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Threshold and Asymptotic Behavior of the N/D Equations

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Two important problems involved in obtaining solutions of partial wave dispersion relations (by the N/D method) are having (i) the correct threshold behavior, and (ii) an acceptable high energy behavior. Various physical and numerical approximations have been made to insure (i) and (ii). We numerically investigate the sensitivity of the solutions of the N/D equations to these approximations. For this purpose, we consider $J = 1$ π-π scattering, employing elastic unitarity and assuming that the left hand cut is dominated by the exchange of the ρ resonance. Two significant features we find are: (a) The values of the cutoffs needed to product a resonance are quite sensitive to the input "strength" of the left hand cut, e.g., a change of the input width of the ρ by a factor of two changed the value for a "straight cutoff" to produce a resonance at a given energy by a factor of ten. Due to the results of (a) we wish to emphasize the possible danger in employing a single cutoff in the calculations of SU_3 multiplets. (b) If one introduces a pole on the left hand cut in order to insure the threshold behavior (i), then the ranges in values for the cutoffs (to insure (ii)) for which any resonance occurs are *extremely* narrow. On the other hand, a solution in which the phase shift does not become large is insensitive to the position of this pole.

I. INTRODUCTION

Obtaining solutions of partial wave dispersions relations using the N/D formalism is of current interest. Given a partial wave "generalized potential term" B_l or in other words specifying the discontinuities of the partial wave amplitude A_l in the unphysical region, the N/D formalism (*1–3*) permits one to include the unitarity cut in the physical region and calculate the amplitude A_l by

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solving a *linear* integral equation. Two important difficulties enter into the calculations: (i) Insuring that A_l have the correct threshold behavior. (ii) Obtaining an acceptable high energy or asymptotic behavior.

From general quantum mechanical considerations we know that near threshold, a phase shift δ_l with orbital angular momentum l should behave like

$$\delta_l \underset{k \to 0}{\propto} k^{2l+1}$$

where k is the momentum in the center of mass system. Hence we want to (i) force A_l to have the correct threshold behavior. For certain physical problems, the "obvious" choice for B_l behaves badly at high energy so that the resulting integral equation in the N/D formalism is not of the Fredholm type. Now we want the solution A_l for a given B_l to be unique[1]; and thus we want to (ii) force the integral equation to be of the Fredholm type.

The purpose of this paper is to numerically investigate the sensitivity of the solutions to various approximations which have been made to insure the desirable features (i) and (ii).[2]

We consider numerically two types of cutoffs to insure (ii): a straight cutoff on all the integrals, and a "Regge" type cutoff on B_l. To force (i) we consider l subtractions for the integral equation, or we introduce an lth order pole in the unphysical region. In order to concentrate on a problem with relatively few purely kinematical complications, we discuss the elastic scattering of π mesons. In particular we investigate the $J = 1$ partial wave and assume that the generalized potential B is dominated by the exchange of the $I = 1, J = 1$ ρ resonance.

Section II is devoted to a review of the relevant formalism.[3] The calculations and results are presented in Section III. The two most significant features we find from our calculations are: (a) The values of the cutoffs needed to produce a resonance are quite sensitive to the input "strength" of the left hand cut, e.g., a change of the input width of the ρ by a factor of two changed the value for a "straight cutoff" to produce a resonance at a given energy by a factor of ten. Due to the results of (a) we wish to emphasize the possible danger in employing a single cutoff in the calculations of an SU_3 multiplet. (b) If one introduces a pole on the left hand cut in order to insure the threshold behavior (i), then the ranges in values for the cutoffs (to insure (ii)) for which any resonance occurs are extremely narrow. On the other hand, a solution in which the phase shift does not become large is insensitive to the position of this pole.

[1] For one exceptional case there exists a unique solution of the N/D equations with a non-Fredholm kernel. For details see ref. *4*.

[2] For a formal discussion of existence and uniqueness see ref. *5*.

[3] For a more extensive treatment and references the reader should consult ref. *3*.

II. REVIEW OF PARTIAL WAVE DISPERSION RELATIONS

A. Analytic Properties of the Partial Wave Equations

Consider the system shown in Fig. 1. The usual scalar variables s, t, u:

$$\begin{aligned} s &= (p_1 + p_2)^2, \\ t &= (p_1 - p_3)^2, \\ u &= (p_1 - p_4)^2, \end{aligned} \tag{1}$$

with[4]

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2,$$

are used to denote the 3 processes or "channels"

$$\begin{aligned} s&: a + b \to c + d, \\ t&: a + \bar{c} \to \bar{b} + d, \\ u&: a + \bar{d} \to \bar{b} + c. \end{aligned} \tag{2}$$

which are related by "crossing" or the substitution rule.[5] As we are interested in a study of the sensitivity of the solutions of the N/D equations to various physical assumptions and numerical approximations we shall concentrate on a problem with relatively few purely kinematical complications. We analyze the problem of two spinless bosons of equal mass scattering elastically. As isotopic spin presents no major complications, we specifically discuss the scattering of π mesons on π mesons (so that the processes (2) are all $\pi - \pi$ scattering).

The Mandelstam representation for this scattering in a given isotopic spin state I is[6]

$$\begin{aligned} A^I = \frac{1}{\pi^2} \int ds'\, dt'\, \frac{\rho_{st}^I(s', t')}{(s' - s)(t' - t)} + \frac{1}{\pi^2} \int ds'\, du'\, \frac{\rho_{su}^I(s', u')}{(s' - s)(u' - u)} \\ + \frac{1}{\pi^2} \int dt'\, du'\, \frac{\rho_{tu}^I(t', u')}{(t' - t)(u' - u)}. \end{aligned} \tag{3}$$

The functions appearing in the integrands in (3) are the (real) double spectral discontinuities which, in principle, determine the complete dynamics of the system.

[4] Our units are such that $\hbar = c = m_\pi = 1$.

[5] The "TCP" processes are also linearly related to the same analytic functions. For details consult ref. *3*, p. 11.

[6] A superscript is used to denote isospin. In the amplitude, the first variable is also used to denote the channel whereas in the absorptive parts the channel is denoted by subscripts.

