Title

NOTE OF THE MAGNETIC-MOMENT ADIABATTC INVARIANT FOE PARTICLE MOTION IN A DIPOLE MAGNETIC FIELD

Author

Heckman, Harry H.

Publication Date

1966-12-13

UCRL-17308
Preprint

University of California

Ernest O. Lawrence Radiation Laboratory

NOTE ON THE MAGNETIC-MOMENT ADIABATIC INVARIANT FOR PARTICLE MOTION IN A DIPOLE MAGNETIC FIELD

Berkeley, California

DISCLAIMER

Letter to be published in
Journal of Geophysical Research

UCRL-17308
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

NOTE ON THE MAGNETIC-MOMENT ADIABATIC INVARIANT FOR PARTICLE MOTION IN A DIPOLE MAGNETIC FIELD

Harry H. Heckman

December 13, 1966

Note on the Magnetic-Moment Adiabatic Invariant for Particle Motion in a Dipole Magnetic Field

Harry H. Heckman

Lawrence Radiation Laboratory
University of California
Berkeley, California

December 13, 1966

ABSTRACT

The magnetic-moment adiabatic-invariant series is evaluated through second order in the expansion parameter $\epsilon = \rho/L$, i.e. the ratio of the gyroradius $\rho = pc/eB$ to the McIlwain shell parameter, for the particular case of a particle at the equatorial plane of a dipole magnetic field. The resulting expression for the magnetic-moment series is compared with computed values of the magnetic moment for particle orbits in a dipole magnetic field obtained by numerically integrating the equations of motion. Orbits have been calculated for particles with $1.22\times10^{-2} < \epsilon < 8.29\times10^{-2}$. Under conditions of stable particle motion, the numerical results verify the constancy of the magnetic-moment series through $O(\epsilon^2)$ to accuracies σ (S. D.) = 0.026 to 3.2% depending on the value of ϵ.

Note on the Magnetic-Moment Adiabatic Invariant for Particle Motion in a Dipole Magnetic Field

Harry H. Heckman

Lawrence Radiation Laboratory
University of California
Berkeley, California

December 13, 1966

Quantities that are adiabatically conserved during particle motion are asymptotic series of the form: constant = $A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots$ [Northrop, 1963]. The magnetic-moment adiabatic invariant of motion for a particle in a dipole magnetic field can therefore be expressed as: constant = $M_0 + \epsilon M_1 + \epsilon^2 M_2 + \cdots$, where $M_0 = mv_\perp^2/2B$, and the expansion parameter ϵ is the ratio of the gyroradius at the median plane, $\rho = pc/eB$, to the McIlwain shell parameter, L.

To date theoretical calculations on the higher-order terms in the magnetic-moment series have been principally limited to the constant and linear term in ϵ [Kruskal, 1958; Dragt, 1964; and Mozer, 1966]. Gardner [1966] gives the following expression for the magnetic-moment adiabatic invariant series through the second order, i.e. $O(\epsilon^2)$, for the particular case of a particle at the equatorial plane of an axially symmetric field:

$$\begin{aligned}\text{constant} = \; & v_\perp^2/B - \epsilon\,(B'/B^3)[v^2 + v_\parallel^2]\, v_\phi \\ & -\epsilon^2(B''/2B^4)[v_\phi^2 v_\parallel^2 + (v_\phi^2 + v_\perp^2/4)(v^2 + v_\parallel^2)] \\ & +\epsilon^2(B'^2/B^5)[\tfrac{1}{2}(3v_\phi^2 + v_\parallel^2)(v^2 + v_\parallel^2) + 3v_\perp^4/8] \\ & +\epsilon^2(B'/2rB^4)[v_\phi^2(v^2 + 2v_\parallel^2) - v_\perp^2 v_\parallel^2 - 5v_\perp^2(v^2 + v_\parallel^2)/4].\end{aligned} \tag{1}$$

Here we define $v_\perp = v \sin\alpha$, $v_\parallel = v \cos\alpha$, $v_\phi = r\dot{\phi} = v \sin\alpha \sin\lambda$, and $B' = dB/dr$, etc.

