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Multisymplectic Geometry with Boundaries

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Abstract

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Geometric approaches form the foundation of modern classical mechanics. The prototypical example of a geometric method in mechanics is symplectic geometry applied to the Hamiltonian formulation of a system of particles. Extending this approach to field theories leads to unattractive features, such as an infinite-dimensional phase space and loss of manifest covariance. These deficiencies are particularly glaring in general relativity, where manifest covariance is closely tied to the fundamental symmetries of the theory.

Recent progress on covariant Hamiltonian approaches for field theories has led to the development of multisymplectic geometry. Multisymplectic geometry generalizes the symplectic geometry of particle systems to covariant fields, producing a finite-dimensional phase space and retaining manifest covariance. The symplectic 2-form common to symplectic geometry generalizes to the multisymplectic 5-form. The Euler-Lagrange equations for the field can be written in geometric language using the 5-form in a way that is formally identical to the geometric form of Hamilton's equations in particle mechanics. The resulting approach is a powerful geometric tool for understanding classical field theories.

In this dissertation, we improve upon the current approach to performing a 3+1 decomposition (also known as space-time split) of multisymplectic geometry. We clarify the relationship between multisymplectic geometry, its 3+1 decomposition, and the traditional symplectic approach to field theory. The key observation is that there exist two intermediate phase spaces between the multisymplectic phase space and the traditional symplectic phase space. We show how a proper understanding of the geometry of these intermediate spaces clarifies aspects of the traditional symplectic formulation. Our improved 3+1 decomposition allows us to easily handle the case when the spatial manifold (in our space-time split) has a boundary. By careful consideration of what happens to the theory at the boundary, we can arrive at appropriate boundary conditions and boundary modifications to various 3+1 quantities. This is the first such decomposition of the multisymplectic phase space with boundaries in the literature.

Lastly, we develop a multisymplectic formalism for general relativity. Our approach here is new, and gives great insight into the geometric structure of the theory. In the course of developing multisymplectic general relativity, we introduce local Lorentz transformations as an additional gauge symmetry. We show how reducing by this symmetry after 3+1 decomposition leads to the usual symplectic approach to general relativity.

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Contents

Chapter 1

Introduction

Geometric methods are at the heart of modern approaches to classical mechanics. [1–5] Symplectic and presymplectic geometry provide powerful frameworks for analyzing Hamiltonian dynamics. At their heart is the observation that by introducing a Hamiltonian vector field X_H and a closed 2-form ω (both on an appropriately defined phase space for the system), Hamilton's equations can be rephrased in geometric language as $i_{X_H}\omega = -\mathrm{d}H$ or even as $i_{X_H}\omega = 0$ (where in the latter case the Hamiltonian is absorbed into the definition of ω). We can then draw general conclusions about systems based on the structure of H and ω in a coordinate-free manner. We will demonstrate this in chapter 2 in the context of particle mechanics.

There has traditionally been a draw-back in extending the Hamiltonian formalism (symplectic or presymplectic) to field theories over spacetime, since doing so requires specifying a time direction and a introducing a spatial slicing (see [6–10] for discussion and examples of this approach). These choices break the manifest covariance of the theory, though see 3.3 for our alternative perspective. We can avoid these difficulties by using multisymplectic geometry, [5, 11] which allows us to construct a geometric (Hamiltonian) approach to field theories while maintaining manifest covariance.

The roots of this approach date back to DeDonder [12] and Weyl [13]. We can demonstrate the DeDonder-Weyl approach using a scalar field with Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$. In the usual Hamiltonian approach we define a momentum $\pi = \partial\mathcal{L}/\partial\dot{\phi}$, which requires making a choice for time in order to define $\dot{\phi}$. In the DeDonder-Weyl approach we introduce momenta for each spacetime direction $\pi^\mu = \partial\mathcal{L}/\partial(\partial_\mu\phi)$. We can then write a covariant Hamiltonian $\mathcal{H} = \pi^\mu \partial_\mu \phi - \mathcal{L}$ and manifestly covariant Hamilton's equations

$$\partial_\mu \phi = +\frac{\partial \mathcal{H}}{\partial \pi^\mu}, \tag{1.1a}$$

$$\partial_\mu \pi^\mu = -\frac{\partial \mathcal{H}}{\partial \phi}. \tag{1.1b}$$

Multisymplectic geometry takes the DeDonder-Weyl description as a starting point to develop a geometric description of the theory, much like symplectic geometry uses the standard Hamiltonian approach as a starting point. We can introduce a multisymplectic phase space as a bundle over spacetime with coordinates (ϕ, π^μ, x^μ). On this phase space, we have the DeDonder-Weyl Hamiltonian $\mathcal{H}$ and a closed multisymplectic 5-form ω. Just like there is a

Hamiltonian vector field tangent to solution curves in (pre)symplectic geometry, there is a 4-dimensional distribution tangent to solution surfaces in multisymplectic geometry. These surfaces represent field configurations over space-time, and we use them to write (1.1) in geometric form as $i_Y\omega|_{\bar{R}} = 0$ for all vector fields Y (here $\bar{R}$ is the solution surface in the multisymplectic phase space). We will describe the multisymplectic approach in detail in chapter 3.

In chapter 4, we apply the multisymplectic formalism to general relativity. To facilitate this, we use a tetrad description of gravity (see for example [14, 15]). The tetrad approach gives a theory equivalent to the usual GR in the absence of matter, which is the case we consider. In the presence of matter, fermions can potentially generate torsion (depending on the way they couple to the gravitational field) but the effect is likely too small to be detectable by current experiments (see [15] for some discussion). The multisymplectic formalism provides a nice avenue for exploring GR, as the geometric structure respects full diffeomorphism invariance. We then introduce a foliation of spacetime and derive a presymplectic description of the theory. The tetrad formalism has an extra local SO(3,1) gauge invariance not present in ordinary GR, so we demonstrate a version of (presymplectic) reduction that eliminates this symmetry and reduces us to the usual ADM Hamiltonian approach. [16] This provides a clean and direct path from a fully covariant geometric description of GR to the 3+1 decomposed Hamiltonian description of ADM.

Chapter 2

Hamiltonian Formalism for Particle Mechanics

In this chapter we discuss the Hamiltonian formalism and its associated geometry for a system with finite degrees of freedom. We loosely refer to this as particle mechanics. For systems with infinite degrees of freedom (field theories) see chapter 3.

Hamilton's equations define a vector field on the phase space of the system. We can recast the equations in geometric language using this vector field and the symplectic geometry of the phase space. We review the details of this approach in section 2.1. There are some key drawbacks to the symplectic approach, particularly when dealing with time-dependent systems or time-dependent symmetry transformations. To handle time-dependence, it is useful to extend the traditional phase space by a time axis and incorporate the Hamiltonian into a presymplectic structure on this extended phase space. We review this approach based on presymplectic geometry in section 2.2. Finally, in section 2.3 we introduce the Hamilton-Pontryagin formalism, which incorporates aspects of the Lagrangian formalism directly into the phase space. This extension can help clarify aspects of the theory, particularly the Legendre transformation and constraints.

2.1 Symplectic Geometry

2.1.1 The Symplectic Formalism

We now review the symplectic formalism of particle mechanics. For further details, see Arnold [1] (particularly chapters 8-10) and references therein or Abraham & Marsden [2] (particularly chapters 1-3) and references therein. Consider a system described by a set of configuration variables q^i. These q^i are local coordinates on a configuration manifold, which we denote by Q. We assume the equations of motion for the system are Euler-Lagrange equations of the action

$$S = \int L(q^i, \dot{q}^i)\,\mathrm{d}t, \tag{2.1}$$

where $L(q^i, \dot{q}^i)$ is a time-independent Lagrangian. The $\dot{q}^i$ are components of a tangent vector to Q, so we interpret the Lagrangian as a function on the tangent bundle TQ, $L : TQ \to \mathbb{R}$:

$(q^i, \dot{q}^i) \mapsto L(q^i, \dot{q}^i)$.

We obtain the Hamiltonian formalism from the Lagrangian one by performing a Legendre transformation. That is, we define the momenta

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{2.2}$$

and the Hamiltonian

$$H(q^i, p_i) = p_i \dot{q}^i - L, \tag{2.3}$$

with the convention that repeated indices are summed over and where we solve (2.2) for the $\dot{q}^i$ in order to substitute them into (2.3). We will later discuss what happens when (2.2) cannot be solved for the $\dot{q}^i$. We interpret the p_i as components of a 1-form on Q so that (2.2) defines a map from the tangent bundle to the cotangent bundle : $TQ \to T^*Q : (q^i, \dot{q}^i) \mapsto (q^i, p_i)$. Furthermore, (2.3) defines the Hamiltonian as a function on the cotangent bundle, $H : T^*Q \to \mathbb{R} : (q^i, p_i) \mapsto H(q^i, p_i)$. The cotangent bundle is thus the phase space of the system (the space on which the Hamiltonian dynamics takes place).

The Legendre transformation converts the second-order Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i}, \tag{2.4}$$

into a set of first-order Hamilton's equations,

$$\dot{q}^i = +\frac{\partial H}{\partial p_i} \tag{2.5a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.5b}$$

These can be interpreted geometrically by defining the 2-form $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i \in \Omega^2(T^*Q)$. This 2-form is closed ($\mathrm{d}\omega = 0$) and non-degenerate (if $\omega(X, Y) = 0$ for a vector Y on T^*Q and all vectors X on T^*Q, then Y is the zero vector), and thus a symplectic form. We now write (2.5) using this symplectic form,

$$i_{X_H}\omega = -\mathrm{d}H, \tag{2.6}$$

where X_H is a vector field on T^*Q. Any X_H that solves (2.6) is called a Hamiltonian vector field (corresponding to the Hamiltonian H). To demonstrate the equivalence of (2.6) and (2.5), note that in local coordinates (q^i, p_i) on T^*Q, integral curves of X_H are represented by functions : $\mathbb{R} \to T^*Q : t \mapsto (q^i(t), p_i(t))$.[1] Using these functions, we can write X_H as

$$X_H = \dot{q}^i(t)\,\frac{\partial}{\partial q^i} + \dot{p}_i(t)\,\frac{\partial}{\partial p_i}. \tag{2.7}$$

Plugging this form of X_H into (2.6) explicitly reproduces (2.5).

[1] To avoid introducing additional notation, we will always assume curves are defined over all of $\mathbb{R}$.

This symplectic structure ω on T^*Q can be defined canonically. A point of T^*Q is a pair (q, α) where q labels a point in Q and α labels a 1-form (on Q) based at $q \in Q$. We define the canonical 1-form $\theta \in \Omega^1(T^*Q)$ via

$$\theta|_{(q,\alpha)}(X) = \alpha|_q(\pi_* X), \tag{2.8}$$

where $\pi : T^*Q \to Q : (q, \alpha) \mapsto q$ is the bundle projection map. We now define the canonical symplectic 2-form as $\omega = \mathrm{d}\theta \in \Omega^2(T^*Q)$. In local coordinates this canonical 2-form agrees with the one used in (2.6). To see this explicitly, let q^i be local coordinates on Q and let p_i be the coordinate components of $\alpha \in \Omega^1(Q)$ (that is, $\alpha = p_i\,\mathrm{d}q^i$). This induces local coordinates (q^i, p_i) on T^*Q, and in these coordinates $\theta = p_i\,\mathrm{d}q^i$ and $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$, as above.

2.1.2 Momentum Maps

Momentum maps are the modern approach to symmetries and conserved quantities. Here we review their basic properties. See Arnold [1], appendix 5 for a brief overview and Abraham & Marsden [2] (particularly chapter 4) or Marsden & Ratiu [3] for more details.

Suppose a Lie group G acts on the phase space of our system (T^*Q) with a group action Φ,

$$\Phi : G \to \mathrm{Diff}(T^*Q) \tag{2.9a}$$

$$g \mapsto \Phi_g : T^*Q \to T^*Q, \text{ with } \Phi_g\Phi_h = \Phi_{gh}. \tag{2.9b}$$

An element of the Lie algebra $\xi \in \mathfrak{g}$ induces a vector field on T^*Q defined by

$$X_\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^*_{\exp(t\xi)}, \tag{2.10}$$

where both sides are understood to act on functions on T^*Q. See [2], [3], or [17] for a discussion of induced vector fields. Here t is simply a parameter to facilitate the definition of the induced vector field; it is not to be confused with the time variable used when discussing the evolution of the system (under the Hamiltonian flow).

We will be specifically interested in group actions which preserve the symplectic form ω,

$$\Phi^*_g\omega = \omega \quad \forall g \in G. \tag{2.11}$$

Most physical symmetries in mechanics are of this form, and we often refer to the group G as a symmetry group. The induced vector fields of an action by a symmetry group satisfy

$$\pounds_{X_\xi}\omega = 0, \tag{2.12}$$

which follows from (2.11) by writing $g = \exp t\xi$ and differentiating both sides with respect to t.

