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Abstract

Previous research has shown sketching to be useful to students solving math problems. The present study examines which aspects of middle school students' sketching are related to, or predict, successfully answering math problems. The effects of individual differences in cognitive factors – working memory, spatial ability, and prior math knowledge – on answer accuracy are also analyzed. Stepwise regression analysis indicates that prior math knowledge and the inclusion of numerical representations of key problem relationships in sketches positively predict answer accuracy, whereas including irrelevant relationships in a sketch is associated with lower answer accuracy. Methodological implications for future research are discussed.

Keywords: sketching; working memory; spatial ability; diagrams; middle school; mathematics

Introduction

Sketching or drawing is an important technique students use to solve math problems (Jee et al., 2014; Van Meter & Garner, 2005). Prior research has shown sketching to facilitate reasoning (Ruchti & Bennett, 2013) and communication of students' thinking (Haltiwanger & Simpson, 2013). Calculus teachers frequently include sketching in their instruction, as do students when solving calculus problems (Haciomeroglu, Aspinwall, & Presmeg, 2010). However, in the few types of representations identified in students' drawing – often schematic versus pictorial representations – it is not clear which one(s) seem to be the most useful for successful problem solving and math learning. Some research indicates that pictorial sketches, which include extraneous and decorative detail, are associated with lower answer accuracy than sketches that focus on schematic relationships in the problem (e.g. Hegarty & Kozhevnikov, 1999; Van Garderen & Montague, 2003). Edens and Potter (2008) examined upper elementary school students' sketches, finding that more schematic-like representations were positively associated with answer success compared to more pictorial sketches. In their study, sketches were rated as schematic versus pictorial, with the two categories as opposite ends of a single dimension. Thus, it is not clear whether pictorial representations themselves, rather than extraneousness or lack of attention to the components of the problem, are detrimental to problem solving.

In addition, individual differences in cognitive ability might also affect how different types of drawing influence students' problem solving in math. Working memory, for example, has been shown to affect the amount of new information that can be integrated into a working mental model (Seufert, Schutze, & Brunken, 2009), the quality of which may affect whether a student could represent the model diagrammatically. Prior content knowledge has also been shown to affect the extent to which representations of a problem contain extra details; those with more expertise tend to include more "embellishments" in their problem representations (Chi, Feltovich, & Glaser, 1981). And research on spatial ability suggests that answering certain problems requires mental transformations not provided by text or visualization alone (Trafton & Trickett, 2001).

The present study set out to identify associations between types of representation in student's drawing and performance in problem solving. In particular, we examined middle-school students' drawing while they engaged in math problem solving, identified different representation strategies in these drawings – words, numbers or numerical operations, schematic/diagrammatic relationships, or decorative pictures – and explored whether there is any association between the prevalent representational strategies of drawings and students' performance in solving math problems. We also sought to test whether the effects of representation strategies operate independently of the effects of cognitive factors. Thus, we sought to answer the following research questions: does using one type of representation predict or relate to answer accuracy, what effect does representing irrelevant components have on answer accuracy, and what relationships do cognitive abilities have to answer accuracy?

Methods

Participants

Participants included 47 sixth graders from two diverse middle schools in the mid-Atlantic US. The sample was 60% female and was racially diverse (53% White, 15% Black, 13% Asian, 11% Latino, 4% Middle Eastern, 4% declining to identify their race, and 11% more than one race). Participants were also socioeconomically diverse, with 23% coming from families where neither parent had earned a Bachelor's degree, and 47% coming from families where at least one parent had earned a Bachelor's degree or higher (30% of the sample did not know the level of education their parent/guardian had obtained). The mean age for the students was 11.5 years.

