(L-2)(L+2)(L-1)L(L+1)\right]^{-1/2}. \tag{37}$$

Appendix IV describes the $\mathcal{D}^{L}_{MM'}(\phi, \theta, 0)$ functions utilized above.

Utility of Tests

Although it is difficult for one to make generalizations as to the desirability of using one test rather than another in particular experimental situations, it is clear from the forms of the above J-dependent functions that experimental evaluations of these test functions can be made only for certain values of L, and hence only for certain initial (tensor) polarizations. Further, the tests that ignore the higher ℓ wave of decay are applicable only in situations in which the "Q value" of decay is low enough that little contribution can be expected from the higher ℓ wave. The following statements summarize the relationships between the test functions and particular data characteristics. (The latter of course may not be known until the tests have been at least partially completed!)

If the lower ℓ wave of X decay does not predominate - all test functions except those of Eqs. (29) and (32) are applicable.

If only even-L tensor polarizations of the X exist (L=0 being the normalization, present even in the absence of any form of polarization; L = 2 "polarization" being alignment; L = 4 "polarization" being the expectation value of S_z^4 or a similar-rank tensor; etc.), tests given by Eqs. (29) through (31) are applicable.

If only odd-L tensor polarizations of the X exist (L = 1 polarization being the usual "vector polarization"; L = 3 polarization being $\langle S_z^3 \rangle$ or a similar quantity; etc.), tests of Eqs. (32) through (37) are applicable. (The L = 0 test also holds.)

If no polarization of any sort exists (L = 0 yielding the only nonzero moment), only Eq. (29) or the earlier Eqs. (22) through (24) provide tests.

If only vector polarization (L=1) exists, only Eq. (35) furnishes a rigorous test (in addition to that for L = 0).

If only alignment (L=2) exists, Eq. (30) or (31) provides a rigorous test (in addition to that for L = 0).

If only L=3 moments exist, Eqs. (35) and (37) may be used as rigorous tests.

There are as many independent tests for a given L (and a given test function) as there are permissible values of M (from -L to +L). (In practice, the tests are made separately for real and imaginary parts of all L, M moments with $M \geqslant 0$.)

If the spin J of "particle" X is 1/2, only the quantities I, $I\langle T_{20}\rangle$, $I\langle T_{10}\rangle$, $I\langle T_{11}\rangle$, $I\langle T_{30}\rangle$, and $I\langle T_{31}\rangle$ can be nonzero, as $\rho_Z]_{\lambda\lambda'}$ is zero for $|\lambda|$ or $|\lambda'| > 1/2$. Parity of a spin-1/2 X may be found by comparison of the L, M = 1, 0 moment in $I\langle T_{11}\rangle$ with the corresponding moment in $I\langle T_{10}\rangle$, or by comparison of a moment of $I\langle T_{31}\rangle$ with the corresponding one in $I\langle T_{30}\rangle$, etc. [See Eqs. (26) and (5).]

In the course of analysis, it may be useful to study the odd-L moments from the $I\langle T_{2m}\rangle$ distributions and the even-L moments from $I\langle T_{1m}\rangle$ or $I\langle T_{3m}\rangle$ distributions; these are proportional to $2\,\mathrm{Im}\,A_\lambda A^*_{\lambda'}$, i.e., to terms like a^*c or b^*d, and thus give a measure of the interference of the higher ℓ wave.

VI. APPLICATIONS

Some of the tests described above are being applied to the decay sequence $\Xi^*(1820) \to \Xi^*(1530) + \pi$, $\Xi^*(1530) \to \Xi + \pi$, $\Xi \to \Lambda + \pi$. Unfortunately, the number of useful events is small and the background is appreciable.[21] Other processes to which the formalism for spin J decay into spin 3/2 might be applicable are (a) higher lying $N^* \to N^*_{33}$ and (b) $Y^*(1815) \to Y^*(1385)$.

