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Wong's equations and the small x effective action in QCD

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Abstract

We propose a new form for the small x effective action in QCD. This form of the effective action is motivated by Wong's equations for classical, colored particles in non–Abelian background fields. We show that the BFKL equation, which sums leading logarithms in x, is efficiently reproduced with this form of the action. We argue that this form of the action may be particularly useful in computing next-to- leading-order results in QCD at small x.

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1   Introduction

One of the more interesting open questions in QCD is the behavior of cross–sections at very high energies [1]. In the last decade, a kinematic window has opened up at colliders where $Q^2 \gg \Lambda_{QCD}^2$ but $x = Q^2/s \ll 1$. The physics in this regime is non–perturbative because the field strengths at small x are large. However, it is also weak coupling physics since $\alpha_S(Q^2) \ll 1$. Further, since the density of partons is large at small x, classical field methods are applicable [2].

An effective field theory approach can be used to study the physics of small x modes in QCD [3, 4, 5] [1] . The small x effective action is obtained by successively integrating out the more static modes at larger values of x. The measure for this action is represented by a weight corresponding to the color charge density of the higher x modes. As one integrates out higher x modes, the form of the action is maintained, while the weight satisfies a Wilsonian non–linear renormalization group (RG) equation [5]. If the parton density is not too large, the RG equation can be linearized, and the resulting equation is the well known BFKL equation. The BFKL equation is a renormalization group equation that sums the leading logarithms in $\alpha_S \ln(1/x)$ [8]. In the double log limit of small x and large Q^2, the Wilson RG can be simplified, and one obtains a series in inverse powers of Q^2, where the leading term is the small x DGLAP equation [9] and the first sub–leading term agrees with the expression derived by Gribov, Levin and Ryskin [10], and by Mueller and Qiu [11].

The effective action approach therefore reproduces the standard linear evolution equations of perturbative QCD in the limit of low parton densities. The truly interesting and unknown regime however is the non–linear regime of high parton densities where one might hope to predict novel phenomena [12, 13, 14]. What the correct effective action is in the high density regime should therefore be a matter of some interest.

In this paper, we will discuss an alternative gauge invariant form to the gauge invariant action discussed in Ref. [5]. The motivation for this form of the effective action came from

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[1] For alternative approaches, see for example, Ref. [6] and references therein, and Ref. [7] and references therein. These will not be discussed in this paper.

our recent work in formulating a many body world line formalism for the one loop effective action in QCD [15]. Briefly, the difference between the two actions is in the term describing the coupling of the small x gauge field modes to the large x modes represented by a color charge density ρ. In the work of Jalilian–Marian, Kovner, Leonidov, and Weigert (JKLW), this term is expressed as

$$S_{int}^{JKLW} \sim \mathrm{Tr}\left(\rho\, W_{\infty,-\infty}\right)\,,$$

where W is an adjoint matrix corresponding to a path ordered exponential of the gauge field A^- in the light cone direction x^+. We propose instead that this term be

$$S_{int} \sim \mathrm{Tr}\left(\rho\, \ln W_{\infty,-\infty}\right)\,,$$

replacing $W \to \ln W$ in the effective action.

The earlier form of S_{int} was chosen primarily because it is a gauge invariant generalization of the coupling between hard and soft modes. The motivation for the latter form comes from the background field method and the eikonal approximation. The one loop effective action, in the background field method, can be expressed as $\ln[\det(D^2)] \to \mathrm{Tr}\ln[D^2]$, where D is the usual covariant derivative. If, for instance, one integrated out hard fermions in the soft background gauge field, the eigenvalues of the determinant would correspond to solutions of the Dirac equation in the eikonal approximation. These correspond to path ordered phases of the component of the soft gauge field, conjugate to the hard current, in the fundamental representation [16]. Similarly, performing an eikonal separation of hard and soft gauge fields, one obtains path ordered exponentials (in the adjoint representation) of the soft gauge fields (see, for example, appendix B of the first paper in Ref. [5]). Since the effective action is the logarithm of the determinant, one can thus anticipate the appearance of the logarithm of the path ordered phase in the effective action. This form of the effective action is also gauge invariant. We will show later that the $\ln(W)$ action has the nice feature that one can derive the BFKL equation from it efficiently–certain terms that one needs to argue to be zero in the W form of the effective action are absent in the $\ln(W)$ action.

The subsequent discussion is organized as follows. In section 2, we will discuss the form of the small x effective action discussed in Ref. [5]. In section 3, we will discuss Wong's equations and motivate an alternative form for the small x effective action. We will show that the form of the action that we propose is also consistent with Wong's equations and that the two different currents arising from the two actions correspond to different boundary conditions for solving Wong's equations. In section 4, we will show that our form of the effective action also reproduces the BFKL equation. We end this paper with a brief summary in section 5. Some technical details are contained in three appendices.

2 The Small x Effective Action

In this section, we will review the effective action and Wilson renormalization group approach to small x QCD as developed in Refs. [3, 4, 5]. We refer the reader to these papers for more details.