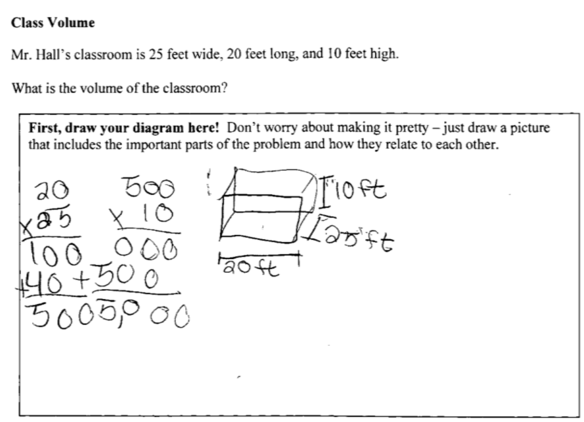
Procedure

Students were recruited from their homeroom classes. After obtaining parental consent and student assent, participants provided basic demographic information, completed 4 of the paper and pencil math stimuli and completed a computer-based task for visuo-spatial working memory during two individual sessions of approximately 25 minutes each. Students received one pen and one pencil each as compensation for participation. Students also participated in a group testing session in which they completed the paper and pencil measure of dynamic spatial ability (mental rotation test) and a test of prior math knowledge.

Measures

Math Stimuli Four math stimuli were constructed on the topics characteristic of middle school mathematics curricula; these topics included basic arithmetic/mathematical reasoning, algebra, area, and volume. Each stimulus included instructional text and a challenging problem that is amenable to solving with diagrams. Students were directed to "make a diagram," and researchers emphasized that the diagrams did not "have to be pretty," but that they should include the important parts of the problem, and should help them solve it. See Figure 1 for a sample stimulus.

Figure 1: Sample stimulus and student work.



Prior Knowledge in Mathematics To assess students' mathematics conceptual knowledge, we used 21 researcher-constructed items that measured students' prior knowledge of the topics/skills required to complete the mathematics stimuli accurately. For example, one item asks participants to identify whether an expression represents area, perimeter, volume, or none of these for given dimensions of a rectangular prism (e.g., four times seven – length times height). This measure was not timed. Scores ranged from 29% to 100% (M=64%). Internal consistency was computed using Cronbach's alpha; α=.70.

Visuo-spatial Working Memory Participants' working memory was assessed using a computerized and spatial version of a complex-span working memory task. In this task, participants recall the order and location of squares in a grid while making decisions about the symmetry of images presented. Previous research shows the test to be reliable using Cronbach's alpha; $\alpha = .80$ (Unsworth, Heitz, Schrock, & Engle, 2005).

Spatial Ability Participants' spatial ability was assessed using the Mental Rotation Test from the Primary Mental Ability Battery (Thurstone, 1974). Participants select from four shapes one that, when rotated and combined with a target shape, creates a square. Colom et al. (2004) report reliability using Cronbach's alpha: $\alpha = .73$.

Diagram coding scheme A fine-grained coding scheme was developed to capture the participants' sketches in as much detail as possible. This first required that the problems themselves be broken down into their basic components. For each problem, faculty and graduate students in Temple University's College of Education were asked to write out all their work to answer the question. Their solution steps were broken down into the necessary and sufficient components of the problem (e.g. each individual dimension of a rectangular prism), and the necessary and sufficient relationships between components (e.g., the area of one plane equals one dimension times another dimension). These key elements and relationships were compiled as a coding rubric for each problem, which was used to score each participant's sketch.

In order to explore the relationship between representational strategies and successfully solving the problem, each element/relationship was scored by the type of representation: as words, as numbers or numerical operations, as schematic/diagrammatic relationships, or as a decorative picture. Each instance of a representation of a key element or relationship was given credit based on its representation type. For example, a participant received credit for both a word representation and a numerical representation if they wrote "length times width times height equals volume" and also wrote the equation for volume with the numerical dimensions listed in the problem.

Any element or relationship represented – in any form – that was not necessary and sufficient to solve the problem was scored as "irrelevant." These did not include arithmetical errors or diagrammatic size inconsistencies, which, if part of a necessary and sufficient representation, were considered relevant and credited by representation type.

Participants' scores for each representation type – words, numbers/numerical operations, diagrammatic relationships, or picture – were averaged across all problems. The sums of participants' irrelevant elements/relationships were also averaged across all problems.

