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ABSTRACT

We develop analytical methods for the prediction of the reactivity lifetime and burnup of nuclear fuels.

The analysis applies to those nuclear fuels whose changes in composition with time are due solely to neutron-absorption processes, so that the composition of any fuel species is a function only of the integrated flux time of its irradiation exposure. Reactivity lifetime can then be expressed as a function of the appropriate average flux time of the fuel at the end of the irradiation. We can then calculate local and average burnup of the irradiated fuel without necessarily specifying the magnitude of the irradiation flux or the power program of irradiation.

A generalized perturbation method is developed which allows us to calculate the above results, and which takes into account the spatial variation in neutron flux within the reactor and changes in this spatial variation during irradiation resulting from changes in fuel composition. Tabulated functions allow hand computations for the batch irradiation of fixed fuel in cylindrical or spherical reactors that have a uniform initial fuel loading. Such functions also apply to radial mixing and graded irradiation.

The perturbation method is most easily applied to the one-group diffusion model, but the method is extended to the multigroup model with only slight modification for reactors with energy-independent boundary conditions.

An exact analytical solution for the reactivity lifetime and fuel burnup has been developed for continuous fueling schemes, if the one-group model applies and if the characteristic excess neutron production of the fuel varies as a quadratic function of the flux time of irradiation exposure. A comparative study of various continuous fueling schemes has been made for a fuel with typical properties (see Sec. III-6).

The validity of the approximate solutions is determined by comparing results of the second-order perturbation method with the exact solutions of the same equations. Numerical computations on high-speed digital computers have been used to obtain exact solutions of those equations which could not be solved analytically by means of elliptic functions.

The computational procedure here developed allows survey studies comparing the performances of various fuels and various reactor designs. Also, it predicts the magnitude of the errors to be expected when we use various neutron behavior models (one-group, two-group, continuous slowing down) in more elaborate computations on high-speed digital computers.

Although attention has been focused mainly on the properties related to the variations of the flux in the equivalent homogeneous fuel-moderator cell, corrections for lumped fuel have also beeen investigated.

I. INTRODUCTION

1. Statement of the Problem

The rapid development of power reactors during the last decade has focused considerable attention on the changes that occur in the properties of nuclear fuel and nuclear reactors during long-term irradiation.

The variation and the control of the excess reactivity of the reactor, and the burnup (quantity of power released per unit weight) of the fuel at end of life, have been given special consideration. The comprehensive study by Benedict and Pigford[B1] is a useful tool for qualitative and quantitative estimates of these properties; they made a straight forward hand-computation method possible by using a simple model to describe the neutron behavior; such a model neglects the changes in the spatial and energy distribution of the neutrons which occur during irradiation.

The effects of these changes have been included in many numerical studies which the development of high-speed digital computers has made possible on a large scale. Such detailed analyses are of prime value for design studies of specific reactor systems. However, general investigations involve scores of parameters and require the use of computation techniques which represent a compromise between simplicity and accuracy.

Our purpose in this thesis is to develop a method of fuel-cycle analysis that accounts for the changes of the spatial distribution of the neutron flux, yet remains simple enough to be hand computed. Thus, we make available a set of formulae for further studies of power density distribution, of heat exchange problems, and of optimization of fuel-cycle parameters.

2. Review of Previous Works

Fuel-cycle computations rely on the reactor statics theory; i.e., they rely on the solution of equations describing the equilibrium behavior of the neutrons in a reactor of given composition. Knowing the cross sections of the various nuclides present in the reactor, we can compute changes in fuel composition, at least during a differential increment of the irradiation time. Thus, the refinement of a fuel-cycle computation depends on the adequacy of the model used to describe the spatial and energy distribution of the neutrons and on the accuracy of the extrapolation of the time-dependent properties.

While numerical studies can nowadays be performed on high-speed digital computers for extremely sophisticated models,[N1] methods seeking an analytical approximation of the characteristics of fuel cycles have been developed for the simpler models which follow.

2.1 Zero-Dimensional Model

Many of the fundamental characteristics of fuel cycles can be determined by calculation of the composition changes that occur in a local section of fuel as it is irradiated. This method, usually referred to as the constant-flux or zero-dimensional approximation, was used in basic papers presented at the First Geneva Conference on the Peaceful Uses of Atomic Energy by Dunworth,[D1] Lewis,[L1] Spinrad,[S1] and Weinberg.[W1] They have assumed that the flux energy spectrum does not vary with the irradiation and with the position in the reactor, in which case the zero-dimensional model would exactly describe the behavior during irradiation of a well mixed fuel (e.g., a solution of UO_2SO_4 in D_2O).

This model describes also the irradiation of a fuel in a uniform flux (i.e., a constant flux, or zero-dimensional flux with respect to spatial coordinate). Some useful results are developed below.

2.1.1. <u>Changes of fuel composition during irradiation</u>. Consider a local section of a fuel of given initial composition which is irradiated in a thermal flux $\phi(t)$. By assuming that the radioactive decay of a nuclide either takes place without delay or is negligible,[B1] the atomic concentrations N of the various nuclides are the solution of a set of linear first-order differential equations where the independent variable is the flux time

$$\theta = \int_0^t \phi(t)dt \; . \qquad (1)$$

This neglects the changes of the neutron-flux energy spectrum in the reactor and assumes that we have chosen suitable effective thermal cross sections; such cross sections may be determined according to the method of Westcott[W2] or according to the summaries of cross sections averaged over a Wigner-Wilkins spectrum;[A1] they can take into account epithermal absorptions that depend very much on concentration, like those occurring in Pu^{240}.[C1]

The most comprehensive studies of the fuel-composition changes have been completed for U^{235}-U^{238} and U^{235}-thorium fuels;[B1] we refer to these studies for the detailed analytical solution of the corresponding sets of linear first-order equations for nuclide concentrations as a function of flux time. As an example, the simpler fuel, pure U^{235}, would be treated as follows.

2.1.2. <u>Highly enriched uranium fuel</u>. The U^{235} undergoes the reaction

$$
{}_{92}U^{235} + {}_0n^1 \begin{array}{l} \nearrow \text{fission products} + \nu_0 n^1, \quad \text{probability: } \dfrac{1}{1 + \alpha_{25}} \\[2ex] \searrow {}_{92}U^{236}, \qquad\qquad\qquad \text{probability: } \dfrac{\alpha_{25}}{1 + \alpha_{25}} \end{array}
$$

where $\alpha_{25} = \dfrac{\sigma(n, \gamma)}{\sigma(n, f)}$ is the capture-to-fission ratio in U^{235}.

The reaction rate is the product of the concentration N, the microscopic absorption cross section σ, and the flux $\phi(t)$.

The concentration of U^{235} then obeys the equation

$$\frac{dN_{25}}{dt} = - N_{25}\,\sigma_{25}\,\phi\,. \tag{2}$$

The solution subject to $N_{25} = N_{25}^0$ at $t = 0$ is

$$N_{25} = N_{25}^0\, e^{-\sigma_{25}\theta}\,. \tag{3}$$

The concentration of U^{236} is determined by the equation

$$\frac{dN_{26}}{dt} = \frac{\alpha_{25}}{1+\alpha_{25}}\, N_{25}\,\sigma_{25}\,\phi - N_{26}\,\sigma_{26}\,\phi\,. \tag{4}$$

The solution subject to $N_{26} = 0$ at $t = 0$ is

$$N_{26} = N_{25}^0\, \frac{\sigma_{25}\alpha_{25}}{(\sigma_{25}-\sigma_{26})(1+\alpha_{25})} \left(e^{-\sigma_{26}\theta} - e^{-\sigma_{25}\theta}\right). \tag{5}$$

The neutron burnout of U^{236} is usually negligible, and Eq. (5) is replaced by

$$N_{26} = N_{25}^0\, \frac{\alpha_{25}}{1+\alpha_{25}} \left(1 - e^{-\sigma_{25}\theta}\right). \tag{6}$$

The fission products are classed into two groups:

(a) Low cross-section fission products whose neutron burnout and decay are negligible:

$$N_F = N_{25}^0\, \frac{1}{1+\alpha_{25}} \left(1 - e^{-\sigma_{25}\theta}\right). \tag{7}$$

(b) High cross-section fission products (e.g., Xe^{135} and Sm^{149}) that build up rapidly to steady-state concentrations. Their macroscopic cross section is taken as equal to a constant fraction r of the macroscopic cross section of U^{235}:

$$N_{Xe}\,\sigma_{Xe} + N_{Sm}\,\sigma_{Sm} = r\, N_{25}\,\sigma_{25}\,. \tag{8}$$

In a reactor loaded with pure U^{235}, and with no burnable poison present, the concentration of all nuclides but the above mentioned remains constant. The subscript P will refer to all permanent absorbers; their macroscopic cross section is $N_P \sigma_P$.

2.1.3. Complex fuels. For low-enrichment fuels containing fertile materials, Benedict and Pigford[B1] have shown that the variation of the concentration N_i of each nuclide can be represented by a sum of exponentials plus a linear function of the flux time:

$$N_i = C_{i,0} + C_{i,1}\,\theta + \sum_j C_{i,j}\, e^{-\sigma_j \theta}. \qquad (9)$$

The above equation is clearly valid for highly enriched uranium fuel. For reactors loaded with U^{235}-U^{238} or U^{235}-Th fuels, formulae giving the coefficients $C_{i,j}$ have been derived.[B1]

For our present purposes, it will be assumed that the variation of the nuclide concentrations are given functions of the flux time to which the fuel has been irradiated. In order to develop a hand computation procedure, it is most convenient to use Eq. (9) directly. The function $N_i(\theta)$ could also be expanded in power series of the flux time.[M1] Polynomial fits of the functions $N_i(\theta)$ are usually used for machine computations.

2.1.4. Changes of the flux energy spectrum. The flux energy spectrum changes as the irradiation proceeds. These changes of the flux energy spectrum can be studied by evaluating the change of the neutron temperature (for a Maxwell-Boltzmann energy spectrum) or by solving the equations of Wilkins or of Wigner-Wilkins for various isotopic compositions. A simple procedure for solving the nuclide-concentration equations is as follows:

(a) Given an initial composition of the mixture, one computes the flux energy spectrum. One can then compute an effective cross section for each nuclide by taking the average of its energy-dependent cross section over the flux energy spectrum.

(b) One uses the just computed effective cross sections in the equations for the nuclide concentrations, and solves for the concentration changes during a small time increment.

(c) One repeats steps 1 and 2, using the latest composition of the mixture for the determination of a new flux energy spectrum.

The above procedure is used, for instance, in FUELCYC, a program for computations on high-speed digital computers.[S3] The depression of the flux (see Appendix C) in cylindrical fuel elements has also been taken into account by Kushneriuk[K1] and Greebler.[G1]

2.2. One-Group Diffusion Equation

Papers[L4, P1, G2] that were presented at the Second Geneva Conference on the Peaceful Uses of Atomic Energy used the one-group diffusion equation in order to describe the spatial dependence of the neutron distribution. While Gotto[G2] used high-speed digital computers to repeatedly solve this equation while irradiation proceeds, most studies[L4, M2, P1, Z1] sought an approximation to the values of reactivity lifetime and burnup by using first-order perturbation theory. In the following paragraphs we present the one-group diffusion equation and the first-order perturbation with the notations used in this thesis.

Besides finite-differences methods,[G2] synthesis methods have recently been used in numerical computations on high-speed digital computers;[B4] another method, which represents the flux changes as a power series in the irradiation time, has been developed by Chambré.[C3] However, prior to the present work, only first-order perturbation computations have been used in hand computations of reactivity lifetime and burnup.

2.2.1. The modified one-group diffusion equation.

The neutron balance in the reactor is described locally by the diffusion equation written in terms of the thermal flux:

$$D\Delta\phi + (\nu_{th} \Sigma_f - \Sigma_a)\,\phi = \Sigma_E \phi\,, \qquad (10)$$

where:

(1) D is the diffusion coefficient. It is assumed constant, for its value is determined mainly by the properties of the reactor that do not change on irradiation.

(2) Σ_E is the positive macroscopic cross section of the control absorber adjusted in order to keep the reactor just critical.

(3) ν_{th} is the average number of thermal neutrons produced per fission. It takes into account the fast fission effect and the losses of neutrons (by leakage and absorption) during the slowing-down process. When the Fermi age slowing-down model is used in order to obtain the expression of ν_{th}, Eq. (10) is known as the modified one-group age-diffusion equation.[W3]

(4) Σ_f is the macroscopic thermal-fission cross section.

(5) $\Sigma_a \phi$ is the thermal neutron-absorption rate per unit volume, excluding the absorption in the control absorber.

One defines

$$\nu_{th} \Sigma_f - \Sigma_a = DB_m^2 , \tag{11}$$

where B_m^2 will be referred to as the material buckling[W3] of the reactor without control absorber. It is a function of the nuclide concentrations N_i.

The irradiation to which the fuel has been exposed in the reactor, according to a given fuel scheduling scheme, is to be characterized by a parameter T. If the flux ϕ were known, the concentrations N_i could be determined as a function of the position $\vec{x}$ in the reactor and of this parameter T which describes the extent of the irradiation.

The material buckling is then the result of an operation on the flux ϕ; this operation will be described later for various fuel scheduling schemes.

2.2.2. <u>Boundary conditions.</u> The flux ϕ, a function of the position $\vec{x}$ and of the irradiation parameter T, is the solution of Eq. (10). It is subject to either of the following boundary conditions:

(1)
$$\phi = 0 \tag{12}$$

at the extrapolated boundary of the reactor.

(2)
$$\phi + \delta_\ell \frac{\partial \phi}{\partial n} = 0 \tag{13}$$

at the physical boundary of the core.

The linear extrapolation distance δ_ℓ is constant if the properties of the reflector are independent of the irradiation. The condition that δ_ℓ be much smaller than the physical dimension of the reactor core is prescribed in order to have a good representation of the diffusion of the neutron with a one-group (thermal flux) diffusion equation. Furthermore, flux changes are much smaller in a well-reflected reactor than in a bare reactor, since the reflector tends to flatten the distribution of the neutrons in the core and thus diminishes nonuniformity of the fuel composition.

2.2.3. Unperturbed reactor equation. The unperturbed equation is the equation for the thermal flux in a reactor of uniform composition, because its solutions are well known for various geometries;[S4] we have

$$\Delta\phi + B^2\phi = 0. \tag{18}$$

With the linear homogeneous boundary conditions [Eq. (12) or (13)], Eq. (18) possesses a complete set of orthogonal eigenfunctions corresponding to eigenvalues B_k^2 all positive (the operator $(-\Delta)$ is positive-definite). The physical restriction that ϕ should be positive over the volume of the reactor requires the choice of the smallest eigenvalue B_0^2;[C2] B_0 is known as the geometrical buckling of the reactor core.

The unperturbed equation then is

$$\Delta\phi_0 + B_0^2 \phi_0 = 0 . \tag{19}$$

2.2.4. Excess neutron production. Equations (10) and (11) are combined into

$$-D(\Delta\phi + B_0^2 \phi) = (DB_m^2 - DB_0^2 - \Sigma_E)\,\phi \tag{20}$$

in order to distinguish between the perturbation term on the right side and the unperturbed expression on the left side. Equation (20) would be exactly equivalent to the unperturbed equation, Eq. (19), if the control absorber were locally adjusted according to

$$\Sigma_E = DB_m^2 - DB_0^2 \tag{21}$$

to compensate for the local excess reactivity of the fuel.

One defines

$$\nabla_\phi = DB_m^2 - DB_0^2 , \tag{22}$$

which has the dimension of a macroscopic cross section and will be refered to as the excess neutron production.B1, P1

The subscript ϕ indicates that ∇_ϕ is the result of an operation on the flux ϕ in the sense that, if the flux were known, one could write explicitly

$$\nabla_\phi = F(\vec{x}, T) , \tag{23}$$

where T is a parameter characteristic of the extent of the irradiation of the fuel.

Equations (20) and (22) finally yield

$$-D(\Delta\phi + B_0^2 \phi) = (\nabla_\phi - \Sigma_E)\, \phi . \tag{24}$$

2.2.5. First-order perturbation. As long as the perturbation term in the right side of Eq. (24) is small compared to $DB_0^2\phi$ in the left side, one would introduce a smaller error by replacing in the right side of Eq. (24) the actual flux ϕ by its approximate value ϕ_0. A better approximation to the flux will then be the solution of the equation

$$-D(\Delta\phi + B_0^2 \phi) = (\nabla_\phi - \Sigma_E)\, \phi_0 . \tag{25}$$

The solution ϕ of this equation is the first corrective term in the perturbation-theory series expansion of the solution of Eq. (24), as well as the first term of a suite of iterations (cf. Eq. (II-22)).

In general, Eq. (25) could be used to obtain approximate solutions for the thermal flux in a reactor with nonuniform fuel loading and control absorption, when this nonhomogeneity does not induce important variations of the diffusion coefficient D.

However, Eq. (25) is mostly suitable for the study of the changes with irradiation of a flux which would satisfy exactly the unperturbed equation, at least at zero irradiation.

2.2.6. Criticality condition; reactivity lifetime. Equation (25) is a linear nonhomogeneous differential equation and the corresponding homogeneous equation has a nontrivial solution ϕ_0; Eq. (25) has a solution only if its right side is orthogonal to this nontrivial solution ϕ_0. This yields the criticality condition*

$$\left\langle (\nabla_{\phi_0} - \Sigma_E)\, \phi_0,\ \phi_0 \right\rangle = 0\ . \tag{26}$$

In the case of batch irradiation of the fuel, Σ_E is set equal to zero at the end of the reactivity lifetime. For steady-state fuel scheduling schemes, Σ_E is always zero. The corresponding equation

$$\left\langle (\nabla_{\phi_0})\, \phi_0,\ \phi_0 \right\rangle = 0\ , \tag{27}$$

which has been widely used after Pigford,[P1] is the result of a first order perturbation theory (also called a statistical weighting procedure by Lewis[L4]). In the present approximation, no assumption needs to be made concerning the distribution of the control absorber in the reactor. For a reactor with a uniformly distributed control absorber we have

$$\Sigma_E = \frac{\left\langle (\nabla_{\phi_0})\, \phi_0, \phi_0 \right\rangle}{\left\langle \phi_0, \phi_0 \right\rangle}\ . \tag{28}$$

According to Eq. (9), the excess neutron production in a reactor cell irradiated to a uniform flux time θ is the following function of this flux time:

$$\nabla(\theta) = d_0 + d_1\theta + \sum_j d_j e^{-\sigma_j \theta}\ . \tag{29}$$

When the extent of the irradiation of the fuel in the reactor, in accordance with a given fuel scheduling scheme, is characterized by a parameter T, Eq. (28) can be written

---

*See Appendix A for definition of the following scalar product notation.

$$\Sigma_E = d_0 + d_1 f_2 T + \sum_j d_j\, E_2(\sigma_j T)\,. \tag{30}$$

The coefficient f_2 and the function $E_2(w)$ have been tabulated for various fuel scheduling schemes in spherical, cylindrical, and parallelopiped reactors.[B1, S2, M1]

The coefficients d_1, d_0, and d_j are linear functions of the initial nuclide concentrations; by setting $\Sigma_E = 0$ in Eq. (30), a straightforward hand computation determines the enrichment of the fuel which must be loaded in a given reactor in order to obtain an irradiation T.

2.2.7. Burnup of the fuel. The burnup is defined here as that fraction of the original fuel which has been converted into fission products. It is a function of the nuclide concentrations and, like the excess neutron production ∇_ϕ, it is the result of an operation on the flux. The burnup of the fuel, β_ϕ, is a function of the position x in the reactor and of the irradiation T. The average burnup $\langle 1, \beta_\phi \rangle$, is a function of the irradiation T only.

According to Eq. (9), the burnup in a reactor cell, irradiated to a uniform flux time θ, is a function of the flux time, and we have

$$\beta(\theta) = b_0 + b_1\theta + \sum_j b_j\, e^{-\sigma_j \theta}\,. \tag{31}$$

A first approximation to the burnup is obtained by replacing in the expression for β_ϕ the actual value of the flux ϕ by its approximate value ϕ_0. In the fuel-cycle study of Benedict and Pigford[B1] this approximation is used in calculating local and average burnup of fuel irradiated according to a given fuel scheduling scheme in a reactor of simple geometrical shape; this average is

$$\langle 1, \beta_{\phi_0} \rangle = b_0 + b_1 f_0 T + \sum_j b_j E_0(\sigma_j T)\,, \tag{32}$$

where T characterizes the extent of the irradiation, and is usually the maximum flux time, and the coefficient f_0 and the function $E_0(w)$ have been tabulated.

The final burnup is evaluated when the fuel has been irradiated to such an extent that the reactor is just critical with all control absorbers removed. This burnup is determined by Eq. (32), where from Eq. (30) one takes for T the root of $\Sigma_E(T) = 0$.

The above results give only a first-order approximation; in the present thesis, we seek a better approximation by adding a second-order term to correct for the flux changes that occur during irradiation.

2.3. Multigroup Diffusion Equations

It is only in a bare, uniformly loaded reactor that the spatial distribution of the neutron can be described separately from the energy distribution. However, in fuel-cycle analysis we are often concerned with reactors where nonuniformity in composition arises because of the nonuniform burnup of the fuel. The nonuniformity may also be caused by a reflector, by a breeding blanket, or by control rods. The problems of the spatial variation of the flux-spectrum are then usually studied by breaking up the energy range from fission to thermal into several groups of equivalent monoenergetic neutrons.

For example, the two-group diffusion equations have been widely used to study the characteristics of fuel cycles. An approximate solution has been derived by Murray[M3] and studies of the reactivity lifetime have been performed by Wolfe[W4] who compared the results of the first-order perturbation theory with those of exact computations. Wolfe points out that the thermal flux shape does not change significantly if the effect of burnup on neutron balance in a small core region is small compared to the neutron leakage from this region. The perturbation method is then appropriate for the study of highly enriched reactors which have a large leakage of neutrons. However, the analytical study of flux changes by the perturbation method is greatly complicated because the operator that acts on the two-dimensional vector flux (fast and thermal fluxes) is not self-adjoint. Thus, most numerical studies have used the finite-differences method in order to solve the multigroup diffusion equations. A great improvement in speed of calculations has been achieved by the pseudomultigroup treatment used by the FUELCYC computer program.[S3]

2.4. Fuel Scheduling Schemes

In early studies, the irradiation of thoroughly mixed fuel (zero-dimensional model) was primarily considered. The incentive to produce nuclear power at competitive costs has since focused considerable attention on various fuel and control management schemes that might yield large burnups of the fuel. The following describes some of these fuel scheduling schemes, which have been intensively studied since Lewis' early investigation on batch irradiation of fixed fuel[L2] as well as on continuous refueling schemes.[L3]

2.4.1. Batch irradiation of the fuel.

Fuel is initially loaded in the reactor and remains fixed in position throughout the operation of the reactor. Control poison is adjusted in order to keep the reactor just critical at all times. The fuel is discharged batchwise when the reactivity of the fuel has decreased to such an extent that the reactor becomes subcritical with all control poison removed.

The results of the first-order perturbation method do not take into account the distribution of the control absorber in the reactor. This distribution has been assumed uniform in earlier studies. Numerical analyses relating to specific reactor systems have shown that a better neutron economy can be achieved with nonuniform distribution of the control,[S5] of the fuel,[I1] or of both.[J1]

The burnup of the fuel can also be increased by interchanging the fuel rods throughout the irradiation in a cylindrical reactor with sufficient frequency to keep the composition uniform radially (radial mixing [P1]).

2.4.2. Steady-state fuel scheduling schemes.

The fuel is continuously fed to and discharged from the reactor at the frequency at which the reactor remains just critical without control poison. With a reactor shaped as a right cylinder of arbitrary cross section loaded with fuel contained in elements (rods or channels) parallel to the axis, one can consider graded irradiation or steady axial movements of the fuel.

(1) Graded irradiation. The fuel rods, fixed in place in the reactor, are discharged individually when burned up to a specified extent, and replaced by fresh rods.

(2) Unidirectional. The fuel is moved steadily in channels through the reactor; it is fed at one end of the reactor and discharged at the other.

(3) Bidirectional. The fuel is moved in opposite directions in adjacent channels, half of the fuel being charged and half of it being discharged at each end plane.

(4) Out-in. The fuel is loaded at the outer surface of the reactor and moved steadily toward the center, where it is discharged.

3. Method of Perturbation Analysis to be Used

The first-order perturbation theory has been widely used to obtain an approximation to the eigenvalue of various eigenvalue problems of the reactor theory, but very few studies have also attempted to obtain an approximation to the eigenfunction itself and to determine higher-order perturbation corrections.

In general, perturbation theory can give a series expansion of all the quantities of interest, such as reactivity and neutron distribution. It has been pointed out (see Ref. W3, Chapter XVI) that the perturbation series may converge very poorly and that its first few terms may be misleading; however, for linear problems, modified perturbation methods[M5] have been developed with which the convergence of the successive approximations can be realized in all cases.

The application of these modified perturbation methods to nonlinear problems is so complex analytically as to be unwieldy for the purposes of this thesis. Therefore, after a development of an approximate calculational procedure with the help of conventional perturbation methods, it will be necessary[W4, B1] to check the validity of the results thus obtained by comparing them with those given by more accurate methods.

In the present thesis, a perturbation method of solution of the irradiation-dependent one-group diffusion equation is developed. The solution (eigenfunction) of the first-order perturbation theory describes the flux changes throughout the irradiation. The second-order perturbation yields then a corresponding correction to the reactivity lifetime. The successive terms of the perturbation series can be obtained according to the general scheme developed in Sec. II.2. The main characteristics of the method as applied to the study of the flux changes and of the corresponding correction of the reactivity life-time are outlined below.

3.1. Flux Changes

The flux can always be written as the product of a normalized flux-shape $u(\vec{x}, t)$ by a magnitude $f(t)$, where

$$\phi(\vec{x}, t) = f(t)\, u(\vec{x}, t) \; . \tag{33}$$

The magnitude factor varies according to the time-dependent power program of the reactor and is chosen so that $u(\vec{x}, t)$ satisfies some given normalization condition.

In the present study, it is assumed that the nuclide concentrations are functions of the local flux time only; at a given time, ∇_ϕ in Eq. (24) is a function only of the irradiation T to which the fuel has been previously exposed and is independent of the magnitude of the flux at this given time. The magnitude of the flux $\phi(\vec{x}, t)$ [the solution of Eq. (24)] is not determined by this equation.

The operation ∇_ϕ on ϕ can then be replaced by a corresponding operation ∇_u on u, with a suitable definition of the irradiation parameter T. Then Eq. (24) reads

$$-D(\Delta u + B_0^2 u) = (\nabla_u - \Sigma_E)\, u \; . \tag{34}$$

Since we seek the departure of the flux-shape u from its unperturbed value u_0, the flux-shape change

$$u_1 = u - u_0 \tag{35}$$

is conveniently chosen to be orthogonal to the unperturbed flux shape u_0.

This is equivalent to a normalization of the flux-shape

$$\langle u, u_0 \rangle = \text{constant} = \langle u_0, u_0 \rangle . \tag{36}$$

The first approximation to the flux-shape change u_1^* is then defined as the solution of Eq. (25)

$$-D(\Delta u_1^* + B_0^2 u_1^*) = (\nabla_{u_0} - \Sigma_E)\, u_0 , \tag{37}$$

which satisfies the normalization condition

$$\langle u_1^*, u_0 \rangle = 0 . \tag{38}$$

The left side of Eq. (37) is a linear differential operator with constant coefficients acting on the unknown function u_1^* and the right side is a known function. The criticality condition, Eq. (26), ensures that Eq. (37) has a solution and then, the solution satisfying the normalization condition, Eq. (38), is unique.

Although Eq. (37) is rather simple, its solution will not usually be expressible in a closed analytical form, but, in general, it could be easily expanded in an infinite series of the eigenfunctions of the operator in the left side of Eq. (37). This operator is self-adjoint, positive-definite, and the equation

$$-(\Delta v_k + B_0^2 v_k) = (B_k^2 - B_0^2)\, v_k \tag{39}$$

defines a complete set of orthogonal eigenfunctions v_k corresponding to positive eigenvalues $B_k^2 - B_0^2$. These are well known in the usual geometries of cylinder, sphere, and parallelopiped, where one obtains Bessel functions or trigonometric functions.

Replacing u_1^* by its expansion

$$u_1^* = \sum_k a_k(T)\, v_k(\vec{x}) , \text{ where } k \neq 0, \tag{40}$$

and taking the scalar product of the both sides of Eq. (37) with an eigenfunction v_k yields

$$a_k(T) = \frac{\langle (\nabla_{u_0} - \Sigma_E)\, u_0, v_k \rangle}{D(B_k^2 - B_0^2)\langle v_k, v_k \rangle} . \tag{41}$$

When the excess neutron production is represented by Eq. (29), we will show that the coefficients $a_k(T)$ are determined by

$$D(B_k^2 - B_0^2)\, a_k(T) = d_1 f_{2,k}\, T + \sum_j d_j\, E_{2,k}(\sigma_j T) \,. \tag{42}$$

The coefficients $f_{2,k}$ and the functions $E_{2,k}(w)$ will be tabulated for various fuel scheduling schemes and reactors of different shapes. The hand computation of the flux changes is similar to that of the reactivity changes, Eq. (30), and requires little additional effort, since only the first few coefficients a_k are significant in most cases (least-squares approximation).

The smaller the coefficient $a_k(T)$, the smaller the flux-changes. Thus, in Eq. (41), the magnitude of the change in the excess-neutron production ∇u_0 shall be compared to the term $D(B_k^2 - B_0^2)$. For bare reactors, $D(B_k^2 - B_0^2)$ is proportional to the initial leakage DB_0^2; for reflected reactors, it can still be compared to the leakage from a bare reactor of the same dimension as the core. To illustrate the later statement, compare the following results for two reactor cores shaped as infinite slabs of thickness $2Z$ (see Sec. II.3.3):

(a) Bare: $B_k = (k + \frac{1}{2})\pi/Z$, $D(B_1^2 - B_0^2) = \frac{3}{4} D\, (\frac{\pi}{Z})^2$

(b) Perfectly reflected: $B_k = k\,\pi/Z$, $D(B_1^2 - B_0^2) = D(\frac{\pi}{Z})^2$

3.2. Reactivity Changes

Consider the equation for the flux shape, Eq. (34), and the corresponding unperturbed equation

$$\Delta u_0 + B_0^2 u_0 = 0 \,. \tag{43}$$

The scalar product of the left side of Eq. (34) with the solution u_0 of Eq. (43) vanishes because the Laplacian operator Δ is self-adjoint. Thus the condition

$$\left\langle (\nabla_u - \Sigma_E)\, u, u_0 \right\rangle = 0 \tag{44}$$

must be satisfied by any function u solution of Eq. (34). For a given u, the value of Σ_E is defined by Eq. (44) and this value is unique, since u_0 is unique except for a multiplicative constant which does not modify Eq. (44). This property has been used[Z1] in order to justify Eq. (26): assuming that the flux shape is very close to the initial flux shape, one replaces u by u_0 in Eq. (44).

When the initial flux shape u_0 is added to the flux-shape change u_1^*, Eq. (44) defines a corresponding approximation Σ_E' for the control absorber. Then Σ_E' is determined by

$$\left\langle \left(\nabla_{u_0 + u_1^*} - \Sigma_E' \right) (u_0 + u_1^*),\ u_0 \right\rangle = 0 \,. \tag{45}$$

3.3. Normalization Condition

The first approximation of the flux-shape changes has been defined in the preceeding section as the particular solution u_1^* of the first-order perturbation equation, Eq. (37).

However, if a_0 is an arbitrary constant, Eq. (37) has a general solution

$$u_1 = a_0 u_0 + u_1^* \,, \tag{46}$$

where u_0 is the solution of the homogeneous equation, Eq. (43).

The function u_1 is then defined uniquely only if an additional condition (normalization condition) uniquely determines the coefficient a_0. Equation (38) is such a normalization condition, which is widely used in quantum mechanics;[S6] it has been applied to fuel-cycle computations by Hinman.[H1]

When others were in need of a normalization condition, they made the assumption that the thermal power of the reactor is constant in time.[G2] With the present notations, this is

$$\left\langle \Sigma_f, \phi \right\rangle = \text{constant.}$$

The above equation is an additional condition on the flux equivalent to a normalization condition on the flux shape.

For generality's sake, Chambré[C2] has considered subsequently that the thermal power is an arbitrary function of the time, defined by a given power program.

However, such a specification of the power program is not necessary for our present purposes. If the local nuclide concentrations are functions only of the flux time to which the fuel has been irradiated and are independent of the magnitude of the actual flux, then the burnup of the fuel is a function only of the total power which has been produced by a fuel element; this burnup is independent of the program according to which this power has been produced.

Thus, the normalization condition is arbitrary, but the values of the control absorber and the flux shape are always related by Eq. (44).

In the present thesis, the use of conveniently chosen normalization conditions (to be specified in Sec. II.2.2.) has greatly simplified the computations.

4. Summary of Studies to be Made

4.1. Batch Irradiation of Unmixed Fuel

The perturbation theory is applied first to the study of the batch irradiation of unmixed fuel. The spatial distribution of the neutron is described by the solution of the one-group diffusion equation. The changes of the properties of a local section of the fuel, given by the zero-dimensional model, are assumed to be a known function of the irradiation.

The perturbation theory replaces the reactor equation by an infinite set of simpler (linear) partial differential equations; the solution to each of these linear equations is obtained by using eigenfunction techniques and is properly normalized.

The method is then applied specifically to the flux and reactivity changes in a reactor where the control absorber and the initial fuel are uniformly loaded. A calculation procedure is developed, which yields an approximation of the flux changes and of the corresponding correction to the burnup of the fuel.

Calculations are made amenable to hand computations by tabulating sets of auxiliary functions.

A parametric study of the fuel-cycle characteristics in bare reactors loaded with highly enriched uranium is then performed according to this computation procedure. The results are compared with those obtained by solving the one-group diffusion equation by the finite-differences method using high-speed digital computers.

4.2. Continuous Fueling

The perturbation theory is also applied to the study of various fuel scheduling schemes which are described by an ordinary differential equation (one-dimensional). The solution of the corresponding linear equations defined by perturbation theory can now be obtained by using (besides the eigenfunction expansion technique) the Fourier series or Green's function techniques.

The following fuel scheduling schemes are considered:

(a) Radial mixing of the fuel, with uniformly distributed control or with control absorber localized in the midplane of the reactor.

(b) Graded iraddiation of the fuel.

(c) Steady axial movements of the fuel.

The equations describing all these fuel scheduling schemes (except the first) can be put in the form of a differential equation which does not explicitly contain the variable. They can be solved by two quadratures, and have a solution in terms of known functions (elliptic functions) if the nuclide concentrations vary as a quadratic function of the flux time to which a local section of fuel has been irradiated.

Thus exact solutions of the one-group diffusion equation are obtained. They give a basis for a comparison of the various fuel scheduling schemes as well as a basis for a test of the accuracy of the calculation procedure developed by perturbation methods.

II. BATCH IRRADIATION

1. Description and Equations

The fuel is initially loaded in the reactor, and the control absorber is adjusted to make the reactor just critical. In general, the initial fuel and the control absorber are not uniformly distributed over the volume of the reactor.

The fuel remains unmixed during its irradiation in the reactor and, at a time t after startup, it has been irradiated to a flux-time θ defined by

$$\theta(x,t) = \int_0^t \phi(x,t)\,dt. \tag{1}$$

Given the physical properties of an homogenized cell containing the fuel, the moderator, the coolant, and the structural materials, we find that the excess neutron production, ∇_ϕ, varies with the irradiation as a known function $\nabla(\theta)$ of the flux time θ. If the initial composition of the reactor is nonuniform, the excess neutron production is furthermore an explicit function of the position $\nabla(x,\theta)$.

The thermal flux ϕ, a function of the position x and of the time t, is given at every instant by the solution of the one-group diffusion equation

$$-D(\Delta\phi + B_0^2\phi) = [\nabla(\theta) - \Sigma_E]\,\phi\,. \tag{2}$$

The boundary condition for the thermal flux is

$$\delta_\ell \frac{\partial\phi}{\partial n} + \phi = 0\,. \tag{3}$$

In the above equation, $\frac{\partial\phi}{\partial n}$ is the normal derivative of the flux and δ_ℓ is the linear extrapolation distance. We evaluate ϕ and $\frac{\partial\phi}{\partial n}$ at the physical boundary of the reactor core.

The Laplacian operator Δ, acting on the linear manifold of the functions which satisfy such a linear homogeneous boundary condition, is self-adjoint with respect to the scalar product

$$\langle f, g\rangle = \frac{1}{V}\int_V f(x)\,g(x)\,dV. \tag{4}$$

The macroscopic cross section of the control absorber required to keep the reactor just critical is Σ_E. It varies with time, and is also a function of the position, $\Sigma_E(x,t)$, in case of nonuniform distribution of the control absorber.

Let v_0 be the solution of the equation

$$\Delta v_0 + B_0^2 v_0 = 0 . \tag{5}$$

The scalar product of the left side of Eq. (2) by v_0 vanishes, since the Laplacian operator is self-adjoint. A condition for Eq. (2) to have a solution is then

$$\left\langle \frac{\nabla(\theta) - \Sigma_E}{D} \phi, v_0 \right\rangle = 0 . \tag{6}$$

Given the function ϕ, the above equation determines the magnitude of the control absorber required to keep the reactor just critical; the spatial distribution of the control absorber, however, must be specified by a given control management procedure.

If we know the flux time $\theta(x,t)$ to which the fuel has been irradiated, at any given time, then Eq. (2) as well as the boundary condition, Eq. (3), is linear and homogeneous with respect to the flux ϕ; any multiple of the flux is still a solution, i.e., the magnitude of the flux is arbitrary. It will be convenient to select a normalized solution $u(x,t)$.

The normalization condition will be represented by the symbol

$$N(u) = 0 . \tag{7}$$

The flux time can still be represented by a simple integral of the flux shape u with a suitable definition of the variable which defines the irradiation.

The flux can be written

$$\phi(x,t) = f(t)\, u(x,t) . \tag{8}$$

If $f(t)$ is chosen at every instant so that $u(x,t)$ satisfies the normalization condition, the flux time is

$$\theta(x,t) = \int_0^t f(t)\, u(x,t)\, dt . \tag{9}$$

Now, instead of characterizing the irradiation by the time t after startup, a new variable T is defined by

$$T = \int_0^t f(t)\,dt \;. \tag{10}$$

Thus, Eq. (9) becomes

$$\theta(x, T) = \int_0^T u(x, T)\,dT. \tag{11}$$

For instance, if one wishes to normalize the flux shape to unity at the center of the reactor one uses

$$N(u) = u(0,t) - 1 = 0 \;; \tag{12}$$

then Eq. (8) shows that f(t) is the magnitude of the flux at the center of the reactor, since one can always write

$$\phi(x,t) = \phi(0,t)\, u(x,t) \;. \tag{13}$$

Therefore, the irradiation is characterized by the central flux time

$$T = \int_0^t \phi(0,t)\,dt \tag{14}$$

instead of the time t.

In fact, the time t is just a "dummy variable" since, for the present purposes, the extent of the irradiation in the reactor would be best defined by the amount of control absorber which is still required to make the reactor critical. The end-of-life condition is that the reactor be just critical when all the control absorber has been removed: that is, we have

$$\boldsymbol{\Sigma}_E \equiv 0 \tag{15}$$

or, according to Eq. (6),

$$\left\langle \nabla(\theta)\,\phi, v_0 \right\rangle = 0 \;. \tag{16}$$

The final burnup of the fuel, and the end-of-life properties must be evaluated for the value T_f of the variable T for which Eq. (15) is satisfied.

For poorly reflected reactors (see the discussion at the end of Sec. I.3.3.), the problem can be stated in dimensionless form by dividing both sides of Eq. (2) by DB_0^2; we have

$$\frac{1}{B_0^2}\Delta\phi + \left[1 + \frac{\nabla(\theta) - \Sigma_E}{DB_0^2}\right]\phi = 0 \ . \tag{17}$$

In the perturbation method, as presented in the following section, we consider that the term $\left[\nabla(\theta) - \Sigma_E\right] / DB_0^2$ is a perturbation term which is small compared to 1.

We define the following dimensionless quantities

$$g(\theta) = \frac{\nabla(\theta)}{DB_0^2} \ , \tag{18}$$

and

$$c = \frac{\Sigma_E}{DB_0^2} \ . \tag{19}$$

The above quantities will still be referred to as the neutron-excess production and the control absorption, respectively.

The problem is to find the function u(x, T) and the magnitude of the function c(x, T) that satisfy

$$-\left[\frac{1}{B_0^2}\Delta u(x,T) + u(x,T)\right] = \left\{g\left[\int_0^T u(x,T)\,dT\right] - c(x,T)\right\}u(x,T), \tag{20}$$

where u is the function of position x and of irradiation T normalized by a condition, Eq. (7), and where u satisfies a linear homogeneous boundary condition.

2. Perturbation Theory

2.1. Perturbation Method

The purpose of the perturbation method is to replace the equation of the problem, Eq. (20), by a set of simpler equations which determine the successive approximations to the function u and to the control c; these are expanded in a power series of a perturbation parameter ϵ:

$$c = c(x, T) = c_0(x, T) + \epsilon c_1(x, T) + \epsilon^2 c_2(x, T) + \cdots = \sum_{s=0}^{\infty} c_s \epsilon^s \qquad (21)$$

$$u = u(x, T) = u_0(x, T) + \epsilon u_1(x, T) + \epsilon^2 u_2(x, T) + \cdots = \sum_{p=0}^{\infty} u_p \epsilon^p \qquad (22)$$

The perturbation parameter is chosen in such a way that the equation

$$-\left(\frac{1}{B_0^2} \Delta u + u\right) = \epsilon[g(\theta) - c]\, u \qquad (23)$$

reproduces the given equation, Eq. (20), when $\epsilon = 1$, and reduces, when $\epsilon = 0$, to the simple equation

$$\Delta u + B_0^2 u = 0 \,. \qquad (24)$$

The above "unperturbed equation" has a known nontrivial solution $u_0(x)$, and the first approximation of the flux time is

$$\theta_0 = \int_0^T u_0(x)\, dT = T\, u_0(x) \,. \qquad (25)$$

The introduction of the perturbation parameter ϵ is simply a device to trace the order of magnitude of the various terms. In Eqs. (21) and (22), it is assumed that the solution of Eq. (23) can be expanded in power series of ϵ. The smaller ϵ becomes, that is, the smaller the magnitude of $[g(\theta) - c]$, the faster the convergence of the perturbation series.

We find that c can be a function of the position x, as well as of the irradiation T, and the function $g(\theta)$ can depend explicitly upon x; this is the general case, where the control absorption, the initial fuel loading, and/or the diffusion coefficient D are functions of position; the initial flux itself is then expressed by means of a perturbation series.

However, the perturbation method is best suited to study the changes occurring in a reactor whose initial composition is uniform, since the initial flux is then the solution of the unperturbed equation, Eq. (24), and this ensures a fast convergence of the perturbation series, at least for small irradiations.

Corresponding to the expansion for the flux, Eq. (22), the power-series expansion for the flux time is

$$\theta = \sum_{p=0}^{\infty} \theta_p \epsilon^p, \tag{26}$$

where, according to the Eqs. (11) and (22),

$$\theta_p = \theta_p(x, T) = \int_0^T u_p(x, T)\, dT. \tag{27}$$

The expansion in power series of $g(\theta)$ is obtained by expanding $g(\theta)$ in a Taylor series in the neighborhood of θ_0; thus, we have

$$g(\theta) = g(\theta_0) + \left(\sum_{p=1}^{\infty} \theta_p \epsilon^p\right) g'(\theta_0) + \cdots + \frac{1}{n!}\left(\sum_{p=1}^{\infty} \theta_p \epsilon^p\right)^n g^{(n)}(\theta_0) + \cdots \tag{28}$$

Replacing the various quantities in Eq. (23) by their expansions in powers of ϵ, one equates to zero the coefficients of the successive powers of ϵ, thus obtaining the following equations:

$$-\left(\frac{1}{B_0^2}\ \Delta u_0 + u_0\right) = 0\ , \tag{29}$$

$$-\left(\frac{1}{B_0^2}\ \Delta u_1 + u_1\right) = [\,g(\theta_0) - c_0]\ u_0\ , \tag{30}$$

$$-\left(\frac{1}{B_0^2}\ \Delta u_2 + u_2\right) = [\,g(\theta_0) - c_0]\ u_1 + [\,\theta_1 g'(\theta_0) - c_1]\ u_0, \tag{31}$$

$$-\left(\frac{1}{B_0^2}\ \Delta u_3 + u_3\right) = [\,g(\theta_0) - c_0]\ u_2$$

$$+ [\,\theta_1 g'(\theta_0) - c_1]\ u_1 + [\,\theta_2 g'(\theta_0) + \frac{\theta_1^2}{2}\, g''(\theta_0) - c_2]\ u_0\ , \tag{32}$$

etc.

Defining the self-adjoint linear operator L by

$$Lu = -\left(\frac{1}{B_0^2}\ \Delta u + u\right), \tag{33}$$

the equation for the general term u_n of the perturbation series has the form

$$Lu_n = F_n(u_0, \theta_0, c_0, u_1, \theta_1, c_1,\ \cdots\ , u_{n-1}, \theta_{n-1}, c_{n-1}). \tag{34}$$

The above equation is a nonhomogeneous linear differential equation for u_n; it has a solution satisfying the boundary condition if and only if its right side F_n is orthogonal to the nontrivial solution of the self-adjoint homogeneous equation

$$Lu_0 = 0\ . \tag{35}$$

Then, one chooses c_{n-1} to satisfy the condition

$$\left\langle F_n, u_0\right\rangle = 0\ . \tag{36}$$

The procedure for solving the system of equations, Eqs. (29) through (32), is as follows:

Having determined u_p and θ_p up to $p = n-1$, and c_s up to $s = n-2$, then c_{n-1} is determined uniquely by Eq. (36). The right side of Eq. (34) is now a known function of x and T, and since Eq. (36) is satisfied, Eq. (34) has a solution that can be obtained by standard techniques for the Helmholtz equations. However, the solution u_n is not unique; if u_n^* is a particular solution of Eq. (34), any function u_n defined by

$$u_n = u_n^* + a_{0,n} u_0 \tag{37}$$

is still a solution for any arbitrary value of the coefficient $a_{0,n}$, since u_0 satisfies the homogeneous equation, Eq. (35).

In the following, the particular solution noted u_n^* is defined as the unique solution of Eq. (34) which is orthogonal to u_0. Thus, by definition we have

$$\left\langle u_n^*, u_0 \right\rangle = 0 \, . \tag{38}$$

One notes that u_n^* is the particular solution which has the minimum norm in a Hilbert space where the norm of a function is defined by

$$\| u \| = \sqrt{\left\langle u, u \right\rangle}. \tag{39}$$

The solution u_n of Eq. (34) is determined uniquely if and only if an additional equation allows a unique determination of the coefficient $a_{0,n}$; this is the purpose of the normalization condition.

If one is given a normalization condition for the function u, Eq. (7), the perturbation method then replaces u by its expansion, Eq. (22), and requires that the normalization condition be satisfied for every order of ϵ. This yields a normalization condition for every function u_n.

For instance, if one were to choose the normalization given by Eq. (12) and replace u by its expansion, then if one requires that the coefficient of every power of ϵ vanish, Eq. (12) yields

$$u_0(0) - 1 = 0 , \tag{40}$$

and

$$u_n(0, T) = 0, \text{ where } n \geqslant 1 , \tag{41}$$

and, in the present notations, the unique solution u_n would be

$$u_n(x, T) = u_n^*(x, T) - u_n^*(0, T) u_0(x) . \tag{42}$$

2.2. Normalization Condition

The perturbation method ensures that, for an arbitrarily chosen normalization condition on the function u, the magnitude of the corrections u_n and c_n are of the order ϵ^n. However, an appropriate choice of the normalization condition can improve the accuracy of an approximation which uses only a few terms of the perturbation series.[†]

The following investigates the effect of the normalization condition, that is, the dependence of the successive approximations of u and c upon the coefficients $a_{0,n}$.

An asterisk will denote that a function is orthogonal to u_0; a particular solution u_n with n asterisks denotes the nth-order correction to the flux shape determined uniquely by the normalization condition $\left\langle u_0, u\right\rangle = 0$; i.e., $a_{0,j} = 0$, $j = 1, 2, \cdots, n$.

As long as a normalization condition has not been fixed, $a_{0,n}$ is an arbitrary function of T. Its integral is noted $A_{0,n}$, i.e.,

$$A_{0,n}(T) = \int_0^T a_{0,n}(T)dT . \tag{43}$$

The first approximation to the flux shape and to the flux time are $u_0 = u_0(x)$ and $\theta_0 = u_0 T$, respectively.

2.2.1. First-order perturbation.

The following equation

$$Lu_1^* = [\, g(u_0T) - c_0]\, u_0 = F_1(u_0, \theta_0, c_0) \tag{44}$$

has a unique solution u_1^* if and only if c_0 satisfies

$$\left\langle F_1(u_0, \theta_0, c_0),\ u_0\right\rangle = 0. \tag{45}$$

[†] The effect is similar to considering $1/(1+\epsilon)$ instead of $1-\epsilon$.

