Title

Isoperimetric and Sobolev problems under various curvature assumptions

ISBN
9798186385356

Author
Ricci, Fabio

Publication Date
2026-05-11

Peer reviewed|Thesis/dissertation

University of California

Santa Barbara

Isoperimetric and Sobolev problems under various curvature assumptions

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy

in

Mathematics

by

Fabio Ricci

Committee in charge:

Professor Guofang Wei, Chair

Professor Katy Craig

Professor Xianzhe Dai

June 2026

The dissertation of Fabio Ricci is approved.

Professor Katy Craig

Professor Xianzhe Dai

Professor Guofang Wei, Chair

May 2026

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Dedicated to my parents

Acknowledgements

This thesis would not have been possible without the support of my family, friends, advisor, collaborators, teachers, and the broader geometric analysis community.

First and foremost, I would like to express my deepest gratitude to my advisor, Guofang Wei. I am incredibly thankful for her patience, encouragement, and constant support throughout my time at UCSB. She has been an inspiring mentor, and I feel very fortunate to have had the opportunity to learn from her. Her guidance, generosity, and trust have shaped both my mathematical development and my experience as a graduate student.

A very special thank you goes to Jihye Lee. Our collaboration has been one of the main reasons I have grown as a mathematician. I am grateful for her patience, generosity, and the many mathematical conversations we have shared. It is exciting that we are still collaborating, and I look forward to a long-term collaboration and to many new problems to solve together.

I would also like to thank the Department of Mathematics at UC Santa Barbara for its continuous support. I have felt extremely lucky to be part of such a welcoming and supportive department. In particular, I am grateful for the many ways in which the department helped take away the bureaucratic worries of graduate school, allowing me to focus first on mathematical research and then on teaching.

I am especially grateful to Guofang Wei, Xianzhe Dai, and Matthias Wink for their numerous topics courses in geometry. These courses were among the places where I learned the most, and I deeply appreciate the effort they put into teaching me the tools I needed for my work. Year after year, they brought new ideas, new perspectives, and new problems into the classroom, and these experiences had a lasting influence on me.

I would also like to thank Xin Zhou for introducing me to Riemannian geometry.

That early exposure played an important role in shaping my mathematical interests and in setting me on the path toward geometric analysis.

I would also like to thank Davit Harutyunyan for his generous time during office hours, for the many conversations in his office, and for solving problems together. His analysis class was one of the most exciting classes I took at UCSB, and it had a lasting influence on the way I think about analysis. I am also grateful to Katy Craig for introducing me to optimal transport very early in my graduate studies through her topics class, for her kindness and assistance with my job applications, and for organizing such a wonderful summer school.

I am also deeply grateful to Flavio Vetrano, Dieter Armbruster, and Erik Katsavounidis for their support and help with my PhD applications. I would like to thank Gioacchino Antonelli for many meaningful conversations.

I am also grateful to Bennett Chow and Ayush Khaitan for welcoming me into the exciting new project on AI and the formalization of mathematics. It has been exciting to realize that we are on the same page about the future of mathematics and mathematical research, and I am looking forward to learning from them and contributing to this new direction.

I would also like to thank Ronan Conlon for inviting me to give my first talk outside UCSB and for the wonderful dinners together.

I would like to thank Jiayin Pan, Jie Qing, Longzhi Lin, and Dimitri Navarro from UC Santa Cruz for welcoming me into my next mathematical adventure. I am very grateful for their kindness and trust, and I am excited for what comes next.

I would also like to thank Chris Ograin for the wonderful conversations about teaching and for his support in helping me develop my teaching experience. I am very grateful to Matt Porter, for whom I TAed many times. It has always been a pleasure to work together. I also thank Denis Labutin, for whom I TAed only a few times, but with whom I had many great conversations about PDE. I would also like to thank Zheng Liu, Peter

Garfield, Chris Ograin, Jitendra Rathore, Denis Labutin, and Felipe Linares, all of whom I had the pleasure to work with during my time at UCSB.

A special thank you goes to Medina Price for her support, kindness, and help with all the bureaucracy. I am also grateful for the wonderful ocean-view office, which I will always miss.

I would also like to thank Darren Long, Gustavo Ponce, Gunhee Cho, and Mihai Putinar for many meaningful conversations and interactions throughout my time at UCSB. A special thank you goes to Darren Long, who would have been the best person in the math department to be stuck with in our old elevator for a few hours.

I want to thank my lovely girlfriend, Maria, for her incredible support, especially during the postdoc applications and during the final stretch of the PhD. Her encouragement, patience, and love have meant more than I can say.

A special thank you also goes to Stijn Pletinckx. I met Stijn through the Very Applied Fluid Dynamics Seminar, and we very quickly built a wonderful friendship. I am grateful to Stijn for our inspiring conversations and for sharing so many ideas. I have always been looking forward to our discussions. I am also grateful to both Stijn and Joel for their help in coorganizing the seminar. Finally, I am grateful to Flavio Malagutti and Djordje Nicolic for the great dinners, adventures, and friendship.

I would like to thank all my other friends who made these years so meaningful: Malik, Ricky Ellie, Ilya, Steve, Connor, Andres, Zoe, Michael, Jack, Jose, Matthew, Andre, Yusen, Paige, Melody, Leslie, Irene, Nick, Tin Yao, Robert, Alan, Aaron, Gianluca, Samuele, Nico, Evan, Jimmy, Clive, Jake, and Chris. I have felt so lucky to have so many good friends that, when traveling for conferences, one of the hardest decisions was often deciding whom to visit.

I am also grateful to everyone who participated in the Very Applied Fluid Dynamics Seminar. I started this project during the first quarter of my first year, simply as a way to socialize and bring a few people together. Over time, it grew into something much

larger, bringing together people from many different departments. Through this event, we were able to spark scientific collaborations, but most importantly, I met some of my closest friends from different departments here at UCSB.

The Very Applied Fluid Dynamics Seminar.

Finally, a special thanks goes to Scott and the team at Draughtsmen Aleworks. Thank you for hosting the Very Applied Fluid Dynamics Seminar so many times. I am looking forward to celebrating with you on June 11th.

Fabio Ricci

805-284-3607 • fabioricci@ucsb.edu • hal9009math.github.io

Research Interests

Geometric analysis, comparison geometry, geometric inequalities, optimal transport, machine learning, and mathematical formalization.

Employment

2026–2029 **Math Fellow**, University of California, Santa Cruz.

Education

2019–2026 **Ph.D. in Mathematics**, University of California, Santa Barbara.

2019–2021 **M.A. in Mathematics**, University of California, Santa Barbara.

2011–2014 **B.Sc. in Pure Mathematics**, Sapienza University of Rome, Department of Mathematics Guido Castelnuovo. Grade: $110/110$ with honors.

Publications and Preprints

2026 **Geometric inequalities with quadratic curvature decay**, joint with J. Lee. In preparation.

2025 **Isoperimetric profile function comparisons with integral Ricci curvature bounds**, joint with J. Lee. To appear in *Proceedings of the American Mathematical Society*. arXiv:2403.15973.

2024 **The log-Sobolev inequality for submanifolds in manifolds with asymptotically nonnegative intermediate Ricci curvature**, joint with J. Lee. *Journal of Geometric Analysis* **34**, 141. doi:10.1007/s12220-024-01581-1.

Awards and Achievements

2024, 2025 Direct Reading Program Mentor's Choice Award, UC Santa Barbara, Department of Mathematics.

2024 Academic Senate Outstanding Teaching Award, UC Santa Barbara.

2023 UCSB Grad Slam Finals People's Choice Award, UC Santa Barbara, Graduate Division.

2023 UCSB Grad Slam Preliminary Round Winner and Finalist, UC Santa Barbara, Graduate Division.

2021 Outstanding Teaching Assistant Award, UC Santa Barbara, Department of Mathematics.

2021, 2022	Nomination for Academic Senate Outstanding Teaching, UC Santa Barbara.

Invited Talks

Mar. 2026	**Isoperimetric and Sobolev problems under various curvature conditions**. *AMS Special Session: Topics in Geometric Analysis*, Boise State University, Boise, ID.
Jan. 2026	**Isoperimetric and Sobolev problems under various curvature conditions**. Geometry Seminar, Department of Mathematics, UC Santa Cruz, Santa Cruz, CA.
Jan. 2025	**Isoperimetric and Sobolev problems under various curvature conditions**. *Joint Mathematics Meetings*, Seattle, WA.
Oct. 2024	**Isoperimetric and Sobolev problems under various curvature conditions**. *AMS Special Session: Topics in Geometric Analysis*, UC Riverside, Riverside, CA.
Feb. 2024	**Why the Earth is almost flat: understanding curvature**. Math Matters Series, Bakersfield College, Bakersfield, CA.
Nov. 2023	**Log-Sobolev inequalities via the ABP method**. Geometry Seminar, Department of Mathematics, UC Santa Barbara, Santa Barbara, CA.
Sept. 2023	**Log-Sobolev inequalities under intermediate curvature conditions**. Geometry Seminar, Department of Mathematics, UT Dallas, Dallas, TX.
Apr. 2023	**Log-Sobolev inequalities under intermediate curvature conditions**. BIRS Workshop: A Unified View of Quasi-Einstein Manifolds, Banff International Research Station, Canada.
June 2023	**Optimal transport and isoperimetric inequalities**. UC Santa Barbara Donor Event, Goleta and Los Angeles, CA.

Conferences Attended

2026	*When Analysis Meets Geometry*, UC Santa Cruz; *Soul and Shape*, University of Oklahoma; *AMS Western Sectional Meeting*, Boise State University; *Joint Mathematics Meetings*, Washington, DC.
2025	*Texas Geometry and Topology Conference*, UT Dallas; *Joint Mathematics Meetings*, Seattle.
2024	*Recent Progress on Geometric Analysis*, SLMath; *AMS Western Sectional Meeting*, UC Riverside; *Frontiers in Geometric Analysis*, UC Santa Cruz.
2023–2026	*Southern California Geometric Analysis*, UC Irvine and UC San Diego.
2023	*A Unified View on Quasi-Einstein Metrics*, Banff International Research Station; *Joint Mathematics Meetings*, Boston.

Teaching Experience

Instructor **Instructor of Record**, UC Santa Barbara: Vector Calculus with Applications, 4 times; Introduction to Logic, 1 time; Differential Equations with Applications, 2 times.

TA **Teaching Assistant**, UC Santa Barbara, 2019–present: Introduction to Proof Writing/Discrete Mathematics, 19 quarters; Differential Equations, 2 quarters; Linear Algebra, 1 quarter; Integral Calculus, 1 quarter; Partial Differential Equations, 3 quarters.

Undergraduate Mentoring

2025 **Hyperbolic Land**, Direct Reading Program. Students: Maria Segall, Kevin Petersen, and Sarah Sault.

2024 **Curvature Done Optimally**, Direct Reading Program. Student: Merrick Hua.

2023 **Earth is Locally Flat**, Direct Reading Program. Student: Jeremy Lauro.

Travel Grants Awarded

2025–2026 Doctoral Travel Grant, UC Santa Barbara.

2024 Academic Senate Award, UC Santa Barbara; Graduate Student Association Travel Grant, UC Santa Barbara.

Joint Mathematics Meetings travel support.

Summer Schools

2025 Optimal Transport in Honor of Luigi Ambrosio, Northwestern University.

2025 Optimal Transport Summer School, UC Santa Barbara.

2023 Summer School in Differential Geometry, VIASM, Hanoi.

2022 Geometric Flows, MSRI, Greece.

Other Skills and Achievements

2025 Quantum Computing Boot Camp, Erdős Institute.

2013–2018 Algorithm developer for assistive technologies in disability, Walden Technology.

2016 Diploma in classical guitar and solfeggio, Conservatorio Santa Cecilia, Rome.

2014 Certified sommelier, Associazione Italiana Sommelier.

2013 FIV sailing assistant instructor at Circolo Velico Ventotene.

Abstract

Isoperimetric and Sobolev problems under various curvature assumptions

by

Fabio Ricci

Geometric inequalities are one of the main ways in which curvature controls the analysis and geometry of a Riemannian manifold. Sobolev-type inequalities compare a function with its derivatives and are closely connected to volume growth, heat flow, and concentration phenomena. Isoperimetric inequalities compare the volume of a region with the area of its boundary and give a geometric measure of how efficiently a space encloses volume. In comparison geometry, such inequalities are often understood by comparing a manifold or submanifold with a model space, such as Euclidean space, a sphere, or a space form of constant curvature. This thesis studies two problems in this direction: logarithmic Sobolev inequalities for submanifolds and comparison theorems for isoperimetric profile functions. In both cases, the goal is to extend classical comparison principles under weaker or more flexible curvature assumptions. The results are obtained in joint work with Jihye Lee.

The first part of the thesis concerns logarithmic Sobolev inequalities for submanifolds. Logarithmic Sobolev inequalities have been extensively studied in analysis, geometry, and probability, beginning in Euclidean and Gaussian settings and later extending to Riemannian manifolds and submanifolds. In the submanifold setting, these inequalities reflect not only the behavior of a function and its gradient, but also the mean curvature of the submanifold and the geometry of the ambient space. A pioneering result of Brendle established a logarithmic Sobolev inequality for compact submanifolds, placing the

classical Euclidean theory in a geometric framework. We extend this result to ambient manifolds satisfying weaker curvature assumptions than those previously required. We also prove a version under an asymptotic curvature condition, allowing the comparison principle to persist under a controlled large-scale weakening of the hypothesis. As an application, we obtain nonexistence results for closed minimal submanifolds in ambient manifolds with Euclidean volume growth.

The second part of the thesis concerns isoperimetric comparison. The isoperimetric problem asks for the least possible boundary area among regions of a prescribed volume. A natural way to organize this problem is through the isoperimetric profile function, which records this optimal boundary area for every volume. Explicit lower bounds for the isoperimetric profile are isoperimetric inequalities, while comparison theorems for the profile compare the geometry of a manifold with that of a model space. Classical profile comparison theorems are known under pointwise Ricci curvature lower bounds. We extend these comparison results to integral Ricci curvature bounds, where curvature is controlled in an averaged sense rather than pointwise. The resulting estimates compare the isoperimetric profile of the manifold with that of the corresponding model space, with quantitative error terms measuring the deviation from the classical comparison setting. When this deviation vanishes, the classical comparison theorems are recovered. We also obtain corresponding results for the relative isoperimetric problem inside convex bodies.

Together, these results place both parts of the thesis within comparison geometry. They show that fundamental Sobolev-type and isoperimetric inequalities continue to hold under curvature hypotheses beyond the traditional settings. In the logarithmic Sobolev problem, the curvature condition is adapted to the geometry of the submanifold and admits an asymptotic version; in the isoperimetric-profile problem, pointwise Ricci control is replaced by integral curvature control. Thus the thesis develops comparison principles for geometric inequalities in broader curvature regimes.

Contents

Chapter 1

Introduction

One of the most elegant themes in geometric analysis is the interplay between the underlying geometry of a space and the analytical objects defined upon it. This relationship is made rigorous and quantitative through geometric inequalities. They relate quantities such as volume, boundary area, gradients, entropy, and mean curvature, and they often reveal how close a space is to a model geometry. A first model example is the classical isoperimetric inequality in Euclidean space: if $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, then

$$|\partial\Omega| \geq n\omega_n^{1/n} |\Omega|^{\frac{n-1}{n}} ,$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and equality holds precisely for balls. This inequality says that balls enclose volume most efficiently. Although it is a geometric statement, it has direct analytic consequences. Through the coarea formula, it leads to Sobolev-type inequalities, such as

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \, dx\right)^{\frac{n-1}{n}} \leq C_n \int_{\mathbb{R}^n} |\nabla u| \, dx,$$

for suitable compactly supported functions u. Thus isoperimetric and Sobolev inequalities are two expressions of the same general principle: geometry controls analysis.

Logarithmic Sobolev inequalities are another important member of this family. In their simplest Euclidean form, they control the entropy of a function by its energy. For instance, if $f \geq 0$ is smooth on $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} f\,dx = 1$, then the sharp Euclidean logarithmic Sobolev inequality can be written as

$$\int_{\mathbb{R}^n} f \log f\,dx \leq \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\,dx - \Big(n + \frac{n}{2}\log(4\pi)\Big).$$

The sharp constant is chosen so that equality is attained by the Gaussian density. While these inequalities take their simplest form in flat space, they naturally extend to Riemannian manifolds. In this setting, geometric inequalities reveal how the curvature of a manifold influences its analysis. As a result, the basic entropy-energy comparison is modified by the curvature of the ambient space, by volume growth, and, for submanifolds, by the mean curvature.

On a Riemannian manifold, these inequalities are often comparison theorems. The philosophy of comparison geometry is that a curvature assumption should allow one to compare a given manifold with a model space such as Euclidean space, a round sphere, or a simply connected space form of constant curvature. A fundamental example is the Lévy–Gromov isoperimetric inequality. If (M^n, g) is closed and satisfies $\operatorname{Ric}_g \geq (n-1)g$, then its normalized isoperimetric profile is bounded below by that of the round sphere:

$$h_1(\beta, g) \geq h_1(\beta, g_{\mathbb{S}^n}), \qquad \beta \in (0,1). \tag{1.1}$$

Here

$$h_1(\beta, g) := \inf\left\{\frac{|\partial\Omega|}{|M|} : \Omega \subset M \text{ is a smooth domain with } \frac{|\Omega|}{|M|} = \beta\right\}.$$

This is a model comparison theorem: a Ricci curvature lower bound forces the isoperimetric behavior of M to be no worse than that of the corresponding sphere [35, 11]. Similar comparison principles appear throughout geometric analysis, relating curvature assumptions to volume growth, heat kernel estimates, Sobolev inequalities, eigenvalue estimates, and rigidity phenomena.

For the unnormalized isoperimetric problem, the main object is the isoperimetric profile function. For a Riemannian manifold (M^n, g), define

$$h_2(\beta, g) := \inf \{|\partial\Omega| : \Omega \subset M \text{ is a smooth domain with } |\Omega| = \beta\}. \tag{1.2}$$

Thus an explicit lower bound

$$h_2(\beta, g) \geq F(\beta)$$

is an isoperimetric inequality. On the other hand, comparison theorems for the profile compare $h_2(\beta, g)$ with the corresponding profile of a model space. For example, if $\mathrm{Ric} \geq (n-1)k$, then the classical profile comparison theorem of Morgan–Johnson and Ni–Wang gives

$$h_2(\beta, g) \leq h_2(\beta, g_k), \tag{1.3}$$

where $(\mathbb{M}^n_k, g_k)$ is the simply connected space form of constant curvature k [59, 63]. The direction of the comparison depends on the normalization and on the model being used, but in each case the profile gives a quantitative way to compare the manifold with a model geometry.

The thesis focuses on two problems in this comparison-geometric direction. The first concerns logarithmic Sobolev inequalities for compact submanifolds of complete non-compact ambient manifolds. In this setting, the curvature assumption is chosen to reflect the dimension and codimension of the submanifold, rather than requiring full non-negative sectional curvature of the ambient space. The precise condition is the k-intermediate Ricci curvature condition, whose definition appears in (1.4) below; its asymptotic version is formulated in (1.5) and (1.6). The second concerns comparison theorems for the isoperimetric profile function under integral Ricci curvature bounds, where the deviation from a pointwise Ricci lower bound is measured in an averaged sense; see (1.10). These two directions are technically different, but they share the same guiding idea: one seeks to preserve a model geometric inequality while replacing the classical curvature hypothesis by a weaker or more flexible one. The results described in this thesis are obtained in two joint works with Jihye Lee [54, 53].

We first discuss the logarithmic Sobolev side. Logarithmic Sobolev inequalities have been studied extensively in analysis, geometry, and probability. They appeared in early work of Stam [79] and Federbush [31], and Gross's foundational work [36] placed them in their modern form by relating them to hypercontractivity of semigroups; see also [37]. Since then, they have become a central tool in concentration of measure, heat kernel estimates, entropy methods, and geometric analysis [52, 65, 64]. In the Euclidean setting, the logarithmic Sobolev inequality can be viewed as a sharp form of the principle that entropy is controlled by energy, with equality governed by Gaussian functions.

For submanifolds, the inequality must also see the extrinsic geometry. The classical Michael–Simon Sobolev inequality [58] shows that the usual Sobolev inequality on a submanifold involves not only the gradient of a function, but also the mean curvature of the submanifold. Ecker [30] established logarithmic Sobolev inequalities in the Euclidean submanifold setting, and Brendle later proved a sharp logarithmic Sobolev inequality for compact submanifolds of Euclidean space [16]. In one common normalization, Brendle's inequality says that if $\Sigma^n \subset \mathbb{R}^{n+m}$ is compact without boundary and $f > 0$ is smooth, then

$$\begin{aligned}
&\int_\Sigma f \left(\log f + n + \frac{n}{2}\log(4\pi)\right) dV - \int_\Sigma \frac{|\boldsymbol{\nabla} f|^2}{f}\, dV - \int_\Sigma f\,|H|^2\, dV \\
&\leq \\
&\qquad \left(\int_\Sigma f\, dV\right) \log\left(\int_\Sigma f\, dV\right),
\end{aligned}$$

where H is the mean curvature vector of Σ. This formula is a useful model for the first part of the thesis: the Euclidean logarithmic Sobolev inequality is modified by a mean curvature term, and the goal is to understand what ambient curvature assumptions allow the same structure to persist.

A major development behind several recent results is the use of the Alexandrov–Bakelman–Pucci method. In its geometric form, the method starts from an auxiliary elliptic equation and uses a contact set together with a Jacobian estimate for a naturally defined map. The method was introduced into the Riemannian setting by Cabré [20, 19] and was subsequently used in several geometric inequalities [87, 82]. Brendle's work [14, 16] showed that this approach is especially effective for Sobolev and Michael–Simon type inequalities on non-compact manifolds and for submanifolds in curved ambient spaces. What makes the method well suited to comparison geometry is that curvature enters through the control of a Jacobian along geodesics. Thus the method naturally leads to the question of which curvature assumptions are actually needed for the relevant Jacobian estimates.

For submanifolds of higher codimension, non-negative sectional curvature of the ambient space is a strong assumption. The geometry of the submanifold only samples certain averages of sectional curvatures along directions determined by the tangent and normal bundles. This points toward intermediate Ricci curvature, a family of curvature conditions interpolating between sectional curvature and Ricci curvature. This notion goes back to work of Wu [85] and has since played an important role in comparison geometry and topology, including work of Shen [78], Guijarro and Wilhelm [39, 40, 38], Reiser and Wraith [71], and others. It also has a natural connection with optimal transport through the formulation of Ketterer and Mondino [51]. From the point of view of geometric inequalities, Ma and Wu [57] and Wang [81] showed that Brendle's Sobolev and Michael–Simon–Sobolev inequalities can be extended under suitable intermediate Ricci curvature assumptions. One contribution of this thesis is to show that the logarithmic Sobolev inequality fits into the same picture.

We also consider an asymptotic version of this curvature condition, defined precisely below in (1.5) and (1.6). The general idea of asymptotically non-negative curvature is that curvature may fail to be non-negative everywhere, but its failure is controlled at large distances. This framework was developed for sectional curvature by Abresch [1] and Kasue [49], and for Ricci curvature by Abresch–Gromoll [2] and Zhu [90]; see also [70, 89]. It has proved useful because many comparison phenomena remain valid when the curvature defect decays sufficiently fast. In the logarithmic Sobolev part of the thesis, we extend Brendle's inequality from the classical sectional-curvature setting to an intermediate Ricci curvature setting, and further to an asymptotically non-negative version. This gives a logarithmic Sobolev comparison theorem whose curvature hypothesis is adapted to the submanifold geometry and whose error terms record the large-scale weakening of the curvature condition.

Let us now place the profile comparison (1.3) in its broader context. Classical profile comparison theorems are based on pointwise curvature lower bounds. The Lévy–Gromov inequality (1.1) compares the normalized isoperimetric profile of a manifold with positive Ricci curvature to that of the round sphere [35, 11]. Morgan and Johnson proved (1.3) by deriving a differential inequality for the isoperimetric profile [59]; related arguments and refinements were developed by Bayle [8], Bayle–Rosales [9], and Ni–Wang [63], the latter using viscosity methods. More recently, profile comparison has also been studied in non-smooth settings with lower Ricci curvature bounds [5]. These results show that the isoperimetric profile is not merely a way of organizing isoperimetric inequalities: it is itself a geometric quantity governed by curvature comparison.