By numerically integrating the equations of motion for a particle in a dipole magnetic field, [McCracken et al., 1962, and Heckman and Brady, 1966] we have evaluated the magnetic-moment adiabatic invariant for particle orbits at the equatorial plane. The purpose of this note is to compare these numerical results with equation 1.

Following Northrop [1963], we utilize the fact that the canonical angular momentum $p_\phi = \gamma m r v_\phi + e r A_\phi$ is a constant of the motion, in order to express the magnetic moment adiabatic invariant series, equation 1, for a dipole field in the form

$$C(\alpha, \lambda) = \text{constant} = \cos^2\alpha[a_0 - a_1\epsilon \sin\alpha \sin\lambda + a_2\epsilon^2 \sin^2\alpha \sin^2\lambda + \frac{3}{4}\epsilon^2 (1 - 8\cos^2\alpha)] , \qquad (2)$$

where the coefficients derived from equation 1 are $a_0 = 1$, $a_1 = 6$, and $a_2 = 51/2$. In this equation, α is the angle between the particle's velocity vector and the local magnetic field, i.e. the pitch angle, and λ is the phase angle between the radius vector $\underset{\sim}{r}$ and $\underset{\sim}{v}_\perp$ (Fig. 1).

Representative orbit calculations for particles with $1.22\times 10^{-2} < \epsilon < 8.29\times 10^{-2}$ were carried out in a dipole magnetic field having a magnetic moment $M = 0.3132$ gauss $r_e^3 \approx M_{earth}$. The corresponding proton kinetic energies at $L = 1.45\ r_e$ are $62.5 < T < 1600$ MeV. The starting points for the orbit calculations were in the equatorial plane at a radial distance $L = 1.45\ r_e$. The initial equatorial pitch angle in each case was given by $\cos\alpha_0 = 0.7448$.

Figure 2 presents the results of these calculations, where we have plotted the quantity $\cos^2\alpha_0/\cos^2\alpha$ vs $\sin\alpha \sin\lambda$. The calculated points for the phase and pitch angles are shown for 10 to 15 consecutive traversals of the equatorial plane. The numerical data generate smooth, well-defined loci indicative of stable particle motion [Garren et al., 1958]. For values of $\epsilon \leq 2.6 \times 10^{-2}$, the numerical results are in good agreement with theory. At higher values of ϵ, however, systematic differences are apparent between the numerical data and the adiabatic theory. Unstable, nonadiabatic motion, characterized by large, irregular fluctuations in the phase-pitch-angle points (not shown) developed between $\epsilon = 7.53 \times 10^{-2}$ (1400 MeV) and 8.29×10^{-2} (1600 MeV).

As demonstrated by Dragt [1965], particle motion in a dipole field may exhibit long-term stability even though $M_0 = mv_\perp^2/2B$ may vary by ± 40% in the equatorial plane. This behavior is illustrated in Fig. 2, where the zero-order approximation of the magnetic-moment series, $C(\alpha_0, 0) = \cos^2 \alpha_0$, can vary up to ≈ 50% from the true value of $C(\alpha, \lambda)$ for the particular orbits we have examined.

Using the numerically computed quantities $\cos\alpha$ and $\sin\alpha \sin\lambda$ at the median plane, we find that the magnetic-moment series, $C(\alpha, \lambda)$, equation 2, deviates from constancy by 0.026 to 3.2% (S. D.), depending upon ϵ. In order to examine the upper limits of the computation errors involved in the evaluation of $C(\alpha, \lambda)$, we adjusted the coefficients a_0, a_1, and a_2 by the method of least-squares to fit the data points shown in Fig. 2 and reevaluated equation 2.[1] The standard deviation of the resultant $C(\alpha, \lambda)_{LS}$ from constancy is 0.1 to 0.3 of that observed for $C(\alpha, \lambda)$. The results of these calculations are summarized in Table I. The actual values of $C(\alpha, \lambda)$ and $C(\alpha, \lambda)_{LS}$ vs $\sin\alpha \sin\lambda$ at the equatorial plane of the dipole are shown in Figs. 3 and 4, respectively.