The general solution of the first-order perturbation equation, Eq. (30), is then

$$u_1 = u_1^* + a_{0,1} u_0 . \tag{46}$$

The corresponding correction to the flux time is given, according to Eqs. (27) and (43), by

$$\theta_1 = \theta_1^* + A_{0,1}\, u_0 . \tag{47}$$

2.2.2. <u>Second-order perturbation.</u> Given c_0, u_1 and θ_1, the following equation

$$Lu_2^* = F_2(u_0, \theta_0, c_0, u_1, \theta_1, c_1) ,$$

or, more explicitly,

$$Lu_2^* = [\, g(u_0T) - c_0]\, u_1 + [\, \theta_1 g'(u_0T) - c_1]\, u_0, \tag{48}$$

has a unique solution u_2^* if and only if c_1 satisfies

$$\left\langle F_2(u_0, \theta_0, c_0, u_1, \theta_1, c_1),\ u_0 \right\rangle = 0 . \tag{49}$$

Replacing, on the right side of Eq. (48), the functions u_1 and θ_1 by u_1^* and θ_1^*, there results the following equation whose particular solution orthogonal to u_0 is denoted by u_2^{**}

$$Lu_2^{**} = F_2(u_0, \theta_0, c_0, u_1^*, \theta_1^*, c_1^*) \tag{50}$$

where c_1^* satisfies

$$\left\langle F_2(u_0, \theta_0, c_0, u_1^*, \theta_1^*, c_1^*),\ u_0 \right\rangle = 0 . \tag{51}$$

Then a function u_2^{**} exists, and it is unique and independent of $a_{0,1}$ (and of $A_{0,1}$).

The general solution u_2 of the second-order perturbation equation, Eq. (31), and the control absorption c_1 can now be obtained from u_2^{**} and c_1^*. Equation (52) is obtained below by substracting Eq. (50) from Eq. (48) and by using Eqs. (44) through (47):

$$L(u_2^* - u_2^{**}) = (u_1 - u_1^*)\,[\,g(u_0T) - c_0] + (\theta_1 - \theta_1^*)g'(u_0T)u_0 - (c_1 - c_1^*)u_0,$$

$$L(u_2^* - u_2^{**}) = a_{0,1}\, Lu_1^* + A_{0,1}\frac{\partial}{\partial T}[\,Lu_1^* + c_0u_0] - (c - c_1^*)\,u_0\;,$$

and finally

$$L\left\{u_2^* - u_2^{**} - \frac{\partial}{\partial T}\,[A_{0,1}u_1^*]\right\} = -\,[\,c_1 - c_1^* - A_{0,1}\frac{\partial c_0}{\partial T}\,]\,u_0\,. \tag{52}$$

Equation (52) has a solution if its right side is orthogonal to u_0; that is, if c_1 satisfies the condition

$$\langle c_1, u_0, u_0\rangle = \langle c_1^*, u_0, u_0\rangle - A_{0,1}\langle \frac{\partial c_0}{\partial T}\;u_0, u_0\rangle. \tag{53}$$

When the expression $(c_1 - c_1^* - A_{0,1}\,\frac{\partial c_0}{\partial T})$ is independent of the spatial variable x, (for instance, when the control absorber is uniformly distributed), the condition that the right side of Eq. (52) be orthogonal to u_0 requires that it vanish; then Eq. (53) becomes simply,

$$c_1 = c_1^* + A_{0,1}\frac{dc_0}{dT}\;, \tag{54}$$

and the second-order perturbations of the flux shape and of the flux time are now given by Eq. (52) as

$$u_2 = u_2^{**} + \frac{\partial}{\partial T}\,[\,A_{0,1}\,u_1^*\,] + a_{0,2}u_0\;, \tag{55}$$

and

$$\theta_2 = \theta_2^{**} + A_{0,1}u_1^* + A_{0,2}u_0\,. \tag{56}$$

2.2.3. Third-order perturbation. Given $c_0, u_1, \theta_1, c_1, u_2,$ and θ_2, the following equation

$$Lu_3^* = F_3(u_0, \theta_0, c_0, u_1, \theta_1, c_1, u_2, \theta_2, c_2), \tag{57}$$

or, more explicitly,

$$Lu_3^* = [\,g(u_0T) - c_0]\;u_2 + [\,\theta_1 g'(u_0T) - c_1]\,u_1 + [\,\theta_2 g'(u_0T) + \frac{\theta_1^2}{2}\,g''(u_0T) - c_2]\,u_0$$

has a unique solution u_3^* if and only if c_2 satisfies

$$\left\langle F_3(u_0, \theta_0, c_0, u_1, \theta_1, c_1, u_2, \theta_2, c_2),\ u_0 \right\rangle = 0\ . \tag{58}$$

Replacing, on the right side of Eq. (57), the functions u_1, θ_1, c_1, u_2, and θ_2 by u_1^*, θ_1^*, c_1^*, u_2^{**}, and θ_2^{**}, one defines a function u_3^{***}, orthogonal to u_0, as the solution of the equation

$$Lu_3^{***} = F_3(u_0, \theta_0, c_0, u_1^*, \theta_1^*, c_1^*, u_2^{**}, \theta_2^{**}, c_2^{**}), \tag{59}$$

where c_2^{**} satisifies

$$\left\langle F_3(u_0, \theta_0, c_0, u_1^*, \theta_1^*, c_1^*, u_2^{**}, c_2^{**}),\ u_0 \right\rangle = 0\ . \tag{60}$$

Then a function u_3^{***} exists, and it is unique and independent of $a_{0,1}$ and $a_{0,2}$ (and of $A_{0,1}$ and $A_{0,2}$).

The general solution u_3 of the third-order perturbation equation, Eq. (32), and the control absorption c_2, can now be obtained from u_3^{***} and c_2^{**}. Equation (61) is obtained below by subtracting Eq. (59) from Eq. (57) and by using Eqs. (54) through (56), as well as the equations which yielded Eq. (52). One has assumed the control absorber to be uniformly distributed. We have obtained

$$\begin{aligned} L \left\{ u_3^* - u_3^{***} - \frac{\partial}{\partial T}\left[A_{0,1} u_2^{**} + \frac{A_{0,1}^2}{2} \frac{\partial u_1^*}{\partial T} + A_{0,2} u_1^* \right] \right\} \\ = \left\{ -c_2 + c_2^{**} + A_{0,1} \frac{\partial c_1^*}{\partial T} + \frac{A_{0,1}^2}{2} \frac{\partial^2 c_0}{\partial T^2} + A_{0,2} \frac{\partial c_0}{\partial T} \right\} u_0\ . \end{aligned} \tag{61}$$

By assuming the control absorber to be uniformly distributed, Eq. (61) has a solution if the right side is orthogonal to u_0, i. e., if it vanishes; this yields

$$c_2 = c_2^{**} + A_{0,1} \frac{dc_1^*}{dT} + \frac{A_{0,1}^2}{2} \frac{d^2 c_0}{dT^2} + A_{0,2} \frac{dc_0}{dT}, \tag{62}$$

and the third-order perturbation of the flux shape and of the flux time are given by

$$u_3 = u_3^{***} + \frac{\partial}{\partial T} \left[A_{0,1} u_2^{**} + \frac{A_{0,1}^2}{2} \frac{\partial u_1^*}{\partial T} + A_{0,2} u_1^* \right] + a_{0,3} u_0 , \qquad (63)$$

and

$$\theta_3 = \theta_3^{***} + A_{0,1} u_2^{**} + \frac{A_{0,1}^2}{2} \frac{\partial u_1^*}{\partial T} + A_{0,2} u_1^* + A_{0,3} u_0 . \qquad (64)$$

In general, if $a_{0,n}$ (and then $A_{0,n}$) is of the order ϵ^n, one will obtain equations for u_n, θ_n and c_n, such that each term of the equation is of the order ϵ^n.

In the above equations, Eqs. (43) through (64), the symbol T represents a "dummy variable," which would be defined uniquely, according to Eqs. (8) and (10), by choosing a normalization condition. In each of the two following paragraphs, the variable T is defined according to normalization conditions which are of interest for solving Eq. (20) by the perturbation method.

2.2.4. <u>Eigenfunction normalization</u>. Given the first approximation to the flux-shape, u_0, and the corresponding approximation to the flux-time, $\theta_0 = u_0 T$, one defines the variable T by the condition that $u_0 T$ is the best approximation to the actual flux-time θ. Defining the norm of a function according to Eq. (39), the best approximation can be defined by the condition that the norm of the correction term $(\theta - u_0 T)$ be a minimum. This requires that $(\theta - u_0 T)$ be orthogonal to u_0, i.e., that

$$\left\langle (\theta - u_0 T),\ u_0 \right\rangle = 0. \qquad (65)$$

The flux time θ will then be

$$\theta = T u_0 + \theta^* , \qquad (66)$$

and Eq. (65) becomes

$$\left\langle \theta^*, u_0 \right\rangle = 0 . \qquad (67)$$

The variable T is the scalar quantity defined uniquely as a function of the irradiation (i.e., of the flux time to which the fuel has been irradiated) by

$$T = \frac{\langle \theta, u_0 \rangle}{\langle u_0, u_0 \rangle} . \tag{68}$$

Replacing θ by its expansion in the perturbation series given in Eq. (26), Eq. (65) yields

$$A_{0,n} = \frac{\langle \theta_n, u_0 \rangle}{\langle u_0, u_0 \rangle} = 0 \ , \ n \geqslant 1 \ . \tag{69}$$

The successive terms of the perturbation series are then the functions which have been noted $u_1^*, \theta_1^*, c_1^*, u_2^{**}, \theta_2^{**}, c_2^{**}, u_3^{***}$, etc.

2.2.5. Eigenvalue normalization. Given the first approximation to the control absorber, c_0, one defines the variable T by the condition that c_0 is the best approximation to the actual value of the control absorption c. Usually, it is even possible to define T so that c_0 is the exact value of the control absorption. This means that one seeks the solution of the reactor equation, Eq. (20), when the fuel has been irradiated to such an extent that the reactor is just critical with a given amount, c, of control absorber. Therefore, for a specified value of c, the auxiliary variable T is defined by

$$c = c_0(T), \tag{70}$$

where the function $c_0(T)$ is given by Eq. (45).

Replacing c by its expansion, Eq. (21), Eq. (70) yields

$$c_n = 0, \text{ where } n \geqslant 1. \tag{71}$$

This determines, in general, a unique value for each of the coefficients $A_{0,n}$. For instance, Eqs. (54) and (62) yield

$$A_{0,1} = - c_1^* \bigg/ \frac{dc_0}{dT} , \tag{72}$$

and

$$A_{0,2} = - \left[c_2^{**} + A_{0,1} \frac{dc_1^*}{dT} + \frac{A_{0,1}^2}{2} \frac{d^2 c_0}{dT^2} \right] \bigg/ \frac{dc_0}{dT} . \tag{73}$$

However, the normalization condition, Eq. (71), cannot be used when the variable T takes a value T_m for which the function $c_0(T)$ is stationary, i.e., when

$$\frac{dc_0}{dT} = 0 \ . \tag{74}$$

One must then add to the first approximation, $c_0(T_m)$, at least a first corrective term $c_1^*(T_m)$.

The next correction to the control absorption, $c_2(T_m)$, is given by Eq. (62). If the discriminant of the quadratic equation for $A_{0,1}$,

$$c_2(T) = c_2^{**} + A_{0,1} \frac{dc_1^*}{dT} + \frac{A_{0,1}^2}{2} \frac{d^2c_0}{dT^2} = 0 \ , \tag{75}$$

is positive (case 1), there exist two real values of $A_{0,1}$ that satisfy the equation

$$c_2(T_m) = 0 \ . \tag{76}$$

If the discriminant is negative (case 2), it is impossible to satisfy the above equation, Eq. (76), but it is possible to minimize the absolute value of c_2 by choosing the following value for $A_{0,1}$

$$A_{0,1} = - \frac{dc_1^*}{dT} \Big/ \frac{d^2c_0}{dT^2} \ . \tag{77}$$

The physical meaning of the above results can be understood as follows: One assumes that the control absorption varies with the irradiation parameter as a function $c(T)$ which has a maximum c_m. The corresponding function $c_0(T)$, which has a maximum $c_0(T_m)$, has been plotted in Fig. II-1. If $c_0(T_m) > c_m$, the reactor containing an amount of control absorber corresponding to $c_0(T_m)$ could not be made critical at any irradiation. When one adds the correction term $c_1^*(T_m)$ to $c_0(T_m)$, there exist either (a) two values of the irradiation T, each of which corresponds to a just critical reactor containing an amount of control absorber $c_0(T_m) + c_1^*(T_m)$, (case 1), or (b) no value of T for which criticality can be achieved (case 2).

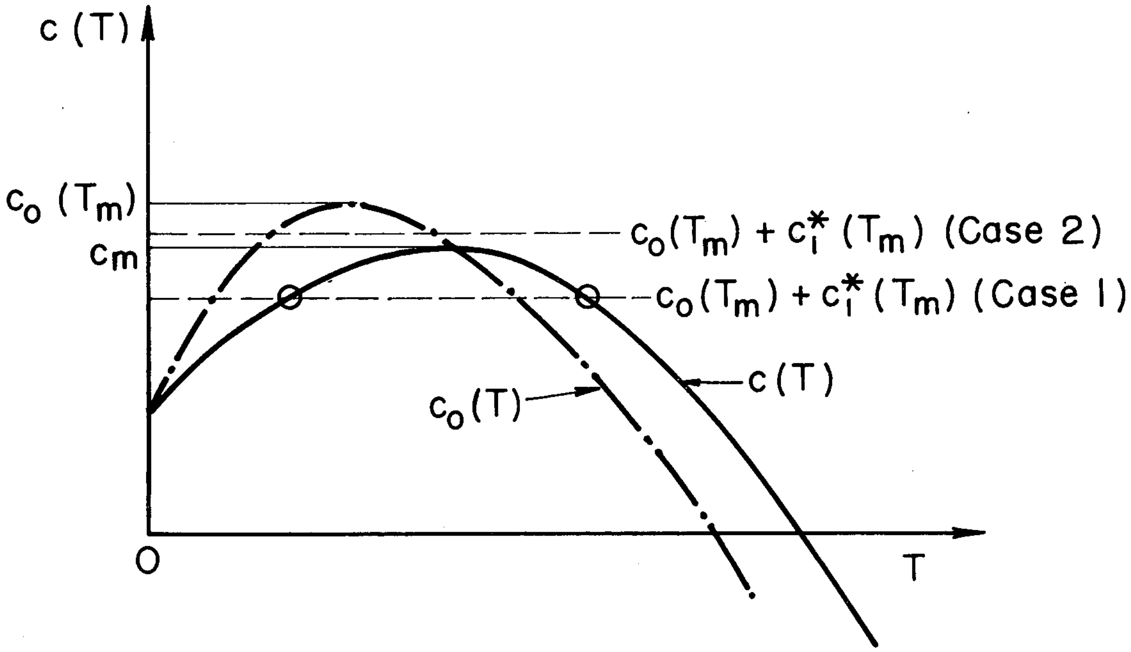

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Fig. II-1. Variation of the control absorber with irradiation.

2.3. Eigenfunction Expansion

The solution of the nonhomogeneous linear differential equation, Eq. (34), can be obtained by using the eigenfunction expansion technique. This technique is developed below and the expansion of the solution of the first-order perturbation to the flux shape is obtained.

The first-order perturbation equation is given by Eqs. (25) and (30). It reads

$$-\left(\frac{1}{B_0^2}\Delta u_1 + u_1\right) = [g(u_0T) - c_0]\, u_0 . \tag{78}$$

This equation has a solution u_1 if, and only if, the right side is orthogonal to the solution u_0 of the corresponding homogeneous equation, Eq. (24). According to Eq. (45), the magnitude of the first approximation to the control term c_0 is then determined by

$$\left\langle [g(u_0T) - c_0]\, u_0, u_0 \right\rangle = 0 , \tag{79}$$

or by

$$\left\langle c_0\, u_0, u_0 \right\rangle = \left\langle g(u_0T)\, u_0, u_0 \right\rangle . \tag{80}$$

This ensures that Eq. (78) has a solution; however, it is only possible to obtain this solution in a closed analytical form for one-dimensional problems with very simple expressions for the function $g(\theta)$.

In general, the solution u_1 can always be expanded in an infinite series of the eigenfunctions of the eigenvalue equation

$$\Delta v + B^2 v = 0 , \tag{81}$$

since this equation possesses a complete set of eigenfunctions satisfying the linear homogeneous boundary condition, Eq. (3). These eigenfunctions are the functions v_k which satisfy

$$\Delta v_k + B_k^2 v_k = 0 . \tag{82}$$

Because the Laplacian operator Δ is self-adjoint (Green's theorem), these eigenfunctions are orthogonal, i.e.,

$$\left\langle v_k, v_\ell \right\rangle = 0 \text{ if } k \neq \ell . \tag{83}$$

The eigenvalues B_k^2 are all positives; u_0, the nonnegative solution of the unperturbed equation, Eq. (24), is a multiple of the eigenfunction v_0 corresponding to the smallest eigenvalue B_0^2.

For plane, cylindrical, and spherical geometries, the functions v_k are trigonometric functions, Bessel functions, or products of these functions. They are then normalized to unity at the center of the volume, by the condition

$$v_k(0) = 1 . \tag{84}$$

The solution u_1 of Eq. (78) is then sought as an expansion of the form

$$u_1(x, T) = \sum_{k=0}^{\infty} a_{k,1}(T)\, v_k(x) . \tag{85}$$

The property of orthogonality of the eigenfunctions v_k yields a simple technique for evaluating the coefficients $a_{k,1}$.

In Eq. (78), one replaces u_1 by its expansion, Eq. (85); making use of Eq. (82), one obtains

$$\sum_{\ell=0}^{\infty} \frac{B_\ell^2 - B_0^2}{B_0^2}\, a_{\ell,1}\, v_\ell = [\, g(u_0 T) - c_0]\, u_0 . \tag{86}$$

The coefficient $a_{k,1}$ is then obtained by taking the scalar product of both sides of Eq. (86) by the eigenfunction v_k; because of the orthogonality of the functions v_k, Eq. (83), one is left with

$$\frac{B_k^2 - B_0^2}{B_0^2}\, a_{k,1} \left\langle v_k, v_k \right\rangle = \left\langle [\, g(u_0 T) - c_0]\, u_0, v_k \right\rangle . \tag{87}$$

The criticality condition, which determines c_0 by Eq. (80), ensures that Eq. (87) is satisfied for $k = 0$.

The coefficient $a_{0,1}$ is left undetermined, and the coefficients $a_{k,1}$, where $k \neq 0$, are the functions of the irradiation parameter T defined by Eq. (87) as

$$a_{k,1}(T) = \frac{B_0^2}{B_k^2 - B_0^2} \frac{\left\langle [g(u_0 T) - c_0]\, u_0, v_k \right\rangle}{\left\langle v_k, v_k \right\rangle} . \tag{88}$$

The particular solution u_1^*, which is orthogonal to the unperturbed flux shape u_0, is then determined uniquely by the expansion

$$u_1^*(x, T) = \sum_{k=1}^{\infty} a_{k,1}(T) v_k(x) . \tag{89}$$

The general solution u_1, of the first-order perturbation equation, Eq. (78), is given by Eq. (46) where the coefficient $a_{0,1}$ must be defined by a normalization condition. For instance, if one were to require that the flux shape be normalized to unity at the center of the reactor, i.e., that according to Eq. (41) the function u be normalized to zero at the center, then the corresponding first-order perturbation of the flux shape would have the expansion

$$u_1(x, T) = \sum_{k=1}^{\infty} a_{k,1}(T) \left[v_k(x) - v_0(x) \right] , \tag{90}$$

where the functions v_k satisfy the condition expressed by Eq. (84).

If we are given the analytic expression of the function $g(\theta)$, and once we know the eigenfunctions v_k and the eigenvalues B_k^2, then each coefficient $a_{k,1}$ (where $k \neq 0$) can be evaluated as a function of the irradiation T if we perform the integrations indicated by the scalar products in Eq. (88). The numerical computations are developed in the following sections. One will note the important simplifications resulting from the fact that the first-order perturbation of the flux shape, u_1, is a linear function of $g(\theta)$.

The first-order perturbation of the flux time is obtained by integration of the first-order perturbation of the flux shape according to Eq. (27). Thus, the expansion of θ_1 in the eigenfunctions v_k is given by

$$\theta_1(x, T) = \sum_{k=0}^{\infty} A_{k,1}(T)\, v_k(x), \tag{91}$$

with

$$A_{k,1}(T) = \int_0^T a_{k,1}(T)\, dt . \tag{92}$$

The second-order perturbation equation, Eq. (48), has a solution if c_1 satisfies Eq. (49). Given u_1^* and θ_1^* by their eigenfunction expansions, c_1 is determined according to Eq. (53), where $\langle c_1^* u_0, u_0 \rangle$ is obtained as follows: Eqs. (48) and (51) yield .

$$\left\langle [\, g(u_0T) - c_0]\; u_1^*, u_0 \right\rangle + \left\langle [\, \theta_1^* g'(u_0T) - c_1^*\,] u_0, u_0 \right\rangle = 0 . \tag{93}$$

Using Eq. (27), the above equation can be rewritten as

$$\left\langle c_1^* u_0, u_0 \right\rangle = \frac{\partial}{\partial T} \left[\left\langle \theta_1^* g(u_0T), u_0 \right\rangle \right] - \left\langle c_0 u_1^*, u_0 \right\rangle . \tag{94}$$

Finally, using the eigenfunction expansions, Eqs. (89) and (91), one obtains

$$\left\langle c_1^* u_0, u_0 \right\rangle = \sum_{k=1}^{\infty} \left\{ \frac{\partial}{\partial T} \left[A_{k,1}(T) \left\langle v_k g(u_0T), u_0 \right\rangle \right] - a_{k,1}(T) \left\langle c_0 v_k, u_0 \right\rangle \right\} . \tag{95}$$

The second-order perturbation equation, Eq. (48), can now be solved by the eigenfunction expansion technique used to solve the first-order perturbation equation. More generally, the <u>n</u>th-order correction to the flux shape, u_n^*, solution of the general Eq. (34), can be expanded in an infinite series of the eigenfunctions v_k given by

$$u_n^* = \sum_{k=1}^{\infty} a_{k,n}(T)\, v_k(x), \tag{96}$$

where the coefficients $a_{k,n}$ are defined by

$$a_{k,n} = \frac{B_0^2}{B_k^2 - B_0^2} \cdot \frac{\langle F_n, v_k \rangle}{\langle v_k, v_k \rangle} . \tag{97}$$

The normalization condition given by Eqs. (70) and (71) will be used in the computations which are developed in the following sections. This best fits the problem of finding the properties of the reactor at the end of reactivity lifetime (end of life) since the final value T_f of the irradiation parameter is then the root of

$$c_0(x, T_f) \equiv 0 . \tag{98}$$

It has been shown in the preceding section that Eq. (71) is a valid normalization condition, except when the parameter T takes a value T_m which satisfies Eq. (74). At $T = T_m$, the removal of control poison needed to continue the irradiation of the fuel in a critical reactor is stationary to a first approximation. It is then very unlikely that $T_f = T_m$, since one would miss an extremely favorable opportunity to increase the life time of the reactor.

3. Application: Uniform Fuel Loading, Uniform Control

3.1. Calculation Procedure

The following study is limited to the approximation afforded by a second-order perturbation method.

The control absorber needed to keep the reactor just critical will be assumed uniformly distributed over the volume of the reactor throughout the irradiation. Since c is now a function of the irradiation variable T only, c can be taken out of the brackets that represent the scalar products in the preceding equations; when the brackets enclose only two different eigenfunctions, the result vanishes according to Eq. (83). Also, u_0 will be replaced everywhere by v_0, since u_0 and v_0 are the solutions of the same equation, Eq. (5) or (29), and can be taken to be equal.

To simplify the notation, we have written a_k and A_k instead of $a_{k,1}$ and $A_{k,1}$, since these are the only coefficients which are considered hereafter.

Under these assumptions, one obtains the following results. Equation (80) becomes

$$c_0(T) = \frac{\left\langle g(v_0 T)\, v_0, v_0 \right\rangle}{\left\langle v_0, v_0 \right\rangle} , \tag{100}$$

and Eq. (88) becomes

$$a_k(T) = \frac{B_0^2}{B_k^2 - B_0^2} \cdot \frac{\left\langle g(v_0 T)\, v_0, v_k \right\rangle}{\left\langle v_k, v_k \right\rangle} \text{ , where } k \neq 0. \tag{101}$$

Besides using $A_k(T)$, defined by Eq. (92), one will also use the dimensionless coefficient $\alpha_k(T)$ defined by

$$\alpha_k(T) = A_k(T)/T = \frac{1}{T} \int_0^T a_k(T) dt . \tag{102}$$

Equation (95) becomes

$$c_1^*(T) = \sum_{k=1}^{\infty} \frac{d}{dT} \left[T\, \alpha_k(T) \left\langle v_k g(v_0 T), v_0 \right\rangle \right]. \tag{103}$$

The normalization condition, Eq. (72), then determines $A_0(T)$ uniquely; we have

$$\alpha_0(T) = A_0(T)/T , \tag{104}$$

and

$$a_0(T) = \frac{dA_0(T)}{dT} . \tag{105}$$

One of the most important characteristics of the fuel irradiated in a reactor is its burnup. For the present purposes, we are given the variation of the local burnup of the fuel with the irradiation; i.e., we are given a function $\beta(\theta)$ which will be referred to as the local burnup. Its derivative is noted $\beta'(\theta)$.

The perturbation method expands the function $\beta(\theta)$ in an infinite series as it has been done for the function $g(\theta)$, Eq. (28).

To the present approximation, the variation of the burnup with position and irradiation is given by

$$\beta(x, T) = \beta(\theta) = \beta(v_0 T) + \theta_1 \beta'(v_0 T) . \qquad (106)$$

The central burnup is $\beta(0, T)$, obtained by evaluating the right side of the above equation for $x = 0$.

The average burnup will be noted $\langle \beta \rangle$, since it is the scalar product of the function β by the function unity, according to the definition of the scalar product in Appendix A.

The first approximation of the average burnup $\langle \beta \rangle_0$ is given by

$$\langle \beta \rangle_0 = \langle \beta(v_0 T) \rangle . \qquad (107)$$

The correction term corresponding to the correction θ_1 of the flux time is defined according to Eq. (106) by

$$\langle \beta \rangle_1 = \langle \theta_1, \beta'(v_0 T) \rangle . \qquad (108)$$

The calculation proceeds as follows:

To a reactor of given geometry, there corresponds a given set of eigenfunctions v_k, of eigenvalues B_k^2, and of coefficients h_k defined by

$$h_k = \frac{\langle v_k, v_k \rangle}{\langle v_0, v_0 \rangle} . \qquad (109)$$

For a given fuel irradiated in such a reactor, i.e., for given functions $g(\theta)$ and $\beta(\theta)$, one first computes the following dimensionless functions of the irradiation parameter T:

$$s_k(T) = \frac{\langle g(v_0 T) v_0, v_k \rangle}{\langle v_k, v_k \rangle} , \qquad (110)$$

$$S_k(T) = \frac{1}{T} \int_0^T s_k(T)\, dT = \frac{1}{T} \frac{\left\langle \left[\int_0^T g(v_0 T)\, dT \right] v_0, v_k \right\rangle}{\langle v_k, v_k \rangle} , \qquad (111)$$

$$\left\langle \beta \right\rangle_0 = \left\langle \beta(v_0 T) \right\rangle, \tag{112}$$

and

$$T \left\langle v_k, \beta'(v_0 T) \right\rangle, \tag{113}$$

where, as a first approximation, v_0T has been set equal to θ. Then, the first approximation of the flux shape is $u_0(x) = v_0(x)$, and Eqs. (100) and (110) give the first approximation of the control absorption as

$$c_0(T) = s_0(T) . \tag{114}$$

The first approximation of the flux shape change is

$$u_1^*(x, T) = \sum_{k=1}^{\infty} a_k(T)\, v_k(x), \tag{115}$$

and Eqs. (101) and (110) yield

$$a_k(T) = \frac{B_0^2}{B_k^2 - B_0^2}\, s_k(T) \text{ , where } k \neq 1. \tag{116}$$

The corresponding correction of the control absorption is given by Eqs. (103), (109) and (110) as

$$c_1^*(T) = \sum_{k=1}^{\infty} h_k \frac{d}{dT} [\, T\, a_k(T)\, s_k(T)] \ . \tag{117}$$

At end of life, the irradiation parameter is the root of the equation

$$s_0(T) = 0 \ . \tag{118}$$

The first approximation of the flux time is $\theta_0(x, T) = v_0(x)T$ and the corrective term is

$$\theta_1(x, T) = T \sum_{k=0}^{\infty} a_k(T)\, v_k(x) \ . \tag{119}$$

Equations (72) and (104) yield

$$\alpha_0(T) = - c_1^*(T) \Big/ T \; \frac{ds_0(T)}{dT} \, , \tag{120}$$

and Eqs. (91), (102), and (111) yield

$$\alpha_k(T) = \frac{B_0^2}{B_k^2 - B_0^2} \, S_k(T), \text{ where } k \neq 0 \, . \tag{121}$$

The first approximation of the average burnup is $\langle \beta \rangle_0 = \langle \beta(v_0 T) \rangle$, and Eqs. (108) and (119) give the corrective term as

$$\langle \beta \rangle_1 = \sum_{k=0}^{\infty} \alpha_k(T) \left[T \langle v_k, \beta'(v_0 T) \rangle \right] \, . \tag{122}$$

The summations over the countable infinite sequence of subscripts k will be restricted to a sum of the terms corresponding to the first few values of k.

The functions given by Eqs.(110) through (113) must be evaluated for each fuel-cycle analysis. However, the next section defines some auxiliary functions which, once tabulated, will make this evaluation available to hand computations for any given fuel irradiated in reactors of simple geometrical shapes.

For hand-computation purposes, it is convenient to evaluate the derivative of a function f(T) by the formula

$$\frac{df}{dT} \approx \frac{f(T_1) - f(T_2)}{T_1 - T_2} \, . \tag{123}$$

Accordingly, the computation of the right side of Eqs. (117) and (120) requires the determination of only the functions $s_k(T)$ and $S_k(T)$.

For instance, the coefficient $\alpha_0(T)$, defined by Eqs. (120), (117), and (121), will be computed according to

$$\alpha_0(T) = - \frac{1}{T} \sum_{k=1}^{\infty} \frac{B_0^2}{B_k^2 - B_0^2} \, h_k \, \frac{T_1 S_k(T_1) s_k(T_1) - T_2 S_k(T_2) s_k(T_2)}{s_0(T_1) - s_0(T_2)} \, . \tag{124}$$

Equation (124) is exactly satisfied when T_1 and T_2 tend simultaneously toward T, if we assume that $\alpha_0(T)$ is a continuous function of T and that the derivative of $s_0(T)$ is nonzero.

3.2. Definition of the Auxiliary Functions

The fuel-cycle analysis developed by Benedict and Pigford[B1] shows how the neutron excess production and the burnup of the fuel can be represented by a sum of exponential functions of the flux time and an additional linear term; i.e.,

$$g(\theta) = g_0 + g_1\theta + \sum_j g_j e^{-\sigma_j \theta} , \tag{125}$$

and

$$\beta(\theta) = b_0 + b_1\theta + \sum_j b_j e^{-\sigma_j \theta} , \tag{126}$$

In the case of a uniform initial fuel loading, the case which will be considered here, the coefficients g and b are constants determined by the properties of the equivalent homogeneous mixture which is initially loaded in the reactor; they are given by standard formulae.[B1]

Auxiliary functions and coefficients are defined below and will be tabulated in the following sections. The functions defined by Eqs. (110) through (113) are then evaluated according to Eqs. (132), (133), (135), and (141), respectively.

Equations (110) and (125) yield the following representation of the functions $s_k(T)$ as linear functions of the coefficients g:

$$s_k(T) = g_0 \frac{\langle v_0, v_k \rangle}{\langle v_k, v_k \rangle} + g_1 \frac{\langle v_0^2, v_k \rangle}{\langle v_k, v_k \rangle} T + \sum_j g_j \frac{\langle e^{-\sigma_j T v_0} v_0, v_k \rangle}{\langle v_k, v_k \rangle} . \tag{127}$$

Then we define a function of a variable w, $E_{2,k}(w)$, by

$$E_{2,k}(w) = \frac{\langle e^{-wv_0} v_0, v_k \rangle}{\langle v_k, v_k \rangle} , \tag{128}$$

and a coefficient $f_{2,k}$ by

$$f_{2,k} = \frac{\left\langle v_0^2, v_k \right\rangle}{\left\langle v_k, v_k \right\rangle}. \tag{129}$$

Because of the orthogonality of the eigenfunctions, Eq. (83), one can write

$$\frac{\left\langle v_0, v_k \right\rangle}{\left\langle v_k, v_k \right\rangle} = \delta_{0,k} , \tag{130}$$

where $\delta_{i,j}$ is the Kronecker symbol defined by the property

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j . \end{cases} \tag{131}$$

With these definitions, Eq. (127) becomes

$$s_k(T) = g_0 \ \delta_{0,k} + g_1 f_{2,k} T + \sum_j g_j E_{2,k}(\sigma_j T) . \tag{132}$$

The function $S_k(T)$ is then obtained by integration and division by T, according to Eq. (111), and one obtains

$$S_k(T) = g_0 \ \delta_{0,k} + g_1 \ \frac{f_{2,k}}{2} \ T + \sum_j g_j E_{3,k}(\sigma_j T), \tag{133}$$

with the corresponding definition of a function $E_{3,k}(w)$ by

$$E_{3,k}(w) = \frac{1}{w} \int_0^w E_{2,k}(w)\, dw = \frac{1}{w} \ \frac{\left\langle (1-e^{-wv_0}), v_k \right\rangle}{\left\langle v_k, v_k \right\rangle}. \tag{134}$$

Equations (122) and (126) give the first approximation to the average burnup by

$$\left\langle \beta(v_0 T) \right\rangle = b_0 + b_1 f_0 T + \sum_j b_j E_0(\sigma_j T) , \tag{135}$$

where one has defined the coefficient f_0 by

$$f_0 = \langle 1, v_0 \rangle, \tag{136}$$

and the function $E_0(w)$ by

$$E_0(w) = \langle 1, e^{-wv_0} \rangle. \tag{137}$$

Tables of the coefficient f_0 and of the function $E_0(w)$ are given by Benedict and Pigford.[B1]

Using Eq. (126), the functions defined by Eq. (113) become

$$T \langle v_k, \beta'(v_0T) \rangle = b_1 \langle 1, v_k \rangle T - \sum_j b_j \sigma_j T \langle v_k, e^{-\sigma_j T v_0} \rangle \tag{138}$$

One defines a coefficient $f_{1,k}$ by

$$f_{1,k} = \langle 1, v_k \rangle, \tag{139}$$

and a function $E_{1,k}(w)$ by

$$E_{1,k}(w) = w \langle e^{-wv_0}, v_k \rangle, \tag{140}$$

and obtains

$$T \langle v_k, \beta'(v_0T) \rangle = b_1 f_{1,k} T - \sum_j b_j E_{1,k}(\sigma_j T). \tag{141}$$

The above defined auxiliary functions $E_{1,k}(w)$, $E_{2,k}(w)$, $E_{3,k}(w)$, and coefficients $f_{1,k}$, and $f_{2,k}$, will be tabulated for given sets of eigenfunctions v_k corresponding to bare and reflected reactors of simple geometrical shapes. The numerical study of the irradiation-dependent characteristics of a reactor which has been initially uniformly loaded with any fuel [i.e., for any set of coefficients g and b in Eqs. (125) and (126)] will then become amenable to hand computations.

The three following sections will be concerned with reactor cores shaped as slabs, spheres, or finite circular cylinders.

The quantity μ_k denotes the coefficient defined as

$$\mu_k = \langle v_k, v_k \rangle, \tag{142}$$

and Eq. (109) then yields

$$h_k = \mu_k/\mu_0 \ . \tag{143}$$

The functions $E_{1,k}(w)$ from Eq. (140), and $E_{3,k}(w)$ from Eq. (134), are related by the equation

$$w\mu_k \, E_{3,k}(w) = f_{1,k} - E_{1,k}(w)/w \ . \tag{144}$$

The function $E_0(w)$ and the coefficient $f_0 = f_{1,0}$, as well as the function $E_{2,0}(w)$ and the coefficient $f_{2,0}$ [that Sola denotes by $E_2(w)$ and f_2], have already been computed.S2

The following approximations hold for small w:

$$\begin{aligned} E_{1,k}(w) &= f_{1,k}w \ , \\ E_{2,k}(w) &= \delta_{0,k} - f_{2,k}w \ , \\ \text{and} \quad E_{3,k}(w) &= \delta_{0,k} - \tfrac{1}{2} f_{2,k}w \ . \end{aligned} \tag{145}$$

3.3. The Slab Reactor

The reactor core is a slab of thickness 2Z, and is symmetrical about its midplane xy. It is infinite, or perfectly reflected, along the directions x and y. The composition of the core, and then the flux, is a function of the time and of the axial coordinate z only. The eigenfunction equation, Eq. (82), becomes

$$\frac{d^2 v_k}{dz^2} + B_k^2 v_k = 0 \ , \tag{146}$$

with the boundary condition

$$v + \delta_Z \frac{dv}{dz} = 0 \quad \text{at} \quad z = Z \ , \tag{147}$$

and the symmetry condition

$$\frac{dv}{dz} = 0 \quad \text{at} \quad z = 0 \ .$$

Equation (146) has an infinite number of eigenfunctions

$$v_k(z) = \cos B_k z \ . \tag{148}$$

The corresponding eigenvalues, B_k^2, are such that the boundary condition, Eq. (147), is satisfied; they are given by the positive roots of

$$\cos B_k Z - \delta_Z B_k \sin B_k Z = 0 . \tag{149}$$

These results will be expressed in dimensionless form as follows: The change of variable,

$$\zeta = z/Z \quad , \quad 0 \leqslant \zeta \leqslant 1 , \tag{150}$$

yields

$$v_k(\zeta) = \cos \gamma_k \zeta , \tag{151}$$

with

$$B_k^2 = \gamma_k^2/Z^2 . \tag{152}$$

The ratio of the linear extrapolation distance δ_Z to the half thickness of the slab Z is noted as

$$\epsilon_Z = \delta_Z/Z . \tag{153}$$

The eigenvalue γ_k is then defined according to Eq. (149), as the $(k+1)^{th}$ root of the equation

$$\cos \gamma_k - \epsilon_Z \gamma_k \sin \gamma_k = 0 . \tag{154}$$

For a bare reactor with negligible extrapolation distance, where $\epsilon_Z = 0$, one obtains

$$\gamma_k = (2k+1)\pi/2 . \tag{156}$$

For a given positive value of ϵ_Z, the roots of Eq. (1954) can be obtained from tabulations or graphical representations of the function $\gamma \tan \gamma$.[J2] There is an infinite number of roots which satisfy the inequality

$$k\pi < \gamma_k < (2k+1)\pi/2 . \tag{157}$$

For very large values of k and for positive ϵ_Z, γ_k tends toward $k\pi$.

The scalar product of two functions $f(\zeta)$ and $g(\zeta)$, Eq. (4), is now:

$$\langle f, g \rangle = \int_0^1 f(\zeta)\, g(\zeta)\, d\zeta \tag{158}$$

The auxiliary functions and coefficients can be evaluated as follows:

$$\mu_k = \int_0^1 \cos^2 \gamma_k\zeta \, d\zeta = \frac{1}{2}\left(1 + \frac{\sin 2\gamma_k}{2\gamma_k}\right), \tag{159}$$

$$h_k = \mu_k / \mu_0 = \left(1 + \frac{\sin 2\gamma_k}{2\gamma_k}\right) \Big/ \left(1 + \frac{\sin 2\gamma_0}{2\gamma_0}\right), \tag{160}$$

$$f_{1,k} = \int_0^1 \cos \gamma_k\zeta \, d\zeta = \frac{\sin \gamma_k}{\gamma_k}, \tag{161}$$

and

$$f_{2,k} = \frac{1}{\mu_k} \int_0^1 \cos^2 \gamma_0\zeta \cos \gamma_k\zeta \, d\zeta$$

$$= \frac{1}{2\mu_k} \left\{ \frac{\sin \gamma_k}{\gamma_k} + \frac{1}{2} \left[\frac{\sin(2\gamma_0+\gamma_k)}{2\gamma_0+\gamma_k} + \frac{\sin(2\gamma_0-\gamma_k)}{2\gamma_0-\gamma_k} \right] \right\}. \tag{162}$$

The latter result can be simplfied by using Eq. (154) to yield

$$f_{2,k} = \frac{4\gamma_0^2}{4\gamma_0^2-\gamma_k^2} \left(\frac{\sin \gamma_k/\gamma_k}{1 + \frac{\sin 2\gamma_k}{2\gamma_k}} \right). \tag{163}$$

The functions $E_{1,k}(w)$, $E_{2,k}(w)$ and $E_{3,k}(w)$ defined respectively by Eqs. (140), (128), and (134) become:

$$E_{1,k}(w) = w \int_0^1 e^{-w \cos \gamma_0\zeta} \cos \gamma_k\zeta \, d\zeta \tag{164}$$

$$E_{2,k}(w) = \frac{1}{\mu_k} \int_0^1 e^{-w \cos \gamma_0\zeta} \cos \gamma_0\zeta \cos \gamma_k\zeta \, d\zeta \tag{165}$$

$$E_{3,k}(w) = \frac{1}{w\mu_k} \int_0^1 \left(1 - e^{-w \cos \gamma_0\zeta}\right) \cos \gamma_k\zeta \, d\zeta \tag{166}$$

In order to establish a table of these functions, one has to evaluate numerically the integrals on the right side of the above equations. This has been done for the case of a bare reactor with $\epsilon_Z = 0$. The integrals on the right side of Eqs. (164) and (165) are evaluated by Gauss quadratures. The derivatives of the functions to be integrated increase with γ_k, and if one uses a Gauss quadrature formula with a fixed number of points, the accuracy decreases when k increases. The computations have been performed on a high-speed digital computer according to a Fortran program which uses a 4, 6, 8, or 10-point Gauss quadrature formula. The 10-point formula gives the values of the auxiliary functions up to k = 5, accurate to the fourth decimal place (Tables II-1 through II-3).