The second contribution of the thesis is to extend this profile-comparison viewpoint to integral Ricci curvature bounds. Instead of requiring a pointwise lower bound $\mathrm{Ric} \geq (n-1)k$, one measures the part of the Ricci curvature below the model value $(n-1)k$ using the integral curvature defect defined later in (1.10). When this curvature defect vanishes, the pointwise theory is recovered; when it is small but non-zero, the curvature is controlled in an averaged sense. To put this in context, two foundational results in comparison geometry are the Bishop–Gromov relative volume comparison and the

Laplacian comparison theorem. If $\mathrm{Ric} \geq (n-1)k$, then the volume ratio

$$\frac{\operatorname{vol} B_r(x)}{v(n,k,r)} \qquad \text{is monotonically nonincreasing in } r,$$

where $v(n,k,r)$ denotes the volume of a ball of radius r in the simply connected space form of constant curvature k, and the Laplacian of the distance function satisfies

$$\Delta\, d(\cdot, x) \leq (n-1)\,\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}$$

pointwise on $M \setminus (\{x\} \cup \mathrm{Cut}(x))$,where sn_k is the standard comparison function defined later in (3.6). These two results control volume doubling, precompactness, and heat kernel estimates, and they underlie much of the structure theory of manifolds with lower curvature bounds. A natural question is whether such comparison results persist when the pointwise assumption $\mathrm{Ric} \geq (n-1)k$ is replaced by an integral condition.

This program began with Gallot's work on isoperimetric inequalities under integral Ricci bounds [33] and was developed substantially by Petersen and Wei [66], who proved that the Bishop–Gromov monotonicity survives in an approximate sense under L^p curvature conditions with $p > n/2$: the volume ratio is no longer monotone, but an appropriate power of it is almost monotone, with an additive error controlled by the integral curvature defect. Their work also established the mean curvature and Laplacian comparison theorems in the integral setting, which provide the basic comparison tools. Further developments include work of Petersen–Sprouse [67], Aubry [6], Chen–Wei [23], Dai–Wei–Zhang [25], and Chahine [22]; related questions have also appeared in connection with stability problems such as the positive mass theorem [4].

Integral curvature bounds are natural from the comparison-geometric point of view because they separate the model comparison from the exact pointwise curvature condition. The price is that the classical monotonicity statements acquire error terms, and these errors must be propagated through the differential inequalities that govern the profile. In this thesis we prove comparison estimates for the isoperimetric profile under integral Ricci curvature bounds in both non-positive and positive model-curvature regimes. The estimates recover the Morgan–Johnson and Ni–Wang comparison (1.3) when the integral curvature defect is zero, and in general they quantify the deviation from the classical comparison in terms of the size of the curvature defect. We also obtain corresponding relative isoperimetric profile comparisons for convex bodies.

The two parts of the thesis therefore fit into a single narrative. Both begin with a classical geometric inequality that can be understood through comparison with a model space. Both ask whether the comparison survives when the curvature hypothesis is adjusted to a broader setting. In the logarithmic Sobolev problem, the adjustment is to use curvature assumptions adapted to the codimension of the submanifold, including an asymptotic version. In the isoperimetric-profile problem, the adjustment is to replace pointwise Ricci control by integral control of the curvature defect. The common goal is to enlarge the range of comparison geometry while retaining inequalities whose constants, model terms, and error terms have clear geometric meaning.

1.0.1 Main Results

We now state the main results of this thesis. We begin with the results on logarithmic Sobolev inequalities under intermediate Ricci curvature conditions, and then turn to the comparison results for the isoperimetric profile function under integral Ricci curvature bounds.

Logarithmic Sobolev inequality under intermediate Ricci curvature.

Let (M^{n+m}, g) be a complete non-compact Riemannian manifold of dimension $n+m$. For $1 \leq k \leq n+m-1$, given a k-dimensional subspace $P \subset T_xM$ at $x \in M$ and any tangent vector $v \in T_xM$ with $v \perp P$, the *k-intermediate Ricci curvature* with respect to P in the direction of v is defined as

$$\mathrm{Ric}_k(P, v) := \sum_{i=1}^{k} \mathrm{Sec}(v, e_i)\ |v|^2\,, \tag{1.4}$$

where $\{e_1, \dots, e_k\}$ is an orthonormal basis of P. When $k = n+m-1$, this reduces to the Ricci curvature, and when $k=1$, it reduces to the sectional curvature. The manifold M is said to have *asymptotically non-negative* Ric_k if there exist a base point $o \in M$ and a non-negative non-increasing continuous function $\lambda\colon [0,\infty) \to [0,\infty)$ with

$$b_0(\lambda) := \int_0^\infty t\,\lambda(t)\,dt < \infty, \qquad b_1(\lambda) := \int_0^\infty \lambda(t)\,dt < \infty, \tag{1.5}$$

such that

$$\mathrm{Ric}_k(P, v) \geq -k\,\lambda(d(o,x))\,|v|^2 \tag{1.6}$$

for all appropriate P and v. When $\lambda \equiv 0$, this reduces to the usual non-negative Ric_k condition, and $b_0 = b_1 = 0$.

We also use the Gaussian factor

$$P(t) := (4\pi)^{-\frac{n+m}{2}} \int_{\mathbb{R}^{n+m}} e^{-\frac{(|z|+t)^2}{4}}\,dz. \tag{1.7}$$

Our first main result extends Brendle's logarithmic Sobolev inequality [14, 16] to this setting. The choice $k = \min\{n-1, m-1\}$ is natural, as the codimension of the submanifold determines the partial average of sectional curvatures entering the inequality.

Theorem 1.1 ([54])

Let M^{n+m} be a complete, non-compact manifold of dimension $n+m$ with asymptotically non-negative Ric_k and Euclidean volume growth, where $k = \min\{n-1, m-1\}$. Assume that Σ^n is an n-dimensional compact submanifold without boundary. Then, for any positive smooth function f, we have

$$\begin{aligned}
&\int_\Sigma f\Bigg(\log f + \log\big(P(4b_1 n)\theta_h\big) + \frac{n}{2}\log(4\pi) + n + 4b_1^2 n^2 \\
&\qquad\qquad + (n+m-1)\log\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\Bigg) dV(x) - \int_\Sigma \frac{|\nabla f|^2}{f}\,dV - \int_\Sigma f\,|H|^2\,dV \\
&\leq \int_\Sigma f\,dV(x)\left(\log\int_\Sigma f(x)\,dV(x)\right),
\end{aligned} \tag{1.8}$$

where θ_h is the asymptotic volume ratio of M associated to the comparison function determined by λ, H is the mean curvature vector of Σ, and $r_0 = \max_{x\in\Sigma} d(o,x)$.

The additional terms involving b_0, b_1, and $P(4b_1 n)$ quantify the effect of the asymptotic weakening of the curvature hypothesis in (1.6). When $\lambda \equiv 0$, these terms reduce to the non-negative curvature case and θ_h becomes the usual asymptotic volume ratio. Setting $\lambda \equiv 0$ in Theorem 1.1 yields the following cleaner statement.

Corollary 1.2

Let M^{n+m} be a complete, non-compact manifold of dimension $n+m$ with Euclidean volume growth and $\mathrm{Ric}_k \geq 0$, where $k = \min\{n-1, m-1\}$. Let Σ^n be an n-dimensional compact submanifold without boundary. Then, for any positive smooth function f, we have

$$\begin{aligned}
&\int_\Sigma f\left(\log f + n + \frac{n}{2}\log(4\pi) + \log\theta\right) dV - \int_\Sigma \frac{|\boldsymbol{\nabla} f|^2}{f}\, dV - \int_\Sigma f\,|H|^2\, dV \\
&\quad \leq \left(\int_\Sigma f\, dV\right) \log\left(\int_\Sigma f\, dV\right),
\end{aligned} \tag{1.9}$$

where θ is the asymptotic volume ratio of M and H is the mean curvature vector of Σ.

We believe the k-intermediate Ricci curvature condition may be optimal for this inequality. Evidence for this comes from one of our corollaries, concerning the existence of closed minimal submanifolds, which fails under the weaker assumption of mere non-negative Ricci curvature. Determining the truly optimal curvature condition remains an important open question.

Isoperimetric profile comparisons under integral Ricci curvature. We now turn to isoperimetric profile comparisons. Let (M^n, g) be a Riemannian manifold and let $\rho(x)$ denote the smallest eigenvalue of the Ricci tensor at $x \in M$. Given $k \in \mathbb{R}$, we define the

Ricci curvature defect and the *integral Ricci curvature* by

$$\begin{aligned} \mathrm{Ric}_-^k(x) &:= \max\{0,\ (n-1)k - \rho(x)\}, \\ \left|\left|\mathrm{Ric}_-^k\right|\right|_p &:= \left(\int_M \left(\mathrm{Ric}_-^k(x)\right)^p d\mathrm{vol}\right)^{\frac{1}{p}}. \end{aligned} \tag{1.10}$$

The condition $\left|\left|\mathrm{Ric}_-^k\right|\right|_p = 0$ is equivalent to the pointwise bound $\mathrm{Ric} \geq (n-1)k$. When $\left|\left|\mathrm{Ric}_-^k\right|\right|_p$ is small but non-zero, the Ricci curvature is allowed to dip below $(n-1)k$ in a controlled L^p sense.

Recall the isoperimetric profile $h_2(\beta, g)$ from (1.2). Our comparison theorems relate the profile of (M^n, g) to that of the model space form $(\mathbb{M}_k^n, g_k)$ of constant curvature k. We state two results: the first for $k \leq 0$, including the non-compact case, and the second for $k > 0$ under a diameter and smallness condition.

Theorem 1.3 ([53])

Let (M^n, g) be a Riemannian manifold, $p > \frac{n}{2}$ and $k \leq 0$ be given. When $k = 0$, assume that $\mathrm{diam}(M) = d < \infty$. Then for any $\beta \in (0, |M|)$, we have

$$\begin{aligned} h_2(\beta, g) - h_2(\beta, g_k) \leq &\left(\frac{(n-1)(2p-1)}{2p-n} \left|\left|\mathrm{Ric}_-^k\right|\right|_p\right)^{\frac{1}{2}} \beta^{\frac{2p-1}{2p}} \\ &+ f(\beta, n, p, k, d, \left|\left|\mathrm{Ric}_-^k\right|\right|_p), \end{aligned}$$

where $(\mathbb{M}_k^n, g_k)$ is a complete simply connected space of constant curvature k and $f \equiv 0$ when $\left|\left|\mathrm{Ric}_-^k\right|\right|_p = 0$, recovering the classical comparison (1.3).

Theorem 1.4 ([53])

Let (M^n, g) be a Riemannian manifold, $p > \frac{n}{2}$ and $k > 0$ be given. Assume additionally that $d := \mathrm{diam}(M) < \frac{\pi}{2\sqrt{k}}$. Then there exists $\varepsilon > 0$ such that if $\left|\left|\mathrm{Ric}_-^k\right|\right|_p < \varepsilon$, then for any $\beta \in (0, |M|)$, we have

$$h_2(\beta, g) - h_2(\beta, g_k) \leq \left(\frac{(n-1)(2p-1)}{2p-n} \left|\left|\mathrm{Ric}_-^k\right|\right|_p \right)^{\frac{1}{2}} \beta^{\frac{2p-1}{2p}} + f(\beta, n, p, k, d, \left|\left|\mathrm{Ric}_-^k\right|\right|_p),$$

where $(\mathbb{M}_k^n, g_k)$ is a complete simply connected space of constant curvature k and $f \equiv 0$ when $\left|\left|\mathrm{Ric}_-^k\right|\right|_p = 0$, recovering the classical comparison (1.3).

These results highlight both the potential and the difficulty of working with integral curvature bounds. Controlling curvature in the L^p sense is significantly weaker than imposing pointwise conditions, and it introduces new technical challenges in the analysis of volume comparison and isoperimetric inequalities. Our proofs build upon the Bishop–Gromov volume comparison and Laplacian comparison results of Petersen and Wei [66], whose additive error terms persist in our estimates. When the integral curvature term vanishes, both theorems recover the classical pointwise comparison of Morgan and Johnson [59] and the viscosity-based results of Ni and Wang [63]. What is new is the ability to quantify and control the error when the Ricci curvature defect in (1.10) is nonzero, thereby extending the scope of these comparison theorems beyond the traditional pointwise regime.

Together, Theorems 1.1–1.4 expand the range of comparison geometry for geometric inequalities. They show that classical Sobolev-type and isoperimetric comparisons persist under broader curvature regimes, with the effect of the weakened hypotheses measured by explicit geometric and analytic quantities.

1.0.2 Overview of the thesis

The thesis is organized as follows.

In Chapter 2, we introduce the Alexandrov–Bakelman–Pucci (ABP) method in the Riemannian setting. We discuss the construction of the transport map via a PDE in divergence form on a contact set, the associated matrix Riccati equation along geodesics, and the resulting monotonicity formula for the Jacobian. We explain how this framework, originally developed by Cabré [20, 19] and further advanced by Brendle [14], provides a unified approach to Sobolev-type and isoperimetric inequalities on manifolds with curvature bounds.

In the same chapter, we prove our Michael–Simon type logarithmic Sobolev inequality under intermediate Ricci curvature conditions. After introducing the necessary background on k-intermediate Ricci curvature and the asymptotically non-negative framework, we present the proof of Theorem 1.1 and Corollary 1.2. We explain how the monotonicity formula for the Jacobian of the transport map interacts with the intermediate curvature condition, discuss the role of the Euclidean volume growth assumption and the optimality of the curvature condition, and derive several consequences of the main inequality. The content of this chapter is based on [54].

In Chapter 3, we study the isoperimetric profile function under integral Ricci curvature bounds. We begin by presenting basic properties of integral curvature conditions and surveying the relevant comparison results, including the Bishop–Gromov volume comparison and Laplacian comparison theorems of Petersen and Wei [66]. We then prove our comparison theorems for the isoperimetric profile, Theorem 1.3 and Theorem 1.4, discussing the structure of the error terms, the role of the diameter assumption in the positive curvature case, and the recovery of the classical comparison theorems when the integral curvature vanishes. The content of this chapter is based on [53].

Chapter 2

The ABP Method and the Logarithmic Sobolev Inequality

The Alexandrov–Bakelman–Pucci (ABP) method is a PDE-based technique that has proven remarkably effective for establishing a wide range of geometric inequalities, including isoperimetric, Sobolev, logarithmic Sobolev, Michael–Simon, eigenvalue, and Willmore-type inequalities, by reducing them to three analytical ingredients: a suitable elliptic equation, a surjectivity argument for the induced gradient map, and a pointwise Jacobian bound via the arithmetic-geometric mean inequality. For a comprehensive survey of the method and its geometric applications, see Brendle [13]. In this chapter, we present the method first in its Euclidean setting, where the key ideas can be seen most clearly and without the additional technicalities introduced by curvature, and then describe its extension to Riemannian manifolds. We begin with the classical isoperimetric inequality, discuss the Sobolev and logarithmic Sobolev inequalities, and give a detailed ABP proof of the global logarithmic Sobolev inequality on $\mathbb{R}^n$, which serves as the model for the curved setting. We then survey the manifold ABP framework and the results it has produced, and finally present the full proof of our main result [54]: the sharp logarithmic Sobolev inequality for submanifolds under asymptotically non-negative k-intermediate

Ricci curvature.

2.1 The ABP Method in Euclidean Space

2.1.1 The Classical Isoperimetric Inequality

We present a proof of the classical isoperimetric inequality in $\mathbb{R}^n$ using the ABP method, following the approach introduced by Cabré [20, 19] and later surveyed by Brendle [14]. This method, which relies only on the existence and regularity theory for linear elliptic PDEs, serves as the analytical foundation for the results developed in subsequent sections. We emphasize the structure of the proof, particularly the surjectivity of the gradient map on the contact set, as this step is precisely where curvature enters in the Riemannian generalization.

Theorem 2.1 (Classical isoperimetric inequality)
Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Then

$$|\partial\Omega|^n \geq n^n \omega_n \, |\Omega|^{n-1} \, ,$$

where $\omega_n = |B_1^n|$ denotes the volume of the unit ball in $\mathbb{R}^n$. Equality holds if and only if Ω is a ball.

The proof proceeds in several steps. We first set up an auxiliary Neumann problem, then analyze the gradient map on a natural contact set, and finally combine a surjectivity argument with the arithmetic-geometric mean inequality.

The Neumann problem. Set $c = \frac{|\partial\Omega|}{|\Omega|}$ and consider the Neumann problem

$$\Delta u = c \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 1 \quad \text{on } \partial\Omega, \tag{2.1}$$

where ν denotes the outward-pointing unit normal to $\partial\Omega$. The compatibility condition for solvability of (2.1) requires

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, dS,$$

which, by our choice of c, reduces to $c\,|\Omega| = |\partial\Omega|$. This is satisfied by construction, and classical elliptic regularity theory [34] guarantees the existence of a smooth solution u, unique up to an additive constant.

The contact set. We define the *lower contact set* of u by

$$\Gamma^- = \left\{x \in \overline{\Omega} : u(y) \geq u(x) + p \cdot (y - x) \text{ for some } p \in \mathbb{R}^n \text{ and all } y \in \overline{\Omega}\right\}.$$

At any interior point $x \in \Gamma^- \cap \Omega$, the supporting hyperplane condition forces $\nabla u(x) = p$ and $D^2 u(x) \geq 0$, i.e., the Hessian is non-negative definite. The contact set is thus the locus where the graph of u is touched from below by an affine function, and the gradient map $\Phi\colon x \mapsto \nabla u(x)$ is well-defined on it.

Remark 2.1. The contact set plays a role analogous to that of the subdifferential in convex analysis. On $\Gamma^- \cap \Omega$, the function u is "locally convex" in the sense that its Hessian is non-negative definite. This is the key structural feature that makes the AM-GM estimate in Lemma 2.3 below possible.

The following surjectivity result is the heart of the proof. It shows that the gradient map on the contact set captures at least the open unit ball.

Lemma 2.2 (Surjectivity of the gradient map)
Let u be a solution of (2.1). Then $B_1(0) \subset \nabla u(\Gamma^- \cap \Omega)$.

Proof. Let $e \in \mathbb{R}^n$ with $|e| < 1$. Consider the function $\varphi(x) = u(x) - e \cdot x$ on $\overline{\Omega}$. Since $\overline{\Omega}$ is compact, φ attains its minimum at some point $x_0 \in \overline{\Omega}$. We claim that $x_0 \in \Omega$, i.e., the minimum is attained in the interior.

Suppose for contradiction that $x_0 \in \partial\Omega$. Then the outward normal derivative of φ at

x_0 must satisfy

$$\frac{\partial \varphi}{\partial \nu}(x_0) \leq 0.$$

On the other hand,

$$\frac{\partial \varphi}{\partial \nu}(x_0) = \frac{\partial u}{\partial \nu}(x_0) - e \cdot \nu(x_0) = 1 - e \cdot \nu(x_0) \geq 1 - |e| > 0,$$

which is a contradiction. Therefore $x_0 \in \Omega$.

Since x_0 is an interior minimizer, we have $\nabla \varphi(x_0) = 0$, i.e., $\nabla u(x_0) = e$. Moreover, for all $y \in \overline{\Omega}$,

$$u(y) - e \cdot y = \varphi(y) \geq \varphi(x_0) = u(x_0) - e \cdot x_0,$$

which gives $u(y) \geq u(x_0) + e \cdot (y - x_0)$. This means $x_0 \in \Gamma^- \cap \Omega$ with $p = e$. Since e was an arbitrary point in $B_1(0)$, we conclude that $B_1(0) \subset \nabla u(\Gamma^- \cap \Omega)$.

$\square$

Remark 2.2. Lemma 2.2 is the most delicate step to in the manifold setting. The proof relies on two ingredients: the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 1$, which prevents the minimum from being attained on the boundary, and the flatness of $\mathbb{R}^n$, which ensures that the affine function $e \cdot x$ is globally well-defined and compatible with the supporting hyperplane construction.

When one moves to a Riemannian manifold (M^n, g), the role of the affine function $e \cdot x$ is replaced by the distance function or, more precisely, by geodesic-based constructions. In Brendle's work [14], the gradient map ∇u is replaced by a map Φ defined using the exponential map, and the surjectivity of Φ onto a suitable region relies on analyzing a matrix Riccati equation along geodesics. It is in this Riccati analysis that the curvature of the manifold enters: non-negative sectional (or intermediate Ricci) curvature provides the monotonicity estimates needed to control the Jacobian of Φ.

The second ingredient is the pointwise Jacobian estimate on the contact set.

Lemma 2.3 (AM-GM estimate on the contact set)

At any point $x \in \Gamma^- \cap \Omega$, we have

$$\det D^2 u(x) \leq \left(\frac{c}{n}\right)^n,$$

where $c = \frac{|\partial\Omega|}{|\Omega|}$.

Proof. At $x \in \Gamma^- \cap \Omega$, the Hessian $D^2u(x)$ is non-negative definite. Let $\lambda_1, \ldots, \lambda_n \geq 0$ be its eigenvalues. Since $\Delta u(x) = c$, we have $\lambda_1 + \cdots + \lambda_n = c$. The arithmetic-geometric mean inequality gives

$$\det D^2 u(x) = \lambda_1 \cdots \lambda_n \leq \left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right)^n = \left(\frac{c}{n}\right)^n. \qquad \square$$

Remark 2.3. In the Riemannian setting, the AM-GM inequality is replaced by a more refined Jacobian estimate that accounts for the curvature of the manifold. The Laplacian condition $\Delta u = c$ becomes a divergence-form equation, and the eigenvalue bound is obtained through a monotonicity formula for the Jacobian of the transport map along geodesics. The curvature of the manifold contributes additional terms to this monotonicity formula via the Riccati equation, and the sign of these terms is controlled by the assumed curvature condition.

We are now in a position to complete the proof of the isoperimetric inequality.

Proof of Theorem 2.1. By Lemma 2.2, $B_1(0) \subset \nabla u(\Gamma^- \cap \Omega)$. The area formula (change of variables for ∇u) gives

$$|B_1| \leq \int_{\Gamma^- \cap \Omega} \det D^2 u \, dx.$$

Applying the pointwise estimate of Lemma 2.3 and using the fact that $\Gamma^- \cap \Omega \subset \Omega$, we obtain

$$\omega_n = |B_1| \leq \left(\frac{c}{n}\right)^n |\Omega|.$$

Substituting $c = |\partial\Omega| \,/\, |\Omega|$ yields

$$\omega_n \leq \frac{|\partial\Omega|^n}{n^n \, |\Omega|^n} \cdot |\Omega| = \frac{|\partial\Omega|^n}{n^n \, |\Omega|^{n-1}},$$

which is equivalent to $|\partial\Omega|^n \geq n^n\omega_n |\Omega|^{n-1}$.

Equality is achieved when Ω is a ball B_R: since $|\partial B_R| = n\omega_n R^{n-1}$ and $|B_R| = \omega_n R^n$, one verifies directly that $|\partial B_R|^n = n^n\omega_n |B_R|^{n-1}$.

Conversely, suppose that equality holds in the isoperimetric inequality, that is,

$$|\partial\Omega|^n = n^n\omega_n |\Omega|^{n-1} .$$

Recall that the proof established the chain of inequalities

$$|B_1| \leq \int_{\Gamma_v} \det D^2 v \, dx \leq \int_{\Gamma_v} \left(\frac{\Delta v}{n}\right)^n dx = \left(\frac{|\partial\Omega|}{n\,|\Omega|}\right)^n |\Gamma_v| \leq \left(\frac{|\partial\Omega|}{n\,|\Omega|}\right)^n |\Omega| . \tag{2.2}$$

If the isoperimetric inequality is an equality, then

$$|B_1| = \left(\frac{|\partial\Omega|}{n\,|\Omega|}\right)^n |\Omega| ,$$

so all four quantities in (2.2) must coincide. The last inequality becoming an equality forces the contact set to have full measure: $|\Gamma_v| = |\Omega|$.

Equality in the arithmetic-geometric mean inequality, applied to the eigenvalues of $D^2 v(x)$ at each point of Γ_v, forces all eigenvalues to be equal. Since $\Delta v = |\partial\Omega| \,/\, |\Omega|$ is constant, writing $c = |\partial\Omega| \,/(n\,|\Omega|)$ gives

$$D^2 v(x) = c\, I \qquad \text{for every } x \in \Gamma_v.$$

Since Γ_v has full measure in Ω and $v \in C^{2,\gamma}(\bar{\Omega})$, continuity extends this to all of Ω:

$$D^2 v \equiv c\, I \qquad \text{throughout } \Omega.$$

It follows that v must be a quadratic,

$$v(x) = \frac{c}{2}\,|x - x_0|^2 + a,$$

for some point $x_0 \in \mathbb{R}^n$ and constant $a \in \mathbb{R}$, with gradient $\nabla v(x) = c(x - x_0)$. The surjectivity established earlier, $B_1 \subseteq \nabla v(\Gamma_v)$, now reads

$$B_1 \subseteq c(\Omega - x_0),$$

which gives $\Omega \supseteq B_{1/c}(x_0)$. On the other hand, the condition $|\nabla v| \leq 1$ on Γ_v yields

$$c\,|x - x_0| \leq 1 \qquad \text{for a.e. } x \in \Omega,$$

so $\Omega \subseteq \overline{B_{1/c}(x_0)}$. Combining these two inclusions,

$$\Omega = B_{1/c}(x_0) = B_R(x_0), \qquad R = \frac{1}{c} = \frac{n\,|\Omega|}{|\partial\Omega|},$$

so Ω is a ball (see [18]).