Since $\mathrm{d}\omega = 0$, it follows from Cartan's formula ($\pounds_X = i_X\mathrm{d} + \mathrm{d}i_X$) and (2.12) that

$$\mathrm{d}i_{X_\xi}\omega = 0. \tag{2.13}$$

Thus $i_{X_\xi}\omega$ is locally exact and there exists, locally, a function $J_\xi : T^*Q \to \mathbb{R}$ which satisfies

$$i_{X_\xi}\omega = -\mathrm{d}J_\xi. \tag{2.14}$$

The negative sign is conventional. In our examples, (2.14) holds globally, so we restrict attention to that case.

In most cases of interest, J_ξ is linear in ξ. In this case, it is standard to express (2.14) in terms of a map from T^*Q into the dual of the Lie algebra $\mathfrak{g}^*$. This map is called the momentum map [1]- [3] and is denoted $J : T^*Q \to \mathfrak{g}^*$. Let p label a point of T^*Q and let $\langle\cdot,\cdot\rangle$ be the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ (the dual Lie algebra element goes on the left and is evaluated on the Lie algebra element that goes on the right). Then J is defined by

$$\langle J(p), \xi\rangle = J_\xi(p), \tag{2.15}$$

and (2.14) can be written

$$i_{X_\xi}\omega = -\mathrm{d}\langle J, \xi\rangle, \tag{2.16}$$

with the dependence on points of T^*Q being implicit. An alternative view, which will prove particularly useful in the context of field theory, is to regard the momentum map as a map from the Lie algebra $\mathfrak{g}$ into the space of functions on T^*Q. That is, we can define $\tilde{J} : \mathfrak{g} \to \Omega^0(T^*Q) : \xi \mapsto J_\xi$. Henceforth, we write both J and $\tilde{J}$ as simply J, as the distinction will be clear from context. We will often refer to J_ξ as the momentum map as well.

In order to connect the momentum map with the more familiar concept of conserved Noether charge, we need to look at symmetries which also preserve the Hamiltonian. Specifically, for any $\xi \in \mathfrak{g}$ whose induced vector field X_ξ satisfies $\pounds_{X_\xi}H = \mathrm{d}H(X_\xi) = 0$, we have

$$0 = \mathrm{d}H(X_\xi) = \omega(X_\xi, X_H) = -\mathrm{d}J_\xi(X_H) = -\pounds_{X_H}J_\xi. \tag{2.17}$$

Thus J_ξ is invariant under the Hamiltonian flow and determines a conserved Noether charge.

It will be convenient to formally solve (2.14) as follows.

$$\mathrm{d}J_\xi = -i_{X_\xi}\omega = -i_{X_\xi}\mathrm{d}\theta = -\pounds_{X_\xi}\theta + \mathrm{d}i_{X_\xi}\theta, \tag{2.18}$$

where we used Cartan's identity. Now $\pounds_{X_\xi}\omega = \pounds_{X_\xi}\mathrm{d}\theta = \mathrm{d}\pounds_{X_\xi}\theta = 0$ since the Lie and exterior derivatives commute. Thus locally $\pounds_{X_\xi}\theta = \mathrm{d}\alpha_\xi$ for some function α_ξ. Plugging back into (2.18), we have

$$\mathrm{d}J_\xi = \mathrm{d}i_{X_\xi}\theta - \mathrm{d}\alpha_\xi \implies \mathrm{d}\left(J_\xi - i_{X_\xi}\theta + \alpha_\xi\right) = 0, \tag{2.19}$$

and thus

$$J_\xi = i_{X_\xi}\theta - \alpha_\xi + C_\xi, \tag{2.20}$$

where C_ξ is a constant that may depend on ξ. For the special case of point transformations (group actions on Q that are lifted to group actions on T^*Q), we have $\pounds_{X_\xi}\theta = 0 \implies \alpha_\xi = 0$ and

$$J_\xi = i_{X_\xi}\theta + C_\xi, \tag{2.21}$$

with the choice of C_ξ fixed by co-adjoint equivariance (see [2,3]).

2.2 Presymplectic Geometry

2.2.1 The Presymplectic Formalism

In the previous section, we considered systems described by time-independent Lagrangians. Let us now consider a system described by a time-dependent Lagrangian. That is, we now assume the equations of motion for the system are Euler-Lagrange equations of the action

$$S = \int L(q^i, \dot{q}^i, t)\, \mathrm{d}t. \tag{2.22}$$

We now interpret the Lagrangian as a function on $TQ \times \mathbb{R}$, $L : TQ \times \mathbb{R} \to \mathbb{R} : (q^i, \dot{q}^i, t) \mapsto L(q^i, \dot{q}^i, t)$. We will call $TQ \times \mathbb{R}$ the "extended tangent bundle" (the tangent bundle extended by a time axis). See [1] for some details.

To obtain the Hamiltonian formalism, we define the momenta p_i via (2.2) and the Hamiltonian via

$$H(q^i, p_i, t) = p_i \dot{q}^i - L, \tag{2.23}$$

as before. Now the Hamiltonian is time-dependent and should be interpreted as a function on $T^*Q \times \mathbb{R}$, $H : T^*Q \times \mathbb{R} \to \mathbb{R} : (q^i, p_i, t) \mapsto H(q^i, p_i, t)$. See [1], particularly chapter 9, sections 44-45 for details on this and what follows. In the time-dependent case, the Legendre transformation, (2.2), is a map $: TQ \times \mathbb{R} \to T^*Q \times \mathbb{R} : (q^i, \dot{q}^i, t) \mapsto (q^i, p_i, t)$ into the "extended cotangent bundle" $T^*Q \times \mathbb{R}$. When the Lagrangian, and hence Hamiltonian, is time-dependent, it is the extended cotangent bundle that is the phase space of the theory. We will sometimes refer to such a phase space (for a system with a time-dependent Hamiltonian) as an "extended phase space" [1] to emphasize its difference from the phase space considered in the symplectic context.

Hamilton's equations take the same form, (2.5), as before, but the geometric interpretation is different. Define the closed 2-form $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}H \wedge \mathrm{d}t \in \Omega^2(T^*Q \times \mathbb{R})$. This 2-form is degenerate (it is a 2-form on the odd-dimensional space $T^*Q \times \mathbb{R}$). A closed, degenerate 2-form such as this is termed "presymplectic" (see Gotay et. al. [18] for the definition, but note they had a different application in mind). Using this presymplectic ω, we write Hamilton's equations as

$$i_{X_H}\omega = 0, \tag{2.24}$$

where X_H is now a vector field on $T^*Q \times \mathbb{R}$. Note that Hamiltonian vector fields in the presymplectic setting lie in the kernel of ω.

We will refer to formulations of mechanics where H is incorporated into ω as "homogeneous" (we use this terminology due to the homogeneous form of (2.24)). Formulations of mechanics where H is carried around as a separate structure from ω (as is typical in symplectic geometry) will be called "inhomogeneous". The distinction between homogeneous and inhomogeneous is separate from the distinction between presymplectic and symplectic, though there is often significant overlap between the categories.

The solutions to (2.24) are integral curves of X_H. In local coordinates (q^i, p_i, t) on $T^*Q \times \mathbb{R}$, these curves are represented by functions $: \mathbb{R} \to T^*Q \times \mathbb{R} : \lambda \mapsto (q^i(\lambda), p_i(\lambda), t(\lambda))$. The parameter of the curve, λ, is arbitrary. This freedom to reparameterize is clear from (2.24). At a point, X_H is determined only to within an arbitrary constant (rescaling a vector in the kernel leaves it in the kernel). On the other hand, in an inhomogeneous setting, we work with solutions X_H to (2.6). These cannot be arbitrarily rescaled (due to the inhomogeneous form of (2.6)) and thus their integral curves cannot be reparameterized (they must use the physical time t). So in a homogeneous setting, solutions Hamilton's equations are unparameterized curves while in an inhomogeneous setting, solutions are parametrized curves.

We now show explicitly that (2.24) reproduces Hamilton's equations, (2.5). We use the coordinate representation of the integral curves $(q^i(\lambda), p_i(\lambda), t(\lambda))$ to write

$$X_H = \frac{\mathrm{d}q^i}{\mathrm{d}\lambda}\frac{\partial}{\partial q^i} + \frac{\mathrm{d}p_i}{\mathrm{d}\lambda}\frac{\partial}{\partial p_i} + \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\partial}{\partial t}. \tag{2.25}$$

We now compute

$$i_{X_H}\omega = \frac{\mathrm{d}p_i}{\mathrm{d}\lambda}\,\mathrm{d}q^i - \frac{\mathrm{d}q^i}{\mathrm{d}\lambda}\,\mathrm{d}p_i - [\mathrm{d}H(X_H)]\,\mathrm{d}t + \frac{\mathrm{d}t}{\mathrm{d}\lambda}\,\mathrm{d}H. \tag{2.26}$$

Using this and $\mathrm{d}H = (\partial H/\partial q^i)\,\mathrm{d}q^i + (\partial H/\partial p_i)\,\mathrm{d}p_i + (\partial H/\partial t)\,\mathrm{d}t$, (2.24) becomes

$$\left[\frac{\mathrm{d}p_i}{\mathrm{d}\lambda} + \frac{\partial H}{\partial q^i}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right]\mathrm{d}q^i + \left[-\frac{\mathrm{d}q^i}{\mathrm{d}\lambda} + \frac{\partial H}{\partial p_i}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right]\mathrm{d}p_i \left[-\mathrm{d}H(X_H) + \frac{\partial H}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right]\mathrm{d}t, \tag{2.27}$$

which gives the equations

$$\frac{\mathrm{d}p_i}{\mathrm{d}\lambda} = -\frac{\partial H}{\partial q^i}\frac{\mathrm{d}t}{\mathrm{d}\lambda}, \tag{2.28a}$$

$$\frac{\mathrm{d}q^i}{\mathrm{d}\lambda} = \frac{\partial H}{\partial p_i}\frac{\mathrm{d}t}{\mathrm{d}\lambda}, \tag{2.28b}$$

$$\mathrm{d}H(X_H) = \frac{\partial H}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}\lambda}. \tag{2.28c}$$

The last of these, (2.28c), is a consequence of the first two. Explicitly,

$$\mathrm{d}H(X_H) = \frac{\partial H}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}\lambda} + \frac{\partial H}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}\lambda} + \frac{\partial H}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}\lambda}, \tag{2.29}$$

and plugging in (2.28a) and (2.28b) gives

$$\mathrm{d}H(X_H) = \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial p_i}\frac{\mathrm{d}t}{\mathrm{d}\lambda} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^i}\frac{\mathrm{d}t}{\mathrm{d}\lambda} + \frac{\partial H}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\partial H}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}\lambda}, \tag{2.30}$$

which is just (2.28c).

We see that if $\mathrm{d}t/\mathrm{d}\lambda \neq 0$, we can divide both sides of (2.28a) and (2.28b) by it to obtain Hamilton's equations, (2.5). When $\mathrm{d}t/\mathrm{d}\lambda = 0$, we see that also $\mathrm{d}p/\mathrm{d}\lambda = 0$ and $\mathrm{d}q/\mathrm{d}\lambda = 0$. Thus (2.24) reproduces Hamilton's equations whenever we let the physical time t advance (hence $\mathrm{d}t/\mathrm{d}\lambda \neq 0$).

We do not attempt to define ω canonically in the homogeneous setting because it contains the Hamiltonian, and so is dependent on the dynamics of the system (rather than just the structure of the cotangent bundle as in the symplectic setting). Rather, we will show in section 2.3 that the 1-form $\theta = p_i\,\mathrm{d}q^i - H\,\mathrm{d}t$ arises naturally from a variational principle, and the 2-form $\omega = \mathrm{d}\theta$ then follows.

2.2.2 Momentum Maps

Now we revisit the concept of momentum maps to put it in a presymplectic context, making the contrast with what we did earlier in the symplectic context. Our Lie group G now acts on extended phase space $(T^*Q \times \mathbb{R})$ with a group action Φ,

$$\Phi : G \to \mathrm{Diff}(T^*Q \times \mathbb{R}) \tag{2.31a}$$

$$g \mapsto \Phi_g : T^*Q \times \mathbb{R} \to T^*Q \times \mathbb{R}, \text{ with } \Phi_g \Phi_h = \Phi_{gh}. \tag{2.31b}$$

We define the induced vector field X_ξ for $\xi \in \mathfrak{g}$ as before (see (2.10)) and again consider actions satisfying $\Phi_g^* \omega = \omega$ for all $g \in G$.

This time, ω contains the Hamiltonian. Thus, in order for G to be a symmetry group it must preserve the dynamics in a certain sense. It is insufficient for a symmetry group G to have Φ_g preserve only $\mathrm{d}p_i \wedge \mathrm{d}q^i$ as it did in the symplectic setting. A group G would be a symmetry group if Φ_g preserved, for example, $\mathrm{d}p_i \wedge \mathrm{d}q^i$ as well as H and t. More general symmetry actions are also possible, such as when Φ_g preserves ω as a whole, but not any of its parts separately.