We start with the following action [5] which is the gauge invariant generalization of McLerran-Venugopalan effective action first proposed in [3]. In the infinite momentum frame, and in Light Cone gauge $A^+ = 0$, one can write

$$
\begin{aligned}
S & = -\frac{1}{4} \int d^4x G_a^{\mu\nu} G_{\mu\nu}^a + i \int d^2x_\perp F[\rho^a(x_\perp)] \\
& + \frac{i}{N_c} \int d^2x_\perp dx^- \delta(x^-) \operatorname{Tr}\left(\rho(x_\perp) W_{\infty,-\infty}[A^-](x^-, x_\perp)\right),
\end{aligned}
\tag{1}
$$

where W is the Wilson line in the adjoint representation along the x^+ axis

$$
W_{\infty,-\infty}[A^-](x^-, x_\perp) = \hat{P} \exp\left[ig \int_{-\infty}^{\infty} dx^+ A_a^-(x^+, x^-, x_\perp) T_a\right].
\tag{2}
$$

and the label $\hat{P}$ denotes the path-ordered exponential.

Taking the saddle point of the effective action, we obtain the Yang–Mills equations

$$
D_\mu G_a^{\mu\nu} = \delta^{\nu+} J_a^+,
\tag{3}
$$

with the current

$$
J_a^+(x) = \frac{g}{N_c}\delta(x^-)\operatorname{Tr}\left[T_a W_{x^+,-\infty}[A^-]\rho(x_\perp)W_{\infty,x^+}[A^-]\right]
\tag{4}
$$

satisfying the boundary condition

$$J_a^+(x^+ = -\infty) = \frac{g}{N_c}\delta(x^-)\text{Tr}\left[T_a\rho(x_\perp)W_{\infty,-\infty}[A^-]\right] \tag{5}$$

The first term in the expansion of the Wilson line in the action is

$$S_{\text{int}} = -g\int d^4x\, A^-\rho(x_\perp)\delta(x^-) \tag{6}$$

used in [3].

To derive the general evolution equation, one first solves the classical equations of motion, computes quantum fluctuations in the background of the classical field (semi-classical approximation), and separates these fluctuations according to their longitudinal momentum as

$$A_\mu^a(x) = b_\mu^a(x) + \delta A_\mu^a(x) + a_\mu^a(x), \tag{7}$$

where $b_\mu^a(x)$ is the solution of the classical equations of motion, $\delta A_\mu^a(x)$ is the fluctuation field containing longitudinal momentum modes k^+ that are constrained to be $p^+ < k^+ < P^+$. The upper cutoff P^+ is the longitudinal momentum of the fast moving charges while the lower cutoff p^+ is the momentum scale of the soft fluctuations. These cut-offs are chosen to be such that $\alpha_S\ln(P^+/p^+) \ll 1$ since quantum fluctuations give rise to such logarithms [4]. This constraint thus requires that the fluctuations with momentum modes $p^+ < k^+ < P^+$ are small, and can therefore be integrated out to obtain the effective action for the soft (in longitudinal momenta alone!) fields a^μ. This procedure can be iterated as one goes to smaller x leading to a Wilsonian RG equation [5].

The physics underlying this procedure is simple. One starts with some initial color charge density at large x represented by ρ. In order to compute a quantity with this action, one averages over all color configurations represented by the statistical weight

$$Z = \exp\{-F[\rho]\}\,.$$

We then integrate out the hard fluctuations with the constraint discussed above. This changes the color charge density and the statistical weight for their configurations. The

soft fluctuations, with logarithmic accuracy, "see" the induced charge density as a part of the color charge density to which they are coupled. As one goes to smaller and smaller x (longer and longer wavelength gluons) one correspondingly includes more of the hard fluctuations in the color charge density. One obtains the following renormalization group equation for the change of the statistical weight Z with x [5]:

$$\frac{dZ}{d\ln(1/x)} = \alpha_S \left[\frac{1}{2} \frac{\delta^2}{\delta\rho_\mu \delta\rho_\nu} \left(Z\chi_{\mu\nu} \right) - \frac{\delta}{\delta\rho_\mu} \left(Z\sigma_\mu \right) \right] , \tag{8}$$

where $\sigma[\rho]$ and $\chi[\rho]$ are respectively one and two point functions obtained by integrating over δA for fixed ρ. The one point function σ includes the virtual corrections to $F[\rho]$ while the two point function χ includes the real contributions to $F[\rho]$. Both of these can be computed explicitly from the small fluctuations propagator in the classical background field. In the weak field limit, the functions σ and χ simplify, and the resulting renormalization group equation is the BFKL equation.

In the following section, we will interpret the color charge density ρ of the hard (large k^+) modes as the density of classical color charges moving in the field of the soft modes. Such an interpretation arises naturally when one computes the one loop effective action in QCD for soft modes using the background field method [15, 18, 19]. One expects therefore that these classical charges must satisfy Wong's equation for the motion of color charges in a non–Abelian background field. These equations are discussed below where a new form of the effective action is proposed.