VII. ACKNOWLEDGMENTS

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APPENDICES

I. Helicity Amplitudes

The general form of a helicity amplitude is

$$A_\lambda = (-)^{\lambda-S} \sum_\ell a_\ell \, C(J S \ell; \lambda, -\lambda) \tag{38}$$

for the decay of a particle of spin J into one with spin S plus a spinless boson, where J and S may be either integral or half-integral spins. The helicity amplitudes may be constructed without the use of Eq. (38) for spin J → [spin J plus spinless boson] by taking combinations of irreducible tensors in the spin space of dimensionality 2J+1. In analogy with the construction of the transition matrix for decay into a spin-1/2 object, the matrix for decay into spin J is required by invariance arguments to be[22]

$$A = \sum_L^{2J} \sum_M (p^\dagger_{LM} \, T_{LM} \times \text{complex coefficient}), \tag{39}$$

where p_{LM} represents an irreducible tensor formed from components of decay momentum. In the helicity representation, only the p_{L0} terms are nonzero; these in fact become constants because $p_Z = p$. With the absorption of p_{L0} factors, the transition matrix becomes, in the helicity representation,

$$A = a + c\, T_{20} \text{ for one parity}$$

$$\text{and } A' = b\, T_{10} + d\, T_{30} \text{ for the opposite parity of decay.} \tag{40}$$

<u>However</u>, when the initial and final spins differ in a decay, factors dependent on initial spin modify the various elements of A; and these must be calculated by a prescription similar to that of Eq. (38). Decay into final spin 1/2 is an exception to this statement, as there is only one initial-spin factor which is common to $A_{1/2}$ and $A_{-1/2}$, and this is absorbed in the normalization of the amplitudes.

II. Relations Among $n^{(2\lambda)}_{L m}$ Coefficients

The derivation of general expressions for

$$n^{(2\lambda)}_{L m} \equiv (-)^{J-\lambda'} [(2J+1)/4\pi]^{1/2} C(J\,J\,L;\, \lambda, -\lambda')\,; \quad \lambda' = \lambda - m \qquad (41)$$

in terms of $n^{(1)}_{L0}$ is useful because comparison of experimental distributions containing these coefficients may test for J, the spin of X.[9] The identity $C(J\,J\,L;\, \lambda_1 \lambda_2) = (-)^{2J+L} C(J\,J\,L;\, \lambda_2 \lambda_1)$ permits derivation of relations between the λ and $-\lambda'$ forms of $n^{(2\lambda)}_{L0}$ and also relations between the $n^{(2\lambda)}_{L m}$ coefficients in the (λ, λ') and $(-\lambda', -\lambda)$ elements of ρ_Z [Eq. (14)]:

$$n^{(-2\lambda)}_{L0} = (-)^{2J+L+1}\, n^{(2\lambda)}_{L0}$$

$$n^{(-1)}_{L1} = (-)^{2J+L}\, n^{(3)}_{L1}$$

$$n^{(1)}_{L2} = (-)^{2J+L+1}\, n^{(3)}_{L2} \quad . \qquad (42)$$

Recursion relations for Clebsch-Gordan coefficients (p. 39, Edmonds, Ref. 3) may be utilized to obtain the following:

$$n^{(3)}_{L0} = (1/X)\, \{X - L(L+1) + (J+1/2)^2 [1 + (-)^{2J+L}]\}\, n^{(1)}_{L0} \qquad (43)$$

which becomes $\quad n^{(3)}_{L0} = (1/X)[X - L(L+1)]\, n^{(1)}_{L0} \quad$ for even L

and $\quad n^{(3)}_{L0} = (1/X)[3J(J+1) - 1/4 - L(L+1)]\, n^{(1)}_{L0}$ for odd L,

where $X = (J+3/2)(J-1/2)$. Further, by use of the same recursion relations,

$$n^{(1)}_{L1} = -(2J+1)[L(L+1)]^{-1/2}\, n^{(1)}_{L0}\, [1+(-)^{2J+L}]/2 \qquad (44)$$

$$n^{(3)}_{L1} = X^{-1/2}\, \{L(L+1) - (J+1/2)^2 [1+(-)^{2J+L}]\}\, [L(L+1)]^{-1/2}\, n^{(1)}_{L0} \qquad (45)$$

$$n^{(3)}_{L2} = X^{-1/2}(J+1/2) \quad \{1+(-)^{2J+L} - L(L+1)\}\, [(L+2)(L-1)L(L+1)]^{-1/2} n^{(1)}_{L0} \qquad (40)$$

$$n^{(3)}_{L3} = -X^{+1/2}\, [1+(-)^{2J+L}]\, [(L+3)(L-2)]^{-1/2}\, n^{(3)}_{L2} \qquad (46)$$

III. Matrix Forms for T_{LM} Tensors

The T_{LM} may be defined in matrix form (in the representation where T_{L0} is diagonal) by the following:

$$T_{LM}\Big]_{mm'} \equiv C(J\,L\,J;\, m'\,M) \text{ with } m' + M = m.$$

The matrix forms for all required T_{LM}'s for spin-1/2 and spin-3/2 systems are given below. Only those with positive M are presented, as $T_{L,-M} = (-)^M T^{\dagger}_{LM}$.