$\square$

Remark 2.4. We close this subsection by highlighting the duality between the ABP approach and optimal transport as pointed in [14] and [13]. In the optimal transport proof of the isoperimetric inequality (see, e.g., [24]), one solves the Monge–Ampère equation $\det D^2 u = 1$ for a convex function u and concludes $\Delta u \geq n$ by AM-GM. In the ABP approach, the logic is reversed: one solves $\Delta u = c$ and concludes $\det D^2 u \leq (c/n)^n$ on the contact set by AM-GM. Both approaches yield the isoperimetric inequality, but the ABP method has the advantage of requiring only linear PDE theory. This makes it more amenable to generalization to curved settings, where the Monge–Ampère approach faces additional difficulties.

2.1.2 The Sobolev and Logarithmic Sobolev Inequalities

The isoperimetric inequality (Theorem 2.1) admits a natural functional extension. Rather than comparing the volume and perimeter of a domain directly, one can ask for an inequality involving a test function f and its gradient. This leads to the Sobolev inequality.

Theorem 2.4 (Sobolev inequality in $\mathbb{R}^n$)

Let D be a compact domain in $\mathbb{R}^n$ with smooth boundary. Let f be a smooth positive function on D. Then

$$\int_D |\nabla f| + \int_{\partial D} f \geq n\,|B^n|^{\frac{1}{n}} \left(\int_D f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}},$$

where B^n denotes the open unit ball in $\mathbb{R}^n$.

Choosing $f \equiv 1$ in Theorem 2.4 reduces the left-hand side to $|\partial D|$ and the right-hand side to $n\,|B^n|^{1/n}\,|D|^{(n-1)/n}$, so the isoperimetric inequality is recovered as a special case. Conversely, the Sobolev inequality can be proved using the coarea formula together with the isoperimetric inequality applied to level sets of f. This equivalence is classical and we omit the details.

What is more relevant for us is that the Sobolev inequality also admits a direct proof via the ABP method. The argument parallels the one given for Theorem 2.1: the Laplace equation $\Delta u = c$ is replaced by a weighted divergence-form equation $\operatorname{div}(f\nabla u) = \text{RHS}$ with the same Neumann condition $\langle \nabla u, \nu\rangle = 1$ on ∂D, and the surjectivity and AM-GM steps carry through with minor modifications. We refer to Cabré [19] and Brendle [14] for the complete argument.

Beyond the Sobolev inequality, there is a logarithmic variant that plays a central role in this thesis. Logarithmic Sobolev inequalities control the entropy of a function in terms of its energy, and they arise naturally in the study of concentration of measure, heat kernel estimates, and information theory (see Chapter 1 for a broader discussion). In the Euclidean submanifold setting, the sharp result is due to Brendle [16].

Theorem 2.5 (Logarithmic Sobolev inequality in $\mathbb{R}^{n+m}$, Brendle [16])
Let Σ be a compact n-dimensional submanifold of $\mathbb{R}^{n+m}$ without boundary. Let f be a positive smooth function on Σ. Then

$$\int_\Sigma f\left(\log f + n + \frac{n}{2}\log(4\pi)\right) - \int_\Sigma \frac{|\boldsymbol{\nabla} f|^2}{f} - \int_\Sigma f\,|H|^2 \leq \left(\int_\Sigma f\right)\log\left(\int_\Sigma f\right),$$

where H denotes the mean curvature vector of Σ.

Remark 2.5. By the substitution $f = (4\pi)^{-n/2}e^{-|x|^2/4}\,\varphi$, Theorem 2.5 can be recast as an inequality with respect to the Gaussian measure $d\gamma = (4\pi)^{-n/2}e^{-|x|^2/4}\,d\mathrm{vol}$ on Σ, in which the submanifold geometry enters through the shifted mean curvature $H + \frac{1}{2}x^{\perp}$. In this form it sharpens Ecker's logarithmic Sobolev inequality for submanifolds [30] and connects to the classical inequality of Gross [36] for the Gaussian measure on $\mathbb{R}^n$.

The proof of Theorem 2.5 again fits into the ABP scheme, though the PDE and the geometry of the transport map are more involved. Since Σ has no boundary, there is no Neumann condition; instead, the divergence-form equation

$$\mathrm{div}_\Sigma(f\,\boldsymbol{\nabla} u) = f\log f - \frac{|\boldsymbol{\nabla} f|^2}{f} - f\,|H|^2 \quad \text{on } \Sigma$$

is solvable by the Fredholm alternative (the right-hand side integrates to zero by the normalization assumption). The gradient map $\boldsymbol{\nabla} u$ is then lifted to a map Φ from a subset of the normal bundle of Σ into $\mathbb{R}^{n+m}$, defined using the ambient exponential map, and the surjectivity and Jacobian estimates proceed as before. See [16, 14] for the full details.

In the next subsection, we give a detailed proof of the global logarithmic Sobolev inequality on $\mathbb{R}^n$ (Theorem 2.6 below), following Wang, Xia, and Zhang [80]. This inequality is distinct from Theorem 2.5, which concerns compact submanifolds: it applies to the non-compact setting $\Sigma = \mathbb{R}^n$ without boundary and involves a Gaussian weight rather than a boundary term. Nevertheless, the proof shares the same ABP architecture, an elliptic PDE, a surjectivity argument on a contact set, and an AM-GM Jacobian bound and it is the first application of this technique in a non-compact setting.

2.1.3 The Global Logarithmic Sobolev Inequality in $\mathbb{R}^n$

We now present a complete ABP-based proof of the classical global logarithmic Sobolev inequality on $\mathbb{R}^n$ using the ABP method.

Theorem 2.6 (Euclidean logarithmic Sobolev inequality)
Let $f \geq 0$ be a smooth function on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} f\,dx = 1$. Then

$$\int_{\mathbb{R}^n} f \log f\,dx \leq \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\,dx - \left(n + \frac{n}{2}\log(4\pi)\right). \tag{2.3}$$

Without the normalization $\int f = 1$, the inequality takes the following equivalent form: for any $f \geq 0$ with $0 < \int_{\mathbb{R}^n} f\,dx < \infty$,

$$\int_{\mathbb{R}^n} f \log f\,dx - \int_{\mathbb{R}^n} f\,dx \log \int_{\mathbb{R}^n} f\,dx \leq \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\,dx - \left(n + \frac{n}{2}\log(4\pi)\right)\int_{\mathbb{R}^n} f\,dx. \tag{2.4}$$

Indeed, setting $M = \int f\,dx$ and $\tilde{f} = f/M$ and applying (2.3) to $\tilde{f}$ yields (2.4) after multiplying through by M.

Both formulations are equivalent to the Gaussian logarithmic Sobolev inequality. We define the Gaussian probability measure on $\mathbb{R}^n$ by $d\gamma = \gamma(x)\,dx$, where $\gamma(x) = (4\pi)^{-n/2}e^{-|x|^2/4}$.

Corollary 2.7 (Gaussian form)
For any $\varphi \geq 0$ with $\int_{\mathbb{R}^n} \varphi\,d\gamma = 1$,

$$\int_{\mathbb{R}^n} \varphi \log \varphi\,d\gamma \leq \int_{\mathbb{R}^n} \frac{|\nabla \varphi|^2}{\varphi}\,d\gamma. \tag{2.5}$$

Proof. Given $\varphi \geq 0$ with $\int \varphi\,d\gamma = 1$, set $f = \varphi\,\gamma$. Then $\int f\,dx = 1$ and

$$\nabla f = \gamma\left(\nabla\varphi - \frac{x}{2}\varphi\right),$$

so that

$$\frac{|\nabla f|^2}{f} = \gamma\,\frac{\left|\nabla\varphi - \frac{x}{2}\varphi\right|^2}{\varphi} = \gamma\left(\frac{|\nabla\varphi|^2}{\varphi} - x\cdot\nabla\varphi + \frac{|x|^2}{4}\varphi\right).$$

Also,

$$f \log f = \varphi\, \gamma \,(\log \varphi + \log \gamma) = \varphi\, \gamma \log \varphi + \varphi\, \gamma \left(-\frac{n}{2} \log(4\pi) - \frac{|x|^2}{4} \right).$$

Substituting into (2.3), the $\frac{|x|^2}{4}\varphi$ terms cancel. For the remaining cross term, we integrate by parts:

$$-\int_{\mathbb{R}^n} x \cdot \nabla\varphi \, d\gamma = \int_{\mathbb{R}^n} \varphi \, \mathrm{div}_\gamma(x) \, d\gamma, \quad \text{where } \mathrm{div}_\gamma(x) = \mathrm{div}(x) + x \cdot \nabla \log \gamma = n - \frac{|x|^2}{2}.$$

All extra terms cancel, yielding (2.5). □

Remark 2.6. The sharp constant $n + \frac{n}{2}\log(4\pi)$ in (2.3) is precisely the entropy–Fisher information gap evaluated at the Gaussian density γ:

$$n + \frac{n}{2}\log(4\pi) = \int_{\mathbb{R}^n} \frac{|\nabla\gamma|^2}{\gamma} \, dx - \int_{\mathbb{R}^n} \gamma \log \gamma \, dx.$$

In other words, the Gaussian saturates the inequality. The clean form (2.5), free of any explicit constant, reflects that the Gaussian measure is the natural background for the logarithmic Sobolev inequality: the constant has been absorbed into the measure.

Reduction to a PDE problem

We now describe the ABP proof of Theorem 2.6. Assume $\int_{\mathbb{R}^n} f \, dx = 1$ and suppose we can find a function $u \colon \mathbb{R}^n \to \mathbb{R}$ and a constant α satisfying:

(i) **(PDE)** $\mathrm{div}(f\nabla u) = f \log f - \dfrac{|\nabla f|^2}{f} + \alpha f \quad \text{in } \mathbb{R}^n,$

(ii) **(Growth)** $\dfrac{u(x)}{|x|} \to +\infty \quad \text{as } |x| \to \infty,$

(iii) **(Divergence-free)** $\displaystyle\int_{\mathbb{R}^n} \mathrm{div}(f\nabla u) \, dx = 0.$

Integrating the PDE over $\mathbb{R}^n$ and using the divergence-free condition gives

$$0 = \int_{\mathbb{R}^n} f \log f \, dx - \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, dx + \alpha,$$

so that

$$\alpha = \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, dx - \int_{\mathbb{R}^n} f \log f \, dx. \tag{2.6}$$

Thus the logarithmic Sobolev inequality (2.3) is equivalent to the lower bound

$$\alpha \geq n + \frac{n}{2} \log(4\pi). \tag{2.7}$$

The main difficulty in the non-compact setting is the existence of a solution (u, α) satisfying all three conditions. The key observation, due to Wang, Xia, and Zhang [80], is that the Gaussian density provides a canonical solution to the PDE.

Lemma 2.8 (The Gaussian extremizer)
Let $f_0(x) = (4\pi)^{-n/2} e^{-|x|^2/4}$, $u_0(x) = \frac{1}{2}|x|^2$, and $\alpha_0 = n + \frac{n}{2}\log(4\pi)$. Then (u_0, α_0) solves the PDE, satisfies the growth and divergence-free conditions.

Proof. We have $\nabla u_0 = x$ and $\Delta u_0 = n$. Also $\log f_0 = -\frac{n}{2}\log(4\pi) - \frac{|x|^2}{4}$, so $\nabla \log f_0 = -\frac{x}{2}$ and $\nabla f_0 = -\frac{x}{2} f_0$, giving $\frac{|\nabla f_0|^2}{f_0} = \frac{|x|^2}{4} f_0$. The left-hand side of the PDE is

$$\operatorname{div}(f_0 \nabla u_0) = \operatorname{div}(f_0\, x) = \nabla f_0 \cdot x + f_0 \operatorname{div}(x) = -\frac{|x|^2}{2} f_0 + n f_0 = \left(n - \frac{|x|^2}{2}\right) f_0.$$

The right-hand side is

$$\begin{aligned} f_0 \log f_0 - \frac{|\nabla f_0|^2}{f_0} + \alpha_0 f_0 &= \left(-\frac{n}{2}\log(4\pi) - \frac{|x|^2}{4}\right) f_0 - \frac{|x|^2}{4} f_0 + \left(n + \frac{n}{2}\log(4\pi)\right) f_0 \\ &= \left(n - \frac{|x|^2}{2}\right) f_0. \end{aligned}$$

Both sides agree. The growth condition is immediate since $u_0(x)/|x| = |x|/2 \to \infty$. For the divergence-free condition,

$$\int_{\mathbb{R}^n} \operatorname{div}(f_0 \nabla u_0)\, dx = \int_{\mathbb{R}^n} \left(n - \frac{|x|^2}{2}\right) f_0 \, dx = n - \frac{1}{2}\mathbb{E}[|X|^2] = n - \frac{1}{2}(2n) = 0,$$

since f_0 is the density of $\mathcal{N}(0, 2I)$ and thus $\mathbb{E}[|X|^2] = 2n$. $\square$

Using this canonical solution, one perturbs f by setting $f_\varepsilon = \frac{1}{1+\varepsilon}(f+\varepsilon f_0)$ for $\varepsilon > 0$ and solves the PDE for f_ε by writing $u = u_0 + w$, where w satisfies a linear elliptic equation with bounded coefficients. The existence of w and the verification of the growth and divergence-free conditions are carried out in detail in [80, Section 2]. For our purposes, we take the existence of $(u_\varepsilon, \alpha_\varepsilon)$ as given and focus on the ABP argument that yields the lower bound (2.7).

The ABP argument

Assume (u, α) satisfies conditions (i)–(iii) above for some $f > 0$ with $\int f = 1$. We define the contact set and the gradient map as follows.

Definition 2.1. The *contact set* of u is $U = \{x \in \mathbb{R}^n : D^2u(x) \geq 0\}$, and the *gradient map* is $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $\Phi(x) = \nabla u(x)$.

Lemma 2.9 (Surjectivity)
We have $\Phi(U) = \mathbb{R}^n$.

Proof. Fix $\xi \in \mathbb{R}^n$ and consider $q(x) = u(x) - \xi \cdot x$. The growth condition $u(x)/|x| \to +\infty$ ensures that $q(x) \to +\infty$ as $|x| \to \infty$, so q attains a global minimum at some $\bar{x} \in \mathbb{R}^n$. At this minimizer, $\nabla q(\bar{x}) = 0$ gives $\nabla u(\bar{x}) = \xi$, and $D^2q(\bar{x}) \geq 0$ gives $D^2u(\bar{x}) \geq 0$, so $\bar{x} \in U$ and $\Phi(\bar{x}) = \xi$. $\square$

Remark 2.7. In contrast to the isoperimetric case (Lemma 2.2), where surjectivity onto the unit ball B_1 was achieved using the Neumann condition $\frac{\partial u}{\partial \nu} = 1$ to prevent the minimum from reaching the boundary, here the domain is all of $\mathbb{R}^n$ and there is no boundary. Instead, the superlinear growth of u ensures that $q(x) \to +\infty$ at infinity, guaranteeing a global interior minimizer. This is precisely the role of the growth condition (ii).

The next step is to obtain a pointwise bound on the Jacobian $\det D^2u$ on the contact set.

Lemma 2.10 (Jacobian bound)

On the contact set U, we have

$$e^{-\frac{1}{4}|\nabla u|^2} \det D^2 u \le f\, e^{\alpha - n}. \tag{2.8}$$

Proof. Expanding the PDE $\operatorname{div}(f\nabla u) = f\Delta u + \nabla f \cdot \nabla u$ and dividing by f gives

$$\Delta u + \frac{\nabla f}{f} \cdot \nabla u = \log f - \frac{|\nabla f|^2}{f^2} + \alpha.$$

Rearranging,

$$\Delta u = \log f - |\nabla \log f|^2 + \alpha - \nabla \log f \cdot \nabla u.$$

We complete the square: setting $A = \nabla \log f$ and $B = \nabla u$,

$$-|A|^2 - A \cdot B = -\left|A + \tfrac{1}{2}B\right|^2 + \tfrac{1}{4}|B|^2 \le \tfrac{1}{4}|B|^2,$$

so that

$$\Delta u \le \log f + \alpha + \frac{1}{4}|\nabla u|^2. \tag{2.9}$$

On U, the Hessian $D^2 u$ is non-negative definite with eigenvalues $\lambda_1, \dots, \lambda_n \ge 0$ satisfying $\lambda_1 + \cdots + \lambda_n = \Delta u$. By the AM-GM inequality,

$$\det D^2 u = \prod_{i=1}^{n} \lambda_i \le \left(\frac{\Delta u}{n}\right)^n.$$

Since $\log t \le t - 1$ applied with $t = \Delta u / n$ gives $\left(\frac{\Delta u}{n}\right)^n \le e^{\Delta u - n}$, we obtain $\det D^2 u \le e^{\Delta u - n}$. Combining with (2.9),

$$\Delta u - n \le \log f + \alpha - n + \frac{1}{4}|\nabla u|^2,$$

and therefore

$$e^{-\frac{1}{4}|\nabla u|^2} \det D^2 u \le e^{-\frac{1}{4}|\nabla u|^2} e^{\Delta u - n} \le f\, e^{\alpha - n}. \qquad \square$$

Proof of Theorem 2.6. We carry out the proof for $f_\varepsilon = \frac{1}{1+\varepsilon}(f + \varepsilon f_0)$ and pass to the limit at the end. Let $(u_\varepsilon, \alpha_\varepsilon)$ be the solution provided by [80, Proposition 2.2]. By Lemma 2.9, $\Phi(U) = \mathbb{R}^n$. Using the area formula and Lemma 2.10,

$$\begin{aligned}
1 &= (4\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-|\xi|^2/4}\, d\xi \\
&\le (4\pi)^{-n/2} \int_U e^{-|\nabla u(x)|^2/4} \det D^2 u(x)\, dx \\
&\le (4\pi)^{-n/2} \int_U f_\varepsilon(x)\, e^{\alpha_\varepsilon - n}\, dx \\
&\le (4\pi)^{-n/2}\, e^{\alpha_\varepsilon - n} \int_{\mathbb{R}^n} f_\varepsilon\, dx \\
&= (4\pi)^{-n/2}\, e^{\alpha_\varepsilon - n}.
\end{aligned}$$

In the first inequality, we used $\Phi(U) = \mathbb{R}^n$ together with the change of variables $\xi = \nabla u(x)$. It follows that $e^{\alpha_\varepsilon - n} \ge (4\pi)^{n/2}$, i.e.,

$$\alpha_\varepsilon \ge n + \frac{n}{2} \log(4\pi).$$

By the identity (2.6) applied to f_ε and the divergence-free condition, this gives the logarithmic Sobolev inequality for f_ε. Passing to the limit $\varepsilon \to 0$ by dominated convergence yields (2.3) for f. □

Remark 2.8. The proof exhibits the same three-step structure as the isoperimetric case: an appropriate elliptic PDE, a surjectivity lemma for the gradient map, and a pointwise Jacobian bound via AM-GM. The key differences in the non-compact setting are: the surjectivity relies on superlinear growth of u rather than a Neumann boundary condition, and the Jacobian bound involves an additional Gaussian weight $e^{-|\nabla u|^2/4}$ arising from the completing-the-square step. This Gaussian weight is what connects the ABP proof to the Gaussian measure and explains why the sharp constant involves $\log(4\pi)$.

2.1.4 Brendle's Logarithmic Sobolev Inequality for Submanifolds

In this subsection we describe the proof of the sharp logarithmic Sobolev inequality for compact submanifolds of Euclidean space (Theorem 2.5), due to Brendle [16]. While the global logarithmic Sobolev inequality on $\mathbb{R}^n$ proved in the previous subsection involves a single gradient map $\Phi = \nabla u\colon \mathbb{R}^n \to \mathbb{R}^n$, the submanifold case requires a richer construction: the map Φ is defined on the total space of the normal bundle of Σ and takes values in the full ambient space $\mathbb{R}^{n+m}$. This is the same construction that, when adapted to curved ambient spaces, yields the results described in our work [54] and the subsequent developments surveyed in Section 2.2.

Setup. Let $\Sigma^n \subset \mathbb{R}^{n+m}$ be a compact connected submanifold without boundary, with second fundamental form II and mean curvature vector H. Let $f > 0$ be smooth on Σ. By scaling, one may assume

$$\int_\Sigma f \log f \, d\mathrm{vol} - \int_\Sigma \frac{|\boldsymbol{\nabla} f|^2}{f} \, d\mathrm{vol} - \int_\Sigma f \, |H|^2 \, d\mathrm{vol} = 0. \tag{2.10}$$

Since Σ is compact and connected, the operator $u \mapsto \mathrm{div}_\Sigma(f \boldsymbol{\nabla} u)$ has one-dimensional kernel (the constants) and cokernel. The normalization (2.10) ensures that the right-hand side

$$\mathrm{div}_\Sigma(f \boldsymbol{\nabla} u) = f \log f - \frac{|\boldsymbol{\nabla} f|^2}{f} - f \, |H|^2 \tag{2.11}$$

integrates to zero, so a smooth solution u exists by the Fredholm alternative.

The normal bundle map. The key departure from the codimension-zero case is the definition of the transport map. One considers the total space of the normal bundle $U = \big\{(x, y) : x \in \Sigma,\, y \in T_x^\perp \Sigma\big\}$ and defines a map $\Phi\colon U \to \mathbb{R}^{n+m}$ by

$$\Phi(x, y) = \boldsymbol{\nabla} u(x) + y.$$

Since $\boldsymbol{\nabla} u(x) \in T_x\Sigma$ and $y \in T_x^\perp\Sigma$ are orthogonal, we have $|\Phi(x,y)|^2 = |\boldsymbol{\nabla} u(x)|^2 + |y|^2$. The contact set is defined as

$$A = \left\{(x,y) \in U : D^2_\Sigma u(x) - \langle \mathrm{II}(x), y\rangle \geq 0\right\},$$

where $\langle \mathrm{II}(x), y\rangle$ denotes the symmetric bilinear form $(\xi,\eta) \mapsto \langle \mathrm{II}(\xi,\eta), y\rangle$ and the inequality means non-negative definiteness.

Remark 2.9. When $m = 0$ (the codimension-zero case), the normal bundle is trivial, $y = 0$, $H = 0$, and A reduces to $\{x \in \mathbb{R}^n : D^2u(x) \geq 0\}$, which is the contact set U from Definition 2.1. For $m \geq 1$, the condition $D^2_\Sigma u(x) - \langle \mathrm{II}(x), y\rangle \geq 0$ couples the intrinsic Hessian of u with the extrinsic geometry of Σ through the second fundamental form.

The three main lemmas of the proof mirror the three steps of the codimension-zero argument—surjectivity, Jacobian computation, and Jacobian bound—but now adapted to the normal bundle setting.

Lemma 2.11 (Surjectivity, [16, Lemma 3])
$\Phi(A) = \mathbb{R}^{n+m}$.

The proof is short and elegant. Given $\xi \in \mathbb{R}^{n+m}$, one minimizes the function $w(x) = u(x) - \langle x, \xi\rangle$ over the compact manifold Σ. At a minimizer $\bar{x}$, the first-order condition $\boldsymbol{\nabla} w(\bar{x}) = 0$ gives $\boldsymbol{\nabla} u(\bar{x}) = \xi^\top$ (the tangential component), so $\xi = \boldsymbol{\nabla} u(\bar{x}) + \bar{y}$ where $\bar{y} = \xi^\perp \in T^\perp_{\bar{x}}\Sigma$. The second-order condition $D^2_\Sigma w(\bar{x}) \geq 0$ yields $D^2_\Sigma u(\bar{x}) - \langle \mathrm{II}(\bar{x}), \xi\rangle \geq 0$, and since $\langle \mathrm{II}(\bar{x}), \xi\rangle = \langle \mathrm{II}(\bar{x}), \bar{y}\rangle$ (the tangential part of ξ does not contribute), this shows $(\bar{x}, \bar{y}) \in A$ with $\Phi(\bar{x}, \bar{y}) = \xi$.