3 Wong's equations and an alternative effective action

In Ref. [15], we developed a many body formalism for the one loop effective action in QCD. We employed the world line formalism [18, 20] to re–write the path ordered exponential as a quantum mechanical path integral over world lines. The equations of motion for the corresponding point particle Lagrangian satisfies Wong's equations for the motion of a classical charged particle in a non–Abelian background field [17]. These are

$$p^\mu = m\frac{dx^\mu}{d\tau} = mv^\mu \tag{9}$$

$$\frac{dp^{\mu}}{d\tau} = v_{\nu} Q^{a} G_{a}^{\mu\nu} \tag{10}$$

$$D^{\nu} G_{\nu\mu} = j_{\mu} \tag{11}$$

where

$$j_{\mu}(x) = \int d\tau\, Q(\tau)\, v_{\mu}(\tau)\, \delta^{4}\left[x - z(\tau)\right]. \tag{12}$$

and

$$\dot{Q} = -ig\left[Q, v_{\mu} A^{\mu}\right] \tag{13}$$

The generalization to a system of particles is straightforward.

Without explicitly going over to the world line approach, one can write down the following many-body classical action

$$S_{\text{Wong}} = -\frac{1}{4} \int d^{4}x\, G_{\mu\nu}^{a} G_{a}^{\mu\nu} - \sum_{I=1}^{K} \int d\tau\, m_{0}^{I}\, \sqrt{v_{\mu}^{I} v_{I}^{\mu}} + \frac{i}{N_{c}} \sum_{I=1}^{K} \text{Tr}\left\{Q_{I} \ln W_{I}\right\}, \tag{14}$$

where K is the number of Wong's particles and I is the particle label. Also

$$W_{I} = \hat{P} \exp\left(ig \int_{-\infty}^{\infty} d\tau\, v_{\mu}^{I} A_{a}^{\mu}(x_{I}^{\mu}(\tau)) T_{a}^{I}\right). \tag{15}$$

This action is gauge invariant under gauge transforms U that satisfy the constraint $U(\infty) = U(-\infty)$. We will define "$\ln W$" shortly. As shown in appendix A, the Wong equations in Eq. (11) can be derived from Eq. (14) above.

In an infinite momentum frame (relevant for the small x problem), the momenta of the particles are not dynamical. They are static light cone sources–$v^{\mu} = \delta^{\mu+}$. The kinetic part of the action in S_{Wong} therefore drops out to yield

$$S_{\text{Wong}} = -\frac{1}{4} \int d^{4}x\, G_{\mu\nu}^{a} G_{a}^{\mu\nu} + \frac{i}{N_{c}} \sum_{I=1}^{K} \text{Tr}\left\{Q_{I} \ln W_{I}\right\}. \tag{16}$$

We assume now that the initial $x_{I}^{-} = 0$ is the same for all particles. In the infinite momentum frame, $P^{+} \to \infty$, this assumption is justified because the particles can be

viewed as being confined to a Lorentz contracted sheet in the transverse plane of width $1/P^+$. This implies that the particles can be labeled using their transverse positions $x_\perp^I$ only. Using

$$\rho_a(x_\perp) = \sum_{I=1}^{K} \delta(x_\perp - x_\perp^I) Q_a^I, \tag{17}$$

one can assume $\rho^a(x_\perp)$ to be continuous (and large). One can therefore make an educated guess that the coarse grained effective action of the wee parton modes will be

$$S_{\ln W} = -\frac{1}{4} \int d^4x\, G_{\mu\nu}^a G_a^{\mu\nu} + \frac{i}{N_c} \int d^2x_\perp \, \mathrm{Tr}\left\{ \rho(x_\perp)\, \ln W(x_\perp) \right\} \tag{18}$$

where now

$$W(x_\perp) = \hat{P} \exp\left(ig \int_{-\infty}^{\infty} dx^+ \, A_a^-(x^+, 0, x_\perp) \, T_a \right). \tag{19}$$

Just as in Eq. (1), the action $S_{\ln W}$ should contain an identical functional $F[\rho]$ representing the likelihood of different ρ configurations. This term will only be implicit in what follows since it is not relevant to the concerns of this paper.

We will now show explicitly that the charge obtained from the action $S_{\ln W}$ is Hermitean and traceless, and therefore an element of the Lie algebra.

We first define the log of an operator as the power series

$$\ln W = \ln(1 - (1 - W)) \equiv -\sum_{k=1}^{\infty} \frac{1}{k}(1 - W)^k. \tag{20}$$

Taking the functional derivative of $\ln W$ with respect to A gives

$$\frac{\delta}{\delta A_\mu^a} \ln W = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{s=0}^{k-1} (1 - W)^s \frac{\delta W}{\delta A_\mu^a} (1 - W)^{k-s-1} \tag{21}$$

After some straightforward algebra, this can be written as

$$\frac{\delta}{\delta A_\mu^a} \ln W = \int_0^1 d\lambda \, \frac{1}{1 - (1 - W)\lambda} \frac{\delta W}{\delta A_\mu^a} \frac{1}{1 - (1 - W)\lambda} \tag{22}$$

Then from the relation

$$J_a^\mu = -\frac{i}{N_c} \mathrm{Tr}\left(\rho \frac{\delta}{\delta A_\mu^a} \ln W \right), \tag{23}$$

8

we find that the color charge is given by

$$
\begin{aligned}
Q(x^+) &= \int d^3x \, J^+(x) \\
&= \int d^3x \, W(x^+, -\infty) \left[\int_0^1 d\lambda \frac{1}{B(\lambda)} \rho \frac{1}{B(\lambda)} \right] W(\infty, x^+)
\end{aligned}
\tag{24}
$$

where we used the shorthand

$$
B(\lambda) \equiv 1 - (1 - W)\lambda
\tag{25}
$$

and

$$
W(x_f^+, x_i^+) = \hat{P} \exp \left(ig \int_{x_i^+}^{x_f^+} dx^+ \, A_a^-(x^+, x^-, x_\perp) T_a \right)
\tag{26}
$$

We also defined $W \equiv W(\infty, -\infty)$.