Given a symmetry group G, its induced vector fields X_ξ satisfy $\pounds_{X_\xi} \omega = 0$ as before, which implies $\mathrm{d} i_{X_\xi} \omega = 0$ as before (since $\mathrm{d}\omega = 0$ still holds). Thus we still have $i_{X_\xi} \omega = -\mathrm{d} J_\xi$ as in (2.14). Furthermore, we can, just as before, define a momentum map J as either a map $: T^*Q \times \mathbb{R} \to \mathfrak{g}^*$ or a map $: \mathfrak{g} \to \Omega^0(T^*Q \times \mathbb{R})$. A key difference between momentum maps in the symplectic setting and the homogeneous (presymplectic) setting is that the latter are always conserved. Specifically,

$$\pounds_{X_H} J_\xi = i_{X_H} \mathrm{d} J_\xi = -i_{X_H} i_{X_\xi} \omega = \omega(X_H, X_\xi) = 0, \tag{2.32}$$

so J_ξ is constant on solutions. This is a consequence of G (in the homogeneous setting) preserving not just a symplectic form, but the dynamics as well.

2.2.3 Constrained Systems

In section 2.1.1 we limited our discussion to the case when (2.2) could be solved for the $\dot{q}^i$. When this is not the case, we are dealing with a constrained system. Dirac [6] provides some of the earliest discussion on constrained systems with an eye towards applications to gauge field theories. More comprehensive reviews of the topic can be found in, for example, [19]. Gotay et. al. [18] introduced a geometric formulation for constrained systems, which we will briefly summarize in this section. Their formulation is inhomogeneous, but also presymplectic. Later we show how to reformulate Gotay's ideas for a homogeneous setting.

For a constrained system, the Legendre transformation (2.2) is not invertible. We will take this usual formula for the Legendre transformation and interpret it as a map $F : TQ \to T^*Q : (q^i, \dot{q}^i) \mapsto (q^i, p_i)$. This map is sometimes called the fiber map [2], since it maps fibers of TQ to fibers of T^*Q. We can define the Hamiltonian as in (2.3), but with the understanding that H is defined on TQ:

$$H(q^i, \dot{q}^i) = p_i \dot{q}^i - L(q^i, \dot{q}^i), \tag{2.33}$$

where $p_i = \partial L / \partial \dot{q}^i$. In order to define a Hamiltonian on T^*Q, we would like to push forward $H(q^i, \dot{q}^i)$ using F. Since F is not invertible we cannot, in general, use it to push forward functions from TQ to T^*Q. However, the Hamiltonian can be pushed forward under F because for each point $(q^i, p_i) \in T^*Q$, H is constant along the pre-image $F^{-1}(q^i, p_i)$. To see this, note that

$$\mathrm{d}H = \dot{q}^i\,\mathrm{d}p_i + p_i\,\mathrm{d}\dot{q}^i - \frac{\partial L}{\partial q^i}\,\mathrm{d}q^i - \frac{\partial L}{\partial \dot{q}^i}\,\mathrm{d}\dot{q}^i = \dot{q}^i\,\mathrm{d}p_i - \frac{\partial L}{\partial q^i}\,\mathrm{d}q^i, \tag{2.34}$$

and a vector X tangent to the pre-image $F^{-1}(q^i, p_i)$ satisfies $\mathrm{d}q^i(X) = \mathrm{d}p_i(X) = 0$. Hence $\mathrm{d}H(X) = 0$ for any vector tangent to the pre-image, from which it follows that H is constant on the pre-image of any point of T^*Q. This argument requires the preimage to be a connected smooth manifold, which is the case in most applications. We thus define $H(q^i, p_i)$ by "pushing forward" $H(q^i, \dot{q}^i)$ under F.

This only defines H on the image of F and not on the full T^*Q. This image is called the primary constraint submanifold and is the starting point for the Hamiltonian approach to constrained systems. This manifold could be symplectic, but more commonly it is presymplectic (ω restricted to it is degenerate). When it is presymplectic, we must take care when using Hamilton's equations, (2.6). To simplify the discussion, let $M = T^*Q$ and $C_1 = \mathrm{img}(F) \subset M$ be the primary constraint manifold. Define the map $\tilde{\omega} : TC_1 \to T^*C_1 : X \mapsto \omega|_{C_1}(X, \cdot)$, from which it follows that Hamilton's equations are $\tilde{\omega}(X_H) = -\mathrm{d}H$. Since $\omega|_{C_1}$ is (generally) presymplectic, $\tilde{\omega}$ is not full-rank. Thus $\mathrm{d}H$ might not be in the image of $\tilde{\omega}$. This means that some points of C_1 could have no solutions for X_H and thus no Hamiltonian vector field. Such points hold no physical meaning and must be excluded leaving a submanifold $C_2 \subset C_1$.

Every point of C_2 has at least one Hamiltonian vector X_H, but for some points of C_2 it may happen that no such vectors are tangent to C_2. This is a problem, since following any Hamiltonian vector from such a point will result in leaving C_2 in an infinitesimal time step, after which no Hamiltonian vector fields exist to continue a solution curve. Thus, starting at a point where no Hamiltonian vectors are tangent to C_2 produces a solution with no time evolution, and hence such points must also be excluded as unphysical. We can phrase this mathematically as excluding points of C_2 where $\mathrm{d}H$ is not in the image $\tilde{\omega}(TC_2)$ (no vectors in the preimage $\tilde{\omega}^{-1}(\mathrm{d}H)$ are in TC_2). This defines a submanifold $C_3 \subset C_2$ from which we exclude points where $\mathrm{d}H$ is not in the image $\tilde{\omega}(TC_3)$ for the same reason as above, and so forth. This procedure, called the constraint algorithm, terminates on some final constraint manifold C on which a Hamiltonian vector X_H tangent to C exists for all points. Once we have identified C, Hamilton's equations take the usual form $i_{X_H}\omega = -\mathrm{d}H$, but we only attempt to solve them on C with X_H tangent to C.

Solutions to Hamilton's equations (on C) might not be unique as ω can have a kernel which is tangent to C. This indicates the presence of a gauge symmetry (a la Dirac) since there are quantities that are not uniquely determined from initial data. Conversely, if a system is known to have a gauge symmetry, it is likely that it is a constrained system with Hamilton's equations governed by presymplectic geometry.

2.3 The Hamilton-Pontryagin Approach to Mechanics

Here we introduce the Hamilton-Pontryagin formulation of mechanics (see [20] and references therein). This naturally leads to a homogeneous formulation, though a slightly different one than introduced in section 2.2. The Hamilton-Pontryagin approach has several advantages over the formulations introduced above. Firstly, the phase space includes both the momentum and velocity variables, allowing us to easily consider dynamics on either TQ or T^*Q if the need arises. Secondly, for constrained systems there is no longer a primary constraint manifold generated by the Legendre transform. There is in fact no Legendre transformation at all. All constraints follow from the homogeneous constraint algorithm, as we will show. Thirdly, the Hamilton-Pontryagin approach derives the appropriate 1-form θ directly from the Lagrangian (the 2-form follows as $\omega = \mathrm{d}\theta$). This gives the approach a closer connection to the physics. In particular, the phase space is determined based on the physical field present. This is in contrast to traditional approaches where the phase space is taken to be a cotangent bundle (or extended cotangent bundle) and the 2-form ω is taken as the canonical 2-form on the cotangent bundle, whose relation to the physics can be more indirect. We will see the benefits of using the physical fields to determine the phase space most clearly when applying the Hamilton-Pontryagin formulation to gauge field theories in the next chapter.

We will present the Hamilton-Pontryagin approach in some detail. This is mainly in preparation for applying it to field theories in the next chapter, where the Hamilton-Pontryagin approach will lead us to the multisymplectic geometry of field theories.

2.3.1 Hamilton-Pontryagin Formalism

Suppose our system of interest is governed by the action (2.22):

$$S[q^i(t)] = \int L(q^i, \dot{q}^i, t)\,\mathrm{d}t. \tag{2.35}$$

Introduce the "lifted" action

$$\tilde{S}[q^i(t), v^i(t), p_i(t)] = \int \left[L(q^i, v^i, t) - p_i(v^i - \dot{q}^i)\right]\,\mathrm{d}t. \tag{2.36}$$

Note that this new action is a functional of the coordinates q^i, the velocities v^i, and the momenta p_i, the last of which also act as Lagrange multipliers enforcing the condition $v^i = \dot{q}^i$. The action is termed "lifted" because the usual velocity condition $v^i = \dot{q}^i$ has been lifted and the action accepts more general curves. The condition is, of course, restored on solution curves (those satisfying stationarity of the lifted action). It is useful to parameterize the curves by an arbitrary parameter λ rather than t, so that the lifted action reads

$$\tilde{S}[q^i(\lambda), v^i(\lambda), p_i(\lambda), t(\lambda)] = \int \left[L(q^i, v^i, t)\frac{\mathrm{d}t}{\mathrm{d}\lambda} - p_i\left(v^i\frac{\mathrm{d}t}{\mathrm{d}\lambda} - \frac{\mathrm{d}q^i}{\mathrm{d}\lambda}\right)\right]\,\mathrm{d}\lambda. \tag{2.37}$$

In this form, $\tilde{S}$ is explicitly a functional of curves $\gamma : \mathbb{R} \to P$, where P has coordinates (q^i, v^i, p_i, t) and is formally a fiber bundle over $Q \times \mathbb{R}$ with fiber $T_qQ \times T_q^*Q$ over each point (q, t), as shown in figure 2.3.1. Alternatively we may view P as a bundle over $\mathbb{R}$ with the Pontryagin bundle $TQ \oplus T^*Q$ as the fiber (see [20] for details on the definition and notation of the Pontryagin bundle). We will see that, in addition to supporting the action functional, P plays the role of the phase space in the Hamilton-Pontryagin approach, without the need to perform a Legendre transformation.

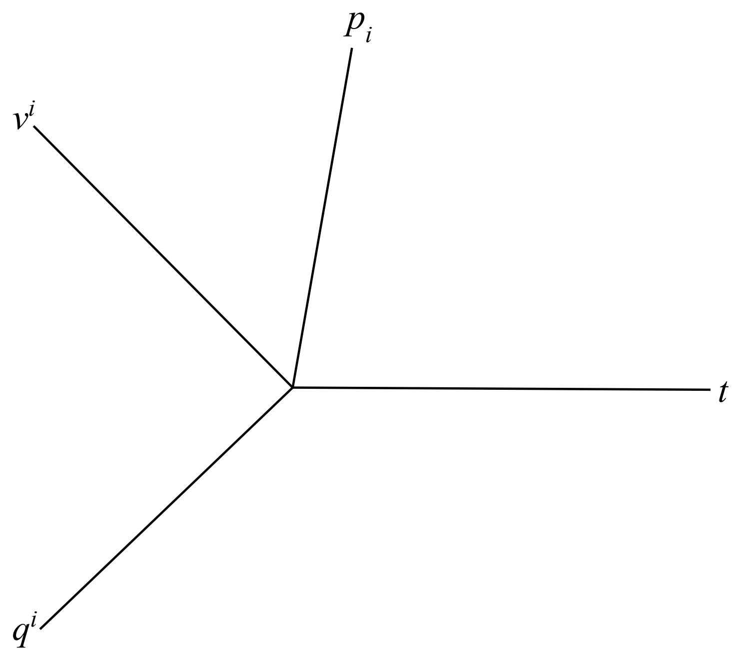

Figure 2.1: Schematic of the Hamilton-Pontryagin phase space. The q^i axis represents the configuration space Q. Both velocity and momenta fibers are present, unifying the Lagrangian and Hamiltonian perspectives.

We now show that the variational principle for the lifted action produces equations which are equivalent to the original equations of motion (those coming from (2.22)). The variation of $\tilde{S}$,

$$\delta \tilde{S} = \int \mathrm{d}\lambda \left[\frac{\partial L}{\partial q^i} \frac{\mathrm{d}t}{\mathrm{d}\lambda} \delta q^i + \frac{\partial L}{\partial v^i} \frac{\mathrm{d}t}{\mathrm{d}\lambda} \delta v^i + \frac{\partial L}{\partial t} \frac{\mathrm{d}t}{\mathrm{d}\lambda} \delta t - \frac{\mathrm{d}L}{\mathrm{d}\lambda} \delta t \right. \\ \left. - \delta p_i \left(v^i \frac{\mathrm{d}t}{\mathrm{d}\lambda} - \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} \right) - p_i \frac{\mathrm{d}t}{\mathrm{d}\lambda} \delta v^i - \frac{\mathrm{d}p_i}{\mathrm{d}\lambda} \delta q^i + \frac{\mathrm{d}(p_i v^i)}{\mathrm{d}\lambda} \delta t \right], \tag{2.38}$$

leads to the equations

$$\delta q^i : \qquad \frac{\partial L}{\partial q^i} \frac{\mathrm{d}t}{\mathrm{d}\lambda} - \frac{\mathrm{d}p_i}{\mathrm{d}\lambda} = 0, \tag{2.39a}$$

$$\delta v^i : \qquad \left(\frac{\partial L}{\partial v^i} - p_i \right) \frac{\mathrm{d}t}{\mathrm{d}\lambda} = 0, \tag{2.39b}$$

$$\delta p_i : \qquad v^i \frac{\mathrm{d}t}{\mathrm{d}\lambda} - \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} = 0, \tag{2.39c}$$

$$\delta t : \qquad \frac{\partial L}{\partial t} \frac{\mathrm{d}t}{\mathrm{d}\lambda} - \frac{\mathrm{d}(p_i v^i - L)}{\mathrm{d}\lambda} = 0. \tag{2.39d}$$

When $\frac{\mathrm{d}t}{\mathrm{d}\lambda} \neq 0$ we can divide through by it and combine (2.39a)-(2.39c) into (2.4) (the last equation, (2.39d), is in this case a consequence of the other three and is equivalent to the well-known result $\frac{\partial H}{\partial t} = \frac{\mathrm{d}H}{\mathrm{d}t}$). We will generally not discuss the case $\frac{\mathrm{d}t}{\mathrm{d}\lambda} = 0$. It represents what we call a vertical motion (vertical over $P \to \mathbb{R}$).