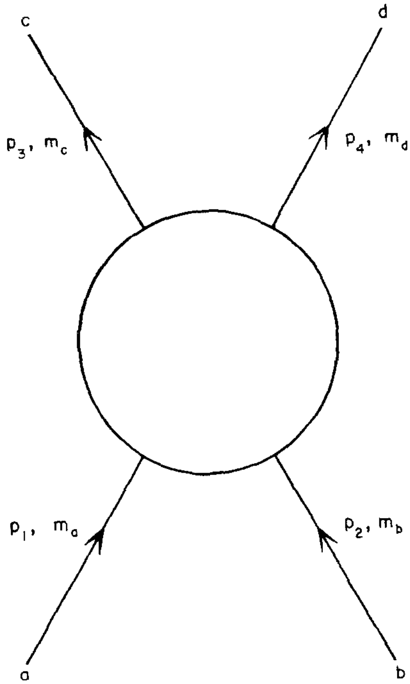

FIG. 1. Two particle scattering process

If we choose a particular channel, say s (where s is the center of mass energy squared and t the invariant momentum transfer), then the relation (3) can be written as

$$A^I(s, t) = \frac{1}{\pi} \int \frac{A_t^I(s, t')\, dt'}{t' - t}\, dt + \frac{1}{\pi} \int \frac{A_u^I(s, u')}{u' - u}\, du' \tag{4}$$

where $A_{t(u)}^I$ is the absorptive part of the amplitude in the $t(u)$ channel. In (3) and (4) we have neglected to write possible subtraction and single integral terms as it is the purpose of the N/D method to determine these from just the knowledge of the double spectral functions alone. Introducing the momentum k and the cosine of the scattering angle z for the s channel in the center of mass system, we have

$$\begin{aligned} s &= 4(k^2 + 1), \\ t &= -2k^2(1 - z), \\ u &= -2k^2(1 + z). \end{aligned} \tag{5}$$

Using (5) we may project out the partial waves from (4):

$$A_l^I(s) = \tfrac{1}{2}\int_{-1}^{1} A^I(s,t)P_l(z)\,dz$$
$$= \frac{1}{\pi}\int A_t^I(s,t')\left(\frac{s-4}{2}\right)^{-1} Q_l\left(1+\frac{2t'}{s-4}\right)dt' \qquad (6)$$
$$+\frac{1}{\pi}\int A_u^I(s,u')\left(\frac{s-4}{2}\right)^{-1} Q_l\left(-1-\frac{2u'}{s-4}\right)du'$$

From (6) we read off the analytic properties of $A_l^I(s)$. Both $A_t^I(s, t')$ and $A_u^I(s, u')$ have a cut along the positive real s axis for $s > 4$, i.e.,

$$A_t^I(s,t') = \frac{1}{\pi}\int \frac{\rho_{st}(s',t')}{s'-s}\,ds' \qquad (7)$$

The functions Q_l introduce a cut along the negative s axis running from 0 to $-\infty$.

Had we considered more complicated kinematics of unequal mass particles, the analytic structure of the partial wave amplitudes would have acquired some complications. We would still have the right hand cut discontinuities discussed above. However, the cut due to the Q_l functions would include detached segments along the real axis and circular cuts in the complex s plane (*6*). If none of the masses is too large compared to the others as well as to masses of possible intermediate particles (which we shall consider later), the right hand cut is disjoint from the cuts due to the Q_l functions (which we shall from now on call the left hand cut). Then there exists a region of the real axis in which $A_l^I(s)$ is analytic which permits analytic continuation between the upper and lower regions of the complex s plane.

B. Determination of the Discontinuities

For s not in the interval $(4, \infty)$, $A_t^I(s, t')$ is an analytic function in s. The function Q_l has a discontinuity (*7*) such that

$$\frac{1}{2i}\left[Q_l\left(1+\frac{2t'}{s-4+i\epsilon}\right) - Q_l\left(1+\frac{2t'}{s-4-i\epsilon}\right)\right]$$
$$= \frac{\pi}{2}P_l\left(1+\frac{2t'}{s-4}\right)\theta(-s-t'-4). \qquad (8)$$

Then the left hand discontinuity depending on $A_t^I(s, t')$ is

$$(s-4)^{-1}\int_{-(s-4)}^{0} A_t^I(s,t')P_l\left(1+\frac{2t'}{s-4}\right)dt'$$
$$= \frac{1}{2}\int_{-1}^{1} A_t^I(s,z')P_l(z')\,dz' \equiv A_{t,l}^I(s), \qquad (9)$$

i.e., the lth partial wave in the s channel of the absorptive part of the t channel amplitude. Utilizing crossing symmetry we have, e.g.,

$$A_t{}^I(s, t) = X_{t,s}^{II'} A_s^{I'}(t, s),$$
$$A_u{}^I(s, u) = X_{u,s}^{II'} A_s^{I'}(u, s), \tag{10}$$

where X is a numerical isotopic spin crossing matrix (3).

To obtain the right hand discontinuities we employ unitarity. In the physically accessible region for scattering in the s channel, i.e., $s > 4$, $A_l{}^I(s)$ has the form

$$A_l{}^I(s) = \frac{\eta_l{}^I(s) \exp(2i\delta_l{}^I(s)) - 1}{2i\rho(s)},$$
$$\rho = \left(\frac{s - 4}{s}\right)^{1/2} \tag{11}$$

where the factor $\eta_l (\equiv e^{-2\delta_l}$ with δ the imaginary part of the phase shift) determines the total inelastic cross section for the lth partial wave σ_R

$$\sigma_{Rl}^I(s) = \pi k^2(2l + 1)(1 - (\eta_l{}^I(s))^2). \tag{12}$$

The discontinuity of $A_l{}^I(s)$ is equal to its imaginary part:

$$\frac{1}{2i}[A_l{}^I(s + i\epsilon) - A_l{}^I(s - i\epsilon)]$$
$$= \left\{\rho(s) \mid A_l{}^I(s) \mid^2 + \frac{1 - (\eta_l{}^I(s))^2}{4\rho(s)}\right\} \theta(s - 4). \tag{13}$$

Hence with "proper[2] asymptotic behavior" we have

$$A_l{}^I(s) = B_l{}^I(s) + \frac{1}{\pi} \int_4^\infty \frac{\operatorname{Im} A_l{}^I(s')}{s' - s} ds' \tag{14}$$

where

$$B_l{}^I(s) \equiv \frac{1}{\pi} \int_{-\infty}^0 \frac{ds'}{s' - s} [A_{t,l}^I(s') + A_{u,l}^I(s')]. \tag{15}$$

Thus in principle if we knew $\eta_l{}^I(s)$ we would have an infinite system of integral equations to determine the amplitude since, e.g., $A_{t,l}^I(s)$ as given by (9) is related to the s channel amplitude by the crossing relations (10).