Remark 2.10. In the isoperimetric proof (Lemma 2.2), surjectivity onto B_1 was achieved by the Neumann boundary condition. In the global log-Sobolev proof (Lemma 2.9), surjectivity onto all of $\mathbb{R}^n$ was achieved by superlinear growth of u. Here, surjectivity onto all of $\mathbb{R}^{n+m}$ is achieved by the compactness of Σ: a continuous function on a compact set always attains its minimum, and the decomposition $\xi = \xi^\top + \xi^\perp$ into tangential and normal components does the rest. The normal directions are "free," in the sense that the map Φ is automatically surjective in the fiber directions—it is the tangential surjectivity that requires the contact set argument.

Lemma 2.12 (Jacobian formula, [16, Lemma 4])
For each $(x, y) \in U$,

$$\det D\Phi(x, y) = \det \left(D^2_\Sigma u(x) - \langle \mathrm{II}(x), y \rangle \right) .$$

This follows from the block structure of $D\Phi$: the tangential-to-tangential block is $D^2_\Sigma u(x) - \langle \mathrm{II}(x), y \rangle$, the normal-to-normal block is the identity, and the off-diagonal blocks cancel. In particular, on the contact set A, the Jacobian is nonnegative.

Lemma 2.13 (Jacobian bound, [16, Lemma 5])
For each $(x, y) \in A$,

$$0 \leq e^{-\frac{|\Phi(x,y)|^2}{4}} \det D\Phi(x, y) \leq f(x)\, e^{-\frac{|2H(x)+y|^2}{4} - n}. \tag{2.12}$$

The proof follows the same completing-the-square strategy as in Lemma 2.10. Dividing the PDE (2.11) by f and rearranging gives

$$\Delta_\Sigma u - \langle H, y \rangle = \log f - |\nabla_\Sigma \log f|^2 + \alpha - \nabla_\Sigma \log f \cdot \nabla_\Sigma u - \langle H, y \rangle ,$$

where the right-hand side is bounded above by $\log f + \frac{1}{4}(|\boldsymbol{\nabla} u|^2 + |y|^2) - \frac{1}{4} |2H + y|^2$ after completing the square. The AM-GM inequality $\det B \leq e^{\operatorname{tr} B - n}$ for the nonnegative definite matrix $B = D^2_\Sigma u - \langle \mathrm{II}, y \rangle$ (whose trace is $\Delta_\Sigma u - \langle H, y \rangle$) then yields the bound.

Proof of Theorem 2.5. The three lemmas are combined through the area formula applied to $\Phi\colon U \to \mathbb{R}^{n+m}$:

$$\begin{aligned}
1 &= (4\pi)^{-\frac{n+m}{2}} \int_{\mathbb{R}^{n+m}} e^{-\frac{|\xi|^2}{4}}\, d\xi \\
&\le (4\pi)^{-\frac{n+m}{2}} \int_\Sigma \left(\int_{T_x^\perp \Sigma} e^{-\frac{|\Phi(x,y)|^2}{4}} \, |\det D\Phi(x,y)| \; \mathbf{1}_A(x,y)\, dy \right) d\mathrm{vol}(x) \\
&\le (4\pi)^{-\frac{n+m}{2}} \int_\Sigma f(x) \left(\int_{T_x^\perp \Sigma} e^{-\frac{|2H(x)+y|^2}{4} - n}\, dy \right) d\mathrm{vol}(x).
\end{aligned}$$

The inner integral over the m-dimensional normal fiber is a Gaussian integral:

$$\int_{T_x^\perp \Sigma} e^{-\frac{|2H(x)+y|^2}{4}}\, dy = (4\pi)^{m/2},$$

since the integrand is a Gaussian centered at $-2H(x)$ with covariance $2I_m$, and a Gaussian integral is invariant under translation. This cancels the $(4\pi)^{-m/2}$ factor, leaving

$$1 \le (4\pi)^{-n/2}\, e^{-n} \int_\Sigma f \, d\mathrm{vol},$$

which gives $n + \frac{n}{2}\log(4\pi) \le \log \int_\Sigma f\, d\mathrm{vol}$. Combined with the normalization (2.10), this yields Theorem 2.5.

Remark 2.11. Several features of this proof are worth highlighting, as they reappear in the Riemannian setting:

(i) The map $\Phi(x,y) = \boldsymbol{\nabla} u(x) + y$ uses the ambient Euclidean structure to combine tangential and normal components. On a Riemannian manifold, this is replaced by $\Phi(x,y) = \exp_x(\boldsymbol{\nabla} u(x) + y)$, where the exponential map of the ambient manifold provides the nonlinear analogue. The Jacobian of Φ then involves the Jacobian of the exponential map, which is controlled by the curvature of the ambient space.

(ii) The Gaussian integral over the normal fiber produces the factor $(4\pi)^{m/2}$ that absorbs the codimension. On a manifold with nonnegative sectional curvature, the Bishop–Gromov inequality $\mathrm{Jac}\exp_x \le 1$ ensures that the normal fiber integral is *bounded above* by the Euclidean value, so the same cancellation occurs.

(iii) The contact set condition $D^2_\Sigma u - \langle \mathrm{II}, y\rangle \geq 0$ couples the PDE (through $D^2_\Sigma u$) with the extrinsic geometry (through II). On a curved manifold, the Hessian comparison from Lemma 2.14 provides an additional correction to this condition, and the k-intermediate Ricci curvature enters through the bounds on the partial traces of $D^2(d_y^2/2)$.

2.2 The ABP Method on Riemannian Manifolds

The Euclidean proofs presented in the previous section rely on the flatness of the ambient space in two essential ways: the affine functions $e \cdot x$ used in the surjectivity argument are globally well-defined, and the Hessian of the squared distance function $|x|^2/2$ is the identity. On a Riemannian manifold, neither of these holds, and the ABP method must be reformulated. The key idea, due to Cabré [20], is to replace affine functions by *concave paraboloids* built from the squared Riemannian distance. We begin by describing this construction, as it motivates all subsequent developments.

2.2.1 The manifold contact set

Let (M^n, g) be a complete Riemannian manifold with nonnegative sectional curvature. The starting point is the following Hessian comparison, which is a consequence of the Rauch comparison theorem.

Lemma 2.14 (Hessian comparison, see e.g. [20, Lemma 3.1])
Let (M^n, g) be a complete Riemannian manifold with nonnegative sectional curvature. For any $y \in M$ and any $x \notin \mathrm{Cut}(y)$,

$$D^2\left(\frac{d_y^2}{2}\right)(x)(\xi,\xi) \leq |\xi|^2 \qquad \text{for all } \xi \in T_xM,$$

where $d_y = d(\cdot, y)$ denotes the Riemannian distance to y.

In $\mathbb{R}^n$, the squared distance function $|x-y|^2/2$ has Hessian equal to the identity, so Lemma 2.14 says that nonnegative sectional curvature makes the squared distance *less convex* than in the flat case. This is precisely the bound needed to control the Jacobian of the transport map.

In the Euclidean ABP argument (Subsection 2.1.1), the contact set Γ^- consists of points where the graph of u is touched from below by an affine function $a + e \cdot x$. On a manifold, there are no global affine functions, but for each $y \in M$ the function $C - d_y^2/2$ serves as a "concave paraboloid" that plays the same role. Given a function u on M and a parameter $R > 0$, one considers for each $y \in B_R$ the function

$$w_y(x) = R^2 u(x) + \frac{d(x,y)^2}{2},$$

which is the sum of the rescaled solution and the squared distance. Minimizing w_y over a suitable ball produces a contact point x at which the paraboloid $C - d_y^2/2$ touches $R^2 u$ from below. The following result, which is the heart of Cabré's approach, shows that the transport map defined via the exponential map is surjective, and that its Jacobian is controlled by the operator applied to u.

Lemma 2.15 (Cabré [20, Lemma 4.1])

Let u be a smooth function on B_{7R} satisfying $u \geq 0$ in $B_{7R} \setminus B_{5R}$ and $\inf_{B_{2R}} u \leq 1$. Define the map $\phi \colon B_{7R} \to M$ by

$$\phi(z) = \exp_z \nabla(R^2 u)(z).$$

Then:

(i) For every $y \in B_R$, there exists $x \in B_{5R}$ with $u(x) \leq 6$ such that w_y attains its minimum at x and

$$y = \exp_x \nabla(R^2 u)(x). \tag{2.13}$$

In particular, $B_R \subset \phi(\{u \leq 6\} \cap B_{5R})$.

(ii) The area formula and the Jacobian bound yield

$$|B_R| \leq \int_{\{u \leq 6\} \cap B_{5R}} \operatorname{Jac} \phi \, dV \leq \frac{1}{c_0^n} \int_{\{u \leq 6\} \cap B_{5R}} \left\{ (n^{-1} R^2 L u + C_0)_+ \right\}^n \, dV,$$

where $L = \operatorname{tr}(D^2 u \circ A)$ is a uniformly elliptic operator with ellipticity constants c_0, C_0.

The identity (2.13) is the manifold analogue of the Euclidean relation $y = x + R^2 \nabla u(x)$ that appeared in Lemma 2.2: the linear shift $x \mapsto x + p$ is replaced by the exponential map $x \mapsto \exp_x(p)$. The Jacobian bound in (ii) is obtained by combining Lemma 2.14 with the AM-GM inequality (as in Lemma 2.3) and the Bishop–Gromov inequality $\operatorname{Jac} \exp_x(\xi) \leq 1$, which holds under nonnegative Ricci curvature. The factorization

$$\operatorname{Jac} \phi(x) = \operatorname{Jac} \exp_x(\nabla v(x)) \cdot \left| \det D^2 \left(v + \frac{d_y^2}{2} \right)(x) \right|$$

(where $v = R^2 u$ and $y = \phi(x)$) makes the role of the Hessian comparison transparent: the matrix $D^2(v + d_y^2/2)(x)$ is nonnegative definite at the contact point, and its trace is bounded by $R^2 L u(x) + C_0$ thanks to Lemma 2.14.

Remark 2.12. In the Euclidean case, $d_y^2/2 = |x - y|^2/2$ has Hessian equal to the identity, so $D^2(v + d_y^2/2) = D^2 v + \mathrm{Id}$, and the contact condition reduces to $D^2 v \geq -\mathrm{Id}$, which is the usual notion of semiconvexity. On a general manifold, the contact condition involves the full Hessian $D^2(v + d_y^2/2)$, and the curvature enters through the bound on $D^2(d_y^2/2)$. This is the fundamental reason why the ABP method on manifolds requires curvature hypotheses.

2.2.2 Survey of results

Cabré's foundational work [20] used the contact set construction described above to prove a global Krylov–Safonov Harnack inequality for nondivergent elliptic operators on manifolds with nonnegative sectional curvature. Wang and Zhang [82] subsequently extended the ABP estimate to manifolds with a Ricci curvature lower bound, and Xia and Zhang [87] applied the manifold ABP framework to prove the Heintze–Karcher inequality and new Minkowski-type inequalities. In a different direction, Cabré, Ros-Oton, and Serra [21] used ABP in convex cones of $\mathbb{R}^n$ to establish sharp weighted isoperimetric inequalities with a natural class of homogeneous weights.

Brendle's results on manifolds. The results surveyed in Section 2.1 all concern submanifolds of Euclidean space. The passage to curved ambient spaces was accomplished by Brendle [14], who proved a sharp Sobolev inequality on complete non-compact manifolds with nonnegative Ricci curvature, depending on the asymptotic volume ratio, and a sharp Michael–Simon inequality for submanifolds in manifolds with nonnegative sectional curvature. Earlier, Brendle [15] had used the same technique to resolve a conjecture of Almgren by proving the sharp isoperimetric inequality for minimal submanifolds of $\mathbb{R}^{n+m}$ of codimension at most two. An alternative proof of the Michael–Simon–Sobolev inequality via optimal transport was given by Brendle and Eichmair [17]. Agostiniani, Fogagnolo, and Mazzieri [3] independently obtained sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature using nonlinear potential theory; Brendle's ABP method later provided an alternative proof of their main result. The ABP framework for geometric inequalities is surveyed comprehensively in [13].

Intermediate Ricci curvature. For submanifolds of codimension $m \geq 2$, Brendle's Michael–Simon inequality requires nonnegative sectional curvature, which is strictly stronger than nonnegative Ricci curvature. A natural question is whether this can be relaxed to a condition adapted to the codimension. Ma and Wu [57] showed that Brendle's Sobolev and Michael–Simon inequalities extend to manifolds with nonnegative k-intermediate Ricci curvature, where $k = \min\{n-1, m-1\}$, and Wang [81] obtained a related result via optimal transport. In our work [54], joint with Jihye Lee, we proved that the sharp logarithmic Sobolev inequality also holds under this intermediate curvature condition, and moreover extends to the asymptotically nonnegative setting (Theorem 2.16). Recently, Ji and Kwong [45] established a Fenchel–Willmore inequality with curvature index $k = \min\{n, m-1\}$.

Asymptotically nonnegative curvature. Another direction of generalization allows the curvature to be negative in compact regions, provided it decays at a controlled rate. Yi and Zheng [88] extended the logarithmic Sobolev inequality to manifolds with nonnegative sectional curvature, and Dong, Lin, and Lu [27, 29] further generalized both the Sobolev and logarithmic Sobolev inequalities to manifolds with asymptotically nonnegative sectional curvature. The same authors [28] treated the weighted setting of asymptotically nonnegative Bakry–Émery Ricci curvature, building on Johne's [48] earlier work in the nonnegative case. More recently, Impera, Rimoldi, and Veronelli [43] introduced the elliptic Kato constant of the negative part of the Ricci curvature as a tool for establishing isoperimetric inequalities in the presence of genuine negative curvature, and Impera, Pigola, Rimoldi, and Veronelli [44] developed a localized ABP technique that proves isoperimetric and Michael–Simon inequalities on manifolds whose sectional curvature is nonnegative only outside a compact set.

Non-compact submanifolds. Extending the ABP method to non-compact submanifolds introduces analytical difficulties beyond the curvature question: the PDE may lack a solution, the transport map may fail to be surjective, and the divergence theorem needs justification. As discussed in Section 2.1.3, Wang, Xia, and Zhang [80] addressed these issues by perturbing the data with the Gaussian extremizer, proving an optimal logarithmic Sobolev inequality for complete non-compact properly embedded self-shrinkers. Independently, Balogh and Kristály [7] established sharp logarithmic Sobolev inequalities on non-compact Euclidean submanifolds via optimal transport, characterizing the equality case as isometry to $\mathbb{R}^n$.

Free boundary and capillary problems. The ABP method has also been adapted to settings with boundary conditions. Liu, Wang, and Weng [55] proved optimal relative isoperimetric inequalities for minimal submanifolds with free boundary on a convex support surface, as well as relative Michael–Simon and logarithmic Sobolev inequalities. Fusco, Julin, Morini, and Pratelli [32] solved the capillary isoperimetric problem for hypersurfaces with general contact angle $\theta \in (0, \pi)$ outside arbitrary convex sets, and Jia, Wang, Xia, and Zhang [46] proved Willmore-type inequalities in unbounded convex sets with a sharp constant involving the asymptotic volume ratio.

Further extensions. Pham [69] applied the ABP method to prove a logarithmic Sobolev inequality for closed minimal submanifolds of the sphere, a Sobolev-type inequality for positive symmetric 2-tensors, and an inequality for quermassintegrals. Wu, Yi, and Zheng [86] generalized the Michael–Simon–Sobolev inequality to symmetric positive definite $(0, 2)$-tensor fields on submanifolds with nonnegative sectional curvature. Lu [56] established the optimal L^p logarithmic Sobolev inequality on manifolds with nonnegative Ricci curvature with sharp constants depending on the asymptotic volume ratio. In the non-smooth setting, Han [41] extended the ABP estimate to metric measure spaces with synthetic Ricci curvature lower bounds (RCD spaces) via optimal transport.

The breadth of these developments illustrates the versatility of the ABP contact set technique introduced in Section 2.1. In the next section, we present the full proof of our contribution to this program: the extension of the logarithmic Sobolev inequality to manifolds with non-negative and asymptotically non-negative k-intermediate Ricci curvature.

2.3 The Logarithmic Sobolev Inequality Under Intermediate Ricci Curvature

In this section we present the proof of the sharp logarithmic Sobolev inequality for submanifolds of complete non-compact Riemannian manifolds with asymptotically non-negative k-intermediate Ricci curvature and Euclidean volume growth. This is joint work with Jihye Lee [54], extending results of Brendle [16], Yi–Zheng [88], and Dong–Lin–Lu

[27].

2.3.1 Intermediate Ricci curvature

Let (M^n, g) be a complete non-compact Riemannian manifold. For $1 \leq k \leq n-1$, let P be a k-dimensional subspace of T_xM at $x \in M$. Given any unit tangent vector $v \in T_xM$ with $v \perp P$, the *k-intermediate Ricci curvature* (or *kth-Ricci curvature*) with respect to P in the direction of v is defined as

$$\mathrm{Ric}_k(P, v) = \sum_{i=1}^{k} \mathrm{Sec}(v, e_i)\,|v|^2\,,$$

where $\{e_1, \ldots, e_k\}$ is an orthonormal basis of P. The k-intermediate Ricci curvature interpolates between sectional and Ricci curvature: $\mathrm{Ric}_1 \geq 0$ is equivalent to non-negative sectional curvature, while $\mathrm{Ric}_{n-1} \geq 0$ is the usual non-negative Ricci curvature condition. A Riemannian manifold has non-negative k-Ricci curvature if $\mathrm{Ric}_k(P, v) \geq 0$ for any $x \in M$, any k-dimensional subspace $P \subset T_xM$, and any unit tangent vector $v \in T_xM$ perpendicular to P. This condition exhibits a monotonicity property: if $k_1 \leq k_2$, then $\mathrm{Ric}_{k_1} \geq 0$ implies $\mathrm{Ric}_{k_2} \geq 0$. (We note that some papers [81, 51] require the stronger condition that $\mathrm{Ric}_k(P, v) \geq 0$ holds for all v, not only those perpendicular to P.)

Historical background. The notion of k-intermediate Ricci curvature appears implicitly in the work of Bishop and Crittenden [12] and was first studied systematically by Hartman [42] in connection with oscillation criteria for second-order differential systems. The foundational results for manifolds with $\mathrm{Ric}_k \geq 0$ were established by Sha [76, 77], who introduced the notion of p-convexity and proved handle-attachment results, and by Wu [85], who studied manifolds of "partially positive curvature" and proved Frankel-type intersection theorems for compact submanifolds. Shen [78] subsequently obtained volume growth estimates, finite topological type results, and vanishing theorems for complete open manifolds of non-negative kth-Ricci curvature. Wilhelm [83] proved restrictions on the fundamental group under positive intermediate curvature, and Guijarro and Wilhelm [38, 40] later developed a “softer connectivity principle” and focal radius bounds that extended classical results of Frankel and Wilking [84] to the intermediate curvature setting.

In recent years, there has been a surge of activity on k-intermediate Ricci curvature, particularly from the point of view of geometric inequalities and existence of positively curved metrics. Ketterer and Mondino [51] characterized lower bounds of k-intermediate Ricci curvature via optimal transport, providing a synthetic formulation. Mouillé [62, 61] studied torus actions on manifolds with positive intermediate Ricci curvature and proved symmetry rank bounds, and Kennard and Mouillé [50] obtained maximal symmetry rank results. Domínguez-Vázquez, González-Álvaro, and Mouillé [26] constructed infinite families of manifolds with positive Ric_k for small k, and Reiser and Wraith [71, 73, 72] established Gromov-type Betti number bounds and proved existence of positive intermediate Ricci curvature on fibre bundles and connected sums.

The study of geometric inequalities under $\mathrm{Ric}_k \geq 0$ was initiated by Ma and Wu [57], who extended Brendle's Sobolev and Michael–Simon inequalities to this setting, and independently by Wang [81], who obtained a related result via optimal transport. In our work [54], joint with Jihye Lee, we proved that the sharp logarithmic Sobolev inequality also holds under the k-intermediate Ricci curvature condition, and moreover extends to the asymptotically nonnegative setting. More recently, Ji and Kwong [45] established a Fenchel–Willmore inequality with the curvature index $k = \min\{n, m-1\}$. Chahine [22] studied volume estimates for tubes around submanifolds using integral intermediate curvature bounds. A comprehensive and regularly updated bibliography of results on intermediate Ricci curvature has been compiled by Mouillé [60].

Asymptotically non-negative curvature. The notion of asymptotically non-negative sectional curvature was introduced by Abresch [1] to extend Toponogov's triangle comparison theorem beyond the constant curvature bound setting. The key insight is that if the sectional curvature satisfies $\mathrm{Sec} \geq -\lambda(d(o, \cdot))$ for a non-increasing function λ with

$\int_0^\infty t\lambda(t)\,dt < \infty$, then geodesic triangles can still be compared with triangles in a model surface of revolution whose Gaussian curvature matches $-\lambda(r)$. This integrability condition ensures that the negative curvature decays faster than $1/t^2$, and Abresch used it to prove that such manifolds have finite topological type and a controlled number of ends. The threshold is sharp: Jiang and Yang [47] later constructed manifolds of positive Ricci curvature with only quadratic curvature decay ($\mathrm{Sec} \geq -C/r^2$, which fails the integrability condition) that have infinite Betti numbers.

Kasue [49] developed the analytic infrastructure for this class, constructing a natural compactification via Busemann functions and proving that harmonic functions with polynomial growth conditions span finite-dimensional spaces, extending results of Li and Tam from the $\mathrm{Sec} \geq 0$ case. Abresch and Gromoll [2] then shifted focus to Ricci curvature, introducing the condition $\mathrm{Ric} \geq -(n-1)\lambda(d(o,\cdot))$ with the same integrability condition. Their main contribution—the *excess estimate*—provides a pointwise bound on the excess function $e(x) = d(x, q_+) + d(x, q_-) - d(q_+, q_-)$ in terms of the distance to the geodesic segment, and has become a foundational tool in the Cheeger–Colding theory of Ricci limit spaces.

The volume comparison theorem for this setting was proved using the analogue of the Bishop–Gromov inequality with the constant curvature model replaced by a variable curvature model (see e.g. Zhu [90]). Specifically, if $h(t)$ solves the ODE $h'' = \lambda(t)h$ with $h(0) = 0$, $h'(0) = 1$, then the ratio $|B_r(o)| / (n\omega_n \int_0^r h^{n-1}(t)\,dt)$ is non-increasing in r. This ensures that the asymptotic volume ratio θ_h is well-defined, and Zhu showed that $h(t) \leq e^{b_0}t$, so the model volume growth is at most polynomial. When $\lambda \equiv 0$, this reduces to the classical Bishop–Gromov inequality. Zhu's comparison is the key technical ingredient in the proof of Theorem 2.16: it provides the Gaussian volume ratio estimates of Lemmas 2.20–2.22 and controls the $t \to \infty$ limit in the final step.

Further structural results were obtained by Bazanfaré [10], who proved a generalized Toponogov theorem and a rigidity result for manifolds with asymptotically non-negative curvature, and by Zhang [89], who studied open manifolds with asymptotically non-negative Ricci curvature and large volume growth. Yi and Zheng [88] and Dong, Lin, and Lu [27, 29] applied Zhu's comparison theorem in the context of geometric inequalities, extending the Sobolev and logarithmic Sobolev inequalities to this setting. Our work [54] further extends these results to the intermediate curvature setting, as described in the next subsection.

To state our result in its most general form, we use the following definition.

Definition 2.2 (Abresch [1]). A complete non-compact Riemannian manifold (M, g) with base point o has *asymptotically non-negative* Ric_k if there exists a non-negative, non-increasing function $\lambda\colon [0, \infty) \to [0, \infty)$ such that:

(i) $b_0(\lambda) = \displaystyle\int_0^\infty t\lambda(t)\, dt < \infty$,

(ii) $\mathrm{Ric}_k(P, v) \geq -k\lambda(d(o, p))$ at each point $p \in M$, for any k-dimensional subspace $P \subset T_pM$ and any unit tangent vector v perpendicular to P.

We also set $b_1(\lambda) = \displaystyle\int_0^\infty \lambda(t)\, dt < \infty$, which is implied by condition (i).

When $\lambda \equiv 0$, this reduces to $\mathrm{Ric}_k \geq 0$. The class of manifolds with asymptotically non-negative Ric_k includes those with $\mathrm{Ric}_k \geq 0$ outside a compact set as well as asymptotically flat manifolds. The monotonicity property persists: if $k_1 \leq k_2$, then $\mathrm{Ric}_{k_1} \geq -k_1\lambda$ implies $\mathrm{Ric}_{k_2} \geq -k_2\lambda$.