[1] The signs of the least-squares coefficients a_0, a_1, and a_2 agree with theory, equation 2. Upon fitting the data to third order, i. e. $O(\epsilon^3)$, in $\sin\alpha \sin\lambda$, we find the sign of a_3 is negative.

Errors of integration introduced fractional changes in the particle velocity of $(1.5\pm0.9)\times10^{-4}$ during the course of each computation--indicative of the overall accuracy of the calculations. As a final check on the computational accuracy, the 125-MeV orbit calculations were traced backwards in time from terminus to starting point. We found that, while the phase and pitch angles are not precisely duplicated, the quantities $\cos^2\alpha_0/\cos^2\alpha$ continue to fall on the loci of data points (Fig. 2) to an accuracy of $(3\pm2)\times10^{-3}\%$. We conclude, therefore, that these numerical results are subject to small computational errors, thereby validating a comparison with the adiabatic theory. The magnetic-moment series, equation 2, predicted by theory has been verified to $O(\epsilon^2)$ for orbits at the median plane of a dipole field for $\epsilon \lesssim 2.6\times10^{-2}$, within the accuracies of our numerical results. It is clear, however, that $C(\alpha,\lambda)$ varies systematically from theory for $\epsilon \gtrsim 2.6\times10^{-2}$. Also, the standard deviation of both $C(\alpha,\lambda)$ and $C(\alpha,\lambda)_{LS}$ from their mean values increases monotonically with ϵ, i.e., with kinetic energy, which is consistent with increasing nonadiabatic behavior.

ACKNOWLEDGMENTS

I wish to thank Mr. Victor Brady of the Mathematics and Computing Group, for his assistance in carrying out the computer programming, and Dr. Theodore Northrop for his most helpful discussions on the subtleties of adiabatic theory. This work was performed under the auspices of the U. S. Atomic Energy Commission.

REFERENCES

Dragt, A. J., Trapped orbits in a magnetic dipole, Rev. Geophys., 3, 255-290, 1964.

Gardner, C. S., Second-order magnetic moment of a charged particle in an axially symmetric static field, PPL-AF-4, Plasma Physics Laboratory, Princeton University, May 1966.

Garren, A., R. J. Riddell, L. Smith, G. Bing, L. R. Henrich, T. G. Northrop, and J. E. Roberts, in Proceedings of the Second United Nations International Conference on Peaceful Uses of Atomic Energy, Geneva, 1958, Vol. 31, p. 65, United Nations, New York, 1958.

Heckman, H. H., and V. O. Brady, Effective atmospheric losses for 125-MeV protons in the South Atlantic anomaly, J. Geophys. Res., 71, 2791-2798, 1966.

Kruskal, M., The gyration of a charged particle, Project Matterhorn Rept. PM-S-33 (NYO-7903), Princeton University, March 1958.

McCracken, K. G., U. R. Rao, and M. A. Shea, The trajectories of cosmic rays in a high degree simulation of the geomagnetic field, Technical Report 77, Massachusetts Institute of Technology, August 1962.

Mozer, F. S., Proton trajectories in the radiation belts, J. Geophys. Res., 71, 2701-2708, 1966.

Northrop, T. G., The Adiabatic Motion of Charged Particles, Interscience Publishers, New York, 1963, and private communication, 1966.

TABLE I. Standard deviation $\sigma = \overline{C}^{-1}\left(\sum_i \frac{[C_i - \overline{C}]^2}{n-1}\right)^{\frac{1}{2}}$ of the equatorial magnetic-moment adiabatic series, equation 2, using theoretical and least-squares (LS) coefficients a_0, a_1, and a_2.