One of the authors coded all participant sketches, and a second coder was trained on data other than those used for calculating inter-rater statistics and recoded 35% of the corpus. Intra-class correlations were used to calculate inter-rater reliability; correlations for each stimulus were greater than .76.

Answer Accuracy Answer accuracy was scored as its own dimension, independent of irrelevant features included in the sketches, and representational strategies. This permitted testing of the effects of correctness separately from the effects of relevant or irrelevant features, and from different representational strategies. Student answers for each stimulus were rated as 0 (incorrect), 1 (partially-correct), and 2 (correct). Scores for each problem were then averaged across all problems the participant completed. Two of the authors rated student answers; ratings were found to be highly reliable, with intra-class correlations for each stimulus greater than .90.

Table 1: Correlations between representation strategies, cognitive factors, and answer accuracy

	Working Memory	Mental Rotation	Prior Math Knowledge	Answer Accuracy
Element representation type				
Numerical	.129	.248~	.374*	.465**
Words	-.068	.049	-.115	-.257~
Diagram	.141	.105	.049	-.129
Picture	.124	.050	.072	.089
Irrelevant Elements	.340*	-.048	.142	-.006
Relationship representation type				
Numerical	.198	.233	.452**	.553**
Words	-.233	.001	-.070	-.197
Diagram	.270~	.297*	.264~	.253~
Picture	.197	.159	.252~	.090
Irrelevant Relationships	-.101	-.367*	-.190	-.471**
Working Memory	--	.556**	.405**	.192
Mental Rotation		--	.364*	.227
Prior Math Knowledge			--	.494**

~ $p < .1$, *$p < .05$, ** $p < .01$

Results

Bivariate correlations were conducted to explore the relationships between representational strategies, inclusion of irrelevant features, cognitive factors, and answer accuracy.

Representational strategies for key problem elements were largely not significantly related to cognitive factors or answer accuracy. The exceptions were for numerical representations, which were correlated with spatial ability, prior math knowledge, and answer accuracy. Irrelevant representations of elements were significantly correlated with working memory, though why this should be the case is unclear (see table 1).

Of the strategies coded for representing relationships, only the use of words bore no significant relationship to any cognitive factors or answer accuracy, though the correlations were in a consistent direction. Numerical representations of relationships were strongly related to answer accuracy (r =.553, $p < .01$) and prior math knowledge ($r = .452$, $p < .01$). Correlations between diagrammatic representations of relationships and were only marginally significant for answer accuracy and cognitive factors, with the exception of spatial ability, to which they bore a statistically significant relationship ($r = .297$, $p < .05$). Of particular relevance to our research questions were the significant inverse correlations between representing irrelevant relationships and answer accuracy ($r = -.471$, $p < .01$), and spatial ability ($r = -.367$, $p < .05$).

Correlational analysis of cognitive factors and answer accuracy indicated that, at least for this relatively small sample, only prior math knowledge was significantly related to answer accuracy ($r = .494$, $p < .01$), though both spatial ability and working memory were significantly related to prior math knowledge (respectively, $r = .364$, $p < .05$; and $r = .405$, $p < .01$).

Stepwise multiple regression analysis was then conducted to examine the predictive values of representational strategies for key problem relationships and cognitive abilities on answer accuracy. Stepwise regression, rather than simple linear regression, was chosen for these analyses given the exploratory nature of the diagram coding scheme, and the inter-relationships among independent variables. However, the choice was made to focus regression analyses on relationship representations, rather than element representations or the combination of both, because the inclusion of relationships presupposes the inclusion of corresponding problem elements, i.e. calculating the volume of a rectangular prism necessarily involves each individual dimension.