For this case (where $\epsilon_Z = 0$) the various coefficients required for the computation of the flux shape, flux time and burnup have the following expressions:

$$\gamma_k = (2k + 1)\pi/2 \,, \tag{167}$$

$$\frac{B_0^2}{B_k^2 - B_0^2} = \frac{\gamma_0^2}{\gamma_k^2 - \gamma_0^2} = \frac{1}{(2k + 1)^2 - 1} \,, \tag{168}$$

$$\mu_k = \frac{1}{2} \quad \text{and} \quad h_k = 1, \quad \text{for any } k, \tag{169}$$

$$f_{1,k} = (-1)^k \frac{2}{(2k + 1)\pi} \,, \tag{170}$$

and

$$f_{2,k} = (-1)^{k+1} \frac{8}{(2k - 1)(2k + 1)(2k + 3)\pi} \,. \tag{171}$$

Table II-1. Bare slab reactor: functions $E_{1,k}(w)$.

w	k=0	k=1	k=2	k=3	k=4	k=5
0.10	0.0589	-0.0212	0.0127	-0.0091	0.0071	-0.0058
0.20	0.1089	-0.0421	0.0254	-0.0182	0.0141	-0.0116
0.30	0.1512	-0.0627	0.0380	-0.0272	0.0212	-0.0173
0.40	0.1868	-0.0826	0.0505	-0.0362	0.0282	-0.0231
0.50	0.2163	-0.1019	0.0629	-0.0452	0.0353	-0.0289
0.60	0.2407	-0.1204	0.0751	-0.0541	0.0422	-0.0346
0.70	0.2605	-0.1381	0.0871	-0.0630	0.0492	-0.0403
0.80	0.2764	-0.1549	0.0989	-0.0717	0.0561	-0.0460
0.90	0.2888	-0.1707	0.1105	-0.0804	0.0630	-0.0517
1.00	0.2982	-0.1856	0.1217	-0.0890	0.0698	-0.0574
1.10	0.3051	-0.1996	0.1327	-0.0974	0.0766	-0.0630
1.20	0.3097	-0.2126	0.1434	-0.1058	0.0833	-0.0686
1.30	0.3124	-0.2247	0.1538	-0.1140	0.0900	-0.0742
1.40	0.3135	-0.2359	0.1638	-0.1221	0.0966	-0.0797
1.50	0.3132	-0.2462	0.1735	-0.1300	0.1031	-0.0852
1.60	0.3117	-0.2556	0.1828	-0.1378	0.1096	-0.0906
1.70	0.3093	-0.2642	0.1919	-0.1454	0.1159	-0.0960
1.80	0.3060	-0.2720	0.2005	-0.1529	0.1222	-0.1014
1.90	0.3020	-0.2791	0.2088	-0.1602	0.1284	-0.1067
2.00	0.2975	-0.2854	0.2167	-0.1674	0.1345	-0.1119
2.10	0.2925	-0.2911	0.2243	-0.1743	0.1406	-0.1171
2.20	0.2872	-0.2960	0.2315	-0.1811	0.1465	-0.1223
2.30	0.2816	-0.3004	0.2384	-0.1877	0.1523	-0.1274
2.40	0.2758	-0.3042	0.2450	-0.1942	0.1581	-0.1324
2.50	0.2698	-0.3074	0.2512	-0.2004	0.1637	-0.1374
2.60	0.2638	-0.3102	0.2570	-0.2065	0.1692	-0.1423
2.70	0.2577	-0.3124	0.2625	-0.2124	0.1747	-0.1471
2.80	0.2516	-0.3143	0.2677	-0.2180	0.1800	-0.1519
2.90	0.2456	-0.3157	0.2726	-0.2236	0.1852	-0.1567
3.00	0.2396	-0.3167	0.2772	-0.2289	0.1903	-0.1613
3.10	0.2337	-0.3173	0.2815	-0.2340	0.1953	-0.1659
3.20	0.2278	-0.3177	0.2855	-0.2390	0.2002	-0.1704
3.30	0.2221	-0.3177	0.2892	-0.2438	0.2050	-0.1749
3.40	0.2165	-0.3174	0.2926	-0.2484	0.2096	-0.1792
3.50	0.2110	-0.3169	0.2958	-0.2528	0.2142	-0.1835
3.60	0.2057	-0.3161	0.2987	-0.2570	0.2186	-0.1878
3.70	0.2005	-0.3152	0.3014	-0.2611	0.2229	-0.1919
3.80	0.1954	-0.3140	0.3039	-0.2650	0.2271	-0.1960
3.90	0.1905	-0.3127	0.3061	-0.2687	0.2312	-0.2000
4.00	0.1857	-0.3112	0.3081	-0.2723	0.2352	-0.2039
4.10	0.1811	-0.3095	0.3099	-0.2757	0.2391	-0.2078
4.20	0.1767	-0.3077	0.3115	-0.2789	0.2429	-0.2116
4.30	0.1723	-0.3058	0.3129	-0.2820	0.2465	-0.2153
4.40	0.1682	-0.3038	0.3141	-0.2849	0.2501	-0.2189
4.50	0.1641	-0.3017	0.3151	-0.2877	0.2535	-0.2225
4.60	0.1602	-0.2995	0.3160	-0.2903	0.2568	-0.2259
4.70	0.1565	-0.2972	0.3167	-0.2928	0.2600	-0.2293
4.80	0.1528	-0.2949	0.3173	-0.2951	0.2631	-0.2327
4.90	0.1493	-0.2925	0.3177	-0.2973	0.2661	-0.2359
5.00	0.1460	-0.2901	0.3180	-0.2994	0.2690	-0.2391

Table II-2. Bare slab reactor: functions $E_{2,k}(w)$.

w	k=0	k=1	k=2	k=3	k=4	k=5
0.10	0.9188	-0.0158	0.0024	-0.0008	0.0004	-0.0002
0.20	0.8444	-0.0293	0.0048	-0.0016	0.0007	-0.0004
0.30	0.7762	-0.0409	0.0071	-0.0024	0.0011	-0.0006
0.40	0.7138	-0.0507	0.0094	-0.0032	0.0015	-0.0008
0.50	0.6567	-0.0590	0.0116	-0.0040	0.0018	-0.0010
0.60	0.6043	-0.0658	0.0137	-0.0047	0.0022	-0.0012
0.70	0.5563	-0.0715	0.0156	-0.0055	0.0025	-0.0014
0.80	0.5122	-0.0761	0.0175	-0.0062	0.0029	-0.0016
0.90	0.4719	-0.0797	0.0192	-0.0069	0.0032	-0.0018
1.00	0.4349	-0.0825	0.0209	-0.0076	0.0036	-0.0019
1.10	0.4009	-0.0846	0.0224	-0.0083	0.0039	-0.0021
1.20	0.3697	-0.0861	0.0237	-0.0089	0.0042	-0.0023
1.30	0.3411	-0.0870	0.0250	-0.0096	0.0045	-0.0025
1.40	0.3148	-0.0874	0.0261	-0.0102	0.0049	-0.0027
1.50	0.2907	-0.0874	0.0272	-0.0107	0.0052	-0.0028
1.60	0.2685	-0.0870	0.0281	-0.0113	0.0055	-0.0030
1.70	0.2482	-0.0864	0.0289	-0.0118	0.0057	-0.0032
1.80	0.2295	-0.0854	0.0296	-0.0123	0.0060	-0.0034
1.90	0.2123	-0.0843	0.0302	-0.0127	0.0063	-0.0035
2.00	0.1964	-0.0830	0.0308	-0.0131	0.0065	-0.0037
2.10	0.1819	-0.0815	0.0312	-0.0135	0.0068	-0.0038
2.20	0.1685	-0.0799	0.0316	-0.0139	0.0070	-0.0040
2.30	0.1561	-0.0781	0.0318	-0.0143	0.0073	-0.0041
2.40	0.1448	-0.0763	0.0321	-0.0146	0.0075	-0.0043
2.50	0.1343	-0.0745	0.0322	-0.0149	0.0077	-0.0044
2.60	0.1247	-0.0726	0.0323	-0.0151	0.0079	-0.0046
2.70	0.1158	-0.0707	0.0323	-0.0154	0.0081	-0.0047
2.80	0.1076	-0.0687	0.0323	-0.0156	0.0083	-0.0048
2.90	0.1000	-0.0668	0.0322	-0.0158	0.0085	-0.0049
3.00	0.0930	-0.0648	0.0321	-0.0160	0.0086	-0.0051
3.10	0.0866	-0.0629	0.0319	-0.0161	0.0088	-0.0052
3.20	0.0807	-0.0610	0.0317	-0.0163	0.0089	-0.0053
3.30	0.0752	-0.0591	0.0314	-0.0164	0.0091	-0.0054
3.40	0.0701	-0.0572	0.0312	-0.0165	0.0092	-0.0055
3.50	0.0654	-0.0554	0.0309	-0.0165	0.0093	-0.0056
3.60	0.0610	-0.0536	0.0305	-0.0166	0.0094	-0.0057
3.70	0.0570	-0.0518	0.0302	-0.0166	0.0095	-0.0058
3.80	0.0533	-0.0501	0.0298	-0.0166	0.0096	-0.0059
3.90	0.0498	-0.0484	0.0294	-0.0167	0.0097	-0.0059
4.00	0.0466	-0.0468	0.0290	-0.0166	0.0098	-0.0060
4.10	0.0437	-0.0452	0.0286	-0.0166	0.0098	-0.0061
4.20	0.0409	-0.0437	0.0282	-0.0166	0.0099	-0.0062
4.30	0.0384	-0.0422	0.0278	-0.0165	0.0099	-0.0062
4.40	0.0360	-0.0408	0.0273	-0.0165	0.0100	-0.0063
4.50	0.0338	-0.0394	0.0269	-0.0164	0.0100	-0.0063
4.60	0.0318	-0.0380	0.0264	-0.0163	0.0101	-0.0064
4.70	0.0299	-0.0367	0.0260	-0.0162	0.0101	-0.0064
4.80	0.0281	-0.0354	0.0255	-0.0161	0.0101	-0.0065
4.90	0.0265	-0.0342	0.0250	-0.0160	0.0101	-0.0065
5.00	0.0249	-0.0330	0.0246	-0.0159	0.0101	-0.0066

Table II-3. Bare slab reactor: functions $E_{3,k}(w)$.

w	k=0	k=1	k=2	k=3	k=4	k=5
0.10	0.9588	-0.0081	0.0012	-0.0004	0.0002	-0.0001
0.20	0.9199	-0.0154	0.0024	-0.0008	0.0004	-0.0002
0.30	0.8832	-0.0220	0.0036	-0.0012	0.0006	-0.0003
0.40	0.8485	-0.0280	0.0048	-0.0016	0.0007	-0.0004
0.50	0.8158	-0.0334	0.0059	-0.0020	0.0009	-0.0005
0.60	0.7849	-0.0382	0.0070	-0.0024	0.0011	-0.0006
0.70	0.7556	-0.0426	0.0081	-0.0028	0.0013	-0.0007
0.80	0.7279	-0.0465	0.0092	-0.0032	0.0015	-0.0008
0.90	0.7016	-0.0500	0.0102	-0.0036	0.0016	-0.0009
1.00	0.6768	-0.0531	0.0112	-0.0039	0.0018	-0.0010
1.10	0.6532	-0.0559	0.0121	-0.0043	0.0020	-0.0011
1.20	0.6309	-0.0584	0.0131	-0.0047	0.0022	-0.0012
1.30	0.6097	-0.0605	0.0139	-0.0050	0.0023	-0.0013
1.40	0.5895	-0.0624	0.0148	-0.0054	0.0025	-0.0014
1.50	0.5704	-0.0641	0.0156	-0.0057	0.0027	-0.0015
1.60	0.5522	-0.0656	0.0163	-0.0060	0.0028	-0.0015
1.70	0.5349	-0.0668	0.0170	-0.0063	0.0030	-0.0016
1.80	0.5185	-0.0679	0.0177	-0.0067	0.0032	-0.0017
1.90	0.5028	-0.0688	0.0183	-0.0070	0.0033	-0.0018
2.00	0.4879	-0.0695	0.0190	-0.0073	0.0035	-0.0019
2.10	0.4736	-0.0701	0.0195	-0.0076	0.0036	-0.0020
2.20	0.4601	-0.0706	0.0201	-0.0078	0.0038	-0.0021
2.30	0.4471	-0.0710	0.0206	-0.0081	0.0039	-0.0022
2.40	0.4348	-0.0712	0.0210	-0.0084	0.0041	-0.0023
2.50	0.4229	-0.0714	0.0215	-0.0086	0.0042	-0.0023
2.60	0.4117	-0.0715	0.0219	-0.0089	0.0043	-0.0024
2.70	0.4009	-0.0715	0.0223	-0.0091	0.0045	-0.0025
2.80	0.3905	-0.0714	0.0226	-0.0093	0.0046	-0.0026
2.90	0.3806	-0.0713	0.0230	-0.0096	0.0047	-0.0027
3.00	0.3712	-0.0711	0.0233	-0.0098	0.0049	-0.0027
3.10	0.3621	-0.0709	0.0236	-0.0100	0.0050	-0.0028
3.20	0.3534	-0.0706	0.0238	-0.0102	0.0051	-0.0029
3.30	0.3450	-0.0703	0.0241	-0.0104	0.0052	-0.0030
3.40	0.3370	-0.0699	0.0243	-0.0105	0.0053	-0.0030
3.50	0.3293	-0.0695	0.0245	-0.0107	0.0055	-0.0031
3.60	0.3219	-0.0691	0.0246	-0.0109	0.0056	-0.0032
3.70	0.3148	-0.0687	0.0248	-0.0110	0.0057	-0.0032
3.80	0.3080	-0.0682	0.0249	-0.0112	0.0058	-0.0033
3.90	0.3014	-0.0677	0.0250	-0.0113	0.0059	-0.0034
4.00	0.2951	-0.0672	0.0252	-0.0114	0.0060	-0.0034
4.10	0.2890	-0.0667	0.0252	-0.0116	0.0061	-0.0035
4.20	0.2831	-0.0662	0.0253	-0.0117	0.0061	-0.0036
4.30	0.2775	-0.0656	0.0254	-0.0118	0.0062	-0.0036
4.40	0.2720	-0.0651	0.0254	-0.0119	0.0063	-0.0037
4.50	0.2667	-0.0645	0.0255	-0.0120	0.0064	-0.0038
4.60	0.2616	-0.0640	0.0255	-0.0121	0.0065	-0.0038
4.70	0.2567	-0.0634	0.0255	-0.0122	0.0066	-0.0039
4.80	0.2520	-0.0628	0.0255	-0.0123	0.0066	-0.0039
4.90	0.2474	-0.0622	0.0255	-0.0124	0.0067	-0.0040
5.00	0.2430	-0.0617	0.0255	-0.0124	0.0068	-0.0040

3.4. The Spherical Reactor

The reactor core is a sphere of radius R. The flux and the composition of the fuel are symmetrical about the center of the sphere. They are functions of the time and of the radial coordinate r only.

The eigenfunction equation, Eq. (82), becomes

$$\frac{d^2v_k}{dr^2} + \frac{2}{r}\frac{dv_k}{dr} + B_k^2 v_k = 0. \tag{172}$$

Each eigenfunction must be spherically symmetric and satisfy the following boundary condition on the external surface

$$v_k + \delta_R \frac{dv_k}{dr} = 0, \quad \text{at} \quad r = R. \tag{173}$$

The spherically symmetric solution normalized to 1 at the center, is

$$v_k(r) = \frac{\sin B_k r}{B_k r}, \tag{174}$$

and the eigenvalue B_k^2 must be such that the boundary condition, Eq. (173), is satisfied; B_k is then a root of

$$\sin B_k R + \delta_R B_k (\cos B_k R - \frac{\sin B_k R}{B_k R}) = 0. \tag{175}$$

The results are expressed in dimensionless form as follows: The change of variable

$$\rho = r/R, \quad \text{where} \quad 0 \leqslant \rho \leqslant 1, \tag{176}$$

yields

$$v_k(\rho) = \frac{\sin \omega_k \rho}{\omega_k \rho}, \tag{177}$$

with

$$B_k^2 = \omega_k^2/R^2. \tag{178}$$

The ratio of the linear extrapolation distance δ_R to the radius of the sphere R is

$$\epsilon_R = \delta_R/R. \tag{179}$$

The eigenvalue ω_k is defined, according to Eq. (175), as the $(k+1)^{th}$ root of

$$(1-\epsilon_R)\sin\omega_k+\epsilon_R\,\omega_k\cos\omega_k=0\,. \tag{180}$$

For any positive value of ϵ_R, this equation has an infinite number of roots, which can be obtained from tabulations or graphical representations of the function $\omega\cot\omega$.[J2] For very large k, these roots tend toward $(k+\frac{1}{2})\pi$, which are the roots of the equation $\cos\omega_k=0$.

For $\epsilon_R=0$, the eigenvalues are simply

$$\omega_k=(k+1)\pi\,. \tag{181}$$

The scalar product of two functions $f(\rho)$ and $g(\rho)$, Eq. (4), is now

$$\left\langle f,g\right\rangle=\int_0^1 f(\rho)\,g(\rho)\,3\rho^2\,d\rho\,. \tag{182}$$

The auxiliary coefficients can be evaluated as follows:

$$\mu_k=\int_0^1\left(\frac{\sin\omega_k\rho}{\omega_k\rho}\right)^2 3\rho^2=\frac{3}{2\omega_k^2}\left(1-\frac{\sin 2\omega_k}{2\omega_k}\right), \tag{183}$$

$$f_{1,k}=\int_0^1\frac{\sin\omega_k\rho}{\omega_k\rho}\,3\rho^2\,d\rho=\frac{3}{\omega_k^2}\left(-\cos\omega_k+\frac{\sin\omega_k}{\omega_k}\right), \tag{184}$$

and

$$f_{2,k}=\frac{1}{\mu_k}\int_0^1\left(\frac{\sin\omega_0\rho}{\omega_0\rho}\right)^2\frac{\sin\omega_k\rho}{\omega_k\rho}\,3\rho^2\,d\rho\,. \tag{185}$$

Letting the symbol Si denote the sine integral function which is tabulated by Jahnke and Emde,[J2] the above equation becomes

$$f_{2,k}=\frac{\omega_k}{\omega_0^2}\;\frac{\mathrm{Si}(\omega_k)-\frac{1}{2}[\,\mathrm{Si}(\omega_k+2\omega_0)+\mathrm{Si}(\omega_k-2\omega_0)]}{1-\frac{\sin 2\omega_k}{2\omega_k}}\,. \tag{186}$$

The functions $E_{1,k}(w)$, $E_{2,k}(w)$ and $E_{3,k}(w)$ defined respectively by Eqs. (140), (128), and (134) become:

$$E_{1,k}(w) = w \int_0^1 (e^{-w \sin \omega_0 \rho / \omega_0 \rho}) \frac{\sin \omega_k \rho}{\omega_k \rho} 3\rho^2 d\rho, \tag{187}$$

$$E_{2,k}(w) = \frac{1}{\mu_k} \int_0^1 (e^{-w \sin \omega_0 \rho / \omega_0 \rho}) \frac{\sin \omega_0 \rho}{\omega_0 \rho} \frac{\sin \omega_k \rho}{\omega_k \rho} 3\rho^2 d\rho, \tag{188}$$

and

$$E_{3,k}(w) = \frac{1}{w\mu_k} \int_0^1 (1 - e^{-w \sin \omega_0 \rho / \omega_0 \rho}) \frac{\sin \omega_k \rho}{\omega_k \rho} 3\rho^2 d\rho . \tag{189}$$

These integrals have been evaluated by numerical computations using a 10-point Gauss quadrature formula, as in the case of the slab reactor. The results for $\epsilon_R = 0$ are given in Tables II.4 through II.6.

For this case, where $\epsilon_R = 0$, the various coefficients required for the computations of the flux shape, flux time and burnup have the following expressions:

$$\omega_k = (k + 1)\pi , \tag{190}$$

$$\frac{B_0^2}{B_k^2 - B_0^2} = \frac{\omega_0^2}{\omega_k^2 - \omega_k^2} = \frac{1}{(k+1)^2 - 1} = \frac{1}{k(k+2)} , \tag{191}$$

$$\mu_k = \frac{3}{2(k+1)^2 \pi^2} , \tag{192}$$

$$h_k = \frac{1}{(k+1)^2} , \tag{193}$$

$$f_{1,k} = (-1)^k \frac{3}{(k+1)^2 \pi^2} , \tag{194}$$

Table II-4. Bare spherical reactor: functions $E_{1,k}(w)$.

w	k=0	k=1	k=2	k=3	k=4
0.10	0.0289	-0.0076	0.0034	-0.0019	0.0012
0.20	0.0551	-0.0151	0.0068	-0.0038	0.0024
0.30	0.0787	-0.0226	0.0101	-0.0057	0.0036
0.40	0.1000	-0.0300	0.0135	-0.0076	0.0049
0.50	0.1192	-0.0372	0.0168	-0.0095	0.0061
0.60	0.1365	-0.0442	0.0201	-0.0114	0.0073
0.70	0.1521	-0.0511	0.0234	-0.0132	0.0085
0.80	0.1661	-0.0577	0.0267	-0.0151	0.0097
0.90	0.1786	-0.0642	0.0299	-0.0170	0.0109
1.00	0.1898	-0.0704	0.0331	-0.0188	0.0121
1.10	0.1998	-0.0764	0.0363	-0.0207	0.0133
1.20	0.2087	-0.0822	0.0394	-0.0225	0.0145
1.30	0.2167	-0.0878	0.0425	-0.0243	0.0157
1.40	0.2237	-0.0931	0.0455	-0.0262	0.0169
1.50	0.2299	-0.0982	0.0485	-0.0280	0.0180
1.60	0.2353	-0.1031	0.0514	-0.0297	0.0192
1.70	0.2401	-0.1078	0.0543	-0.0315	0.0204
1.80	0.2442	-0.1122	0.0571	-0.0333	0.0215
1.90	0.2478	-0.1164	0.0598	-0.0350	0.0227
2.00	0.2508	-0.1205	0.0625	-0.0367	0.0238
2.10	0.2534	-0.1243	0.0651	-0.0384	0.0250
2.20	0.2556	-0.1279	0.0677	-0.0401	0.0261
2.30	0.2574	-0.1313	0.0702	-0.0417	0.0272
2.40	0.2588	-0.1345	0.0726	-0.0434	0.0284
2.50	0.2599	-0.1375	0.0750	-0.0450	0.0295
2.60	0.2607	-0.1404	0.0773	-0.0466	0.0306
2.70	0.2613	-0.1431	0.0796	-0.0481	0.0317
2.80	0.2617	-0.1456	0.0817	-0.0497	0.0327
2.90	0.2618	-0.1480	0.0839	-0.0512	0.0338
3.00	0.2617	-0.1502	0.0859	-0.0527	0.0349
3.10	0.2615	-0.1523	0.0879	-0.0541	0.0359
3.20	0.2611	-0.1542	0.0898	-0.0556	0.0370
3.30	0.2605	-0.1560	0.0917	-0.0570	0.0380
3.40	0.2599	-0.1577	0.0935	-0.0584	0.0390
3.50	0.2591	-0.1593	0.0952	-0.0598	0.0400
3.60	0.2582	-0.1607	0.0969	-0.0611	0.0410
3.70	0.2573	-0.1620	0.0985	-0.0624	0.0420
3.80	0.2562	-0.1632	0.1001	-0.0637	0.0430
3.90	0.2551	-0.1644	0.1016	-0.0650	0.0439
4.00	0.2539	-0.1654	0.1030	-0.0662	0.0449
4.10	0.2527	-0.1663	0.1044	-0.0674	0.0458
4.20	0.2514	-0.1672	0.1058	-0.0686	0.0467
4.30	0.2501	-0.1679	0.1070	-0.0697	0.0476
4.40	0.2487	-0.1686	0.1083	-0.0709	0.0485
4.50	0.2473	-0.1692	0.1094	-0.0720	0.0494
4.60	0.2459	-0.1697	0.1106	-0.0730	0.0503
4.70	0.2445	-0.1702	0.1117	-0.0741	0.0511
4.80	0.2430	-0.1706	0.1127	-0.0751	0.0520
4.90	0.2415	-0.1709	0.1137	-0.0761	0.0528
5.00	0.2400	-0.1712	0.1146	-0.0771	0.0536

Table II-5. Bare spherical reactor: functions $E_{2,k}(w)$.

w	k=0	k=1	k=2	k=3	k=4
0.10	0.9403	-0.0403	0.0068	-0.0031	0.0018
0.20	0.8846	-0.0760	0.0142	-0.0063	0.0037
0.30	0.8326	-0.1075	0.0219	-0.0095	0.0056
0.40	0.7840	-0.1353	0.0299	-0.0129	0.0075
0.50	0.7386	-0.1595	0.0380	-0.0163	0.0094
0.60	0.6962	-0.1807	0.0462	-0.0197	0.0113
0.70	0.6565	-0.1991	0.0543	-0.0232	0.0133
0.80	0.6194	-0.2149	0.0623	-0.0268	0.0153
0.90	0.5846	-0.2285	0.0702	-0.0304	0.0173
1.00	0.5521	-0.2400	0.0779	-0.0340	0.0193
1.10	0.5216	-0.2496	0.0853	-0.0376	0.0213
1.20	0.4931	-0.2575	0.0925	-0.0412	0.0233
1.30	0.4663	-0.2640	0.0994	-0.0448	0.0254
1.40	0.4412	-0.2691	0.1060	-0.0483	0.0274
1.50	0.4177	-0.2730	0.1122	-0.0519	0.0294
1.60	0.3956	-0.2758	0.1181	-0.0554	0.0315
1.70	0.3749	-0.2777	0.1237	-0.0588	0.0335
1.80	0.3554	-0.2787	0.1289	-0.0622	0.0355
1.90	0.3372	-0.2789	0.1338	-0.0655	0.0376
2.00	0.3200	-0.2785	0.1384	-0.0688	0.0396
2.10	0.3038	-0.2774	0.1426	-0.0720	0.0416
2.20	0.2886	-0.2759	0.1465	-0.0751	0.0435
2.30	0.2743	-0.2738	0.1500	-0.0781	0.0455
2.40	0.2608	-0.2714	0.1533	-0.0811	0.0474
2.50	0.2481	-0.2686	0.1562	-0.0839	0.0493
2.60	0.2361	-0.2655	0.1589	-0.0866	0.0512
2.70	0.2249	-0.2622	0.1613	-0.0893	0.0531
2.80	0.2142	-0.2586	0.1634	-0.0918	0.0549
2.90	0.2042	-0.2548	0.1652	-0.0943	0.0567
3.00	0.1947	-0.2509	0.1668	-0.0966	0.0585
3.10	0.1858	-0.2468	0.1682	-0.0988	0.0603
3.20	0.1774	-0.2426	0.1693	-0.1009	0.0620
3.30	0.1694	-0.2384	0.1702	-0.1030	0.0636
3.40	0.1618	-0.2341	0.1709	-0.1049	0.0653
3.50	0.1547	-0.2297	0.1714	-0.1067	0.0668
3.60	0.1479	-0.2254	0.1718	-0.1084	0.0684
3.70	0.1415	-0.2210	0.1719	-0.1100	0.0699
3.80	0.1355	-0.2166	0.1719	-0.1115	0.0714
3.90	0.1297	-0.2122	0.1718	-0.1129	0.0728
4.00	0.1243	-0.2078	0.1715	-0.1142	0.0742
4.10	0.1191	-0.2035	0.1710	-0.1154	0.0755
4.20	0.1142	-0.1992	0.1705	-0.1165	0.0768
4.30	0.1096	-0.1949	0.1698	-0.1176	0.0781
4.40	0.1052	-0.1907	0.1690	-0.1185	0.0793
4.50	0.1010	-0.1866	0.1682	-0.1193	0.0804
4.60	0.0970	-0.1825	0.1672	-0.1201	0.0815
4.70	0.0932	-0.1785	0.1661	-0.1208	0.0826
4.80	0.0896	-0.1745	0.1650	-0.1214	0.0836
4.90	0.0862	-0.1706	0.1638	-0.1219	0.0846
5.00	0.0829	-0.1668	0.1625	-0.1223	0.0856

Table II-6. Bare spherical reactor: functions $E_{3,k}(w)$.

w	k=0	k=1	k=2	k=3	k=4
0.10	0.9698	-0.0206	0.0034	-0.0015	0.0009
0.20	0.9410	-0.0396	0.0069	-0.0031	0.0018
0.30	0.9134	-0.0571	0.0106	-0.0047	0.0028
0.40	0.8871	-0.0732	0.0144	-0.0063	0.0037
0.50	0.8619	-0.0881	0.0183	-0.0080	0.0046
0.60	0.8377	-0.1018	0.0223	-0.0097	0.0056
0.70	0.8147	-0.1144	0.0263	-0.0113	0.0066
0.80	0.7925	-0.1260	0.0303	-0.0130	0.0075
0.90	0.7713	-0.1367	0.0343	-0.0148	0.0085
1.00	0.7510	-0.1465	0.0383	-0.0165	0.0095
1.10	0.7315	-0.1554	0.0422	-0.0183	0.0105
1.20	0.7128	-0.1636	0.0461	-0.0200	0.0114
1.30	0.6949	-0.1711	0.0499	-0.0218	0.0124
1.40	0.6777	-0.1779	0.0537	-0.0236	0.0134
1.50	0.6611	-0.1841	0.0574	-0.0253	0.0144
1.60	0.6452	-0.1898	0.0610	-0.0271	0.0154
1.70	0.6299	-0.1949	0.0645	-0.0289	0.0164
1.80	0.6152	-0.1995	0.0680	-0.0306	0.0174
1.90	0.6010	-0.2037	0.0713	-0.0324	0.0184
2.00	0.5874	-0.2075	0.0745	-0.0341	0.0195
2.10	0.5743	-0.2108	0.0777	-0.0358	0.0205
2.20	0.5616	-0.2138	0.0807	-0.0375	0.0215
2.30	0.5495	-0.2165	0.0837	-0.0392	0.0225
2.40	0.5377	-0.2188	0.0865	-0.0409	0.0235
2.50	0.5264	-0.2209	0.0892	-0.0426	0.0245
2.60	0.5154	-0.2226	0.0919	-0.0442	0.0255
2.70	0.5049	-0.2242	0.0944	-0.0459	0.0264
2.80	0.4947	-0.2255	0.0968	-0.0474	0.0274
2.90	0.4848	-0.2265	0.0991	-0.0490	0.0284
3.00	0.4753	-0.2274	0.1014	-0.0506	0.0294
3.10	0.4661	-0.2281	0.1035	-0.0521	0.0304
3.20	0.4572	-0.2286	0.1055	-0.0536	0.0313
3.30	0.4486	-0.2290	0.1075	-0.0550	0.0323
3.40	0.4403	-0.2292	0.1093	-0.0565	0.0332
3.50	0.4323	-0.2293	0.1111	-0.0579	0.0342
3.60	0.4244	-0.2292	0.1128	-0.0593	0.0351
3.70	0.4169	-0.2291	0.1144	-0.0606	0.0360
3.80	0.4096	-0.2288	0.1159	-0.0619	0.0369
3.90	0.4025	-0.2284	0.1173	-0.0632	0.0378
4.00	0.3956	-0.2280	0.1187	-0.0645	0.0387
4.10	0.3889	-0.2274	0.1200	-0.0657	0.0396
4.20	0.3824	-0.2268	0.1212	-0.0669	0.0405
4.30	0.3761	-0.2261	0.1223	-0.0681	0.0413
4.40	0.3700	-0.2253	0.1234	-0.0692	0.0422
4.50	0.3641	-0.2245	0.1244	-0.0703	0.0430
4.60	0.3583	-0.2237	0.1253	-0.0714	0.0438
4.70	0.3527	-0.2227	0.1262	-0.0724	0.0446
4.80	0.3473	-0.2218	0.1270	-0.0735	0.0454
4.90	0.3420	-0.2208	0.1278	-0.0744	0.0462
5.00	0.3368	-0.2197	0.1285	-0.0754	0.0470

and

$$f_{2,k} = \frac{(k+1)}{\pi} \left\{ \mathrm{Si}\,(k+1)\pi - \frac{1}{2}\left[\mathrm{Si}\,(k+3)\pi + \mathrm{Si}\,(k-1)\pi\right] \right\} . \tag{195}$$

3.5. The Cylindrical Reactor

The reactor core is a cylinder of radius R and of height $2Z$. It is symmetrical about the axis and about the midplane of the cylinder.

The corresponding Helmholtz equation is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \times \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} + B^2 v = 0 . \tag{201}$$

The origin of the coordinate system is taken at the center of the cylinder. By using the technique of separation of the variables, the general solution of Eq. (201), which satisfies the symmetry condition, is obtained as the following product of a Bessel function of the first kind and of zero order by a cosine function:

$$v = J_0(B_R r) \cos (B_Z z), \tag{202}$$

with

$$B^2 = B_R^2 + B_Z^2 . \tag{203}$$

Letting the linear extrapolation distance be a constant δ_R on the outer radius of the cylinder and a constant δ_Z on the end plane, the boundary condition, Eq. (3), becomes

$$v + \delta_Z \frac{\partial v}{\partial z} = 0 \quad \text{at} \quad z = Z , \tag{204}$$

and

$$v + \delta_R \frac{\partial v}{\partial r} = 0 \quad \text{at} \quad r = R . \tag{205}$$

The function v satisfies the boundary condition at the outer radius of the core if B_R is a root of

$$J_0(B_R R) - \delta_R B_R J_1(B_R R) = 0 ; \tag{206}$$

v also satisfies the boundary condition at the end plane if B_Z is a root of Eq. (149).

The change of variables,

$$\eta = r/R \text{ where } 0 \leqslant \eta \leqslant 1 , \tag{207}$$

and

$$\zeta = z/Z \text{ where } 0 \leqslant \zeta \leqslant 1 , \tag{208}$$

yields

$$v_k = J_0(j_\ell \eta) \cos (\gamma_m \zeta), \tag{209}$$

with

$$B_k^2 = j_\ell^2 / R^2 + \gamma_m^2/Z^2 . \tag{210}$$

The ratio of the linear extrapolation distance to the corresponding dimension of the reactor core will be noted ϵ, thus obtaining

$$\epsilon_R = \delta_R/R , \tag{211}$$

and

$$\epsilon_Z = \delta_Z/Z . \tag{212}$$

According to Eq. (206), j_ℓ is now defined as the $(\ell + 1)^{th}$ positive root of

$$J_0(j_\ell) - \epsilon_R \, j_\ell \, J_1(j_\ell) = 0 , \tag{213}$$

and γ_m is the $(m + 1)^{th}$ positive root of

$$\cos \gamma_m - \epsilon_Z \sin \gamma_m = 0 . \tag{214}$$

The above equations, Eqs. (213) and (214), have an infinite number of solutions, which are studied, for instance, in reference (C5).

In the following, the subscript k stands for the pair of non-negative integers ℓ and m.

The scalar product of two functions $f(\eta,\zeta)$ and $g(\eta,\zeta)$, Eq. (4), is now

$$\langle f, g \rangle = \int_0^1 2\eta d\eta \int_0^1 d\zeta \, f(\eta,\zeta) \, g(\eta,\zeta) . \tag{215}$$

This is, in general, a double integral, but becomes the product of two simple integrals if the function fg can be expressed as the product of a function of η only by a function of ζ only. For instance, the coefficients $f_{1,k}$, Eq. (139), and μ_k, Eq. (143), are given by

$$f_{1,k} = \int_0^1 J_0(j_\ell \eta)\, 2\eta\, d\eta \int_0^1 \cos \gamma_m \zeta\, d\zeta\ , \tag{216}$$

and

$$\mu_k = \int_0^1 J_0^2\, (j_\ell \eta)\, 2\eta\, d\eta \int_0^1 \cos^2 \gamma_m \zeta\, d\zeta\ . \tag{217}$$

The above integrations are readily performed, thus obtaining

$$f_{1,k} = \frac{2 J_1(j_\ell)}{j_\ell} \left(\frac{\sin \gamma_m}{\gamma_m} \right) , \tag{218}$$

and

$$\mu_k = [\, J_1^2(j_\ell) + J_0^2(j_\ell)] \left(\frac{1}{2} + \frac{\sin 2\, \gamma_m}{4\, \gamma_m} \right) . \tag{219}$$

Given the set of the functions v_k corresponding to given values of ϵ_R and ϵ_Z, the auxiliary functions $E_{1,k}(w)$, $E_{2,k}(w)$, and $E_{3,k}(w)$ can be computed according to the following procedure:

First, one defines the functions

$$S_{2,k}(w) = \left\langle e^{-wv_0} v_0, v_k \right\rangle , \tag{220}$$

and

$$S_{1,k}(w) = \left\langle (1 - e^{-wv_0}), v_k \right\rangle = \int_0^w S_{2,k}(w)\, dw\ . \tag{221}$$

Then, using Eqs. (139), (142), (220), and (221), the auxiliary functions, defined by Eqs. (140), (128), and (134) respectively, are given by

$$E_{1,k}(w) = w\,[\,f_{1,k} - S_{1,k}(w)]\ , \tag{222}$$

$$E_{2,k}(w) = S_{2,k}(w)/\mu_k , \tag{223}$$

and

$$E_{3,k}(w) = S_{1,k}(w)/(\mu_k w)\ . \tag{224}$$

To evaluate $S_{1,k}(w)$ and $S_{2,k}(w)$, one must evaluate numerically the double integrals represented by the bracketing operations in Eqs. (220) and (221); $S_{1,k}(w)$ could also be evaluated simply by integrating $S_{2,k}(w)$ with respect to w, once having set up a table of the latter function.

However, the direct computation of $S_{2,k}(w)$ requires the evaluation of a double integral for each value of w. We have therefore preferred to expand the function $S_{2,k}(w)$ in powers of w, because the computation of the coefficients of the power series requires only the evaluation of simple integrals.

The function e^{-wv_0} is represented as follows by the uniformly convergent series

$$e^{-wv_0} = \sum_{n=0}^{\infty} \frac{(-w)^n}{n!}\, v_0^n\ , \tag{225}$$

and the functions $S_{1,k}(w)$ and $S_{2,k}(w)$ are represented by

$$S_{2,k}(w) = \sum_{n=1}^{\infty} \frac{(-w)^{n-1}}{(n-1)\,!} \left\langle v_0^n , v_k \right\rangle , \tag{226}$$

and

$$S_{1,k}(w) = -\sum_{n=1}^{\infty} \frac{(-w)^n}{n!} \left\langle v_0^n , v_k \right\rangle . \tag{227}$$

The coefficients which appear in the above equations are defined as

$$D_{k,n} = \left\langle v_0^n , v_k \right\rangle = \int_0^1 J_0^n(j_0\eta) J_0(j_\ell \eta)\, 2\eta\, d\eta \int_0^1 \cos^n \gamma_0 \zeta \cos \gamma_m \zeta\, d\zeta\ . \tag{228}$$

The first integral in the right side of the above equation is

$$B_{\ell,n} = \int_0^1 J_0^n (j_0 \eta) J_0 (j_\ell \eta)\, 2\eta\, d\eta\,, \tag{229}$$

and it will be evaluated by using a p-point Gauss quadrature formula ($p = 4, 6, 8,$ or 10).

The integrals

$$C_{m,n} = \int_0^1 \cos^n(\gamma_0 \zeta) \cos(\gamma_m \zeta)\, d\zeta \tag{230}$$

are given by standard formulae.[G3]

Since the functions $J_0(j_\ell \eta)$ [or $\cos(\gamma_m \zeta)$] are eigenfunctions of a self-adjoint one-dimensional Helmholtz equation (Sturm-Liouville equation), the corresponding orthogonality property reads

$$B_{\ell,1} = 0, \text{ where } \ell \neq 0 \tag{231}$$

and

$$C_{m,1} = 0, \text{where } m \neq 0\,. \tag{232}$$

With the above notations, the coefficients $f_{2,k}$ defined by Eq. (129) become

$$f_{2,k} = \frac{B_{\ell,2} C_{m,2}}{\mu_k}\,. \tag{233}$$

The computations outlined above, Eqs. (220) through (233), have been performed on a high-speed digital computer for the case of a bare reactor with negligible extrapolation distances, i. e., where

$$\epsilon_Z = \epsilon_R = 0\,. \tag{234}$$

Equation (213) then defines j_ℓ as the $(\ell + 1)^{\text{th}}$ root of the Bessel function of the first kind and zeroth order; the corresponding numerical values are given to ten decimal places by Watson.[W5] The following table gives j_ℓ for the first few values of ℓ:

ℓ	0	1	2	3	4	5
j_ℓ	2.405	5.520	8.654	11.692	14.931	18.071

In general, j_ℓ can be represented by the following series:[J2]

$$j_{\ell-1} = \pi \left[(\ell - \tfrac{1}{4}) + \frac{0.051}{4\ell-1} - \frac{0.053}{(4\ell - 1)^3} + \cdots \right] \tag{235}$$

Equation (214), where $\epsilon_Z = 0$, yields

$$\gamma_m = (2m + 1)\pi/2 . \tag{236}$$

The eigenvalues B_k^2 are given by Eq. (210), the subscript k corresponding to a given pair of values of the integers ℓ and m, and the value $k = 0$ corresponding to $\ell = 0$ and $m = 0$. One obtains:

$$\frac{B_0^2}{B_k^2 - B_0^2} = \frac{j_0^2/R^2 + \gamma_0^2/Z^2}{(j_\ell^2/R^2 + \gamma_m^2/Z^2) - (j_0^2/R^2 + \gamma_0^2/Z^2)} . \tag{237}$$

This is a function of the ratio R/Z only; one sets

$$R/Z = \lambda \tag{238}$$

and obtains

$$\frac{B_0^2}{B_k^2 - B_0^2} = \frac{j_0^2 + \gamma_0^2\lambda^2}{(j_\ell^2 - j_0^2) + (\gamma_m^2 - \gamma_0^2)\lambda^2} . \tag{239}$$

Equations (219), (143), and (218), respectively, now yield

$$\mu_k = \frac{1}{2} J_1^2 (j_\ell) , \quad \text{for any } m, \tag{240}$$

$$h_k = J_1^2(j_\ell)/J_1^2(j_0) \quad \text{for any } m, \tag{241}$$

and

$$f_{1,k} = \left[\frac{2J_1(j_\ell)}{j_\ell}\right] \frac{(-1)^m}{(2m+1)\pi/2} . \tag{242}$$

The coefficients $C_{m,n}$, Eq. (230), are now defined by

$$C_{m,n} = \frac{2}{\pi} \int_0^{\pi/2} \cos(2m+1)x \cos^n x \, dx . \tag{243}$$

The standard formula[D2] yields first

$$C_{0,n} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+3}{2})} , \tag{244}$$

where the symbol Γ denotes the factorial function.

All the coefficients $C_{m,n}$ can then be easily evaluated by the recurrence formula, Eq. (245), which is derived as follows:

$$\frac{\pi}{2}(C_{m,n} + C_{m,n+1}) = \int_0^{\pi/2} 2\cos(2m+2)x \cos^{n+1} x \, dx,$$

and

$$\frac{\pi}{2}(C_{m,n} - C_{m,n+1}) = \int_0^{\pi/2} 2\sin(2m+2)x \sin x \cos^n x \, dx.$$

After integrating by parts, the later integral yields

$$C_{m,n} - C_{m,n+1} = \frac{2m+2}{n+1}(C_{m,n} + C_{m+1,n}),$$

and then we get the recurrence formula

$$C_{m+1,n} = \frac{n+1-2(m+1)}{n+1+2(m+1)} C_{m,n} . \tag{245}$$

The coefficients $B_{\ell,n}$, Eq. (229), have been computed by evaluating numerically the corresponding integrals by means of a 10-point Gauss quadrature formula.

The power series, Eqs. (226) and (227), then give the values of the functions $S_{1,k}(w)$ and $S_{2,k}(w)$ for any value of w, and the tables of the functions $E_{1,k}(w)$, $E_{2,k}(w)$ and $E_{3,k}(w)$ have been set up according to Eqs. (222), (223), and (224). Besides Tables II-7 through II-9, the following values of the coefficients h_k, $f_{1,k}$ and $f_{2,k}$ have been obtained for bare cylindrical reactors.

k	h_k	$f_{1,k}$	$f_{2,k}$
$\ell = 0$, m = 0	1.0000	0.2749	0.6139
$\ell = 0$, m = 1	1.0000	-0.0916	0.1228
$\ell = 0$, m = 2	1.0000	0.0550	-0.0175
$\ell = 1$, m = 0	0.4296	-0.0785	0.2646
$\ell = 1$, m = 1	0.4296	0.0262	0.0529
$\ell = 1$, m = 2	0.4296	-0.0157	-0.0076
$\ell = 2$, m = 0	0.2734	0.0399	-0.0405
$\ell = 2$, m = 1	0.2734	-0.0133	-0.0081
$\ell = 2$, m = 2	0.2734	0.0080	0.0012

4. Parametric Study of U^{235} Fueled Reactor

The calculation procedure developed in the preceding sections is now applied to the study of the flux changes which take place in a bare reactor where fuel consisting of substantially pure U^{235} is irradiated batchwise.

The local excess neutron production and the burnup of the fuel (fraction of initial U^{235} atoms destroyed) are represented by the following dimensionless functions of the local flux time θ:

$$g(\theta) = g_0 + g_{25}\, e^{-\sigma_{25}\theta}, \tag{250}$$

and

$$\beta(\theta) = 1 - e^{-\sigma_{25}\theta}. \tag{251}$$

Table II-7. Bare cylindrical reactor: functions $E_{l,k}(w)$.

w	ℓ=0, m=0	ℓ=0, m=1	ℓ=0, m=2	ℓ=1, m=0	ℓ=1, m=2	ℓ=2, m=0
0.10	0.02618	-0.00915	0.00550	-0.00784	0.00262	0.00399
0.20	0.04990	-0.01826	0.01099	-0.01564	0.00524	0.00798
0.30	0.07137	-0.02729	0.01646	-0.02336	0.00788	0.01196
0.40	0.09080	-0.03619	0.02191	-0.03097	0.01054	0.01593
0.50	0.10837	-0.04495	0.02734	-0.03845	0.01322	0.01987
0.60	0.12423	-0.05354	0.03273	-0.04577	0.01592	0.02380
0.70	0.13854	-0.06195	0.03809	-0.05292	0.01864	0.02769
0.80	0.15144	-0.07015	0.04339	-0.05988	0.02139	0.03155
0.90	0.16304	-0.07813	0.04865	-0.06665	0.02416	0.03538
1.00	0.17347	-0.08589	0.05385	-0.07321	0.02695	0.03916
1.10	0.18283	-0.09341	0.05899	-0.07955	0.02976	0.04290
1.20	0.19120	-0.10070	0.06407	-0.08568	0.03258	0.04659
1.30	0.19869	-0.10774	0.06908	-0.09159	0.03542	0.05024
1.40	0.20536	-0.11455	0.07401	-0.09728	0.03826	0.05382
1.50	0.21130	-0.12111	0.07887	-0.10275	0.04112	0.05735
1.60	0.21656	-0.12743	0.08366	-0.10800	0.04398	0.06083
1.70	0.22120	-0.13351	0.08836	-0.11303	0.04684	0.06424
1.80	0.22528	-0.13935	0.09298	-0.11786	0.04970	0.06759
1.90	0.22886	-0.14496	0.09751	-0.12247	0.05255	0.07087
2.00	0.23197	-0.15034	0.10196	-0.12687	0.05540	0.07409
2.10	0.23465	-0.15550	0.10633	-0.13108	0.05824	0.07724
2.20	0.23696	-0.16044	0.11060	-0.13509	0.06106	0.08032
2.30	0.23891	-0.16517	0.11478	-0.13891	0.06387	0.08333
2.40	0.24055	-0.16969	0.11888	-0.14254	0.06666	0.08628
2.50	0.24190	-0.17401	0.12289	-0.14600	0.06943	0.08915
2.60	0.24298	-0.17813	0.12680	-0.14928	0.07217	0.09196
2.70	0.24383	-0.18206	0.13063	-0.15239	0.07490	0.09469
2.80	0.24446	-0.18581	0.13436	-0.15535	0.07759	0.09736
2.90	0.24489	-0.18939	0.13801	-0.15814	0.08026	0.09995
3.00	0.24515	-0.19279	0.14156	-0.16078	0.08290	0.10248
3.10	0.24524	-0.19602	0.14503	-0.16328	0.08550	0.10493
3.20	0.24519	-0.19910	0.14841	-0.16564	0.08808	0.10732
3.30	0.24501	-0.20202	0.15171	-0.16787	0.09062	0.10964
3.40	0.24471	-0.20480	0.15491	-0.16996	0.09312	0.11189
3.50	0.24431	-0.20743	0.15804	-0.17194	0.09559	0.11408
3.60	0.24380	-0.20992	0.16108	-0.17379	0.09802	0.11620
3.70	0.24322	-0.21229	0.16403	-0.17553	0.10041	0.11826
3.80	0.24255	-0.21453	0.16691	-0.17717	0.10277	0.12025
3.90	0.24181	-0.21665	0.16970	-0.17870	0.10508	0.12218
4.00	0.24101	-0.21865	0.17242	-0.18012	0.10736	0.12405
4.10	0.24016	-0.22054	0.17506	-0.18146	0.10960	0.12586
4.20	0.23925	-0.22233	0.17762	-0.18270	0.11179	0.12761
4.30	0.23831	-0.22401	0.18011	-0.18386	0.11395	0.12930
4.40	0.23732	-0.22559	0.18252	-0.18493	0.11607	0.13093
4.50	0.23630	-0.22708	0.18486	-0.18593	0.11814	0.13251
4.60	0.23525	-0.22849	0.18714	-0.18685	0.12018	0.13403
4.70	0.23417	-0.22980	0.18934	-0.18770	0.12217	0.13550
4.80	0.23307	-0.23103	0.19148	-0.18848	0.12413	0.13692
4.90	0.23196	-0.23219	0.19355	-0.18919	0.12604	0.13829
5.00	0.23082	-0.23327	0.19555	-0.18984	0.12792	0.13961

Table II-8. Bare cylindrical reactor: functions $E_{2,k}(w)$.

w	ℓ=0, m=0	ℓ=0, m=1	ℓ=0, m=2	ℓ=1, m=0	ℓ=1, m=1	ℓ=2, m=0
0.20	0.88530	-0.02190	0.00349	-0.04711	-0.00868	0.00837
0.30	0.83358	-0.03104	0.00520	-0.06671	-0.01176	0.01267
0.40	0.78526	-0.03912	0.00688	-0.08399	-0.01414	0.01700
0.50	0.74010	-0.04624	0.00851	-0.09917	-0.01590	0.02131
0.60	0.69787	-0.05248	0.01010	-0.11245	-0.01713	0.02556
0.70	0.65838	-0.05794	0.01163	-0.12402	-0.01788	0.02973
0.80	0.62143	-0.06268	0.01310	-0.13403	-0.01823	0.03380
0.90	0.58684	-0.06677	0.01451	-0.14263	-0.01823	0.03775
1.00	0.55445	-0.07027	0.01585	-0.14996	-0.01793	0.04155
1.10	0.52411	-0.07325	0.01713	-0.15615	-0.01738	0.04521
1.20	0.49568	-0.07575	0.01834	-0.16131	-0.01662	0.04871
1.30	0.46902	-0.07781	0.01949	-0.16554	-0.01567	0.05205
1.40	0.44402	-0.07949	0.02056	-0.16894	-0.01458	0.05521
1.50	0.42056	-0.08082	0.02158	-0.17158	-0.01336	0.05820
1.60	0.39855	-0.08184	0.02252	-0.17356	-0.01205	0.06102
1.70	0.37787	-0.08258	0.02340	-0.17493	-0.01067	0.06367
1.80	0.35845	-0.08306	0.02422	-0.17577	-0.00923	0.06614
1.90	0.34019	-0.08332	0.02498	-0.17612	-0.00775	0.06844
2.00	0.32303	-0.08338	0.02568	-0.17606	-0.00624	0.07057
2.10	0.30689	-0.08326	0.02632	-0.17561	-0.00473	0.07254
2.20	0.29170	-0.08298	0.02690	-0.17483	-0.00321	0.07436
2.30	0.27740	-0.08256	0.02743	-0.17376	-0.00170	0.07601
2.40	0.26393	-0.08202	0.02791	-0.17242	-0.00020	0.07752
2.50	0.25124	-0.08136	0.02835	-0.17086	0.00127	0.07888
2.60	0.23928	-0.08062	0.02873	-0.16910	0.00272	0.08010
2.70	0.22800	-0.07979	0.02907	-0.16718	0.00413	0.08119
2.80	0.21736	-0.07889	0.02937	-0.16510	0.00550	0.08215
2.90	0.20732	-0.07792	0.02962	-0.16290	0.00684	0.08299
3.00	0.19783	-0.07691	0.02984	-0.16059	0.00813	0.08371
3.10	0.18888	-0.07585	0.03002	-0.15820	0.00938	0.08432
3.20	0.18041	-0.07475	0.03017	-0.15573	0.01058	0.08482
3.30	0.17241	-0.07362	0.03028	-0.15321	0.01173	0.08522
3.40	0.16484	-0.07247	0.03036	-0.15064	0.01283	0.08553
3.50	0.15767	-0.07130	0.03042	-0.14803	0.01389	0.08575
3.60	0.15089	-0.07011	0.03044	-0.14540	0.01490	0.08589
3.70	0.14447	-0.06892	0.03044	-0.14276	0.01585	0.08595
3.80	0.13838	-0.06771	0.03042	-0.14011	0.01677	0.08593
3.90	0.13261	-0.06651	0.03037	-0.13746	0.01763	0.08584
4.00	0.12714	-0.06531	0.03030	-0.13481	0.01845	0.08568
4.10	0.12195	-0.06411	0.03022	-0.13218	0.01922	0.08547
4.20	0.11702	-0.06291	0.03011	-0.12957	0.01994	0.08519
4.30	0.11235	-0.06172	0.02999	-0.12698	0.02063	0.08487
4.40	0.10790	-0.06055	0.02985	-0.12441	0.02127	0.08449
4.50	0.10368	-0.05938	0.02970	-0.12187	0.02187	0.08407
4.60	0.09966	-0.05823	0.02953	-0.11936	0.02242	0.08360
4.70	0.09584	-0.05709	0.02935	-0.11689	0.02294	0.08310
4.80	0.09221	-0.05596	0.02916	-0.11446	0.02343	0.08256
4.90	0.08875	-0.05485	0.02896	-0.11206	0.02387	0.08198
5.00	0.08545	-0.05376	0.02875	-0.10970	0.02429	0.08138

Table II-9. Bare cylindrical reactor: functions $E_{3,k}(w)$.

w	ℓ=0, m=0	ℓ=0, m=1	ℓ=0, m=2	ℓ=1, m=0	ℓ=1, m=1	ℓ=2, m=0
0.10	0.97000	-0.00591	0.00088	-0.01273	-0.00248	0.00205
0.20	0.94134	-0.01138	0.00175	-0.02449	-0.00464	0.00414
0.30	0.91394	-0.01644	0.00262	-0.03537	-0.00653	0.00627
0.40	0.88774	-0.02112	0.00347	-0.04541	-0.00815	0.00841
0.50	0.86268	-0.02545	0.00432	-0.05467	-0.00953	0.01056
0.60	0.83869	-0.02944	0.00515	-0.06322	-0.01070	0.01271
0.70	0.81572	-0.03313	0.00597	-0.07110	-0.01168	0.01484
0.80	0.79372	-0.03654	0.00677	-0.07836	-0.01248	0.01696
0.90	0.77263	-0.03968	0.00755	-0.08503	-0.01312	0.01905
1.00	0.75241	-0.04256	0.00831	-0.09117	-0.01362	0.02111
1.10	0.73302	-0.04522	0.00906	-0.09680	-0.01399	0.02314
1.20	0.71442	-0.04766	0.00978	-0.10197	-0.01424	0.02512
1.30	0.69655	-0.04991	0.01048	-0.10670	-0.01439	0.02707
1.40	0.67940	-0.05196	0.01117	-0.11103	-0.01444	0.02897
1.50	0.66292	-0.05384	0.01183	-0.11498	-0.01441	0.03082
1.60	0.64707	-0.05556	0.01247	-0.11859	-0.01431	0.03262
1.70	0.63184	-0.05713	0.01308	-0.12186	-0.01413	0.03437
1.80	0.61719	-0.05856	0.01368	-0.12484	-0.01390	0.03606
1.90	0.60308	-0.05986	0.01425	-0.12753	-0.01362	0.03771
2.00	0.58951	-0.06103	0.01481	-0.12996	-0.01328	0.03930
2.10	0.57643	-0.06209	0.01534	-0.13215	-0.01291	0.04084
2.20	0.56383	-0.06305	0.01585	-0.13410	-0.01251	0.04232
2.30	0.55168	-0.06391	0.01635	-0.13585	-0.01207	0.04375
2.40	0.53997	-0.06467	0.01682	-0.13740	-0.01161	0.04512
2.50	0.52867	-0.06536	0.01727	-0.13878	-0.01112	0.04645
2.60	0.51777	-0.06596	0.01770	-0.13998	-0.01062	0.04772
2.70	0.50724	-0.06649	0.01812	-0.14102	-0.01010	0.04894
2.80	0.49708	-0.06694	0.01852	-0.14192	-0.00956	0.05011
2.90	0.48726	-0.06734	0.01889	-0.14268	-0.00902	0.05123
3.00	0.47777	-0.06768	0.01926	-0.14331	-0.00847	0.05230
3.10	0.46859	-0.06796	0.01960	-0.14383	-0.00791	0.05332
3.20	0.45972	-0.06819	0.01993	-0.14424	-0.00735	0.05430
3.30	0.45113	-0.06837	0.02024	-0.14455	-0.00679	0.05523
3.40	0.44282	-0.06851	0.02054	-0.14477	-0.00623	0.05612
3.50	0.43478	-0.06860	0.02082	-0.14490	-0.00567	0.05696
3.60	0.42698	-0.06866	0.02109	-0.14495	-0.00512	0.05776
3.70	0.41943	-0.06868	0.02134	-0.14493	-0.00456	0.05852
3.80	0.41212	-0.06867	0.02158	-0.14484	-0.00401	0.05924
3.90	0.40502	-0.06863	0.02180	-0.14468	-0.00347	0.05993
4.00	0.39814	-0.06857	0.02202	-0.14447	-0.00293	0.06057
4.10	0.39147	-0.06847	0.02222	-0.14420	-0.00240	0.06118
4.20	0.38499	-0.06835	0.02241	-0.14388	-0.00188	0.06176
4.30	0.37871	-0.06821	0.02259	-0.14352	-0.00136	0.06230
4.40	0.37260	-0.06805	0.02275	-0.14311	-0.00085	0.06281
4.50	0.36667	-0.06787	0.02291	-0.14267	-0.00036	0.06329
4.60	0.36091	-0.06767	0.02305	-0.14219	0.00013	0.06373
4.70	0.35531	-0.06746	0.02319	-0.14168	0.00061	0.06415
4.80	0.34987	-0.06723	0.02332	-0.14114	0.00108	0.06454
4.90	0.34457	-0.06699	0.02343	-0.14057	0.00154	0.06490
5.00	0.33942	-0.06674	0.02354	-0.13997	0.00200	0.06524

Bare reactors of the same initial (uniform) composition and the same geometrical buckling correspond to the same parameters g_0 and g_{25}, whatever the shape of the reactor. The excess neutron production is a monotonically decreasing function of the irradiation, i.e., $g_{25} > 0$, and $g_0 < 0$.