Spin 1/2

$$T_{10} = (1/\sqrt{3}) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad T_{11} = (2/3)^{1/2} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Spin 3/2

$$T_{10} = (1/15)^{1/2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \qquad T_{11} = -(2/15)^{1/2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{20} = (1/5)^{1/2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{21} = (2/5)^{1/2} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad T_{22} = (2/5)^{1/2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{30} = (1/35)^{1/2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad T_{31} = (4/35)^{1/2} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad T_{32} = (2/7)^{1/2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{33} = (4/7)^{1/2} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The T_{LM} are orthogonal tensors in the sense that $\mathrm{Tr}\, T_{LM} T^{\dagger}_{L'M'} = \delta_{LL'} \delta_{MM'} \times (2J+1)/(2L+1)$.

IV. The $\mathcal{D}^L_{MM'}(\alpha,\beta,\gamma)$ Functions

The general expression for $\mathcal{D}^L_{MM'}(\alpha,\beta,\gamma)$ is $e^{-iM\alpha} d^L_{MM'}(\beta)\, e^{-iM'\gamma}$, where $d^L_{MM'}(\beta)$ is a polynomial in $\sin\beta$ and $\cos\beta$ (if indices are integral) and the α, β, and γ arguments represent Euler angles. Only $d^L_{MM'}$ functions with integral indices are required for the analysis described in the text. All such functions may be readily derived from the well-known spherical harmonics through the use of the following relations (found in Jacob and Wick[4]):

$$d^L_{M0}(\theta) = [4\pi/(2L+1)]^{1/2}\, Y_{LM}(\theta,0)$$

$$d^L_{M1}(\theta) = [L(L+1)]^{-1/2}\{-M(1+\cos\theta)\, d^L_{M0}(\theta)/\sin\theta - [(L-M)(L+M+1)]^{1/2} d^L_{M+1,0}(\theta)\}$$

$$d^L_{MM'}(\theta) = (-)^{M-M'} d^L_{M'M}(\theta)$$

$$d^L_{-M,M'}(\theta) = (-)^{L+M'} d^L_{MM'}(\pi-\theta)$$

$$\begin{aligned}2[(L+M')(L+M'-1)]^{1/2} d^L_{MM'}(\theta) = {} & [(L+M)(L+M-1)]^{1/2}(1+\cos\theta)\, d^{L-1}_{M-1,M'-1}(\theta) \\ & + 2(L^2-M^2)^{1/2}\sin\theta\, d^{L-1}_{M,M'-1}(\theta) \\ & + [(L-M)(L-M-1)]^{1/2}(1-\cos\theta)\, d^{L-1}_{M+1,M'-1}(\theta).\end{aligned}$$

FOOTNOTES AND REFERENCES

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1. N. Byers and S. Fenster, Phys. Rev. Letters 11, 52 (1963).
2. For discussion of the density matrix, see L. Wolfenstein, Ann. Rev. Nucl. Sci. 6, 43 (1956), and U. Fano, Rev. Mod. Phys. 29, 74 (1957).
3. For descriptions of the $\mathcal{D}^J_{MM'}(\alpha, \beta, \gamma)$ functions and the T_{LM} tensors, see A. R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1957) or M. E. Rose, Elementary Theory of Angular Momentum (John Wiley & Sons, Inc., New York, 1957). For $\mathcal{D}^J_{MM'}$ tabulations, see ref. 4 or App. IV.
4. M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959) show that the matrix element connecting a J, M representation of a state with a helicity representation is $[(2J+1)/4\pi]^{1/2}\mathcal{D}^{J*}_{M\lambda}(\phi, \theta, -\phi)$. The third argument is not observable in processes discussed here.
5. The first expression in Eq. (4) follows from Eq. (B5) of Jacob and Wick, Ref. 4. These Clebsch-Gordan coefficients are written in the form used by Jacob and Wick, $C(j_1 j_2 j; m_1 m_2)$.
6. See reference 1 and also the unpublished appendix of "Determination of Spin and Decay Parameters of Fermion States," N. Byers and S. Fenster, unpublished report of Dept. of Physics, University of California, Los Angeles, May 27, 1963.
7. These T_{LM} tensors have been previously utilized to describe the spin state of the deuteron in scattering processes. See W. Lakin, Phys. Rev. 98, 139 (1955).