In practice some approximations are made about the "potential" term $B_l{}^I(s)$. We shall discuss these approximations in Section II, D; for now, we assume that $B_l{}^I(s)$ is known. The inelastic factor $\eta_l{}^I(s)$ must also be approximated. This function may be taken from experiment, or one may approximate inelastic unitarity by considering many channel two-body scatterings, or (as is often done when one is interested in relatively low energies) assume that elastic unitarity holds out to infinity. It is this last approximation that we will make, i.e., we take $\eta(s) = 1$ so that dropping the isotopic spin index we have for (14),

$$A_l(s) = B_l(s) + \frac{1}{\pi}\int_4^\infty \frac{\rho(s')\,|\,A_l(s')\,|^2\,ds'}{s'-s}. \tag{16}$$

C. N/D Equations

It is possible to linearize (16) by the N/D method. Define

$$A_l(s) \equiv N_l(s)/D_l(s) \tag{17}$$

where $N_l(s)$ has cuts along the discontinuities of $B_l(s)$, and $D_l(s)$ has the (elastic) unitarity cut:

$$\begin{aligned}\frac{1}{2i}[D_l(s+i\epsilon) - D_l(s-i\epsilon)] &= N_l(s)\ \mathrm{Im}\ (1/A_l(s))\theta(s-4)\\ &= -\rho(s)N_l(s)\theta(s-4) \qquad (18)\\ \frac{1}{2i}[N(s+i\epsilon) - N(s-i\epsilon)] &= \mathrm{Im}\ (B_l(s))D_l(s).\end{aligned}$$

These discontinuities do not specify N and D completely as we do not know their asymptotic behavior. This ambiguity is related to the possible existence of elementary particles which communicate with the $\pi - \pi$ system (8). The simplest assumption to make is that N and D are sufficiently well behaved that no subtractions are necessary and that a knowledge of $B_l(s)$ determines the amplitude uniquely. We have however a freedom of multiplying both N and D by the same nonzero constant and thus we may normalize D at any convenient point, s_0, to unity (the ratio N/D being independent of s_0). Thus using (18), we can write the coupled dispersion relations

$$N_l(s) = \frac{1}{\pi}\int \frac{\mathrm{Im}\ B_l(s')D_l(s')}{s'-s}\,ds', \tag{19}$$

$$D_l(s) = 1 - \frac{(s-s_0)}{\pi}\int_4^\infty \frac{\rho(s')N_l(s')}{(s'-s)(s'-s_0)}\,ds', \tag{20}$$

where the integral in Eq. (19) for N runs over the discontinuities in $B_l\,(s)$. Pole terms which may appear in (3), (4), or (16) are now automatically taken care of, as they appear as zeroes of $D_l(s)$. Thus the N/D method permits one, in principle, to calculate the positions of bound states from the knowledge of the discontinuities of the amplitudes in the physical regions alone.

From general quantum mechanical principles, it is expected that the threshold behavior of $A_l(s)$, see Eq. (11), will be

$$A_l(s) \underset{s\to 4}{\propto} (s-4)^l. \tag{21}$$

Had we put in the exact discontinuities and inelasticity, this behavior should come out automatically. However, we still want to force the correct behavior (21) even with approximate input information. We do this by writing the dispersion relation for N_l with l subtractions:

$$N_l(s) = \frac{(s-4)^l}{\pi} \int \frac{\operatorname{Im} B_l(s')D_l(s')}{(s'-4)^l(s'-s)}\, ds'. \tag{22}$$

Note that the approximate forms of $B_l(s)$ that we will be dealing with have the correct threshold behavior by themselves.

Substituting Eq. (20) for D_l into (22) we have a linear integral equation for N_l :

$$\begin{aligned} N_l(s) = B_l(s) + \frac{(s-4)^l}{\pi} \int_4^\infty \frac{\rho(s')N_l(s')}{s'-s} \\ \cdot \left[\frac{B_l(s')}{(s'-4)^l} - \frac{B_l(s)}{(s-4)^l}\left(\frac{s-s_0}{s'-s_0}\right)\right] ds'. \end{aligned} \tag{23}$$

It is the major purpose of this article to discuss the solution of this integral equation. The only singularities (23) may have (for $B_l(s)$ having the behavior (21)) come from the infinite ranges of integration. It is this singular behavior that causes most of the difficulties and it is the purpose of this article to discuss various methods which have been employed to overcome it; we require (23) to have a unique solution and thus demand that it be an integral equation of the Fredholm type.

D. Approximations for $B_l(s)$

Several types of approximations have been utilized thus far in approximating $B_l(s)$. In the case of complete ignorance about the singularities on the left, this cut may be replaced by a sequence of poles whose position and residues are adjusted to fit empirical data in the scattering region. With this approximatation, Eq. (23) may be reduced to a system of linear algebraic equations. The resulting amplitudes are of the effective range type (*3*).

Another approximation has been to keep only a few partial waves in the direct channel and even though the partial wave diverges outside a small neighborhood of the physical region it is assumed that a small number of these amplitudes still dominate the crossed channels. One is thus faced with a finite set of coupled integral equations; crossing symmetry is made full use of (*9*).

The approximation we shall consider has been called the single particle exchange or resonance approximation (*10*). It consists of assuming that the "crossed" t and u channels are dominated by a resonance or resonances in particular partial waves. In the language of Feymnan diagrams we consider the exchange of elementary particles in the crossed channels. We then use crossing, Eq. (10), to give the absorptive amplitudes in the direct channel and project out the partial waves to give us $B_l(s)$. The N/D equations simply enforce unitarity in the physical region of the direct channel while leaving the left hand singularities unchanged.