There are cases when allowing vertical motions is desirable (for example, in gauge theories since a gauge transformation is a vertical motion), but one must be careful not to introduce solutions to (2.39) that are not solutions to (2.4). Usually, we can avoid such spurious solutions by working with only those points of P where at least one non-vertical solution exists. It then typically follows that even the purely vertical solutions through such points do not violate (2.4) and are potentially physically meaningful.

We can write $\tilde{S}$ in a more compact form by introducing the 1-form $\theta = p\,\mathrm{d}q - H\,\mathrm{d}t \in \Omega^1(P)$, where $H(q^i, v^i, p_i, t) = p_i v^i - L(q^i, v^i, t)$ is the (Hamilton-Pontryagin) Hamiltonian. It reduces to the usual Hamiltonian on solutions, but is distinct from it, particularly since the two Hamiltonians are defined on different spaces. The lifted action can be written

$$\tilde{S}[\gamma] = \int_\gamma p\,\mathrm{d}q - H\,\mathrm{d}t = \int_\gamma \theta, \tag{2.40}$$

where $\gamma : \mathbb{R} \to P$ is any curve. This naturally defines the 1-form θ from which we can obtain the presymplectic 2-form $\omega = \mathrm{d}\theta$. We expect ω to define the dynamics through its kernel ($i_{X_H}\omega = 0$) as in a homogeneous formulation of mechanics. Proving that the dynamics defined by ω is equivalent to (2.39) requires discussing the homogeneous version of the constraint algorithm (the inhomogeneous version was reviewed in section 2.2.3).

2.3.2 Homogeneous Constraint Theory

Suppose we have a manifold M endowed with a presymplectic 2-form ω whose kernel determines the dynamics according to $i_{X_H}\omega = 0$. In the previous section this manifold was P. We assume M comes with the bundle structure $M \to \mathbb{R}$ so that ω has a typical homogeneous form (two terms, of which one has a $\mathrm{d}t$). If the kernel of ω has non-constant rank, care must be taken in defining the dynamics. This is similar to what happens when ω is not full-rank in an inhomogeneous formulation. Namely, we will be working through a sequence of constraints to arrive at a final constraint manifold on which the dynamics make sense.

Define $\tilde{\omega} : TM \to T^*M : X \mapsto \omega(X, \cdot)$ and note that solving Hamilton's equations ($i_{X_H}\omega = 0$) is equivalent to finding the kernel of $\tilde{\omega}$. If there are points of M where $\tilde{\omega}$ has no kernel, solutions to Hamilton's equations do not exist there. Such points must be excluded from M as being unphysical. In addition, at points where a kernel of $\tilde{\omega}$ is non-trivial, it may be purely vertical ($\mathrm{d}t$ vanishes on the kernel). If the only solutions for X_H at a point are vertical, then no solutions from that point can advance in time and thus the point cannot be physically meaningful. We thus need to exclude all points where the kernel of $\tilde{\omega}$ has only a trivial projection under $\pi : M \to \mathbb{R}$ (the projection onto the t-space). This defines the submanifold C_1, from which we then exclude points where $\tilde{\omega}|_{C_1} : TC_1 \to T^*M$ has only a trivial projection under $\pi : C_1 \to \mathbb{R}$. The reason for the tangency requirement is the same as in section 2.2.3 (leaving C_1 will yield no solution immediately after). As with the inhomogeneous constraint algorithm, we repeat this process to find the final constraint manifold C, which everywhere possess a kernel of $\tilde{\omega}|_C$ with non-trivial projection under $\pi : C \to \mathbb{R}$ and with at least one (non-vertical) dimension tangent to C. Hamilton's equations, $i_{X_H}\omega = 0$, can now be solved on C with X_H tangent to C. The solution may turn out to be non-unique because the kernel of ω may have multiple dimensions tangent to C.

2.3.3 Geometric Formulation of the Equations

We now look at the geometrical formulation of the lifted equations (2.39). We use the 2-form $\omega = \mathrm{d}\theta$, with θ defined by (2.40), and consider the equations implied by $i_{X_H}\omega = 0$. It is useful to write X_H (using the (q^i, v^i, p_i, t) coordinates) as

$$X_H = {q^i}' \frac{\partial}{\partial q^i} + {v^i}' \frac{\partial}{\partial v^i} + {p_i}' \frac{\partial}{\partial p_i} + t' \frac{\partial}{\partial t}, \tag{2.41}$$

where the prime ($'$) denotes differentiation by λ (the parameter of a curve X_H is tangent to). Then

$$i_{X_H}\omega = \\ {p_i}' \,\mathrm{d}q^i - {q^i}' \,\mathrm{d}p_i - \left({q^i}' \frac{\partial H}{\partial q^i} + {v^i}' \frac{\partial H}{\partial v^i} + p_i' \frac{\partial H}{\partial p_i} \right) \mathrm{d}t + \left(\frac{\partial H}{\partial q^i} \,\mathrm{d}q^i + \frac{\partial H}{\partial v^i} \,\mathrm{d}v^i + \frac{\partial H}{\partial p_i} \,\mathrm{d}p_i \right) t' = 0. \tag{2.42}$$

Reorganizing the terms,

$$i_{X_H}\omega = \\ \left({p_i}' + \frac{\partial H}{\partial q^i} t' \right) \mathrm{d}q^i + \left(-{q^i}' + \frac{\partial H}{\partial p_i} t' \right) \mathrm{d}p_i + \frac{\partial H}{\partial v^i} t' \,\mathrm{d}v^i - \left({q^i}' \frac{\partial H}{\partial q^i} + {v^i}' \frac{\partial H}{\partial v^i} + p_i' \frac{\partial H}{\partial p_i} \right) \mathrm{d}t = 0 \tag{2.43}$$

The coefficients of all the (independent) one-forms must vanish, which sets conditions on X_H for it to lie in the kernel of ω. One of these conditions (from the coefficient of $\mathrm{d}v^i$) is $t' \frac{\partial H}{\partial v^i} = t'(p_i - \frac{\partial L}{\partial v^i}) = 0$. Since $t' = 0$ represents a vertical kernel, we must have $p_i = \frac{\partial L}{\partial v^i}$. This is not a condition on the vector X_H, but rather a constraint (a condition on the phase space itself). This shows how the Legendre transform is incorporated directly into constraint theory for the Hamilton-Pontryagin approach.

One can see this is the only constraint at this step of the algorithm, as the other conditions are conditions on the vector X_H and not on the phase space. Following the constraint algorithm of the previous section, we now want to make sure that the constraint manifold $C_1 : p_i = \frac{\partial L}{\partial v^i}$ has a non-vertical kernel at least partially tangent to C_1. At points of C_1, $i_{X_H}\omega$ is

$${p_i}' \,\mathrm{d}q^i - {q^i}' \,\mathrm{d}p_i - \left({q^i}' \frac{\partial H}{\partial q^i} + p_i' \frac{\partial H}{\partial p_i} \right) \mathrm{d}t + \left(\frac{\partial H}{\partial q^i} \,\mathrm{d}q^i + \frac{\partial H}{\partial p_i} \,\mathrm{d}p_i \right) t'. \tag{2.44}$$

The kernel is given by

$$p_i{}' + \frac{\partial H}{\partial q^i}\, t' = 0, \tag{2.45a}$$

$$-q^{i\prime} + \frac{\partial H}{\partial p^i}\, t' = 0, \tag{2.45b}$$

$$q^{i\prime}\, \frac{\partial H}{\partial q^i} + p_i'\, \frac{\partial H}{\partial p_i} = 0. \tag{2.45c}$$

Meanwhile, in order for X_H to be tangent to C_1, the components of X_H must satisfy

$$p_i{}' - \frac{\partial^2 L}{\partial q^j\, \partial v^i}\, q^{j\prime} - \frac{\partial^2 L}{\partial v^j\, \partial v^i}\, v^{j\prime} - \frac{\partial^2 L}{\partial t\, \partial v^i}\, t' = 0, \tag{2.46}$$

which can be thought of as "differentiating" the constraint $p_i - \frac{\partial L}{\partial v^i}$ and setting the derivative to zero. We need there to exist at least one non-vertical vector in the kernel (satisfying (2.45)) that is also tangent to C_1 (satisfying (2.46)). Thus we must have at least one non-vertical vector satisfying (plugging (2.45) into (2.46))

$$\left[\frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial q^j\, \partial v^i}\, v^j - \frac{\partial^2 L}{\partial t\, \partial v^i}\right] t' = \frac{\partial^2 L}{\partial v^j\, \partial v^i}\, v^{j\prime}. \tag{2.47}$$

If the matrix $\frac{\partial^2 L}{\partial v^j\, \partial v^i}$ is full-rank, then such a vector exists and the constraint algorithm is complete. Hamilton's equations then take place on the constraint manifold given by $p_i = \frac{\partial L}{\partial v^i}$ and are given by equation (2.45). Note that this geometric formulation of the equations reproduces the equations in (2.39).

If the matrix $\frac{\partial^2 L}{\partial v^j\, \partial v^i}$ is not full-rank, then we need the left-hand side of (2.47) to lie in the image of $\frac{\partial^2 L}{\partial v^j\, \partial v^i}$. If this is true for all points of the first constraint manifold, then the constraint algorithm is complete. Otherwise, we continue with the constraint algorithm, generating further constraints. We will not discuss this case as it does not arise in the specific examples we consider.

2.3.4 The Velocity Variables

Traditional approaches to symplectic and presymplectic geometry use a phase space with coordinates (q^i, p_i) or an extended phase space with coordinates (q^i, p_i, t). Meanwhile, the Hamilton-Pontryagin approach includes additional velocity variables, so the Hamilton-Pontryagin phase space has coordinates (q^i, v^i, p_i, t). Sometimes we want to make contact with the traditional approaches and to do this we need a way to eliminate the velocity variables.

Suppose the Hamilton-Pontryagin constraint algorithm terminates with the constraint manifold given by the Legendre transform $p_i = \partial L/\partial v^i$. In this case the constraint manifold has, at least locally, a non-vertical projection onto (q^i, p_i, t)-space. To see this, note that $\partial p_i/\partial v^j = \frac{\partial^2 L}{\partial v^i\, \partial v^j}$ is full-rank. If we could restrict our theory to the constraint manifold, then we could simply project it onto the extended phase space. However, restricting a theory to a submanifold can produce an inequivalent theory. The original Hamilton equations $i_{X_H}\omega = 0$

take place on the constraint manifold, but with ω defined on the full space. The restricted form $\omega|_C$ can have a larger kernel and thus allow for additional solutions not present in the original theory. In the case at hand, we can explicitly show that restricting to the constraint manifold produces no new solutions.

We take (q^i, p_i, t) as coordinates on the constraint manifold. This is possible because $p_i = \partial L/\partial v^i$ can be solved for $v^i = v^i(q^i, p_i, t)$, and is closely related to the existence of a non-vertical projection onto (q^i, p_i, t)-space. The restricted ω is

$$\omega|_C = \mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}H(q^i, p_i, t) \wedge \mathrm{d}t, \tag{2.48}$$

with the only difference between ω and $\omega|_C$ in the Hamiltonian, where all v^i are replaced with $v^i(q^i, p_i, t)$. Writing

$$X_H = q'^i \frac{\partial}{\partial q^i} + p'_i \frac{\partial}{\partial p_i} + t' \frac{\partial}{\partial t}, \tag{2.49}$$

the kernel of $\omega|_C$ is given by

$$p'_i \, \mathrm{d}q^i - q'^i \, \mathrm{d}p_i - \left(\frac{\partial H}{\partial q^i} q'^i + \frac{\partial H}{\partial p_i} p'_i \right) \mathrm{d}t + \left(\frac{\partial H}{\partial q^i} \mathrm{d}q^i + \frac{\partial H}{\partial p_i} \mathrm{d}p_i \right) t' = 0. \tag{2.50}$$

This leads to the equations

$$p'_i + \frac{\partial H}{\partial q^i} t' = 0, \tag{2.51a}$$

$$-q'^i + \frac{\partial H}{\partial p_i} t' = 0, \tag{2.51b}$$

$$\frac{\partial H}{\partial q^i} q'^i + \frac{\partial H}{\partial p_i} p'_i = 0, \tag{2.51c}$$

which are just the usual Hamilton's equations on the extended phase space, since $H(q^i, p_i, t)$ has the Legendre transform $v^i = v^i(q^i, p_i, t)$ substituted in. These Hamilton equations are equivalent to the original Euler-Lagrange equations just as the Hamilton-Pontryagin equations are and so no new solutions were produced by restricting to the constraint manifold. Furthermore, using the (q^i, p_i, t) coordinates on the constraint manifold makes the projection trivial, so the above results show how to eliminate the velocity variables by projection.