It is easy to check that the current density J^+ satisfies

$$
\frac{\partial J^+}{\partial x^+} = -ig \left[J^+, A^- \right],
\tag{27}
$$

and hence is a solution of the Wong's equation with a "boundary" condition given by

$$
J^+(x^+ = -\infty) = \int_0^1 d\lambda \, B^{-1}(\lambda) \, \rho \, B^{-1}(\lambda) W
\tag{28}
$$

Note that $1/B(\lambda) = B^{-1}(\lambda)$.

To confirm that $J^+(x)$ is an element of the Lie-Algebra, first consider the trace. We have

$$
\mathrm{Tr} \left(J^+(x) \right) = \rho^b \, \mathrm{Tr} \left(W \left[\int_0^1 d\lambda \left(B^{-1}(\lambda) \right)^2 T_b \right] \right),
\tag{29}
$$

since W and B commute. One can show that

$$
\frac{d}{d\lambda} B^{-1}(\lambda) = (-1) \left(B^{-1}(\lambda) \right)^2 (W - 1).
\tag{30}
$$

Consequently,

$$
\begin{aligned}
\mathrm{Tr} \left(J^+(x) \right) &= \rho^b \mathrm{Tr} \left(W \left[\int_0^1 d\lambda \, (B^{-1}(\lambda))^2 \, T_b \right] \right) \\
&= \rho^b \, \mathrm{Tr} \left(W \, (1 - W)^{-1} \left(B^{-1}(1) - B^{-1}(0) \right) T_b \right) \\
&= \rho^b \, \mathrm{Tr} \left(W(1 - W)^{-1}(W^{-1} - 1) \, T_b \right) \\
&= \rho^b \, \mathrm{Tr} \left(T_b \right) = 0.
\end{aligned}
\tag{31}
$$

We shall now show that J is also Hermitean. Consider

$$
\begin{aligned}
(J^+(x))^\dagger &= \rho^b \left(W(x^+, -\infty) \left[\int_0^1 d\lambda\, B^{-1}(\lambda)\, T_b\, B^{-1}(\lambda) \right] W(\infty, x^+) \right)^\dagger \\
&= \rho^b\, W(x^+, \infty) \left[\int_0^1 d\lambda\, (B^{-1}(\lambda))^\dagger\, T_b\, (B^{-1}(\lambda))^\dagger \right] W(-\infty, x^+) \\
&= \rho^b\, W(x^+, -\infty) \left[\int_0^1 d\lambda\, W^\dagger\, (B^{-1}(\lambda))^\dagger\, T_b\, W^\dagger\, (B^{-1}(\lambda))^\dagger \right] W(\infty, x^+). \quad (32)
\end{aligned}
$$

Let us now focus on the term in the square brackets. Since

$$
W^\dagger = W(-\infty, \infty) = W^{-1}, \tag{33}
$$

this term can be re-written as

$$
\int_0^1 d\lambda\, (B^{-1}(\lambda)\, W)^\dagger\, T_b\, (B^{-1}(\lambda)\, W)^\dagger. \tag{34}
$$

Here one has used the relation $W\, B^{-1} = B^{-1}\, W$. Performing the change of variable $\lambda \to 1 - \lambda$, one can show that

$$
(B^{-1}\, W) = (B^{-1})^\dagger. \tag{35}
$$

Thus,

$$
\begin{aligned}
(J^+(x))^\dagger &= \rho^b\, W(x^+, -\infty) \int_0^1 d\lambda \left[(B^{-1}(\lambda)\, W)^\dagger\, T_b\, (B^{-1}(\lambda)\, W)^\dagger \right] W(\infty, x^+), \\
&= \rho^b\, W(x^+, -\infty) \int_0^1 d\lambda \left[B^{-1}(\lambda)\, T_b\, B^{-1}(\lambda) \right] W(\infty, x^+), \\
&= J^+(x). \tag{36}
\end{aligned}
$$

We have now explicitly shown above that J^+ (and hence Q) is both Hermitean and traceless. It is therefore an element of the Lie Algebra. In general, it is possible, if non–trivial, to show that $\ln(W)$ itself is a member of the Lie algebra [21]. The charge obtained from Eq. (4) is also an element of the Lie Algebra. It is easy to see that the color components of the color charge J_a^+ are real and therefore, the color charge matrix defined as $J_\mu = \frac{1}{N_c} J_\mu^a T^a$ is Hermitean and traceless. Both S_W and $S_{\ln W}$ lead to Wong's equations, but with a different current J_μ. This difference is due to imposing different "boundary" conditions at $\tau = -\infty$ when solving the Wong's equations (13) as given by (5) and (28). It should be noted that the boundary condition in (28) is more complicated than (5), and involves the non–trivial task of inverting the operator $B(\lambda)$. It is important to realize that the two different currents may describe different physics.