100 ϵ	T(MeV)	$\sigma(C)$	$\sigma(C)_{LS}$
1.758	125	2.6×10^{-4}	8.1×10^{-5}
3.831	500	3.1×10^{-3}	2.5×10^{-4}
5.961	1000	1.6×10^{-2}	1.3×10^{-3}
7.527	1400	3.2×10^{-2}	3.6×10^{-3}

FIGURE LEGENDS

Fig. 1. Coordinates of the trajectory at the equatorial plane of a magnetic dipole.

Fig. 2. Curves showing phase angles (λ) and pitch angles (α) for particles with $0.0122 \leq \epsilon \leq 0.0753$ at the equatorial plane of a magnetic dipole. Equivalent proton kinetic energies at $L = 1.45\ r_e$ are given. The circles are numerical results; the solid lines are theoretical curves derived from equation 2, with $a_0 = 1$, $a_1 = 6$, and $a_2 = 51/2$.

Fig. 3. The adiabatic constant $C(\alpha, \lambda)$, equation 2, plotted as a function of $\sin\alpha \ \sin\lambda$, using theoretical coefficients $a_0 = 1$, $a_1 = 6$, and $a_2 = 51/2$.

Fig. 4. The adiabatic constant $C(\alpha, \lambda)_{LS}$, where the coefficients a_0, a_1, and a_2 are obtained by a least-squares fit to the computed results (Fig. 2).