In addition to projecting onto the extended phase space with coordinates (q^i, p_i, t) we can also project the constraint manifold onto the extended tangent bundle with coordinates (q^i, v^i, t). Thus we see how the Hamilton-Pontryagin approach encompasses both traditional approaches on the (extended) tangent and (extended) cotangent bundles. When the constraint structure is more complicated, projections from the Hamilton-Pontryagin space may be impossible or disadvantageous. In these cases, the Hamilton-Pontryagin approach needs to be used directly. When working directly with the Hamilton-Pontryagin formalism, it is important to remember that the presence of velocity variables makes Hamiltonians, momentum maps, and other structures behave slightly differently than in traditional symplectic or presymplectic approaches.

Chapter 3

Field Theory and Multisymplectic Geometry

Multisymplectic geometry [4, 5, 11, 14, 21] generalizes the symplectic geometry of Hamiltonian mechanics into a covariant framework useful in field theory. The usual thinking is that Hamiltonian approaches to field theory are not manifestly covariant because they rely on a 3+1 decomposition to produce an infinite-dimensional symplectic phase space. However, multisymplectic geometry preserves manifest covariance while using a finite-dimensional phase space, where the symplectic potential 1-form and symplectic 2-form of symplectic geometry are replaced by a multisymplectic potential 4-form and multisymplectic 5-form (in 4-dimensional spacetime).

Multisymplectic geometry is thus able to combine the best features of the Lagrangian and Hamiltonian formulations of field theory. It also allows us to easily recover both the covariant variational principle and the 3+1 decomposed canonical formulation, clarifying the relationship between the two approaches, simplifying various derivations, and giving us deeper insight into the structure of the theory.

Much of what has traditionally been done using the Lagrangian or Hamiltonian approaches to field theory has by now been brought under the umbrella of multisymplectic geometry. See [5], [22] and references therein for an overview of what has been accomplished thus far. For aspects related to quantization see for example [23], [14] and references therein.

The novel element we present is the use of multisymplectic geometry to study boundary terms of field theories. In particular, when performing a 3+1 decomposition, a modification to the usual Hamiltonian and symplectic structure may be necessary if the 3-dimensional hypersurface has a 2-dimensional boundary. We refer to all such modifications as boundary terms. Their existence has long been known in specific examples in the canonical formulation of field theory [16], [24], [10], but we provide a multisymplectic foundation for organizing and generalizing these boundary terms.

Fundamental work on the role of boundary terms in multisymplectic geometry in the context of a covariant Legendre transformation has been carried out by Kijowski [24]. Our approach will differ in that we explicitly describe the phase spaces involved in the presence of boundaries and connect the boundary terms to multisymplectic momentum maps (as introduced in [22]). In order to discuss the boundary terms in the context of momentum maps, we have found it advantageous to introduce what we feel is a simpler and clearer approach to the 3+1 decomposition in multisymplectic geometry, and to clarify the definitions of the different phase spaces that result. The novel ideas in this thesis include a more geometrical approach to the 3+1 decomposition, a characterization of the phase spaces encountered and their interrelations, and an understanding of boundary terms from the perspective of multisymplectic momentum maps.

The organization of this chapter is as follows. In section 3.1 we review multisymplectic geometry using the example of a real scalar field. In 3.2 we review the notion of multisymplectic momentum maps and relate them to the more familiar concepts of symmetries and conserved currents. In 3.3 we review the 3+1 decomposition and the associated (pre)symplectic geometry, taking advantage of insights provided by the multisymplectic perspective. In 3.4 we introduce two phase spaces that naturally occur when 3+1 decomposing a multisymplectic formulation. We refer to these as the *large boundary phase space* and *small boundary phase space*. Our version of these phase spaces is new, but they have a close relationship with the phase spaces of parametrized theories studied in the past [6, 25–27]. In 3.5 we introduce an important map, the I*-map*, between differential forms on the multisymplectic phase space and the large (and small) boundary phase space. We use the properties of this map (discussed in appendix B) to show that the large and small boundary phase spaces carry a presymplectic structure. In 3.6 we use the I-map to connect multisymplectic geometry with the variational principle. In 3.7 we consider the equations of motion on the boundary phase spaces, and show that they are formally similar to Hamilton's equations as used in the canonical formulation of field theory. In 3.8 we discuss momentum maps from the perspective of the large and small boundary phase spaces. In 3.9 we consider what happens to our phase spaces if the 3-dimensional surface of our 3+1 decomposition has a boundary (or non-trivial fall-off conditions on the fields). We discuss the relevant modifications necessary to 3+1 decompose the multisymplectic formalism in that setting. In 3.10 we discuss what happens to momentum maps in the presence of boundaries. Lastly, in 3.11 we apply our theoretical considerations to some field theories of physical interest.

3.1 Multisymplectic Geometry

In this section, we review multisymplectic geometry using the example of a scalar field. In section 3.11 we will provide examples of other physical theories, including electromagnetism and Yang-Mills, and in 4 we consider gravity. These field theories (as well as most others of physical interest) come with their own characteristic additional structure, which it is useful to incorporate into the multisymplectic treatment. We thus discuss the multisymplectic structure of theories on a case-by-case basis rather than attempting to provide a single framework for all of them. For a more unified discussion, as well as additional mathematical details, see [5, 11, 21, 22].

In this chapter we will not use the Hamilton-Pontryagin approach. If we were to use a Hamilton-Pontryagin approach, many of the details would remain unchanged except that the Legendre transform would define a constraint submanifold rather than a map into the phase space.

Let M be the full 4-dimensional spacetime manifold, with metric $g_{\mu\nu}$ of signature $(-,+,+,+)$. Let $R \subset M$ be a 4-dimensional region. The dynamics of a real scalar field in

R can be described by the action

$$S = \int\limits_R \mathcal{L}(\phi, \partial_\mu \phi, x^\mu)\, \mathrm{d}^4x = \int\limits_R \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \sqrt{|g|} \mathrm{d}^4 x. \tag{3.1}$$

Several comments are in order regarding this action. First, we restrict our attention to a region $R \subset M$ so as to formulate the theory in a local neighborhood in spacetime (such local formulations arise in cavity QED, for instance). Second, we view the metric $g_{\mu\nu}$ as a given function of the spacetime coordinates x^μ, and not as an independent field. Lastly, we have picked a particular form of the Lagrangian for simplicity, but in fact a large class of Lagrangians for the scalar field will yield the same multisymplectic structure.

We introduce the multi-momentum π^μ via the covariant Legendre transformation,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}. \tag{3.2}$$

For the choice of Lagrangian in (3.1), $\pi^\mu = -\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi$. We can use this equation to eliminate $\partial_\mu \phi$ from the theory in favor of π^μ. We can thus use the fields ϕ and π^μ rather than ϕ and $\partial_\mu \phi$ to describe the scalar field.

The multisymplectic phase space (see Figure 3.1) P is a fiber bundle over M, with a closed "multisymplectic 5-form" ω. Note that the multisymplectic phase space is finite dimensional, in contrast with the infinite dimensional phase spaces that occur in a traditional Hamiltonian framework based on a 3+1 decomposition. The coordinates on each fiber can be taken to be ϕ and the multi-momentum π^μ. Thus P is a 9-dimensional bundle consisting of 5-dimensional fibers over a 4-dimensional spacetime.

We choose coordinates x^μ on M. These coordinates label the fibers of P, and so (ϕ, π^μ, x^μ) are a full set of coordinates on P. In these coordinates, the bundle projections map $\pi : P \to M$ is then $\pi(\phi, \pi^\mu, x^\mu) = x^\mu$. This implies that we may regard the x^μ coordinates on P as pull-backs by the bundle projection map of the x^μ coordinates on M, and this will turn out to be a useful point of view. We also introduce a convenient set of differential forms

$$\mathrm{d}^4x = \frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta} \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta, \tag{3.3a}$$

$$\mathrm{d}^3x_\mu = \frac{1}{3!} \epsilon_{\mu\alpha\beta\gamma} \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma = i_{(\partial/\partial x^\mu)} \mathrm{d}^4x, \tag{3.3b}$$

$$\mathrm{d}^2x_{\mu\nu} = \frac{1}{2!} \epsilon_{\mu\nu\alpha\beta} \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta = i_{(\partial/\partial x^\nu)} \mathrm{d}^3x_\mu, \tag{3.3c}$$

and so on. Here $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita completely antisymmetric symbol with $\epsilon_{0123} = +1$. These forms can be viewed as forms on M or as forms on P by pulling back. Notice that we use the same notation for the coordinates x^μ on M and on P, although the two sets of coordinates are related by a pull-back. It will be clear from context which is intended.

Using the local coordinates (ϕ, π^μ, x^μ) and definitions (3.3) on P we will write what we call the multisymplectic potential as

$$\theta = \pi^\mu \mathrm{d}\phi \wedge \mathrm{d}^3x_\mu - \mathcal{H} \mathrm{d}^4x, \tag{3.4}$$

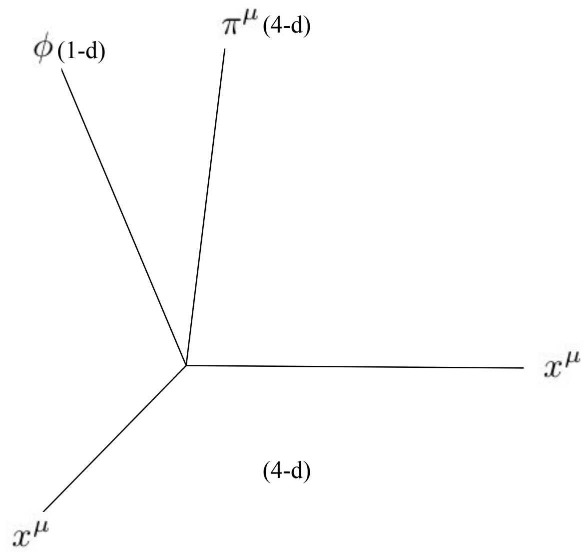

Figure 3.1: The multisymplectic phase space for a scalar field, with coordinate axes drawn in. The field and multi-momentum coordinates are each represented by a single axis, while the x^μ coordinates are represented by a 2-d plane with both axes in the plane carrying the label x^μ.

where $\mathcal{H}$ is the De Donder-Weyl Hamiltonian [12, 13, 28]

$$\mathcal{H} = \pi^\mu \partial_\mu \phi - \mathcal{L}, \tag{3.5}$$

understood to be a function of only ϕ, π^μ, and x^μ. The multisymplectic 5-form on P is then

$$\omega = \mathrm{d}\theta. \tag{3.6}$$

Clearly ω is closed ($\mathrm{d}\omega = 0$).

As will be seen below in (3.7), the equations of motion depend only on ω, so we can always add a closed 4-form to θ without changing the physical content of the theory. We recall that in mechanics modifying the canonical 1-form by a closed 1-form is used to generate a canonical transformation. Here we have the multisymplectic generalization of that.

The multisymplectic phase space P with its multisymplectic potential θ and multisymplectic 5-form ω provides a complete, geometric description of the field theory. In particular, since ω contains the De-Donder Weyl Hamiltonian, the dynamics is already encoded in the geometry. The dynamics is a generalization of Hamilton's equations to multisymplectic geometry. These equations are algebraic and differential conditions that field configurations must satisfy in order to be solutions. We thus need a geometric description of field configurations before we can discuss the equations of motion.

Given a fiber bundle like P where the base space is spacetime and the fibers refer to fields, it is a standard construction to regard a field configuration as a (local) section of the bundle. Thus a field configuration over the region $R \subset M$ can, in multisymplectic geometry, be identified with a section map $\sigma : R \to P$. We use $\bar{R}$ to label the image of $\sigma : R \to P$, which is just the graph in P of the field configuration over R. A section σ with image $\bar{R} \subset P$ is a solution to the equations of motion if and only if it satisfies

$$(i_Y \omega)_{\bar{R}} = 0, \tag{3.7}$$

for all vector fields Y on P. We refer to an $\bar{R}$ that satisfies this condition as a solution surface. This the standard form of Hamilton's equations in the multisymplectic context [5, 14]. In 3.6 we will show that it is equivalent to the Euler-Lagrange equations.

3.2 Momentum Maps

Momentum maps are the modern approach to symmetries and conserved quantities. Their role in the symplectic geometry of particle mechanics is discussed in [2, 3], while their generalization to the multisymplectic geometry of covariant field theories is a principal element of [5, 22, 29]. Here we provide a brief summary of the relevant material. The logic follows that of ordinary momentum maps from symplectic geometry, with which we assume the reader is familiar. At the end of the section, we will connect multisymplectic momentum maps to the conserved currents which stem from Noether's theorem.