10

4 The $\ln W$ action and the BFKL equation

We will now show that the form of the action in Eq. (18) also reproduces the BFKL equation. Since the two actions differ only by the form of the Wilson line term, we will focus on the expansion of the Wilson line term in the two actions. To reproduce the Wilsonian renormalization group evolution, we need to keep terms that are quadratic in the hard fluctuations (the field δA^μ in Eq. (7)). The leading order non-trivial contribution therefore comes from the cubic terms in the action. (The contribution from quartic terms to the evolution is sub–leading in DIS.)

The difference between the two actions is

$$\Delta S \equiv S_W - S_{\ln W} = \text{Tr}\left(\rho\left[W - \ln W\right]\right) \tag{37}$$

where $\rho = \rho^a T_a$ and $\ln W$ is defined as in Eq. (20) to be

$$\ln W \equiv \ln[1 - (1 - W)] = -\text{Tr}\left(\rho\left[(1 - W) + \frac{1}{2}(1 - W)^2 + \frac{1}{3}(1 - W)^3 + \cdots\right]\right). \tag{38}$$

The integration over the spatial variables x^- and $x_\perp$ and the convolution with $\delta(x^-)$ is implicit in the trace above. The difference between the two actions is then

$$\Delta S = \text{Tr}\left(\rho\left[\frac{1}{2}(1 - W)^2 + \frac{1}{3}(1 - W)^3 + \cdots\right]\right), \tag{39}$$

where $1 - W$, from Eq. (2) can be expanded as $1 - W = -igA^- + (g^2/2)\,\hat{P}(A^-)^2 + \cdots$. Again, the integral over x^+ is implicit in the expansion, with the symbol $\hat{P}$ denoting the time ordering in x^+. Potential differences between the two actions will show up at order A^2. At this order [2],

$$\Delta S(A^2) \sim \text{Tr}\left(\rho\,(A^-)^2\right) \sim \rho_a f_{abc} \int dx^+ dy^+ A_b^-(x^+) A_c^-(y^+). \tag{40}$$

This term is identically zero because the integrand is symmetric under both the color exchange $b \leftrightarrow c$ and the co–ordinate exchange $x^+ \leftrightarrow y^+$ while multiplying the totally anti–symmetric structure constant f_{abc}.

---

[2] We use the following conventions for the trace of adjoint matrices: $\text{Tr}(T^a T^b) = N_c \delta_{ab}$ and $\text{Tr}(T^a T^b T^c) = \frac{iN_c}{2} f_{abc}$.

To investigate terms of order A^3, it is convenient to first consider S_W and $S_{\ln W}$ separately. The cubic terms in the expansion of $S_{\ln W}$ are

$$
\begin{aligned}
S_{\ln W}(A^3) &= \frac{g^3}{N_c} \text{Tr}\left(\rho \left[\hat{P}(A^-)^3 - \frac{1}{2} A^- \hat{P}(A^-)^2 - \frac{1}{2}\hat{P}(A^-)^2 A^- + \frac{1}{3}(A^-)^3 \right]\right) \\
&= \frac{g^3}{N_c} \text{Tr}\rho \int dx^+ dy^+ dz^+ A^-(x^+) A^-(y^+) A^-(z^+) \\
&\quad \times \left[\theta(x^+ - y^+)\theta(y^+ - z^+) - \frac{1}{2}\theta(x^+ - y^+) - \frac{1}{2}\theta(y^+ - z^+) + \frac{1}{3} \right] \qquad (41)
\end{aligned}
$$

After some algebra (performed in appendix B) the above can be re-expressed as

$$
\begin{aligned}
S_{\ln W}(A^3) &= \frac{g^3}{6} \rho_a \int dx^+ dy^+ dz^+ A_b^-(x^+) A_c^-(y^+) A_d^-(z^+) \theta(x^+ - y^+)\theta(y^+ - z^+) \\
&\quad \times \left[f^{adn} f^{bcn} - f^{abn} f^{cdn} \right] \qquad (42)
\end{aligned}
$$

The cubic term in S_W is

$$
\begin{aligned}
S_W(A^3) &= \frac{g^3}{N_c} \text{Tr}\rho\, \hat{P}(A^-)^3 \\
&= \frac{g^3}{N_c} \rho_a \int dx^+ dy^+ dz^+ A_b^-(x^+) A_c^-(y^+) A_c^-(z^+) \theta(x^+ - y^+)\theta(y^+ - z^+) \\
&\quad \times \left[\frac{I_2}{6}(f_{adn}f_{bcn} - f_{abn}f_{cdn}) + d_{abcd} \right]. \qquad (43)
\end{aligned}
$$

Here, we have used an identity for the trace of four $SU(3)$ adjoint matrices [22]. For an adjoint representation, $I_2 = N_c$. Also, the totally symmetric tensor d_{abcd} is defined as the symmetrized trace of four $SU(3)$ adjoint matrices. For an explicit form, see Ref. [23]. Note that the f-terms above are identical to those derived from $S_{\ln W}$ in Eq. (42). However, this action also contains the d_{abcd} term that was absent in the $S_{\ln W}$ action.