8. The useful relations, found in Rose and Edmonds (Ref. 3), are

$$\mathcal{D}^{j_1}_{\mu_1 m_1}\mathcal{D}^{j_2}_{\mu_2 m_2} = \sum_j C(j_1 j_2 j;\mu_1\mu_2)C(j_1 j_2 j; m_1 m_2)\,\mathcal{D}^{j}_{\mu_1+\mu_2,\, m_1+m_2}$$

and $\mathcal{D}^{j*}_{m'm}(\alpha\beta\gamma) = (-)^{m'-m}\mathcal{D}^{j}_{-m',\,-m}(\alpha\beta\gamma)$ and also

$\sum_{m_1 m_2} C(j_1 j_2 j; m_1 m_2)\, C(j_1 j_2 j'; m_1 m_2) = \delta_{jj'}$. The first relation here holds also for products of spherical harmonics Y_{LM}, as

$Y_{LM}(\theta,\phi) \propto \mathcal{D}^{L}_{M0}(\phi,\theta,0)^*$.

The $\mathcal{D}^{L}_{MM'}$ functions are described in Jacob and Wick[4] as well as in Rose and Edmonds; some are tabulated in the former reference in terms of simple θ and ϕ functions. For evaluation of Eq. (11), symmetry properties of the T_{LM} (see $T_{L,-M}$ expression in text) and $\mathcal{D}^{L}_{MM'}$ are useful $[\mathcal{D}^{L}_{-M,\mu}(\phi,\theta,0) = (-)^{L+\mu}\mathcal{D}^{L}_{M\mu}(\phi,\pi-\theta,0)^*]$.

9. An alternate definition, more convenient for calculation, is

$$n^{(2\lambda)}_{L\mathcal{m}} \equiv \left[\frac{2L+1}{4\pi}\right]^{1/2} C(JLJ;\lambda-\mathcal{m},\mathcal{m}).$$

10. Taking $\mathrm{Tr}(\rho\, T_{\ell m})$ is equivalent to finding $\langle \chi_n | T_{\ell m} | \chi_n \rangle$ for each spin state n and summing over all spin states with proper weighting.

An alternate derivation of the distributions for particle F may be used which does not demand the calculation of the I $\langle T_{\ell m}\rangle$ quantities for Z. This is the transforming of the density matrix ρ_Z by use of a transition matrix ($\mathcal{M}'$) for the Z $\rightarrow$ F decay; i.e., the calculation of $\rho_F = \mathcal{M}'\rho_Z\mathcal{M}'^{\dagger}$ from the expression for ρ_Z in Eq. (8). The transition matrix $\mathcal{M}'$ here involves the well-known $\mathcal{D}^{3/2}_{mm'}$ functions and the helicity amplitudes for spin 3/2 decay into spin 1/2 plus spin 0. Although this is a more elegant derivation, it does not provide so clearly the means for making spin and parity tests as does the method presented in the text.

11. The θ and ϕ angles must be referred to axes defined by vectors in the X production process. If the normal serves as polar axis, all t_{LM} with odd M are zero. (Fig. 1 describes the coordinate system.)

12. Throughout the text, the notation used in complex expressions has been simplified by omission of the absolute-value signs from squares of amplitudes. Thus, A^2 should be interpreted as $|A|^2$ or A^*A.

13. Prof. Charles Zemach has derived these same distributions by the use of an entirely different formalism. Charles Zemach, (University of California, Berkeley), private communication, 1964.

14. A direct Lorentz transformation means the translating of axes so that their orientation relative to the direction of the usual Lorentz transformation is maintained. See H. P. Stapp, Relativistic Transformation of Spin Directions, University of California Radiation Laboratory Report UCRL-8096, December 1957. (unpublished).

15. Here the first two expressions represent longitudinal and transverse polarization components for Z; i.e., $T_{10} \propto \overline{S} \cdot \hat{Z}$ and $T_{11} \propto (\overline{S} \cdot \hat{x} + i\, \overline{S} \cdot \hat{y})$.

Some relations from Appendix II have been utilized to simplify expressions.