In the $\pi - \pi$ problem—the example we will study in the remainder of this paper—the scattering amplitude is assumed to be dominated by the ρ resonance in the $I = 1, J = 1$ partial wave so that, e.g.,

$$A^1(t, s) \approx \frac{3(t-4)\Gamma P_1[1 + 2s/(t-4)]}{t - m_\rho^2 - i[(t-4)^3/t]^{1/2}\Gamma} . \tag{24}$$

Hence, using (10),

$$A_t^I(s, t) = 3X_{ts}^{I1} \frac{\Gamma^2 P_1[1 + 2s/(t-4)]((t-4)^5/t)^{1/2}}{(t - m_\rho^2)^2 + [(t-4)^3/t]\Gamma^2} . \tag{25}$$

Further, making the narrow width approximation, i.e., $\Gamma \to 0$ we have from (6),

$$B_l^I(s) = X^{I1} \frac{6\Gamma(m_\rho^2 - 4 + 2s)}{s-4} Q_l\left(1 + \frac{2m_\rho^2}{s-4}\right)[1 + (-1)^{l+I}] \tag{26}$$

where

$$X^{I1} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} . \tag{27}$$

As will be discussed below this generalized potential term is of just such a nature that the resulting integral equation (23) taken as it stands is not of the Fredholm type. We shall discuss in detail how various modifications of this discontinuity reflect themselves in the solutions.

III. CALCULATIONS

A. High Energy Behavior

Let us now consider (26) for $l = I = 1$ and use it to generate the kernel of (23). It is an easy exercise to show that the resulting kernel is not L^2 and the integral equation is not of the Fredholm type. One means of modifying (26) to obtain a kernel which yields an integral equation of the Fredholm type is to, in some manner, damp the high energy behavior of (26). The "physical justification" consists of admitting ignorance of the very short range forces, and hoping that the mechanics of an exact theory are such as to actually produce damping. We wish to emphasize that this is at most an intuitive argument since it is quite possible that the exact $B_l(s)$ has a strong oscillatory behavior for large s and (23) may have a unique solution with such a kernel. Any approximate damping is at best an average of what happens in the exact theory. Our calculations will show that the solutions of (23) are not in general insensitive to the cutoff.

A most naive cutoff procedure consists of replacing the upper infinite limit of integration in (23) by a finite one, Λ. The integral equation (23) may now be

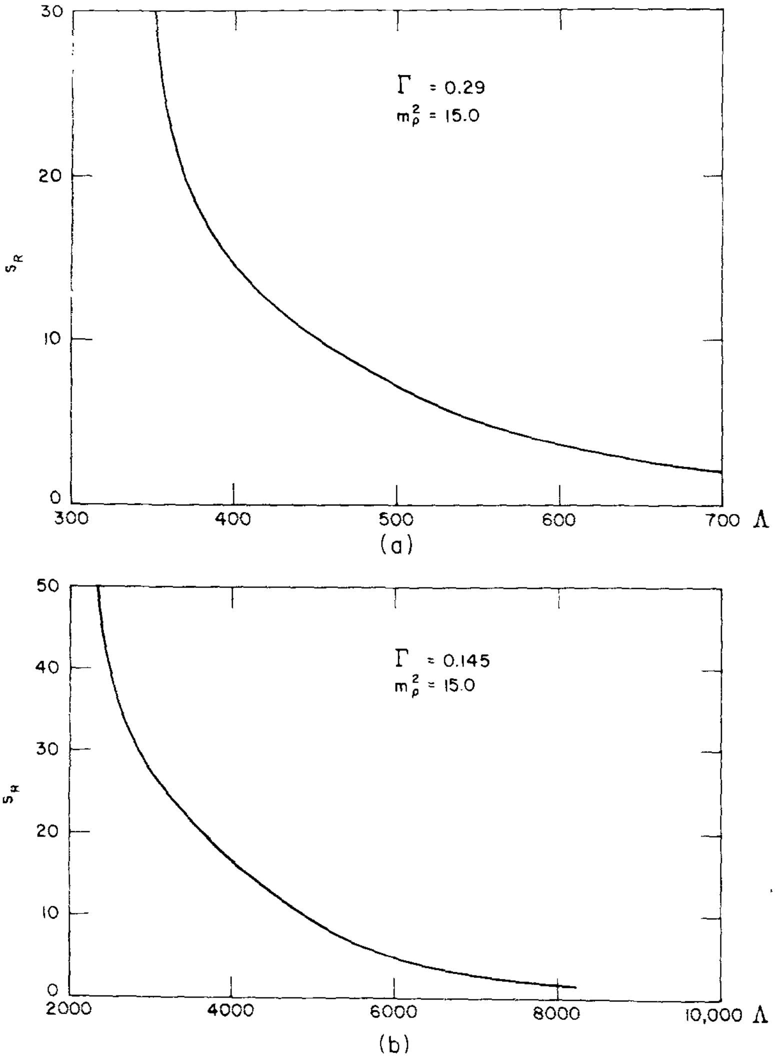

FIG. 2. Plots of s_R (position of the zero of Re $D_{l=1}$ (s) versus the straight cutoff Λ for given input position and width of the exchanged ρ resonance. The correct threshold behavior (21) for A_1 has been forced by making one subtraction at $s = 4$ in the integral equation for N_1.

solved by standard numerical means. We solved (23) by matrix inversion on the Stanford 7090 IBM computer. To show the sensitivity of the solutions as Λ is varied, we plot in Figs 2 and 3 s_R, the position of the zero of the real part of $D_1(s)$, Eq. (20), i.e.

$$\operatorname{Re} D_1(s_R) = 0, \tag{28}$$

as a function of Λ for various positions and strengths of the input ρ force (26). We observe the disturbing feature that the cutoffs needed to produce a resonance at a given position are sensitive to the input "strength" of the left hand cut. For example, from Fig. 3, we see that for an input $\Gamma = 0.29(m_\rho^2 = 29.0)$ we need a