In appendix C, we show that, within the approximations made in the derivation of the small x evolution equation in Ref. [5], the d_{abcd} term does not contribute. Therefore $\Delta S = 0$ to cubic order. One may therefore conclude that BFKL equation can also be obtained from the $S_{\ln W}$ action.

The reason the d_{abcd} term in the S_W action vanishes is because the propagator of the hard modes (and the color sources to which it couples) is static. The static nature of the sources is due to the fact that one ignores the recoil of the color sources as they emit softer

partons. As one goes to a next-to-leading-order calculation, one will have to take recoil effects into account. These would cause the color sources to be time dependent, giving rise to a finite contribution from the d^{abcd} terms in the W action. Conversely, note that the d^{abcd} terms are naturally absent in $\ln W$ action.

The fact that the $S_{\ln W}$ action does not have the d_{abcd} term suggests that the underlying symmetry of the small x dynamics is manifest in this action. The agreement between the two actions is even more remarkable when one considers that the factor $1/6$ in Eq. (43) comes directly from the trace of four adjoint generators, while in the "$\ln W$" action it arises as a consequence of extensive algebraic manipulations.

5   Summary

In this paper, we proposed an alternative form of the small x effective action to the one discussed in Ref. [5]. We showed explicitly that both forms of the effective action are compatible with Wong's equations, albeit with currents that satisfy different boundary conditions. We showed that the the two effective actions agreed up to cubic order in the fields. Consequently, both of them give rise to the BFKL equation. However, in the case of the effective action of Ref. [5], one had to explicitly invoke the kinematic constraint imposed by the static sources–no such constraint was necessary for the action we propose. Differences between the two actions will show up at higher orders when one considers sub–leading corrections to the small x effective action.

Acknowledgments

Appendix A

The equation of motion for color charges in the action (18) has already been derived in Section 3. The derivation is the same for the point particle action Eq. (14). Further, the Euler-Lagrange equation for the position

$$p_\mu^I = m \frac{dx_\mu^I}{d\tau} \tag{44}$$

follows trivially from Eq. (14). Hence, we will only derive here the other remaining equation of motion, namely, that for the momentum p_μ.

The Euler-Lagrange equation for the momentum in Eq. (14) is

$$\dot{p}_\mu^I = -\frac{\delta}{\delta x_I^\mu} S_{\text{Wong}} \ . \tag{45}$$

The right hand side is given by

$$
\begin{aligned}
\frac{\delta S_{\text{Wong}}}{\delta x_I^\mu(\tau)} &= \frac{i}{N_c} \frac{\delta}{\delta x_I^\mu(\tau)} \sum_{J=1}^K \text{Tr} \left\{ Q_0^J \ln W_J \right\} \\
&= \frac{i}{N_c} \int d\tau' \sum_{J=1}^K \frac{\delta A_a^\nu(x_J(\tau'))}{\delta x_I^\mu(\tau)} \frac{\delta}{\delta A_a^\nu(x_J(\tau'))} \text{Tr} \left\{ Q_0^J \ln W_J \right\} \\
&\quad + \frac{i}{N_c} \int d\tau' \sum_{J=1}^K \frac{\delta v_J^\nu(\tau')}{\delta x_I^\mu(\tau)} \frac{\delta}{\delta v_J^\nu(\tau')} \text{Tr} \left\{ Q_0^J \ln W_J \right\}
\end{aligned} \tag{46}
$$

Using Eqs.(23) and (24), we get

$$\frac{\delta S_{\text{Wong}}}{\delta x_I^\mu(\tau)} = -v^\nu \partial_\mu A_\nu^a(x_I(\tau)) Q_a^I(\tau) + \frac{i}{N_c} \int d\tau' \dot{\delta}(\tau' - \tau) \frac{\delta}{\delta v_\mu^I(\tau')} \text{Tr} \left\{ Q_0^I \ln W_I \right\} \tag{47}$$

For simplicity, we'll omit the particle label I from now on. We'll also omit any argument labels (everything should be considered as a function of τ via $x(\tau)$). Applying the method

developed in Section 3, we see that

$$
\begin{aligned}
\frac{\delta S_{\text{Wong}}}{\delta x^\mu} & = -Q_a v^\nu \partial_\mu A_\nu^a - \frac{i}{N_c} \frac{d}{d\tau} \text{Tr} \left\{ Q_0 \frac{\delta}{\delta v_\mu} \ln W \right\} \\
& = -Q_a v^\nu \partial_\mu A_\nu^a + \frac{d}{d\tau} \left\{ A_\mu^a Q_I^a \right\} \\
& = -Q_a v^\nu \partial_\mu A_\nu^a + Q_a v^\nu \partial_\nu A_\mu^a + A_\mu^a \frac{d}{d\tau} Q_a \\
& = -Q_a v^\nu \partial_\mu A_\nu^a + Q_a v^\nu \partial_\nu A_\mu^a - ig A_\mu^a [Q, v^\nu A_\nu]_a \\
& = -Q_a G_{\mu\nu}^a v^\nu
\end{aligned}
\tag{48}
$$

where

$$
G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \, f_{abc} \, A_\mu^b A_\nu^c
\tag{49}
$$