16. This function cannot be interpreted as a true χ^2, but should yield an unbiased estimate of J. An example of the application of a "χ^2" test for variable J is presented in an analysis of the $Y^*(1385)$, Janice B. Shafer and Darrell O. Huwe, Phys. Rev. <u>134</u>, B1372 (1964); the χ^2 of Eq. (19) and Fig. (2) tests the relation $\gamma(2J+1)\, t(1) = [L(L+1)]^{1/2}\, t(2)$, where t(1) and t(2) represent moments from longitudinal and transverse components of polarization, respectively.

17. An example is given in the calculation of $(2J+1)^2$ from moments for decay suggested by M. Ademollo and R. Gatto, Phys. Rev. 133, B531 (1964), or in the calculation of 2J+1 for strong decay suggested by Byers and Fenster, Ref. 1.

18. For $\ell \neq 0$, the $I \langle T_{\ell m} \rangle (\theta, \phi)$ distributions must be found by application of the analyzing expressions of Eqs. (25) and (27) to the data. The more general forms of these equations are

$$\mathcal{G}(\psi, \zeta) = \sum_{\ell}^{2} \sum_{m} n_{\ell 0}^{(1)} \langle T_{\ell m} \rangle Y_{\ell m}^{*}(\psi, \zeta) \text{ with } \ell \text{ even;}$$

$$\mathcal{G}\overline{P} \cdot \hat{F}(\psi, \zeta) = \sum_{\ell}^{3} \sum_{m} n_{\ell 0}^{(1)} \langle T_{\ell m} \rangle Y_{\ell m}^{*}(\psi, \zeta) \text{ with } \zeta \text{ odd;}$$

and $\mathcal{G}(\overline{P} \cdot \hat{x}' + i\overline{P} \cdot \hat{y}') = -\gamma \sum_{\ell} \sum_{m} n_{\ell 1}^{(1)} \langle T_{\ell m} \rangle \mathcal{D}_{m1}^{\ell}(\zeta, \psi, 0) [(2\ell+1)/4\pi]^{1/2}$

with ℓ odd; where $n_{\ell m}^{(1)}$ is to be found from Eq. (12) with J set equal to 3/2. Thus, the $I \langle T_{\ell m} \rangle (\theta, \phi)$ may be found either by fitting the expressions in the text to the experimental distributions in ψ and ζ or by projecting the coefficients (moments) of the orthogonal functions given above. The evaluation of the sum $\sum_i Y_{\ell m}(\psi_i, \zeta_i)$ over all events yields an even-ℓ $n_{\ell 0}^{(1)} \langle T_{\ell m} \rangle$; and the evaluation from F decay of $\sum_i (\hat{p} \cdot \hat{F})_i Y_{\ell m}(\psi_i, \zeta_i)(3/\alpha)$ yields an odd-ℓ $n_{\ell 0}^{(1)} \langle T_{\ell m} \rangle$. The latter is also obtainable experimentally from

$$\sum_i (\hat{p} \cdot \hat{x}' + i\hat{p} \cdot \hat{y}')_i \mathcal{D}_{m1}^{\ell *}(\zeta, \psi, 0) [(2\ell+1)/4\pi]^{1/2} (3/\alpha).$$

(Each of these sums must be understood as taken separately over real and imaginary parts of the functions.)

19. This equation is valid only if the a_ℓ amplitude of higher orbital angular momentum can be ignored relative to the amplitude of lower angular momentum (i.e., $c \ll a$ or $d \ll b$).

In the discussion following, the ℓ designating orbital angular momentum has no connection with the ℓ used as a subscript (the rank) of tensor $T_{\ell m}$.