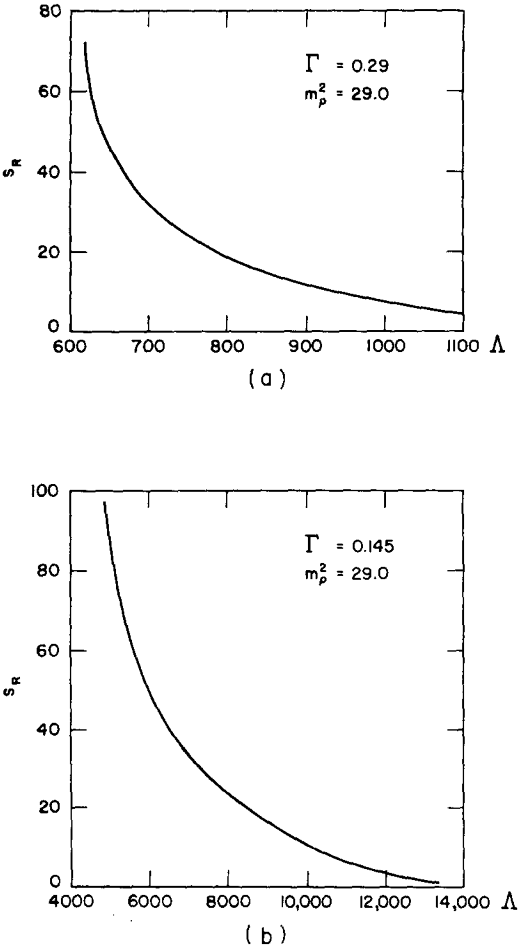

FIG. 3. Same as Fig. 2

$\Gamma = 730$ to get a resonance at $s = 29.0$, whereas for an input $\Gamma = 0.145$ the required $\Lambda(\approx 7400)$ is 10 times larger.

Although the straight cutoff is simplest to apply, it has several bad features. The analytic properties of the resulting amplitudes are mutilated for large s, with at least one possible consequence at small energies. It is found that for certain input parameters, a sought for zero of Re D occurs near the value Λ. As $D(s)$ has a logarithmic branch point at Λ, this function undergoes unreasonable variation over small intervals and this makes the entire procedure somewhat suspect.

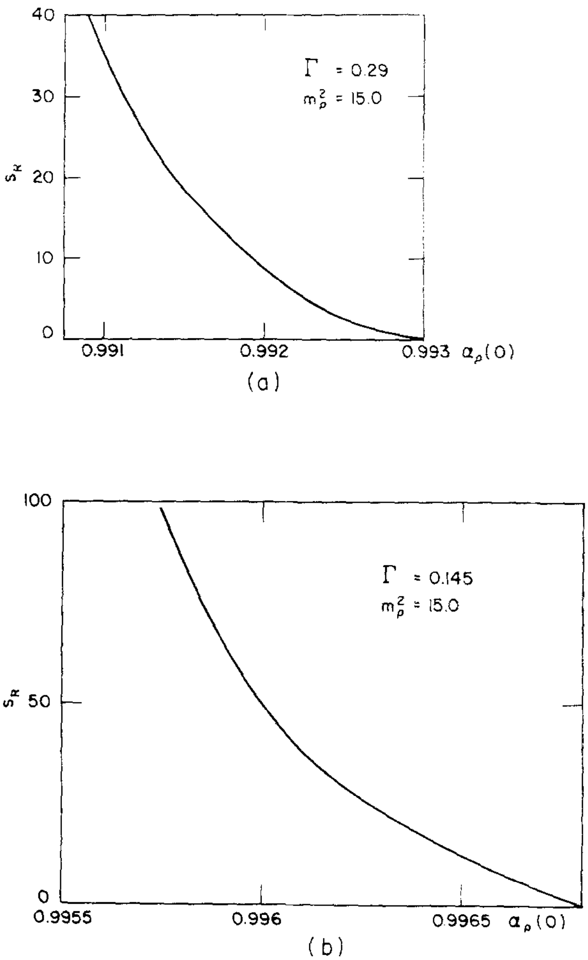

FIG. 4. Plots of s_R versus the Regge cutoff parameter α_ρ (0). Other features are the same as Fig. 2.

A way of avoiding the above difficulties is to introduce a smooth damping function. There exists such a scheme which has considerable physical appeal. One uses an analogy that may exist between potential and relativistic scattering theories, and postulates that resonances lie on Regge trajectories (*11*, *12*). For our case this amounts to replacing (24) by

$$A^1(t, s) = \frac{b_\rho(t)}{\sin \pi\alpha_\rho(t)} \frac{1}{2}\left[P_{\alpha_\rho(t)}\left(-1 - \frac{2s}{t-4}\right) - P_{\alpha_\rho(t)}\left(1 + \frac{2s}{t-4}\right)\right] \quad (29)$$

where b_ρ and α_ρ are the residue and position of the ρ meson Regge pole. We further approximate (29) in such a way as to make it correspond as closely as