This yields the desired result:

$$
\dot{p}_\mu = Q_a \, G_{\mu\nu}^a v^\nu \,.
\tag{50}
$$

Appendix B

We will show here how one gets equation (42) from equation (41). We start with Eq. (41):

$$
\begin{aligned}
S_{\ln W}(A^3) & = \frac{g^3}{N_c} \text{Tr} \Bigg(\rho \int dx^+ dy^+ dz^+ A^-(x^+) A^-(y^+) A^-(z^+) \\
& \qquad \left[\theta(x^+ - y^+) \theta(y^+ - z^+) - \frac{1}{2} \theta(x^+ - y^+) - \frac{1}{2} \theta(y^+ - z^+) + \frac{1}{3} \right] \Bigg) \,.
\end{aligned}
$$

To save space, we will use the following shorthand notation. We shall represent the Light Cone times x^+, y^+, z^+ by $1, 2, 3$, and shall not write the Light Cone Lorentz index '$-$' and the integrations over the Light Cone times explicitly. For example,

$$
\theta_{123} \equiv \theta(x^+ - y^+) \theta(y^+ - z^+) \,.
$$

We shall also neglect the overall coefficient g^3/N_c, including it only at the very last step. With these notations, equation (41) becomes

$$
S_{\ln W}(A^3) = \text{Tr} \left(\rho A_1 A_2 A_3 \left[\theta_{123} - \frac{1}{2} \theta_{12} - \frac{1}{2} \theta_{23} + \frac{1}{3} \right] \right) \,.
\tag{51}
$$

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We will now use identities like $\theta_{12} + \theta_{21} = 1$ to write Eq. (51) as

$$
\begin{aligned}
S_{\ln W}(A^3) &= \operatorname{Tr} \rho A_1 A_2 A_3 \Big[\theta_{123} - \frac{1}{2}\theta_{12}(\theta_{23} + \theta_{32}) - \frac{1}{2}\theta_{23}(\theta_{12} + \theta_{21}) \\
&\quad + \frac{1}{3}(\theta_{123} + \theta_{132} + \theta_{231} + \theta_{213} + \theta_{312} + \theta_{321}) \Big] \\
&= \operatorname{Tr} \rho A_1 A_2 A_3 \Big[-\frac{1}{2}(\theta_{12}\theta_{32} + \theta_{21}\theta_{23}) + \frac{1}{3}(\theta_{123} + \theta_{132} + \theta_{231} + \theta_{213} + \theta_{312} + \theta_{321}) \Big] \\
&= \operatorname{Tr} \rho A_1 A_2 A_3 \Big[-\frac{1}{2}\theta_{12}\theta_{32}(\theta_{13} + \theta_{31}) - \frac{1}{2}\theta_{21}\theta_{23}(\theta_{13} + \theta_{31}) \\
&\quad + \frac{1}{3}(\theta_{123} + \theta_{132} + \theta_{231} + \theta_{213} + \theta_{312} + \theta_{321}) \Big] \\
&= \operatorname{Tr} \rho A_1 A_2 A_3 \Big[-\frac{1}{2}(\theta_{132} + \theta_{312} + \theta_{213} + \theta_{231}) \\
&\quad + \frac{1}{3}(\theta_{123} + \theta_{132} + \theta_{231} + \theta_{213} + \theta_{312} + \theta_{321}) \Big] \\
&= \frac{1}{6} \operatorname{Tr} \rho A_1 A_2 A_3 \Big[\theta_{123} - \theta_{213} + \theta_{321} - \theta_{312} + \theta_{123} - \theta_{132} + \theta_{321} - \theta_{231} \Big] \\
&= \frac{1}{6} \operatorname{Tr} \rho \left(\theta_{123}[A_1, A_2]A_3 + \theta_{321}[A_1, A_2], A_3 + \theta_{123}A_1[A_2, A_3] + \theta_{321}A_1[A_2, A_3] \right)
\end{aligned}
$$

which, after change of variables, can be re–written as

$$
S_{\ln W}(A^3) = \frac{1}{6} \theta_{123} \operatorname{Tr} \left(\rho \left([[A_1, A_2], A_3] + [[A_3, A_2], A_1] \right) \right) . \tag{52}
$$

Using $[T^a, T^b] = if^{abc}T^c$, and restoring all the indices, coefficients, and integration variables, we obtain finally

$$
\begin{aligned}
S_{\ln W}(A^3) &= \frac{g^3}{6} \rho_a \int dx^+ dy^+ dz^+ [A_b^-(x^+)A_c^-(y^+)A_d^-(z^+)\theta(x^+ - y^+)\theta(y^+ - z^+) \\
&\quad \times \Big[f_{adn}f_{bcn} - f_{abn}f_{cdn} \Big] , \tag{53}
\end{aligned}
$$

which is Eq. (42).