When discussing momentum maps, it is important to distinguish between what we refer to as the inhomogeneous and homogeneous formulations of mechanics. Inhomogeneous formulations, such as symplectic geometry, carry two distinct geometric structures: the (symplectic) 2-form $\omega = \mathrm{d}p \wedge \mathrm{d}q$ and the Hamiltonian H. One can have a symmetry of ω which is not a symmetry of H. In an inhomogeneous formulation, Hamilton's equations take the inhomogeneous form

$$i_{X_H} \omega = -\mathrm{d}H, \tag{3.8}$$

so one needs a symmetry of both ω and H to identify a quantity which is conserved on solutions. Homogeneous formulations, such as presymplectic geometry, carry just a single geometric structure: the presymplectic 2-form $\omega = \mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t$. Hamilton's equations take the homogeneous form $i_{X_H} \omega = 0$, and a symmetry of ω alone automatically yields a quantity conserved on solutions. Both the inhomogeneous and homogeneous formulations have various generalizations to field theory. The approach we take to multisymplectic geometry is a (generalized) homogeneous formulation, and so we will discuss momentum maps from the homogeneous perspective.

Those familiar with momentum maps in the context of symplectic geometry will recall that the components of the momentum map are functions (0-forms) on phase space. This is because the symplectic structure is given by a 2-form and the momentum map construction produces a differential form (a component of the momentum map) two degrees lower. Hence we expect that in multisymplectic geometry (where the geometric structure is given by a 5-form) the components of the momentum maps will be 3-forms and not functions.

To summarize the momentum map construction in multisymplectic geometry, we recall that a momentum map is associated with a Lie group acting on the phase space of the theory. Let Φ be the action of a Lie group G on P,

$$\Phi : G \to \mathrm{Diff}(P) \tag{3.9a}$$

$$g \mapsto \Phi_g : P \to P, \text{ with } \Phi_g \Phi_h = \Phi_{gh}. \tag{3.9b}$$

An element of the Lie algebra $\xi \in \mathfrak{g}$ induces a vector field on P defined by

$$X_\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^*_{\exp(t\xi)}, \tag{3.10}$$

where both sides are understood to act on functions on P. See [2, 17] for a discussion of induced vector fields. Here t is simply a parameter to facilitate the definition of the induced vector field; it is not to be confused with the time coordinate on M.

We will be specifically interested in group actions which preserve the multisymplectic form ω,

$$\Phi^*_g \omega = \omega \quad \forall g \in G. \tag{3.11}$$

In this case, G is a symmetry group for the theory. The induced vector fields of an action by a symmetry group satisfy

$$\pounds_{X_\xi} \omega = 0, \tag{3.12}$$

which follows from (3.11) by writing $g = \exp t\xi$ and differentiating both sides with respect to t.

Since $\mathrm{d}\omega = 0$, it follows from Cartan's formula ($\pounds_X = i_X \mathrm{d} + \mathrm{d} i_X$) and (3.12) that

$$\mathrm{d} i_{X_\xi} \omega = 0. \tag{3.13}$$

Thus $i_{X_\xi}\omega$ is locally exact and there exists, locally, a 3-form $J_\xi \in \Omega^3(P)$ which satisfies

$$i_{X_\xi} \omega = -\mathrm{d} J_\xi. \tag{3.14}$$

The negative sign is conventional. In our examples, (3.14) holds globally, so we restrict attention to that case.

In symplectic geometry, it is standard to express an equation of the same form as (3.14) in terms of a map from phase space into the dual of the Lie algebra, $\mathfrak{g}^*$. In multisymplectic geometry, we can express (3.14) in terms of a 3-form valued map from P into $\mathfrak{g}^*$, so long as J_ξ is linear in ξ, as will be the case in our examples. Consider a point $p \in P$ and the 3-form $J_\xi|_p \in \Lambda^3(T_pP)$ (the latter being the space of 3-forms at p). When J_ξ is linear in ξ, we can define a linear function $J|_p : \mathfrak{g} \to \Lambda^3(T_pP)$ via $J|_p(\xi) \equiv J_\xi|_p$. A linear map from $\mathfrak{g}$ to V for any vector space V is an element of $\mathfrak{g}^* \otimes V$. Thus if J_ξ is linear in ξ, we can construct the map $J : P \to \mathfrak{g}^* \otimes \Omega^3(P)$ with $J(p) \equiv J|_p$. This $\mathfrak{g}^*$-valued 3-form J is called the momentum map. We can write this concisely using the natural pairing $\langle \cdot, \cdot \rangle$ between $\mathfrak{g}^*$ and $\mathfrak{g}$:

$$J_\xi|_p = \langle J(p), \xi \rangle . \tag{3.15}$$

Thus, defining J_ξ for all $\xi \in \mathfrak{g}$ via (3.14) also defines $J : P \to \mathfrak{g}^* \otimes \Omega^3(P)$ via (3.15). We will often refer to both J_ξ and J as the momentum map.

From (3.14), we see that the momentum map is defined only up to a closed (locally exact) 3-form. This will become important in the context of boundary terms and boundary conditions, as discussed in section 3.10.

Let us now connect the momentum map with the more familiar notion of a conserved current as described by Noether's theorem. Recall that Noether's theorem requires the equations of motion to hold in order to arrive at a conserved current, so let σ be a section of P over R with image $\bar{R} \subset P$ satisfying the equations of motion (3.7). Note that (3.14) holds everywhere on P, not just on solutions, but we are free to restrict (3.14) to a solution surface $\bar{R}$. Using (3.7) with X_ξ in place of Y we get

$$0 = (\mathrm{d}J_\xi)_{\bar{R}}. \tag{3.16}$$

Pulling this back to spacetime via σ we obtain

$$0 = \mathrm{d}j_\xi = \mathrm{d}(j_\xi^\mu \mathrm{d}^3x_\mu) = \partial_\mu j_\xi^\mu \, \mathrm{d}^4x \implies \partial_\mu j_\xi^\mu = 0, \tag{3.17}$$

where $j_\xi \equiv \sigma^* J_\xi \in \Omega^3(R)$ is the traditional conserved Noether current. Here we have used the identity $\mathrm{d}x^\nu \wedge \mathrm{d}^3x_\mu = \delta^\nu_\mu$, which follows immediately from (3.3).

To be slightly more precise, we obtained the Noether current j_ξ as a 3-form. This is natural since we can then relate the flux integrals of the current to the integrals of j_ξ over various 3-surfaces. It is worth noting that this derivation of the Noether current (as a 3-form) does not require a metric, and doesn't use it if there is one. On the other hand, if there is a metric, we can use it to convert j_ξ into a 1-form using the Hodge star and then use the inverse metric to construct the 4-vector current $g^{-1}(\cdot, *j_\xi) = \frac{j_\xi^\mu}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu}$.

As with (pre)symplectic momentum maps, we can formally solve (3.14). This is similar to the discussion for formally solving (2.14), but now we deal with forms of higher rank.

$$\mathrm{d}J_\xi = -i_{X_\xi}\omega = -i_{X_\xi}\mathrm{d}\theta = -\pounds_{X_\xi}\theta + \mathrm{d}i_{X_\xi}\theta, \tag{3.18}$$

where we used Cartan's identity. Now $\pounds_{X_\xi}\omega = \pounds_{X_\xi}\mathrm{d}\theta = \mathrm{d}\pounds_{X_\xi}\theta = 0$ since the Lie and exterior derivatives commute. Thus locally $\pounds_{X_\xi}\theta = \mathrm{d}\alpha_\xi$ for some 3-form α_ξ. Plugging back into (3.18), we have

$$\mathrm{d}J_\xi = \mathrm{d}i_{X_\xi}\theta - \mathrm{d}\alpha_\xi \implies \mathrm{d}\left(J_\xi - i_{X_\xi}\theta + \alpha_\xi\right) = 0, \tag{3.19}$$

and thus

$$J_\xi = i_{X_\xi}\theta - \alpha_\xi + \mathrm{d}\beta_\xi, \tag{3.20}$$

where β_ξ is a 2-form that may depend on ξ. For the special case of point transformations (group actions on P that are lifted from group actions on M and/or the field ϕ), we have $\pounds_{X_\xi}\theta = 0 \implies \alpha_\xi = 0$ and

$$J_\xi = i_{X_\xi}\theta + \mathrm{d}\beta_\xi, \tag{3.21}$$

with the choice of β_ξ at least partially fixed by co-adjoint equivariance (see [5]).

3.3 Multisymplectic Perspective on the Initial Value Problem

We consider two distinct approaches to the equations of motion: the multisymplectic and presymplectic. The multisymplectic equations of motion are a covariant version of Hamilton's equations given in (3.7). The presymplectic approach is, loosely speaking, the homogeneous version of the *symplectic* approach to the equations of motion. The symplectic approach provides the traditional geometric structure for classical field theories [6–10]. Our goal for much of the rest of the chapter will be to derive this symplectic framework, or rather its closely related presymplectic framework, starting from the multisymplectic framework. The process involves several intermediate steps and uncovers some interesting geometry, clarifying many features of the symplectic approach. In this section we will provide an overview of the steps necessary to bridge the multisymplectic and symplectic approaches, while briefly introducing the relevant geometric concepts. The details will be discussed in subsequent sections.

We first review the traditional Hamiltonian approach to field theory and its associated symplectic structure (see [6–10] for discussion and examples of this approach). To cast the Euler-Lagrange equations into Hamiltonian form requires performing a decomposition of spacetime into space and time, hereafter referred to as the 3+1 decomposition (see [30] and references therein). It is usually held that this decomposition breaks the (manifest) covariance of the field theory, but one can also say that the decomposition maintains covariance while introducing additional geometry, and this is the point of view we will take. The additional geometry consists of a 3-dimensional hypersurface (3-surface) Σ_0, nicely embedded and usually assumed to be spacelike, and a vector field N on M whose flow maps Σ_0 into a family of 3-surfaces Σ_t where t is the parameter of the flow. This vector field is chosen such that all Σ_t are also spacelike and nicely embedded. In the Hamiltonian formulation of general relativity, usually referred to as the ADM formalism, this vector field N is decomposed into its components along the normal to the surfaces (the lapse) and tangent to the surfaces (the shift) [16]. Borrowing the terminology, we refer to N as the lapse-shift vector field, although we will not decompose it with respect to the normal to the surfaces (we will in fact avoid introducing the normal altogether as it tends to mix the dynamics and kinematics, complicating the geometry).

To be explicit, let $A_t^N : M \to M$ be the advance map of N corresponding to the value of the flow parameter t. Then Σ_t is the image of Σ_0 under the map A_t^N. If we introduce local (spatial) coordinates on Σ_0, then we can use A_t^N to transport these coordinates to Σ_t. This gives us local coordinates on all the Σ_t, then using t as a fourth coordinate, we obtain local coordinates on M. In these coordinates, the advance map looks like the translation in the t-coordinate (mapping between points on different Σ_t with the same spatial coordinates) and the lapse-shift vector field is the coordinate vector field $\partial/\partial t$. We will often use this coordinate system to work through examples or to show how a complicated expression might simplify in practice.

In this traditional Hamiltonian approach, Σ_0 is regarded as a submanifold of M (typically a Cauchy surface) and N is to be nowhere vanishing so that the family Σ_t foliates M (at least locally). There are situations in which these assumptions are too restrictive, so later we will make several generalizations to this standard picture. We will reinterpret Σ_0 as the image of a map, rather than a submanifold of M, and then N (at Σ_0) will be allowed to be an arbitrary infinitesimal motion of the map (a tangent vector to the space of maps). In addition to handling cases not covered by the standard picture, these generalizations will be shown to be advantageous from a multisymplectic perspective.

Returning to the standard picture and continuing the example of the scalar field, we take the dynamical variables of Hamilton's equations to be the field configuration ϕ and its conjugate momentum $\pi^0 = \partial\mathcal{L}/(\partial\dot{\phi})$ (the $\mu = 0$ component of (3.2); in most standard references this is denoted simply π). The way we have introduced it, $\mathcal{L}$ is a density rather than a scalar (see (3.1)). It follows that π^0 is also a density. It can be useful to construct a 3-form out of π^0, since 3-forms behave nicely under pull-backs. To do this, we let (t, x^i) be the transported coordinate system on M as described above (so x^i are coordinates on Σ_t). Consider the 3-form $\pi^0\mathrm{d}^3x = \pi^0\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. One can verify that it is invariant under coordinate transformations of the x^i (since π^0 transforms as a density) and so really does define a 3-form on Σ_t.

The fields ϕ and π^0 are defined on each of the surfaces Σ_t and thus for different Σ_t, they are functions with different domains. To be explicit about this, we will denote a field ϕ whose domain is Σ_t by $\phi^{(\Sigma_t)}$ and similarly for π^0. For convenience, we will now replace these with functions $\phi^{(\Sigma)}$, $(\pi^0)^{(\Sigma)}$ defined over a certain "standard region" Σ.