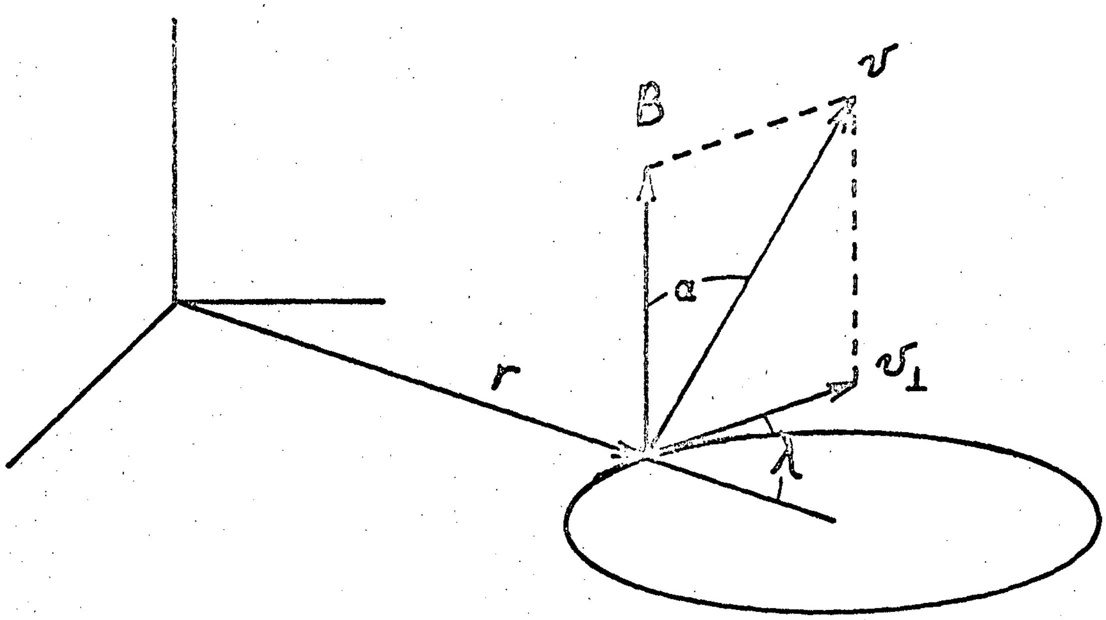

Fig. 1

MUB-14101

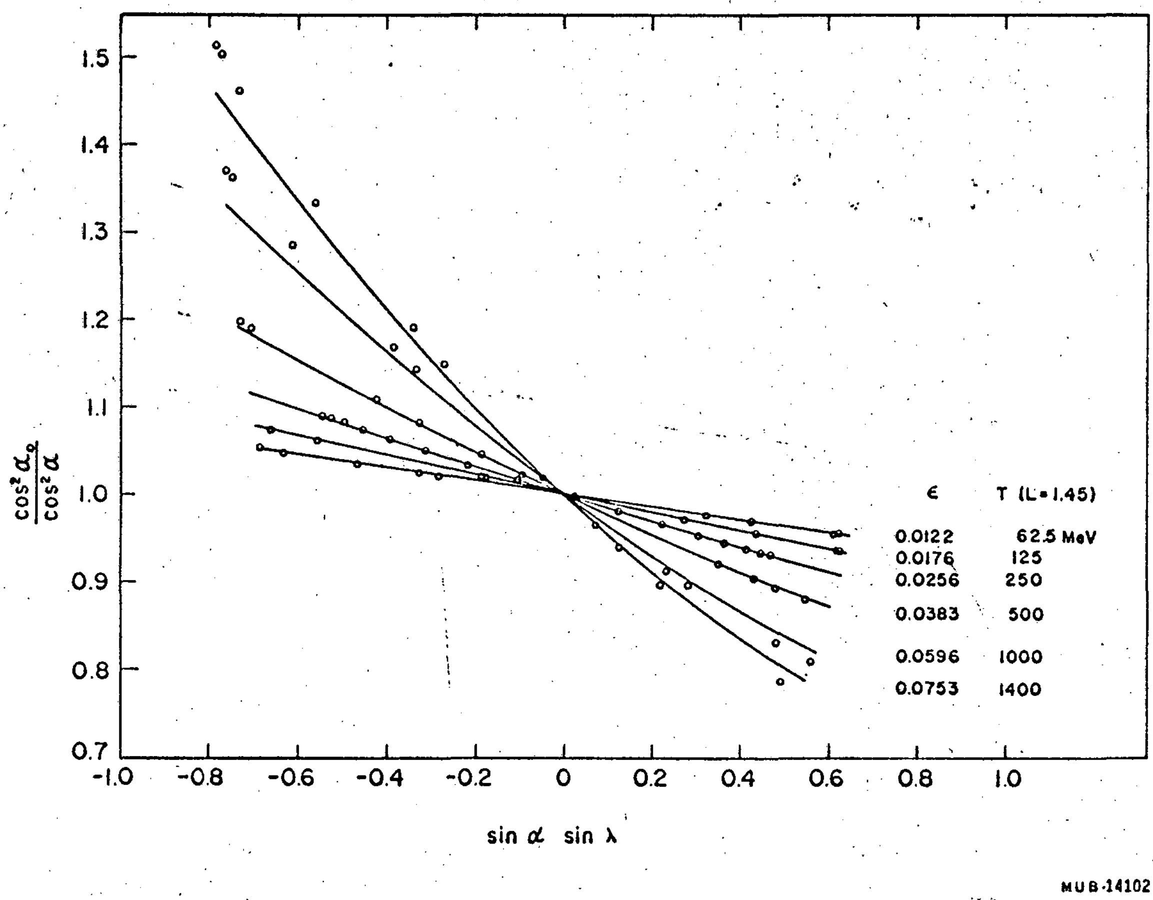
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
$\frac{\cos^2\alpha_0}{\cos^2\alpha}$
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
sin α sin λ
ε T (L=1.45)
0.0122 62.5 MeV
0.0176 125
0.0256 250
0.0383 500
0.0596 1000
0.0753 1400
MUB-14102