To formulate the inequality, we replace the asymptotic volume ratio θ by a modified quantity adapted to the curvature decay. Define $h(t)$ as the unique solution of

$$h''(t) = \lambda(t)h(t), \qquad h(0) = 0, \quad h'(0) = 1. \tag{2.14}$$

The *asymptotic volume ratio with respect to* h is

$$\theta_h = \lim_{r\to\infty} \frac{|B_r(o)|}{n\omega_n \int_0^r h^{n-1}(t)\, dt},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. The corresponding version of Bishop–Gromov, due to Zhu [90], ensures that the ratio is non-increasing in r, so the limit exists. We say M has *Euclidean volume growth* if $\theta_h > 0$. When $\lambda \equiv 0$, we have $h(t) = t$ and θ_h reduces to the usual θ.

We also define, for $t \geq 0$,

$$P(t) = (4\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-\frac{(|x|+t)^2}{4}}\, dx, \tag{2.15}$$

where $N = n + m$. Note that $P(0) = 1$.

2.3.2 Statement of results

Theorem 2.16 (Lee–Ricci [54])

Let M^{n+m} be a complete, non-compact manifold of dimension $n+m$ with asymptotically non-negative Ric_k, where $k = \min\{n-1, m-1\}$, and Euclidean volume growth. Let Σ^n be an n-dimensional compact submanifold without boundary. Then, for any positive smooth function f on Σ,

$$\begin{aligned} &\int_\Sigma f \left(\log f + \log P(4b_1 n)\theta_h + \frac{n}{2}\log(4\pi) + n + 4b_1^2 n^2 + (n+m-1)\log\frac{1+b_0}{e^{2r_0 b_1 + b_0}} \right) dV \\ &\quad - \int_\Sigma \frac{|\boldsymbol{\nabla} f|^2}{f}\, dV - \int_\Sigma f\, |H|^2\, dV \leq \left(\int_\Sigma f\, dV \right) \log \left(\int_\Sigma f\, dV \right), \end{aligned} \tag{2.16}$$

where H is the mean curvature vector of Σ and $r_0 = \max_{x\in\Sigma} d(o,x)$.

When $\lambda \equiv 0$ (i.e., $b_0 = b_1 = 0$), all the error terms vanish and we obtain:

Corollary 2.17

Let M^{n+m} be a complete, non-compact manifold with $\mathrm{Ric}_k \geq 0$ and Euclidean volume growth, where $k = \min\{n-1, m-1\}$. Let Σ^n be a compact submanifold without boundary. Then, for any positive smooth function f,

$$\int_{\Sigma} f\left(\log f + n + \frac{n}{2}\log(4\pi) + \log\theta\right) dV - \int_{\Sigma} \frac{|\nabla f|^2}{f}\, dV - \int_{\Sigma} f\, |H|^2\, dV$$
$$\leq \left(\int_{\Sigma} f\, dV\right) \log\left(\int_{\Sigma} f\, dV\right), \tag{2.17}$$

where θ is the asymptotic volume ratio of M.

Choosing $f \equiv 1$ in Corollary 2.17 yields two immediate consequences.

Corollary 2.18

Let M^{n+m} be a complete, non-compact manifold with $\mathrm{Ric}_k \geq 0$, $k = \min\{n-1, m-1\}$, and Euclidean volume growth. Then there is no closed minimal submanifold of dimension n in M.

Corollary 2.19

Let M^{n+m} be as above. If there exists a closed minimal submanifold of dimension n in M, then the asymptotic volume ratio of M is zero.

Remark 2.13. For hypersurfaces ($m = 1$) under non-negative Ricci curvature, $k = \min\{n-1, 0\} = 0$, so our result does not apply. In that case, Corollaries 2.18 and 2.19 were proved by Agostiniani–Fogagnolo–Mazzieri [3]. For higher dimension and codimension, these corollaries are new even in the non-negative Ric_k setting. When $k = 1$, $\mathrm{Ric}_1 \geq 0$ is equivalent to non-negative sectional curvature, and we recover the corollaries of Yi–Zheng [88]. Note that Corollaries 2.18 and 2.19 do not hold for $\mathrm{Ric} \geq 0$ in total dimension 4: the Eguchi–Hanson metric on TS^2 is Ricci-flat with Euclidean volume growth and has a totally geodesic S^2.

Remark 2.14. Ji and Kwong [45] recently proved a Fenchel–Willmore inequality for submanifolds under $\mathrm{Ric}_k \geq 0$ with the curvature index $k = \min\{n, m-1\}$, they recover Corollaries 2.18 and 2.19 under this assumption.

Remark 2.15. Corollaries 2.18 and 2.19 are derived from Corollary 2.17 (the $\lambda \equiv 0$ case) by setting $f \equiv 1$ and applying the scaling argument from [88, Section 4]. This argument cannot be applied in the asymptotically non-negative setting of Theorem 2.16, because the error terms b_0, b_1, r_0 are not scale-invariant: they blow up under the metric rescaling used in [88].

2.3.3 Asymptotic volume ratio estimates

The following estimates relate the asymptotic volume ratio θ_h to integrals of Gaussian-type functions on M. They are used in the final step of the proof.

Lemma 2.20 ([27, Lemma 2.2])

Let M be a complete non-compact Riemannian manifold of dimension N with non-negative Ricci curvature. Then

$$\theta = \lim_{r\to\infty} \frac{(4\pi)^{-N/2}}{r^N} \int_M e^{-\frac{d(x,p)^2}{4r^2}}\, dV(x)$$

for any $p \in M$.

The proof of Lemma 2.20 is based on the Bishop–Gromov volume comparison. In the asymptotic case, we use the inequality $r \leq h(r)$ (with upper bound $h(r) \leq e^{b_0} r$ from [90]) in place of the flat comparison. However, the presence of $b_0 > 0$ generally precludes replacing r^N by $h(r)^N$ in Lemma 2.20. Following Dong–Lin–Lu [27], the correct replacement is r^N by $r\, h(r)^{N-1}$.

Lemma 2.21 ([27, Lemma 2.2])

Let M be a complete non-compact Riemannian manifold of dimension N with asymptotically non-negative Ricci curvature. Then

$$P(t)\theta_h = \lim_{r\to\infty} \frac{(4\pi)^{-N/2}}{r\, h(r)^{N-1}} \int_M e^{-\left(\frac{d(x,o)}{r}+t\right)^2/4}\, dV(x), \tag{2.18}$$

where o is the base point of M.

Proof. By de l'Hôpital's rule applied to the definition of θ_h,

$$\theta_h = \lim_{r\to\infty} \frac{|B_r(o)|}{N\omega_N \int_0^r h^{N-1}(t)\,dt} = \lim_{r\to\infty} \frac{|\partial B_r(o)|}{N\omega_N h^{N-1}(r)}. \tag{2.19}$$

Now compute:

$$\begin{aligned} \frac{(4\pi)^{-N/2}}{r\,h(r)^{N-1}} \int_M e^{-\left(\frac{d(x,o)}{r}+t\right)^2/4}\,dV(x) &= (4\pi)^{-N/2} \int_0^\infty \frac{|\partial B_{rs}(o)|}{h^{N-1}(r)}\, e^{-(s+t)^2/4}\,ds \\ &= (4\pi)^{-N/2} \int_0^\infty \frac{|\partial B_{rs}(o)|}{h^{N-1}(rs)} \cdot \frac{h^{N-1}(rs)}{h^{N-1}(r)}\, e^{-(s+t)^2/4}\,ds. \end{aligned}$$

Taking $r \to \infty$ and using (2.19) and the monotone convergence theorem,

$$= (4\pi)^{-N/2}\, N\omega_N\, \theta_h \int_0^\infty s^{N-1} e^{-(s+t)^2/4}\,ds = \theta_h\,(4\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-(|x|+t)^2/4}\,dx = P(t)\theta_h. \square$$

Lemma 2.22

Let M be a complete non-compact manifold of dimension N with asymptotically non-negative Ric_k. Then for any Borel map $\psi\colon M \to K$, where $K \subset M$ is compact,

$$\lim_{r\to\infty} \frac{(4\pi)^{-N/2}}{r\,h(r)^{N-1}} \int_M e^{-\left(\frac{d(x,\psi(x))}{r}+t\right)^2/4}\,dV(x) = \lim_{r\to\infty} \frac{(4\pi)^{-N/2}}{r\,h(r)^{N-1}} \int_M e^{-\left(\frac{d(x,o)}{r}+t\right)^2/4}\,dV(x).$$

Proof. This follows from the triangle inequality: $|d(x,\psi(x)) - d(x,o)| \le \max_{z\in K} d(o,z)$, so the difference in the exponents vanishes as $r \to \infty$. $\square$

2.3.4 The PDE and contact set

We now begin the proof of Theorem 2.16. Let M^{n+m} be a complete non-compact manifold with asymptotically non-negative Ric_k and Euclidean volume growth, where $k = \min\{n-1, m-1\}$. Let $\Sigma^n \subset M$ be a compact connected submanifold without boundary, and let $f > 0$ be smooth on Σ. (The disconnected case is handled at the end by summing over components, using $a\log a + b\log b < (a+b)\log(a+b)$ for $a, b > 0$.)

By scaling f, we may assume

$$\int_\Sigma f\log f\,dV - \int_\Sigma \frac{|\boldsymbol{\nabla} f|^2}{f}\,dV - \int_\Sigma f\,|H|^2\,dV = 0. \tag{2.20}$$

Indeed, replacing f by cf for a suitable constant $c > 0$ achieves this, since $\log(cf) = \log c + \log f$ and $\log c$ can be any real number. It then suffices to show

$$\log P(4b_1 n)\theta_h + \frac{n}{2}\log(4\pi) + n + 4b_1^2 n^2 + (n+m-1)\log\frac{1+b_0}{e^{2r_0 b_1 + b_0}} \leq \log\int_\Sigma f\, dV. \quad (2.21)$$

Since Σ is compact, connected, and $f > 0$, the operator $u \mapsto \operatorname{div}_\Sigma(f\boldsymbol{\nabla} u)$ has one-dimensional kernel (the constants) and the normalization (2.20) ensures solvability. By the Fredholm alternative, there exists a smooth function $u\colon \Sigma \to \mathbb{R}$ satisfying

$$\operatorname{div}_\Sigma(f\boldsymbol{\nabla} u) = f\log f - \frac{|\boldsymbol{\nabla} f|^2}{f} - f\,|H|^2 \quad \text{on } \Sigma. \quad (2.22)$$

For each $r > 0$, define the map $\Phi_r\colon T^\perp\Sigma \to M$ by

$$\Phi_r(x,y) = \exp_x(r\boldsymbol{\nabla} u(x) + ry), \quad (2.23)$$

where $T^\perp\Sigma = \big\{(x,y) : x \in \Sigma,\ y \in T_x^\perp\Sigma\big\}$ is the total space of the normal bundle. The *contact set* $A_r \subset T^\perp\Sigma$ is defined as the set of all $(\bar{x},\bar{y}) \in T^\perp\Sigma$ satisfying

$$r\,u(x) + \frac{1}{2}d\big(x, \exp_{\bar{x}}(r\boldsymbol{\nabla} u(\bar{x}) + r\bar{y})\big)^2 \geq r\,u(\bar{x}) + \frac{r^2}{2}\big(\,|\boldsymbol{\nabla} u(\bar{x})|^2 + |\bar{y}|^2\,\big) \quad (2.24)$$

for all $x \in \Sigma$.

Lemma 2.23 (Surjectivity, [88, Lemma 3.2])
For each $r \in (0,\infty)$, $\Phi_r(A_r) = M$.

Proof. Take any $p \in M$. Since Σ is compact, the function $x \mapsto r\,u(x) + \frac{1}{2}d(x,p)^2$ attains its minimum at some $\bar{x} \in \Sigma$. Let $\bar{\gamma}$ be a minimizing geodesic with $\bar{\gamma}(0) = \bar{x}$ and $\bar{\gamma}(r) = p$. Then $\bar{\gamma}$ minimizes the functional $u(\gamma(0)) + \frac{1}{2}\int_0^r |\gamma'(t)|^2\, dt$ among all smooth curves γ with $\gamma(0) \in \Sigma$ and $\gamma(r) = p$. By the first variational formula,

$$\boldsymbol{\nabla} u(\bar{x}) - \bar{\gamma}'(0) \in T_{\bar{x}}^\perp\Sigma,$$

so there exists $\bar{y} \in T_{\bar{x}}^\perp\Sigma$ with $\boldsymbol{\nabla} u(\bar{x}) + \bar{y} = \bar{\gamma}'(0)$. One checks that $\Phi_r(\bar{x},\bar{y}) = p$ and

$(\bar{x}, \bar{y}) \in A_r$ using the geodesic $\bar{\gamma}$. $\square$

The surjectivity lemma yields a map $\psi\colon M \to \Sigma$ defined by $\psi(p) = \bar{x}$, where $\bar{x}$ is the point where $r\,u(x) + \frac{1}{2}d(x,p)^2$ attains its minimum. Using the area formula for $\Phi_r\colon A_r \to M$, together with Lemma 2.23, we obtain

$$\begin{aligned}\int_M & e^{-\left(\frac{d(\psi(p),p)}{2r}+2b_1 n\right)^2}\, dV(p) \\ &\leq \int_\Sigma \left(\int_{T_{\bar{x}}^\perp \Sigma} e^{-\left(\frac{d(x,\Phi_r(x,y))}{2r}+2b_1 n\right)^2} \left|\det D\Phi_r(x,y)\right|\, \chi_{A_r}(x,y)\, dy \right) dV(x). \end{aligned} \tag{2.25}$$

2.3.5 Jacobi fields and the Riccati equation

We now estimate the integrand on the right-hand side of (2.25). Fix $r > 0$ and $(\bar{x}, \bar{y}) \in A_r$. Define the geodesic $\bar{\gamma}(t) = \exp_{\bar{x}}(t\boldsymbol{\nabla} u(\bar{x}) + t\bar{y})$ on $[0, r]$. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_{\bar{x}}\Sigma$ such that $\operatorname{Hess}_\Sigma u(e_i, e_j) - \langle \mathrm{II}(e_i, e_j), \bar{y}\rangle$ is diagonal. Let E_i denote the parallel transport of e_i along $\bar{\gamma}$ for $1 \leq i \leq n$. Choose an orthonormal frame $\{e_{n+1}, \ldots, e_{n+m}\}$ of $T^\perp\Sigma$ near $\bar{x}$ such that $\langle \nabla_{e_i} e_\alpha, e_\beta\rangle = 0$ at $\bar{x}$ for all $1 \leq i \leq n$ and $n + 1 \leq \alpha, \beta \leq n + m$.

For $1 \leq i \leq n$, let $X_i(t)$ be the Jacobi field along $\bar{\gamma}(t)$ with initial conditions

$$\begin{cases} X_i(0) = e_i \\ \langle X_i'(0), e_j\rangle = \operatorname{Hess}_\Sigma u(e_i, e_j) - \langle \mathrm{II}(e_i, e_j), \bar{y}\rangle & \text{for } 1 \leq j \leq n \\ \langle X_i'(0), e_\alpha\rangle = \langle \mathrm{II}(e_i, \boldsymbol{\nabla} u(\bar{x})), e_\alpha\rangle & \text{for } n+1 \leq \alpha \leq n+m. \end{cases} \tag{2.26}$$

For $n + 1 \leq \alpha \leq n + m$, let $X_\alpha(t)$ be the Jacobi field along $\bar{\gamma}(t)$ with

$$X_\alpha(0) = 0 \qquad \text{and} \qquad X_\alpha'(0) = e_\alpha. \tag{2.27}$$

Define the $(n + m) \times (n + m)$ matrices $P(t)$ and $S(t)$ by

$$P_{ij}(t) = \langle X_i(t), E_j(t)\rangle\,, \qquad S_{ij}(t) = \bar{R}(\bar{\gamma}'(t), E_i(t), \bar{\gamma}'(t), E_j(t)),$$

for $1 \leq i, j \leq n + m$, where $\bar{R}$ denotes the Riemann curvature tensor of M. Since each

$X_i(t)$ is a Jacobi field, $P''(t) = -P(t)S(t)$. Define $Q(t) = P^{-1}(t)P'(t)$. Then

$$Q'(t) = -(Q(t))^2 - S(t). \tag{2.28}$$

Partial traces and curvature bounds

Let $\operatorname{tr}_n Q(t) = \sum_{i=1}^{n} Q_{ii}(t)$ and $\operatorname{tr}_m Q(t) = \sum_{\alpha=n+1}^{n+m} Q_{\alpha\alpha}(t)$ denote the tangential and normal partial traces. From (2.28) and the Cauchy–Schwarz inequality,

$$(\operatorname{tr}_n Q)' + \frac{1}{n}(\operatorname{tr}_n Q)^2 \leq -\operatorname{tr}_n S, \qquad (\operatorname{tr}_m Q)' + \frac{1}{m}(\operatorname{tr}_m Q)^2 \leq -\operatorname{tr}_m S, \tag{2.29}$$

where

$$\operatorname{tr}_n S(t) = \sum_{i=1}^{n} \bar{R}(\bar{\gamma}'(t), E_i(t), \bar{\gamma}'(t), E_i(t)), \quad \operatorname{tr}_m S(t) = \sum_{\alpha=n+1}^{n+m} \bar{R}(\bar{\gamma}'(t), E_\alpha(t), \bar{\gamma}'(t), E_\alpha(t)).$$

Curvature decomposition

The curvature assumption cannot be applied directly because $\bar{\gamma}'(t)$ may not be perpendicular to the plane spanned by $E_1(t), \ldots, E_n(t)$. Since $E_i(t)$ is the parallel transport of e_i along $\bar{\gamma}$, we need only consider the angle at $t = 0$. Let a denote the angle between $\bar{\gamma}'(0) = \boldsymbol{\nabla} u(\bar{x}) + \bar{y}$ and the tangent plane $T_{\bar{x}}\Sigma$.

We consider the generic case $\boldsymbol{\nabla} u(\bar{x}) \neq 0$ and $\bar{y} \neq 0$. Now we choose an adapted orthonormal basis as follows (cf. [57]). Set

$$E_1(0) = \frac{\boldsymbol{\nabla} u(\bar{x})}{|\boldsymbol{\nabla} u(\bar{x})|},$$
$$E_{n+1}(0) = \frac{\bar{y}}{|\bar{y}|},$$

and now extend this to an orthonormal frame $\{E_1(0), \ldots, E_{n+m}(0)\}$ by choosing $\{E_2(0), \ldots, E_n(0)\}$ as an orthonormal basis of $\{\boldsymbol{\nabla} u(\bar{x})\}^\perp \cap T_{\bar{x}}\Sigma$

and $\{E_{n+2}(0), \ldots, E_{n+m}(0)\}$ as an orthonormal basis of $\{\bar{y}\}^{\perp} \cap T^{\perp}_{\bar{x}}\Sigma$. With this choice,

$$\frac{\bar{\gamma}'(0)}{|\bar{\gamma}'(0)|} = \cos(a)\, E_1(0) + \sin(a)\, E_{n+1}(0),$$

so $E_i(0) \perp \bar{\gamma}'(0)$ for all $i \neq 1, n+1$. We then parallel transport this frame along $\bar{\gamma}$ to obtain $E_i(t)$ for all t.

For the tangential partial trace $\operatorname{tr}_n S(t) = \sum_{i=1}^{n} \bar{R}(\bar{\gamma}', E_i, \bar{\gamma}', E_i)$, we separate the $i = 1$ term from the rest. Since $\bar{\gamma}' = |\boldsymbol{\nabla} u(\bar{x})|\, E_1 + |\bar{y}|\, E_{n+1}$ and $R(E_1, E_1, \cdot, \cdot) = 0$ by the symmetries of the curvature tensor,

$$\bar{R}(\bar{\gamma}', E_1, \bar{\gamma}', E_1) = |\bar{y}|^2\, \bar{R}(E_{n+1}, E_1, E_{n+1}, E_1) = \sin^2(a)\, |\bar{\gamma}'(0)|^2\, \operatorname{Sec}(E_{n+1}, E_1).$$

For $i = 2, \ldots, n$, since $E_i \perp \bar{\gamma}'$,

$$\bar{R}(\bar{\gamma}', E_i, \bar{\gamma}', E_i) = |\bar{\gamma}'(0)|^2\, \operatorname{Sec}\left(\frac{\bar{\gamma}'}{|\bar{\gamma}'|}, E_i\right).$$

Summing, and noting that

$$\operatorname{Sec}(\bar{\gamma}'/\, |\bar{\gamma}'|\,, E_{n+1}) = \operatorname{Sec}(E_1, E_{n+1})$$

since the 2-plane $\operatorname{Span}(\bar{\gamma}'/\, |\bar{\gamma}'|\,, E_{n+1})$ coincides with $\operatorname{Span}(E_1, E_{n+1})$, we obtain

$$\begin{aligned} \operatorname{tr}_n S(t) = {} & \sin^2(a)\, |\bar{\gamma}'(0)|^2 \operatorname{Ric}_{\,n}\left(\frac{\bar{\gamma}'(t)}{|\bar{\gamma}'(t)|}, P_1(t)\right) \\ & + \cos^2(a)\, |\bar{\gamma}'(0)|^2 \operatorname{Ric}_{\,n-1}\left(\frac{\bar{\gamma}'(t)}{|\bar{\gamma}'(t)|}, P_2(t)\right), \end{aligned} \tag{2.30}$$

where $P_1(t) = \operatorname{Span}(E_2(t), \ldots, E_{n+1}(t))$ is an n-dimensional subspace and

$$P_2(t) = \operatorname{Span}(E_2(t), \ldots, E_n(t))$$

is an $(n-1)$-dimensional subspace. Applying the curvature bound $\operatorname{Ric}_{\,k} \geq -k\lambda(d(o, \cdot))$

with $k = \min\{n-1, m-1\}$,

$$\begin{aligned}
\operatorname{tr}_n S(t) &\geq -n\sin^2(a)\left|\bar\gamma'(0)\right|^2 \lambda(d(o,\bar\gamma(t))) \\
&\qquad - (n-1)\cos^2(a)\left|\bar\gamma'(0)\right|^2 \lambda(d(o,\bar\gamma(t))) \\
&= \big(\cos^2(a) - n\big)\left|\bar\gamma'(0)\right|^2 \lambda(d(o,\bar\gamma(t))),
\end{aligned}$$

since $-n\sin^2(a) - (n-1)\cos^2(a) = -n(\sin^2(a) + \cos^2(a)) + \cos^2(a) = \cos^2(a) - n$. Thus

$$\operatorname{tr}_n S(t) \geq (\cos^2(a) - n)\left|\bar\gamma'(0)\right|^2 \lambda(d(o,\bar\gamma(t))). \tag{2.31}$$

Similarly,

$$\operatorname{tr}_m S(t) \geq (\sin^2(a) - m)\left|\bar\gamma'(0)\right|^2 \lambda(d(o,\bar\gamma(t))). \tag{2.32}$$

These bounds also hold in the degenerate cases.

When $\boldsymbol{\nabla} u(\bar x) = 0$, the angle is $a = \pi/2$ since $\bar\gamma'(0) = \bar y \in T^\perp_{\bar x}\Sigma$. In this case $\bar\gamma'$ is perpendicular to $P_3(t) = \operatorname{Span}(E_1(t), \ldots, E_n(t))$, so

$$\begin{aligned}
\operatorname{tr}_n S(t) &= \left|\bar\gamma'(0)\right|^2 \operatorname{Ric}_n\left(\frac{\bar\gamma'(t)}{|\bar\gamma'(t)|}, P_3(t)\right) \\
&\geq -n\left|\bar\gamma'(0)\right|^2 \lambda(d(o,\bar\gamma(t))) \\
&= (\cos^2(a) - n)\left|\bar\gamma'(0)\right|^2 \lambda(d(o,\bar\gamma(t))).
\end{aligned}$$

For the normal trace, setting $E_{n+1} = \bar y / |\bar y|$, the term $\bar R(\bar\gamma', E_{n+1}, \bar\gamma', E_{n+1}) = 0$ since E_{n+1} is parallel to $\bar\gamma'$, so

$$\begin{aligned}
\operatorname{tr}_m S(t) &= \sum_{\alpha=n+2}^{n+m} \bar R(\bar\gamma'(t), E_\alpha(t), \bar\gamma'(t), E_\alpha(t)) \\
&= \left|\bar\gamma'(0)\right|^2 \operatorname{Ric}_{m-1}\left(\frac{\bar\gamma'(t)}{|\bar\gamma'(t)|}, P_4(t)\right) \\
&\geq -(m-1)\left|\bar\gamma'(0)\right|^2 \lambda(d(o,\bar\gamma(t))) \\
&\geq (\sin^2(a) - m)\left|\bar\gamma'(0)\right|^2 \lambda(d(o,\bar\gamma(t))),
\end{aligned}$$

where $P_4(t) = \mathrm{Span}(E_{n+2}(t), \ldots, E_{n+m}(t))$ and the last step uses $\sin^2(\pi/2) - m = 1 - m = -(m-1)$.