Appendix C

In this appendix, we will show that, within the approximations made in the Wilson renormalization group approach, the totally symmetric d_{abcd}–term in Eq. (43) vanishes. Using the field decomposition Eq. (7), we get

$$
\begin{aligned}
S_{cubic}^{sym} &= \frac{g^3}{N_c} d^{abcd} \rho_a \int dx^+ dy^+ dz^+ \theta(x^+ - y^+)\theta(y^+ - z^+) \\
&\quad \Big[a_b^-(x^+)A_c^-(y^+)A_d^-(z^+) + A_b^-(x^+)a_c^-(y^+)A_d^-(z^+) + A_b^-(x^+)A_c^-(y^+)a_d^-(z^+) \Big] .
\end{aligned}
$$

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Renaming the variables in the second and third terms above gives

$$
S_{cubic}^{sym} = \frac{g^3}{N_c} d^{abcd} \rho_a \int dx^+ dy^+ dz^+ a_b^-(x^+) A_c^-(y^+) A_d^-(z^+)
$$
$$
\left[\theta(x^+ - y^+)\theta(y^+ - z^+) + \theta(y^+ - x^+)\theta(x^+ - z^+) + \theta(z^+ - y^+)\theta(y^+ - x^+) \right].
$$

Re–write the sum of θ–functions above as follows:

$$
\theta(x^+ - y^+)\theta(y^+ - z^+) + \theta(y^+ - x^+)\theta(x^+ - z^+) + \theta(z^+ - y^+)\theta(y^+ - x^+) =
$$
$$
\theta(x^+ - y^+)\theta(y^+ - z^+)\theta(x^+ - z^+) + \theta(y^+ - x^+)\theta(x^+ - z^+)\theta(y^+ - z^+) +
$$
$$
\theta(z^+ - y^+)\theta(y^+ - x^+) =
$$
$$
\theta(y^+ - z^+)\theta(x^+ - z^+) + \theta(z^+ - y^+)\theta(y^+ - x^+) =
$$
$$
\theta(y^+ - z^+)\theta(x^+ - z^+) - \theta(z^+ - y^+)\theta(x^+ - y^+) + \theta(z^+ - y^+)
$$

The first two terms in the last line are anti–symmetric with respect to change of $y^+ \to z^+$, and multiply the product $d^{abcd} A_c^-(y^+) A_d^-(z^+)$ which is totally symmetric. They therefore vanish, and the expression in S_{cubic}^{sym} reduces to

$$
S_{cubic}^{sym} = \frac{g^3}{N_c} d^{abcd} \rho_a \int dx^+ dy^+ dz^+ \, \theta(z^+ - y^+) a_b^-(x^+) A_c^-(y^+) A_d^-(z^+). \tag{54}
$$

This term can further be written as

$$
\int dx^+ dy^+ dz^+ \theta(z^+ - y^+) a_b^-(x^+) A_c^-(y^+) A_d^-(z^+) =
$$
$$
\int dx^+ dy^+ dz^+ \left[1 - \theta(y^+ - z^+) \right] a_b^-(x^+) A_c^-(y^+) A_d^-(z^+). \tag{55}
$$

Consider the "1" term on the right hand side:

$$
\int dx^+ dy^+ dz^+ a_b^-(x^+) A_c^-(y^+) A_d^-(z^+) = \int dx^+ a_b^-(x^+) dy^+ dz^+ A_c^-(y^+) A_d^-(z^+) \sim
$$
$$
\int dx^+ a_b^-(x^+) d(y^+ - z^+) d(y^+ + z^+) A_c^-(y^+) A_d^-(z^+). \tag{56}
$$

When we integrate over hard fluctuations, the term $A_c^-(y^+) A_d^-(z^+)$ will become the hard fluctuations propagator $G^{--}(y^+ - z^+)$. After integrating this propagator over the $d(y^+ - z^+)$ variable, it will give an overall factor p^-. Since $\rho \sim \delta(p^-)$ (this would break down when

considering NLO corrections), the "1" term vanishes because $p^-\delta(p^-) = 0$. The integrand of Eq. (54) is then

$$\int dx^+ dy^+ dz^+ \theta(z^+ - y^+) a_b^-(x^+) A_c^-(y^+) A_d^-(z^+) =$$
$$- \int dx^+ dy^+ dz^+ \theta(y^+ - z^+) a_b^-(x^+) A_c^-(y^+) A_d^-(z^+). \tag{57}$$

Since the LHS of the above is invariant under $z^+ \leftrightarrow y^+$, $c \leftrightarrow d$, the above identity would require it to be '−' of itself, and therefore equal to zero. Thus the d_{abcd}–term in the cubic piece of the action vanishes, and we are left with

$$
\begin{aligned}
S_{cubic} &= \frac{g^3}{N_c} \rho_a \int dx^+ dy^+ dz^+ A_b^-(x^+) A_c^-(y^+) A_d^-(z^+) \\
&\times \frac{I_2}{6} \theta(x^+ - y^+) \theta(y^+ - z^+) \Big[f^{adn} f^{bcn} - f^{abn} f^{cdn} \Big],
\end{aligned} \tag{58}
$$

which is identical to cubic term in the expansion of $S_{\ln W}$.

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