Fig. 2

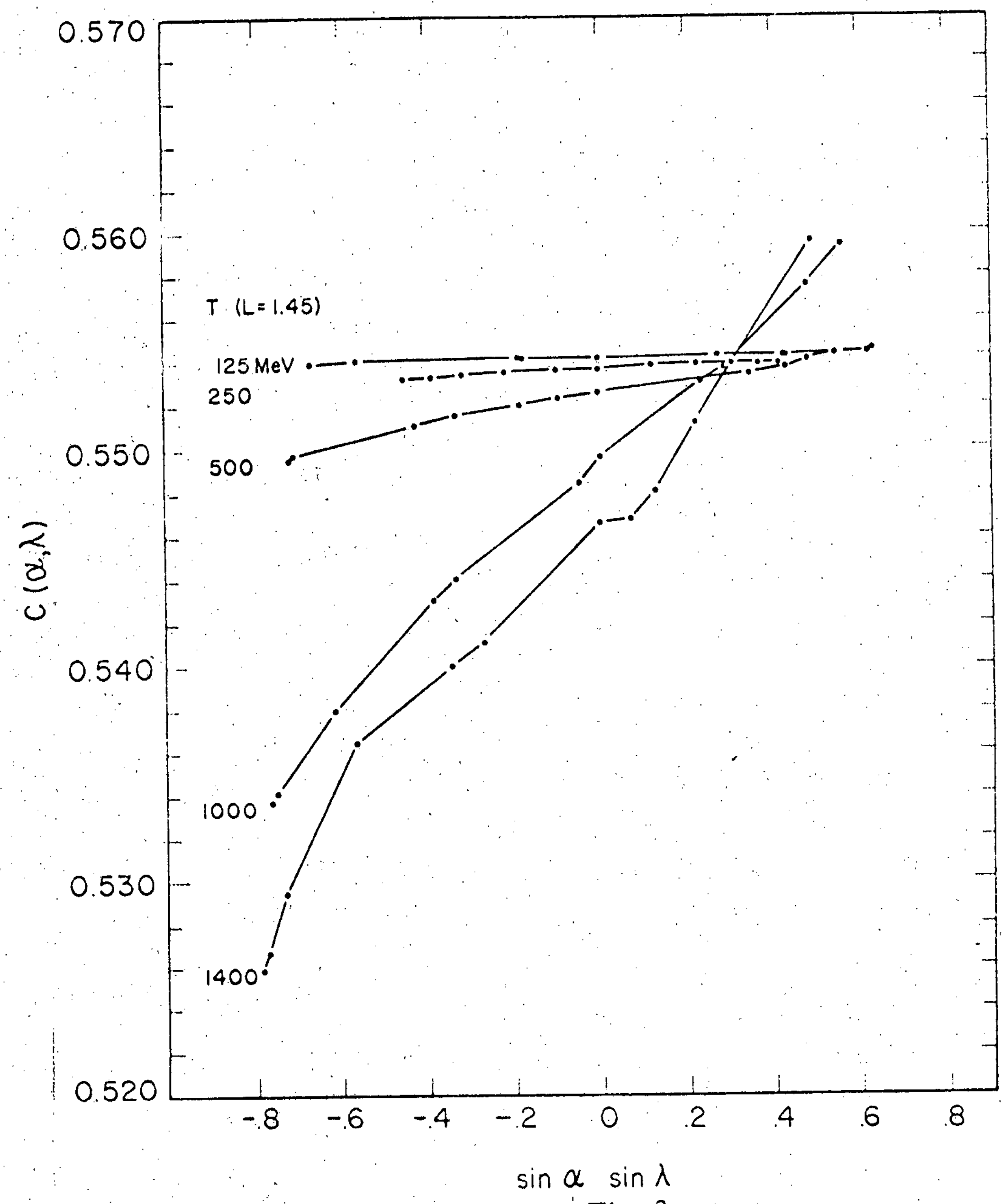

Fig. 3

MUB-14103