When $\bar{y} = 0$, the angle is $a = 0$ since $\bar{\gamma}'(0) = \boldsymbol{\nabla} u(\bar{x}) \in T_{\bar{x}}\Sigma$; setting $E_1(t) = \bar{\gamma}'(t)/|\bar{\gamma}'(t)|$, one verifies that (2.31) and (2.32) hold by the same reasoning.

Combining (2.29) with (2.31)–(2.32) and using the triangle inequality $d(o, \bar{\gamma}(t)) \geq |d(o,\bar{x}) - d(\bar{x}, \bar{\gamma}(t))|$ together with the definition of A_r, we obtain: for $(\bar{x}, \bar{y}) \in A_r \setminus \{(0,0)\}$,

$$\begin{cases} (\mathrm{tr}_n\, Q)' + \dfrac{1}{n}(\mathrm{tr}_n\, Q)^2 \leq (n - \cos^2(a))\, |\bar{\gamma}'(0)|^2\, \lambda\left(\left|d(o,\bar{x}) - t\sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2 + |\bar{y}|^2}\right|\right) \\ (\mathrm{tr}_m\, Q)' + \dfrac{1}{m}(\mathrm{tr}_m\, Q)^2 \leq (m - \sin^2(a))\, |\bar{\gamma}'(0)|^2\, \lambda\left(\left|d(o,\bar{x}) - t\sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2 + |\bar{y}|^2}\right|\right). \end{cases} \tag{2.33}$$

2.3.6 Comparison ODE and Jacobian bound

Tangential comparison

Define $\phi(t) = e^{\frac{1}{n}\int_0^t \sum_{i=1}^n Q_{ii}(\tau)\, d\tau}$. From (2.33),

$$\begin{cases} \phi''(t) \leq \frac{n-\cos^2(a)}{n}\, |\bar{\gamma}'(0)|^2\, \lambda\left(\left|d(o,\bar{x}) - t\sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2 + |\bar{y}|^2}\right|\right) \phi(t) \\ \phi'(0) = \frac{1}{n}(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y}\rangle) \\ \phi(0) = 1. \end{cases} \tag{2.34}$$

To apply comparison, define $\psi_1(t)$ and $\psi_2(t)$ as solutions to

$$\begin{cases} \psi_1''(t) = \frac{n-\cos^2(a)}{n}\, |\bar{\gamma}'(0)|^2\, \lambda\left(\left|d(o,\bar{x}) - t\sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2 + |\bar{y}|^2}\right|\right) \psi_1(t) \\ \psi_1(0) = 0, \quad \psi_1'(0) = 1 \end{cases} \tag{2.35}$$

and

$$\begin{cases} \psi_2''(t) = \frac{n-\cos^2(a)}{n} \left|\bar{\gamma}'(0)\right|^2 \lambda \left(\left| d(o,\bar{x}) - t\sqrt{\left|\nabla u(\bar{x})\right|^2 + \left|\bar{y}\right|^2} \right| \right) \psi_2(t) \\ \psi_2(0) = 1, \quad \psi_2'(0) = 0. \end{cases} \tag{2.36}$$

The function $\psi(t) = \psi_2(t) + \frac{1}{n}(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y}\rangle)\psi_1(t)$ satisfies the same ODE as ϕ with the same initial conditions, so the Sturm comparison theorem gives $\phi'(t)/\phi(t) \leq \psi'(t)/\psi(t)$, and hence

$$\operatorname{tr}_n Q(t) = n \cdot \frac{\phi'(t)}{\phi(t)} \leq n \cdot \frac{\psi'(t)}{\psi(t)} = \frac{\psi_2' + \frac{1}{n}\big(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y}\rangle\big)\psi_1'}{\psi_2 + \frac{1}{n}\big(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y}\rangle\big)\psi_1}. \tag{2.37}$$

Normal comparison

Define $\bar{\phi}(t) = t\, e^{\frac{1}{m}\int_0^t \sum_{\alpha=n+1}^{n+m} (Q_{\alpha\alpha}(\tau) - 1/\tau)\, d\tau}$. Then

$$\begin{cases} \bar{\phi}''(t) \leq \frac{m-\sin^2(a)}{m} \left|\bar{\gamma}'(0)\right|^2 \lambda \left(\left| d(o,\bar{x}) - t\sqrt{\left|\nabla u(\bar{x})\right|^2 + \left|\bar{y}\right|^2} \right| \right) \bar{\phi}(t) \\ \bar{\phi}(0) = 0, \quad \bar{\phi}'(0) = 1. \end{cases} \tag{2.38}$$

Let $\bar{\psi}$ solve

$$\begin{cases} \bar{\psi}''(t) = \frac{m-\sin^2(a)}{m} \left|\bar{\gamma}'(0)\right|^2 \lambda \left(\left| d(o,\bar{x}) - t\sqrt{\left|\nabla u(\bar{x})\right|^2 + \left|\bar{y}\right|^2} \right| \right) \bar{\psi}(t) \\ \bar{\psi}(0) = 0, \quad \bar{\psi}'(0) = 1. \end{cases} \tag{2.39}$$

The comparison gives $\operatorname{tr}_m Q(t) = m \cdot \bar{\phi}'(t)/\bar{\phi}(t) \leq m \cdot \bar{\psi}'(t)/\bar{\psi}(t)$.

Determinant bound

Since $\frac{d}{dt}\log(\det P(t)) = \operatorname{tr} Q(t) = \operatorname{tr}_n Q(t) + \operatorname{tr}_m Q(t)$, the two comparison estimates yield

$$\frac{d}{dt}\log(\det P(t)) \leq n \cdot \frac{\psi'}{\psi} + m \cdot \frac{\bar{\psi}'}{\bar{\psi}} = \frac{d}{dt}\log(\psi^n \bar{\psi}^m).$$

Integrating from ε to t and taking $\varepsilon \to 0$,

$$\det P(t) \leq \psi(t)^n \, \bar{\psi}(t)^m = \left(\psi_2(t) + \frac{1}{n}(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y} \rangle)\psi_1(t) \right)^n \bar{\psi}(t)^m. \tag{2.40}$$

Standard comparison theorems give the following bounds:

$$\begin{cases} \dfrac{\psi_2}{\psi_1} \leq \dfrac{2b_1(n - \cos^2(a))}{n} \sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2 + |\bar{y}|^2} + \dfrac{1}{t} \\ \psi_1(t) \leq t \, e^{\frac{n - \cos^2(a)}{n}(2b_1 d(o,\bar{x}) + b_0)} \\ \bar{\psi}(t) \leq t \, e^{\frac{m - \sin^2(a)}{m}(2b_1 d(o,\bar{x}) + b_0)}. \end{cases} \tag{2.41}$$

Substituting into (2.40),

$$\begin{aligned} \det P(t) &\leq \left(\frac{\psi_2(t)}{\psi_1(t)} + \frac{1}{n}(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y} \rangle) \right)^n \psi_1(t)^n \, \bar{\psi}(t)^m \\ &\leq \left(2b_1 \sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2 + |\bar{y}|^2} + \frac{1}{t} + \frac{1}{n}(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y} \rangle) \right)^n \\ &\qquad \cdot t^n \, e^{(n - \cos^2(a))(2b_1 d(o,\bar{x}) + b_0)} \cdot t^m \, e^{(m - \sin^2(a))(2b_1 d(o,\bar{x}) + b_0)} \\ &\leq \left(2b_1 \sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2 + |\bar{y}|^2} + \frac{1}{t} + \frac{1}{n}(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y} \rangle) \right)^n \\ &\qquad \cdot t^{n+m} \, e^{(n+m-1)(2b_1 d(o,\bar{x}) + b_0)} \\ &\leq \left(2b_1 \sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2 + |\bar{y}|^2} + \frac{1}{t} + \frac{1}{n}(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y} \rangle) \right)^n t^{n+m} \, e^{\alpha}, \end{aligned} \tag{2.42}$$

where $\alpha = (n + m - 1)(2r_0 b_1 + b_0)$ and $r_0 = \max_{x \in \Sigma} d(o, x)$.

2.3.7 The completing-the-square lemma

Lemma 2.24

For $(x, y) \in A_r$,

$$\Delta_\Sigma u(x) - \langle H(x), y \rangle \leq \log f(x) + \frac{|\boldsymbol{\nabla} u|^2 + |y|^2}{4} - \frac{|2H(x) + y|^2}{4}. \tag{2.43}$$

Proof. From the PDE (2.22), dividing by f,

$$\Delta_\Sigma u - \langle H(x), y\rangle = \log f - \frac{|\nabla f|^2}{f^2} - |H(x)|^2 - \frac{\langle \nabla f, \nabla u\rangle}{f} - \langle H(x), y\rangle . \tag{2.44}$$

Now complete the square. For the gradient terms:

$$\frac{|\nabla f|^2}{f^2} + \frac{\langle \nabla f, \nabla u\rangle}{f} = \frac{|2\nabla f + f\nabla u|^2}{4f^2} - \frac{|\nabla u|^2}{4}. \tag{2.45}$$

For the mean curvature terms:

$$|H(x)|^2 + \langle H(x), y\rangle = \frac{|2H(x) + y|^2}{4} - \frac{|y|^2}{4}. \tag{2.46}$$

Substituting (2.45) and (2.46) into (2.44),

$$\begin{aligned}\Delta_\Sigma u - \langle H(x), y\rangle &= \log f + \frac{|\nabla u(x)|^2 + |y|^2}{4} - \frac{|2\nabla f + f\nabla u|^2}{4f^2} - \frac{|2H(x) + y|^2}{4} \\ &\le \log f + \frac{|\nabla u(x)|^2 + |y|^2}{4} - \frac{|2H(x) + y|^2}{4}.\end{aligned}$$

□

2.3.8 Jacobian estimate and proof of the main theorem

Lemma 2.25

For $(\bar{x}, \bar{y}) \in A_r \setminus \{(0,0)\}$,

$$\begin{aligned}|\det D\Phi_t(\bar{x}, \bar{y})| \le{}& e^{\alpha + \frac{n}{t} - n - 4b_1^2 n^2}\, t^{m+n}\, f(\bar{x}) \\ &\cdot \exp\left(\left(\frac{d(\bar{x}, \Phi_t(\bar{x}, \bar{y}))}{2t} + 2b_1 n\right)^2\right) \exp\left(-\frac{|2H(\bar{x}) + \bar{y}|^2}{4}\right).\end{aligned} \tag{2.47}$$

Proof. Combining the determinant bound (2.42) with Lemma 2.24, we obtain

$$\begin{aligned}\det P(t) \le \Bigg(& 2b_1\sqrt{|\nabla u(\bar{x})|^2 + |\bar{y}|^2} + \frac{1}{t} \\ &+ \frac{1}{n}\left(\log f + \frac{|\nabla u|^2 + |\bar{y}|^2}{4} - \frac{|2H(\bar{x}) + \bar{y}|^2}{4}\right)\Bigg)^n t^{n+m}\, e^{\alpha}.\end{aligned}$$

By the inequality $\lambda \le e^{\lambda-1}$,

$$\begin{aligned}|\det D\Phi_t(\bar{x},\bar{y})| &\le \exp\Bigg(2b_1 n\sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2+|\bar{y}|^2}+\frac{n}{t}+\log f \\ &\qquad\qquad + \frac{|\boldsymbol{\nabla} u|^2+|\bar{y}|^2}{4}-\frac{|2H+\bar{y}|^2}{4}-n\Bigg)\, t^{n+m}\, e^{\alpha} \\ &= e^{\alpha+\frac{n}{t}-n}\, t^{m+n}\, f(\bar{x}) \\ &\qquad \cdot \exp\left(2b_1 n\sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2+|\bar{y}|^2}+\frac{|\boldsymbol{\nabla} u|^2+|\bar{y}|^2}{4}-\frac{|2H+\bar{y}|^2}{4}\right).\end{aligned}$$

To handle the remaining exponential, we complete the square in the $\sqrt{\cdot}$ terms:

$$\begin{aligned}2b_1 n\sqrt{|\boldsymbol{\nabla} u|^2+|\bar{y}|^2}+\frac{|\boldsymbol{\nabla} u|^2+|\bar{y}|^2}{4} &= \left(\frac{\sqrt{|\boldsymbol{\nabla} u|^2+|\bar{y}|^2}}{2}+2b_1 n\right)^2-4b_1^2 n^2 \\ &= \left(\frac{d(\bar{x},\Phi_t(\bar{x},\bar{y}))}{2t}+2b_1 n\right)^2-4b_1^2 n^2,\end{aligned}$$

where we used $d(\bar{x},\Phi_t(\bar{x},\bar{y})) \le t\sqrt{|\boldsymbol{\nabla} u(\bar{x})|^2+|\bar{y}|^2}$. This gives (2.47). $\square$

Proof of Theorem 2.16. Combining (2.25) with Lemma 2.25,

$$\begin{aligned}\int_M e^{-\left(\frac{d(\psi(p),p)}{2t}+2b_1 n\right)^2} dV(p) &\le \int_\Sigma \left(\int_{T_x^\perp\Sigma} e^{\alpha+\frac{n}{t}-n-4b_1^2 n^2}\, t^{m+n}\, f(x)\, e^{-\frac{|2H(x)+y|^2}{4}}\, dy\right) dV(x) \\ &= t^{n+m}\, e^{\frac{n}{t}-n+\alpha-4b_1^2 n^2}\, (4\pi)^{m/2}\int_\Sigma f(x)\, dV(x),\end{aligned}$$

where the inner integral $\int_{T_x^\perp\Sigma} e^{-|2H(x)+y|^2/4}\, dy = (4\pi)^{m/2}$ is a Gaussian integral over the m-dimensional normal fiber. Thus

$$\begin{aligned}&\frac{(4\pi)^{-(n+m)/2}}{t\,(h(t))^{n+m-1}}\int_M e^{-\left(\frac{d(\psi(p),p)}{2t}+2b_1 n\right)^2} dV(p) \\ &\qquad \le \frac{(4\pi)^{-n/2}\, t^{n+m-1}}{(h(t))^{n+m-1}}\, e^{\frac{n}{t}-n+\alpha-4b_1^2 n^2}\int_\Sigma f\, dV. \qquad (2.48)\end{aligned}$$

Taking $t\to\infty$: the left-hand side converges to $P(4b_1 n)\theta_h$ by Lemmas 2.21 and 2.22. For the right-hand side, we use

$$\lim_{t\to\infty}\frac{t^{n+m-1}}{(h(t))^{n+m-1}} = \left(\lim_{t\to\infty}\frac{t}{h(t)}\right)^{n+m-1} = \left(\lim_{t\to\infty}\frac{1}{h'(t)}\right)^{n+m-1} \le \frac{1}{(1+b_0)^{n+m-1}},$$

where the last step uses $h'(t) \geq 1$ and $\lim_{t\to\infty} h'(t) \leq 1 + b_0$. We obtain

$$P(4b_1 n)\theta_h \leq (4\pi)^{-n/2}\, e^{-n+\alpha-4b_1^2 n^2} \cdot \frac{1}{(1+b_0)^{n+m-1}} \int_\Sigma f\, dV. \tag{2.49}$$

Taking logarithms,

$$\log P(4b_1 n)\theta_h + \frac{n}{2}\log(4\pi) + n - \alpha + 4b_1^2 n^2 + (n+m-1)\log(1+b_0) \leq \log \int_\Sigma f\, dV. \tag{2.50}$$

Substituting $\alpha = (n+m-1)(2r_0 b_1 + b_0)$, the left-hand side becomes

$$\log P(4b_1 n)\theta_h + \frac{n}{2}\log(4\pi) + n + 4b_1^2 n^2 + (n+m-1)\log \frac{1+b_0}{e^{2r_0 b_1 + b_0}},$$

which is exactly (2.21). Multiplying by f and integrating over Σ,

$$\begin{aligned}\int_\Sigma f\left(\log P(4b_1 n)\theta_h + \frac{n}{2}\log(4\pi) + n + 4b_1^2 n^2 + (n+m-1)\log \frac{1+b_0}{e^{2r_0 b_1 + b_0}}\right) dV \\ \leq \left(\int_\Sigma f\, dV\right) \log\left(\int_\Sigma f\, dV\right).\end{aligned}$$

Adding the normalization (2.20) yields the full inequality (2.16). The disconnected case follows by applying the inequality to each connected component and using $a\log a + b\log b < (a+b)\log(a+b)$. $\square$

Chapter 3

Isoperimetric Profile function comparison

Isoperimetric inequalities quantify the fundamental relationship between the volume of a domain and the area of its boundary, and their study on Riemannian manifolds has a long history going back to the work of Lévy, Gromov, Bérard–Besson–Gallot, and Morgan–Johnson, among others. The central object is the isoperimetric profile function, which records the minimal boundary area among all domains of a given volume. Under a pointwise Ricci curvature lower bound Ric $\geq (n-1)k$, sharp comparison theorems for the isoperimetric profile were established by Morgan and Johnson [59] and Ni and Wang [63], who showed that the profile is controlled by that of the simply connected space form of constant curvature k. In this chapter, we extend these comparison results to the setting of L^p integral Ricci curvature bounds, following our joint work with Jihye Lee [53]. We begin with a survey of the comparison geometry under integral Ricci curvature, the Petersen–Wei mean curvature and volume comparisons, the Chen–Wei improvement, the Petersen–Sprouse Heintze–Karcher inequality, and Aubry's generalized Bonnet–Myers theorem, which form the technical foundation for our results. We then define the isoperimetric profile functions h_1 and h_2, state the main theorems, and give their complete proofs. A noteworthy feature of our h_2 comparison is that it applies in the non-compact setting $k < 0$ without assuming finite diameter, which is made possible by the bounded constant in the Chen–Wei volume comparison.

3.1 Comparison Geometry under Integral Ricci Curvature

The comparison theorems surveyed in this section replace the classical pointwise lower bound $\mathrm{Ric} \geq (n-1)k$ with an L^p condition on the negative part of the Ricci curvature, and they constitute the technical foundation for the isoperimetric profile comparisons proved later in this chapter. We begin by recalling the relevant definitions, then trace the development of the theory from Gallot's pioneering work through the Petersen–Wei mean curvature comparison, which is the central result from which all subsequent estimates flow.

Let (M^n, g) be a complete Riemannian n-manifold. Denote by $\rho(x)$ the smallest eigenvalue of the Ricci tensor at $x \in M$. For a given constant $k \in \mathbb{R}$, we define the *Ricci curvature deficit below* $(n-1)k$ as

$$\mathrm{Ric}_{-}^{k}(x) = \max\{0,\, (n-1)k - \rho(x)\},$$

and the *integral Ricci curvature* as

$$\left\|\mathrm{Ric}_{-}^{k}\right\|_{p} = \left(\int_{M} (\mathrm{Ric}_{-}^{k}(x))^{p}\, \mathrm{dvol}\right)^{1/p}.$$

We also use the *local* version

$$\left\|\mathrm{Ric}_{-}^{k}\right\|_{p}(r) = \sup_{y \in M} \left(\int_{B_y(r)} (\mathrm{Ric}_{-}^{k})^{p}\, \mathrm{dvol}\right)^{1/p},$$

which measures the worst-case integral curvature excess on balls of radius r. When $\mathrm{Ric} \geq (n-1)k$ pointwise, all of these quantities vanish and the classical comparison theorems apply. The integral theory studies what happens when they are merely small.

Throughout, the critical exponent is $p > n/2$. This threshold is sharp: Gallot [33] constructed manifolds M_j with bounded diameter and $\|K^-\|_{n/2} \le \varepsilon$, but with second Betti numbers $b_2(M_j) \to \infty$, showing that for $p \le n/2$ no meaningful geometric conclusions follow.

Gallot's pioneering work. The systematic study of integral curvature conditions in comparison geometry was initiated by Gallot [33], who worked on compact manifolds with bounded diameter. Using a modification of the Bochner method combined with Sobolev embedding, Gallot proved that if $\|\mathrm{Ric}_-^k\|_p$ is sufficiently small for $p > n$, then the Cheeger isoperimetric constant, the Sobolev constants, and the first eigenvalue $\lambda_1(M)$ are all controlled in terms of $\mathrm{vol}(M)$, $\mathrm{diam}(M)$, and the L^p curvature norm. In particular, he obtained Gromov–Lévy type isoperimetric inequalities with additive error terms. Gallot's approach required the stronger exponent $p > n/2$ for several of his estimates and did not yield a *relative* volume comparison; both limitations were addressed by the subsequent work of Petersen and Wei.

Mean curvature comparison under integral Ricci curvature. The central result of the integral curvature theory is an L^{2p} estimate for the excess mean curvature of geodesic spheres, due to Petersen and Wei [66]. In geodesic polar coordinates centered at $x \in M$, the mean curvature $m(r,\theta) = \Delta r$ of the geodesic sphere $\partial B_x(r)$ satisfies the Riccati inequality

$$m' + \frac{m^2}{n-1} \le -\mathrm{Ric}\,(\partial_r, \partial_r).$$

Recall that on the simply connected space form (M_k^n, g_k) of constant curvature k, the mean curvature is $\bar{m}(r) = (n-1)\,\mathrm{sn}_k'(r)/\mathrm{sn}_k(r)$, where sn_k is as defined in (3.6). Define the *mean curvature excess* $m_+^k(r,\theta) = (m(r,\theta) - \bar{m}(r))_+$. The Riccati inequality, combined with the curvature decomposition $\mathrm{Ric}\,(\partial_r, \partial_r) \ge (n-1)k - \mathrm{Ric}_-^k$, yields a differential inequality for m_+^k. The key step is to multiply this inequality by $(m_+^k)^{2p-2} A(r,\theta)$, where

$A(r,\theta)$ is the volume element in polar coordinates satisfying $A'/A = m$, integrate along geodesic rays over the unit sphere S^{n-1}, and apply Hölder's inequality. The condition $p > n/2$ is essential: it ensures the convergence of the integral $\int_0^r v(n,k,t)^{-1/(2p)}\,dt$ near $t = 0$, where $v(n,k,t) \sim t^n$ is the model volume element. This procedure yields the following fundamental estimate.

Theorem 3.1 (Mean curvature comparison, Petersen–Wei [66])
Let (M^n, g) be a complete Riemannian manifold, $p > n/2$, and $k \le 0$. Then

$$\left\|m_+^k\right\|_{2p}(r) := \sup_{y\in M}\left(\int_{B_y(r)} (m_+^k)^{2p}\,\mathrm{dvol}\right)^{1/(2p)} \le \left(\frac{(n-1)(2p-1)}{2p-n}\,\left\|\mathrm{Ric}_-^{\,k}\right\|_p(r)\right)^{1/2}.$$

This is the integral analogue of the classical pointwise Laplacian comparison $\Delta r \le \bar{m}(r)$, which holds when $\mathrm{Ric} \ge (n-1)k$. Under integral bounds, the pointwise inequality is replaced by an L^{2p} estimate: the mean curvature excess m_+^k need not vanish, but its L^{2p} norm is controlled by the square root of the L^p norm of the Ricci deficit.

For positive curvature $k > 0$, the model mean curvature $\bar{m}(r)$ changes sign at $r = \pi/(2\sqrt{k})$, which causes the above Riccati-based argument to break down in the second half of the geodesic interval. This difficulty was resolved by Petersen and Sprouse [67] using an integrating factor technique, and later refined by Aubry [6], who introduced the factor $\sin^{4p-n-1}(\sqrt{k}\,r)$ to obtain a mean curvature estimate valid on the full interval $(0, \pi/\sqrt{k})$.

Relative volume comparison. From the mean curvature estimate, Petersen and Wei [66] derived a relative volume comparison that generalizes the Bishop–Gromov inequality. The classical Bishop–Gromov theorem states that if $\mathrm{Ric} \ge (n-1)k$, then the volume ratio $\mathrm{vol}\, B_r(x)/v(n,k,r)$ is monotonically nonincreasing in r, where $v(n,k,r)$ denotes the volume of a ball of radius r in the space form M_k^n. Under integral curvature bounds, one obtains "almost monotonicity" of the $(1/2p)$-th power of this ratio.

Theorem 3.2 (Relative volume comparison, Petersen–Wei [66])

Let (M^n, g) be a complete Riemannian manifold, $p > n/2$, and $k \leq 0$. Then for $0 < r < R$,

$$\left(\frac{\operatorname{vol} B_R(x)}{v(n,k,R)}\right)^{1/(2p)} - \left(\frac{\operatorname{vol} B_r(x)}{v(n,k,r)}\right)^{1/(2p)} \leq C(n,p,k,R) \left\|\operatorname{Ric}_-^k\right\|_p (R)^{1/(2p)}.$$

When $\left\|\operatorname{Ric}_-^k\right\|_p = 0$, the error vanishes and the classical monotonicity is recovered. However, the original Petersen–Wei constant $C(n,p,k,R)$ diverges as $R \to \infty$ when $k < 0$, because the model volume element $A_k(s)$ grows exponentially. This divergence renders the estimate ineffective for large balls on non-compact manifolds with negative curvature bounds.