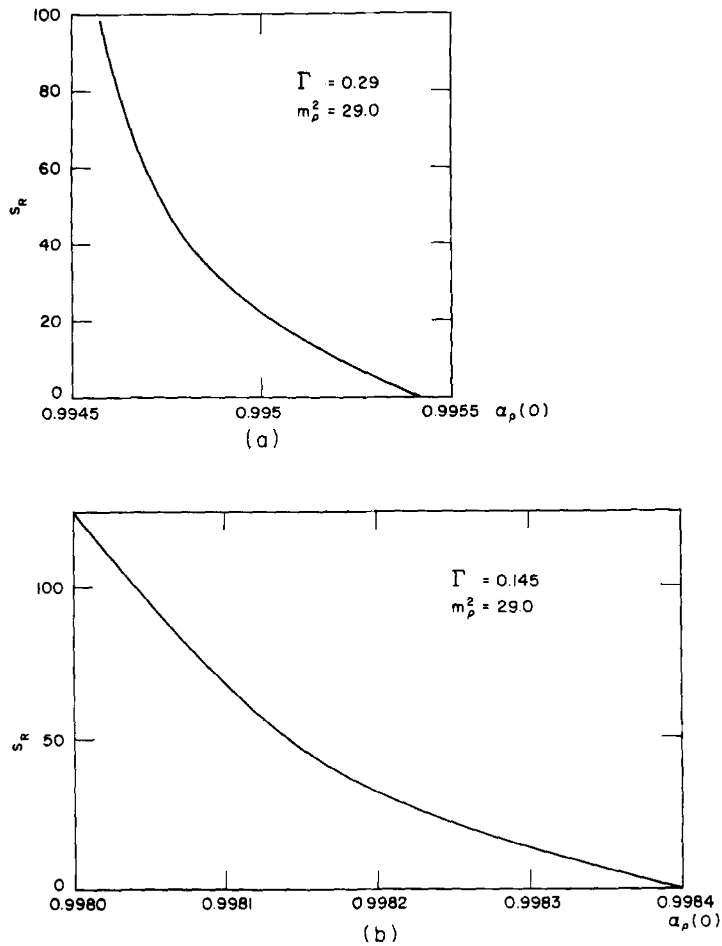

FIG. 5. Same as Fig. 4

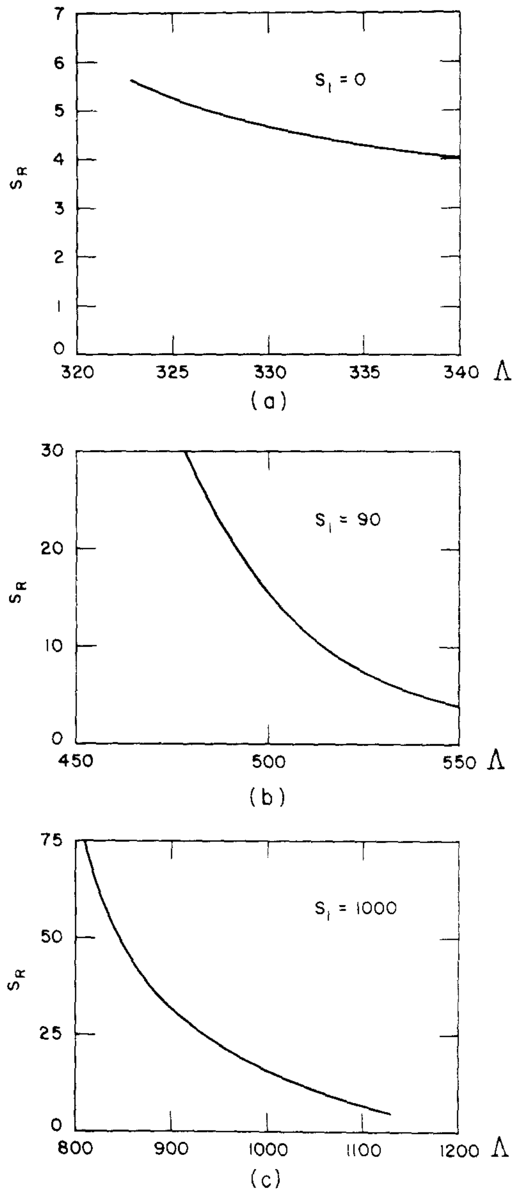

FIG. 6. Plots of s_R versus Λ with input parameters $\Gamma = 0.2$ and $m_\rho^2 = 29.0$ for various positions s_1 of the extra pole, Eq. (31), which was introduced (instead of the subtraction in N_1) in order to insure the correct threshold behavior.

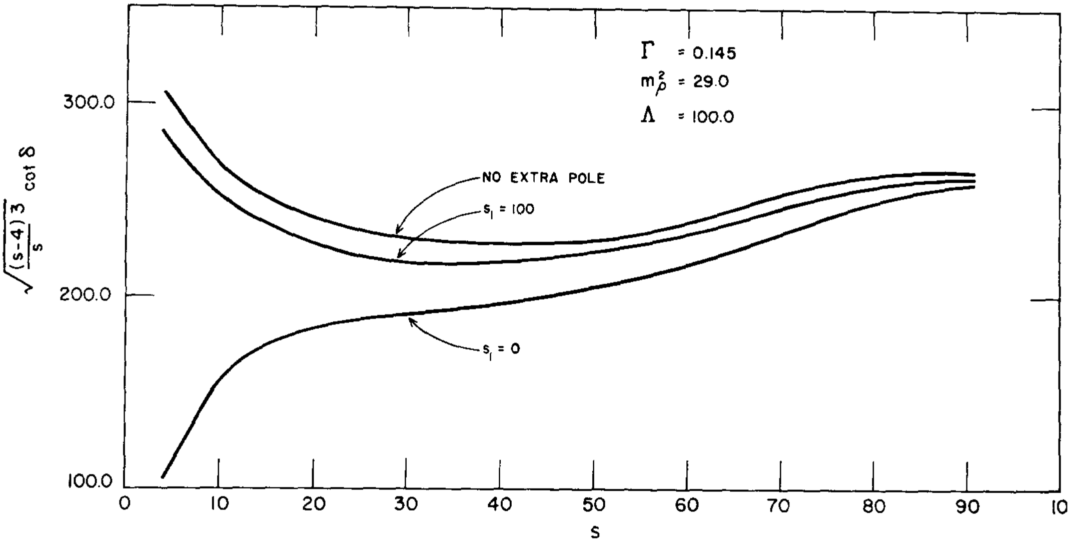

FIG. 7. Plots of $[(s-4)^3/s]^{1/2}\cot\delta$ versus s with input parameters $\Gamma = 0.145$, $m_\rho^2 = 29.0$ and $\Lambda = 100.0$ for "extra" pole positions $s_1 = 0$ and 100, and the case of no extra pole but a subtraction in N at $s = 4$. This graph demonstrates the insensitivity of the solution of s_1 for a weak solution, i.e., one for which the phase shift never becomes large.

possible to (26) (*13–15*). (The details of this approximation are given in ref. *15*.) The resulting $B_l^1(s)$ for odd integer l,

$$B_l^1(s) = \frac{6\Gamma}{s-4}(m_\rho^2 - 4 + 2s)Q_l\left(1 + \frac{2m_\rho^2}{s-4}\right)\left(\frac{s}{4}\right)^{\alpha_\rho(0)-1}, \tag{30}$$

differs from (26) by the factor $(s/4)^{\alpha_\rho(0)-1}$. As long as $\alpha_\rho(0) < 1$, the resulting equations for $l = 1$ are of the Fredholm type. An investigation of the sensitivity of the position of the zero of $\operatorname{Re} D(s)$, s_R, to $\alpha_\rho(0)$ is shown in Figs. 4 and 5.