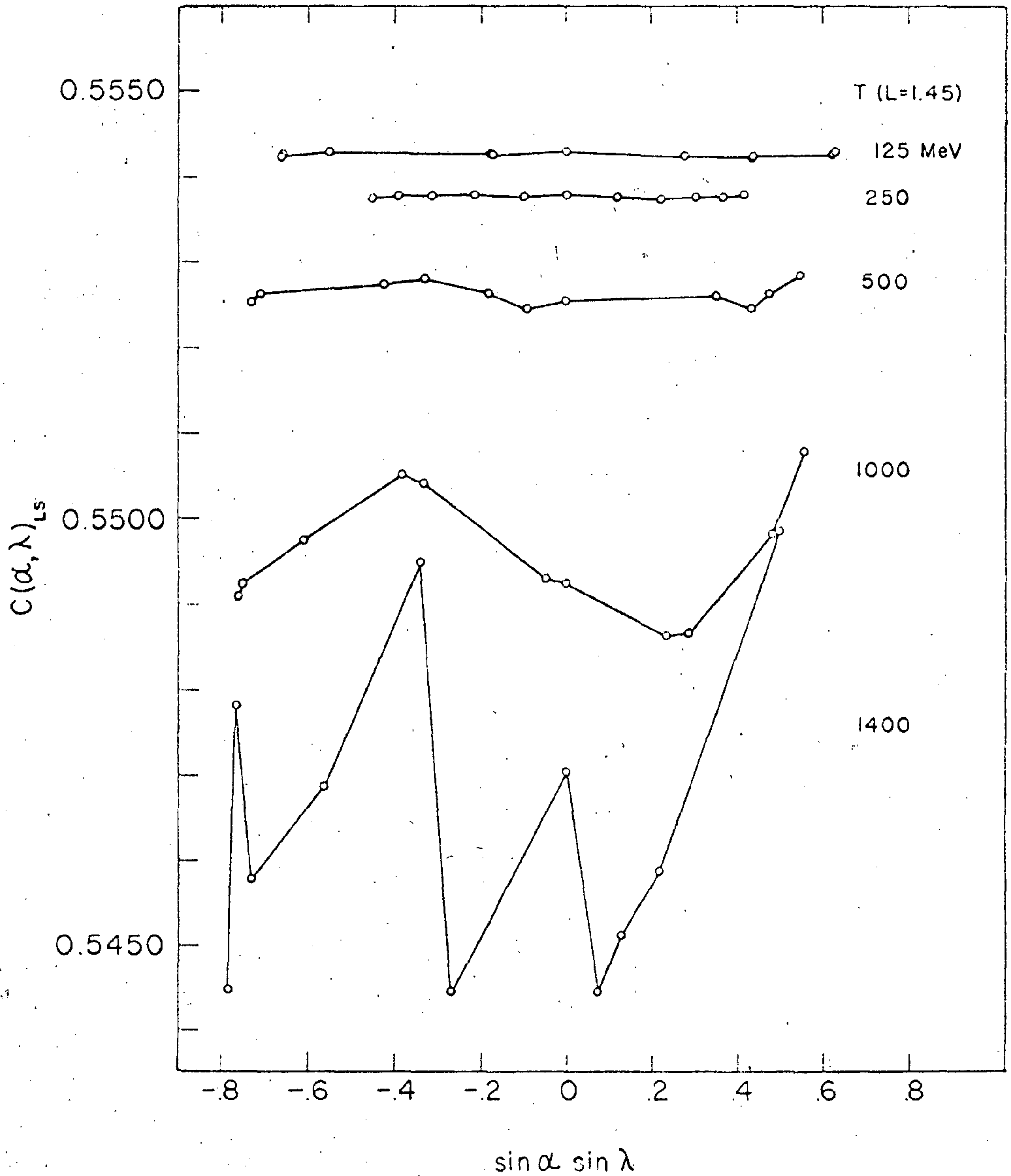

Fig. 4

MUB-14104