Let Σ be a manifold diffeomorphic to Σ_0. Since Σ_0 is an embedded submanifold, there exists an embedding map $f_{\Sigma_0} : \Sigma \to M$ with $\mathrm{img}(f_{\Sigma_0}) = \Sigma_0$. With the help of this embedding and the advance map A_t^N of the lapse-shift vector field N, we can pull back all field configurations $\phi^{(\Sigma_t)}$ and $(\pi^0)^{(\Sigma_t)}$ to field configurations $\phi^{(\Sigma)}$ and $(\pi^0)^{(\Sigma)}$ over Σ. Explicitly,

$$\phi^{(\Sigma)} = (A_t^N \circ f_{\Sigma_0})^*\phi^{(\Sigma_t)}, \tag{3.22a}$$

$$(\pi^0)^{(\Sigma)}\mathrm{d}^3\sigma = (A_t^N \circ f_{\Sigma_0})^* \left[(\pi^0)^{(\Sigma_t)}\mathrm{d}^3x\right], \tag{3.22b}$$

where $\mathrm{d}^3\sigma = \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2 \wedge \mathrm{d}\sigma^3$ and σ^i are coordinates on Σ (the density $(\pi^0)^{(\Sigma)}$ is defined with respect to the σ^i). Note that these fields are $t-$dependent. From now on we will omit the superscript (Σ) for fields defined over Σ.

We may now describe the dynamics on an infinite-dimensional phase space whose coordinates are the field configurations (over Σ) ϕ and π^0. Specifically, time dependent fields $\phi(t)$ and $\pi^0(t)$ over Σ can be interpreted as specifying a parametrized curve in an infinite-dimensional phase space with coordinates (ϕ, π^0). Conversely, a parametrized curve $(\phi(t), \pi^0(t))$ in this phase space can be used to reconstruct the field evolution on spacetime:

$$\phi^{(\Sigma_t)} = (A_t^N \circ f_{\Sigma_0})^{-1\,*}[\phi(t)], \tag{3.23a}$$

$$(\pi^0)^{(\Sigma_t)}\mathrm{d}^3x = (A_t^N \circ f_{\Sigma_0})^{-1\,*}[(\pi^0(t))\mathrm{d}^3\sigma]. \tag{3.23b}$$

We will soon show that this infinite-dimensional phase space possesses a symplectic structure, and so we will refer to it as the symplectic phase space (for the scalar field). Note the contrast between the infinite-dimensional symplectic phase space and the finite-dimensional multisymplectic one.

When we need to talk about a vector X on the symplectic phase space, we introduce its components with respect to the (ϕ, π^0) coordinates as follows. Let $F[\phi, \pi^0]$ be a functional of the fields ϕ and π^0 (this is a function on the symplectic phase space). We write

$$XF = \int_\Sigma \left[(\delta\phi)_X \frac{\delta F}{\delta \phi} + (\delta\pi^0)_X \frac{\delta F}{\delta \pi^0} \right] \mathrm{d}^3\sigma, \tag{3.24}$$

which defines $((\delta\phi)_X, (\delta\pi^0)_X)$ as the coordinate components of X. The (possibly weakly, see [2, 18] and references therein) symplectic structure on this phase space can be written with the help of this coordinate representation as

$$\Omega(X,Y) = \int_\Sigma \left[(\delta\pi^0)_X (\delta\phi)_Y - (X \leftrightarrow Y) \right] \mathrm{d}^3\sigma. \tag{3.25}$$

Hamilton's equations take the usual form (see (3.8)) $i_{X_H}\Omega = -\mathrm{d}H$ where $H = \int_\Sigma \big(\pi^0 \dot{\phi} - \mathcal{L}\big)\, \mathrm{d}^3\sigma$ is the 3+1 Hamiltonian. This completes the review of the symplectic geometry of field theory.

We turn now to the presymplectic framework for field theory, which is a simple generalization of the symplectic framework just presented. We extend the infinite-dimensional symplectic phase space with a 1-dimensional t axis (the parameter labeling the family Σ_t). The coordinates on this extended phase space are now (ϕ, π^0, t). As before, we may describe the dynamics using a parametrized curve. The coordinates of such a curve on this extended phase space are $(\phi(\lambda), \pi^0(\lambda), t(\lambda))$ and this translates into an evolution of the fields

$$\phi^{(\Sigma_t)} = [A_t^N \circ f_{\Sigma_0}]^{-1\,*}[\phi(\lambda(t))], \tag{3.26a}$$

$$(\pi^0)^{(\Sigma_t)} \mathrm{d}^3 x = [A_t^N \circ f_{\Sigma_0}]^{-1\,*}[\pi^0(\lambda(t)) \mathrm{d}^3\sigma]. \tag{3.26b}$$

We require t to be an invertible function of λ. One can check that this evolution is invariant under reparameterizations, so on the extended phase space unparameterized curves are sufficient to represent dynamics (a generic feature of a homogeneous approach). The extended phase space has a presymplectic structure which we can write with the help of the coordinate representation introduced above (3.25):

$$\Omega(X,Y) = \left[\int_\Sigma \big((\delta\pi^0)_X (\delta\phi)_Y \big)\, \mathrm{d}^3\sigma - (\delta H)_X (\delta t)_Y \right] - [X \leftrightarrow Y], \tag{3.27}$$

where we use notation for vectors as in (3.25), and $(\delta H)_X \equiv XH = \int_\Sigma \big[(\delta\phi)_X \frac{\delta H}{\delta\phi} + (\delta\pi^0)_X \frac{\delta H}{\delta\pi^0} \big]\, \mathrm{d}^3\sigma + (\delta t)_X \frac{\partial H}{\partial t}$. Hamilton's equations can now be written $i_{X_H}\Omega = 0$. This is our homogeneous, presymplectic formulation of Hamiltonian field theory. It has advantages over the symplectic formulation, since it more easily handles time-dependent Hamiltonians and time-dependent symmetry transformations. A time-dependent Hamiltonian is a generic feature of scalar field theory in curved spacetime (unless the lapse-shift agrees with a Killing vector field of the background metric) and so we prefer a geometric formulation capable of handling this case. Time-dependent symmetry transformations are ubiquitous in gauge theories, and so we should aim to have a framework capable of supporting such transformations. We will see that such a homogeneous presymplectic formulation is quite natural from the standpoint of multisymplectic geometry, and we will show how to replace standard notions in symplectic geometry (such as Poisson brackets) with their presymplectic analogues.

We have reviewed symplectic Hamiltonian field theory and put it into a presymplectic framework. We now present an overview of the derivation of this presymplectic formulation from multisymplectic geometry. This will involve introducing two additional phase spaces, intermediate between the multisymplectic phase space and the final presymplectic phase space. We refer to these as the large and small boundary phase spaces. In the rest of this section, we introduce the geometric concepts necessary for understanding the large and small boundary phase spaces, and give some sense of what these phase spaces are.

We first address the additional geometry introduced above when performing a 3+1 decomposition, namely, the embedded submanifold Σ_0 and the lapse-shift vector field N. We have found it advantageous to relax the requirement that Σ_0 be an embedded submanifold and consequently we do not henceforth require the advance map of N to produce a foliation. Relaxing the assumptions on Σ_0 amounts to replacing the embedding map f_{Σ_0} with a general map $f : \Sigma \to \Sigma_0$ where Σ is some standard 3-dimensional manifold and $\Sigma_0 = \text{img}(f)$ is not necessarily a submanifold of M. Instead of integrating over a submanifold of M, we will now pull back by the map f and perform integration over the manifold Σ.

Since we now regard Σ_0 as the image of a map rather than a submanifold, we will do the same for all the Σ_t, regarding them as images of a family of maps f_t, which all have the same domain Σ. We thus replace the family of maps $A_t^N \circ f_{\Sigma_0}$ responsible for foliating M with the family of maps f_t. We will not require the images of these maps (Σ_t) to be submanifolds of M nor to form a foliation. The lapse-shift vector field which served to identify nearby surfaces Σ_t and $\Sigma_{t+\delta t}$ will generalize to an identification of nearby maps f_t and $f_{t+\delta t}$. Such an object is called a vector field along the map f_t, and it is no longer a vector field on M, as the images of our maps may have self-intersections or not form a foliation. We will see later that this vector field along a map can also be interpreted as a tangent vector to the space of maps.

We can see the advantages of these generalizations by considering some simple cases of field evolution in Minkowski spacetime. First, consider a "slab" of Minkowski spacetime located between $t = t_0$ and $t = t_1$. The boundary of this slab consists of a pair of 3-surfaces that can be evolved by varying t_0 and t_1. When $t_0 = t_1$, it seems the boundary has coalesced into a single 3-surface, but this is not the case. Rather, the boundary continues to be a map from two disjoint copies of $\mathbb{R}^3$, but the images of the two copies now overlap. Second, consider the classical evolution associated with the Unruh effect. This involves evolving along the boost vector field. The boost vector field vanishes along a plane passing through the origin of Minkowski spacetime, and thus the family of 3-surfaces generated by the boost's advance map does not form a foliation (the surfaces intersect where the boost vector field vanishes). In both cases, our generalized 3+1 decomposition easily handles the scenarios, while the standard formulation is found to be lacking.

We are now in a position to reinterpret field configurations also as general maps. Recall from 3.2 that it is standard to interpret field configurations as sections of a bundle of fields. In the present case we are interested in field configurations of ϕ and π^0, so we may introduce a bundle E over M with two-dimensional fibers having coordinates ϕ and π^0. A section $\sigma : M \to E$ of this bundle represents a field configuration of ϕ and π^0 (over M). Now given one of the maps f_t into spacetime, we may construct the map $F_t = \sigma \circ f_t$ which maps Σ into

E. We may consider this map F_t as representing a field configuration over $\Sigma_t = \text{img}(f_t)$ in the sense that the section σ in the composition is only evaluated at points of $\text{img}(f_t) = \Sigma_t$. This interpretation of field configurations as maps fits nicely with multisymplectic theory, where it is common to take integrations on spacetime and lift them to integrations on a bundle of fields. Since we have switched to using maps (rather than surfaces) to perform integration, it is natural from the multisymplectic perspective to use maps on the field bundle as well.

One advantage of viewing field configurations as maps is that we can directly use them to construct a phase space. Recall that when introducing the standard symplectic and presymplectic pictures we needed to pull back field configurations to a standard region in order to make the construction geometrically clear. Since our maps are already defined over a standard region, we no longer need to use pull backs for clarity. We simply take the space of maps (into E) as our phase space of field configurations. So far we have only regarded maps which are compositions of the form $F_t = \sigma \circ f_t$ as specifying field configurations. Restricting our phase space to such composition maps is an awkward construction, particularly when $\Sigma_t = \text{img}(f_t)$ changes topology for different t. We resolve the issue by allowing all maps into E to be in our phase space. We will use such phase spaces of maps extensively in the rest of the paper. In particular, the large and small boundary phase spaces are spaces of maps much like the ones described here. We will delve into the details of such phase spaces in subsequent sections, but let us point out one interesting feature that emerges. By dropping the requirement that our maps be compositions, we allow "multi-valued field configurations" (field configurations which are not described by sections of a field bundle, but rather general maps into the bundle). Despite the presence of these multivalued field configurations, we nevertheless need to allow for general maps because without them, our phase spaces would not be manifolds (even in simple $0 + 1$ dimensional examples).

We summarize what we have done so far. We began by reviewing the standard symplectic construction of field theory, paying careful attention to geometric clarity. We extended these considerations to the presymplectic (homogeneous) formulation of field theory. We then discussed the need to expand the standard pictures, and ultimately recast them using the language of general maps. Our original goal was to derive the standard symplectic picture starting from multisymplectic geometry, but we see now that it is sufficient to derive a version of the presymplectic homogeneous formulation involving general maps, rather than restricting to embeddings. The other 3+1 formulations discussed in this section can be seen as special cases of it. In the next several sections we will describe how to obtain such a version of presymplectic field theory starting from multisymplectic geometry. We will show that the geometry of presymplectic field theory is a direct consequence of multisymplectic geometry, combined with the additional geometry inherent in a 3+1 decomposition. In this way, we will derive the relevant features of a presymplectic (and hence, symplectic) formulation rather than assume them.

3.4 Large and Small Boundary Phase Spaces

We now introduce the two intermediate phase spaces involved in our 3+1 decomposition of P. As in 3.3, we let Σ be a 3-dimensional manifold, which loosely-speaking will represent the space part of our spacetime decomposition. When Σ is closed ($\partial\Sigma = \emptyset$), we define what we call the "large boundary phase space,"

$$\mathcal{P} \equiv C^\infty(\Sigma, P), \tag{3.28}$$

as the space of smooth maps from Σ into the multisymplectic phase space P (see Figure 3.2). $\mathcal{P}$ is not to be confused with P; $\mathcal{P}$ is infinite-dimensional while P is finite-dimensional. Introducing coordinates σ^i on Σ and (ϕ, π^μ, x^μ) on P, we see that coordinates on $\mathcal{P}$ are functions $(\phi(\sigma), \pi^\mu(\sigma), x^\mu(\sigma))$. These functions represent field configurations of ϕ and π^μ over the region specified by $x^\mu(\sigma)$. Notice the coordinates on $\mathcal{P}$ include not only the field configurations, but also the coordinates $x^\mu(\sigma)$ of the embedding, which makes $\mathcal{P}$ similar to the phase spaces of parametrized field theories [6, 25–27].