Chen–Wei's improved volume comparison. This limitation was resolved by Chen and Wei [23], who proved that a key constant in the volume ratio differential inequality can be taken to be exactly 1 for all values of k and r, eliminating the divergence. The resulting volume comparison, which is the version used later in this chapter, takes the following form.

Theorem 3.3 (Improved volume comparison, Chen–Wei [23])
Let (M^n, g) be a complete Riemannian manifold, $p > n/2$, and $k \leq 0$. Then for any $x \in M$,

$$|B_x(r)| \leq \left(1 + C(n,p,k,r) \left\|\operatorname{Ric}_-^k\right\|_p (r)^{1/2}\right)^{2p} |B_{\bar{x}}(r)|,$$

where $B_{\bar{x}}(r)$ denotes the ball of radius r in (M_k^n, g_k) and the constant is

$$C(n,p,k,r) = \left(\frac{(n-1)(2p-1)}{2p-n}\right)^{1/2} \int_0^r \left(\frac{1}{|B_{\bar{x}}(t)|}\right)^{1/(2p)} dt.$$

When $k < 0$, this constant is bounded for all r; we denote

$$C(n,p,k,d) = \lim_{r \to \infty} C(n,p,k,r).$$

For $k = 0$, compactness (finite diameter) is needed since C increases with r.

The structural form matches Petersen and Wei's original result, but the constant

$C(n, p, k, R)$ now *remains bounded as* $R \to \infty$ when $k < 0$. This is the essential improvement: it enables meaningful volume comparison for arbitrarily large balls on non-compact manifolds. Among its applications, Chen and Wei obtained optimal estimates for the volume entropy: under the condition $\bar{k}(p,k) \le \delta$, one has $h(M) \le (n-1)\sqrt{-k} + c(n,p,k,d)\,\delta^{1/2}$, recovering the classical bound as $\delta \to 0$ for each fixed $p > n/2$. Aubry's earlier volume entropy estimate [6] achieved this only as $\delta \to 0$ *and* $p \to \infty$ simultaneously, precisely because of the divergent constant in the Petersen–Wei inequality.

The Heintze–Karcher inequality under integral Ricci curvature. Later in this chapter, the proof of the comparison for the normalized isoperimetric profile function h_1 relies on the Heintze–Karcher inequality, which controls the volume enclosed by a hypersurface in terms of its area and mean curvature. The classical Heintze–Karcher inequality requires a pointwise Ricci lower bound. Its generalization to the integral curvature setting was obtained by Petersen and Sprouse.

Theorem 3.4 (Heintze–Karcher under integral curvature, Petersen–Sprouse [67, Theorem 4.1])
Let (M^n, g) be a closed Riemannian manifold with $\mathrm{diam}(M) < D$, $p > n/2$, and $k > 0$. Let $H \subset M$ be a hypersurface with constant mean curvature η dividing M into domains $\Omega_\pm$, and let $\bar{\Omega}_\pm$ be the corresponding regions bounded by the geodesic sphere of mean curvature $|\eta|$ in the space form S^n_k. For any $\alpha > 1$, there exists $\varepsilon = \varepsilon(n, p, \alpha, k) > 0$ such that if $\left\|\mathrm{Ric}_-^{\,k}\right\|_p \le \varepsilon$, then

$$\mathrm{vol}(\Omega_\pm) \le \alpha \cdot \frac{\mathrm{area}(H)}{\mathrm{area}(\bar{H})} \cdot \mathrm{vol}(\bar{\Omega}_\pm).$$

By adding the two inequalities, one obtains $\mathrm{vol}(M) \le \alpha \cdot \frac{\mathrm{area}(H)}{\mathrm{area}(\bar{H})} \cdot \mathrm{vol}(\bar{B})$, where $\bar{B}$ is the ball of radius $\mathrm{diam}(M)$ on S^n_k.

The classical Heintze–Karcher inequality corresponds to $\alpha = 1$ under pointwise Ric $\ge$

$(n-1)k$. Under integral curvature smallness, the multiplicative factor α can be taken arbitrarily close to 1, with the required smallness ε depending on α.

The Bonnet–Myers theorem under integral curvature. The final ingredient needed for the isoperimetric profile comparisons in the positive curvature regime is a generalization of the Bonnet–Myers theorem. The classical theorem states that $\mathrm{Ric} \geq (n-1)k$ for $k > 0$ implies $\mathrm{diam}(M) \leq \pi/\sqrt{k}$ and, in particular, compactness. The diameter bound was first extended to the integral curvature setting by Petersen and Sprouse [67]: for any $\delta > 0$ there exists $\varepsilon > 0$ such that $\left\|\mathrm{Ric}_-^k\right\|_p (R) < \varepsilon$ implies $\mathrm{diam}(M) < \pi/\sqrt{k} + \delta$. The proof uses the excess function $e(x) = d(x, y^-) + d(x, y^+) - d(y^-, y^+)$ for points with $d(y^-, y^+) > \pi/\sqrt{k} + \delta$; the integral mean curvature estimate gives an upper bound for Δe, and a generalized maximum principle (based on Sobolev embedding and Moser iteration) yields a contradiction. Aubry [6] subsequently completed the picture by establishing finiteness of the fundamental group under the same hypotheses.

Theorem 3.5 (Generalized Bonnet–Myers, Aubry [6, Theorem 1.2])
Let $k > 0$ and $p > n/2$. There exists $\varepsilon = \varepsilon(n,p) > 0$ such that if (M^n, g) is a closed Riemannian manifold with

$$\left(\frac{1}{\mathrm{vol}(M)} \int_M (\mathrm{Ric}_-^k)^p \, \mathrm{dvol} \right)^{1/p} \leq \varepsilon,$$

then $\mathrm{diam}(M) \leq \pi/\sqrt{k} + C(n,p)\, \varepsilon^{1/2}$ and $\pi_1(M)$ is finite.

Later in this chapter, Aubry's theorem is applied as the first step: it ensures that a manifold with small integral Ricci curvature below $(n-1)k$ for $k > 0$ is compact, so that the normalized isoperimetric profile function h_1 is well-defined. It also guarantees a uniform diameter bound, needed in the subsequent application of the Petersen–Sprouse Heintze–Karcher inequality (Theorem 3.4).

Local Sobolev constants and further developments. Beyond the comparison theorems used directly in this thesis, the integral curvature theory has developed into a comprehensive analytic toolkit. Petersen and Wei [68] obtained a generalized maximum principle, excess estimates with sharp Hölder exponents, and an almost splitting theorem under integral curvature bounds, all derived from the L^{2p} mean curvature estimate. A significant advance came with the work of Dai, Wei, and Zhang [25], who proved the first local Sobolev constant estimate for integral Ricci curvature *without* a non-collapsing assumption: if the scale-invariant curvature quantity $\kappa(p,1)$ is sufficiently small, then a local $(1,1^*)$-Sobolev inequality holds on $B_r(x)$. From this single inequality they derived two-sided Gaussian heat kernel bounds, gradient estimates for harmonic functions, and L^2 Hessian estimates for parabolic approximation of distance functions, extending the Colding–Naber machinery to the collapsed integral curvature setting.

Isoperimetric constant estimates in the integral curvature setting were also explored by Gallot [33] and Dai, Wei, and Zhang [25]. More recently, the integral Ricci comparison framework has been extended in two directions: to the Kato-class setting by Carron, Rose, and Stollmann, where the L^p condition on $\mathrm{Ric}^{\,k}_{-}$ is replaced by a strictly weaker condition involving the heat kernel, and to synthetic curvature-dimension conditions, where Ketterer showed that sequences with L^p Ricci deficit converging to zero subconverge to metric measure spaces satisfying the condition $\mathrm{CD}(K,N)$ of Lott–Sturm–Villani.

The results surveyed in this section—specifically, the Petersen–Wei mean curvature comparison (Theorem 3.1), the Chen–Wei improved volume comparison (Theorem 3.3), the Petersen–Sprouse Heintze–Karcher inequality (Theorem 3.4), and Aubry's generalized Bonnet–Myers theorem (Theorem 3.5)—are the four ingredients that enter into the proofs of the isoperimetric profile comparisons later in this chapter.

3.2 Isoperimetric Profile Functions

Let (M^n, g) be an n-dimensional Riemannian manifold. The *isoperimetric profile function* $h_2(\beta, g)$ is defined on the interval $(0, |M|)$ by

$$h_2(\beta, g) := \inf\big\{|\partial\Omega| \;\big|\; \Omega \text{ is a smooth domain in } M \text{ with } |\Omega| = \beta\big\}. \tag{3.1}$$

Here, $|\Omega|$ denotes the n-dimensional volume and $|\partial\Omega|$ denotes the $(n-1)$-dimensional area of $\partial\Omega$. The case $|M| = \infty$ is allowed. A region Ω in M with $|\partial\Omega| = h_2(\beta, g)$ and $|\Omega| = \beta$ is called an *isoperimetric region* of volume β. The existence and regularity of such regions are well understood; for detailed discussions, see Chapter 6 of [75].

In the context of a compact Riemannian manifold M, the *normalized isoperimetric profile function* $h_1(\beta, g)$ is defined for any $\beta \in (0, 1)$ by

$$h_1(\beta, g) := \inf\left\{\frac{|\partial\Omega|}{|M|} \;\middle|\; \Omega \text{ is a smooth domain in } M \text{ with } \frac{|\Omega|}{|M|} = \beta\right\}.$$

The function $h_1(\beta, g)$ is Hölder continuous and satisfies the asymptotic

$$\lim_{\beta\to 0} \frac{h_1(\beta, g)}{\beta^{\frac{n-1}{n}}} = n\left(\frac{\omega_n}{|M|}\right)^{1/n}, \tag{3.2}$$

where ω_n denotes the n-dimensional volume of the unit ball in $\mathbb{R}^n$. For a compact manifold M, the following relationship holds:

$$h_1(\beta, g) = \frac{h_2(\beta\,|M|\,, g)}{|M|}. \tag{3.3}$$

Classical comparison results. The pointwise comparison for h_2 was originally proven by Morgan and Johnson [59] and later given an alternative proof by Ni and Wang [63].

Theorem 3.6 (Morgan–Johnson [59], Ni–Wang [63])

Let (M^n, g) be a complete Riemannian manifold. For a given $k \in \mathbb{R}$, assume that

Ric $\geq (n-1)k$. Then for $\beta \in (0, |M|)$,

$$h_2(\beta, g) \leq h_2(\beta, g_k), \tag{3.4}$$

where (M_k^n, g_k) is a complete simply connected space of constant curvature k. If equality ever holds somewhere, then (M^n, g) is isometric to (M_k^n, g_k).

The Lévy–Gromov isoperimetric inequality [35] can be reformulated as $h_1(\beta, g) \geq h_1(\beta, g_1)$, where g_1 is the standard metric on the unit sphere. Combining this with the relation (3.3) and the inequality (3.4) yields (Corollary 3.2 in [63]):

$$h_1(\beta, g_1) \leq h_1(\beta, g) \leq \frac{|S^n|}{|M|} \cdot h_1\left(\frac{|M|}{|S^n|}\beta,\, g_1\right). \tag{3.5}$$

Define the usual comparison functions

$$\mathrm{sn}_k(t) = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, t), & k > 0, \\ t, & k = 0, \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}\, t), & k < 0. \end{cases} \tag{3.6}$$

Main results. The main results of this chapter extend the above comparison theorems to the setting of integral Ricci curvature. The first two results concern the h_2 profile function, extending the inequality (3.4) to a quantitative estimate with error controlled by $\left\|\mathrm{Ric}_-^k\right\|_p$. The first pertains to the non-compact setting $k \leq 0$, which is particularly noteworthy due to the scarcity of results for integral curvature on non-compact manifolds. In the compact setting $k > 0$, the result can be used to assess the degree to which the second inequality in (3.5) must be relaxed when transitioning from pointwise to integral curvature conditions.

Theorem 3.7

Let (M^n, g) be a Riemannian manifold, $p > n/2$, and $k \leq 0$ be given. When $k = 0$, assume that $\operatorname{diam}(M) = d < \infty$. Then for any $\beta \in (0, |M|)$,

$$h_2(\beta, g) - h_2(\beta, g_k) \leq \left(\frac{(n-1)(2p-1)}{2p-n} \left\| \operatorname{Ric}_-^k \right\|_p \right)^{1/2} \beta^{\frac{2p-1}{2p}} + f(\beta, n, p, k, d, \left\| \operatorname{Ric}_-^k \right\|_p),$$

where (M_k^n, g_k) is a complete simply connected space of constant curvature k and $f \equiv 0$ when $\left\| \operatorname{Ric}_-^k \right\|_p = 0$, recovering (3.4).

Theorem 3.8

Let (M^n, g) be a Riemannian manifold, $p > n/2$, and $k > 0$ be given. Assume additionally that $\operatorname{diam}(M) < \frac{\pi}{2\sqrt{k}}$. Then there exists $\varepsilon > 0$ such that if $\left\| \operatorname{Ric}_-^k \right\|_p < \varepsilon$, then for any $\beta \in (0, |M|)$,

$$h_2(\beta, g) - h_2(\beta, g_k) \leq \left(\frac{(n-1)(2p-1)}{2p-n} \left\| \operatorname{Ric}_-^k \right\|_p \right)^{1/2} \beta^{\frac{2p-1}{2p}} + f(\beta, n, p, k, d, \left\| \operatorname{Ric}_-^k \right\|_p),$$

where (M_k^n, g_k) is a complete simply connected space of constant curvature k and $f \equiv 0$ when $\left\| \operatorname{Ric}_-^k \right\|_p = 0$, recovering (3.4).

Remark 3.1. The function f is increasing in β and vanishes at $\beta = 0$. In the case $k = 0$, it can be written down explicitly:

$$f = \left[\left(1 + C(n, p, k, d) \left\| \operatorname{Ric}_-^k \right\|_p^{1/2} \right)^{2p/n} - 1 \right] n \, (\omega_n)^{1/n} \, \beta^{\frac{n-1}{n}}.$$

The next two results involve the h_1 isoperimetric profile function in the positive curvature regime $k > 0$.

Theorem 3.9

Let (M^n, g) be an n-dimensional Riemannian manifold, $p > n/2$. For any $k > 0$ and $\alpha > 1$, there exists $\delta = \delta(n, p, \alpha, k) > 0$ such that if $\left\| \operatorname{Ric}_-^k \right\|_p \leq \delta$, then the isoperimetric profile function $h_1(\beta, g)$ is a positive super solution of

$$\alpha \psi \left(k + \left(\frac{\psi'}{n-1} \right)^2 \right)^{\frac{n-1}{2}} = \frac{1}{\int_0^{d'} \operatorname{sn}_k^{n-1} t \, dt}, \tag{3.7}$$

where $d' = \min\{\pi, \mathrm{diam}(M)\}$.

This generalization extends Theorem 4.1 of Ni and Wang [63] and retrieves it by taking the limit $\alpha \to 1^+$. Compactness is not assumed, as it follows from the generalized Bonnet–Myers theorem of Aubry [6]. Combining the Petersen–Sprouse estimate [67] with the approach of Ni and Wang yields the following quantitative Lévy–Gromov type estimate.

Theorem 3.10

Let (M^n, g) be an n-dimensional Riemannian manifold, $p > n/2$. For any $k > 0$ and $\alpha > 1$, there exists $\delta = \delta(n, p, \alpha, k) > 0$ such that if $\left\|\mathrm{Ric}_-^{\,k}\right\|_p \leq \delta$, then

$$h_1(\beta, g) \geq L\, h_1(\beta, g_k) - \varepsilon(n, d', \alpha, k),$$

where $d' = \min\left\{\frac{\pi}{\sqrt{k}}, \mathrm{diam}(M)\right\}$, g_k is the standard metric on S_k^n, and the constants are

$$L = \left(\frac{\gamma_n}{\lambda^k_{n,d'}}\right)^{1/n}, \qquad \gamma_n = \int_0^{\pi/\sqrt{k}} \mathrm{sn}_k^{n-1} t\, dt,$$

$$\lambda^k_{n,d'} = \int_0^{d'} \mathrm{sn}_k^{n-1} t\, dt, \qquad \varepsilon(n, d', \alpha, k) = \frac{(\alpha-1)L}{\alpha} \cdot \frac{1}{\int_0^{\pi} \sin(\sqrt{k}\, t)\, dt}.$$

Since $L \geq 1$, this improves the classical Lévy–Gromov isoperimetric inequality (the first inequality in (3.5)). For a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$ for $k > 0$, $\left\|\mathrm{Ric}_-^{\,k}\right\|_p = 0$ and $\lim_{\alpha \to 1^+} \varepsilon(n, d', \alpha, k) = 0$, so the following corollary is immediate.

Corollary 3.11

Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$ for $k > 0$. Then for any $\beta \in (0, 1)$,

$$h_1(\beta, g) \geq L\, h_1(\beta, g_k),$$

which is Bérard–Besson–Gallot's estimate.

3.3 Comparisons for the h_2 Isoperimetric Profile

This section contains the proofs of Theorem 3.7 and Theorem 3.8, followed by their extension to the relative isoperimetric setting.

3.3.1 Proof of Theorem 3.7 and Theorem 3.8

The proof begins with two preliminary observations on positive increasing functions, which will be used to compare the radius of a ball of a given volume on the manifold with the corresponding radius on the model space.

Proposition 3.12

Let $A \in C^\infty([0, 2r])$ be positive on $(0, 2r)$ and strictly increasing on the subset $(0, r] \subseteq [0, 2r]$. Let $1 \leq \alpha \leq 2$ and assume that for all $s \in [0, r]$, $A(r+s) \geq A(r-s)$. If

$$\int_0^{\bar{r}} A(s)\, ds \leq \alpha \int_0^r A(s)\, ds,$$

then $\bar{r} \leq \alpha r$.

Proof. When $\alpha = 1$, since A is positive, $\int_0^{\bar{r}} A(s)\, ds \leq \int_0^r A(s)\, ds$ implies $\bar{r} \leq r$. Now consider the case $1 < \alpha \leq 2$. Assume $\bar{r} > \alpha r$ to get a contradiction. Then

$$\int_0^{\bar{r}} A(s)\, ds > \int_0^{\alpha r} A(s)\, ds = \int_0^r A(s)\, ds + \int_r^{\alpha r} A(s)\, ds.$$

Let $B(s) = A(r-s)$ on $[0, r]$. Then $B'(s) = -A'(r-s) < 0$, so B is strictly decreasing. Since $1 < \alpha \leq 2$, $(\alpha - 1)s < s$ and so $B(s) < B((\alpha-1)s)$. Thus,

$$(\alpha - 1)B(s) < (\alpha - 1)B((\alpha - 1)s)$$

for $0 < s < r$. Integrating,

$$(\alpha-1)\int_0^r B(s)\,ds < \int_0^r (\alpha-1)B((\alpha-1)s)\,ds = \int_0^{(\alpha-1)r} B(t)\,dt$$
$$= \int_0^{(\alpha-1)r} A(r-t)\,dt \le \int_0^{(\alpha-1)r} A(r+t)\,dt = \int_r^{\alpha r} A(s)\,ds.$$

Thus,

$$\int_0^{\bar{r}} A(s)\,ds > \int_0^r A(s)\,ds + (\alpha-1)\int_0^r B(s)\,ds$$
$$= \int_0^r A(s)\,ds + (\alpha-1)\int_0^r A(r-s)\,ds = \alpha\int_0^r A(s)\,ds,$$

which contradicts the assumption. Thus $\bar{r} \le \alpha r$. □

The next proposition mirrors the previous one for the scenario $k < 0$. The computation is carried out explicitly in the space form of constant sectional curvature $k = -1$; the general case follows straightforwardly. Since $\sinh(s)$ is increasing everywhere, the assumption $\alpha \le 2$ is not needed.

Proposition 3.13

Denote $A(r) = \sinh^{n-1}(r)$ with $r \in [0, +\infty)$, let $\alpha \ge 1$, and assume that

$$\int_0^{\bar{r}} A(s)\,ds \le \alpha \int_0^r A(s)\,ds.$$

Then $\bar{r} \le \alpha^{1/n} r$.

Proof. Define the function $f(r) = \frac{\int_0^r A(s)\,ds}{r^n}$. The assumption can be rewritten as $f(\bar{r}) \le \alpha f(r)\frac{r^n}{\bar{r}^n}$. By directly computing the derivative, f is strictly increasing. Consider the case $0 < r < \bar{r}$. Then $f(r) < f(\bar{r}) \le \alpha f(r)\frac{r^n}{\bar{r}^n}$, implying $1 < \alpha\frac{r^n}{\bar{r}^n}$, which gives $\bar{r}^n < \alpha r^n$. The case $\bar{r} \le r$ follows easily since $\alpha \ge 1$, yielding $\bar{r}^n \le \alpha r^n$ and thus the desired conclusion $\bar{r} \le \alpha^{1/n} r$. □

Remark 3.2. In the case $k = 0$, let $A(s) = s^{n-1}$. By direct computation, $\int_0^{\bar{r}} A(s)\,ds \le \alpha\int_0^r A(s)\,ds$ implies $\bar{r} \le \alpha^{1/n} r$.

Proof of Theorem 3.7 and Theorem 3.8. The proof strategy entails considering the difference between the measures of the boundaries of balls in the manifold and those in the model space. This offers a straightforward upper bound for the difference of the h_2 functions. To estimate this difference, the derivatives are computed explicitly, leading to the emergence of a mean curvature term, which is controlled by the Petersen–Wei mean curvature comparison (Theorem 3.1).