Given the correlations observed in Table 1, numerical representations, diagrammatic, and irrelevant representations of relationships were used in the regression analysis, as were all three cognitive factors. The final model was reached in three steps with no variables removed, and contained three out of the six variables examined: numerical representations of relationships, prior math knowledge, and irrelevant representations of relationships – each entered respectively in the three steps. The regression equation was statistically significant, $F(3, 46) = 11.377$, $p < .001$, with the model accounting for 40% of the variance in accuracy ($R = .442$, $R^2 = .404$). Answer accuracy was predicted both by irrelevant relationships and numerical relationships, in opposite directions but of equal effect size in our analyses. That is, a higher average number of numerical representations predicts higher answer accuracy, and a higher average number of irrelevant relationships represented predicts lower answer accuracy. Prior math knowledge was also significantly predictive of answer accuracy (See Table 2 for raw and standardized regression coefficients).

Table 2: Stepwise regression analysis for cognitive factors and relationship representations

Variable	*B*	*SE (B)*	*β*	*t*	*Sig.*
Entered					
Numerical Representations	.138	.070	.280	1.972	.055
Prior Math Knowledge	.037	.015	.314	2.462	.018
Irrelevant Relationships	-.215	.099	-.281	-2.183	.035
Excluded					
Diagram Representations	.043			.346	.731
Mental Rotation	-.072			-.551	.585
Working Memory	-.022			-.178	.859
Intercept	.369	.235		1.572	.123
R^2			.442		
S.E.E.			.373		

Discussion

The present study sought to explore the relationships between cognitive abilities and representational strategies in middle school students' sketches, and their answer accuracy when solving math problems. Contrary to previous research on cognitive ability correlates of math performance, mental rotation ability and visuo-spatial working memory were not related to answer accuracy. Prior math knowledge, however, was moderately correlated with answer accuracy. Several interpretations are possible: that prior math knowledge is particularly important for these problems, or that working memory and spatial ability are less salient factors for math performance for this age group. Future research exploring different kinds of math problems, and with students of different ages, would clarify the role of prior knowledge relative to other cognitive factors.

One of the main questions this study sought to address was the effect of pictorial representations of key problem elements and relationships on answer accuracy; prior research indicated a negative relationship between pictorial representations and answer performance. Yet, as the results tentatively suggest, it may not be pictorial representations per se, but misunderstanding the operative relationships that predicts lower performance. For example, a sketch may properly construe the key relationships of a problem, but contain additional decorative details that do not interfere with conceptual understanding – a possibility indicated by the marginally significant correlation between pictorial representations of relationships and prior math knowledge. In other words, those who possessed sufficient prior knowledge of math may not have been hindered by adding decorative details.

The results also suggest that the inclusion of irrelevant relationships bears a relatively distinct connection to answer accuracy. Irrelevant features were scored as a separate dimension from both overall answer accuracy and different representation strategies, meaning that discerning and representing relevant relationships may reflect a different dimension of mathematical problem solving. However, irrelevant representations' inverse relationship with spatial ability may indicate difficulty grasping and manipulating abstract relationships. In other words, those who scored higher on a spatial ability measure may have had less trouble conceptualizing the ways the problem elements were related to each other.

This latter interpretation is supported by the relationships between diagrammatic representations of relationships, all three cognitive factors, and answer accuracy. That is, students who can grasp and manipulate – and therefore diagram – operative relationships of a math problem tend to score higher on spatial ability, working memory, and prior math knowledge. An important question, then, is whether the benefits of diagramming, *before* calculations, on answer accuracy obtain for all levels of cognitive ability. To test such a possibility, studies are needed that compare diagramming to other promising active learning methods, such as self-explanation, and against a control group prompted to do neither. These comparisons would help determine whether the cognitive ability correlates of successful diagramming, or the forced attention to the relationships of a problem, contribute to answer accuracy. However, the present study did not entail such comparisons, and its scope is therefore limited to the relationships between representational strategies and cognitive factors.

Another limitation of this study was the lack of distinction between correct and incorrect representations in participants' sketches. The results indicate that relevant numerical representations of relationships predict answer accuracy. This makes sense because some amount of relevant numerical representations (i.e. calculations) are necessary to solve math problems. But the diagram scoring scheme used in the study made no distinction between representations with arithmetical errors and those without. It was therefore not possible to determine whether the relationship between answer accuracy and a higher number of numerical representations reflects additional necessary calculations after noticing arithmetical errors, or simply that a certain amount of calculations are necessary to solve the problems. To correct this, diagram scoring schemes should account for correct and incorrect – but relevant – representations. It would then be possible to differentiate between those who required many additional calculations than the minimum necessary, and those who diligently performed every calculation on paper.