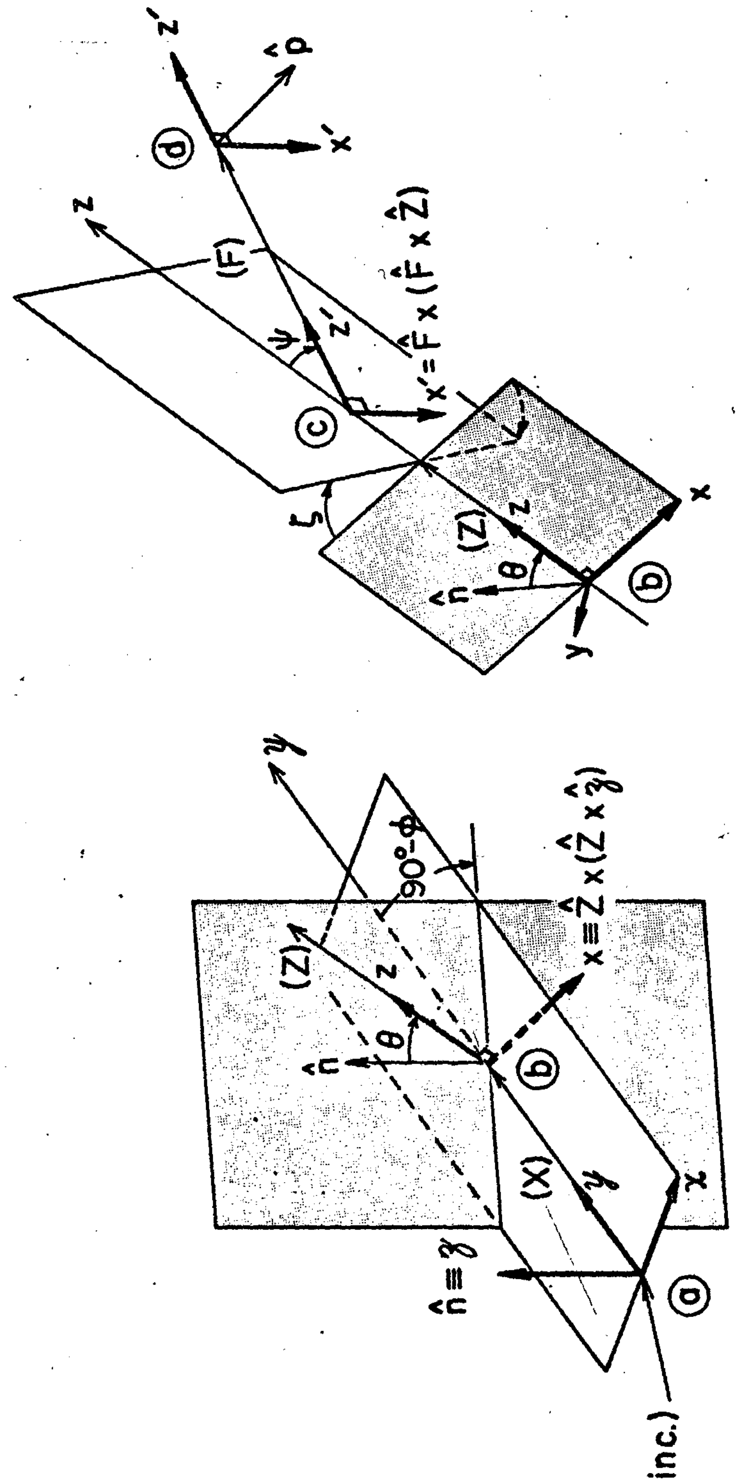
20. See appendix of J. Button-Shafer and D. W. Merrill, "Properties of the Ξ^- Hyperon," Lawrence Radiation Laboratory Report UCRL-11884, December 1964 (unpublished).

21. As no general formalism exists for the treatment of interference or background problems, the experimenter confronted with these problems can at best (a) throw away events in portions of resonance bands showing interference (by using strong-decay symmetry and splitting an X resonance band at $\hat{Z} \cdot \hat{X} = 0$); (b) treat background near resonance separately and compare results; and (c) try to find tests least sensitive to background.

22. This may also be compared with the use of spin- and momentum-space tensors to form invariant terms for scattering matrices (see L. Wolfenstein and J. Ashkin, Phys. Rev. 85, 947 (1952).

FIGURE LEGEND

Fig. 1. The angles θ, ϕ and ψ, ζ describing the directions of "particles" Z and F, respectively, are defined in this figure. The (a) designates the production c.m. frame; (b) designates the rest frame of "particle" X; (c) refers to the rest frame of Z; and (d) refers to the rest of frame of F. The identity of each particle is given by the letter in parentheses near the vector representing its direction. The vector $\hat{n}$ is the normal to the production plane. The use of $\hat{n}$ rather than another vector, say $\hat{X}$, as the z axis is a matter of convenience.[11] The x, y, and z axes may be prescribed in any way from the incident and outgoing (X) directions in the production system. For simplicity of notation, the normalization of vector products is not shown; "x ≡ " means "axis x lies along." All vectors drawn within the boundaries of a plane are to be viewed as lying in that plane. ("Direct" Lorentz transformations, or parallel-axis transfers, are used to move reference axes from one frame to the next.)



MUB-5644

Fig. 1