4.1. Zero-Dimensional Model

If the fuel were thoroughly mixed throughout the irradiation, or if it were fixed in position and irradiated in a spatially uniform flux, the local flux time θ would be independent of the position.

The flux time at the end of life is then determined by use of the vanishing control condition to be the root of the equation $g(\theta) = 0$; i.e.,

$$1 - e^{-\sigma_{25}\theta} = \frac{g_0 + g_{25}}{g_{25}}. \tag{252}$$

The corresponding (average and also maximum) burnup β is the value of the function $\beta(\theta)$; eliminating $e^{-\sigma_{25}\theta}$ between Eqs. (251) and (252) yields

$$\beta = \frac{g_0 + g_{25}}{g_{25}}. \tag{253}$$

In the following sections, the burnup obtainable at the end of batch irradiation of the unmixed fuel is studied as a function of the two parameters $(g_0 + g_{25})/g_{25}$ and g_{25}; by rewriting Eq. (250) as

$$g(\theta) = g_{25}\left[\frac{g_0 + g_{25}}{g_{25}} - (1 - e^{-\sigma_{25}\theta})\right],$$

the parameter $(g_0 + g_{25})/g_{25}$ is the value of the burnup obtained by using the zero-dimensional model, and the parameter g_{25} characterizes the magnitude of the function $g(\theta)$, which is proportional to the initial excess reactivity and inversely proportional to the initial leakage.

The first-order perturbation neglects the changes of the flux shape throughout the irradiation. It yields an approximation of the final burnup which is a function of the parameter $(g_0 + g_{25})/g_{25}$ only. The computation method is that developed in Benedict and Pigford.[B1]

The second-order perturbation gives a first approximation of the flux-changes and a corresponding correction to the final burnup, which are computed according to the method developed in the preceding sections. The parameter g_{25} determines the magnitude of the flux change.

The results thus obtained apply quantitatively to any fuel, the properties of which vary with the irradiation according to Eqs. (250) and (251); the results apply qualitatively to any fuel for which the excess neutron production is a monotonically decreasing function of the irradiation.

4.2. First Order Perturbation

For the more realistic model of an unmixed fuel irradiated in a spatially varying flux, the first approximation of the control absorption is given by Eqs. (114) and (132) as

$$c_0(T) = s_0(T) = g_0 + g_{25} E_{2,0}(\sigma_{25} T). \qquad (254)$$

Instead of T, it is here convenient to define a dimensionless irradiation variable w by

$$w = \sigma_{25} T , \qquad (255)$$

where T is the first approximation of the flux time at the center of the reactor.

The value of the irradiation variable at end of life satisfies the vanishing control condition. Setting $c_0 = 0$ in Eq. (254), one obtains

$$1 - E_{2,0}(w) = \frac{g_0 + g_{25}}{g_{25}} . \qquad (256)$$

Since the flux shape is assumed equal to $v_0(x)$ throughout the irradiation, the flux time to which the fuel has been irradiated at end of life is

$$\sigma_{25} \theta_0(x, w) = w v_0(x) . \qquad (257)$$

The average burnup of the fuel is then given by Eqs. (107) and (135) as

$$\left\langle \beta \right\rangle_0 = 1 - E_0(w) , \qquad (258)$$

and the central (maximum) burnup is simply

$$\beta(w) = 1 - e^{-w}; \tag{259}$$

$\langle \beta \rangle_0$ and $\beta(w)$ are then the functions of $(g_0 + g_{25})/g_{25}$ determined by Eqs. (256), (258) and (259); they are shown in Fig. II-2 for reactors of various geometries. Tabulated values of the functions $E_{2,0}(w)$ and $E_0(w)$ have been used which can be found in various sources with the following notations:

Geometry	Reference	$E_0(w)$	$E_{2,0}(w)$
Slab	(B1)	$E_0^G(w)$	
	Table II-2		$E_{2,0}(w)$
Sphere	(S2)	$E_0^{(1)}(w,0)$	$E_2^{(1)}(w,0)$
	Table II-5		$E_{2,0}(w)$
Cylinder	(S2)	$E_0^{(2)}(w,0)$	$E_2^{(2)}(w,0)$
	(B1)	$E_0^B(w,0)$	$E_2^B(w,0)$
	Table II-8		$E_{2,0}(w)$
Parallelopiped	(S2)	$E_0^{(3)}(w,0)$	$E_2^{(3)}(w,0)$

Figure II-2 shows that the maximum and average burnup are respectively larger and smaller than the corresponding value for mixed fuel (zero-dimensional model). The flatter the initial flux, i.e., the smaller the ratio of the maximum to the average flux, the better the zero-dimensional approximation.

The average burnup is smaller for unmixed fuel since the fuel is more depleted at the center where it has a larger importance.

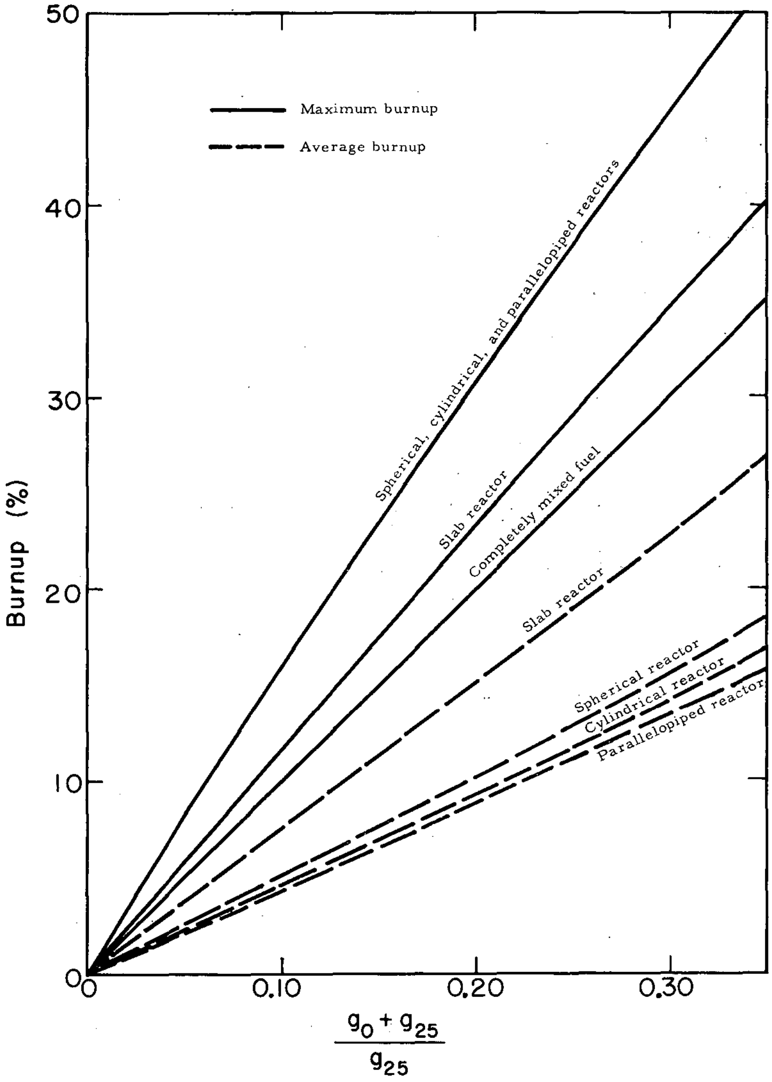

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Fig. II-2. Burnup as predicted by first-order perturbation method for U^{235} fueled reactors.

4.3. Second-Order Perturbation

Using Eqs. (250), (251), and (255), we find that Eqs. (132), (133), and (141) become

$$s_k(w) = g_{25}\, E_{2,k}(w), \quad \text{where } k \geqslant 1, \tag{260}$$

$$S_k(w) = g_{25}\, E_{3,k}(w), \quad \text{where } k \geqslant 1, \tag{261}$$

and

$$T\left\langle v_k, \beta'(v_0 T)\right\rangle = E_{1,k}(w), \quad \text{where } k \geqslant 0 . \tag{262}$$

The flux changes are then given by Eqs. (115) and (116) as

$$u_1^*(x,w) = g_{25} \sum_{k=1}^{\infty} \left(\frac{B_0^2}{B_k^2 - B_0^2}\right) E_{2,k}(w) , \tag{263}$$

where a value of w corresponds to an amount of control absorber $c_0 + c_1^*$; c_0 is given by Eq. (254); and Eqs. (121) and (117) yield

$$c_1^*(w) = g_{25}^2 \sum_{k=1}^{\infty} \frac{B_0^2}{B_k^2 - B_0^2}\, h_k \frac{d}{dw}\left\{ w\, E_{2,k}(w)\, E_{3,k}(w)\right\} . \tag{264}$$

Tables II-2, II-5 and II-8 giving the functions $E_{2,k}(w)$ show that, for $g_{25} > 0$, the coefficients of the lowest eigenfunctions ($k = 1$, or $\ell = 1$, $m = 0$ and $\ell = 0$, $m = 1$) in Eq. (263) are negative; this is easily seen to correspond to a flattening of the flux; on the other hand, $c_1^*(w)$ is positive.

Thus, since the fuel is more depleted at the center, the flux flattens. However, this flattening of the flux has let the fuel be slightly less depleted at the center than if it had been irradiated in a flux of constant shape. The importance of the fuel is then slightly increased, and this results in a higher overall reactivity than predicted by the first-order perturbation, that is, more control absorber is needed to keep the reactor critical ($c_1^* > 0$).

At end of life, using the value of the parameter w determined by Eq. (255), the first approximation of the flux time, Eq. (257), shall be added to the correction given by Eqs. (119) through (121). The corrective term for the average burnup can then be written

$$\left\langle \beta \right\rangle_1 = g_{25} \sum_{k=1}^{\infty} \frac{B_0^2}{B_k^2 - B_0^2} N_k(w) . \tag{265}$$

The functions $N_k(w)$ are defined, according to Eqs. (120) through (122) and Eqs. (260) through (264), by

$$\begin{aligned} N_k(w) = {} & E_{1,k}(w) E_{3,k}(w) \\ & - E_{1,0}(w) h_k \frac{\frac{d}{dw}\left[w \, E_{2,k}(w) \, E_{3,k}(w) \right]}{w \frac{d}{dw}\left[E_{2,0}(w) \right]} . \end{aligned} \tag{266}$$

As expected, the result of the second-order perturbation theory is a linear function of g_{25}, i. e., a linear function of the magnitude of the perturbation $g(\theta)$.

The following sections give the numerical results for slab, spherical and cylindrical reactors. One finds that the burnup is larger than predicted by the first-order perturbation theory. This effect has been explained[B1] by noting that since the flux is flatter, the burnup is more uniform. More precisely, one notes that for the same average burnup, the fuel is comparatively less depleted at the center where its importance is larger.

4.4. One-Dimensional Problems: Slab and Sphere

Although reactors shaped as slabs or spheres are not commonly used, the theory of slab reactors is of importance for its application to some continuous fueling schemes (cf. Sec. III), and spherical reactors can be used as a first approximation in the study of cylindrical reactors.

For a bare slab reactor, the eigenfunction expansion is an ordinary Fourier series. According to Eqs. (168) and (263), the approximation of the flux shape is given by

$$u_0(\zeta) + u_1^*(\zeta, w) = \cos\frac{\pi}{2}\zeta + g_{25} \sum_{k=1}^{\infty} \frac{E_{2,k}(w)}{2k(2k+2)} \cos(2k+1)\frac{\pi}{2}\zeta , \tag{270}$$

and the average and central values of the flux are proportional to

$$\langle u \rangle = \frac{2}{\pi}\left[1 + g_{25}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k(2k+1)(2k+2)}E_{2,k}(w)\right], \qquad (271)$$

and

$$u(0) = 1 + g_{25}\sum_{k=1}^{\infty}\frac{E_{2,k}(w)}{2k(2k+2)}. \qquad (272)$$

Table II-2 shows the rapid convergence of the series in the right side of the above equations; using the results of Sec. III.4.1, one can prove that, for large k, $E_{2,k}(w)$ is of an order of magnitude $1/k^3$. For $w = 1.0$, Table II-2 yields the following results:

$$u(\zeta, w) = \cos\frac{\pi}{2}\zeta + g_{25}\left[-0.0103\cos 3\frac{\pi}{2}\zeta + 0.0009\cos 5\frac{\pi}{2}\zeta - 0.0002\cos 7\frac{\pi}{2}\zeta + \cdots\right], \qquad (273)$$

$$\langle u \rangle \approx \frac{2}{\pi}(1 + 0.0037\, g_{25}), \qquad (274)$$

and

$$u(0) \approx 1 - 0.0096\, g_{25}. \qquad (275)$$

The flux shape flattens more and more as irradiation increases, and this can be shown to result in an increase of the burnup at end of life over the value predicted by the first-order perturbation theory.

4.5. Cylindrical Reactor

The flux changes are given by Eq. (263) with the functions $E_{2,k}(w)$ tabulated in Table II-8. Unlike the reactors considered above, the cylindrical reactor corresponds to a flux shape which is a function not only of w and g_{25}, but also of the ratio of the radial to axial dimensions; that is, according to Eq. (239), the coefficients $B_0^2/(B_k^2 - B_0^2)$ are now functions of a parameter $\lambda = R/Z$. If either one of the radial or axial dimensions tends toward an infinite value while the buckling remains constant, the flux changes become larger and larger. For instance, if the radius becomes very large, the coefficients of all the eigenfunctions $J_0(j_\ell \eta) \cos\frac{\pi}{2}\zeta$ tend to become infinite; this is not in contradiction with the results obtained for a slab or cylindrical but perfectly reflected reactor, since the boundary conditions now yield $J_0(j_0\eta)$ as the initial radial flux shape instead of the radially uniform flux in an infinite slab reactor.

The dependence of the flux changes upon the shape factor of the cylindrical reactor shows clearly the nonseparability of the flux; that is, only for the uniform reactor is the flux-shape the product of a function of the radial coordinate by a function of the axial coordinate z.

When the radius becomes infinitely large, i.e., when the buckling in the r direction (see Reference W3, p. 204) becomes infinitely small, the radial-flux changes become extremely large as irradiation proceeds since the diffusion of neutrons along the radial coordinate r is too small to balance the loss of reactivity of the fuel.

For all practical purposes, the ratio λ of radial to axial dimensions (physical dimensions of the bare reactor core) is close to unity. The following table gives some typical values of the coefficients $B_0^2/(B_k^2-B_0^2)$:

	$\ell=0$, $m=1$	$\ell=0$, $m=2$	$\ell=1$, $m=0$	$\ell=1$, $m=1$	$\ell=1$, $m=2$	$\ell=2$, $m=0$	$\ell=2$, $m=1$	$\ell=2$, $m=2$
R/Z=1	0.4180	0.1393	0.3342	0.1857	0.0983	0.1194	0.0929	0.0643
R/Z=2	0.1983	0.0661	0.6340	0.1510	0.0598	0.2265	0.1057	0.0517

Inspection of Table II-8 giving $E_{2,k}(w)$ shows that the coefficients of the eigenfunctions

$$v_k(x) = J_0(j_\ell \eta) \cos(\gamma_m \zeta) \tag{276}$$

rapidly become negligible when the value of $(\ell + m)$ increases. The most important correction to the flux shape and to the flux time arises from the eigenfunctions

$$J_0(j_0\eta) \cos(\gamma_1 \zeta) \tag{277}$$

and

$$J_0(j_1 \eta) \cos (\gamma_0 \zeta) . \tag{278}$$

The coefficients of these eigenfunctions are proportional to the corresponding auxiliary functions $E_{2,k}(w)$, in Eq. (263), and they are negative when g_{25} is positive. This corresponds to a flattening of the flux shape in the axial and radial directions.

To a given value of $(g_0 + g_{25})/g_{25}$, there corresponds a value w of the irradiation parameter at end of life, in Eq. (256), and a first approximation of the average burnup $\langle \beta \rangle_0$, in Eq. (258). The flattening of the flux then yields a positive correction $\langle \beta \rangle_1$ proportional to the parameter g_{25}. In the following paragraphs, we study the variation of $\langle \beta \rangle_1 / g_{25}$ as a function of w and λ.

4.5.1. Variation of $\langle \beta \rangle_1 / g_{25}$ with the irradiation parameter w. This variation is determined by the functions $N_k(w)$. For small w, Eqs. (145) and (266) yield

$$N_k(w) \approx \left[3 \frac{f_{1,0}}{f_{2,0}} h_k f_{2,k} - f_{1,k} \right] \frac{f_{2,k}}{2} w^2 = \frac{w^2}{2} \left(\frac{d^2 N_k}{dw^2} \right)_{w=0} . \tag{279}$$

The values of $\frac{1}{2} \left(\frac{d^2 N_k}{dw^2} \right)_{w=0}$ are given in the following table:

$\ell=0$, $m=1$	$\ell=0$, $m=2$	$\ell=1$, $m=0$	$\ell=1$, $m=1$	$\ell=1$, $m=2$	$\ell=2$, $m=0$	$\ell=2$, $m=1$
0.01576	0.00069	0.03059	0.00011	-0.00004	0.00110	-0.00004

The above numerical values indicate clearly that, for typical values of the coefficients $B_0^2/(B_k^2 - B_0^2)$, the correction terms due to the various eigenfunctions decrease rapdily as the value of $(\ell+m)$ increases. The following computations will be performed by taking into account only the first eigenfunctions corresponding to $\ell=0, m=1$ and $\ell=1, m=0$.

The functions $N_k(w)$ have been evaluated, from the tabulations of the functions $E_{1,k}(w)$, $E_{2,k}(w)$ and $E_{3,k}(w)$, by hand computations of the derivatives appearing in Eq. (266), using three or five-point formulae.[H2]

The following table gives the functions corresponding to $\ell=0, m=1$ and to $\ell=1, m=0$, which have been denoted by $N_2(w)$ and $N_4(w)$, respectively.

	0.1	0.2	0.3	0.4	0.5
$N_2(w)$	0.00015	0.00054	0.00111	0.00182	0.00262
$N_4(w)$	0.00028	0.00102	0.00213	0.00350	0.00502

	0.6	0.7	0.8	0.9	1.0
$N_2(w)$	0.00349	0.00439	0.00532	0.00623	0.00714
$N_4(w)$	0.00666	0.00836	0.01008	0.01179	0.01346

Using the above table of the functions $N_2(w)$ and $N_4(w)$, the value of $\langle \beta \rangle_1 / g_{25}$ as a function of w has been plotted on Fig. II-3.

4.5.2. Variation of $\langle \beta \rangle_1 / g_{25}$ with the shape factor $\lambda = R/Z$. Since the effects of the higher-order eigenfunctions are very small, this study will be performed by assuming that the flux changes can be represented by only the two eigenfunctions corresponding to $\ell=0, m=1$ and $\ell=1, m=0$. Equations (239) and (265) then give

$$\frac{\langle \beta \rangle_1}{g_{25}} = \frac{j_0^2 + \lambda^2 \gamma_0^2}{\lambda^2 (\gamma_1^2 - \gamma_0^2)} N_2(w) + \frac{j_0^2 + \lambda^2 \gamma_0^2}{j_1^2 - j_0^2} N_4(w) \; . \tag{280}$$

For simplicity's sake, a new shape factor y is defined by the following equation

$$y = \frac{\gamma_0^2}{j_0^2} \lambda^2 = \frac{\gamma_0^2}{j_0^2} \frac{R^2}{Z^2} \; . \tag{281}$$

Then Eq. (280) becomes

$$\frac{\langle\beta\rangle_1}{g_{25}} = \left(1+\frac{1}{y}\right)\frac{\gamma_0^2}{\gamma_1^2-\gamma_0^2}\,N_2(w) + (1+y)\,\frac{j_0^2}{j_1^2-j_0^2}\,N_4(w). \qquad (282)$$

For positive values of y, the above function of y has a unique minimum which is the root of

$$\frac{\partial\langle\beta\rangle_1}{g_{25}\,\partial y} = -\frac{1}{y^2}\,\frac{\gamma_0^2}{\gamma_1^2-\gamma_0^2}\,N_2(w) + \frac{j_0^2}{j_1^2-j_0^2}\,N_4(w)=0. \qquad (283)$$

Thus, y_M is the following function of w:

$$y_M(w)=\sqrt{\frac{\dfrac{\gamma_0^2}{\gamma_1^2-\gamma_0^2}\,N_2(w)}{\dfrac{j_0^2}{j_1^2-j_0^2}\,N_4(w)}}\;, \qquad (284)$$

and the corresponding minimum value of the correction to the burnup is

$$\langle\beta\rangle_{1,\,M}= g_{25}\left(\sqrt{\frac{\gamma_0^2}{\gamma_1^2-\gamma_0^2}\,N_2(w)} + \sqrt{\frac{j_0^2}{j_1^2-j_0^2}\,N_4(w)}\right)^2 \qquad (285)$$

For small values of the irradiation variable w, Eq. (279) and the numerical values of the corresponding table give:

$$y_M(0) = \sqrt{\frac{\dfrac{\gamma_0^2}{\gamma_1^2-\gamma_0^2}\left(\dfrac{d^2N_2}{dw^2}\right)_0}{\dfrac{j_0^2}{j_1^2-j_0^2}\left(\dfrac{d^2N_4}{dw^2}\right)_0}} = \sqrt{0.275} = 0.524 \qquad (286)$$

and $R/Z = 1.11$.

This is just slightly larger than the value of the shape factor corresponding to the minimum leakage in a bare cylindrical reactor (minimum volume for the uniformly loaded reactor), which is given by

$$y = 0.5 \quad \text{and} \quad R/Z = 1.0855 \ . \tag{287}$$

Between $w = 0$ and $w = 1$, the value of $y_M(w)$, Eq. (284), increases slightly, but does not differ more than 5% from its initial value.

The correction to the burnup varies with the factor shape y; its variation can be studied as follows: First, Eqs. (282) and (285) yield

$$\frac{\langle \beta \rangle_1 - \langle \beta \rangle_{1,M}}{g_{25}} = \left(\sqrt{\frac{\gamma_0^2}{\gamma_1^2 - \gamma_0^2} \, \frac{N_2(w)}{y}} - \sqrt{\frac{j_0^2}{j_1^2 - j_0^2} \, y N_4(w)} \right)^2 \ . \tag{288}$$

Then, Eq. (284) yields

$$\frac{\langle \beta \rangle_1 - \langle \beta \rangle_{1,M}}{g_{25}} = \frac{\gamma_0^2}{\gamma_1^2 - \gamma_0^2} \, N_2(w) \, \frac{[\, y_M(w) - y \,]^2}{y \, y_M^2(w)} \ . \tag{289}$$

Thus, for each value of the irradiation parameter w, i.e., for each value of $(g_0 + g_{25})/g_{25}$, there are two values for y which yield the same value of the final burnup; they are related by the equation

$$y_1 y_2 = y_M^2(w) \ .$$

The curves representing the variation of the second-order correction to the burnup, Fig. II-3, have been labeled with the two values of the ratio R/Z which correspond to the above equation when the small variations of $y_M(w)$ are neglected, i.e., when $y_M(w) = y_M(0) = 0.524$.

4.6. Discussion of the Results

The foregoing sections show the flattening of the flux predicted by one-group theory and the resulting increase of the average burnup over its value predicted by first-order perturbation theory when a reactor is loaded with highly enriched fuel; for cylindrical reactors, the average burnup is a function of the shape of the reactor core, and, in the present case, is minimum for reactors whose radius is about 1.1 times the height.

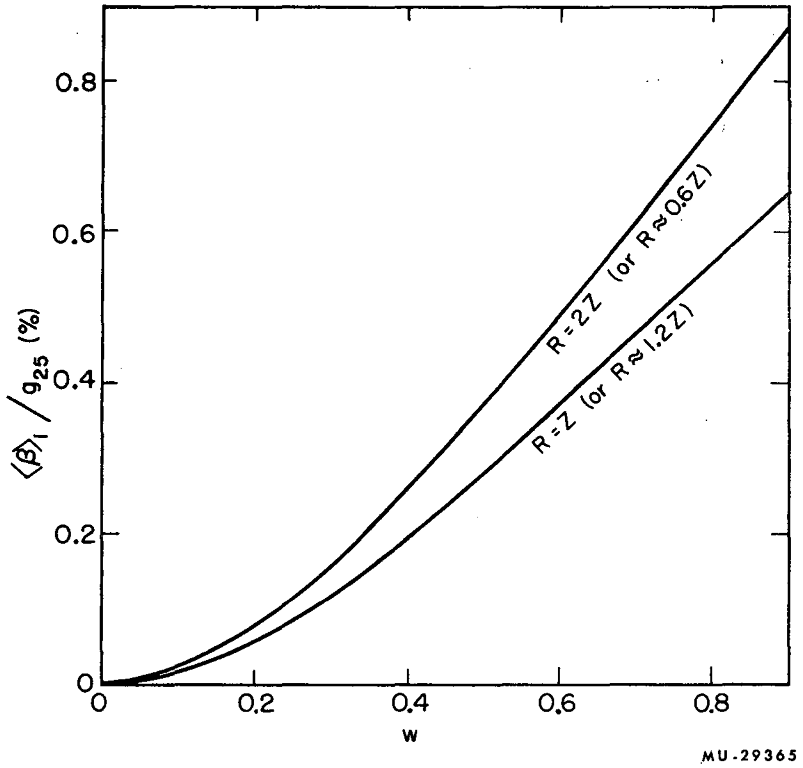

Fig. II-3. Second-order perturbation correction to the average burnup in U^{235} fueled cylindrical reactors.

The values of the average burnup at the end of batch irradiation of fixed fuel in a bare cylindrical reactor whose height and diameter are equal are shown in Fig. II-4. The fuel is pure U^{235}, and its excess neutron production varies according to Eq. (250). Thus, $(g_0+g_{25})/g_{25}$ is the value of the burnup corresponding to completely mixed fuel, and g_{25} then characterizes the magnitude of the perturbation of the flux shape; the evaluation of the parameters from the physical properties of the uniformly loaded (initial) reactor is detailed in Sec. IV. 1.

According to the second-order perturbation method, the flux changes are proportional to g_{25}. The curve labeled $g_{25} = 0$ corresponds to the first-order perturbation theory (unperturbed flux shape); the second-order perturbation results are obtained by adding to it the product of g_{25} and the correction $\langle\beta\rangle_1/g_{25}$ computed in the preceding section (cf. Fig. II-3); the correspondence between the irradiation variable w and the parameter $(g_0+g_{25})/g_{25}$ is given by Eq. (256).

As stated in Sec. I.3, the results of the second-order perturbation method shall be compared with those obtained by more accurate computations. Thus, the reactor equations, Eq. (20), has been solved by numerical methods on a high-speed digital computer. The program used for these computations is a modified version of the FUELCYC code developed at MIT.[S3] Equation (20) is solved repeatedly for successive values of the time t. It is written in matrix form by use of the finite difference method and, at each time step, is solved by the Crout reduction technique.

Results of the finite differences computations are shown in Fig. II-4. A sufficient number of time steps has been chosen to make the error negligible; the results are then functions of the number of mesh points used in the finite differences representation of the Laplacian operator. This number of mesh points is limited by the capacity of the memory of digital computers. Computations have been performed with 49 and 100 mesh points. If the error were exactly proportional to the square of the mesh size, or inversely proportional to the number of mesh points, the exact results would differ from the curve corresponding to 100 mesh points by the same amount the latter curve differs from the curve corresponding to 49 mesh points.

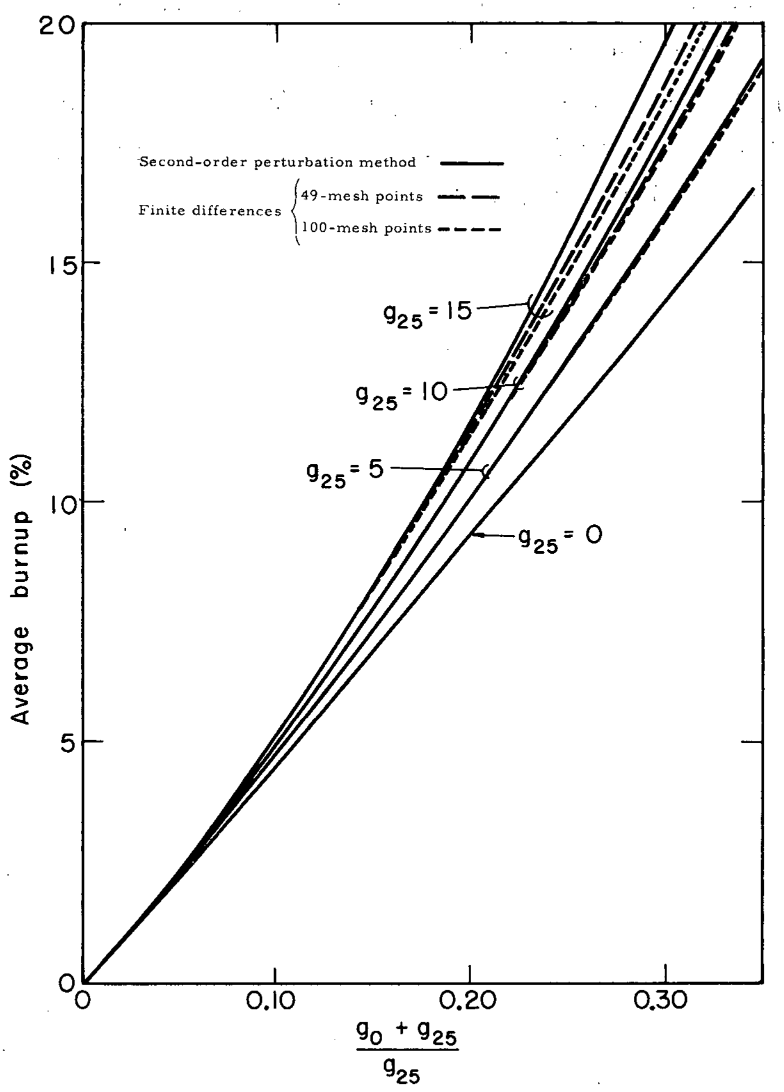

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Fig. II-4. Average burnup at end of life as a function of the parameters g_0 and g_{25} in U^{235} fueled cyclindrical reactors.

The second-order perturbation is seen, in Fig. II-4, to be very accurate for small values of the average burnup and for small values of the perturbation parameter g_{25}. It is as accurate as finite-differences computations with 49 mesh points for average burnups up to 20% when $g_{25} = 5$, or up to 10% when $g_{25} = 15$. Application to a specific reactor is given in Sec. IV. 1.2.3.

III. CONTINUOUS FUELING

In this section, we consider a reactor core shaped as a right cylinder of arbitrary cross section. The fuel can be irradiated according to the fuel management procedures described in the introduction: these procedures are radial mixing, graded irradiation, and various steady axial movements of the fuel. These fuel management procedures, as defined in this section, have in common the characteristic that the average composition of the fuel at a given position in the reactor is independent of the radial coordinates.

1. Equations and Methods

1.1. Equation for the Axial Flux-Shape

If the average composition of the fuel is independent of the radial coordinates x and y, so is the local production of excess neutrons represented by the symbol ∇_ϕ in Eq. (I-24). One assumes that the control absorber Σ_E, if needed, is also independent of the radial coordinates. One uses then the method of separation of variables, writing the flux as the following product:

$$\phi(x, y, z, t) = f(x, y, t)\, h(z, t) \ . \tag{1}$$

The basic reactor equation, Eq. (I-24), can be written as the sum of the two following equations:

$$-D\left(\frac{\partial^2 h}{\partial z^2} + B_Z^2 h\right) = (\nabla_\phi - \Sigma_E)\, h \ , \tag{2}$$

and

$$-D\,(\Delta f + B_R^2 f) = 0 \ , \tag{3}$$

with

$$B_0^2 = B_R^2 + B_Z^2 \ . \tag{4}$$

If the boundary condition on the curved surface of the cylinder is independent of the axial coordinate z and of the irradiation, Eq. (3) can be solved to yield an axial flux shape f(x, y) as well as a radial buckling B_R^2 constant throughout the irradiation.

Dividing Eq. (2) by DB_Z^2, one obtains the following equation for the axial flux shape:

$$-\left(\frac{1}{B_Z^2}\ \frac{\partial^2 h}{\partial z^2} + h\right) = \frac{\nabla_\phi - \Sigma_E}{DB_Z^2}\ h\ . \tag{5}$$

Let the reactor core be of height 2Z and choose the origin of the coordinate system in the midplane xy. The function h(z, t) shall satisfy the boundary condition

$$h \pm \delta_\ell\ \frac{dh}{dz} = 0 \quad \text{for} \quad z = \pm Z\ . \tag{6}$$

We use the following notations:

$$\gamma_0^2 = B_Z^2/Z^2\ , \tag{7}$$

$$\zeta = z/Z\ , \tag{8}$$

$$h(z,t) = u(\zeta,t)\ , \tag{9}$$

and

$$\epsilon_Z = \delta_\ell/Z\ . \tag{10}$$

The axial flux-shape is then the solution $u(\zeta, t)$ of the equation

$$-\left(\frac{1}{\gamma_0^2}\ \frac{\partial^2 u}{\partial \zeta^2} + u\right) = \frac{\nabla_u - \Sigma_E}{DB_Z^2}\ u\ , \tag{11}$$

with boundary conditions

$$u \pm \epsilon_Z\ \frac{\partial u}{\partial \zeta} = 0 \quad \text{for} \quad \zeta = \pm 1\ . \tag{12}$$

For a reactor symmetrical about its midplane, one considers only the interval $0 \leqslant \zeta \leqslant 1$, and the boundary conditions become

$$\frac{\partial u}{\partial \zeta} = 0\ ,\ \zeta = 0\ , \quad \text{and} \quad u + \epsilon_Z \frac{\partial u}{\partial \zeta} = 0,\quad \zeta = 1\ . \tag{13}$$

The symbol ∇_u in the right side of Eq. (11) represents the result of an operation on the function u, an operation which will be made explicit for each fuel scheduling scheme.

The perturbation method seeks the solution of Eq. (11) as the sum of an infinite series of functions, each of which is the solution of a linear differential equation. Some general methods for solving such a linear differential equation are given in the following sections; they could also be used for solving Eq. (11) by the perturbation-iteration technique.[M5]

Finally, elliptic functions are introduced; they yield the exact solution of Eq. (11) when the variation of the excess neutron production $\nabla(\theta)$ is a quadratic function of the flux time to which the fuel has been irradiated.

1.2. The Linearized Equations

The left side of Eq. (11) can be considered as a linear operation on the function u; it will be written Lu, where L is a linear operator. Likewise, the right side of Eq. (11) can be considered as a nonlinear operation on the function u.

There could have been many other ways to write Eq. (11) as an equality between the results of the operation of a linear operator on the left side, and of a nonlinear operator on the right side; however, the perturbation method, as applied to the present problem, requires that the right side of Eq. (11) vanish for the just critical homogeneous reactor. Thus, L is defined by

$$L = -\left(\frac{1}{\gamma_0^2}\frac{\partial^2}{\partial\zeta^2} + 1\right) \tag{14}$$

in such a way that the equation

$$Lu_0 = 0 \tag{15}$$

has a nontrivial solution u_0 satisfying the boundary condition in Eq. (12). This solution is

$$u_0 = \cos \gamma_0\zeta, \tag{16}$$

where γ_0 is the smallest positive root of the equation,

$$\cos \gamma_0 - \epsilon_Z \gamma_0 \sin \gamma_0 = 0 . \tag{17}$$

According to the perturbation method, the solution of Eq. (11) will be sought as the infinite series

$$u(\zeta, t) = u_0(\zeta, t) + u_1(\zeta, t) + u_2(\zeta, t) \cdots \tag{18}$$

where each term is the solution of a linear differential equation

$$Lu_n(\zeta, t) = F_n(\zeta, t) . \tag{19}$$

The following presents some general methods which will be considered for solving such an equation.

1.3. Green's Function

The following results are the application to the present problem of a theorem stated by Friedman (see Ref. F1, p. 170).

The self-adjoint homogeneous differential equation

$$Lu(\zeta) = F(\zeta), \text{ where } 0 \leq \zeta \leq 1 , \tag{20}$$

with linear homogeneous boundary conditions, has a solution if and only if $F(\zeta)$ is orthogonal to the nontrivial solution $u_0(\zeta)$ of the corresponding homogeneous equation; i.e., if,

$$\int_0^1 F(x) u_0(x) dx = 0 . \tag{21}$$

Let $V(\zeta)$ be a function satisfying the equation $LV(\zeta) = 0$ (with arbitrary boundary conditions) and let $V(\zeta)$ be independent of $u_0(\zeta)$.

Then, a particular solution of Eq. (20) is the function $U(\zeta)$ defined by

$$U(\zeta) = \frac{u_0(\zeta)}{J} \int_0^\zeta V(x) F(x) dx - \frac{V(\zeta)}{J} \int_0^\zeta u_0(x) F(x) dx, \tag{22}$$

where J is a constant which is easily determined from Eq. (20) in which one replaces $u(\zeta)$ by $U(\zeta)$ (J can also be determined from its definition as the "conjunct" of the two functions u_0 and V).

Letting L be defined by Eq. (14), with the boundary conditions for symmetrical reactor, Eq. (13), one can then take $V(\zeta) = \sin \gamma_0 \zeta$; the particular solution $U(\zeta)$, Eq. (22), is given by

$$U(\zeta) = \gamma_0 \left[\cos \gamma_0 \zeta \int_0^{\zeta} \sin \gamma_0 x F(x)\, dx - \sin \gamma_0 \zeta \int_0^{\zeta} \cos \gamma_0 x\, F(x)\, dx \right].$$

We see that $U(\zeta)$ vanishes as well as its first derivative at $\zeta = 0$, and the general solution $u(\zeta)$ of Eq. (20) is the sum of this particular solution and of an arbitrary multiple of the solution $\cos \gamma_0 \zeta$ of the homogeneous equation, Eq. (15). Thus,

$$u(\zeta) = u(0) \cos \gamma_0 \zeta - \gamma_0 \int_0^{\zeta} \sin \gamma_0 (\zeta - x) F(x) dx . \qquad (23)$$

An additional condition (normalization condition) is required in order to define uniquely the arbitrary constant $u(0)$, which is here the value of the function $u(\zeta)$ at $\zeta = 0$.

The above result could be used in order to obtain the solution to the linear equation, Eq. (19), when the function $F_n(\zeta, t)$ is simple enough for the above integrals to be evaluated analytically.

This result could also be used in order to determine the solution of the nonlinear equation, Eq. (11), by an iteration procedure, since Eqs. (11) and (23) are equivalent to

$$u(\zeta, t) - u(0,t) \cos \gamma_0 \zeta = - \gamma_0 \int_0^{\zeta} \sin \gamma_0 (\zeta - x) \frac{\nabla_u - \Sigma_E}{DB_Z^2} u(x,t)\, dx \quad (24)$$

and

$$\int_0^1 \cos \gamma_0 x \, \frac{\nabla_u - \Sigma_E}{DB_Z^2} \, u(x,t) dx = 0. \qquad (25)$$

Assume that one has obtained an approximate solution $u^{(n)}(\zeta, t)$ and the corresponding explicit expressions for ∇_u and Σ_E, noted by $\nabla_u^{(n)}$ and $\Sigma_E^{(n)}$ respectively, which satisfy

$$\int_0^1 \cos \gamma_0 x \, \frac{\nabla_u^{(n)} - \Sigma_E^{(n)}}{DB_Z^2} \, u^{(n)}(x, t)dx = 0 \, , \tag{26}$$

then the iteration method yields the next approximation $u^{(n+1)}(\zeta, t)$ as

$$\begin{aligned} u^{(n+1)}(\zeta, t) - u^{(n+1)}(0, t)\cos \gamma_0 \zeta \\ = - \gamma_0 \int_0^\zeta \sin \gamma_0 (\zeta - x) \, \frac{\nabla_u^{(n)} - \Sigma_E^{(n)}}{DB_Z^2} \, u^{(n)}(x, t)\, dx. \end{aligned} \tag{27}$$

The values of $u^{(n+1)}(0, t)$ and $\Sigma_E^{(n+1)}$ are related by the following equation:

$$\int_0^1 \cos \gamma_0 x \, \frac{\nabla_u^{(n+1)} - \Sigma_E^{(n+1)}}{DB_Z^2} \, u^{(n+1)}(x, t)dx = 0. \tag{28}$$

For instance, if one seeks the solution of Eq. (11) for which the reactor is just critical without control absorber, i. e., if one lets $\Sigma_E \equiv 0$, then Eq. (28) is just a normalization condition which determines the value of $u^{(n+1)}(0, t)$.

In section III.4.3, the solution of the first-order perturbation equation is obtained in closed form by use of the Green's function. Equations (191) and (192) compare the results thus obtained with those of an eigenfunction expansion.

1.4. Eigenfunction Expansion

In general, the function $F(\zeta)$ on the right side of Eq. (20) is not simple enough for the integral in Eq. (23) to be evaluated analytically; one then seeks the solution u by using the eigenfunction method.

We developed this method during the study of the batch irradiation of unmixed fuel. In Sec. II, the case of a slab reactor corresponds just to Eq. (20) where the operator L is defined by Eq. (14) together with the boundary conditions for symmetrical reactor, Eq. (13). With the present notations, the result is

$$u(\zeta) = a_0 \cos \gamma_0 \zeta + \sum_{k=1}^{\infty} \frac{\gamma_0^2}{\gamma_k^2 - \gamma_0^2} \cos \gamma_k \zeta \, \frac{\int_0^1 F(x) \cos \gamma_k x \, dx}{\int_0^1 \cos^2 \gamma_k x \, dx} \tag{30}$$

where a_0 is an arbitrary constant, $F(x)$ satisfies Eq. (21), and γ_k is the $(k+1)^{th}$ root of the equation

$$\cos \gamma_k - \epsilon_Z \gamma_k \sin \gamma_k = 0 \, . \tag{31}$$

An iteration procedure for solving the nonlinear equation, Eq. (11), could be derived with the help of Eq. (30), just as had been done with the help of Eq. (23).

1.5. Fourier Expansion

The foregoing eigenfunction expansion has the disadvantage of requiring one to first solve a transcendental equation, Eq. (31), in order to obtain the eigenvalues γ_k and the corresponding eigenfunctions $\cos \gamma_k \zeta$.

This section shows how to obtain the solution of Eq. (20) as a Fourier series of trigonometric functions over a correctly enlarged interval. Only the case of a symmetrical boundary condition, Eq. (13), will be developed below.

First, one defines ζ_1' as the first positive root of the equation $u_0(\zeta) = 0$.

The function $u_0(\zeta)$ can then be defined as the nontrivial solution of Eq. (15) which satisfies,

(1) $$\frac{du}{d\zeta} = 0 \, , \quad \text{at} \quad \zeta = 0 \, , \tag{32}$$

and

(2) $$\begin{cases} \text{either } u + \epsilon_Z \dfrac{du}{d\zeta} = 0 \, , \text{ at } \zeta = 1 \, , & (33) \\ \text{or} \quad u(\zeta_1') = 0 \, . & (34) \end{cases}$$

Given a positive value of ϵ_Z, γ_0 is determined uniquely by Eq. (17) and ζ_1' is then given by

$$\zeta_1' = \pi/2\,\gamma_0 > 1 \; . \tag{35}$$

The above equivalent boundary conditions, Eqs. (33) or (34), are defined in reactor theory[W3] as follows:

(1) The linear extrapolation distance, δ_ℓ, is the distance beyond Z (the physical boundary of the reactor core) at which the tangent to the neutron flux goes to zero. To fix the value of δ_ℓ [or ϵ_Z, Eq. (10)] implies the use of the boundary condition given by Eq. (33).

(2) The reflector saving δ is the distance beyond Z at which the analytic continuation of the flux goes to zero. The degree of truncation of the initial flux will be defined by the coefficient ϵ, where

$$\epsilon = \frac{Z' - Z}{Z'} = \frac{\delta}{Z+\delta} \tag{36}$$

and the change of variable, Eq. (8), yields

$$\zeta_1' = \frac{1}{1-\epsilon} \; . \tag{37}$$

For a homogeneous reactor, ϵ and ϵ_Z are related by Eq. (35) in which one replaces ζ_1' and γ_0 by their values obtained from Eq. (17) and (37).

The above definitions are represented in Fig. III-1.

Consider now the solution $w(\zeta)$ of the following equation:

$$-\left(\frac{1}{\gamma_0^2}\frac{d^2w}{d\zeta^2} + w\right) = F(\zeta)\,H(1-\zeta), \quad 0 \leq \zeta \leq \zeta_1', \tag{38}$$

with

$$\frac{dw}{d\zeta} = 0, \; \zeta = 0 \quad \text{and} \quad w(\zeta_1') = 0 \; . \tag{39}$$

The symbol H, in Eq. (38) represents the Heaviside step function; i.e.,

$$H(1-\zeta) = \begin{cases} 0 & \text{if } \zeta > 1 \; , \\ 1 & \text{if } \zeta < 1 \; . \end{cases} \tag{40}$$

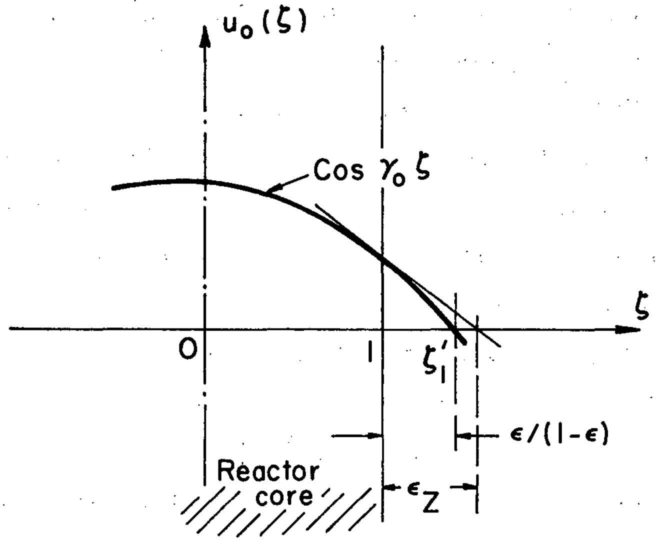

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Fig. III-1. Reflector saving and linear extrapolation distance.

Seeking the function $w(\zeta)$ by Green's function technique, one first requires that the orthogonality condition be satisifed; i.e.,

$$\int_0^{\zeta_1'} F(x)\, H(1-x)u_0(x)dx = \int_0^1 F(x)u_0(x)dx = 0 \,, \qquad (41)$$

which is the same as Eq. (21).

A particular solution $w(\zeta)$ can now be constructed with the same functions $u_0(\zeta)$ and $V(\zeta)$ as in Eq. (22).