B. Threshold Behavior

As l increases, it may easily be seen that the kernel of (23) becomes more and more singular, and the Regge type cutoff (or any smooth cutoff for (26)) is ineffective for $l \geqq 2$. This behavior is due to the fact that we have insisted on making l subtractions of N_l in order to insure the proper threshold behavior (21). A scheme to bypass this difficulty has been suggested which consists of introducing extra poles in the amplitude in the unphysical region. One introduces a function[7]

$$\tilde{A}_l(s) = \left(\frac{s+s_1}{s-4}\right)^l A_l(s) \tag{31}$$

[7] A. Scotti and D. Y. Wong (*14*) introduce a pole of order $l-1$, and make one subtraction in N at threshold.

and writes $\tilde{A}_l(s)$ as N_l/D_l. The equations (19) and (23) for D_l and N_l are modified simply by replacing ρ by ρ_l,

$$\rho_l = \left(\frac{s-4}{s+s_1}\right)^l \rho, \tag{32}$$

and not performing the threshold subtractions in N_l. We present, in Fig. 6, the results for various values of s_1. It should be noted that the region of cutoffs for which a resonance occurs is highly reduced and is very sensitive to the value s_1.

It is worthwhile to look at the situation in a case of weak coupling, i.e., in a case of no resonances or bound states (for any value of s_1). One might expect the sensitivity to s_1 to be small. Indeed, as may be seen from Fig. 7 where we show the variation of the phase shift with s_1, keeping other parameters fixed, the dependence is small.

Although the calculations of resonances are sensitive to almost all parameters that may enter, we wish to stress that the solutions are not unstable, i.e., small variations of the parameters lead to small variations of the solutions. Specifically, the parameter one usually knows best is the mass of the exchanged particle. Slight variations in this mass produce correspondingly small variations in the output, as illustrated in Fig. 8.

C. Numerical Approximations

As a fully numerical solution of the integral equation (23) is often time consuming, certain mathematical approximations are frequently employed. The most common is the so-called determinental (*16*) method which consists of approximating N_l by B_l and solving for D_l by quadrature. One striking disadvantage is that for the multichannel case, the resulting amplitude is not symmetric (and

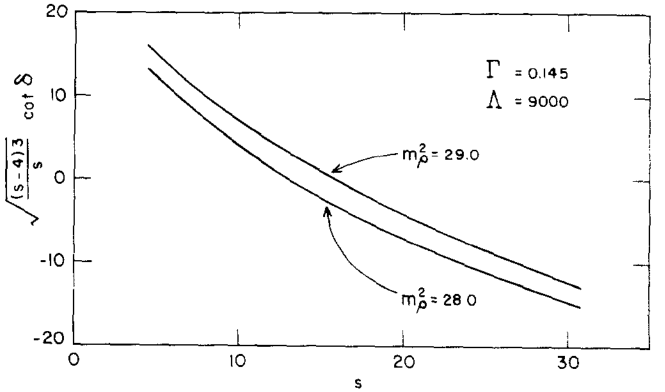

FIG. 8. Plots of $[(s-4)^3/s]^{1/2}$ cot δ versus s with input parameters $\Gamma = 0.145$ and $\Lambda = 9000$ for mass values $m_\rho^2 = 28.0$ and 29.0. This graph demonstrates the stability of the solution to small variations in m_ρ^2.

thus violates time reversal invariance). Even in the one channel case, there is a strong dependence on the choice of the subtraction point s_0 for normalizing D to unity. This dependence is illustrated in Fig. 9. A different approximation has been proposed (*17*) which does not have this subtraction point dependence and is symmetric in the multichannel case. (See ref. *17* for an investigation of this

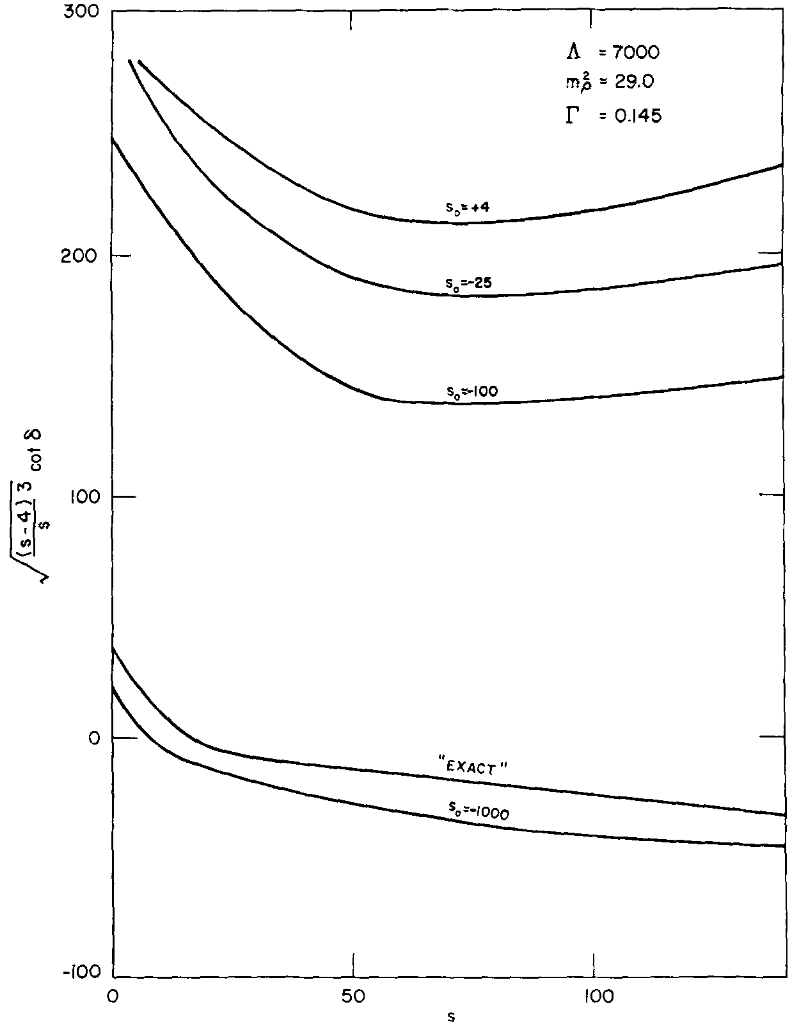

FIG. 9. Plots of $[(s-4)^3/s]^{1/2}\cot\delta$ versus s for the approximate "determinental method" solutions ($N_l = B_l$) for various values of s_0, the subtraction point in D. The "exact" solution is also shown for comparison. The input parameters are $\Lambda = 7000$, $m_\rho^2 = 29.0$, and $\Gamma = 0.145$.