Future studies should also clarify the consistent inverse, though not statistically significant, relationship found between the use of words to represent relationships and working memory and answer accuracy. The direction of the relationship with working memory, which was itself positively related to answer accuracy, hints at the possibility that the use of words to represent relationships was merely a note-taking strategy employed to relieve demands on working memory. Pilot work for this study showed similar results, but the methodology of this study precludes definitive interpretations of this relationship; participants could not designate certain relationships as notes for themselves, rather than components of their representation of the problem.

Therefore, more research is needed before implications for instruction, or for students with lower working memory capacity, become clear.

Overall, these results indicate that knowledge of formulas or arithmetical procedures may be necessary but not entirely sufficient for successfully solving math problems, given the negative effects of failing to grasp the relevant relationships. Beyond prior knowledge of math, and showing lots of calculations, students should also carefully conceptualize math problems' operative relationships. These results also corroborate evidence of a connection between constructing schematic/diagrammatic representations of math problems, and spatial ability, working memory, and prior math knowledge.

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References

Chi, M., Feltovich, P. J., & Glaser, R. (1981). Categorization and representations of physics problems by experts and novices. *Cognitive Science, 5*(2), 121-152.

Colom, R., Escorial, S., & Rebollo, I. (2004). Sex differences on the Progressive Matrices are influenced by sex differences on spatial ability. *Personality and Individual Differences*, *37*(6), 1289-1293.

Edens, K., & Potter, E. (2008). How students "Unpack" the structure of a word problem: Graphic representations and problem solving. *School Science and Mathematics, 108*(5), 184-196.

Haciomeroglu, E. S., Aspinwall, L., & Presmeg, N. C. (2010). Contrasting cases of calculus students' understanding of derivative graphs. *Mathematical Thinking and Learning, 12*(2), 152-176.

Haltiwanger, L., & Simpson, A. M. (2013). Beyond the write answer: Mathematical connections. *Mathematics Teaching In The Middle School, 18*(8), 492-498.

Hegarty, M., & Kozhevnikov, M. (1999). Spatial abilities, working memory, and mechnical reasoning. In J. S. Gero & B. Tversky (Ed.), *Visual and spatial reasoning in design* (pp. 221-241). Cambridge, MA, June, 15-17.

Jee, B., Gentner, D., Forbus, K., Sageman, B., & Uttal, D. (2009). 2009. Drawing on experience: Use of sketching to evaluate knowledge of spatial scientific concepts. *Proceedings of the 31st Annual Conference of the Cognitive Science Society,* Amsterdam, The Netherlands.

Ruchti, W. P., & Bennett, C. A. (2013). Develop reasoning through pictorial representations. *Mathematics Teaching In The Middle School, 19*(1), 30-36.

Seufert, T., Schütze, M., & Brünken, R. (2009). Memory characteristics and modality in multimedia learning: An aptitude—treatment—interaction study. *Learning and Instruction, 19*(1), 28-42.

Thurstone, L. L. (1974). *The Measurement of Values*. Chicago: The University of Chicago Press.

Trafton, J. G., & Trickett, S. B. (2001). A new model of graph and visualization usage. *Proceedings of the 23rd Annual Conference of the Cognitive Science Society,* 1048-1053.

Unsworth, N., Heitz, R. P., Schrock, J. C., & Engle, R. W. (2005). An automated version of the operation span task. *Behavior Research Methods, 37*(3), 498-505.

Van Garderen, D., & Montague, M. (2003). Visual-spatial representation, mathematical problem solving, and students of varying abilities. *Learning Disabilities Research & Practice*, *18*(4), 246-254.

Van Meter, P., & Garner, J. (2005). The promise and practice of learner-generated drawing: Literature review and synthesis. *Educational Psychology Review*, *17*(4), 285-325.