The function $w(\zeta)$ is then, when inside the interval $0 \leqslant \zeta < 1$, identical with the function $u(\zeta)$, Eq. (23). Since the right side of Eq. (38) is a piecewise continuous function, $w(\zeta)$, as well as its first derivative, is continuous on the entire interval $0 \leqslant \zeta \leqslant \zeta_1'$.[F1] Thus, at $\zeta = 1$, it also satisfies Eq. (13).

Then, in the interval $0 \leqslant \zeta \leqslant 1$, the solution $w(\zeta)$ of Eq. (38), with boundary condition Eq. (39), is exactly the solution $u(\zeta)$ of Eq. (20) with boundary condition, Eq. (13).

However, at $\zeta = 1$, the higher derivatives of the function $w(\zeta)$ are not continuous; if one replaces the reactor equations, Eqs. (11) and (13) by

$$-\left(\frac{1}{\gamma_0^2}\,\frac{\partial^2 u}{\partial \zeta^2} + u\right) = \frac{\nabla_u - \Sigma_E}{DB_Z^2}\, u\, H(1-\zeta), \text{ where } 0 \leqslant \zeta \leqslant \zeta_1' \,, \qquad (42)$$

and

$$\frac{\partial u}{\partial \zeta} = 0 \text{ at } \zeta = 0 \,, \text{ and } u(\zeta_1', t) = 0 \,, \qquad (43)$$

then the linear extrapolation distance is assumed constant throughout the irradiation, but not the reflector saving. This corresponds to a reflector the composition of which does not change with irradiation.

One can now use the eigenfunction technique to solve Eq. (38). The eigenfunctions are $\cos \beta_k \zeta$, and the eigenvalue equation is simply

$$\cos \beta_k \zeta_1' = 0 \,. \qquad (44)$$

Equations (35) and (44) yield

$$\beta_k = (2k + 1)\, \gamma_0 \,. \qquad (45)$$

The solution $u(\zeta)$ has then the following expansion

$$u(\zeta) = a_0' \cos \gamma_0 \zeta + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2 - 1} \cos(2k+1)\gamma_0 \zeta \frac{\int_0^1 F(x) \cos(2k+1)\gamma_0 x \, dx}{\int_0^{\zeta_1'} \cos^2(2k+1)\gamma_0 x \, dx} \tag{46}$$

where

$0 \leqslant \zeta \leqslant 1$.

Clearly Eqs. (30) and (46) are identical when $\epsilon_Z = 0$ (and consequently, when $\epsilon = 0$).

Section III.4.2 describes the application of the Fourier expansion technique to the solving of the first-order perturbation equation.

1.6. Elliptic Functions

The reactor equation, Eq. (11), can often be transformed into the following equation which determines the flux time to which the fuel has been irradiated:

$$-\left(\frac{1}{\gamma_0^2} \frac{d^2\theta}{dx^2} + \theta\right) = G(\theta) - I_1 . \tag{50}$$

Here, $G(\theta)$ is a given function of θ, and I_1 is a constant of integration which is determined by the fuel scheduling scheme.

Equation (50) can be solved directly by two integrations; by multiplying both its sides by $\frac{d\theta}{dx}$ and integrating we obtain

$$\left(\frac{d\theta}{dx}\right)^2 = 2\gamma_0^2 \left[I_0 + I_1\theta - \frac{\theta^2}{2} - \int_0^{\theta} G(\theta)\, d\theta \right] = f(\theta), \tag{51}$$

where I_0 is a constant of integration. After requiring that the function $f(\theta)$, defined by the above equality, be non-negative, one obtains

$$\pm x = \int [f(\theta)]^{-\frac{1}{2}} d\theta + I_2 . \tag{52}$$

In general, the above integral cannot be evaluated in terms of tabulated functions, but the following case (see Ref. W6, p. 454) is an exception.

If f(t) is a quartic polynomial with no repeated factors,

$$f(t) = a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4 , \qquad (53)$$

the integral

$$x = \int_a^\theta [f(t)]^{-\frac{1}{2}} dt$$

is equivalent to

$$\theta = a + \frac{\sqrt{f(a)}\gamma'(x) + \frac{1}{2} f'(a)\left[\gamma(x) - \frac{1}{24} f''(a)\right] + \frac{1}{24} f(a) f'''(a)}{2\left[\gamma(x) - \frac{1}{24} f''(a)\right]^2 - \frac{1}{48} f(a) f^{IV}(a)} , \qquad (54)$$

where $\gamma(x) = \gamma(x; G_2, G_3)$ is the Weierstrassian elliptic function formed with the invariants of the quartic f(t), and

$$G_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2 , \qquad (55)$$

and

$$G_3 = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4 . \qquad (56)$$

Whenever possible, the solution will be expressed in terms of the Jacobian elliptic functions which are readily amenable to computations, although that requires that the quartic f(t) be first resolved into factors (see Ref. B3, p. 307).

For most fuel scheduling schemes, the function $G(\theta)$ is the integral of the excess neutron production $g(\theta)$, Eqs. (62) and (67). Then, Eq. (50) has a solution in terms of elliptic functions whenever the excess neutron production, (or the nuclide concentrations) can be fitted to a quadratic polynomial; we have

$$g(\theta) = g_0 + g_1\theta + g_2\theta^2 , \qquad (57)$$

$$G(\theta) = g_0\theta + \frac{g_1}{2}\theta^2 + \frac{g_2}{3}\theta^3 , \qquad (58)$$

and

$$f(\theta) = 2\gamma_0^2\left(I_0 + I_1\theta - \frac{1+g_0}{2}\theta^2 - \frac{g_1}{6}\theta^3 - \frac{g_2}{12}\theta^4\right) . \qquad (59)$$

When $I_1 = 0$ (radial mixing with central control, graded irradiation, bidirectional, in-out), Eq. (51) is in Jacobian normal form if $g_1 = 0$.

Otherwise, one has to find the roots of the polynomial $f(\theta)$; this could be done easily for any steady state fuel scheduling scheme when $g_2=0$; and one would then use the formulae given by Byrd[B3] in order to solve the integral in Eq. (52).

Sections III.4.5, III.5.2, III.5.3 and III.5.4 treat some simple applications of the above discussion. Standard notations[D2] are used consistently to denote the Jacobian elliptic functions sn, cn, and dn, their modulus k, complementary modulus k', and reduced period K.

2. Radial Mixing

2.1. Description and Equations

The fuel is irradiated batchwise while being thoroughly mixed radially; its composition is then a function of the time and of the axial coordinate only. The reactor is kept just critical throughout the irradiation by means of a control absorber the composition of which is also independent of the radial coordinates. The flux is then separable as shown in Eq. (1).

In a plane z = constant, the flux time to which the fuel has been uniformly irradiated is the time integral of the radial average of the flux

$$\theta(z,t) = \int_0^t h(z,t)\left[\frac{1}{S}\int_S f(x,y,t)\, dS\right] dt, \qquad (60)$$

where S is the surface of a normal section of the cylindrical core.

One defines a normalized flux shape $u(\zeta, T)$ and the corresponding irradiation parameter T by

$$\theta(\zeta, T) = \int_0^T u(\zeta, T) dT . \qquad (61)$$

The excess-neutron production denoted by ∇_u in Eq. (11) is now the function $\nabla[\theta(\zeta, T)]$. The following dimensionless functions

$$g(\theta) = \nabla(\theta)/DB_Z^2 \qquad (62)$$

$$c(\zeta, T) = \Sigma_E(\zeta, T)/DB_Z^2 , \qquad (63)$$

will still be referred to as the neutron excess production and the control absorption, respectively. With the above definitions Eq. (11) becomes

$$-\left(\frac{1}{\gamma_0^2}\frac{\partial^2 u(\zeta,T)}{\partial\zeta^2}+u(\zeta,T)\right)=\left\{g\left[\int_0^T u(\zeta,T)\,dT\right]-c(\zeta,T)\right\}u(\zeta,T) \tag{64}$$

The above equation is exactly that which has been found to describe the batch irradiation of unmixed fuel in a slab reactor, Eq. (II-20); an approximate solution can be obtained by the perturbation method developed in Sec. II. The results shall be compared as follows:

Assume that the same fuel is irradiated batchwise in reactors of the same geometrical buckling, B_0^2, according to the following procedures:

<u>Case 1</u>, the fuel is mixed radially throughout its irradiation in a cylindrical reactor.

<u>Case 2</u>, the fuel is fixed in position throughout its irradiation in a slab (or cylindrical but perfectly reflected on its curved surface) reactor core.

The dimensions of the two reactors are not the same, since the height of the first reactor must be $B_0/\sqrt{B_0^2-B_R^2}$ (i.e., B_0/B_Z) times the height of the second; furthermore, the term in the right side of Eq. (64) is, for Case 1, B_0^2/B_Z^2 times what it is for Case 2.

For both cases, the first-order perturbation theory determines the control absorption by the equation

$$\left\langle [g(u_0T)-c]\,u_0,u_0\right\rangle = 0. \tag{65}$$

This yields the same value of the flux time at end of life (c = 0) and the same value for the final burnup of the fuel.

However, the flux changes as given by the solution of the first-order perturbation equation are, in Case 1, B_0^2/B_Z^2 times the flux changes in Case 2, and so are the corrective terms to be added to the flux time and to the burnup of the fuel.

In the following sections we consider nonuniform distributions of the control absorber, and develop a technique for solving exactly the equation corresponding to a control absorber localized at the midplane of the reactor.

2.2. Equation for the Flux-Time. Uniform Control Absorber

Equation (64) can be transformed into an equation which determines the flux-time; integrating it with respect to the irradiation parameter T and using Eq. (61) yields

$$-\left(\frac{1}{\gamma_0^2}\frac{\partial^2\theta}{\partial\zeta^2}+\theta\right)=G(\theta)-\int_0^T c(\zeta,T)\frac{\partial\theta}{\partial T}\,dT\ . \tag{66}$$

The function $G(\theta)$, which represents the total number of excess neutrons which have been produced at a time t after startup, is defined by

$$G(\theta)=\int_0^\theta g(\theta)d\theta\ . \tag{67}$$

The boundary conditions, Eq. (12), which applied to the function $u(\zeta, T)$ also apply to the function $\theta(\zeta, T)$; thus, the right side of Eq. (66) must be orthgonal to the function $u_0(\zeta)$ in Eq. (16); that is

$$\left\langle G(\theta),\ u_0\right\rangle=\int_0^T\left\langle c\ \frac{\partial\theta}{\partial T},\ u_0\right\rangle dT\ . \tag{68}$$

Since the function $u_0(\zeta)$ is also the "importance"W3 of a neutron in the initial (homogeneous) reactor, the above equation states that the total importance of the excess neutrons produced during the irradiation is equal to the total importance of the neutrons absorbed by the control absorber. This is not in contradiction with the existence of a neutron balance which should take into account that fact that the leakage from the reactor did vary during the irradiation and is not exactly balanced by the term DB_0^2 in the definition of the excess neutron production.

Equation (68) shows also that the end of life ($c = 0$) is reached when the total importance of the excess neutron produced during the irradiation attains its maximum value.

The perturbation method could be applied to Eq. (66) instead of Eq. (64) and would clearly yield the same results. The first approximation of the flux time is $\theta_0 = T\, u_0(\zeta)$, solution of the equation $L\theta_0 = 0$. The first-order perturbation equation is then

$$L\theta_1 = G(u_0 T) - u_0(\zeta) \int_0^T c_0(\zeta, T)dT, \tag{69}$$

which has a solution if and only if T and c_0 are related by

$$\left\langle G(u_0 T),\ u_0 \right\rangle = \int_0^T \left\langle c_0\ u_0, u_0 \right\rangle dT. \tag{70}$$

When the control absorber is uniformly distributed, c_0 can be taken out of the bracket, which represents a spatial integral, and Eq. (69) will take the form

$$L\theta_1 = G(u_0 T) - \frac{\left\langle G(u_0 T), u_0 \right\rangle}{\left\langle u_0, u_0 \right\rangle}\ u_0 . \tag{71}$$

The above equation will be helpful in the study of the graded irradiation of the fuel. An approximate solution of this equation is obtained by the calculation procedure developed for the batch irradiation of unmixed fuel in a slab reactor with uniform control absorber [see Eqs. (II-109) through (II-171)] .

2.3. Central Control

Equation (66) can be reduced to a simpler equation, Eq. (50), by letting the control absorber be concentrated at a single axial position, for instance, in the midplane of the reactor. Using the symbol δ to represent the Dirac δ-function, Eq. (66) becomes

$$-\left(\frac{1}{\gamma_0^2} \frac{\partial^2 \theta}{\partial \zeta^2} + \theta\right) = G(\theta) - C(T)\delta(\zeta) , \tag{72}$$

where

$$C(T) = \int_0^T c\,(T)\ \frac{d\theta(0, T)}{dT}\ dT . \tag{73}$$

The above equation reduces to

$$C(T) = \int_0^T c(T)\,dT \tag{74}$$

when one defines the irradiation parameter T as the central flux time (which is a monotonically increasing function of time)

$$T = \theta(0,t) \; . \tag{75}$$

According to Eq. (61), this corresponds to normalizing the flux shape by $u(0,t) = 1$. The function C(T), the total number of neutrons which have been absorbed in the control absorber during the irradiation, is also a monotonically increasing function of t, and could be used as a new variable instead of T. The end of life is determined by the condition

$$c(T) = \frac{dC}{dT} = 0 \; . \tag{76}$$

2.3.1. Exact solution. Equation (72) is the symbolic representation[F1] of

$$-\left(\frac{1}{\gamma_0^2}\,\frac{\partial^2\theta}{\partial\zeta^2} + \theta\right) = G(\theta) \; , \tag{77}$$

with a nonhomogeneous boundary condition at $\zeta = 0$. In the present case, this boundary condition can be obtained simply by integrating Eq. (72) over a small interval $-\epsilon < \zeta < \epsilon$ and letting ϵ tend toward zero; this yields

$$-\frac{1}{\gamma_0^2}\left(\frac{\partial\theta}{\partial\zeta}\bigg|_{+\epsilon} - \frac{\partial\theta}{\partial\zeta}\bigg|_{-\epsilon}\right) = -\,C(T) \; . \tag{78}$$

The function $\theta(\zeta, T)$, the solution of Eq. (72), is continuous at $\zeta = 0$, but its derivative has a jump of magnitude $\gamma_0^2 C(T)$. Thus, if the reactor is symmetrical about $\zeta = 0$, the boundary condition is

$$\frac{\partial\theta}{\partial\zeta} = \frac{1}{2}\gamma_0^2\, C(T), \text{ at } \zeta = 0 \; . \tag{79}$$

The problem can now be stated as follows:

The flux time is the solution $\theta(\zeta,\lambda)$ of Eq. (77) which satisfies the boundary conditions

$$\frac{\partial\theta}{\partial\zeta} = \lambda \text{ at } \zeta = 0, \text{ and } \theta + \epsilon_Z \frac{\partial\theta}{\partial\zeta} = 0 \text{ at } \zeta = 1. \tag{80}$$

An example of such a solution $\theta(\zeta,\lambda)$ is given in Sec. III.4.5. The central flux time T and the parameter λ are related by

$$T = \theta(0,\lambda) . \tag{81}$$

The flux shape, normalized to 1 at the center, is

$$u(\zeta, T) = \frac{\partial\theta(\zeta,\lambda)}{\partial\lambda} \Big/ \frac{d\theta(0,\lambda)}{d\lambda}. \tag{82}$$

Equations (79) and (80) give the total number of neutrons, which have been absorbed by the control absorber throughout the irradiation, as

$$C(T) = 2\lambda/\gamma_0^2 , \tag{83}$$

and the macroscopic cross section of the control absorber required to keep the reactor critical is

$$c(T) = \frac{2}{\gamma_0^2}\frac{d\lambda}{dT} = \left[\frac{1}{2}\gamma_0^2 \frac{d\theta(0,\lambda)}{d\lambda}\right]^{-1} , \tag{84}$$

and at end of life, $c(T) = 0$ determines λ .

In general, however, the analytical representation of the function $\theta(\zeta,\lambda)$ cannot be obtained exactly. An approximate solution of Eq. (72) can then be obtained by the calculation procedure developed below.

2.3.2. <u>Perturbation method</u>. Rewrite Eq. (72) as

$$L\theta = \epsilon[G(\theta) - C(T)\,\delta(x)] , \tag{85}$$

and seek the solution $\theta(\zeta, T)$ by the following power series:

$$\theta(\zeta, T) = \theta_0(\zeta, T) + \epsilon\,\theta_1(\zeta, T) + \epsilon^2\theta_2(\zeta, T) + \cdots , \tag{86}$$

$$C(T) = C_0(T) + \epsilon C_1(T) + \epsilon^2 C_2(T) + \cdots , \qquad (87)$$

and

$$G(\theta) = G(\theta_0) + \epsilon G'(\theta_0)\theta_1 + \cdots . \qquad (88)$$

Equating the coefficients of each power of ϵ in both sides of Eq. (85) yields the successive equations,

$$L\theta_0 = 0 , \qquad (89)$$

$$L\theta_1 = G(\theta_0) - C_0(T)\ \delta(x), \qquad (90)$$

$$L\theta_2 = G'(\theta_0)\ \theta_1 - C_1(T)\ \delta(x), \qquad (91)$$

etc.

The above equations are solved as follows:

(1) The first approximation to the flux time is $\theta_0(\zeta, T) = T\ u_0(\zeta)$.

(2) The value of T is determined by the condition that Eq. (90) have a solution; i. e., that its right side be orthogonal to $u_0(\zeta)$:

$$\left\langle C_0(T)\ \delta(\zeta), u_0 \right\rangle = \frac{1}{2} C_0(T) u_0(0) = \left\langle G(u_0 T), u_0 \right\rangle . \qquad (92)$$

The δ-function, which is generally normalized by $\int_{-1}^{1} \delta(x)dx = 1$, has been normalized, for the symmetrical case here considered, by $\int_0^1 \delta(x)dx = \frac{1}{2}$. According to Eqs. (92), (73), and (67), the amount of control absorber needed to keep the reactor just critical at an irradiation T is given by

$$c_0(T) = \frac{2\left\langle g(u_0 T)u_0, u_0\right\rangle}{u_0(0)u(0,T)} = \frac{2\left\langle u_0, u_0\right\rangle}{u_0(0)u(0,T)}\ s_0(T) . \qquad (93)$$

(3) The general solution of Eq. (90) can then be obtained by Green's function technique, Eq. (23), the eigenfunction expansion, Eq. (30), or by the Fourier series, Eq. (46). For instance, the eigenfunction expansion is

$$\theta_1(\zeta, T) = A_0(T)u_0(\zeta) + \sum_{k=1}^{\infty} \frac{\gamma_k^2}{\gamma_k^2 - \gamma_0^2} \left[\frac{\left\langle G(u_0 T), v_k \right\rangle - \frac{1}{2} C_0(T) v_k(0)}{\left\langle v_k, v_k \right\rangle} \right] v_k(\zeta) \qquad (94)$$

(4) The coefficient $A_0(T)$ in the above expansion is arbitrary. It is related to $C_1(T)$ by the condition that Eq. (91) has a solution, that is,

$$\left\langle C_1(T)\ \delta(\zeta), u_0 \right\rangle = \frac{1}{2} C_1(T)\ u_0(0) = \left\langle G'(\theta_0)\theta_1, u_0 \right\rangle. \tag{95}$$

According to Eqs. (95), (73), and (67), the corresponding correction to the control absorber is

$$c_1(T) = \frac{2\left\langle u_0, u_0\right\rangle}{u_0(0)u(0,T)}\ \frac{d}{dT}\left\{\frac{\left\langle g(u_0T)\,\theta_1, u_0\right\rangle}{\left\langle u_0, u_0\right\rangle}\right\} .$$

Let $\theta_1(\zeta, T) = A_0(T)\ u_0(\zeta) + \theta_1^*(\zeta, T)$; if $s_0(T)$ is defined by Eq. (93) and $a_0(T) = dA_0(T)/dT$, one obtains

$$c_1(T) = \frac{2\left\langle u_0, u_0\right\rangle}{u_0(0)u(0,T)}\left\{\frac{d}{dT}\left[\frac{\left\langle g(u_0T)\,\theta_1^*, u_0\right\rangle}{\left\langle u_0, u_0\right\rangle}\right] + a_0(T)s_0(T) + A_0(T)\frac{ds_0}{dT}\right\} \tag{96}$$

(5) The general solution of Eq. (91) can then be obtained as in (3) above, and one could repeat the foregoing procedure in order to obtain higher terms of the perturbation series in Eq. (86).

A calculation procedure using eigenfunction expansions in (3) and (4) above is as follows:

Let h_k, $s_k(T)$, and $S_k(T)$ be defined by Eqs. (II-109) through (II-111), and note the definition of $G(\theta)$, Eq. (67). By letting $u_0(\zeta) = v_0(\zeta)$, one obtains

$$\theta_1(\zeta, T) = T\sum_{k=0}^{\infty} \alpha_k(T)\ v_k(\zeta), \text{ and } v_k(\zeta) = \cos\ \gamma_k\zeta, \tag{97}$$

where, according to Eqs. (92) and (94),

$$\alpha_k(T) = \frac{\gamma_0^2}{\gamma_k^2 - \gamma_0^2}\left[\, S_k(T) - h_k S_0(T)\,\right], \text{ where } k \neq 0, \tag{98}$$

and $\alpha_0(T) = A_0(T)/T$ is related to $c_1(T)$ by Eq. (96). At end of life, one has $s_0(T) = 0$ from Eq. (93), and one shall set $c_1(T) = 0$. Equation (96) then yields

$$a_0(T) = - \frac{1}{T \frac{ds_0}{dT}} \frac{d}{dT} \left\{ \sum_{k=1}^{\infty} T\, h_k\, a_k(T)\, s_k(T) \right\} \tag{99}$$

Note that if one drops the term $h_k\ S_0(T)$ on the right side of Eq. (98), Eqs. (97) through (99) give the function $\theta_1(\zeta, T)$ corresponding to a uniformly distributed control absorber.

The flux shape is now the derivative of the flux-time, Eq. (61). The burnup of the fuel is given by the same formulae as in Sec. II.3. The auxiliary functions defined and tabulated there make the problem amenable to hand computation when the properties of the fuel vary as a sum of exponential functions of the local flux time; another case (polynomial representations) is developed in Sec. III.4.

3. Graded Irradiation of the Fuel

3.1. Description and Equations

The fuel rods are replaced individually when burned up to a specific extent. The fuel replacement is so scheduled that each small region of the reactor contains fuel rods in all stages of irradiation between zero and maximum, with average local composition independent of the radial position.

The flux is then separable in spatial coordinates and the axial flux shape is the solution of Eq. (11).

The flux time to which the most exposed fuel element, ready to discharge, has been irradiated is denoted by $\theta(\zeta, T)$. It is proportional to the flux shape; let T be the proportionality factor, then

$$\theta(\zeta, T) = T\, u(\zeta, T) . \tag{101}$$

For instance, T would be the maximum irradiation at the middle of the fuel element ready to discharge if the axial flux shape, $u(\zeta, T)$, were normalized to unity at the midplane of the reactor; i.e., if

$$u(0, T) = 1 . \tag{102}$$

The average local concentration $\overline{N}$ of a nuclide, the concentration of which varies with the flux time θ, as a function $N(\theta)$ is,

$$\overline{N} = \frac{1}{\theta}\int_0^{\theta} N(\theta)d\theta \ . \tag{103}$$

In Eq. (11), the average excess neutron production ∇_u is then

$$\nabla_u = \frac{1}{\theta}\int_0^{\theta} \nabla(\theta)\, d\theta, \tag{104}$$

and, since the reactor is just critical without control absorber,

$$\Sigma_E \equiv 0 \ . \tag{105}$$

The equation for the flux shape, Eq. (11), becomes

$$-\left(\frac{1}{\gamma_0^2}\ \frac{\partial^2 u}{\partial \zeta^2} + u\right) = \frac{1}{\theta} G(\theta)u \tag{106}$$

where $G(\theta)$ is defined by Eq. (67).

Multiplying both sides of the above equation by T and using Eq. (101), one obtains

$$-\left(\frac{1}{\gamma_0^2}\ \frac{d^2\theta}{d\zeta^2} + \theta\right) = G(\theta) \ , \tag{107}$$

$$\text{with}\quad \frac{d\theta}{d\zeta} = 0 \ \text{ at } \ \zeta = 0, \ \text{ and } \ \theta + \epsilon_Z \frac{d\theta}{d\zeta} = 0 \ \text{ at } \ \zeta = 1 \ . \tag{108}$$

This is a second-order nonlinear differential equation, θ is a function $\theta(\zeta)$ of the only variable ζ, and the irradiation parameter T is only an auxiliary variable describing the relation between the flux shape and the flux time. For instance, when the flux shape is normalized by Eq. (102), one has

$$T = \theta(0) \ . \tag{109}$$

Taking the scalar product of both sides of Eq. (107) with $u_0(\zeta)$ yields

$$\langle G(\theta), u_0 \rangle = 0 \ .$$

Thus, a fuel rod is discharged when the total "importance"[W3] of all excess neutrons it has produced during its irradiation history has decreased to zero. This illustrates the advantage of "graded irradiation" over "radial mixing" where, according to Eq. (68), the total importance of the excess neutrons produced by the end of life is its maximum value.

3.2. Perturbation Method

The perturbation method can be applied to Eq. (107) as has been done for Eq. (64), if we assume that the function $G(\theta)$ is a small perturbation term compared to θ.

Defining L by Eq. (14) together with the boundary condition, Eq. (13), we write

$$L\theta = \epsilon\, G(\theta) \tag{110}$$

and seek the expansion of the solution θ in power series of the perturbation parameter ϵ; we set

$$\theta = \theta_0 + \epsilon\,\theta_1 + \epsilon^2\theta_2 + \cdots, \tag{111}$$

and a Taylor expansion of the function $G(\theta)$ yields

$$G(\theta) = G(\theta_0) + (\epsilon\theta_1 + \epsilon^2\theta_2 + \cdots)G'(\theta_0) + \frac{1}{2}(\epsilon\theta_1 + \epsilon^2\theta_2 + \cdots)^2 G''(\theta_0) + \cdots . \tag{112}$$

Equating the coefficients of each power of ϵ in both sides of Eq. (110) yields the following set of equations:

$$L\theta_0 = 0\,, \tag{113}$$

$$L\theta_1 = G(\theta_0)\,, \tag{114}$$

$$L\theta_2 = G'(\theta_0)\theta_1, \tag{115}$$

$$L\theta_3 = G'(\theta_0)\theta_2 + \frac{1}{2}\theta_1^2 G''(\theta_0)\,, \tag{116}$$

etc.

Each of the above equations has a solution if and only if its right side is orthogonal to the nontrivial solution of Eq. (113); this yields the following set of conditions:

$$\left\langle G(\theta_0), \theta_0 \right\rangle = 0, \tag{117}$$

$$\left\langle G'(\theta_0)\theta_1, \theta_0 \right\rangle = 0, \tag{118}$$

$$\left\langle [G'(\theta_0)\theta_2 + \tfrac{1}{2}\theta_1^2 G''(\theta_0)], \theta_0 \right\rangle = 0 , \tag{119}$$

etc.

The method of solution is as follows:

The first equation, Eq. (113), has a solution θ_0 which is $u_0(\zeta)$, except for a constant factor; let then

$$\theta_0(\zeta) = T\, u_0(\zeta) . \tag{120}$$

The second equation, Eq. (114), has a solution if Eq. (117) is satisfied; this determines the value of T. The solution is then the sum of a particular solution θ_1^* and of an arbitrary multiple of the nontrivial solution of the corresponding homogeneous equation, Eq. (113); i. e.,

$$\theta_1(\zeta) = \theta_1^*(\zeta) + A_0\, u_0(\zeta) . \tag{121}$$

The third equation, Eq. (115) then has a solution if A_0 is so chosen as to satisfy Eq. (118).

This is,

$$\left\langle G'(u_0T)\, [\theta_1^* + A_0 u_0], \theta_0 \right\rangle = 0 , \tag{122}$$

or, by using Eq. (120),

$$A_0 = -\frac{\left\langle \theta_1^* G'(u_0T), u_0 \right\rangle}{\left\langle u_0 G'(u_0T), u_0 \right\rangle} . \tag{123}$$

One could then find a particular solution θ_2^* of Eq. (115), and repeat the preceding procedure.

The foregoing method of solution is just the same as that developed in Sec. II for the study of batch irradiation of unmixed fuel when the successive correction terms to the flux shape are there so normalized that the control absorption is exactly equal to its first approximation, i. e., when all successive corrections to the control absorption are zero. Accordingly, the calculation procedures are very similar, and use the same results which have been obtained in the case of a uniformly distributed absorber.

The right side of Eq. (114) can be written $G(\theta_0) - \frac{\langle G(u_0T), u_0\rangle}{\langle u_0, u_0\rangle} u_0$ because the last term vanishes according to Eqs. (117) and (120); it is identical with the right side of Eq. (71). Thus, both equations have the same particular solution, orthogonal to u_0, which has the following eigenfunction expansion:

$$\theta_1^*(\zeta, T) = T \sum_{k=1}^{\infty} a_k(T) v_k(\zeta) . \tag{124}$$

However, the coefficient of $v_0(\zeta)$ is different. According to Eqs. (67), (117), and (120)

$$\langle u_0, G(u_0T)\rangle = \int_0^T \langle g(u_0T)u_0, u_0\rangle = 0 , \tag{125}$$

and since T must be positive, it is the root of the equation

$$S_0(T) = \frac{1}{T}\int_0^T \frac{\langle g(u_0T)\, u_0, u_0\rangle}{\langle u_0, u_0\rangle} dT = 0. \tag{126}$$

By replacing $G'(\theta)$ by $g(\theta)$, Eqs. (123) and (124) yield

$$a_0(T) = - A_0(T)/T = - a_k(T) \frac{\langle v_k g(u_0T), u_0\rangle}{\langle u_0 g(u_0T), u_0\rangle}. \tag{127}$$

The flux time to which the fuel of an element ready to discharge has been irradiated and the corresponding average burnup are then calculated as follows:

The functions $s_k(T)$, $S_k(T)$, $\langle \beta\rangle_0$, and $T\langle v_k, \beta'(v_0T)\rangle$ are defined by Eqs. (II-110) through (II-113). They can be computed by Eqs. (II-132), (II-133), (II-135) and (II-141), with the help of the Tables of auxiliary functions and coefficients corresponding to the slab reactor, Tables II-1 through II-3.

Then, the first approximation to the flux time is

$$\theta_0(\zeta, T) = T \cos \gamma_0 \zeta \tag{128}$$

where T is the root of the equation

$$S_0(T) = 0. \tag{129}$$

The corrective term is

$$\theta_1(\zeta, T) = T \sum_{k=0}^{\infty} a_k(T) \cos \gamma_k \zeta , \tag{130}$$

with

$$a_0(T) = - \sum_{k=1}^{\infty} a_k(T) \frac{s_k(T)}{s_0(T)} , \tag{131}$$

and

$$a_k(T) = \frac{\gamma_0^2}{\gamma_k^2 - \gamma_0^2} S_k(T), \text{ where } k \geqslant 1. \tag{132}$$

The corresponding flux shape is easily obtained from Eq. (101). The average burnup of the fuel discharge fuel is

$$\langle \beta \rangle \approx \langle \beta \rangle_0 + \sum_{k=0}^{\infty} a_k(T) \left\{ T \left\langle v_k, \beta'(v_0 T) \right\rangle \right\} . \tag{133}$$

4. Polynomial Representations

In the following we are concerned with the study of flux shape, flux time, and burnup; we assume that the variations of the properties of the fuel with the local flux time θ are functions of θ which have been fitted to polynomials; i.e.,

$$g(\theta) = \sum_{n=0}^{p} g_n \theta^n , \tag{150}$$

and

$$\beta(\theta) = \sum_{n=0}^{q} b_n \theta^n . \tag{151}$$

The coefficients g_n thus defined in the present section should not be confused with those already defined for the exponential representation of the functions $g(\theta)$ and $\beta(\theta)$; for instance, g_0 is now equal to $g(0)$ and $b_0 = \beta(0) = 0$.

The perturbation method, which has been developed in the preceding sections, now yields equations the solutions of which can be obtained in closed form by using Green's function. The following gives the solution of the first-order perturbation equation, Eq. (71), (90), or (114), as obtained by eigenfunction expansions, Fourier series, or Green's function technique. The case where $g(\theta)$ is a quadratic function of θ will be given special consideration, since the approximate solution obtained by the perturbation method can be compared with the exact solution given by elliptic functions.

4.1. Eigenfunction Expansion

The general method of calculation which has been developed by solving the first-order perturbation equation can always be applied. It requires the computation of the following functions:

$$s_k(T) = \frac{\langle g(v_0 T) v_0, v_k \rangle}{\langle v_k, v_k \rangle} = \sum_{n=0}^{p} g_n \, T^n \frac{\langle v_0^{\,n+1}, v_k \rangle}{\langle v_k, v_k \rangle}, \tag{152}$$

$$S_k(T) = \frac{1}{T} \int_0^T s_k(T) dT = \sum_{n=0}^{p} g_n \, \frac{T^n}{n+1} \frac{\langle v_0^{\,n+1}, v_k \rangle}{\langle v_k, v_k \rangle}, \tag{153}$$

$$\langle \beta \rangle_0 = \langle \beta(v_0 T) \rangle = \sum_{n=0}^{q} b_n T^n \langle v_0^{\,n}, 1 \rangle, \tag{154}$$

and

$$T \langle v_k, \beta'(v_0 T) \rangle = \sum_{n=0}^{q} b_n n T^n \langle v_0^{\,n-1}, v_k \rangle. \tag{155}$$

All the coefficients defined by the bracketed terms in the above equations are integrals which could be evaluated analytically; they are denoted as

$$C_{k,n} = \langle v_k, v_0^{\,n} \rangle = \int_0^1 \cos \gamma_k x \cos^n \gamma_0 x \, dx \,. \tag{156}$$

For a bare reactor with negligible extrapolated distance, the coefficients $C_{0,n}$ are given by a single formula and the coefficients $C_{k,n}$ are calculated by a recurrence formula [Eqs.(II-244) and (II-245)].

For n odd, only the first terms up to $k = \frac{n-1}{2}$ are nonzero.

For n even, one obtains formulae of the following type:

$$\left\langle v_0^2, v_k \right\rangle = (-1)^{k+1} \frac{2}{\pi} \frac{2!}{(2k-1)(2k+1)(2k+3)}, \tag{157}$$

and

$$\left\langle v_0^4, v_k \right\rangle = (-1)^{k+1} \frac{2}{\pi} \frac{4!}{(2k-3)(2k-1)(2k+1)(2k+3)(2k+5)} \tag{158}$$

The eigenfunction expansion of the solution of the first-order perturbation equation could then be obtained easily by the formulae previously derived, for instance, Eqs. (97) through (99), or Eqs. (130) through (133).

The above formulae, Eqs. (157) and (158), show the rapid decrease of the coefficients of the successive eigenfunctions when k increases. Table III-1 gives the values of the first few coefficients $C_{k,n}$ corresponding to a bare reactor.

Table III-1. Table of coefficients $2\,C_{k,n} = \dfrac{\left\langle v_k, v_0^{\,n} \right\rangle}{\left\langle v_k, v_k \right\rangle}^{a}$

k \ n	1	2	3	4	5
0	1	$\frac{4}{\pi} \cdot \frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{\pi} \cdot \frac{8}{3 \cdot 5}$	$\frac{10}{16}$
1	0	$\frac{4}{\pi} \cdot \frac{2}{3 \cdot 5}$	$\frac{1}{4}$	$\frac{4}{\pi} \cdot \frac{8}{5 \cdot 7}$	$\frac{5}{16}$
2	0	$\frac{4}{\pi} \cdot \frac{-2}{3 \cdot 5 \cdot 7}$	0	$\frac{4}{\pi} \cdot \frac{8}{5 \cdot 7 \cdot 9}$	$\frac{1}{16}$
3	0	$\frac{4}{\pi} \cdot \frac{2}{5 \cdot 7 \cdot 9}$	0	$\frac{4}{\pi} \cdot \frac{-8}{5 \cdot 7 \cdot 9 \cdot 11}$	0
4	0	$\frac{4}{\pi} \cdot \frac{-2}{7 \cdot 9 \cdot 11}$	0	$\frac{4}{\pi} \cdot \frac{2 \cdot 3 \cdot 4}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13}$	0

[a] $v_k = \cos\,(2k+1)\,\frac{\pi}{2}\,\zeta$

4.2. Fourier Expansion

A particular solution of the first-order perturbation equations, Eq. (69), (90), or (114), can be obtained by applying Eq. (46). Changing the variable ζ to $x = \gamma_0 \zeta$, the limits of integration in Eq. (46) become $\gamma_0 \zeta_1' = \pi/2$ and $\gamma_0 = (1-\epsilon)\pi/2$ according to Eqs. (35) and (37). Thus, if $F(x)$ is then the right side of either Eq. (69), (90), or (114), a particular solution is given by

$$\theta_1(x, T) - A_0'(T) \cos x = \sum_{k=1}^{\infty} \frac{\cos (2k+1)x}{(2k+1)^2 - 1} \frac{\int_0^{(1-\epsilon)\pi/2} F(x) \cos (2k+1)x \, dx}{\int_0^{\frac{\pi}{2}} \cos^2 (2k+1)x \, dx} .$$

The integral in the denominators in the above equation is equal to $\pi/4$. The integrals in the numerators can all be evaluated in terms of the following integral:

$$C_{k,n}' = \frac{2}{\pi} \int_0^{(1-\epsilon)\pi/2} \cos (2k+1)x \cos^n x \, dx .$$

One would then determine the solution of the first-order perturbation equation [i.e., $A_0'(T)$] in order to obtain a solution for the second-order perturbation equation. The integrations which arise then can also be performed with the help of the integrals $C_{k,n}'$.

For a bare reactor, $C_{k,n}'$ is identical with $C_{k,n}$ in Eq. (156), because $\epsilon = 0$ and $\gamma_k = (2k+1)\pi/2$. In general, the above defined coefficients $C_{k,n}'$ could easily be obtained as a power series of ϵ. Reactors, whose reflector saving is small compared with their core dimensions, could then be easily compared with bare reactors by considering the coefficients of the first few powers of ϵ.

4.3. Green's Function

4.3.1. Radial mixing with uniform control. The radial mixing of the fuel with uniform control described by Eq. (71), as well as the graded irradiation described by Eqs. (114) and (117), yield the same equation for the first-order perturbation of the flux time,

$$L\theta_1 = G(\theta_0) - \frac{\langle G(\theta_0), u_0 \rangle}{\langle u_0, u_0 \rangle} u_0$$

$$= \sum_{n=0}^{p} g_n \frac{T^{n+1}}{n+1} (\cos^n \gamma_0 \zeta - K_n) \cos \gamma_0 \zeta , \tag{159}$$

where each coefficient K_n is defined by the condition that the right side of the above equation be orthogonal to $\cos \gamma_0 \zeta$; i.e.,

$$K_n = \frac{\int_0^1 \cos^{n+2} \gamma_0 \zeta \, d\zeta}{\int_0^1 \cos^2 \gamma_0 \zeta \, d\zeta} = \frac{C_{0,n+1}}{C_{0,1}} . \tag{160}$$

Let $H_n(\zeta)$ be a particular solution of the equation,

$$L H_n(\zeta) = (\cos^n \gamma_0 \zeta - K_n) \cos \gamma_0 \zeta . \tag{161}$$

$H_n(\zeta)$ is now given by Eq. (23) as follows:

$$H_n(\zeta) = - \gamma_0 \int_0^{\zeta} \sin \gamma_0 (\zeta - \zeta') \left[\cos^n \gamma_0 \zeta' - K_n \right] \cos \gamma_0 \zeta' \, d\zeta' . \tag{162}$$

With the change of variable,

$$x = \gamma_0 \zeta , \tag{163}$$

one obtains

$$H_n(x) = \cos x \int_0^x \cos^{n+1} x \sin x \, dx - \sin x \int_0^x \cos^{n+2} x \, dx$$

$$- K_n \left[\cos x \int_0^x \cos x \sin x \, dx - \sin x \int_0^x \cos^2 x \, dx \right] ,$$

or

$$H_n(x) = \cos x \; \frac{1-\cos^{n+2}x}{n+2} - \sin x \; Q_{n+2}(x) + \frac{1}{2} K_n \; x \sin x \; , \tag{164}$$

where the integrals

$$Q_{n+2}(x) = \int_0^x \cos^{n+2}x \; dx \tag{165}$$

are given by the following formulae:[G3]

$$n=2p-2 , \quad Q_{2p}(x) = \frac{1}{2^{2p}} \binom{2p}{p} x + \frac{1}{2^{2p-1}} \sum_{r=0}^{p-1} \binom{2p}{r} \frac{\sin 2(p-r)x}{2(p-r)} \tag{166}$$

$$n=2p-1, \quad Q_{2p+1}(x) = \sum_{r=0}^{p} (-1)^r \binom{p}{r} \frac{\sin^{2r+1}x}{2r+1} \; , \quad \text{where} \; \binom{p}{r} = \frac{p!}{r! \; (p-r)!} \; . \tag{167}$$

Then, a particular solution of Eq. (159) is

$$\theta_1(x, T) - \theta_1(0, T)\cos x = \sum_{n=0}^{p} g_n \frac{T^{n+1}}{n+1} \; H_n(x) \; . \tag{168}$$

One notes that the coefficient K_n, Eq. (160), is defined equivalently as follows:

$$K_n = \frac{Q_{n+2}(\gamma_0)}{Q_2(\gamma_0)} = \frac{Q_{n+2}(\gamma_0)}{\frac{1}{2}\gamma_0 \left(1+\frac{\sin 2\gamma_0}{2\gamma_0}\right)} \; . \tag{169}$$

When we have $\gamma_0 = \frac{\pi}{2}$, the coefficient of the term $x \sin x$ in $H_n(x)$ of Eq. (164) vanishes, and one verifies that $H_{2p}(x)$ can be obtained as a finite sum of eigenfunctions $\cos(2k+1)x$.

The first functions $H_n(x)$ are:

$$\underline{n = 0}, \; H_0(x) = 0, \; K_0 = 1 \; , \tag{170}$$

$$\begin{cases} \underline{n = 1,} \\ H_1(x) = \frac{-1}{3}\left(\sin^2 x + 1 - \cos x - \frac{3}{2} K_1\, x \sin x\right), \\ K_1 = \dfrac{\sin \gamma_0\left(1 - \dfrac{\sin^2 \gamma_0}{3}\right)}{\frac{1}{2}\,\gamma_0\left(1 + \dfrac{\sin 2\gamma_0}{2\gamma_0}\right)}, \end{cases} \tag{171}$$

$$\begin{cases} \underline{n = 2,} \\ H_2(x) = \frac{1}{32}\left(\cos 3x - \cos x - 12x \sin x + 16K_2\, x \sin x\right), \\ K_2 = \dfrac{\dfrac{3\gamma_0}{8} + \dfrac{\sin 2\gamma_0}{4} + \dfrac{\sin 4\gamma_0}{32}}{\frac{1}{2}\gamma_0\left(1 + \dfrac{\sin 2\gamma_0}{2\gamma_0}\right)} \end{cases} \tag{172}$$

Thus, Eq. (159), corresponding to radial mixing with uniform control and to graded irradiation of the fuel, has a particular solution given by Eqs. (168) and (164).

4.3.2. Radial mixing with central control. In the case of radial mixing of the fuel with central control, the first-order perturbation Eq. (90) becomes

$$\begin{aligned} L\theta_1(\zeta, T) &= G(u_0 T) - C_0(T)\,\delta(\zeta) \\ &= \sum_{n=0}^{p} g_n \frac{T^{n+1}}{n+1}\left[\cos^{n+1}\gamma_0\zeta - K'_n\,\delta(\zeta)\right] \end{aligned} \tag{173}$$

where each coefficient K'_n is defined by the condition that the right side of the above equation is orthogonal to $\cos \gamma_0 \zeta$. By normalizing the δ-function by $\int_0^1 \delta(x)\, dx = \frac{1}{2}$ and by using Eqs. (156) and (165), one obtains

$$K_n' = 2\int_0^1 \cos^{n+2}\gamma_0\zeta\, d\zeta = 2\, C_{0,n+1} = \frac{2}{\gamma_0}\, Q_{n+2}(\gamma_0)\,. \tag{174}$$

Changing the variable ζ to x, Eq. (163), a particular solution of Eq. (173) is given by

$$\theta_1(x,T) - \theta_1(0,T)\cos x = \sum_{n=0}^{p} g_n \frac{T^{n+1}}{n+1} H_n'(x), \tag{175}$$

where $H_n'(x)$ is a particular solution of the equation

$$LH_n'(x) = \cos^{n+1}x - K_n'\delta(x)\,. \tag{176}$$

Just as with Eq. (116), the above equation is solved by Green's function technique; one obtains

$$H_n'(x) = \cos x\,\frac{1-\cos^{n+2}x}{n+2} - \sin x\, Q_{n+2}(x) + \frac{1}{2}\gamma_0 K_n' \sin x, \tag{177}$$

or, according to Eq. (164);

$$H_n'(x) = H_n(x) + \frac{1}{2}\sin x(\gamma_0 K_n' - K_n x). \tag{178}$$

The solution of the first-order perturbation equation, Eq. (173), is then normalized by the condition that the second-order perturbation equation, Eq. (91), has a solution.

4.4. Examples - Perturbation Method

In the following we develop simple examples that illustrate the use of the techniques of solution developed in the preceding sections, and which yield a parametric study of the effects of the burnup of the fuel.

We consider a bare reactor with negligible extrapolation distance, i.e., with

$$\epsilon_Z = 0\,, \quad \gamma_0 = \frac{\pi}{2}\,. \tag{179}$$

Then the eigenfunction expansion and the Fourier expansion become identical. The various integrals which arise in computations using the perturbation method are readily performed with the help of Table III-1; they are:

$$\int_0^1 \cos\,(2k+1)\frac{\pi}{2}\zeta\,\cos^n\frac{\pi}{2}\zeta\,d\zeta = C_{k,n} = \frac{1}{2}\frac{\langle v_k, v_0{}^n\rangle}{\langle v_k, v_k\rangle},$$

$$\int_0^1 \sin\frac{\pi}{2}\zeta\,\cos^n\frac{\pi}{2}\zeta\,d\zeta = \frac{2}{\pi}\,\frac{1}{n+1},$$

$$\int_0^1 \zeta\sin\frac{\pi}{2}\zeta\;\cos^n\frac{\pi}{2}\zeta\,d\zeta = \frac{2}{\pi}\;\frac{C_{0,n}}{n+1},$$

$$\langle v_k, v_k\rangle = \frac{1}{2},\ h_k = 1,\ \text{and}\ \frac{B_0^2}{B_k^2-B_0^2} = \frac{\gamma_0^2}{\gamma_k^2-\gamma_0^2} = \frac{1}{(2k+1)^2-1}.$$

In the following we assume that the neutron excess production, $g(\theta)$, is represented simply by

$$g(\theta) = g_0 + g_2\theta^2\,. \tag{180}$$

The results are to be compared with those obtained if the fuel were irradiated batchwise, but thoroughly mixed throughout the irradiation. The flux time θ_f to which the fuel has been irradiated uniformly at end of life would then be the root of the equation $g(\theta_f) = 0$; i.e.,

$$\theta_f = \sqrt{-g_0/g_2}\,. \tag{181}$$

Rewriting Eq. (181) as

$$g(\theta) = g_0[\,1 - (\theta/\theta_f)^2]\,, \tag{182}$$

a parametric study is performed by varying the two parameters g_0 and θ_f. The parameter g_0 is the ratio of the initial excess neutron production to the leakage, and is related to the initial reactivity of the reactor.

The flux time to which the fuel has been irradiated is representative of its burnup. Average and maximum flux time will be derived; they correspond exactly to the average and maximum burnup when the function $\beta(\theta)$ is linear.

The chosen form of the function $g(\theta)$ will also allow a simple comparison of the results of the second-order perturbation method with those obtained by exactly solving the nonlinear equation.

The solution of the first order-perturbation equation will be obtained either by eigenfunction expansion or by Green's function technique using the following results:

(a) Green's function. Equations (160) and (174) and Table III-1 yield

$$K = K_0' = 1 , \quad K_2 = K_2' = 2\, C_{0,3} = \frac{3}{4} ;$$

Then, Eq. (172) gives $H_0(\zeta) = 0$, $H_2(\zeta) = \frac{1}{32}(\cos 3\frac{\pi}{2}\zeta - \cos\frac{\pi}{2}\zeta)$,

and Eq. (178) gives $H_0'(\zeta) = \frac{\pi}{4}(1-\zeta)\sin\frac{\pi}{2}\zeta$,

and $H_2'(\zeta) = \frac{3\pi}{16}(1-\zeta)\sin\frac{\pi}{2}\zeta + \frac{1}{32}(\cos 3\frac{\pi}{2}\zeta - \cos\frac{\pi}{2}\zeta)$.