approximation.) We would like to emphasize that although the various approximate solutions to (23) are much faster to use than numerically solving the integral equation none is a reasonable substitute when the actual solution yields a resonance or bound state; this statement becomes stronger and stronger as one deals with more complicated B_l than (26). On the other hand, the determinental method has the decided advantage that in situations (e.g., a sum of single particle exchanges) in which B_l has the correct threshold behavior (21), the partial wave amplitude A_l automatically obeys (21).

D. Discussion

In summary we make the following observations on the sensitivity of the solutions to the N/D equations to the approximations described above in Sections III, A, B, and C. The values of cutoffs needed to produce a resonance are quite sensitive to the input strength of the left hand cut. We saw, e.g., from Fig. 3 that for an input $\Gamma = 0.29$ (and $m_\rho^2 = 29.0$) we needed a straight cutoff $\Lambda = 730$ to get a resonance at $s = 29.0$ whereas for an input $\Gamma = 0.145$ the required cutoff was $\Lambda = 7400$. This result has bearing on a number of different types of problems, e.g., multichannel channel calculations, calculations of SU_3 multiplets, and the N, N^* reciprocal bootstrap calculations. In each of these problems there are a number of cutoffs required; we conclude from the above sensitivity, that it may be dangerous to employ a single cutoff. On the positive side, we observe from Fig. 3 that there is a fairly large region of Λ values for which a resonance can occur. The more physically motivated Regge type cutoff (or any smooth cutoff) has the disadvantage that the threshold behavior (21) for the partial wave amplitude cannot be forced for $l \geqq 2$ except by introducing extra parameters. We see from Fig. 6 that the procedure of introducing an extra pole in the unphysical region to force the behavior (21) greatly increased the sensitivity of the solution to the cutoff parameter. However for a weak solution, i.e., one for which the phase shift never becomes large the extra pole procedure is a reasonable way to insure (21): as seen in Fig. 7, the solution is insensitive to the pole position s_1. Although the calculations of strong or resonant solutions are sensitive to almost all the input parameters, we find that the solutions are not unstable, i.e., small variations of the parameters lead to small variations of the solutions (see, e.g., Fig. 8).

We emphasize that approximate solutions of the integral equation (23) while quite time saving are not very good substitutes for numerically solving the Fredholm equation (see, e.g., the sensitivity of the determinental method to the subtraction point s_0 in D) in the case of *strong solutions*: the more complicated B_l one uses, the stronger the statement becomes.

Finally, it seems worthwhile to make a few qualitative remarks concerning how resonances and bound states occur and how they vary as a function of the coupling constant. We have in mind the situation of a simple "attractive" left hand cut and a partial wave $l \geqq 1$ (no resonance can occur without some sort of longer range repulsion). If we plot the real part of the D function as s varies, we observe that (for the single channel case) it starts positive for large negative s, possibly crosses the zero axis producing a bound state or resonance, reaches a minimum and turns back up, crossing the real axis with a wrong slope to produce a resonance. As the coupling constant is decreased, the first crossing of the axis occurs further and further to the right, and its minimum value gets less and less negative. At a critical coupling constant the minimum occurs on the real axis, and for values of the coupling constant smaller than the critical one, no resonance will appear. We have found it as an empirical fact that the position of the minimum of real part of D is a constant over very large variations of the coupling constant. This fact may be useful as a guide to proper choices of coupling constants to produce desired resonances once a bracketing has been obtained.

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REFERENCES

1. G. F. CHEW AND S. MANDELSTAM, *Phys. Rev.* **119,** 774 (1960).
2. J. L. URETSKY, *Phys. Rev.* **123,** 1459 (1961).
3. G. F. CHEW, "S-Matrix Theory of Strong Interactions." Benjamin, New York, 1962.
4. G. F. CHEW, *Phys. Rev.* **130,** 1264 (1963).
5. G. FRYE AND R. L. WARNOCK, *Phys. Rev.* **130,** 478 (1963).
6. S. W. MCDOWELL, *Phys. Rev.* **116,** 774 (1960).
7. P. M. MORSE AND H. FESHBACH, "Methods of Theoretical Physics," p. 652. McGraw-Hill, New York, 1953.
8. L. CASTILLEJO, R. H. DALITZ, AND F. J. DYSON, *Phys. Rev.* **101,** 453 (1956).
9. G. F. CHEW, S. MANDELSTAM, AND H. P. NOYES, *Phys. Rev.* **119,** 478 (1960).
10. F. ZACHARIASSEN, *Phys. Rev. Letters* **7,** 112 (1961); *ibid.* **7,** 268 (1961).
11. T. REGGE, *Nuovo Cimento* **14,** 951 (1959); *ibid.* **18,** 947 (1960).
12. G. F. CHEW AND S. FRAUTSCHI, *Phys. Rev. Letters* **5,** 580 (1960).
13. D. Y. WONG, *Phys. Rev.* **126,** 1220 (1962).
14. A. SCOTTI AND D. Y. WONG, *Phys. Rev. Letters* **10,** 142 (1963).
15. M. BANDER AND G. L. SHAW, *Phys. Rev.* **135,** B267 (1964).
16. M. BAKER, *Ann. Phys.* (*N.Y.*) **4,** 271 (1958).
17. G. L. SHAW, *Phys. Rev. Letters* **12,** 345 (1964).