Fix a point $x \in M$ and $\beta \in (0, |M|)$. Define a function $I_x(t, g) = |\partial B_x(r)|$, where r is determined by $|B_x(r)| = t$. Then

$$h_2(\beta, g) = \inf_{|\Omega|=\beta} |\partial\Omega| \le I_x(\beta, g).$$

Note that $h_2(\beta, g_k) = I_{\bar{x}}(\beta, g_k)$ for $\bar{x} \in M_k$. Combining,

$$h_2(\beta, g) - h_2(\beta, g_k) \le I_x(\beta, g) - I_{\bar{x}}(\beta, g_k). \tag{3.8}$$

Let $D(t) = I_x(t, g) - I_{\bar{x}}(t, g_k)$. We seek an upper bound for $D(\beta)$. Recall that $I_x(t, g) = |\partial B_x(r)|$ with $|B_x(r)| = t$. Using geodesic polar coordinates centered at x, where $m(r, \theta)$ is the mean curvature and $A(s, \theta)\, d\theta\, ds$ is the volume form,

$$\frac{d}{dr} |\partial B_x(r)| = \frac{d}{dr} \int_{S^{n-1}} A(r, \theta)\, d\theta = \int_{S^{n-1}} m(r, \theta) A(r, \theta)\, d\theta$$

and

$$\frac{dt}{dr} = \frac{d}{dr} |B_x(r)| = \int_{S^{n-1}} A(r, \theta)\, d\theta = |\partial B_x(r)|\,.$$

Thus,

$$I_x'(t, g) = \frac{d}{dt} |\partial B_x(r)| = \frac{\int_{S^{n-1}} m(r, \theta) A(r, \theta)\, d\theta}{|\partial B_x(r)|}$$

and

$$I_{\bar{x}}'(t, g_k) = \frac{\int_{S^{n-1}} \bar{m}(\bar{r}) \bar{A}(\bar{r})\, d\theta}{|\partial B_{\bar{x}}(\bar{r})|} = \bar{m}(\bar{r}),$$

where $B_{\bar{x}}(\bar{r})$ is the ball on M_k with $|B_{\bar{x}}(\bar{r})| = t$. Then

$$\begin{aligned} D'(t) &= I'_x(t,g) - I'_{\bar{x}}(t,g_k) \\ &= \frac{\int_{S^{n-1}} m(r,\theta)A(r,\theta)\,d\theta}{|\partial B_x(r)|} - \bar{m}(\bar{r}) \\ &= \frac{\int_{S^{n-1}} [m(r,\theta) - \bar{m}(r)]A(r,\theta)\,d\theta}{|\partial B_x(r)|} + [\bar{m}(r) - \bar{m}(\bar{r})]. \end{aligned}$$

Let $m^k_+(r,\theta) := (m(r,\theta) - \bar{m}(r))_+$. Then

$$\frac{\int_{S^{n-1}} [m(r,\theta) - \bar{m}(r)]A(r,\theta)\,d\theta}{|\partial B_x(r)|} \le \frac{\int_{S^{n-1}} m^k_+(r,\theta)A(r,\theta)\,d\theta}{|\partial B_x(r)|}.$$

Since $I_x(0,g) = 0$ and $I_{\bar{x}}(0,g_k) = 0$, $D(0) = 0$. Thus,

$$\begin{aligned} D(\beta) = \int_0^\beta D'(t)\,dt &\le \int_0^\beta \frac{\int_{S^{n-1}} m^k_+(r,\theta)A(r,\theta)\,d\theta}{|\partial B_x(r)|}\,dt + \int_0^\beta (\bar{m}(r) - \bar{m}(\bar{r}))\,dt \\ &\le \int_0^{r_0}\int_{S^{n-1}} m^k_+(r,\theta)A(r,\theta)\,d\theta\,dr + \int_0^\beta (\bar{m}(r) - \bar{m}(\bar{r}))\,dt, \end{aligned} \tag{3.9}$$

where r_0 is determined by $\beta = |B_x(r_0)|$. By Hölder's inequality,

$$\begin{aligned} &\int_0^{r_0}\int_{S^{n-1}} m^k_+(r,\theta)\,A(r,\theta)\,d\theta\,dr \\ &\quad\le \left(\int_0^{r_0}\int_{S^{n-1}} (m^k_+)^{2p}\,A(r,\theta)\,d\theta\,dr\right)^{1/(2p)} \left(\int_0^{r_0}\int_{S^{n-1}} A(r,\theta)\,d\theta\,dr\right)^{(2p-1)/(2p)} \\ &\quad= \left(\int_{B_x(r_0)} (m^k_+)^{2p}\,\mathrm{dvol}\right)^{1/(2p)} |B_x(r_0)|^{(2p-1)/(2p)}. \end{aligned}$$

Now the mean curvature comparison under integral Ricci curvature (Theorem 3.1) gives

$$\left\|m^k_+\right\|_{2p}(r) = \sup_{y\in M}\left(\int_{B_y(r)} (m^k_+)^{2p}\,\mathrm{dvol}\right)^{1/(2p)} \le \left(\frac{(n-1)(2p-1)}{2p-n}\left\|\mathrm{Ric}^{\,k}_{-}\right\|_p(r)\right)^{1/2}.$$

Combining these inequalities,

$$\begin{aligned} D(\beta) &\le \left\|m^k_+\right\|_{2p}(r_0)\;|B_x(r_0)|^{(2p-1)/(2p)} + \int_0^\beta (\bar{m}(r) - \bar{m}(\bar{r}))\,dt \\ &\le \left(\frac{(n-1)(2p-1)}{2p-n}\left\|\mathrm{Ric}^{\,k}_{-}\right\|_p(r_0)\right)^{1/2} \beta^{(2p-1)/(2p)} + \int_0^\beta (\bar{m}(r) - \bar{m}(\bar{r}))\,dt. \end{aligned} \tag{3.10}$$

Thus,

$$h_2(\beta, g) - h_2(\beta, g_k) \leq \left(\frac{(n-1)(2p-1)}{2p-n}\left\|\mathrm{Ric}_-^k\right\|_p\right)^{1/2}\beta^{(2p-1)/(2p)} + \int_0^\beta (\bar{m}(r) - \bar{m}(\bar{r}))\, dt. \tag{3.11}$$

To estimate the second term, r and $\bar{r}$ are compared using the improved volume comparison theorem of Chen and Wei [23] (Theorem 3.3):

$$|B_{\bar{x}}(\bar{r})| = |B_x(r)| \leq \left(1 + C(n,p,k,r)\left\|\mathrm{Ric}_-^k\right\|_p(r)^{1/2}\right)^{2p}|B_{\bar{x}}(r)|,$$

where

$$C(n,p,k,r) = \left(\frac{(n-1)(2p-1)}{2p-n}\right)^{1/2}\int_0^r\left(\frac{1}{|B_{\bar{x}}(r)|}\right)^{1/(2p)} dt.$$

When $k < 0$, this constant is bounded for all r, eliminating the need for compactness. We denote $C(n,p,k,d)$ as the limit of the constant as $r \to \infty$. For $k = 0$, compactness is necessary as C increases with r. In the positive case $k > 0$, the diameter restriction is already assumed to apply the mean curvature and volume comparison.

Since $\left\|\mathrm{Ric}_-^k\right\|_p(r)$ is also increasing in r,

$$\int_0^{\bar{r}} \bar{A}(s)\, ds \leq \left(1 + C(n,p,k,d)\left\|\mathrm{Ric}_-^k\right\|_p^{1/2}\right)^{2p}\int_0^r \bar{A}(s)\, ds, \tag{3.12}$$

where $d = \mathrm{diam}(M)$.

Now depending on the sign of k, Proposition 3.12 or Proposition 3.13 is applied. Define $q = 2p$ if $k > 0$ and $q = 2p/n$ if $k \leq 0$. For positive $k > 0$, there exists $\varepsilon > 0$ such that $\left\|\mathrm{Ric}_-^k\right\|_p < \varepsilon$ implies $\left(1 + C(n,p,k,d)\left\|\mathrm{Ric}_-^k\right\|_p^{1/2}\right)^{2p} \leq 2$. Since $\bar{A}(s)$ is a positive increasing function on $(0, r]$, Proposition 3.12 applies. For negative $k < 0$, Proposition 3.13 applies and $\left\|\mathrm{Ric}_-^k\right\|_p$ is not required to be small. In either case,

$$\bar{r} \leq \left(1 + C(n,p,k,d)\left\|\mathrm{Ric}_-^k\right\|_p^{1/2}\right)^q r. \tag{3.13}$$

For $k = 0$, the above inequality follows from a direct computation (Remark 3.2).

Since the mean curvature $\bar{m}$ on the model space is decreasing in r,

$$\bar{m}(r) \leq \bar{m}\left(\frac{\bar{r}}{\left(1 + C(n,p,k,d)\left\|\mathrm{Ric}\,{}_{-}^{k}\right\|_{p}^{1/2}\right)^{q}}\right).$$

Thus,

$$\int_0^\beta (\bar{m}(r) - \bar{m}(\bar{r}))\,dt \leq \int_0^\beta \left(\bar{m}\left(\frac{\bar{r}}{\left(1 + C(n,p,k,d)\left\|\mathrm{Ric}\,{}_{-}^{k}\right\|_{p}^{1/2}\right)^{q}}\right) - \bar{m}(\bar{r})\right)dt.$$

Let

$$f(\beta, n, p, k, d, \left\|\mathrm{Ric}\,{}_{-}^{k}\right\|_{p}) := \int_0^\beta \left(\bar{m}\left(\frac{\bar{r}}{\left(1 + C(n,p,k,d)\left\|\mathrm{Ric}\,{}_{-}^{k}\right\|_{p}^{1/2}\right)^{q}}\right) - \bar{m}(\bar{r})\right)dt.$$

Combining with (3.11),

$$\begin{aligned} &h_2(\beta, g) - h_2(\beta, g_k) \\ &\quad \leq \left(\frac{(n-1)(2p-1)}{2p-n}\left\|\mathrm{Ric}\,{}_{-}^{k}\right\|_{p}\right)^{1/2} \beta^{(2p-1)/(2p)} + f(\beta, n, p, k, d, \left\|\mathrm{Ric}\,{}_{-}^{k}\right\|_{p}). \qquad \square \end{aligned}$$

3.3.2 Relative Isoperimetric Profile Function Comparison

The relative isoperimetric comparison extends the above results to convex bodies. The difference is that the perimeter is considered *relative* to a convex body Ω, which is a convex smooth domain with compact closure in (M^n, g). If a hypersurface splits Ω into two sets, the relative perimeter of each is the surface area of the hypersurface; pieces from the boundary of Ω do not contribute to the relative perimeter. For a more extensive discussion of the relative isoperimetric problem, see Chapter 9 of [74].

Define the *relative isoperimetric profile function* with respect to a convex body Ω as

$$h_2^\Omega(\beta, g) = \inf\left\{|\partial D \cap \Omega| \;\middle|\; D \text{ is a smooth domain in } \Omega \text{ with } |D \cap \Omega| = \beta\right\}.$$

Here $\partial D \cap \partial\Omega$ may be nonempty. Denote by $\mathbb{H}_k^n$ a half space in the simply connected space form (M_k, g_k) of constant sectional curvature k, and denote by $h_2^{\mathbb{H}_k^n}(\beta, g_k)$ its relative isoperimetric function.

Theorem 3.14

Let (M^n, g) be a Riemannian manifold and Ω a convex body on M. Let $p > n/2$ and $k > 0$ be given. Assume that $\operatorname{diam}(M) = d < \frac{\pi}{2\sqrt{k}}$. Then there exists $\varepsilon > 0$ such that if $\left\|\operatorname{Ric}^{\,k}_{-}\right\|_p < \varepsilon$, then for any $\beta \in (0, |\Omega|)$,

$$h_2^{\Omega}(\beta, g) - h_2^{\mathbb{H}^n_k}(\beta, g_k) \leq \left(\frac{(n-1)(2p-1)}{2p-n}\left\|\operatorname{Ric}^{\,k}_{-}\right\|_p\right)^{1/2} \beta^{\frac{2p-1}{2p}} + \tilde{f}(\beta, n, p, k, d, \left\|\operatorname{Ric}^{\,k}_{-}\right\|_p)$$

Theorem 3.15

Let (M^n, g) be a Riemannian manifold and Ω a convex body on M. Let $p > n/2$ and $k \leq 0$ be given. When $k = 0$, assume that $\operatorname{diam}(M) = d < \infty$. Then for any $\beta \in (0, |\Omega|)$,

$$h_2^{\Omega}(\beta, g) - h_2^{\mathbb{H}^n_k}(\beta, g_k) \leq \left(\frac{(n-1)(2p-1)}{2p-n}\left\|\operatorname{Ric}^{\,k}_{-}\right\|_p\right)^{1/2} \beta^{\frac{2p-1}{2p}} + \tilde{f}(\beta, n, p, k, d, \left\|\operatorname{Ric}^{\,k}_{-}\right\|_p).$$

Remark 3.3. In particular, $\tilde{f} = 0$ when $\left\|\operatorname{Ric}^{\,k}_{-}\right\|_p = 0$, which recovers the h_2 comparison theorem under pointwise Ricci curvature lower bounds in [9].

Only the differences from the proof of Theorem 3.7 and Theorem 3.8 are detailed.

Proof of Theorem 3.14 and Theorem 3.15. Now take any $x \in \partial\Omega$. Define $I_x(t, g) = |\partial B_x(r) \cap \Omega|$ when $t = |B_x(r) \cap \Omega|$ and $I_{\bar{x}}(t, g_k) = \left|\partial \bar{B}(\bar{r}) \cap \mathbb{H}^n_k\right|$ when $\left|\bar{B}(\bar{r}) \cap \mathbb{H}^n_k\right| = t$. Let

$$U_t = \left\{\theta \in S^{n-1} \;\middle|\; (t, \theta) \in B_x(r) \cap \Omega\right\},$$

where (t, θ) denotes geodesic polar coordinates centered at x. Convexity of Ω implies that if $a \le b$, then $U_b \subseteq U_a$. Thus,

$$\begin{aligned}
\frac{d}{dr}\left|\partial B_x(r)\cap\Omega\right| &= \lim_{h\to 0}\frac{|\partial B_x(r+h)\cap\Omega| - |\partial B_x(r)\cap\Omega|}{h} \\
&= \lim_{h\to 0}\frac{\int_{U_{r+h}} A(r+h,\theta)\,d\theta - \int_{U_r} A(r,\theta)\,d\theta}{h} \\
&\le \lim_{h\to 0}\frac{\int_{U_r} A(r+h,\theta)\,d\theta - \int_{U_r} A(r,\theta)\,d\theta}{h} \\
&= \int_{U_r} A'(r,\theta)\,d\theta = \int_{U_r} m(r,\theta)A(r,\theta)\,d\theta.
\end{aligned}$$

Also, from $|B_x(r)\cap\Omega| = t$,

$$\frac{dt}{dr} = \frac{d}{dr}\int_0^r\int_{U_s} A(s,\theta)\,d\theta\,ds = \int_{U_r} A(r,\theta)\,d\theta = \left|\partial B_x(r)\cap\Omega\right|.$$

These two inequalities imply

$$I_x'(t,g) = \frac{dI_x}{dr}\frac{dr}{dt} \le \frac{\int_{U_r} m(r,\theta)A(r,\theta)\,d\theta}{|\partial B_x(r)\cap\Omega|}.$$

Take any $\bar{x}$ on the boundary of $\mathbb{H}^n_k$. Then

$$I_{\bar{x}}'(t,g_k) = \frac{\int_{S^{n-1}_+} \bar{m}(\bar{r})\bar{A}(\bar{r})\,d\theta}{\frac{1}{2}\left|\partial B_{\bar{x}}(\bar{r})\right|} = \bar{m}(\bar{r}),$$

where $\bar{r}$ is determined by $|B_{\bar{x}}(\bar{r})\cap\mathbb{H}^n_k| = t$. Defining $D(t) = I_x(t,g) - I_{\bar{x}}(t,g_k)$ as before,

$$D'(t) \le \frac{\int_{U_r}[m(r,\theta) - \bar{m}(r)]A(r,\theta)\,d\theta}{|\partial B_x(r)\cap\Omega|} + [\bar{m}(r) - \bar{m}(\bar{r})].$$

Letting $m^k_+(r,\theta) = (m(r,\theta) - \bar{m}(r))_+$,

$$D(\beta) \le \int_0^{r_0}\int_{U_r} m^k_+(r,\theta)A(r,\theta)\,d\theta\,dr + \int_0^{\beta}(\bar{m}(r) - \bar{m}(\bar{r}))\,dt, \tag{3.14}$$

where r_0 is determined by $\beta = |B_x(r_0)\cap\Omega|$. By Hölder's inequality,

$$\begin{aligned}
\int_0^{r_0}\int_{U_r} m^k_+ A\,d\theta\,dr &\le \left(\int_{B_x(r_0)\cap\Omega}(m^k_+)^{2p}\,\mathrm{dvol}\right)^{1/(2p)}\left|B_x(r_0)\cap\Omega\right|^{(2p-1)/(2p)} \\
&\le \left\|m^k_+\right\|_{2p}(r_0)\,\beta^{(2p-1)/(2p)}.
\end{aligned}$$

For the second term on the right-hand side of (3.14), a volume comparison on a convex set with integral Ricci curvature is needed:

$$|B_x(r)\cap\Omega| \leq \left(1 + C_1(n,p,k,d)\left\|\mathrm{Ric}\,{}^k_-\right\|_p^{1/2}\right)^{2p} |B_{\bar{x}}(r)\cap\mathbb{H}^n_k|\,,$$

where $C_1(n,p,k,d)$ is a constant multiple of $C(n,p,k,d)$ from the proof of Theorem 3.7 and Theorem 3.8. This inequality can be obtained by modifying the proof in Chen and Wei [23]. The remainder of the proof follows the same lines as Theorem 3.7 and Theorem 3.8. □

3.4 Comparisons for the h_1 Isoperimetric Profile

This section contains the proofs of the differential comparison for the h_1 isoperimetric profile function (Theorem 3.9) and the quantitative Lévy–Gromov estimate (Theorem 3.10). The error term arises from the choice of $\alpha > 1$, as the Heintze–Karcher generalization for integral curvature of Petersen and Sprouse (Theorem 3.4) is used to replace the conventional Heintze–Karcher estimate in the argument of Ni and Wang [63]. The pointwise result is recovered by taking $\alpha \to 1^+$.

Proof of Theorem 3.9. The proof begins by applying Aubry's generalized Bonnet–Myers theorem (Theorem 3.5), which establishes that the diameter of manifolds with small integral Ricci curvature is uniformly bounded. This ensures compactness of M and enables the subsequent application of the Petersen–Sprouse Heintze–Karcher inequality (Theorem 3.4).

Choose any $\beta_0 \in (0,1)$ and let U be a small neighborhood of β_0. Within U, let $0 < \psi(\beta) \leq h_1(\beta, g)$ be a smooth function satisfying $\psi(\beta_0) = h_1(\beta_0, g)$. Suppose Ω is an isoperimetric region with volume β_0, meaning $\frac{|\partial\Omega|}{|M|} = h_1(\beta_0, g) = \psi(\beta_0)$. Then Ω has constant mean curvature m on the smooth part of $\partial\Omega$, and by the first variation formula (see the proof of Theorem 2.2 in [63]),

$$m = \psi'(\beta_0).$$

Define

$$r_0 = \max\{\mathrm{dist}(x, \partial\Omega) \mid x \in \Omega\},$$

$$r_1 = \max\{\mathrm{dist}(x, \partial\Omega) \mid x \in M \setminus \Omega\} \le d - r_0.$$

Given $\alpha > 1$ and $p > n/2$, the Heintze–Karcher inequality with integral Ricci curvature bounds (Theorem 3.4) ensures the existence of $\delta(n, p, \alpha, k) > 0$ such that if $\left\|\mathrm{Ric}\,^k_-\right\|_p \le \delta$, then

$$\mathrm{vol}(\Omega) \le \alpha \, \frac{\mathrm{area}(\partial\Omega)}{\mathrm{area}(\partial\bar{\Omega})} \, \mathrm{vol}(\bar{\Omega})$$

and

$$\mathrm{vol}(M \setminus \Omega) \le \alpha \, \frac{\mathrm{area}(\partial\Omega)}{\mathrm{area}(\partial\bar{\Omega})} \, \mathrm{vol}(\bar{B} \setminus \bar{\Omega}),$$

where $\bar{\Omega}$ is the ball of radius r_0 with constant mean curvature m and $\bar{B}$ is the ball of radius $\mathrm{diam}(M)$ on the model manifold S^n_k. Adding the two inequalities,

$$\mathrm{vol}(M) \le \alpha \, \frac{\mathrm{area}(\partial\Omega)}{\mathrm{area}(\partial\bar{\Omega})} \, \mathrm{vol}(\bar{B}).$$

Thus,

$$1 \le \alpha\psi(\beta_0) \cdot \frac{\mathrm{vol}(\bar{B})}{\mathrm{area}(\partial\bar{\Omega})}. \tag{3.15}$$

Now $\left|\partial\bar{\Omega}\right|$ is computed. The volume element on the model manifold S^n_k is $\bar{A}(r) = \mathrm{sn}_k^{n-1}(r)$, and the mean curvature on the boundary of a ball of radius r on S^n_k is $\bar{m}(r) = (n-1)\sqrt{k}\cot(\sqrt{k}\,r)$. Thus,

$$\mathrm{sn}_k(r) = \frac{n-1}{\sqrt{(\bar{m}(r))^2 + (n-1)^2 k}}.$$

Since $\bar{\Omega}$ has radius r_0 and constant mean curvature $m = \bar{m}(r_0) = \psi'(\beta_0)$,

$$\left|\partial\bar{\Omega}\right| = \left|S^{n-1}\right| \mathrm{sn}_k^{n-1}(r_0) = \left(k + \left(\frac{\psi'(\beta_0)}{n-1}\right)^2\right)^{-(n-1)/2}. \tag{3.16}$$

Note that

$$\left|\bar{B}\right| = \left|S^{n-1}\right| \int_0^{d'} \mathrm{sn}_k^{n-1} t \, dt, \tag{3.17}$$

where $d' = \min\{\pi, \mathrm{diam}(M)\}$. Combining (3.15), (3.16), and (3.17),

$$
\begin{aligned}
1 \le \alpha\psi(\beta_0) \cdot \frac{|\bar{B}|}{|\partial\bar{\Omega}|} &= \alpha\psi(\beta_0) \cdot \frac{\int_0^{d'} \mathrm{sn}_k^{n-1} t\, dt}{\mathrm{sn}_k^{n-1}(r_0)} \\
&= \alpha\psi(\beta_0) \left(k + \left(\frac{\psi'(\beta_0)}{n-1} \right)^2 \right)^{(n-1)/2} \int_0^{d'} \mathrm{sn}_k^{n-1} t\, dt.
\end{aligned}
$$

Hence

$$
\alpha\psi(\beta_0) \left(k + \left(\frac{\psi'(\beta_0)}{n-1} \right)^2 \right)^{(n-1)/2} \ge \frac{1}{\int_0^{d'} \mathrm{sn}_k^{n-1} t\, dt}. \qquad \square
$$

Proof of Theorem 3.10. First, there exists $\delta_1 > 0$ such that if $\left\| \mathrm{Ric}\,_-^k \right\|_p < \delta_1$, then M is compact, as ensured by the generalized Bonnet–Myers theorem with integral Ricci curvature bounds (Theorem 3.5). The normalized isoperimetric profile function $h_1(\beta, g)$ is considered on a manifold (M, g) with $\left\| \mathrm{Ric}\,_-^k \right\|_p < \delta = \min\{\delta_1, \delta_2\}$, where δ_2 is as defined in Theorem 3.9.

To obtain a contradiction, assume that there exists $\beta \in (0, 1)$ satisfying

$$
h_1(\beta, g) < L\phi(\beta) - \varepsilon, \tag{3.18}
$$

where $\phi(\beta) = h_1(\beta, g_k)$. Define a function f on $(0, 1)$ by

$$
f(\beta) := \frac{h_1(\beta, g) + \varepsilon}{L\phi(\beta)}.
$$

By the asymptotic (3.2) of h_1 and ϕ,

$$
\lim_{\beta \to 0^+} f(\beta) = \lim_{\beta \to 0^+} \frac{h_1(\beta, g) + \varepsilon}{L\phi(\beta)} = +\infty.
$$

Since h_1 and ϕ are symmetric, $f(1-\beta) = f(\beta)$ and so $\lim_{\beta \to 1^-} f(\beta) = +\infty$. Therefore f attains its minimum λ at some $\beta_0 \in (0, 1)$:

$$
h_1(\beta_0, g) + \varepsilon = \lambda L\phi(\beta_0), \qquad h_1(\beta, g) + \varepsilon \ge \lambda L\phi(\beta) \quad \forall\, \beta \in (0, 1).
$$

Moreover, the assumption (3.18) implies $\lambda < 1$.

Consider a smooth function ψ such that $0 < \psi(\beta) \le h_1(\beta, g)$ near β_0 and $\psi(\beta_0) =$

$h_1(\beta_0, g)$. Then

$$\psi(\beta_0) = \lambda L\phi(\beta_0) - \varepsilon, \qquad \psi'(\beta_0) = \lambda L\phi'(\beta_0). \tag{3.19}$$

By Theorem 3.9 applied with the support function ψ,

$$\frac{1}{\lambda^k_{n,d'}} \leq \alpha\psi \left(k + \left(\frac{\psi'}{n-1} \right)^2 \right)^{(n-1)/2}. \tag{3.20}$$

At β_0, the equations (3.19) and (3.20) imply

$$\frac{1}{\lambda^k_{n,d'}} \leq \alpha(\lambda L\phi - \varepsilon) \left(k + \left(\frac{\lambda L\phi'}{n-1} \right)^2 \right)^{(n-1)/2} < \alpha(L\phi - \varepsilon) \left(k + \left(\frac{L\phi'}{n-1} \right)^2 \right)^{(n-1)/2}, \tag{3.21}$$

where the last inequality follows from $L \geq 1$, $\phi(\beta_0) > 0$, $\lambda < 1$, and $\alpha > 1$. Since ϕ is a solution of the differential equation

$$\phi \left(k + \left(\frac{\phi'}{n-1} \right)^2 \right)^{(n-1)/2} = \frac{1}{\gamma_n}, \tag{3.22}$$

combining (3.21) and (3.22) gives

$$\begin{aligned}
\alpha(L\phi - \varepsilon) \left(k + \left(\frac{L\phi'}{n-1} \right)^2 \right)^{(n-1)/2} &> \frac{\gamma_n}{\lambda^k_{n,d'}} \, \phi \left(k + \left(\frac{\phi'}{n-1} \right)^2 \right)^{(n-1)/2} \\
&= L^n \, \phi \left(k + \left(\frac{\phi'}{n-1} \right)^2 \right)^{(n-1)/2} \\
&= L\phi \left(L^2 k + \left(\frac{L\phi'}{n-1} \right)^2 \right)^{(n-1)/2} \\
&\geq L\phi \left(k + \left(\frac{L\phi'}{n-1} \right)^2 \right)^{(n-1)/2}.
\end{aligned}$$

Thus $\alpha(L\phi - \varepsilon) > L\phi$, and

$$\varepsilon < \frac{(\alpha - 1)L\phi}{\alpha} \leq \frac{\alpha - 1}{\alpha} \cdot \frac{L}{\int_0^\pi \sin(\sqrt{k}\, t)\, dt},$$

which is a contradiction. $\square$

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