(b) Eigenfunctions. Equations (152), (153) and Table III-1 yield:

$$s_0(T) = g_0 + \frac{3}{4} g_2 T^2 , \quad S_0(T) = g_0 + \frac{1}{4} g_2 T^2 ,$$

$$s_1(T) = \frac{1}{4} g_2 T^2 , \qquad S_1(T) = \frac{1}{4} g_2 T^2/3 ,$$

$$s_k(T) = 0 , \ k \geqslant 2, \text{ and } S_k(T) = 0, \ k \geqslant 2 .$$

4.4.1. Example 1: Radial mixing with uniform control. In this case, the solution of the first-order perturbation equation is given in closed form by its eigenfunction expansion which contains only the two first eigenfunctions. One applies the formulae which have been derived in Sec. II:

Eq. (II-121) gives $\alpha_1(T) = \frac{1}{8} S_1(T) = \frac{1}{32} g_2 \frac{T^3}{3}$, and $\alpha_k(T) = 0$ for $k \geqslant 2$;

Eq. (II-117) gives $c_1^*(T) = \frac{d}{dT}[\, T\, \alpha_1(T) s_1(T)] = \frac{d}{dT}[\, \frac{1}{32} g_2 \frac{T^3}{3} \frac{1}{4} g_2 T^2]$

$$= \frac{5}{12\times 32}\, g_2^2\, T^4$$

Eq. (II-120) gives $\alpha_0(T) = -\, c_1^*(T)/T\left(\frac{ds_0}{dT}\right) = -\, \frac{5}{18\times 32}\, g_2\, T^2$.

Then, for a value T of the irradiation, to which corresponds an amount of control absorber $c_0(T) = s_0(T)$, the flux time as given by Eq. (II-119) is

$$\theta(\zeta, T) \approx T\,[\,(1 - \frac{5}{18\times 32} g_2 T^2) \cos\frac{\pi}{2}\zeta + \frac{1}{3\times 32} g_2 T^2 \cos 3\frac{\pi}{2}\zeta]\ ,$$

and the flux shape is its derivative with respect to T; i.e.,

$$u(\zeta, T) \approx (1 - \frac{5}{6\times 32}\, g_2 T^2) \cos\frac{\pi}{2}\zeta + \frac{1}{32}\, g_2\, T^2 \cos 3\frac{\pi}{2}\zeta\ .$$

At end of life, $s_0(T) = 0$ yields

$$T^2 = \frac{4}{3}\left(\frac{g_0}{-g_2}\right), \text{ and } T = \frac{2}{\sqrt{3}}\, \theta_f\ . \tag{183}$$

The final flux time, and its central and average values, are then

$$\theta(\zeta, T) = T\,[\,(1 + \frac{5}{6\times 72} g_0) \cos\frac{\pi}{2}\zeta - \frac{1}{72} g_0 \cos 3\frac{\pi}{2}\zeta\ ]\ , \tag{184}$$

$$\theta(0, T) = T(1 - \frac{1}{6\times 72} g_0) \text{ and } \langle\theta\rangle = \frac{2}{\pi}\, T\ (1 + \frac{7}{6\times 72} g_0)\ ,$$

and the final flux shape, and its central and average values, are

$$u(\zeta, T) = (1 + \frac{5}{6\times 24}\, g_0) \cos\frac{\pi}{2}\zeta - \frac{1}{24}\, g_0 \cos 3\frac{\pi}{2}\zeta\ ,$$

$$u(0, T) = 1 - \frac{1}{6\times 24}\, g_0 \text{ and } \langle u\rangle = \frac{2}{\pi}\,(1 + \frac{7}{6\times 24}\, g_0)\ . \tag{185}$$

4.4.2. Example 2: Radial mixing with central control. The eigenfunction of the solution of the first-order perturbation equation is first obtained as follows:

Equation (98) gives the solution orthogonal to $u_0 = \cos \frac{\pi}{2} \zeta$ as

$$\theta_1^*(\zeta, T) = T \left\{ \frac{1}{8} [S_1(T) - S_0(T)] \cos \frac{3\pi}{2} \zeta - S_0(T) \sum_{k=2}^{\infty} \frac{\cos(2k+1) \frac{\pi}{2} \zeta}{(2k+1)^2 - 1} \right\}. \tag{186}$$

The corresponding term for the flux shape is obtained by derivation with respect to T

$$u_1^*(\zeta, T) = \frac{1}{8} [s_1(T) - s_0(T)] \cos \frac{3\pi}{2} \zeta - s_0(T) \sum_{k=2}^{\infty} \frac{\cos(2k+1) \frac{\pi}{2} \zeta}{(2k+1)^2 - 1}$$

When the approximations of the flux time and the flux shape are

$$\theta(\zeta, T) \approx T \cos \frac{\pi}{2} \zeta + \theta_1^*(\zeta, T) \quad \text{and} \quad u(\zeta, T) \approx \cos \frac{\pi}{2} \zeta + u_1^*(\zeta, T) \ ,$$

the control absorption corresponding to the value T of the irradiation is

$$[c_0(T) + c_1^*(T)] \ u(0, T) \ .$$

Equations (93) and (96) yield:

$$c_0(T) \ u(0, T) = s_0(T)$$

and

$$c_1^*(T) u(0, T) = 2 \frac{d}{dT} \left\{ \int_0^1 (g_0 + g_2 \ T^2 \cos \frac{\pi}{2} \zeta) \, \theta_1^*(\zeta, T) \cos \frac{\pi}{2} \zeta \, d\zeta \right\}$$

$$= 2 \frac{d}{dT} \left\{ g_2 \ T^3 \ \frac{1}{8} [S_1(T) - s_0(T)] \ C_{1,3} \right\}$$

because the coefficients $C_{k,3}$ all vanish for $k \geq 2$. Finally,

$$c_1^*(T) u(0, T) = \frac{-1}{32} g_2 \ T^2 (3 g_0 + \frac{5}{6} g_2 \ T^2) \ .$$

At end of life, however, one can normalize the flux time in order for the irradiation parameter to be the solution of the simpler equation, Eq. (183). The normalization coefficient, $A_0(T)$ in Eq. (94), is then given by Eq. (96) where $s_0(T) = 0$, i.e.,

$$A_0(T) = -\frac{c_1^*(T)u(0,T)}{ds_0/dT} = \frac{T}{48}\left(3g_0 + \frac{5}{6}g_2 T^2\right) = \frac{T}{48} \times \frac{17}{18} g_0 .$$

Since $T^2 = -\frac{4}{3}\frac{g_0}{g_1}$, one also has

$$S_0(T) = \frac{2}{3} g_0 \quad \text{and} \quad S_1(T) = -\frac{1}{9} g_0, \tag{187}$$

and Eqs. (94) and (186) yield the flux time at end of life as

$$\theta(\zeta, T) \approx T \left\{ \left(1 + \frac{1}{24} \times \frac{17}{18} g_0\right) \cos\frac{\pi}{2}\zeta - \frac{7}{8 \times 9} g_0 \cos 3\frac{\pi}{2}\zeta - \frac{2}{3} g_0 \sum_{k=2}^{\infty} \frac{\cos(2k+1)\frac{\pi}{2}\zeta}{(2k+1)^2 - 1} \right\}. \tag{188}$$

The above results will now be obtained in closed form by using Green's function technique. Equation (175) yields

$$\theta_1(\zeta, T) - \theta_1(0, T)\cos\frac{\pi}{2}\zeta = g_0 \, T \, H_0'(\zeta) + g_2 \frac{T^3}{3} H_2'(\zeta)$$

$$= \frac{\pi}{4} T \, S_0(T)\,(1-\zeta)\sin\frac{\pi}{2}\zeta + \frac{1}{32} g_2 \frac{T^3}{3}\left(\cos 3\frac{\pi}{2}\zeta - \cos\frac{\pi}{2}\zeta\right), \tag{189}$$

and Eq. (95) gives the corresponding correction to the total control absorption

$$C_1(T) = 2\int_0^1 \left[g_0 + g_2 T^2 \cos^2\frac{\pi}{2}\zeta \right] \theta_1(\zeta, T)\cos\frac{\pi}{2}\zeta \, d\zeta$$

$$= \theta_1(0, T) s_0(T) + T \, S_0(T)\left(g_0 \frac{1 - C_{0,1}}{2} + g_2 T^2 \frac{1 - C_{0,3}}{4} \right)$$

$$+ \frac{1}{16} g_2 \frac{T^3}{3}\left(-C_{0,1}\, g_0 + g_2 T^2 (C_{1,3} - C_{0,3}) \right);$$

finally,

$$C_1(T) = \theta_1(0, T)s_0(T) + T\ S_0(T)\left[\ \frac{1}{4}g_0 + \frac{5}{32}\, g_2T^2\right] - \frac{1}{32}g_2\,\frac{T^3}{3}\left(g_0 + \frac{1}{2}\, g_2T^2\right)$$

and the correction to the control absorption is the derivative of $C_1(T)$.

At end of life, $s_0(T)$ [the derivative of $T\ S_0(T)$] vanishes. Equation (73) then yields the correction to the control absorption,

$$c_1(T)u(0, T) = \theta_1(0, T)\frac{ds_0(T)}{dT} + \frac{5}{16}g_2\ T^2S_0(T) - \frac{1}{32}\, g_2T^2\left(g_0 + \frac{5}{6}g_2T^2\right) .$$

This corrective term vanishes if $\theta_1(0, T)$ is given by

$$\theta_1(0, T) = \frac{2}{3}\ T\left\{\frac{1}{32}\left(g_0 + \frac{5}{6}\, g_2T^2\right) - \frac{5}{16}S_0(T)\right\} ,$$

and Eqs. (183) and (187) then yield the value of the correction to the final flux time at the center of the reactor as

$$\theta_1(0, T) = -\frac{61}{6\times 72}\, g_0T .$$

Thus, the final flux time is

$$\theta(\zeta, T) \approx T\left[\left(1 - \frac{55}{6\times 72}\, g_0\right)\cos\,\frac{\pi}{2}\,\zeta - \frac{1}{72}\, g_0 \cos\, 3\,\frac{\pi}{2}\,\zeta + g_0\,\frac{\pi}{6}\,(1-\zeta)\sin\frac{\pi}{2}\,\zeta\right] . \qquad (190)$$

The above equation and Eq. (188) represent the same function. The following equations give the central and average flux time at end of life; the exact values from Eq. (190) are compared with the series from Eq. (188):

$$\frac{\pi\langle\theta\rangle}{2T} \approx 1 + \frac{72\pi - 197}{6\times 72}\, g_0 = 1 + 0.0676g_0 , \qquad (191)$$

$$\frac{\pi\langle\theta\rangle}{2T} \approx 1 + g_0\left(\frac{31}{6\times 72} - \frac{3}{2}\sum_{k=2}^{\infty} \frac{(-1)^k}{2k(2k+1)(2k+2)}\right)$$

$$= 1+g_0(0.0718 - 0.0055 + 0.0020 + \cdots),$$

$$\frac{\theta(0, T)}{T} \approx 1 - \frac{61}{6\times 72}\, g_0 = 1 - 0.1412g_0, \text{ and } \frac{\theta(0, T)}{T} \approx 1 - g_0\left(\frac{25}{6\times 72} + \frac{1}{6}\sum_{k=2}^{\infty} \frac{1}{k(k+1)}\right)$$

$$= 1 - g_0(0.058+0.028+0.014+0.008 + \cdots) . \qquad (192)$$

A rather large number of terms shall be kept in the eigenfunction expansion in order to attain high degree of accuracy. This is due to the discontinuity of the derivative of the flux time at $\zeta = 0$.

One notes that the initial (bare) reactor is homogeneous except for the central absorber. This yields

$$u(\zeta, 0) = \cos \frac{\pi}{2} [\sqrt{1 + g_0}\,(\zeta - 1) + 1]/\cos \frac{\pi}{2} (\sqrt{1 + g_0} - 1)$$

and the flux is everywhere positive if $g_0 < 3$. The above perturbation method then gives the initial flux, normalized to 1 at the center, as $\cos \frac{\pi}{2} \zeta + \frac{\pi}{4} g_0 (1 - \zeta) \sin \frac{\pi}{2} \zeta$ and the initial control absorber as $c_0 + c_1 = g_0(1 + g_0/4)$, while the exact value of the initial control is $c = \frac{4}{\pi} \sqrt{1 + g_0}\, \mathrm{tg}\, \frac{\pi}{2} (\sqrt{1 + g_0} - 1) = g_0(1 + g_0/4) + \mathcal{O}(g_0^3)$.

4.4.3. <u>Example 3</u>: <u>Graded irradiation</u>. As for radial mixing with uniform control, the eigenfunction expansion gives the solution in closed form. The computations are straightforward: the flux shape is just proportional to the flux time.

The value of the irradiation parameter at steady state is given by Eq. (129) as

$$S_0(T) = 0\,, \quad T^2 = 4\left(\frac{g_0}{-g_2}\right)\,, \quad \text{and} \quad T = 2\theta_f\,. \tag{193}$$

Equations (131) and (132) yield

$$\alpha_1 = \frac{1}{8} S_1(T) = -g_0/24 \quad \text{and} \quad \alpha_0 = -\alpha_1 \frac{s_1(T)}{s_0(T)} = g_0/48\,,$$

and Eqs. (128) and (130) then yield

$$\theta(\zeta, T) \approx T\left(1 + \frac{1}{48} g_0\right) \cos \frac{\pi}{2} \zeta - \frac{1}{24} g_0 \cos 3 \frac{\pi}{2} \zeta\,.$$

The average and maximum flux time are

$$\theta(0, T) = T\left(1 - \frac{1}{48} g_0\right) \quad \text{and} \quad \langle \theta \rangle = \frac{2}{\pi} T \left(1 + \frac{5}{144} g_0\right)\,. \tag{194}$$

Central and average flux time to which the discharged fuel has been irradiated are plotted in Fig. III-2. They are given in units of the uniform flux time θ_f [Eq. (182)] to which thoroughly mixed fuel could be irradiated in reactors of the same geometrical buckling.

The first-order perturbation[B1] gives the points corresponding to $g_0 = 0$. The second-order perturbation method gives the lines through these points.

4.5. Elliptic Functions Solution Example

Equation (77) or (107) corresponds to Eq. (50) where $I_1 = 0$. Then by setting $g_1 = 0$ according to Eq. (180), Eq. (59) becomes

$$f(\theta) = 2\,\gamma_0^2 \left[I_0 - \frac{1+g_0}{2}\,\theta^2 - \frac{g_2}{12}\,\theta^4 \right].$$

Setting $\tau^2 = \dfrac{1+g_0}{-g_2}$, where $g_0 > 0$ and $g_2 < 0$, we have

$$f(\theta) = \gamma_0^2\,\frac{1+g_0}{6\tau^2}\left(I_0' - 6\tau^2\,\theta^2 + \theta^4 \right).$$

For graded irradiation, as well as for radial mixing with central control, the flux time shall be maximum somewhere in the volume of the reactor. Let T_m be the absolute maximum; then, $f(T_m) = 0$, and

$$f(\theta) = \gamma_0^2\,\frac{1+g_0}{6\tau^2}\left[6\tau^2 (T_m^2 - \theta^2) - (T_m^4 - \theta^4) \right]$$

$$= \gamma_0^2\,\frac{1+g_0}{6\tau^2}\,(T_m^2 - \theta^2)\,(6\,\tau^2 - T_m^2 - \theta^2)$$

$f(\theta)$ shall be positive when θ increases toward T_m; the condition $(1 + g_0) > 0$ yields

$$6\tau^2 \geqslant 2\,T_m^2 \quad \text{or} \quad \frac{T_m^2}{6\tau^2 - T_m^2} \leqslant 1 \,.$$

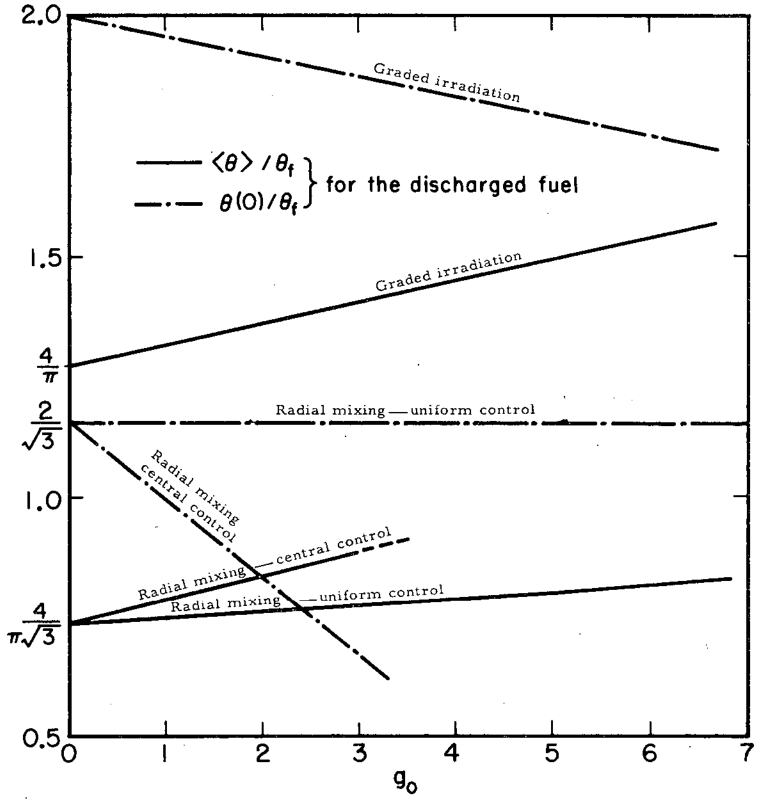

Fig. III-2. Average and central flux time for radial mixing and graded irradiation (perturbation method).

Thus Eq. (51) can then be written as follows:

$$\frac{1}{T_m^2}\left(\frac{d\theta}{dx}\right)^2 = b^2\left[1-\left(\frac{\theta}{T_m}\right)^2\right]\left[1-k^2\left(\frac{\theta}{T_m}\right)^2\right],$$

$$b^2 = \gamma_0^2\,(1+g_0)\,\frac{6\tau^2 - T_m^2}{6\tau^2}, \tag{195}$$

and

$$k^2 = \frac{T_m^2}{6\tau^2 - T_m^2} \leqslant 1 . \tag{196}$$

The general solution of the above equation is the Jacobian elliptic function represented by the symbol sn:

$$\theta(\zeta) = T_m \,\mathrm{sn}\,(a + b\zeta),$$

where T_m and a are determined by the following boundary conditions:

(1) At $\zeta = 0$;

For graded irradiation, $\theta'(0) = 0$ readily gives $a = K$.

For radial mixing with central control, Eq. (79) yields $\theta'(0) = T_m b\,\mathrm{cn}a\,\mathrm{dn}a = \frac{1}{2}\gamma_0^2\,C$. The foregoing equation is a relation between a, T_m, and C, the total number of neutrons which have been absorbed by the control absorber. The end of life is determined by $dC/dT = 0$, where T is any monotonic function of the time (T_m for instance)

(2) At $\zeta = 1$, Eq. (13) yields

$$\mathrm{sn}(a+b) + \epsilon_Z b\,\mathrm{cn}(a+b)\,\mathrm{dn}(a+b) = 0.$$

This is the second relation between a and T_m.

If $\epsilon_Z = 0$, one obtains simply $a + b = 2K$.

For radial mixing with central control, the flux shape is the derivative of $\theta(\zeta)$ with respect to T (or T_m) which involves derivatives of the elliptic sine with respect to the modulus k as well as with respect to the argument.

Example: Graded irradiation of the fuel in a reactor with negligible extrapolation distance. The boundary conditions are now satisfied if $a = b = K$; the flux time is $\theta(\zeta) = T_m \operatorname{sn} K(1 + \zeta)$, and Eqs. (195) and (196) yield

$$k^2 = \frac{T_m^2}{6\tau^2 - T_m^2}, \quad 1 + k^2 = \frac{6\tau^2}{6\tau^2 - T_m^2},$$

and

$$K^2 = \left(\frac{\pi}{2}\right)^2 (1 + g_0) \frac{6\tau^2 - T_m^2}{6\tau^2} = \left(\frac{\pi}{2}\right)^2 (1 + g_0)/(1 + k^2).$$

The quarter period K is the complete elliptic integral of the first kind, a function of the modulus k. Given g_0, one can determine k^2 (or K) by the last of the above equations, which is written as

$$\left[\frac{2}{\pi} K(k^2)\right]^2 (1 + k^2) = 1 + g_0 . \tag{197}$$

The central (and maximum) flux time is then given by

$$T_m^2 = 6\tau^2 \frac{k^2}{1+k^2} = 6 \frac{1 + g_0}{-g_2} \frac{k^2}{1+k^2} = \frac{3}{2} \frac{1 + g_0}{g_0} \left(-4\frac{g_0}{g_2}\right) \frac{k^2}{1+k^2} . \tag{198}$$

The ratio of average to maximum flux time (or flux) is

$$\langle\theta\rangle/T_m = \int_0^1 \operatorname{sn} K(1 + \zeta) d\zeta = \frac{1}{K} \int_0^K \operatorname{sn} x \, dx = \frac{1}{Kk} \log\left(\frac{\operatorname{dn} K - k \operatorname{cn} K}{1 - k}\right),$$

and finally

$$\langle\theta\rangle/T_m = \frac{1}{2Kk} \log\left(\frac{1 + k}{1 - k}\right) = \frac{1}{Kk} \tanh^{-1} k . \tag{199}$$

Figure III-3 represents the variation of the central and average flux time as a function of the parameters g_0 and θ_f in Eq. (182).

We have used tabulated values of the function $K(k^2)$[B3] and computed first the values of g_0 corresponding to a set of values for k^2 in Eq. (197); then, Eqs. (198) and (181) yield

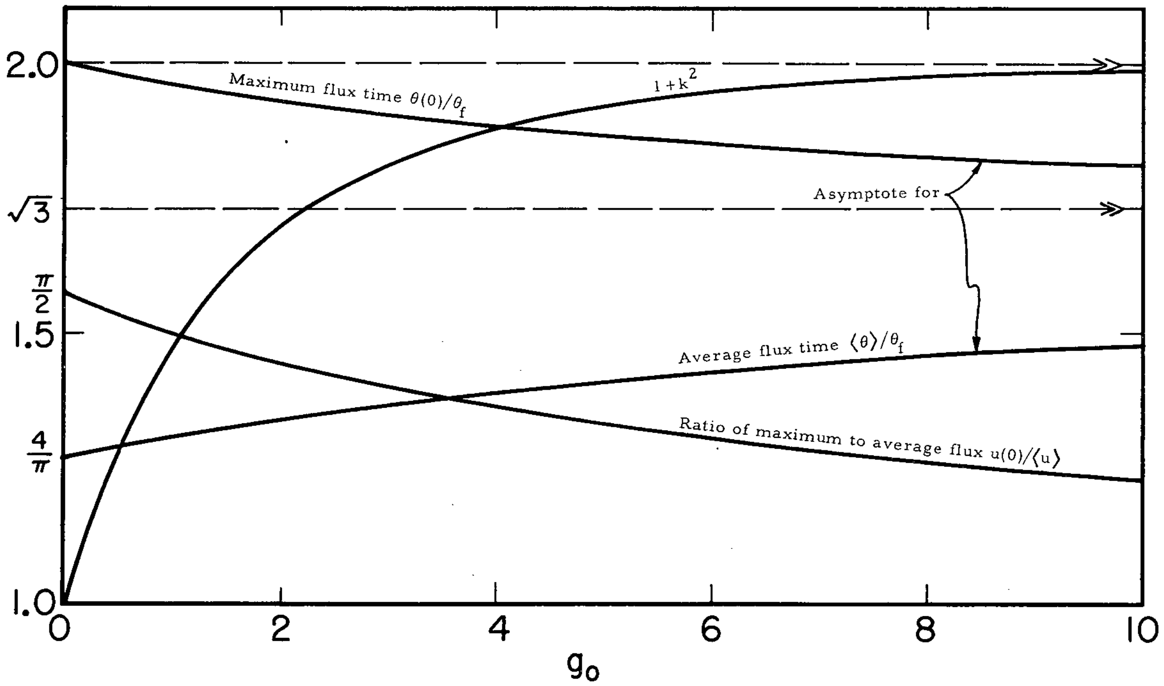

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Fig. III-3. Graded irradiation (elliptic-function solutions). K = modulus of the elliptic functions.

$$\theta(0) = T_m = \theta_f \sqrt{6\,\frac{1+g_0}{g_0} \times \frac{k^2}{1+k^2}}\ .$$

The flux shape $\mathrm{sn}K(1+\zeta)$ is completely defined by the value of k^2; drawings are given by Jahnke and Ende.[J2] The ratio of central (maximum) to average flux, Eq. (199), is representative of the flattening of the flux due to the less reactive fuel at the center.

In the following paragraph, we verify that for $g_0 = 0$ the first-order perturbation is exact and the second-order describes, exactly, small deviations about $g_0 = 0$.

When g_0 tends toward zero, Eq. (197) shows that $k^2 \rightarrow 0$ and $K \rightarrow \pi/2$. Using the series expansion of the function $K(k^2)$, we obtain

$$\frac{2}{\pi} K = 1 + \frac{k^2}{4} + 9\left(\frac{k^2}{8}\right)^2 + \cdots ,$$

and

$$\left(\frac{2}{\pi} K\right)^2 (1+k^2) = 1 + \frac{3}{2} k^2 + 54\left(\frac{k^2}{8}\right)^2 + \cdots ,$$

and Eq. (197) yields $g_0 \approx \frac{3}{2} k^2 (1 + \frac{9}{16} k^2)$.

Then, Eq. (198) gives $T_m \approx 2\theta_f \sqrt{1 - g_0/24} \approx 2\theta_f(1 - g_0/48)$, and Eq. (199) gives $\frac{\pi}{2}\langle\theta\rangle \approx T_m(1 + g_0/18) \approx 2\theta_f(1 + \frac{5}{144} g_0)$.

These are the results already obtained in Eq. (194).

5. Steady Axial Movements

5.1. Description and Equations

The fuel is moved steadily lengthwise through the reactor so that the fuel at position z from the entry plane in each channel has been burned up to the same extent. The composition is uniform radially, and the flux is separable.

The increase in flux time to which the fuel is irradiated when the fuel moves from z to $z + dz$ is the product of the magnitude of the flux at z by the time taken to travel the distance dx, i.e., by the ratio of the distance dz to the velocity. This is independent of the radial coordinate when the fuel moves in each channel with a velocity proportional to the radial flux shape. The flux time θ is then a function of z only, which is related to the axial flux shape u by

$$\theta(\zeta, T) = T\int_{\zeta_0}^{\zeta} u(\zeta, T)d\zeta, \tag{200}$$

where T here denotes a parameter which is a function of the velocity at which the fuel is moved. For instance, if the flux-shape were to be normalized by the condition that its integral from the entry face to the discharge face is equal to unity, T would be the flux time to which the discharge fuel has been irradiated.

Various movements of the fuel will be considered.

(1) Unidirectional: the fuel enters the reactor at $\zeta = -1$ and leaves at $\zeta = 1$.

(2) Bidirectional: part of the fuel enters the reactor at $\zeta = -1$ and leaves at $\zeta = +1$, and the remaining part enters the reactor at $\zeta = 1$ and leaves at $\zeta = -1$.

(3) In-Out: the fuel enters at $\zeta = 0$ and leaves at $\zeta = \pm 1$.

(4) Out-In: the fuel enters at $\zeta = \pm 1$ and leaves at $\zeta = 0$.

In the reactor equation, Eq. (11), one sets $\Sigma_E = 0$ for the steady state, and one sets $\nabla_u = \nabla(\theta)$, where θ is the function defined by Eq. (200) with $\zeta_0 = -1$, $\zeta_0 = 0$, $\zeta_0 = \pm 1$ for unidirectional, in-out and out-in movements respectively.

For bidirectional movement (symmetrical), one has

$$\nabla_u = \frac{1}{2}[\nabla(\theta^+) + \nabla(\theta^-)], \text{ with } \begin{cases} \theta^+ = T\displaystyle\int_{-1}^{\zeta} u(\zeta, T)d\zeta \\ \theta^- = T\displaystyle\int_{\zeta}^{1} u(\zeta, T)d\zeta \end{cases}. \tag{201}$$

The function $u(\zeta, T)$ is symmetrical about $\zeta = 0$; then, by defining

$$T = \int_0^1 u(\zeta, T)dT = \frac{\theta^+ + \theta^-}{2}, \quad \theta = T\int_0^{\zeta} u(\zeta, T)dT = \frac{\theta^+ - \theta^-}{2}, \tag{202}$$

one obtains,

$$\nabla_u = \frac{1}{2}[\nabla(T+\theta)+\nabla(T-\theta)] \ . \tag{203}$$

The reactor equation, Eq. (11), can then be written

$$-\left(\frac{1}{\gamma_0^2}\ \frac{\partial^2 u(\zeta,T)}{\partial\zeta^2} + u(\zeta,T)\right) = g_A(\theta)u(\zeta,T) \ , \tag{204}$$

and each of the four fuel movements considered here is specified by the following definitions:

Movement	(1)Unidirectional	(2)Bidirectional	(3)In-Out	(4)Out-In
$g_A(\theta)$	$g(\theta)$	$\frac{1}{2}[g(T+\theta)+g(T-\theta)]$	$g(\theta)$	$g(\theta)$
Boundary conditions	Eq. (12)	Eq. (13)	Eq. (13)	Eq. (205)
$\theta(\zeta,T)/T$	$\int_{-1}^{\zeta} u(\zeta,T)d\zeta$	$\int_{0}^{\zeta} u(\zeta,T)d\zeta$	$\int_{0}^{\zeta} u(\zeta,T)d\zeta$	$\int_{0}^{\zeta} u(\zeta,T)d\zeta$

The function $g(\theta)$ is defined by Eq. (62) and the last boundary condition is

$$\frac{\partial u}{\partial\zeta} - \epsilon_Z u = 0, \ \text{where } \zeta = 0; \text{ and } \frac{\partial u}{\partial\zeta} = 0 \ , \ \text{where } \zeta = 1. \tag{205}$$

We can then develop the perturbation method, as in the case of batch irradiation of unmixed fuel, writing Eq. (204) as

$$Lu = \epsilon\, g_A(\theta)\, u, \tag{206}$$

and seeking the solution u as a power series in ϵ, where $u = \sum_{n=0}^{\infty} \epsilon^n u_n$.

The functions u_n are then the solution of the following equations:

$$Lu_0 = 0 \ , \tag{207}$$

$$Lu_1 = g_A(\theta_0)u_0 \ , \tag{208}$$

$$Lu_2 = g_A(\theta_0)u_1 + g_A'(\theta_0)\theta_1 u_0 \ , \tag{209}$$

etc.,

where θ_n is obtained from u_n by the integrations indicated above; each function u_n could be normalized, for instance, by the condition that the equation for u_{n+1} has a solution.

However, Eq. (204) is in fact a third-order nonlinear differential equation for the function θ, where

$$-\left(\frac{1}{\gamma_0^2}\ \frac{d^3\theta}{d\zeta^3} + \frac{d\theta}{d\zeta}\right) = g_A(\theta)\frac{d\theta}{d\zeta} \ , \tag{210}$$

with three linear homogeneous boundary conditions given by the following table:

1	2 and 3	4
$\theta(-1)=0$	$\theta(0)=0$	$\theta(0)=0$
$\theta'(-1)-\epsilon_Z\theta''(-1)=0$	$\theta''(0)=0$	$\theta'(0)-\epsilon_Z\theta''(0) = 0$
$\theta'(1)+\epsilon_Z\theta''(1)=0$	$\theta'(1)+\epsilon_Z\theta''(1)=0$	$\theta''(1)=0$

An integration of Eq. (210) with respect to ζ yields Eq. (50) which can be solved by two integrations; the integration constants I_0, I_1, and I_2 are determined by the three boundary conditions. One notes that in cases (2) and (3), $I_1 = 0$.

However, the boundary conditions on the function θ, the solution of Eq. (50), are not generally linear and homogeneous. The application of the perturbation method directly to Eq. (50) instead of to Eq. (206) would then become unnecessarily intricate.[C4]

The following sections give examples where the solution of Eq. (50) is exactly given by elliptic functions. These examples have been chosen because they yield solutions simple enough to allow parametric study.

5.2. Bidirectional Fuel Movement

In this case, elliptic functions yield the solution of the equation for the flux time $\theta = \frac{1}{2}(\theta^{+} - \theta^{-})$ even when the excess neutron production, $g(\theta)$, is a cubic polynomial in θ. Let

$$g(\theta) = g_0 + g_1\theta + g_2\,\theta^2 + g_3\,\theta^3 \,, \qquad (211)$$

then

$$g_A(\theta) = g(T+\theta) + g(T-\theta) = g(T) + \frac{1}{2}\, g''(T)\theta^2 \,. \qquad (212)$$

We note that the sign of $g(T)$ and of $g''(T)$ is not known, while in preceding examples with $g(\theta) = g_0 + g_2\,\theta^2$ we had set $g_0 > 0$ and $g_2 < 0$. There, however, the case $g_0 < 0$, $g_2 > 0$ would correspond to a reactor used as a breeder; any steady-state fuel scheduling scheme could be used for this purpose.

Integrating Eq. (210) with respect to ζ and using the boundary condition at $\zeta = 0$ yields

$$\frac{d^2\theta}{d\zeta^2} = -\gamma_0^2\left\{[\,1 + g(T)]\,\theta + \frac{1}{6}\, g''(T)\theta^3\right\}. \qquad (213)$$

Multiplying by $d\theta/d\zeta$ and integrating again yields

$$\left(\frac{d\theta}{d\zeta}\right)^2 = 2\gamma_0^2\left\{I_0 - \frac{1}{2}[\,1 + g(T)]\,\theta^2 - \frac{1}{24}\, g''(T)\theta^4\right\}.$$

The boundary condition is now $\theta(0) = 0$, and I_0 is determined by $\theta'(1) + \epsilon_Z\theta''(1)$; with $\theta(1) = T$, this yields

$$I_0 = \frac{T^2}{2}[\,1 + g(T)] + \frac{T^4}{24}\, g''(T) + \frac{\epsilon_Z^2}{2\gamma_0^2}[\,\theta''(1)\,]^2, \qquad (214)$$

where $\theta''(1)$ shall be replaced by its expression as a function of T.

In the following we consider bare reactors only, Eq. (179). Then we have

$$\left(\frac{d\theta}{d\zeta}\right)^2 = \gamma_0^2\,(T^2 - \theta^2)[\,1 + g(T) + \frac{1}{12} g''(T)\,(T^2 + \theta^2)]. \qquad (215)$$

This equation has for a solution either $T \operatorname{sn} K\zeta$ or $T \operatorname{cn} K(1-\zeta)$, according to the signs of $[\,1 + g(T)]$ and $g''(T)$.

(1) When we have $g''(T) < 0$, then

$$\theta(\zeta) = T \operatorname{sn} K\zeta\,, \qquad (216)$$

since this function satisfies

$$\left(\frac{d\theta}{d\zeta}\right)^2 = \frac{K^2}{T^2}\,(T^2 - \theta^2)\,(T^2 - k^2\theta^2)\,,$$

and one can find the positive quantities, K^2 and k^2, such that Eq. (215) is identical with the above equation. That is

$$K^2 = \gamma_0^2 \left[1 + g(T) + \frac{T^2}{12} g''(T)\right], \qquad (217)$$

and

$$k^2 = -\frac{T^2}{12}\, g''(T) \Big/ \left[1 + g(T) + \frac{T^2}{12} g''(T)\right]. \qquad (218)$$

We know K^2 and k^2 are positive because the last factor in Eq. (215) shall be positive when $\theta = 0$. Since this factor shall also be positive when θ increases toward T, k^2 is smaller than 1. The complete elliptic integral of the first kind, K, is a function of the modulus k only, i.e., $K(k^2)$ which is tabulated by Byrd.[B3] Thus, Eq. (217) with k^2 given as a function of T by Eq. (218) is the equation which determines T.

(2) When we have $g''(T) > 0$, then

$$\theta = T \operatorname{cn} K(1-\zeta)\,, \qquad (219)$$

since this function satisfies

$$\left(\frac{d\theta}{d\zeta}\right)^2 = \frac{K^2}{T^2}(T^2 - \theta^2)\,(k'^2\,T^2 + k^2\theta^2), \text{ where } k'^2 = 1-k^2\,,$$

and Eq. (215) is identical to the above equation with

$$K^2 k'^2 = \gamma_0^2 \left[1 + g(T) + \frac{T^2}{12} g''(T) \right] ,$$

and

$$k'^2/k^2 = \left[1 + g(T) + \frac{T^2}{12} g''(T) \right] \Big/ \frac{T^2}{12} g''(T) .$$

The above quantities are positive, as is the right side of Eq. (215). Both the above equations are an equality between a function of k on the left side and a function of T on the right side. Given the function $g(T)$, they determine T, k^2 and K^2. For instance, by forming first

$$K^2 k^2 = \gamma_0^2 \frac{T^2}{12} g''(T) , \tag{220}$$

the following equations, similar to Eqs. (217) and (218), are obtained:

$$K^2 = \gamma_0^2 \left[1 + g(T) + \frac{T^2}{6} g''(T) \right] , \tag{221}$$

and

$$k^2 = \frac{T^2}{12} g''(T) \Big/ \left[1 + g(T) + \frac{T^2}{6} g''(T) \right] . \tag{222}$$

5.2.1. Example 1: Linear case: $g(\theta) = g_0 + g_1 \theta$, where $g''(T) \equiv 0$.

Equation (218) or (222) yields $k^2 = 0$, and then $K = \gamma_0 = \frac{\pi}{2}$. Equation (217) or (221) yields simply $g(T) = 0$ or,

$$T = - g_0/g_1 . \tag{223}$$

The flux time to which the fuel has been irradiated when discharged is $\theta^+(1) = 2T$, according to Eqs. (201) and (202). This is twice the final value of the flux time which would be attained if the fuel were irradiated batchwise while being thoroughly mixed.

Equation (216) or (219) yields $\theta = T \sin \frac{\pi}{2} \zeta = T \cos \frac{\pi}{2} (1 - \zeta)$, and the flux shape, $u(\zeta, T)$ is proportional to the derivative of θ with respect to ζ, i.e.,

$$u(\zeta) = \cos \frac{\pi}{2} \zeta \tag{224}$$

as expected, because the composition of the reactor is uniform.

The above results are a function of $-g_0/g_1$, only. If $g_0 > 0$ and $g_1 < 0$, the reactivity of the fuel decreases during the irradiation; this is the usual case. However, one could use the reactor as a breeder by feeding a fuel the reactivity of which increases with irradiation ($g_0 < 0$ and $g_1 > 0$); the operational qualities of the reactor are then reversed; if the velocity of the fuel were decreased, the reactor would become supercritical.

5.2.2. Example: Quadratic case: $g(\theta) = g_0 + g_2\theta^2$, $g''(T) = 2g_2$.

(1) When we have $g_2 < 0$, Eqs. (217) and (218) become

$$K^2 = \gamma_0^2 \left[1 + g_0 + \frac{7}{6} g_2 T^2 \right],$$

and

$$K^2 k^2 = - \gamma_0^2 g_2 \frac{T^2}{6}.$$

The elimination of T^2 yields the following relation between g_0 and k^2 (where $\frac{\pi}{2} = \gamma_0$):

$$K^2(1 + 7k^2) = \gamma_0^2(1 + g_0), \text{ and } T^2 = 6 \frac{1+g_0}{-g_2} \frac{k^2}{1+7k^2}. \tag{225}$$

The flux shape is proportional to the derivative of the flux time, Eq. (216), with respect to ζ; that is,

$$u(\zeta, T) = \text{cn } K\zeta \text{ dn } K\zeta . \tag{226}$$

The flux shape is peaked at the center, because the fuel is more reactive there. The average flux is given by

$$\frac{\langle u \rangle}{u(0)} = \frac{1}{K} < \frac{2}{\pi}. \tag{227}$$

(2) When we have $g_2 > 0$, Eqs. (220) and (211) become

$$K^2 k^2 = \gamma_0^2 g_2 \frac{T^2}{6},$$

and

$$K^2 = \gamma_0^2 \left[1 + g_0 + \frac{4}{3} g_2 T^2 \right] .$$

The elimination of T^2 gives

$$K^2(1 - 8k^2) = \gamma_0^2 (1 + g_0) \quad \text{and} \quad T^2 = 6 \, \frac{1+g_0}{g_2} \, \frac{k^2}{1-8k^2} . \tag{228}$$

The flux shape is proportional to the derivative of the flux time, Eq. (219), with respect to ζ. Normalizing $u(\zeta, T)$ to unity at the center, one obtains

$$u(\zeta, T) = \sin K(1-\zeta) \, \frac{\mathrm{dn}\, K(1-\zeta)}{k'} ,$$

which can be transformed (see Ref. B3, formulae 122) into,

$$u(\zeta, T) = \mathrm{cn}\, K\zeta / \mathrm{dn}^2 K\zeta . \tag{229}$$

The average flux is

$$\frac{\langle u \rangle}{u(0)} = \int_0^1 \frac{\mathrm{cn}\, K\zeta}{\mathrm{dn}^2 K\zeta} \, d\zeta = \frac{1}{K} \, \mathrm{sd}\, K = \frac{1}{Kk'} . \tag{230}$$

The maximum flux is given by $\dfrac{d^2\theta}{d\zeta^2} = 0$. By using Eq. (213), this yields

$$1 + g(T) + \frac{1}{6} g''(T) \, \theta^2 = 1 + g_0 + g_2 T^2 + \frac{1}{3} g_2 \theta^2 = 0,$$

or

$$\frac{\theta^2}{T^2} = -3 \left(\frac{1-8k^2}{6k^2} + 1 \right) = \frac{k^2 - \frac{1}{2}}{k^2} .$$

Then, if $k^2 < 1/2$, the maximum flux is the central flux; if $k^2 > 1/2$, there is a maximum flux given by the above equation, and the flux has a relative minimum at the center.

5.2.3. Example 3: Cubic case: $g(\theta) = g_0 + g_3\,\theta^3$, $g''(T) = 6g_3T$.

This example is treated exactly as was the preceding.

(1) When we have $g_3 < 0$, Eqs. (217) and (218) yield

$$K^2 = \gamma_0^2\left[1 + g_0 + \frac{3}{2}\,g_3T^3\right], \quad K^2k^2 = -\,\gamma_0^2\,\frac{T^3}{2}\,.$$

Thus we have

$$K^2(1 + 3k^2) = \gamma_0^2(1 + g_0) \text{ and } T^3 = 2\,\frac{1+g_0}{-g_3}\,\frac{k^2}{1+3k^2}\,,$$

and

$$u(\zeta, T) = \text{cn } K\zeta \text{ dn } K\zeta \quad \text{and} \quad \frac{\langle u \rangle}{u(0)} = \frac{1}{K}\,.$$

(2) When we have $g_3 > 0$, Eqs. (220) and (221) yield

$$K^2 = \gamma_0^2\left[1 + g_0 + 2g_3T^3\right], \quad K^2k^2 = \gamma_0^2\,g_3T^3/2\,.$$

Thus we have

$$K^2(1 - 4k^2) = \gamma_0^2[\,1 + g_0] \quad \text{and} \quad T^3 = 2\,\frac{1+g_0}{g_3}\,\frac{k^2}{1-4k^2}\,,$$

and

$$u(\zeta, T) = \text{cn } K\zeta/\text{dn}^2\, K\zeta \quad \text{and} \quad \frac{\langle u \rangle}{u(0)} = \frac{1}{Kk'}\,.$$

The maximum flux is off center when $k^2 > \frac{1}{2}$; it is given by

$$1 + g(T) + \frac{1}{6}\,g''(T)\theta^2 = 1 + g_0 + g_3T^3 + g_3T\theta^2 = 0\,.$$

or by

$$\left(\frac{\theta}{T}\right)^2 = -\,\frac{1+g_0}{g_3T^3} - 1 = -\,\frac{1-4k^2}{2k^2} - 1 = \frac{k^2 - \frac{1}{2}}{k^2}\,.$$

The ratio of central to average flux is plotted in Fig. III-4 for the three examples treated above; it is a function of g_0 only. Also plotted is the value of k, the modulus of the corresponding elliptic functions.

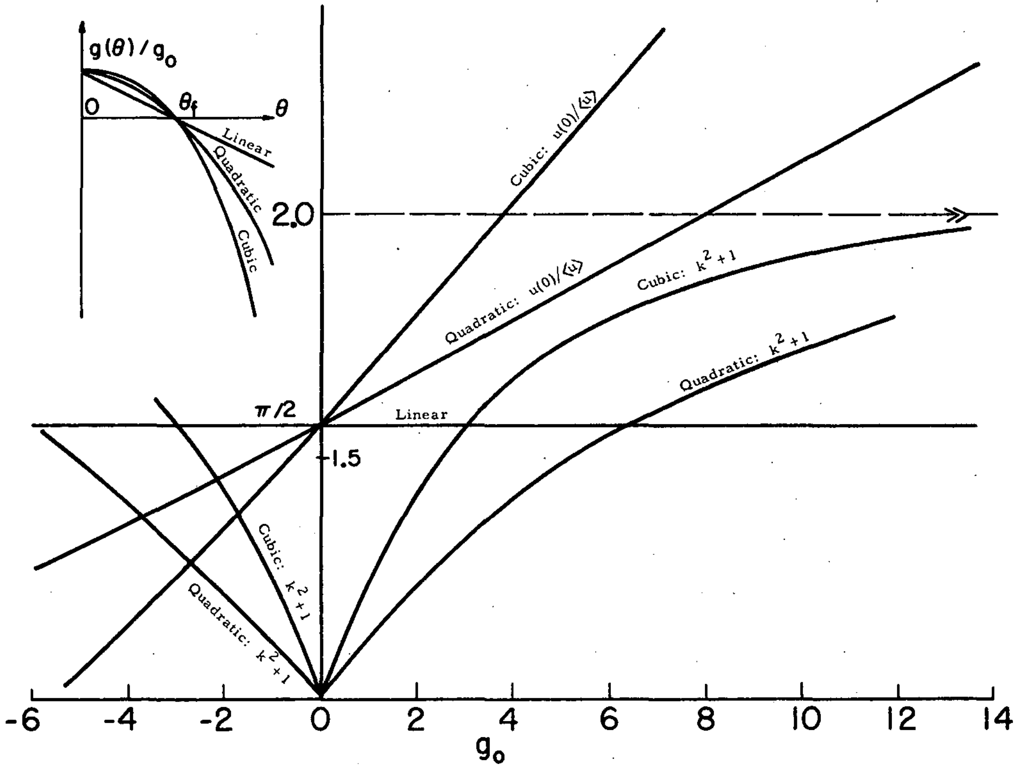

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Fig. III-4. Flux changes resulting from bidirectional fuel movement (elliptic-function solutions).
$u(0)/\langle u \rangle$ = ratio of central to average flux.
k = modulus of the elliptic functions.

The value of T, which is half the flux time to which the fuel is irradiated when discharged, is compared with the flux time θ_f to which thoroughly mixed fuel could be irradiated in reactors of the same geometrical buckling; in the following tabulation we give some typical values of (T/θ_f), which is a monotonically decreasing function of g_0.

	$g_0 = -\infty$	$g_0 = 0$	$g_0 = \infty$
Linear:	1	1	1
Quadratic:	$\left(\frac{6}{7}\right)^{\frac{1}{2}} = 0.925$	$\left(\frac{4}{5}\right)^{\frac{1}{2}} = 0.894$	$\left(\frac{3}{4}\right)^{\frac{1}{2}} = 0.866$
Cubic:	$\left(\frac{2}{3}\right)^{1/2} = 0.873$	$\left(\frac{4}{7}\right)^{1/3} = 0.830$	$\left(\frac{1}{2}\right)^{1/3} = 0.794$

One notes that, although the changes in the flux can be very large (cf. Fig. III-4), the value of T does not vary more than 5% from the value it has when $g_0 = 0$.

5.2.4. Perturbation method. An approximate solution could be obtained by applying the perturbation method to Eq. (206). The following gives the results of the first-order perturbation.

Equation (207) has a solution u_0 proportional to $\cos \gamma_0 \zeta$, which is normalized in order for the average flux to be unity, i.e.,

$$u_0(\zeta) = \frac{\gamma_0}{\sin \gamma_0} \cos \gamma_0 \zeta$$

and the corresponding flux time, Eq. (200), is

$$\theta_0(\zeta, T) = T \frac{\sin \gamma_0 \zeta}{\sin \gamma_0} .$$

The first-order perturbation determines T by the condition that Eq. (208) have a solution; i.e., $\langle g_A(\theta_0) u_0, u_0 \rangle = 0$. With $g_A(\theta)$ given by Eq. (212), one obtains

$$g(T) + \frac{1}{2} g''(T) \frac{\langle \theta_0^2 u_0, u_0 \rangle}{\langle u_0, u_0 \rangle} = 0.$$

The corresponding integrals are evaluated as follows:

$$\frac{\langle \theta_0^2 u_0, u_0 \rangle}{\langle u_0, u_0 \rangle} = T^2 \frac{\int_0^1 \sin^2 \gamma_0 \zeta \cos^2 \gamma_0 \zeta \, d\zeta}{\sin^2 \gamma_0 \zeta \int_0^1 \cos^2 \gamma_0 \zeta \, d\zeta}$$

$$= \frac{T^2}{4 \sin^2 \gamma_0} \times \frac{(1 - \sin 2\gamma_0 / 4\gamma_0)}{(1 + \sin 2\gamma_0 / 2\gamma_0)} \cdot$$

For the three examples treated above, $\gamma_0 = \frac{\pi}{2}$ gives

$$g(T) + \frac{T^2}{8} g''(T) = \rho .$$

(1) <u>Linear case</u>: $g''(T) = 0$; $g_0 + g_1 T = 0$.
(2) <u>Quadratic case</u>: $g''(T) = 2g_2$; $g_0 + \frac{5}{4} g_2 T^2 = 0$.
(3) <u>Cubic case</u>: $g''(T) = 6g_3$; $g_0 + \frac{7}{4} g_3 T^3 = 0$.

which correspond to the exact value of T when g_0 tends toward zero.

5.3. Unidirectional

After two integrations, Eq. (210) becomes Eq. (51), where the two constants of integration, I_0 and I_1, must be determined by the boundary conditions. One will consider the case of a bare reactor with $\epsilon_Z = 0$ and $\gamma_0 = \frac{\pi}{2}$; the boundary conditions at $\zeta = -1$ then yield $I_0 = 0$; when the function $G(\theta)$ is quadratic, Eqs. (59) and (51) become

$$\left(\frac{d\theta}{d\zeta}\right)^2 = 2 \gamma_0^2 \theta (I_1 - \frac{1+g_0}{2} \theta - \frac{g_1}{6} \theta^2 - \frac{g_2}{12} \theta^3) .$$

The flux time $T = \theta(1)$ is a root of the last factor on the right side of the above equation because of the boundary condition at $\zeta = 1$.

In the above equation, one can then use T instead of I_1 as a constant of integration; it becomes

$$\left(\frac{d\theta}{d\zeta}\right)^2 = 2\,\gamma_0^2\theta(T-\theta)\left(\frac{1+g_0}{2} + \frac{g_1}{6}\,T + \frac{g_2}{12}\,T^2 + \left(\frac{g_1}{6} + \frac{g_2}{12}\,T\right)\theta + \frac{g_2}{12}\,\theta^2\right). \tag{231}$$

This equation can be solved by finding first the roots of the quadratic polynomial in the last factor and then by using standard formulae as in the following examples.

5.3.1. Example 1: Linear case: $g(\theta) = g_0 + g_1\,\theta$. Let

$$g_0 > 0,\ g_2 < 0,\ \text{and}\ \tau = \frac{1+g_0}{-g_2}\ ; \tag{232}$$

then Eq. (231) becomes

$$\left(\frac{d\theta}{d\zeta}\right)^2 = \gamma_0^2\,\frac{1+g_0}{3\tau}\,\theta\,(T-\theta)\,(3\tau-T-\theta)\ \text{where}\ 3\tau-T \geqslant T \geqslant \theta \geqslant 0\ .$$

Using the condition $\theta(-1) = 0$, one obtains

$$\gamma_0\sqrt{\frac{1+g_0}{3\tau}}\,(1+\zeta) = \int_0^{\theta} \frac{d\theta}{\theta(T-\theta)\,(3\tau-T-\theta)}$$

and the inversion formulae (see Ref. B3, Eq. 233) give

$$\theta(\zeta) = T\,\mathrm{sn}^2\,b(1+\zeta),\ \text{with}\ b^2 = \gamma_0^2\,\frac{1+g_0}{3\tau}\,\frac{3\tau-T}{4}\ ,$$

$$k^2 = \frac{T}{3\tau-T}\ ,\quad \text{or}\quad T = 3\tau\,\frac{k^2}{1+k^2}\ . \tag{233}$$

The condition that $\theta(\zeta)$ be an increasing function of ζ and the boundary condition $\theta'(1) = 0$ yields

$$\theta(\zeta) = T\,\mathrm{sn}^2\,\tfrac{1}{2}K(1+\zeta) \tag{234}$$

and

$$K^2 = \gamma_0^2(1+g_0)\frac{3\tau-T}{3\tau}\quad \text{or}\quad K^2(1+k^2) = \gamma_0^2\,(1+g_0). \tag{235}$$

The last equation determines K and k^2 as a function of g_0, and Eq. (233) determines T, the flux time at discharge.

Using Eq. (200), one obtains the flux $u(\zeta, T)$ as

$$u(\zeta, T) = K \operatorname{sn} \tfrac{1}{2}K(1+\zeta) \operatorname{cn} \tfrac{1}{2}K(1+\zeta) \operatorname{dn} \tfrac{1}{2}K(1+\zeta)$$

and the corresponding average flux is 1/2. Thus, we have

$$\frac{u(0)}{\langle u \rangle} = K \frac{2k'}{1+k'} . \tag{236}$$

The flux shape, normalized to 1 at the center, can be written

$$u(\zeta, T) = \frac{1+k'}{2k'} [1-k^2 \operatorname{sn}^4 \tfrac{1}{2}K(1+\zeta)] \operatorname{sn} K(1+\zeta). \tag{237}$$

This shows the asymmetry of the flux about the center of the reactor.

5.3.2. Example 2: Quadratic case: $g(\theta) = g_0 + g_2 \theta^2$. Let

$$g_0 > 0, \quad g_1 < 0 , \quad \text{and} \quad \tau^2 = \frac{1+g_0}{-g_2} ; \tag{238}$$

then Eq. (231) becomes

$$\left(\frac{d\theta}{d\zeta}\right)^2 = \gamma_0^2 \frac{1+g_0}{6\tau^2} \theta(T-\theta) (6\tau^2 - T^2 - T\theta - \theta^2) .$$

The last factor is positive if $T^2 < 2\tau^2$. Let its roots be θ_+ and θ_-, then:

$$\frac{1}{2}\left(\sqrt{3(8\tau^2 - T^2)} - T\right) = \theta_+ \geq T \geq \theta \geq 0 \geq \theta_- = -\frac{1}{2}\left(\sqrt{3(8\tau^2 - T^2)} + T\right). \tag{239}$$

Using the condition $\theta(-1) = 0$, one obtains

$$\gamma_0 \sqrt{\frac{1+g_0}{6\tau^2}} (1+\zeta) = \int_0^{\theta} \frac{d\theta}{\sqrt{(\theta_+ - \theta)(T-\theta)\,\theta\,(\theta - \theta_-)}} ,$$

and inversion formulae [see Ref. B3, Eq. (254)] give

$$\theta(\zeta) = T\ \mathrm{sn}^2\, b(1+\zeta)/[\,1 - \frac{T}{\theta_-}\ \mathrm{cn}^2\, b(1+\zeta)]\ ,$$

with

$$b^2 = \gamma_0^2\ \frac{1+g_0}{6\tau^2}\ \frac{\theta_+(T-\theta_-)}{4}\ ,$$

and

$$k^2 = \frac{T(\theta_+-\theta_-)}{\theta_+(T-\theta_-)}\ .$$

The boundary conditions yield $b = \frac{1}{2}K$. We will then eliminate T between the equations

$$K^2 = \gamma_0^2\ (1+g_0)\ \frac{\theta_+(T-\theta_-)}{6\tau^2}\ ,$$

and

$$K^2k^2 = \gamma_0^2\ (1+g_0)\ \frac{T(\theta_+-\theta_-)}{6\tau^2}\ .$$

By noting that θ_+ and θ_-, Eq. (239), satisfy

$$\theta_++\theta_- = -\,T, \quad \text{and}\ \theta_+\theta_- = -\,(6\tau^2-T^2),$$

equations containing only T^2, k^2 and $\lambda=K^2/[\gamma_0^2(1+g_0)]$ are obtained as follows:

$$\lambda^2k^4 = \frac{T^2}{6\tau^2}\ \frac{(\theta_++\theta_-)^2 - 4\theta_+\theta_-}{6\tau^2} = \frac{T^2}{6\tau^2}(4 - \frac{T^2}{2\tau^2})\ ,$$

$$\lambda(1-\frac{1}{2}k^2) = \frac{\frac{1}{2}(\theta_++\theta_-)T - \theta_+\theta_-}{6\tau^2} = 1 - \frac{T^2}{4\tau^2}\ ,$$

and

$$\lambda^2(1-\frac{1}{2}k^2)^2 = 1 - \frac{T^2}{8\tau^2}(4 - \frac{T^2}{2\tau^2})\ .$$

Elimination of T^2 between the first and the last of the above three equations yields

$$K^2\sqrt{1-k^2+k^4} = \gamma_0^2(1+g_0)\ , \tag{240}$$

and

$$T^2 = 4\tau^2\left[1 - \frac{1-\frac{1}{2}k^2}{\sqrt{1-k^2+k^4}}\right]\ . \tag{241}$$

The flux time is then

$$\theta(\zeta, T) = T\,\mathrm{sn}^2\,\frac{1}{2}\,K(1+\zeta)/\left[\,1-\frac{T}{\theta_-}\,\mathrm{cn}^2\,\frac{1}{2}\,K(1+\zeta)\right], \qquad (242)$$

and the flux given by Eq. (200) is

$$u(\zeta, T) = \frac{K\left(1-\frac{T}{\theta_-}\right)\mathrm{sn}\frac{1}{2}K(1+\zeta)\,\mathrm{cn}\,\frac{1}{2}K(1+\zeta)\,\mathrm{dn}\,\frac{1}{2}K(1+\zeta)}{\left[\,1-\frac{T}{\theta_-}\,\mathrm{cn}^2\,\frac{1}{2}K(1+\zeta)\right]^2}\,. \qquad (243)$$

The ratio of average to central flux is then obtained from Eq. (200) as

$$\frac{u(0)}{\langle u\rangle} = \frac{K\left(1-\frac{T}{\theta_-}\right)\frac{2k'}{1+k'}}{\left(1-\frac{T}{\theta_-}\times\frac{k'}{1+k'}\right)^2}\,, \qquad (244)$$

where T/θ_- is obtained from Eqs. (239) and (241).

The flux time at discharge T and the ratio $u(0)/\langle u\rangle$ of central to average flux are plotted in Fig. III-5. For finite g_0, the flux is not maximum at the center of the reactor. Therefore, the ratio $u(0)/\langle u\rangle$, which decreases very rapidly with g_0, is not the ratio of maximum to average flux.

The perturbation method can be applied to obtain an approximation to the flux changes. The first-order perturbation is as follows:

Equation (207) gives the first approximation to the flux shape, $\cos\gamma_0\zeta$. Then Eq. (200) with $\zeta_0 = -1$ gives

$$u_0(\zeta) = \frac{\gamma_0}{2\sin\gamma_0}\cos\gamma_0\zeta,\; \theta_0(\zeta, T) = T\frac{\sin\gamma_0\zeta + \sin\gamma_0}{2\sin\gamma_0}\,.$$

The first-order perturbation determines T by the condition that Eq. (208) have a solution; i.e., $\langle g(\theta_0)u_0, u_0\rangle = 0$. Let

$$g(\theta) = g_0 + g_1\,\theta + g_2\,\theta^2 \qquad (246)$$

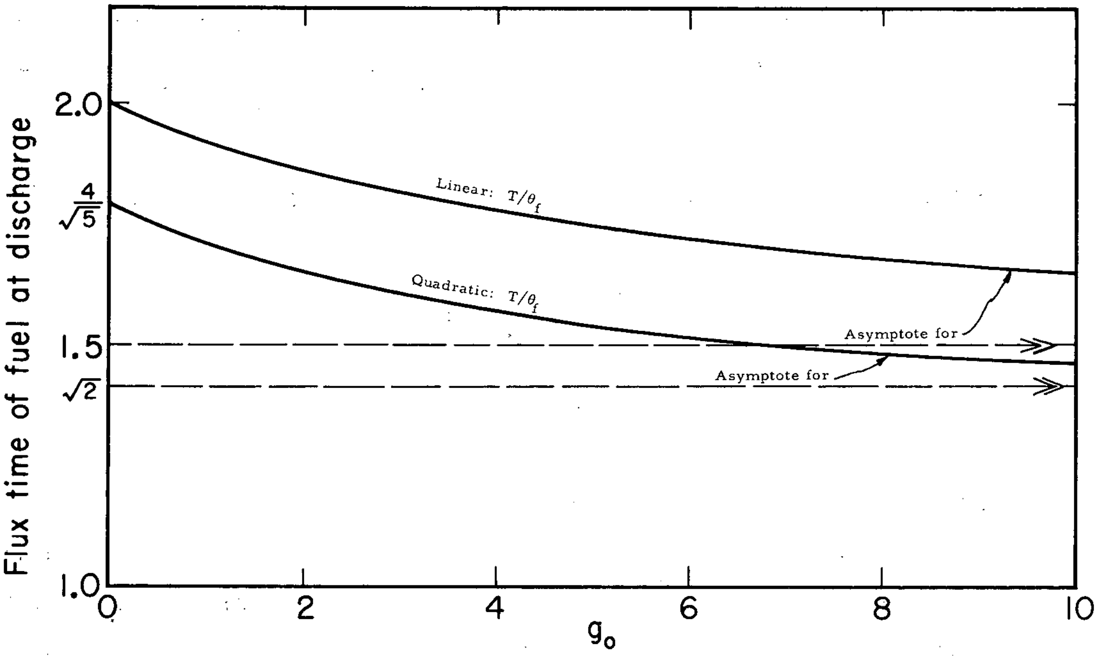

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Fig. III-5. Irradiation resulting from unidirectional fuel movement (elliptic-function solutions).

and

$$g(\theta_0) = g(\tfrac{T}{2}) + \left[g'(\tfrac{T}{2})\right]\frac{T}{2}\,\frac{\sin\gamma_0\zeta}{\sin\gamma_0} + \frac{1}{2}\left[g''(\tfrac{T}{2})\right](\tfrac{T}{2})^2\,\frac{\sin^2\gamma_0\zeta}{\sin^2\gamma_0},$$

then

$$\frac{\langle g(\theta_0)u_0, u_0\rangle}{\langle u_0, u_0\rangle} = g(\tfrac{T}{2}) + f_1\left[g'(\tfrac{T}{2})\right]\frac{T}{2} + \frac{1}{2}f_2\left[g''(\tfrac{T}{2})\right](\tfrac{T}{2})^2, \qquad (247)$$

where the coefficients f_1 and f_2 can be evaluated as follows:

$$f_1 = \frac{\int_{-1}^{1}\sin\gamma_0\zeta\cos^2\gamma_0\zeta\,d\zeta}{\sin\gamma_0\int_{-1}^{1}\cos^2\gamma_0\zeta\,d\zeta} = 0,$$

and

$$f_2 = \frac{\int_{-1}^{1}\sin^2\gamma_0\zeta\cos^2\gamma_0\zeta\,d\zeta}{\sin^2\gamma_0\int_{-1}^{1}\cos^2\gamma_0\zeta\,d\zeta} = \frac{1-\dfrac{\sin 4\gamma_0}{4\gamma_0}}{4\sin^2\gamma_0\left(1-\dfrac{\sin 2\gamma_0}{2\gamma_0}\right)}.$$

When $\gamma_0 = \frac{\pi}{2}$, Eq. (247) becomes

$$g(\tfrac{T}{2}) + \frac{1}{8}\left[g''(\tfrac{T}{2})\right](\tfrac{T}{2})^2 = 0.$$

The two examples treated above give

(1) <u>Linear case:</u> $g_2 = 0$, $g_0 + g_1\frac{T}{2} = 0$.

(2) <u>Quadratic case:</u> $g_1 = 0$, $g_0 + \frac{5}{4}g_2\frac{T^2}{4} = 0$.

The above equations give the exact value of T for $g_0 = 0$.

The first-order perturbation method has been applied by Sola[S2] to this unidirectional fuel movement. Functions giving the first approximation to the value of T have been tabulated in the case where the excess neutron production is represented by a sum of exponentials.[B1] The second-order perturbation method would then yield an eigenfunction expansion of the correction to the flux, the coefficients of which are also given for a bare reactor by modified Bessel functions of the first kind.

5.4. In-Out

A first integration of Eq. (210) yields Eq. (50) with $I_1 = 0$; this is the equation for graded irradiation, and it has the same general solution. The boundary conditions are now $\theta(0) = 0$ and $\theta'(1) + \epsilon_Z \theta''(1) = 0$.

If $\epsilon_Z = 0$, the boundary condition is $\theta(0) = 0$ and $\theta'(1) = 0$ while it was $\theta'(0) = 0$, $\theta(1) = 0$ for graded irradiation. The solution is then exactly the same except for a change of the variable ζ to $1-\zeta$.

5.4.1. Example: $\epsilon_Z = 0$ and $g(\theta) = g_0 + g_2 \theta^2$. This case has been treated for graded irradiation. One obtains the same values of K, k, and T_m, with

$$\theta(\zeta) = T_m \operatorname{sn} K\zeta ,$$

where $T_m = \theta(1)$ is the flux time of the fuel ready to discharge; the fuel is now irradiated to the maximum flux time which is attained only at the center of a rod for graded irradiation.

The flux shape is proportional to the derivative of the flux-time, i.e.,

$$u(\zeta) = \operatorname{cn} K\zeta \operatorname{dn} K\zeta .$$

The flux is much more peaked at the center due to the effect of the more reactive flux at the center. The average flux is now

$$\frac{\langle u \rangle}{u(0)} = \int_0^1 \operatorname{cn} K\zeta \operatorname{dn} K\zeta \, d\zeta = \frac{1}{K} < \frac{2}{\pi} .$$

The perturbation method can be applied to Eq. (206), giving the first approximation to the flux shape and to the flux time as

$$u_0(\zeta) = \frac{\gamma_0}{\sin \gamma_0} \cos \gamma_0 \zeta \quad \text{and} \quad \theta_0(\zeta, T) = T \frac{\sin \gamma_0 \zeta}{\sin \gamma_0} .$$

The condition for Eq. (208) to have a solution is $\langle g(\theta_0) u_0, u_0 \rangle = 0$. When $g(\theta)$ is a quadratic function of θ, this yields

$$\frac{\langle g(\theta_0) u_0, u_0 \rangle}{\langle u_0, u_0 \rangle} = g_0 + f_1 g_1 T + f_2 g_2 T^2 = 0 ,$$

where f_1 and f_2 are evaluated as follows:

$$f_1 = \frac{\int_0^1 \sin \gamma_0 \zeta \cos^2 \gamma_0 \zeta \, d\zeta}{\sin^2 \gamma_0 \int_0^1 \cos^2 \gamma_0 \zeta \, d\zeta} = \frac{2(1 - \cos^3 \gamma_0)}{3\gamma_0 \sin \gamma_0 \left(1 + \frac{\sin 2\gamma_0}{2\gamma_0}\right)} ,$$

and

$$f_2 = \frac{\int_0^1 \sin^2 \gamma_0 \zeta \cos^2 \gamma_0 \zeta \, d\zeta}{\sin^2 \gamma_0 \int_0^1 \cos^2 \gamma_0 \zeta \, d\zeta} = \frac{1 - \frac{\sin 4\gamma_0}{4\gamma_0}}{4 \sin^2 \gamma_0 \left(1 + \frac{\sin 2\gamma_0}{2\gamma_0}\right)} .$$

When $\gamma_0 = \frac{\pi}{2}$, one obtains simply

$$g_0 + \frac{4}{3\pi} g_1 T + \frac{1}{4} g_2 T^2 = 0 .$$

One recognizes the function $S_0(T)$, Eq. (153).

For a bare reactor where $\gamma_0 = \frac{\pi}{2}$, one could obtain an approximation of the flux time by using the same formulae which apply to the graded irradiation of the fuel; however, the flux shape is now the derivative of the flux time with respect to ζ.

6. Discussion of the Results

The average flux time $\langle \theta \rangle$ to which the discharged fuel can be irradiated according to various fuel scheduling schemes is plotted in Fig. III-6 as a function of the parameter g_0. It has been assumed that the excess neutron production is represented by the following function of the irradiation flux time θ:

$$g(\theta) = (\nu_{th} \Sigma_f - \Sigma_a - DB_0^2)/DB_Z^2 = g_0[1-(\theta/\theta_f)^2] ,$$

where θ_f is the uniform flux time to which completely mixed fuel could be irradiated batchwise, and g_0 is the ratio of (a) the excess neutron production corresponding to the fresh fuel to (b) the axial leakage of neutrons in the uniformly loaded reactor. The burnup of the fuel is a monotonically (here linearly) increasing function of the flux time θ.

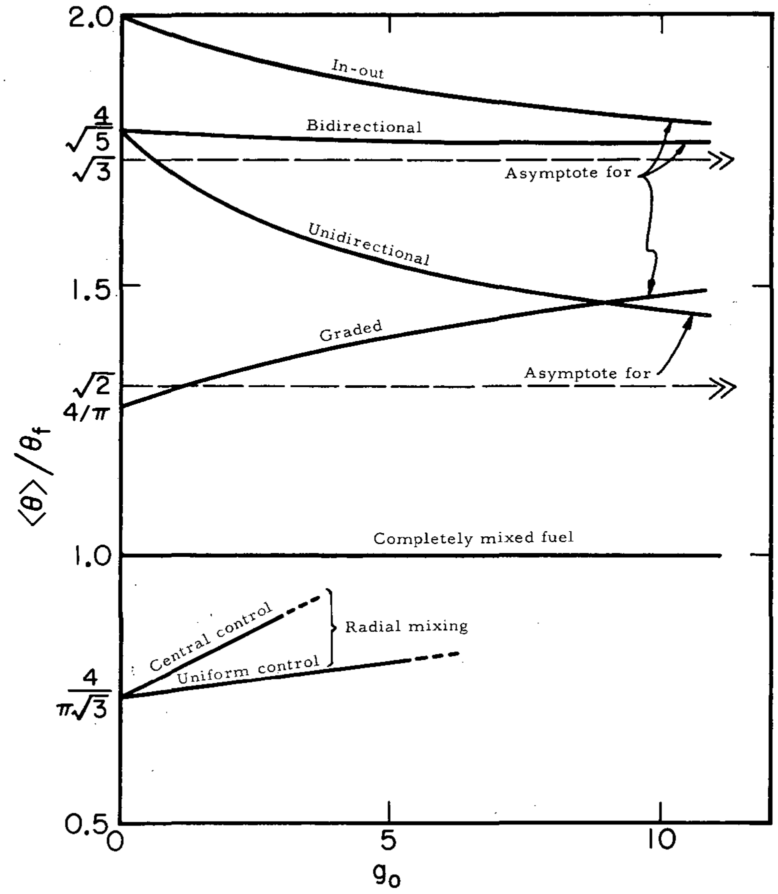

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Fig. III-6. Comparison of average irradiation of the fuel at discharge for various fueling schemes.

The results are interpreted as follows:

(1) The average burnup of the discharged fuel is largest for steady-state fuel scheduling schemes because the reactor always contains some fresh fuel with reactivity greater than average, while only the most irradiated fuel, with reactivity less than average, is discharged. Steady axial movements usually yield better results than graded irradiation where the most irradiated fuel rod still contains some nearly fresh fuel.

(2) According to first-order perturbation theory (i. e., for a given flux shape), the greater the average importance of the fuel, the larger the reactivity of the reactor. The importance function is proportional to the unperturbed flux shape, and thus, larger burnup is obtained with the in-out technique where the fuel at the center of the reactor is relatively more reactive than in the bidirectional or unidirectional fueling schemes. Similarly, radial mixing, in which the less reactive fuel is at the center, yields smaller burnups than completely mixed fuel.

(3) The second-order perturbation theory takes into account the flux-shape changes; the flux shape tends to flatten when the less reactive fuel is at the center, according to the one-group diffusion model. The flattening of the flux, which here occurs for radial mixing and graded irradiation, results in an increase of the average burnup of the fuel; this has been explained in Sec. II.4.3. Thus, as shown on Fig. III-6, the average burnup increases with the parameter g_0 to which the flattening of the flux shape is proportional. On the other hand, for in-out and bidirectional movements, the fuel at the midplane of the reactor is the most reactive; the flux shape peaks, and, on Fig. III-6, the average burnup is seen to decrease with g_0. For unidirectional movement, the flux shape is not symmetrical about the midplane of the reactor, and this skewing of the flux decreases strongly the attainable burnup.

On Fig. III-6, exact results have been obtained by means of elliptic functions for the steady-state fuel scheduling schemes. The accuracy of the perturbation methods is then easily determined, since first-order perturbation approximates the average flux time by a value that corresponds to the unperturbed flux shape ($g_0 = 0$), and since second-order perturbation theory approximates the curves plotted on Fig. III-6 by their tangent at $g_0 = 0$.

The first-order perturbation results are accurate within 5% for bidirectional movement and can be in error of as much as 25% for unidirectional movement.

The exact results here lie between the first-order and the second-order perturbation approximations. The second-order perturbation method emphasizes the effects resulting from the flux-shape changes; it always agrees qualitatively with the exact results, but gives close agreement only for small values of g_0; quantitative agreement decreases as g_0 increases. For graded irradiation when g_0 reaches the value 10, second-order perturbation results are 13% larger and first-order perturbation are 13% smaller than the exact results; at larger values of g_0, first-order perturbation is better than second-order perturbation method. However, only the latter can describe the flux changes.

Elliptic function solutions of the one-group reactor equation should be useful for survey studies of steady-state fuel scheduling schemes: The general method requires that we first approximate the excess neutron production $g(\theta)$ by a second-degree polynomial. However, the uncertainties in the basic nuclear data, discussed by McLeod,[M4] as well as the corrections for lumped fuel (see Appendix C), indicate that a more refined representation of the function $g(\theta)$ may sometimes not be justified. Furthermore, the one group model is shown in the next section to be adequate to describe the diffusion of the neutrons in large thermal reactors (with energy independent boundary conditions) loaded with low enrichment fuels.

Such exact solutions are not available for batch irradiation. However, since the burnup of the fuel is smaller there, the flux changes are, in general, also smaller. Therefore, perturbation methods are more accurate for the same initial reactivity of the fuel.

IV. APPLICATION TO SPECIFIC REACTOR SYSTEMS

Specific reactor systems will be defined by a corresponding reactor which is considered to be at operating temperature with equilibrium xenon and samarium. Such "reference designs" have been developed for large thermal reactors.[B1, P1] In the present section, we present a reference design for a small high-leakage U^{235} light-water reactor. The flux change throughout the irradiation and the average burnup of the fuel at end of life are then determined easily by using the perturbation calculations developed in the preceding sections, according to a one-group model.

The validity of the one-group diffusion model is then investigated. For reactors whose reflectors can be represented by a reflector saving, two-group diffusion calculations can be performed with only a slight modification of the calculation procedure developed in Sec. II.

1. U^{235} Light-Water Reactor

1.1. Reference Design

The reference design reactor is defined as the reactor whose core is uniformly loaded with a charge of fresh fuel; high cross-section fission products (xenon and samarium groups) are present at their equilibrium concentration.

1.1.1. Initial composition. The homogenized composition of the clean reactor core is that of the mixture with which experiments have been performed at Bettis Laboratories:[G4] Uranium (U_3O_8, 93.37% enriched U^{235}), Zirconium, and water. The number densities, in units of 10^{24} atoms/cc, are as follows:

$$N_{25} = 0.0001717, \qquad N_H = 0.03368,$$

$$N_{28} = 0.000012, \quad \text{and} \quad N_{Zr} = 0.02132.$$

1.1.2. Effective thermal cross-sections. These have been obtained from the tables of cross-sections averaged over a Wilkins spectrum[A1] corresponding to a neutron temperature of 500°K and a U^{235} to hydrogen-atom ratio of 0.005.

For 1/v absorbers, the absorption cross sections are given by the formula $\sigma_a = AK$

$A = 0.6087$ $\quad K_{26} = 6.0$ barns, $\quad K_{Zr} = 0.180$ barns,

$K_{28} = 2.73$ barns, and $\quad K_H = 0.332$ barns.

For non 1/v absorbers, one obtains directly:

$\sigma_{a,25} = 386.6$ barns, $\sigma_{f,25} = 328.7$ barns, and $\sigma_{Xe} = 1.979 \times 10^6$ barns.

The average number of neutrons produced per fission and the capture to fission ratio in U^{235} are

$$\nu_{25} = 2.47, \text{ and } \alpha_{25} = 0.1761 .$$

The transfer cross section of hydrogen is $\sigma_{tr,H} = 24.34$ barns. For the other materials, transfer cross sections are computed according to the formula $\sigma_{tr} = \sigma_a + (1-\bar{\mu})\sigma_s$.

	U^{235} or U^{238}	Zirconium	Oxygen
$(1-\bar{\mu})\sigma_s$	9.972	6.155	4.025

The diffusion coefficient is then computed from the total macroscopic transfer cross section Σ_{tr} by the formula

$$D = 1/3\, \Sigma_{tr} = 0.3054 \text{ cm.}$$

1.1.3. High cross-section fission products. The samarium group contains the stable fission products. If Y_{Sm} is the yield per fission of such fission products, then at secular equilibrium

$$Y_{Sm}\, \Sigma_f \phi = N_{Sm} \sigma_{Sm} \phi .$$

The xenon group contains the fission products with a large disintegration constant λ. If the total yield of xenon per fission is Y_X, then at secular equilibrium

$$Y_X \Sigma_f \phi = N_X \sigma_X \phi + \lambda_X N_X \,.$$

The total poisoning ratio, defined by

$$N_X \sigma_X + N_{Sm} \sigma_{Sm} = r\, N_{25} \sigma_{25} \,,$$

is thus given by the formula

$$r = \frac{1}{1+\alpha_{25}} \left[Y_{Sm} + Y_X/(1+\lambda_X/\sigma_X \phi) \right] .$$

The fission yields which have been used in computations performed by Shanstrom are

$$Y_{Sm} = 2.1\% \quad \text{and} \quad Y_X = 6.1\%.$$

The Reactor Handbook[E1] gives

$$\lambda_X = 2.1 \times 10^{-5}\ \text{sec}^{-1} \,.$$

The average flux has been chosen

$$\phi = 3 \times 10^{13}\ \text{neutrons/cm}^2\text{sec}$$

corresponding to an average power density of 52 kW/liter of core. Then we have

$$r = 5.62\%.$$

1.1.4. <u>Neutron balance</u>. The neutron balance is written on the basis of one thermal neutron absorbed in U^{235}. The corresponding production of fast neutrons is

$$\eta_{25} = \nu_{25} \frac{\Sigma_{f,25}}{\Sigma_{a,25}} = \frac{\nu_{25}}{1+\alpha_{25}} \,. \tag{1}$$

Because the amount of U^{238} present is very small, the fast fission effect and the resonance absorption are negligible; thus, the fast fission factor is $\epsilon = 1$, and the resonance escape probability is $p = 1$.

The nonleakage probability during moderation is computed according to the Fermi equation

$$P_{th} = e^{-B_0^2 \tau} \tag{2}$$

For the considered mixture with equal volumes of metal and water, the Fermi age to thermal, as computed by Wilkins,[W7] is

$$\tau = 61 \text{ cm}^2 .$$

The neutron balance is computed for a reactor of geometrical buckling, B_0^2, corresponding to a bare cylindrical reactor of equivalent height and diameter equal to 70 cm where $B_0^2 = 0.006735$.

The fast nonleakage probability is then $P_{th} = 0.6632$.

The absorption by the permanent absorbers is proportional to the sum of the macroscopic cross-sections of Zr, H, and U^{238}. If the mixture were not homogeneous, equivalent homogeneous cross sections could be determined by weighting the cross-section of each nuclide by the corresponding thermal disadvantage factor.[G1, B1]

Neutron Balance

Production of fast neutrons:	η_{25}	=	2.1002
Leakage during moderation:	$(1-P_{th})\eta_{25}$	=	0.7073
Net production of thermal neutrons:	$P_{th}\ \eta_{25}$	=	1.3929
Consumption of thermal neutrons by:			
U^{235} fissions:	$1/(1+\alpha_{25})$	=	0.8503
U^{235} capture:	$\alpha_{25}/(1+\alpha_{25})$	=	0.1497
Absorption by Xe and Sm:	r	=	0.0562
Absorption by permanent absorbers:	$N_P\sigma_P/N_{25}\sigma_{25}$	=	0.1381
Thermal leakage:	$DB_0^2/N_{25}\sigma_{25}$	=	0.0310
Total:		=	1.2253
Absorption by control poison:	$\Sigma_E/N_{25}\sigma_{25}$	=	0.1676
Total consumption of thermal neutrons:		=	1.3929

The reactivity ρ for the reference design, defined as the ratio of the absorption of the thermal neutrons by control absorbers to the production of thermal neutrons, is

$$\rho = \frac{0.1676}{1.3929} = 0.12 .$$

The effective multiplication factor of the reference design without control absorbers is

$$k_{eff} = \frac{1.3929}{1.2253} = 1.14.$$

1.1.5. Effective thermal cross-sections σ_F and σ_{26}. The low cross-section fission products can be considered as $1/v$ absorbers, with a reference cross section $\sigma_F^{2200} = 65$ barns (see Ref. E1, Sec. 2-11). The corresponding cross-section averaged over a Wigner-Wilkins spectrum is

$$\sigma_F = 0.6087 \cdot 65 = 39.6 \text{ barns.}$$

The resonance absorption in U^{236} is taken into account by adding a resonance contribution to a purely thermal cross-section. The rate of absorption in U^{236} is

$$N_{26}\,\sigma_{26}\,\phi = N_{26}\,\sigma_{26}^{th}\,\phi + (1-p)\nu P_1 \Sigma_f \phi, \qquad (3)$$

where $\nu\, P_1\, \Sigma_f \phi$ is the number of neutrons slowed down to resonance energies, per second, and P_1 is the nonleakage probability from fission to resonance. The largest resonance absorption in U^{236} occurs at 2.6 eV, and the leakage from this resonance energy to thermal is negligible; one then lets $P_1 = P_{th}$. The quantity p is the resonance escape probability. Assuming that resonance absorptions occur only in U^{236}, 1-p is the probability for a neutron to be absorbed in U^{236} resonances. The U^{236} is very dilute in hydrogen and the resonance escape probability p is a linear function of the resonance integral I_{eff}^{26}; that is,

$$1-p = \frac{N_{26}\, I_{eff}^{26}}{N_H \sigma_{S,H}} .$$

The scattering cross section $\sigma_{S,H}$ of hydrogen at resonance energies is constant and equal to 20.4 barns;[H4] I_{eff}^{26} is given by Halperin[H3] as 400 barns.

Finally, dividing Eq. (3) by $N_{26}\phi$ and replacing (1-p) by its above expression, and replacing P_1 by P_{th} and replacing $\nu\, \Sigma_f$ by $N_{25}\, \frac{\nu}{1+a_{25}}$ yields

$$\sigma_{26} = \sigma_{26}^{th} + \frac{\nu\, P_{th}}{1+a_{25}}\, \frac{N_{25}\, \sigma_{25}}{N_H\, \sigma_{S,H}}\, I_{eff}^{26} .$$

The above data yield

$$\sigma_{26} = 3.65 + 53.83 = 57.5 \text{ barns.}$$

1.2. Composition Changes During Irradiation

1.2.1. Equations. The nuclide concentrations, excess neutron reactivity, and burnup of the fuel, are expressed as functions of the flux-time θ to which a local section of the fuel has been uniformly irradiated. The corresponding assumptions are stated in Sec. I.2.1.1.

The nuclide concentrations vary with the flux time θ to which the fuel has been irradiated, according to Eqs. (I-3) through (I-8); i.e.,

$$N_{25} = N_{25}^0\, e^{-\sigma_{25}\theta} ,$$

$$N_{Xe}\, \sigma_{Xe} + N_{Sm}\, \sigma_{Sm} = r\, N_{25}\, \sigma_{25} ,$$

$$N_{26} = N_{25}^0\, \frac{1}{1+a_{25}}\, (1-e^{-\sigma_{25}\theta}),$$

and

$$N_F = N_{25}^0\, \frac{a_{25}}{1+a_{25}}\, (1-e^{-\sigma_{25}\theta}) .$$

The burnup $\beta(\theta)$ is defined here as the fraction of U^{235} atoms destroyed; it is simply

$$\beta(\theta) = 1 - e^{-\sigma_{25}\theta} . \qquad (5)$$

The macroscopic cross section for production of thermal neutrons is

$$\nu_{th}\Sigma_f = \nu\epsilon\, P_{th}\, p\, N_{25}\,\sigma_{f,25}$$

$$= \frac{\nu\epsilon\, P_{th}\, p}{1+\alpha_{25}}\, N^0_{25}\sigma_{25}e^{-\sigma_{25}\theta}\ ,$$

and the macroscopic cross section for removal of the thermal neutrons is

$$\Sigma_a + DB_0^2$$

$$= N_{25}\sigma_{25}+N_{Xe}\sigma_{Xe}+N_{Sm}\sigma_{Sm}+N_F\sigma_F+N_{26}\sigma_{26}+N_p\sigma_p+DB_0^2$$

$$=(1+r)N^0_{25}\sigma_{25}e^{-\sigma_{25}\theta} + \frac{\alpha_{25}\sigma_{26}+\sigma_F}{1+\alpha_{25}}\, N^0_{25}(1-e^{-\sigma_{25}\theta})+N_p\sigma_p+DB_0^2\ .$$

The excess neutron production ∇ is then the function of θ defined by

$$\nabla(\theta) = \nu_{th}\Sigma_f - \Sigma_a - DB_0^2\ .$$

According to Eq. (II-18), the dimensionless excess neutron production[*] is

$$g(\theta) = \nabla(\theta)/DB_0^2 = g_p(\theta) - g_r(\theta)\ , \tag{6}$$

with

$$g_p(\theta) = \frac{\nu_{th}\Sigma_f}{DB_0^2} = \frac{N^0_{25}\sigma_{25}}{DB_0^2}\ \frac{\nu\epsilon\, P_{th}\, p}{1+\alpha_{25}}\ e^{-\sigma_{25}\theta}\ , \tag{7}$$

and

$$g_r(\theta) = \frac{N^0_{25}\sigma_{25}}{DB_0^2}\left\{\left[1 + r - \frac{\alpha_{25}\sigma_{26}+\sigma_F}{(1+\alpha_{25})\sigma_{25}}\right] e^{-\sigma_{25}\theta} + \frac{N_p\sigma_p+DB_0^2}{N^0_{25}\sigma_{25}} + \frac{\alpha_{25}\sigma_{26}+\sigma_F}{(1+\alpha_{25})\,\sigma_{25}}\right\} \tag{8}$$

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[*] For use in Sec. III, one should multiply the right side of Eqs. (6) through (8) by B_0^2/B_Z^2 , according to Eq. (III-62).

1.2.2. Fuel-cycle parameters. The burnup and excess neutron production are now functions represented by the sum of a constant and of an exponential function of the flux time. The numerical values of the corresponding parameters g_0 and g_{25} are computed by Eqs. (6) through (8).

The above reference design corresponds to the following values:

$$\left.\begin{aligned} g_p(\theta) &= 44.95\, e^{-\sigma_{25}\theta}, \\ g_r(\theta) &= 8.98 + 30.56\, e^{-\sigma_{25}\theta}, \\ \text{and} \quad g(\theta) &= -\,8.98 + 14.39\, e^{-\sigma_{25}\theta}. \end{aligned}\right\} \quad (9)$$

These functions represent the neutron production, the neutron removal, and the excess neutron production in a reactor loaded with the same amount of U^{235} as is the reference design reactor.

In order to reduce the variation of the excess neutron production of the fuel, it is customary to add in the reactor a "burnable poison" which burns out more quickly than U^{235}. Figure IV-1 shows the variation of the neutron production, the neutron removal and the excess neutron production with and without burnable poison. Analytically, one adds to $g_r(\theta)$ [and subtracts from $g(\theta)$] a term $g_B e^{-\sigma_B\theta}$. In Fig. IV-1, one has chosen for g_B the value which cancels the initial excess neutron production.

1.2.3. Application. The results of the parametric study performed in Sec. II.4 can readily be applied to the above reactors, with

$$(g_0+g_{25})/g_{25} = 0.376, \text{ and } g_{25} = 14.39 .$$

The following values of the average burnup of the fuel at end of life correspond to the bare cylindrical reactor of equal height and diameter with no burnable poison.

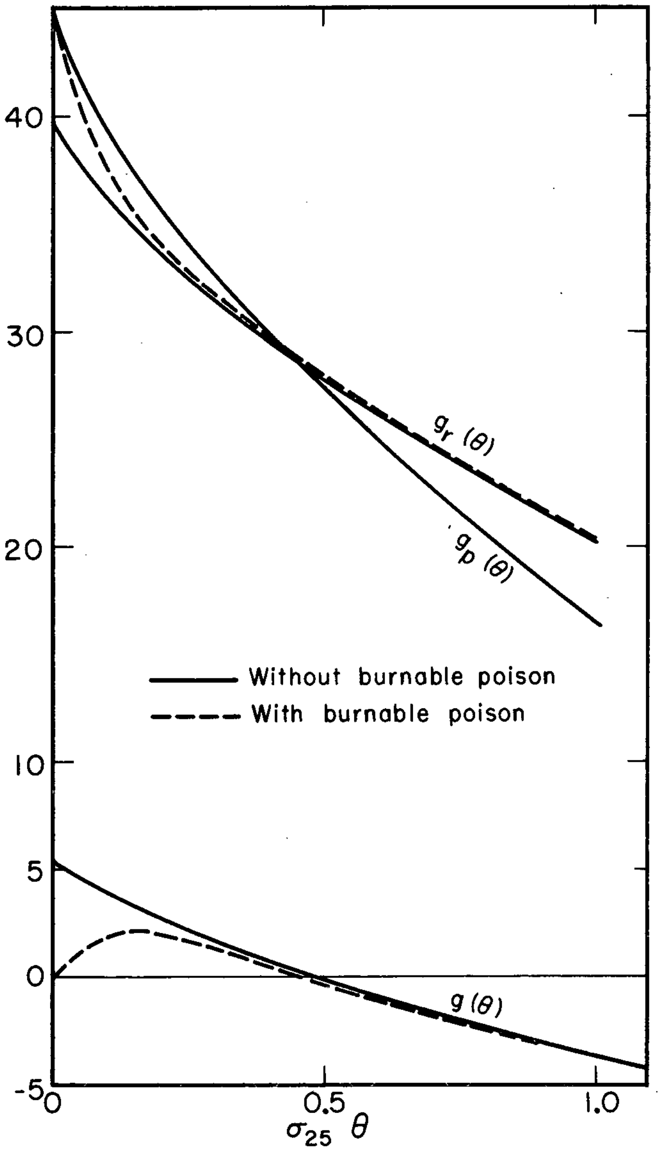

Fig. IV-1. Variation of neutron production, neutron removal, and excess-neutron production with irradiation.

Average burnup at end of batch irradiation of fixed fuel.

Zero-dimensional	Perturbation method		Finite differences		
	1st order	2nd order	35[a]	49[a]	100[a]
0.376	0.182	0.254	0.250	0.244	0.241

Finite differences computations predict an exact value of 0.238. The first-order perturbation, although more accurate than the zero-dimensional approximation, is still in error by 24%. The second-order perturbation yields as accurate a result as does finite differences computations using less than 35 mesh points, and thus should be useful for survey studies.

The above results illustrate the use of the perturbation method. However, they correspond to a one-group analysis which is not adequate in the present case, as will be shown in the next section.

2. Two-Group Diffusion Equations

2.1. Assumptions and Equations

The neutron distribution is now described by a fast flux ϕ_1 and a thermal flux ϕ.

The fast flux is caused by neutrons of all energies above a thermal cutoff energy. These neutrons are produced at a rate $\epsilon\nu\Sigma_f\phi$, and the properties of the fast neutrons can be characterized (see Ref. W3, p. 502) by a diffusion coefficient D_1 and a removal (slowing-down) cross-section approximated by D_1/τ, where τ is the age of the neutron from fission energies to the thermal cutoff energy. Thus, the fast flux satisfies the equation

$$-D_1\,\Delta\phi_1 + \frac{D_1}{\tau}\,\phi_1 = \epsilon\nu\Sigma_f\phi\ . \tag{10}$$

By assuming that all resonance absorptions take place at the thermal cut-off energy, the production rate of thermal neutron is $p\frac{D_1}{\tau}\,\phi_1$, and the thermal flux satisfies the equation

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[a] Number of spatial mesh points; 12 time steps have been used.

$$-D\Delta\phi + (\Sigma_a + \Sigma_E)\,\phi = p\frac{D_1}{\tau}\,\phi_1\ . \tag{11}$$

It is assumed that the diffusion coefficients D and D_1 are constant in the volume of the reactor; otherwise, one should have written div(D grad ϕ) instead of $D\Delta\phi$.

The following applies to bare reactors and to reactors whose reflector can be represented by a reflector saving.* Thermal and fast fluxes are assumed to satisfy the same linear homogeneous boundary condition, Eq. (I-13), on the outer surface of the reactor core. Since both functions ϕ and ϕ_1 satisfy this boundary condition, they can be easily expanded in the eigenfunctions v_k of the corresponding eigenvalue equation, Eq. (I-39).

Consider first the following linear operator

$$L_1\phi_1 = -\tau\,\Delta\phi_1 + \phi_1\ . \tag{12}$$

The operator L_1 is such that the self-adjoint homogeneous equation $L_1\phi_1 = 0$ does not have any nontrivial solution; therefore, it has an inverse L_1^{-1}, and Eq. (10) becomes

$$\frac{D_1}{\tau}\,\phi_1 = L_1^{-1}\,(\epsilon\nu\,\Sigma_f\,\phi)\ . \tag{13}$$

For instance, using the eigenfunction technique yields

$$\frac{D_1}{\tau}\,\phi_1 = \sum_{k=0}^{\infty} \frac{\langle \epsilon\nu\Sigma_f\,\phi,\ v_k\rangle}{(1+B_k^2\,\tau)\,\langle v_k, v_k\rangle}\ v_k\ . \tag{14}$$

An equation for the thermal flux is now obtained from Eqs. (11) and (13); it can be written

$$-D(\Delta\phi + B_0^2\phi) = p\ L_1^{-1}\,(\epsilon\,\nu\,\Sigma_f\phi) - (\Sigma_a + DB_0^2 + \Sigma_E)\ \phi\ , \tag{15}$$

while the one-group diffusion equation reads

---

*Such reactors are considered by the FUELMOVE Computer Code.^{M4}

$$-D(\Delta\phi + B_0^2\phi) = (\epsilon\nu P_{th}p\,\Sigma_f - \Sigma_a - DB_0^2 - \Sigma_E)\,\phi\,. \qquad (16)$$

The one-group model thus replaces the operator L_1^{-1} by a scalar, the probability P_{th} that a fast neutron escapes leakage during the slowing down process. For the uniformly loaded reactor, the two-group model defines the fast non-leakage probability by $P_{th}=1/(1+B_0^2\tau)$. The definition of ν_{th}, Eq. (I-10), is now

$$\nu_{th} = \epsilon\nu P_{th}p = \frac{\epsilon\nu p}{1+B_0^2\tau}\,. \qquad (17)$$

The two-group diffusion equation, Eq. (15), is now set in dimensionless form. As an example, the equation corresponding to batch irradiation of unmixed fuel is derived below by extending the definitions of Sec. II.1.1.

A thermal flux shape u(x, t) and an irradiation variable T are defined as follows by their relation to the thermal flux-time $\theta(x,t)$:

$$\theta(x,t) = \int_0^t \phi(x,T)dt = \int_0^T u(x,T)dT\,. \qquad (18)$$

The macroscopic cross sections Σ_f and Σ_a are assumed to be known functions of the thermal flux time to which the fuel has been irradiated; one defines then two dimensionless functions

$$g_p(\theta) = \frac{\nu_{th}\,\Sigma_f}{DB_0^2} \qquad \text{and} \quad g_r(\theta) = \frac{\Sigma_a + DB_0^2}{DB_0^2}\,, \qquad (19)$$

which represent respectively the production and the removal of thermal neutrons, in units of the initial thermal leakage. The corresponding definition of the dimensionless excess neutron production $g(\theta)$ and control absorption c is

$$g(\theta) = g_p(\theta) - g_r(\theta) \quad \text{and} \quad c = \frac{\Sigma_E}{DB_0^2}\,. \qquad (20)$$

With the above notations, and under the assumption that the resonance absorption p be uniform in the volume of the reactor core, Eqs. (15) and (17) yield

$$-\left(\frac{1}{B_0^2}\,\Delta u + u\right) = (1 + B_0^2\tau)\, L_1^{-1}[\,g_p(\theta)u] - [\,g_r(\theta)+c]\; u\,. \tag{21}$$

Finally, dimensionless coefficients ρ_k are defined by

$$\rho_k = \frac{1 + B_0^2\tau}{1 + B_k^2\tau}\,, \tag{22}$$

and the eigenfunction expansion of the right side of Eq. (21) is obtained from Eqs. (13) and (14). Thus, the thermal flux shape u(x, T) is the solution of the equation

$$-\left(\frac{1}{B_0^2}\,\Delta u + u\right) = \sum_{k=0}^{\infty} \frac{\left\langle [\,\rho_k g_p(\theta) - g_r(\theta) - c]\; u, v_k\right\rangle}{\left\langle v_k, v_k\right\rangle}\, v_k\,. \tag{23}$$

The above equation can also be written:

$$-\left(\frac{1}{B_0^2}\,\Delta u + u\right) = [\,g(\theta) - c]\; u - \sum_{k=1}^{\infty} \omega_k\, \frac{\left\langle g_p(\theta)u, v_k\right\rangle}{\left\langle v_k, v_k\right\rangle}\, v_k, \tag{24}$$

where one has made use of Eq. (20) and defined coefficients ω_k by

$$\omega_k = 1 - \rho_k = \frac{(B_k^2 - B_0^2)\,\tau}{1 + B_k^2\tau} = 1\Big/\left(1 + \frac{1 + B_0^2\tau}{(B_k^2 - B_0^2)\tau}\right). \tag{25}$$

The one-group diffusion equation, Eq. (II-20), would be obtained by setting $\omega_k \equiv 0$. The infinite sum on the right side of Eq. (24) represents the net deficit in the production of thermal neutrons resulting from their diffusion during slowing down; it is orthogonal to the unperturbed flux v_0, and thus it does affect the flux-shape changes. However, the criticality condition is always

$$\left\langle [\,g(\theta) - c]\; u, v_0\right\rangle = 0\,.$$

Multigroup diffusion equations, with energy independent boundary conditions, as well as the continuous slowing down model, still yield Eq. (24). Only the definition of the coefficients ω_k need be modified (see Appendix B).

2.2. Computation Procedure.

The perturbation method can now be used to yield an approximate solution of Eq. (24) with only slight modifications of the procedure developed in the preceding sections.

The flux shape is expanded in the eigenfunctions $v_k(x)$, and its first-order perturbation $u_1(x, T)$ is the solution of Eq. (24) where u and θ are replaced by $v_0(x)$ and $Tv_0(x)$.

2.2.1. Batch irradiation of unmixed fuel with uniform control absorber. In the calculation procedure, we use Eqs. (II-109) through (II-122) except for the following modifications: Eq. (II-116) is replaced by

$$a_k(T) = \frac{B_0^2}{B_k^2 - B_0^2} \left[s_k(T) - \omega_k \, s_{p,k}(T) \right], \text{ where } k \geq 1, \tag{26}$$

and Eq. (II-121) is replaced by

$$\alpha_k(T) = \frac{B_0^2}{B_k^2 - B_0^2} \left[S_k(T) - \omega_k \, S_{p,k}(T) \right], \text{ where } k \geq 1. \tag{27}$$

The function $s_{p,k}(T)$, which can be computed exactly like the function $s_k(T)$, is defined by

$$s_{p,k}(T) = \frac{\langle g_p(v_0 T) \, v_0, v_k \rangle}{\langle v_k, v_k \rangle} \quad \text{and} \quad S_{p,k}(T) = \frac{1}{T} \int_0^T s_{p,k}(T) dT. \tag{28}$$

2.2.2. Continuous fueling. When the composition of a cylindrical reactor core varies only with the axial coordinate z, the radial shape of both the fast and the thermal flux is the same and it does not vary during the irradiation; this requires energy independent boundary conditions on the curved surface of the cylindrical core. The flux is then separable in the axial and radial coordinates.

Thus, considering for instance the graded irradiation of the fuel, only the last of Eqs. (III-128) through (III-133) is modified as follows:

$$a_k(T) = \frac{\gamma_0^2}{\gamma_k^2 - \gamma_0^2} [S_k(T) - \omega_k S_{p,k}(T)] . \qquad (29)$$

2.3. Example 1: U^{235} Light-Water Reactor

Computations have been performed for graded irradiation of the fuel in the U^{235} light-water reactor previously described. According to the note below Eq. (6), the right side of Eq. (9) is multiplied by $B_0^2/B_Z^2 = 3.34$ and one obtains

$$g(\theta) = -30 + 48e^{-\sigma_{25}\theta}, \quad \text{and} \quad g_p(\theta) = 150e^{-\sigma_{25}\theta} .$$

2.3.1. One-group analysis. The one-group analysis is performed according to Eqs. (III-128) through (III-132). The first-order approximation of the flux time at steady state is

$$\sigma_{25}\theta_0(\zeta) = 1.228 \cos\frac{\pi}{2}\zeta$$

and the corresponding value of the average flux time is $\sigma_{25}\langle\theta_0\rangle = 0.782$. The second-order perturbation correction to the flux time is

$$\sigma_{25}\theta_1(\zeta) = 1.228 [0.1184 \cos\frac{\pi}{2}\zeta - 0.3534 \cos 3\frac{\pi}{2}\zeta$$

$$+0.0266 \cos 5\frac{\pi}{2}\zeta - 0.0047 \cos 7\frac{\pi}{2}\zeta$$

$$+0.0013 \cos 9\frac{\pi}{2}\zeta + \cdots] .$$

The average value of the flux time is then

$$\sigma_{25}\langle\theta_0 + \theta_1\rangle = 1.228 \frac{2}{\pi}(1 + 0.242) = 0.971 .$$

This is 24.2% larger than predicted by the first order perturbation theory (the average burnup of the fuel is about 20% larger).

2.3.2. Two-group analysis. The two-group diffusion model results in a modification of the second-order perturbation of the flux-time. The use of Eq. (29) instead of Eq. (III-132) yields

$$\sigma_{25}\theta_1(\zeta) = 1.228\,[\,-0.0358\cos\tfrac{\pi}{2}\zeta + 0.1000\cos 3\tfrac{\pi}{2}\zeta - 0.0296\cos 5\tfrac{\pi}{2}\zeta + 0.0071\cos 7\tfrac{\pi}{2}\zeta - 0.0023\cos 9\tfrac{\pi}{2}\zeta + \cdots\,]\ ,$$

and

$$\sigma_{25}\left\langle\theta_0 + \theta_1\right\rangle = 1.228\,\frac{2}{\pi}\,(1 - 0.076) = 0.722\ .$$

The average flux time is now 7.6% smaller than predicted by the first-order perturbation theory.

2.3.3. Discussion of the results. The flux time at steady state (proportional to the flux shape) is plotted in Fig. IV-2. The very large difference between one-group and two-group results can be explained as follows:

Consider for instance the central region of the reactor, where U^{235} is strongly depleted. According to the one-group model, thermal neutrons are produced in this region proportionally to $\nu_{th}\,\Sigma_f$ and absorbed proportionally to Σ_a. During the irradiation of the fuel, $\nu_{th}\,\Sigma_f$ has decreased more than Σ_a did: there is a net deficit of neutrons, $\nu_{th}\,\Sigma_f - \Sigma_a$, resulting in a minimum in the flux.

According to the two-group model, some fast neutrons, produced at higher rate by fissions in the adjoining regions of the reactor, thermalize in the central region; this increases the net production of thermal neutrons in this region, above the value predicted by one-group theory. This increase can be more than large enough to compensate for a deficit of neutrons proportional to $\nu_{th}\,\Sigma_f - \Sigma_a$, thus resulting in an increase in the flux in the central region.

The continuous slowing down model, Appendix B, yields results which differ even more from those obtained by the one-group analysis, because it increases the fast leakage probability, i. e., the diffusion of the fast neutrons.

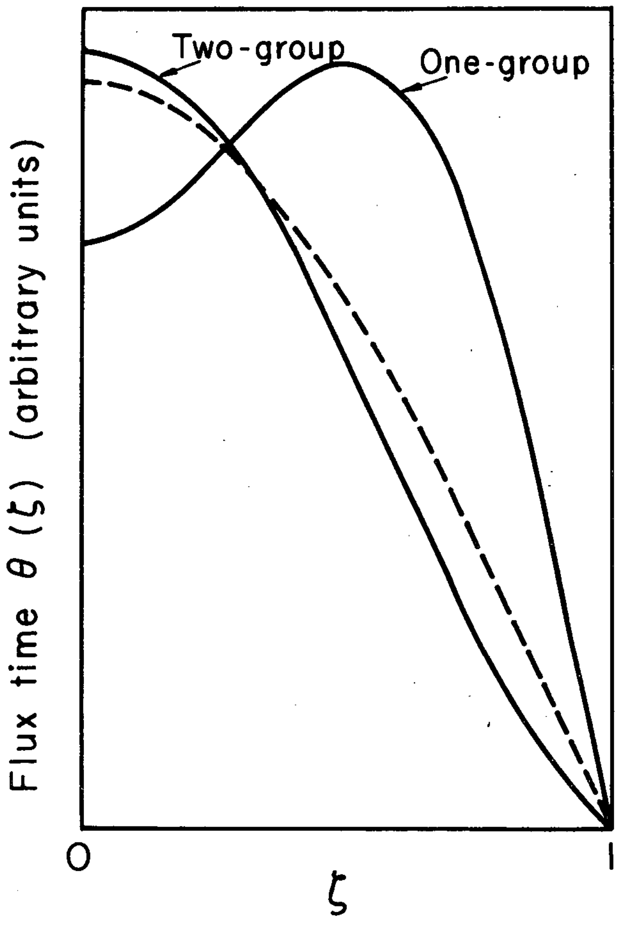

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Fig. IV-2. Spatial variation of the flux time for graded irradiation of U^{235} fuel. Dashed curve represents zeta first-order perturbation values; solid curves represent second-order perturbation values.

Addition of burnable poison to the fuel tends to decrease the flux where it is already smaller; i.e., where this poison has been burned up to a lesser extent. Thus, in Fig. IV-2, the flux shape corresponding to the one-group diffusion model would be closer to the unperturbed flux shape and that corresponding to the two-group diffusion model would be more peaked at the center.

2.4. Example 2: Natural Uranium Heavy-Water Reactor

Consider the graded irradiation of natural uranium fuel in the Candu Reactor. The reference design is to be found in Nuclear Chemical Engineering[B1] together with the formulae for computing the variation of the nuclide concentrations on irradiation. Computations leading to the determination of functions such as the above defined $g(\theta)$, $g_p(\theta)$, and $\beta(\theta)$ are illustrated by Davidson.[D3] We have obtained the following results for the Candu Reactor with the reference design loading of natural uranium:

$$\frac{\nabla(\theta)}{N^*_{25}\sigma_{25}} = -0.2356 - 0.0411\,\theta + 2.2269\,e^{-\sigma_{25}\theta} + 5.2057\,e^{-\sigma_{49}\theta} - 6.1417\,e^{-\sigma_{40}\theta} - 0.9599\,e^{-\sigma_{41}\theta},$$

and

$$\frac{\nu\Sigma_f(\theta)}{N^*_{25}\sigma_{25}} = 1.9064 + 7.1509\,e^{-\sigma_{25}\theta} + 18.4351\,e^{-\sigma_{49}\theta} - 23.2431\,e^{-\sigma_{40}\theta} - 2.1847\,e^{-\sigma_{41}\theta}.$$

The neutron burnout of U^{236} has been neglected and the symbol σ_{49} here represents the exponent denoted as $\gamma\,\sigma_{49}$ by Benedict and Pigford;[B1] when the flux time θ is expressed in neutrons per kilobarns, the various exponents in the above equation are

$$\sigma_{25} = 0.5670,\quad \sigma_{49} = 0.8811,\quad \sigma_{40} = 0.7711,\ \text{and}\ \sigma_{41} = 1.3387.$$

The functions $g(\theta)$ and $g_p(\theta)$ are then obtained according to

$$g(\theta) = \nabla(\theta)/DB_0^2 \quad\text{and}\quad g_p(\theta) = \nu_{th}\Sigma_f/DB_0^2, \tag{30}$$

with the following values (see Ref. B1):

$$DB_0^2/N_{25}^*\sigma_{25} = 0.0392, \; B_0^2\tau = 0.0156, \text{ and } \nu_{th}/\nu = \epsilon P_{th} p = 0.8968.$$

According to Eq. (III-62), for the following second-order perturbation computations performed for graded irradiation of the fuel, one multiplies the right sides of Eq. (30) by $B_0^2/B_Z^2 = 2.7885$.

2.4.1. One-group analysis. We use Eqs. (III-128) through (III-132) to obtain the approximate value of the flux time at steady state; the first-order perturbation result is

$$\theta_0(\zeta) = 3.065 \cos \frac{\pi}{2}\zeta ,$$

and the corresponding value of the average flux time is $\langle\theta_0\rangle = 0.1951$. The second-order perturbation correction to the flux time is

$$\theta_1(\zeta) = 3.065 \,(0.1133 \cos \frac{\pi}{2}\zeta - 0.2828 \cos 3\frac{\pi}{2}\zeta + 0.0124 \cos 5\frac{\pi}{2}\zeta - 0.0004 \cos 7\frac{\pi}{2}\zeta + \cdots),$$

and the average value of the flux time at steady state is now

$$\langle\theta_0 + \theta_1\rangle = 3.065 \frac{2}{\pi} (1 + 0.2101) = 0.2361 \text{ neutrons/kilobarn.}$$

Thus, it is 21% larger than the value predicted by the first-order perturbation theory.

2.4.2. Two-group analysis. When Eq. (III-132) is replaced by Eq. (29), the second-order perturbation correction of the flux time becomes

$$\theta_1(\zeta) = 3.065 \,(0.1134 \cos \frac{\pi}{2}\zeta - 0.2827 \cos 3\frac{\pi}{2}\zeta + 0.0136 \cos 5\frac{\pi}{2}\zeta - 0.0015 \cos 7\frac{\pi}{2}\zeta - 0.0004 \cos 9\frac{\pi}{2}\zeta + \cdots).$$

The average value of the flux time is now

$$\langle\theta_0 + \theta_1\rangle = 3.065 \frac{2}{\pi} (1 + 0.2105) = 0.2362 \text{ neutrons/kilobarns.}$$

It is again 21% larger than the value predicted by the first-order perturbation theory.

2.4.3. Discussion of the results. One-group and two-group analyses yield results which agree to the third decimal; the agreement is well within the accuracy required in the above fuel cycle computations. This is of importance because one-group analyses are simpler; it suggests the usefulness of the analytical solutions which could be obtained in terms of elliptic functions (cf. Sec. III.6). The agreement of one-group and two-group results (for energy-independent boundary conditions) is due to the following:

(1) The fast-leakage probability is small; thus, the coefficients ω_k in Eq. (24) or (26) are small.

(2) The variation of the total fission cross section on irradiation is small because of the production of fissile isotopes of plutonium; a constant fission cross section $[g_p(\theta) = \text{constant}]$ would correspond to $s_{p,k}(T) = 0$ in Eq. (26).

The second-order perturbation method yields here a correction of 21% of the average flux time predicted by the first-order perturbation theory. A similar correction has been obtained by Shanstrom,[S3] (p. 205) in the case of graded irradiation of low enrichment uranium fuel in the Yankee Atomic Electric Reactor.

V. CONCLUSIONS

1. Perturbation Method

The first-order perturbation theory had previously been developed as an important tool for fuel-cycle analysis, because it yields results in an analytical form which can be easily used for parametric studies.

It is improved as follows by the present use of a generalized perturbation theory:

(a) The flux changes can now be described and their effects on reactivity lifetime and burnup of the fuel taken into account.

(b) A successive approximations scheme is obtained; since the equation giving each approximation can be chosen in the simplest possible form, computations are relatively easy. The accuracy obtained by using first- and second-order approximations has been determined for a wide variation of the physical parameters. The second-order approximation gives more accurately the average quantities, like the reactivity lifetime and the average burnup of the fuel, than the local quantities, like the flux shape (see Sec. II.4.7).

(c) The effect of a variation of any physical parameter can be studied analytically. Many practical cases are now amenable to hand computations by the perturbation method. This should be of particular importance in survey studies of various fuel-cycling schemes. This method also suggests a simple approach towards programing such studies on digital computers. Functions have been tabulated which allow the study of fuel burnup and reactivity lifetime for one- and two-dimensional reactors to the second-order approximation.

2. Fuel-Cycle Results

Application of the one-group and two-group perturbation methods to typical reactor problems has illustrated the importance of an accurate description of neutron transport with respect to energy and space for small thermal reactors with large fast leakage. In such case, the one-group model may result in large errors in the predicted flux shape at the end of irradiation. However, the one-group model yields useful and reasonable results for large thermal reactors with relatively little leakage.

Parametric studies emphasize the importance of the neutron leakage in keeping small the variations of the spatial distribution of the neutrons during the irradiation. Large leakage favors the use of first-order perturbation methods. The first-order perturbation predicts a smaller or a larger burnup according to the reactor type and the fuel scheduling scheme considered; its results are not always on the conservative side.

Our study has indicated that there could exist an optimum ratio of radius to height of a cylindrical reactor core such that maximum reactivity lifetime and fuel burnup takes place for a given initial fuel loading.

An exact analytical solution for the reactivity lifetime and fuel burnup has been developed for continuous fueling schemes, provided the one-group model applies and provided the characteristic excess neutron production of the fuel varies as a quadratic function of the flux time of irradiation exposure. A comparative study of various continuous fueling schemes has been made for fuel with typical properties (see Sec. III-6).

NOMENCLATURE

a_k	Coefficients of the eigenfunction expansion of u, Eq. (I-40)
A_k	Coefficients of the eigenfunction expansion of θ, Eq. (II-92)
b_j	Coefficients in the function representing the burnup, Eq. (I-31)
B_k^2	Eigenvalue of Helmholtz Equation, Eq. (I-39)
B_0	Geometrical buckling, Eq. (I-19)
c	Dimensionless control absorption, Eq. (II-19) or (III-63)
C	Time integral of the dimensionless control absorption, Eq. (III-13)
$C_{m,n}$	Coefficients defined by Eq. (II-243)
D	Diffusion coefficient, Eq. (I-10)
E	Auxiliary functions defined by the following equations in Sec. II: $E_0(w)$, Eq. (137); $E_{1,k}(w)$, Eq. (140); $E_{2,k}(w)$, Eq. (128); $E_{3,k}(w)$, Eq. (134)
f	Auxiliary coefficients defined by the following equations in Sec. II: f_0, Eq. (138); $f_{1,k}$, Eq. (139); $f_{2,k}$, Eq. (129)
f(t)	Flux magnitude factor, Eq. (I-33)
F	Right side of Eq. (II-34) or (III-19)
g	Dimensionless excess neutron production, Eq. (II-18) or (IV-62)
g_j	Coefficient in the function $g(\theta)$, Eq. (II-125) or Eq. (III-150)
G	Integral excess neutron production, Eq. (III-67)
h_k	Dimensionless coefficients, Eq. (II-109)
j_ℓ	Eigenvalue, $(\ell+1)^{th}$ root of Eq. (II-213)

k, k'	Modulus and complementary modulus of Jacobi elliptic functions
K	Complete elliptic integral of first kind
L	Linear operator, Eq. (II-33)
N_i	Nuclide concentration
$N_k(w)$	Functions defined by Eq. (II-266)
p	Resonance escape probability
P_{th}	Nonleakage probability for fast neutrons
r	Poisoning ratio in Secs. I and IV; Eq. (I-8)
r	Radical coordinate, in Sec. II
R	Radius of the reactor core
$s_k(T)$	Function defined by Eq. (II-110)
$S_k(T)$	Function defined by Eq. (II-111)
t	Time after startup of the reactor
T	Irradiation parameter
u	Flux shape, Eq. (I-33)
v_k	Eigenfunction of Helmholtz Equation, Eq. (I-39)
w	Dimensionless irradiation variable, Eq. (II-255)
x, y, z	Spatial coordinates
Z	Half thickness of a slab or half height of a cylindrical reactor core

Greek Letters

α_{25}	Capture to fission ratio in U^{235}, Eq. (I-4)
α_k	Coefficients in the eigenfunction expansion of the flux time Eq. (I-102)
β	Burnup, Eq. (I-31)
γ_k	Eigenvalue, $(k+1)^{th}$ root of Eq. (II-152)
$\delta(x)$	Dirac function, Eq. (IV-72)
$\delta_\ell, \delta_R, \delta_Z$	Linear extrapolation distance, Eq. (I-13)
$\delta_{i,j}$	Kronecker symbol, Eq. (II-131)
Δ	Laplacian operator
∇	Excess neutron production, Eq. (I-22)
ϵ	Perturbation parameter in Secs. II and III, Eq. (II-23)
ϵ	Fast fission factor in Sec. IV
ϵ_R, ϵ_Z	Ratio of linear extrapolation distance to core dimensions
ζ	Reduced axial coordinate, Eq. (II-150)
η	Reduced radial coordinate, Eq. (II-207)
θ	Flux time, Eq. (I-1)
θ_f	Flux time to which completely mixed fuel could be irradiated batchwise
μ_k	Coefficients defined by Eq. (II-142)
ν	Average number of neutrons produced per fission
ν_{th}	Average number of thermal neutrons produced per fission, Eq. (I-10)
ρ	Radial coordinate (II-176)
ρ_k	Coefficients for multigroup analysis, Eq. (IV-22)

σ_i	Microscopic cross section
$\Sigma_a, \Sigma_E, \Sigma_f$	Macroscopic cross sections, Eq. (I-10)
τ	Fermi age, in Sec. IV
ϕ	Neutron flux, Eq. (I-1)
ω_k	Coefficients for multigroup analysis, in Sec. IV, Eq. (IV-24)

Subscripts

i, j, k, l, m, n, p	have been used as integer subscripts; k always refers to the $(k+1)^{th}$ solution of the Helmholtz equation, Eq. (I-39)
R	Radial
Z	Axial

Note: Symbols which are used only in the section where they are defined are not listed in this nomenclature.

Equations have been numbered consecutively with arabic numbers starting with one at the beginning of each chapter. Roman numerals always refer to section numbers. Equations referenced out of their proper section are preceded by their correct section number [e. g., Eq. (II-121) would be so referenced in Section III] .

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APPENDIX

A. Scalar Product of Functions; Adjoint Operators

The scalar product of two functions f and g defined in a volume V is defined and denoted as

$$\langle f, g\rangle = \frac{1}{V}\int_V f(x)\, g(x) dV \ .$$

According to this notation, the average value of a function f in the volume V is

$$\langle 1, f\rangle .$$

Two functions are said to be orthogonal if their scalar product is equal to zero.

A linear operator L acting on functions defined in a volume V and satisfying some given boundary conditions on the surface S of V is self-adjoint if, for any two such functions u and v, one has

$$\langle u, \ Lv\rangle = \langle Lu, v\rangle .$$

The Laplacian operator Δ, acting on functions which satisfy linear homogeneous boundary conditions, is self-adjoint with respect to the above defined scalar product. This property is a consequence of Green's formula

$$\int_V (u \ \Delta v) dV = \int_V (v \ \Delta \ u) dV + \int_S (u\frac{dv}{dn} - v\frac{du}{dn}) \ dS,$$

since the last integrand on the right side vanishes.

Non-self-adjoint operators arise, for instance, in the matrix representation of the multigroup equations.[W3] An adjoint operator L^* is then defined[F2] by the equation

$$\langle u, L^* v\rangle = \langle Lu, v\rangle .$$

B. Continuous Slowing Down; Fermi Age Equation

Let τ be the "age" of a neutron[W1] and $q(x, \tau)$ the number of neutrons per unit volume slowing down past the energy corresponding to the age τ in unit time. Then, if the slowing down properties of the reactor do not vary with the position, the function $q(x, \tau)$ satisfies the age equation

$$\Delta q(x, \tau) - \frac{\partial q(x, \tau)}{\partial \tau} = \epsilon \nu \Sigma_f \phi \delta(\tau) . \qquad (1)$$

The function $\delta(\tau)$ is the fission energy spectrum; when all the fissions are assumed to yield neutrons of the same energy (corresponding then to $\tau = 0$), $\delta(\tau)$ is the Dirac δ function, and the above equation can be replaced by

$$\Delta q(x, \tau) - \frac{\partial q(x, \tau)}{\partial \tau} = 0 , \qquad (2)$$

with

$$q(x, 0) = \epsilon \nu \Sigma_f \phi . \qquad (3)$$

By assuming that the slowing down density $q(x, \tau)$ satisfies a boundary condition which is independent of the age τ, we can expand $q(x, \tau)$ in the eigenfunctions v_k of the corresponding eigenvalue equation, Eq. (I-39). Letting

$$q(x, \tau) = \sum_{k=0}^{\infty} q_k(\tau) v_k(x), \qquad (4)$$

Eqs. (2) and (3) yield

$$q_k(\tau) - B_k^2 \frac{dq_k(\tau)}{d\tau} = 0 \qquad (5)$$

with

$$q_k(0) = \frac{\langle \epsilon \nu \Sigma_f \phi, v_k \rangle}{\langle v_k, v_k \rangle} . \qquad (6)$$

The solution of the above equations is

$$q_k(\tau) = e^{-B_k^2 \tau} \frac{\langle \epsilon \nu \Sigma_f \phi, v_k \rangle}{\langle v_k, v_k \rangle} . \qquad (7)$$

The rate of production of thermal neutrons is now $pq(x,\tau_{th})$, where τ_{th} is the age from fission to thermal cutoff energy and p is the resonance escape probability (assumed uniform). Then, Eqs. (4) and (7) yield

$$pq(x, \tau_{th}) = \sum_{k=0}^{\infty} e^{-(B_k^2 - B_0^2)\tau} \frac{\langle \nu_{th} \Sigma_f \phi, v_k \rangle}{\langle v_k, v_k \rangle} v_k(x), \tag{8}$$

where ν_{th} is defined by Eq. (IV-17) and one sets $P_{th} = e^{-B_0^2 \tau_{th}}$, which is the fast nonleakage probability corresponding to the Fermi-age model in the uniformly loaded reactor.

The term $pq(x, \tau_{th})$ corresponds now to the term $p\frac{D_1}{\tau}\phi_1$ in the equation for the thermal flux, Eq. (IV-11). Thus, the coefficients ω_k in Eq. (IV-24) shall now be replaced by the coefficients ω_k^F defined by

$$\omega_k^F = 1 - e^{-(B_k^2 - B_0^2)\tau} . \tag{9}$$

Numerical applications.

U^{235} Reactor. We consider graded irradiation of the fuel in the reactor of Sec. IV. 2.3. Thus,

$$\omega_k^F = 1 - e^{-B_Z^2 \tau (\gamma_k^2 - \gamma_0^2)/\gamma_0^2} , \text{ and } B_Z^2 \tau = 0.1228, \ \frac{\gamma_k^2 - \gamma_0^2}{\gamma_0^2} = 4k(k+1) .$$

Values of ω_k^F are compared below with the values of ω_k given by Eq. (IV-25):

k	1	2	3	4
ω_k	0.4105	0.6761	0.8071	0.8741
ω_k^F	0.6256	0.9475	0.9972	0.9999

The difference between two-group and Fermi - age results is $(\omega_k^F - \omega_k)/\omega_k$ times that between one-group and two-group results; As an example, the following table gives the results for the value of the second-order perturbation of the average flux time to which the fuel can be irradiated; the results are in units of the average flux time given by first-order perturbation method.

	One-group	Two-group	Fermi - age
$\langle \theta \rangle_1 / \langle \theta \rangle_0$	0.242	-0.076	-0.241

This illustrates how important it is to describe as accurately as possible the diffusion of the fast neutrons before any attempt can be made to determine the flux-changes resulting from the nonuniform burn-up of the fuel in a nuclear reactor where fast leakage is large: The correction taking into account the departure of the flux shape from its unperturbed value results in an over or underestimation of the average flux time as large as 25%.

Large thermal reactor. The reference design of the Candu Reactor[B2] corresponds to $B_0^2 \tau = 0.0156$ and $B_Z^2 \tau = 0.0056$.

The fast leakage probability is very small, and so are the coefficients ω_k corresponding to the first (and most important) eigenfunctions. For instance, for graded irradiation, Eqs. (9) and (IV-25) yield values of ω_1^F and ω_1 slightly smaller than 0.05, and $(\omega_1^F - \omega_1)/\omega_1 = 0.037$. Thus, the correction brought about by using the two-group diffusion equation instead of the one-group is likely to be small; furthermore, this correction is accurate within a few percent (here about 3.7%) whatever the model (two-groups or Fermi - age) used to describe the diffusion of the neutrons during slowing down.

C. Lumped Fuels

Most fuel-cycle computations have taken into account the effects due to the lumping of the fuel by means of the following homogenization procedure:

Consider, for instance, a cell consisting of a fuel element and of the adjacent moderator; the cross sections of the nuclides in each region are weighted by the relative value of the average flux in this region. An extremely detailed computational model has been developed by Greebler[G1] according to such an homogenization procedure. The underlying assumption is that the nuclides are uniformly distributed in the volume of the fuel lump.

Here, we investigate the effect of the nonuniform distribution of the nuclides in the fuel lump. The nuclide concentrations and the flux are functions of the position in the fuel lump as well as of the irradiation; thus, when computing the reaction rate of the neutrons with a given nuclide, the average value of the product of the nuclide concentration by the flux differs from the product of the average concentration by the average flux in the fuel lump. This has been taken into account by Ioffe,[I2] who introduced a corrective coefficient, here called "effective concentration coefficient;" its variation with irradiation is investigated.

It is shown that the effects of the nonuniform distribution of the nuclides in a fuel lump can be just as important as the effects of the variation of the flux energy spectrum during irradiation. The chosen example is the natural uranium metal NRX rod, whose behavior on irradiation has been studied in much detail by AERE, Harwell, and AECL.[W8] The effective concentration coefficients computed according to a very simplified procedure here developed allow an accurate evaluation of the effects of the nonuniform distribution of U^{235} and Pu^{239}.

1. Reaction Rates in Fuel Elements: General Considerations.

Let t be the time after the beginning of the irradiation, x be the spatial coordinate, and E be the energy of a neutron. The neutron flux per unit energy interval is denoted by $\Phi(x, E, t)$; let $N(x, t)$ be the concentration of a given nuclide and $\sigma(E)$ its cross section for reaction with neutrons of incident energy E. The local reaction rate is

$$A(x,t) = N(x,t)\int_0^\infty \sigma(E)\,\Phi(x,E,t)dE\ . \qquad (1)$$

The average value of a function f(x) in the volume V of a fuel element is defined and noted as

$$\langle f\rangle = \frac{1}{V}\int_V f(x)dV; \qquad (2)$$

then, the average reaction rate in the fuel element is

$$\langle A\rangle = \left\langle N(x,t)\int_0^\infty \sigma(E)\,\Phi(x,E,t)dE\right\rangle. \qquad (3)$$

To study the effects of the nonuniform distribution of a nuclide in the fuel element, we define an "effective concentration coefficient" ζ by

$$\zeta(t) = \langle A\rangle / \langle N\rangle\left\langle\int_0^\infty \sigma(E)\,\Phi(x,E,t)dE\right\rangle, \qquad (4)$$

where $\zeta(t)$ is the coefficient by which the reaction rate computed according to the homogenization procedure (i.e., letting N(x,t) be approximated by $\langle N\rangle$) shall be multiplied in order to yield the effective reaction rate which takes into account the nonuniform distribution of the nuclide.

For each nuclide, the cross sections $\sigma(E)$ are usually known functions of the energy E.

The nuclide concentrations N(x,t) are the solutions of a set of first-order differential equations; these equations are generally nonlinear because of the dependence of the neutron flux per unit energy interval, $\Phi(x,E,t)$, upon the concentrations N(x,t).

Kushneriuk,[K1] who has determined an approximation of the neutron flux in a cylindrical fuel rod of uniform (average) composition, and Westcott,[W9] who considered a slab of purely absorbing material, have investigated the "hardening" of the neutron energy spectrum within a fuel element and determined the spatial dependence of the quantity $\int_0^\infty \sigma(E)\,\Phi(E,x,t)dE$ in Eq. (1).

It is suggested here that an accurate determination of the neutron flux by unit energy interval, $\Phi(x, E, t)$, in fuel elements of arbitrary shape, be performed on high-speed digital computers with the help of multigrouping multiregion transport codes. Thus, the function $\Phi(x, E, t)$ would be represented in each energy group $E_i < E < E_{i+1}$ by a function $\Phi_i(x, E, t)$ which is separable in space and energy according to the following equation:

$$\Phi_i(x, E, t) = \phi_i(t) h_i(x, t) f_i(E, t); \tag{5}$$

$h_i(x, t)$ is the "flux shape" for the group i; it is conveniently normalized to an average value equal to unity, i. e.,

$$\left\langle h_i(x, t) \right\rangle = 1 \;;$$

$\phi_i(t)$ is the magnitude of the corresponding average flux in the fuel element when the function $f_i(E, t)$ is normalized by

$$\int_{E_i}^{E_{i+1}} f_i(E, t) dE = 1 \;.$$

We can now define an effective cross section associated with the group i as

$$\sigma_i(t) = \int_{E_i}^{E_{i+1}} \sigma(E) f_i(E, t) dE \;. \tag{6}$$

The effective cross section $\sigma_i(t)$ is that corresponding to the average energy spectrum of the flux $\Phi_i(x, E, t)$ in the fuel element; $\sigma_i(t)$ can be obtained by computations performed according to the usual homogenization procedure.^{G1}

Equations (1), (5) and (6) now yield

$$A(x, t) = N(x, t) h_i(x, t) \sigma_i(t) \phi_i(t) \;. \tag{7}$$

Thus, assuming that the flux shapes $h_i(x, t)$ have been obtained from multigroup computations, functions $\zeta_i(t)$ are defined by

$$\zeta_i(t) = \left\langle N(x,t)h_i(x,t)\right\rangle \Big/ \left\langle N\right\rangle \left\langle h_i\right\rangle , \tag{8}$$

and the effective concentration coefficient defined by Eq. (4) is

$$\zeta(t) = \sum_i \zeta_i(t)\sigma_i(t)\, \phi_i(t) \Big/ \sum_i \sigma_i(t)\phi_i(t) . \tag{9}$$

2. Simplified Model for Evaluation of the Effective Concentration Coefficients

Estimates of the effective concentration coefficients will here be obtained by a one-group analysis. The single group will be referred to as the thermal group, and the subscript i used in the above equations is no longer needed. Equations (7), (8) and (9) now read

$$A(x,t) = N(x,t)\; h(x,t)\; \sigma(t)\phi(t),$$

and

$$\zeta(t) = \left\langle N(x,t)h(x,t)\right\rangle \Big/ \left\langle N\right\rangle \left\langle h\right\rangle .$$

Furthermore, the variation of the flux shape with irradiation is neglected, i.e., $h(x,t)$ is replaced by $h(x)$.

With these simplifications, the neutron absorption rates in the U^{235} and the Pu^{239} of a uranium fuel element* can be studied analytically. The subscripts 26 and 49 which will be used hereafter refer respectively to U^{235} and Pu^{239}. Initially, the composition of the fuel element is uniform and it does not contain any plutonium.

Rate of Absorption in U^{235}.

The rate of neutron absorption in U^{235} is now

$$A_{25}(x,t) = N_{25}(x,t)\sigma_{25}(t)h(x)\phi(t), \tag{10}$$

where $\sigma_{25}(t)$ is the effective thermal cross section for absorption in U^{234}; it could be determined according to the homogenization procedure.

The concentration of U^{235} then satisfies the equation

$$\frac{dN_{25}(x,t)}{dt} = - A_{25}(x,t) . \tag{11}$$

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*Similar studies could be performed for a cluster of small fuel elements.

By setting $\sigma_{25} = \sigma_{25}(0)$, a new irradiation (time) variable, the flux time θ is defined by

$$\sigma_{25}\theta = \int_0^t \sigma_{25}(t)\phi(t)dt\ , \tag{12}$$

and the solution of Eq. (11) subject to $N_{25}(x,0) = N_{25}^0$ is

$$N(x,\theta) = N_{25}^0\ e^{-\sigma_{25}\theta\ h(x)}\ . \tag{13}$$

The effective concentration coefficient ζ_{25}, Eq. (4), is now given by

$$\zeta_{25}(\theta) = \frac{\left\langle h(x)\ e^{-\sigma_{25}\theta h(x)}\right\rangle}{\left\langle e^{-\sigma_{25}\theta\, h(x)}\right\rangle\left\langle h\right\rangle}\ . \tag{14}$$

Note that for small irradiations

$$\zeta_{25}(\theta) \approx 1 - \frac{\left\langle h^2\right\rangle - \left\langle h\ ^2\right\rangle}{\left\langle\ h\right\rangle}\ \sigma_{25}\theta\ , \tag{15}$$

and for very large irradiations

$$\zeta_{25}(\infty) = h(0)/\left\langle h\right\rangle , \tag{16}$$

where $h(0)$ denotes the minimum of the flux shape. The minimum usually occurs at the center of the fuel element , and then, $\zeta_{25}(\theta)$ decreases monotonically with irradiation; this decrease is clearly due to the largest depletion of U^{235} at the periphery of the fuel element, where the flux is largest.

The absorption rate in U^{236} and in the fission products could be studied in a very similar way.

Absorption Rate in Pu^{239}.

Because of the important role played by the U^{238} resonance absorptions in the production of Pu^{239}, they should be distinguished from the thermal absorptions.

The thermal absorptions in U^{238} are due to the thermal flux $\phi(t)\ h(x)$ to which corresponds an effective thermal cross section $\sigma_{28}(t)$:

$$A_{28}(x,t) = N_{28}(x,t)\sigma_{28}(t)\phi(t)h(x)\ . \tag{17}$$

Theoretical determinations of the distribution of the resonance absorptions have been performed by Kushneriuk.[K2] For our present purposes, it is convenient to make the following additional simplications which yield a simple expression for the coefficient $\zeta_{49}(\theta)$, Eq. (22):

(a) The depletion of U^{238} is neglected, i.e., $N_{28}(x,t) = N_{28}^0$.

(b) The changes in the flux-energy spectrum are neglected, i.e., $\sigma_{28}(t) = \sigma_{28}$.

(c) The resonance absorptions in U^{238} take place with a given distribution $g(x)$; resonance and thermal neutron capture in the U^{238} present in the fuel element are in a constant ratio R.

Thus, the absorption rate in U^{238} is

$$A_{28}(x,t) = N_{28}^0\sigma_{28}[h(x) + Rg(x)]\phi(t). \tag{18}$$

Let σ_{49} be the effective thermal cross section of Pu^{239}, which is then destroyed at a rate

$$A_{49}(x,t) = N_{49}(x,t)\ \sigma_{49}\phi(t)h(x). \tag{19}$$

The concentration of Pu^{239} is the solution of the equation

$$\frac{dN_{49}(x,t)}{dt} = A_{28}(x,t) - A_{49}(x,t)\ , \tag{20}$$

with $N_{49}(x,0) = 0$. Letting $\theta = \int_0^t \phi(t)\,dt$, one obtains

$$N_{49}(x,0) = \frac{N_{28}^0\ \sigma_{28}}{\sigma_{49}}\ \frac{h(x)+Rg(x)}{h(x)}\ [\,1-e^{-\sigma_{49}\theta h(x)}\,]\ . \tag{21}$$

The effective concentration coefficient $\zeta_{49}(t)$, Eq. (4), is now given by

$$\zeta_{49}(\theta) = \frac{\left\langle [\,h(x) + Rg(x)]\ (1-e^{-\sigma_{49}\theta\ h(x)})\right\rangle}{\left\langle [\,1 + Rg(x)/h(x)]\ (1-e^{-\sigma_{49}\theta\ h(x)})\right\rangle}\ . \tag{22}$$

The function $\zeta_{49}(\theta)$ has the following particular values:

$$\zeta_{49}(0) = \frac{\langle h^2\rangle + R\langle gh\rangle}{\langle h\rangle^2 + R\langle g\rangle\langle h\rangle}, \quad \text{and} \quad \zeta_{49}(\infty) = \frac{\langle h\rangle + R\langle g\rangle}{\langle h\rangle + R\langle g/h\rangle\langle h\rangle}. \tag{23}$$

The flux shapes g(x) and h(x) usually decrease from a maximum value at the surface of the fuel lump to a minimum value about the center of the fuel element; the Pu^{239} production rate is greatest at the periphery of the fuel element where the flux is the largest, and $\zeta_{49}(0) > 1$; similarly, its destruction rate is greatest at the periphery and $\zeta_{49}(\theta)$ decreases on irradiation.

Simple estimates

The determination of the above defined functions $\zeta_{25}(\theta)$ and $\zeta_{49}(\theta)$ require first the determination of the flux shapes g(x) and h(x); only the spatial averages of the first few powers of the flux shape are required to obtain the first few terms of an expansion of $\zeta(\theta)$ in power series of the flux time.

For instance, let the thermal flux shape in a cylindrical fuel element of radius a be known as $h(x) = C\, I_0(Kx)$; then, Eq. (15) yields

$$\zeta_{25}(\theta) \approx 1 - \frac{(K\,a)^4}{192}\,\sigma_{25}\theta, \tag{24}$$

and Eq. (23), with $R = 0$, yields

$$\zeta_{49}(0) \approx 1 + \frac{(K\,a)^4}{192} \quad \text{for} \quad R = 0\ . \tag{25}$$

Similarly, a thermal flux shape $h(x) = C \cosh Kx$ in a fuel element shaped as an infinite slab of half thickness a yields

$$\zeta_{25}(\theta) \approx 1 - \frac{(K\,a)^4}{45}\,\sigma_{25}\theta \quad \text{and} \quad \zeta_{49}(0) \approx 1 + \frac{(K\,a)^4}{45} \quad \text{for} \quad R = 0. \tag{26}$$

Thus, the correction due to the nonuniform distribution of the nuclides in the fuel element is proportional to the fourth power of the dimension of this element. If the thermal flux in the fuel were to be represented by the elementary (diffusion) approximation, Ka would be proportional to the dimension of the fuel element in units of the diffusion length in the fuel; Eqs. (24) through (26) then yield crude but extremely simple estimates of the coefficients ζ.

3. Application to Natural Uranium Rod

Computation of the coefficients $\zeta_{25}(\theta)$ and $\zeta_{49}(\theta)$

Radial distributions of the thermal neutrons and of the resonance absorptions in a natural uranium fuel rod have been measured by Niemuth.[N2] The rod had a radius $a = 1.689$ cm and the ratio R of resonance to thermal captures in U^{238} was reported as 0.575. The reported flux shapes, renormalized to an average value equal to unity, are

$$h(x) = 0.79\, I_0(1.4\, x),$$

and

$$g(x) = 2.133\, I_0(x) - 1.374\, I_0(1.4x) + 0.7477\, \frac{x^4}{1.3662 - x^6}$$

where $x = r/a$, and $0 \leq x \leq 1$.

Numerical integration of the integrals in Eqs. (13) and (22) yield the following results:

For irradiations up to $\sigma_{25}\theta = 2$, the function $\sigma_{25}(\theta)$ is accurately represented by the equation

$$\zeta_{25}(\theta) = 1 - 0.016\, \sigma_{25}\theta.$$

For similar irradiations, $\zeta_{49}(\theta)$ should be represented by a polynomial of at least second degree. The function $\zeta_{49}(\theta) - 1$ is tabulated below for various values of the irradiation and for values of the ratio R, which is the ratio of the resonance to thermal captures in U^{238}

(Table C.1.). It has been assumed that the flux shapes h(x) and g(x) do not change with R.

The resonance absorptions in U^{238} strongly increase the non-uniformity of the Pu^{239} distribution and the effective concentration coefficient ζ_{49}.

Table C.1. 1 - $\zeta_{49}(\theta)$

$\sigma_{49}\theta =$	0.0	0.5	1.0	1.5	2.0	∞
R=0	0.0160	0.0124	0.0094	0.0070	0.0051	0.0000
R=0.3	0.0213	0.0175	0.0144	0.0119	0.0100	0.0048
R=0.5	0.0236	0.0198	0.0167	0.0142	0.0122	0.0070
R=∞	0.0389	0.0349	0.0315	0.0288	0.0268	0.0211

Application to NRX rod

The effects of the irradiation on a natural maximum rod of radius a = 1.727 cm have been extensively studied experimentally and theoretically.[W8, M4] Results obtained according to various computational models are compared below, with the following notations:

(a) $\sigma(\theta)$ is the effective thermal cross section defined by Eq. (6). The values of $\sigma(\theta)/\sigma(0)$ listed below were computed by McLeod (see Ref. M4, Table 5.3) according to the homogenization procedure. $\sigma(\theta)/\sigma(0)$ represents the effect due to the changes of the thermal flux energy spectrum upon the reaction rates.

(b) The coefficients $\zeta_{25}(\theta)$ and $\zeta_{49}(\theta)$ are those computed in Sec. 3. The following numerical values[M4] are used:

$$\sigma_{25}=519.9\text{b},\ \sigma_{49}=1041.3\text{b, and } \sigma_{28}=2.142\text{b},$$

and the initial conversion ratio ICR = 0.77 yields R = 0.35.

The functions $\zeta_{25}(\theta)$ and $\zeta_{49}(\theta)$ are then represented by the following polynomials in θ (neutrons/kilobarn):

$$\zeta_{25}(\theta) = 1 - 0.0083\,\theta,$$

and

$$\zeta_{49}(\theta) = 1.0219 - 0.0083\,\theta + 0.0013\,\theta^2 . \qquad (26)$$

(c) $\bar{\sigma}(t)$ is an average effective cross section defined by

$$\langle A \rangle = \bar{\sigma}(t)\,\phi(t)\,\langle N \rangle . \qquad (27)$$

A one-group (thermal) analysis would then yield

$$\bar{\sigma}(t) = \sigma(\theta)\,\zeta(\theta). \qquad (28)$$

The values of $\bar{\sigma}(t)/\bar{\sigma}(0)$ listed below have been computed by Ward (see Ref. W8, Table A-2) according to a model which takes into account the details of the space-energy dependence of the neutron flux and of the nuclide concentrations.[K1] The correspondence between t and the thermal flux time θ has been approximated by $t = 15\,\theta$ when θ is the thermal flux time in neutrons/kilobarns and t is the irradiation variable used by Ward.[W8]

Tables C.2 and C.3 give the values of $\sigma(\theta)/\sigma(0)$, $\zeta(\theta)/\zeta(0)$ and $\bar{\sigma}(t)/\bar{\sigma}(0)$ thus determined for U^{235} and Pu^{239} .

Table C.2. Effective absorptions in U^{235}

θ	$\frac{\sigma(\theta)}{\sigma(0)} - 1$	$\frac{\zeta(\theta)}{\zeta(0)} - 1$	$\frac{\sigma(\theta)\,\zeta(\theta)}{\sigma(0)\,\zeta(0)}$	$\frac{\bar{\sigma}(t)}{\bar{\sigma}(0)}$
0.1	-0.0000	-0.0008	0.9992	0.9985
0.3	-0.0017	-0.0025	0.9958	0.9953
0.5	-0.0029	-0.0041	0.9939	0.9920
0.7	-0.0035	-0.0058	0.9907	0.9886

Table C.3. Effective absorptions in Pu^{239}

θ	$\frac{\sigma(\theta)}{\sigma(0)} - 1$	$\frac{\zeta(\theta)}{\zeta(0)} - 1$	$\frac{\sigma(\theta)\zeta(0)}{\sigma(0)\zeta(0)}$	$\frac{\bar{\sigma}(t)}{\bar{\sigma}(0)}$
0.1	0.0000	-0.0008	0.9992	0.9895
0.3	0.0021	-0.0023	0.9998	0.9748
0.5	0.0036	-0.0038	0.9998	0.9652
0.7	0.0047	-0.0051	0.9996	0.9578

4. Discussion of the Results

Absorption Rate in U^{235}

Table C.2 shows that the correction introduced by $\zeta_{25}(\theta)$ is larger than that introduced by $\sigma_{25}(\theta)/\sigma_{25}(0)$. Furthermore, Eq. (28) is nearly satisfied. The results are to be interpreted as follows:

(a) Both the hardening of the average flux-energy spectrum and the nonuniform depletion of U^{235} result in a decrease of the absorption rate in U^{235} as irradiation proceeds.

(b) The nonuniform depletion of U^{235} induces larger corrections than does the hardening of the average flux-energy spectrum.

(c) A one-group analysis, where the variations of the flux shape during the irradiation are furthermore neglected, gives a quite accurate description of the relative decreases of the average effective cross section $\bar{\sigma}_{25}(t)$. The usefulness of such a simplified analysis was expected because resonance absorptions in U^{235} are small and because the macroscopic absorption cross section of natural uranium varies little during the irradiation (Σ_a varies less than 10% when the flux time increases from 0 up to 1.7 n/kb.).

Absorption Rate in Pu^{239}

The initial increase of the reaction rate in Pu^{239} due to its nonuniform production by neutron captures in U^{238} is here approximated by $\zeta_{49}(0) - 1 \approx 2.2\%$; it is larger than any subsequent increase resulting solely from the hardening of the average flux-energy spectrum. The results given in Table C.3 are interpreted as follows:

(a) The hardening of the average flux spectrum results in an increase of the absorption rate in Pu^{239} because of the non 1/v behavior of its absorption cross section. The nonuniform consumption of Pu^{239}, on the other hand, results in a decrease of the reaction rate as irradiation proceeds.

(b) The effects of the hardening of the average flux spectrum are overcompensated by the effects of the nonuniform distribution of Pu^{239}. A simple approximation neglecting both those effects would be better than the usual homogenization procedure.

(c) The product $\sigma_{49}(\theta)\,\zeta_{49}(\theta)$, corresponding to a one-group (thermal) analysis, decreases upon irradiation; however, this decrease is by far not as large as the one predicted by a multigroup analysis. The discrepancy is too large to be explained solely by the possibility of inaccuracies in the determination of the flux shapes g(x) and h(x) (see Sec. 3); this discrepancy was expected because of the large absorption in the Pu^{239} resonance at 0.3 eV, which invalidates the use reduction of the thermal flux to a single group of constant flux shape.

A two-group analysis, performed by subdividing the thermal spectrum into a low-energy group and a Pu^{239}-resonance group, would take into account a Pu^{239}-resonance flux shape which varies with irradiation much more than does the overall thermal flux shape. An accurate description of the distribution of the resonance absorptions would then be required in further studies of the effects of the nonuniform distributions of the plutonium isotopes.

5. Conclusion

The reaction rates computed according to an homogenization procedure that takes into account solely the variations of the average flux energy spectrum in a fuel lump are corrected by an effective concentration coefficient which accounts for the nonuniform distribution of the nuclides in fuel lump. When fuel elements have dimensions comparable to, or larger than the neutron diffusion length, any refinement in the determination of the variation of the flux-energy spectrum with irradiation is not justified unless the effects of the nonuniform distribution of the nuclides are also taken into account.

The correction introduced by the effective concentration coefficient has been evaluated according to a simplified method which easily yields accurate results for the reaction rates in the U^{235} of a natural uranium fuel element. More detailed studies are required for an accurate determination of the reaction rates in plutonium isotopes, and the variation of the flux shapes inside the fuel element should also be taken into account in case of highly enriched uranium fuel whose absorption cross section varies strongly during the irradiation.

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