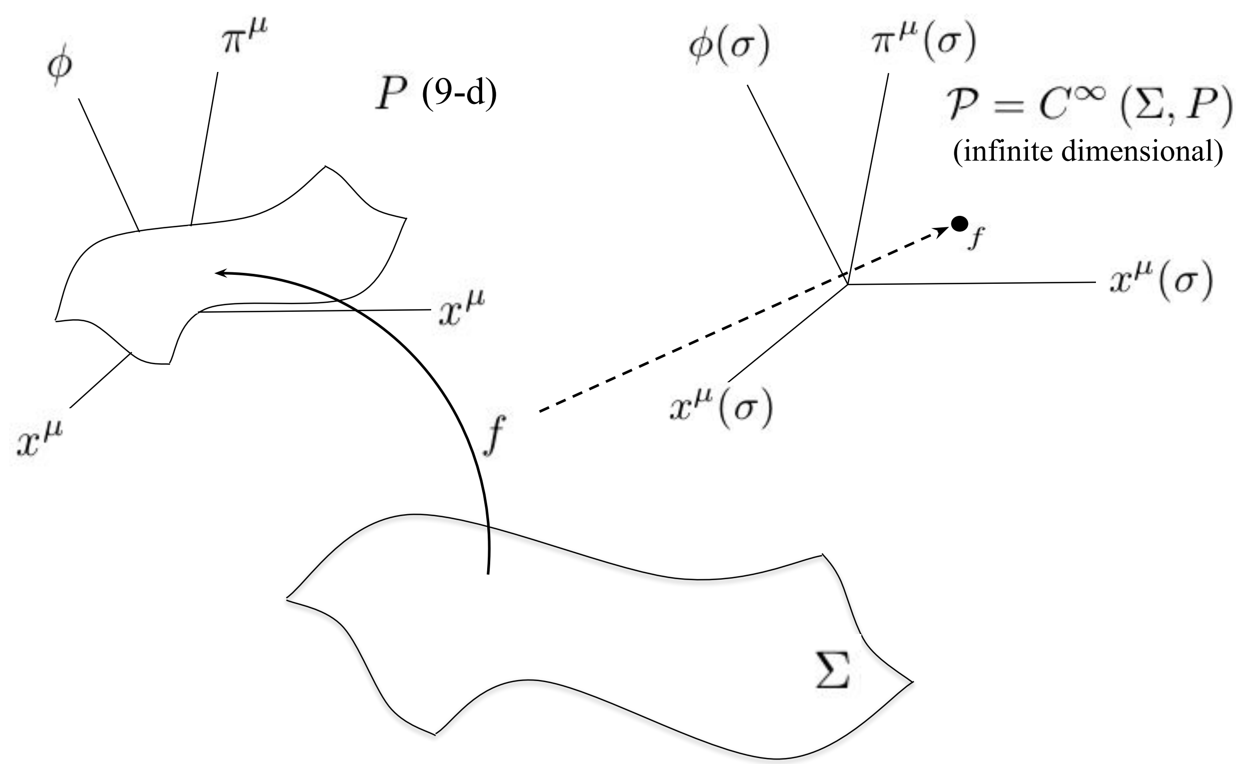

Figure 3.2: The basic structure of the large boundary phase space $\mathcal{P} = C^\infty(\Sigma, P)$. Every smooth map from the standard manifold Σ (bottom) to the multisymplectic phase space P (top left) corresponds to a point in the large boundary phase space (top right). The coordinate axes shown in $\mathcal{P}$ each represent infinite-dimensional spaces of functions.

The large boundary phase space $\mathcal{P}$ inherits some interesting geometry from the multisymplectic phase space P. For convenience, denote $\mathcal{M} \equiv C^\infty(\Sigma, M)$. The projection map $\pi : P \to M$ induces a projection map $\tilde{\pi} : \mathcal{P} \to \mathcal{M} : f \mapsto \pi \circ f$, giving $\mathcal{P}$ the structure of a fiber bundle over $\mathcal{M}$ (see Figure 3.3). Vectors tangent to $\mathcal{P}$ have an interpretation as maps $: \Sigma \to TP$ whose images are collections of vectors on P (see appendix A for details). The multisymplectic 5-form $\omega \in \Omega^5(P)$ induces a presymplectic 2-form $\Omega \in \Omega^2(\mathcal{P})$, as we will explain in the next section. The dynamics on the large boundary phase space is governed by the kernel of this 2-form and is equivalent to the Euler-Lagrange equations. Since $\mathcal{P}$ contains not only fields but also embedding information, the dynamics has more freedom than the usual Hamiltonian formulations of classical fields. The usual formulations are closer to what we call the "small boundary phase space."

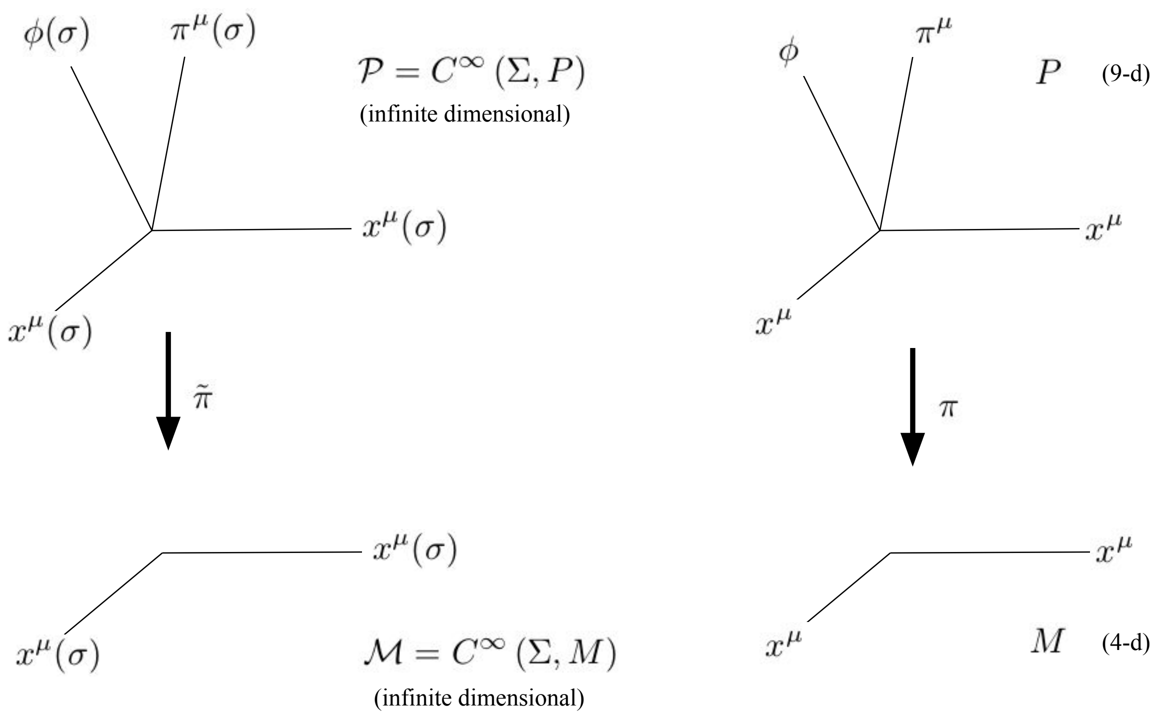

Figure 3.3: The bundle structure of the multisymplectic phase space (right) induces the bundle structure of the large boundary phase space (left). Specifically, given a map $f \in \mathcal{P}$ and the projection $\pi : P \to M$, we can construct the map $F = \pi \circ f \in \mathcal{M}$. This means there is a natural bundle structure $\mathcal{P} \to \mathcal{M}$ with projection map $\tilde{\pi}$ defined by $\tilde{\pi}(f) = F = \pi \circ f$.

Now we will define the small boundary phase space, which we denote by $\mathcal{S}$. The definition of $\mathcal{S}$ involves the base space $\mathcal{M} = C^\infty(\Sigma, M)$, which consists of smooth maps from Σ ("space") into M (spacetime). We remind the reader that the image of Σ need not be a submanifold of M (as discussed in section 3.3).

Now consider a parametrized curve in $C^\infty(\Sigma, M)$, that is $\gamma : \mathbb{R} \to C^\infty(\Sigma, M) : \lambda \mapsto \gamma_\lambda$. Here and throughout we assume, for simplicity and to avoid introducing additional notation, that curves are defined over all of $\mathbb{R}$. Note that γ_λ is a map : $\Sigma \to M$. We may identify the image $\gamma_0(\Sigma) \equiv \Sigma_0$ as an initial value surface and consider the 1-parameter family of surfaces $\Sigma_\lambda \equiv \gamma_\lambda(\Sigma)$ as our 3+1 decomposition of M, much as in 3.3. We define the small boundary phase space $\mathcal{S}$ as a bundle over $\mathbb{R}$, such that the fiber has coordinates $(\phi(\sigma), \pi^\mu(\sigma))$ (it is isomorphic to the fibers of $\mathcal{P} \to C^\infty(\Sigma, M)$). That is, $\mathcal{S}$ is the pull-back bundle [31] $\mathcal{S} = \gamma^*\mathcal{P}$ associated with the map $\gamma : \mathbb{R} \to C^\infty(\Sigma, M)$ (see Figure 3.4). Coordinates on $\mathcal{S}$ are $(\phi(\sigma), \pi^\mu(\sigma), \lambda)$, and a point of $\mathcal{S}$ specifies (in those coordinates) a value of λ and field configurations, which can be interpreted as being over the region Σ_λ.

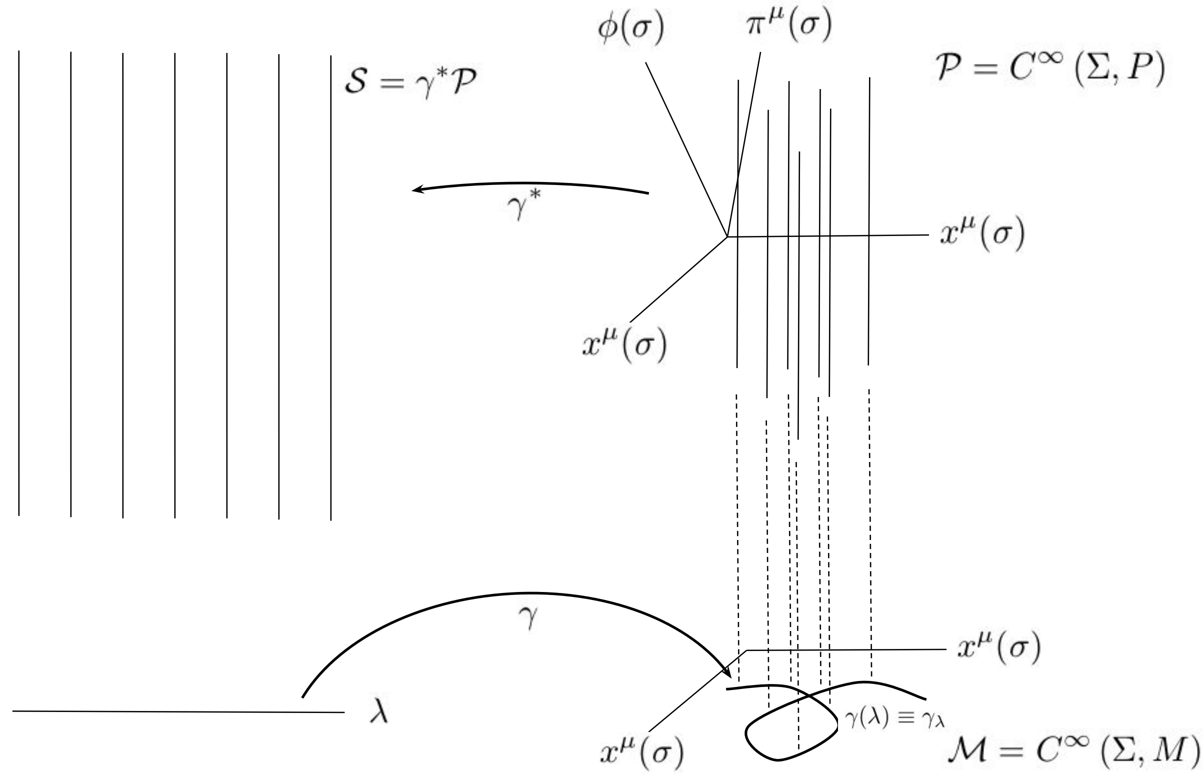

Figure 3.4: Construction of the small boundary phase space. The curve γ maps the real line (bottom left) into $\mathcal{M}$ (bottom right). The part of $\mathcal{P}$ over the image γ_λ (top right) is used to define the pull-back bundle $\mathcal{S}$ over $\mathbb{R}$ (top left).

3.5 Presymplectic Structure of the Large and Small Boundary Phase Spaces

The large boundary phase space inherits its presymplectic 2-form from the multisymplectic 5-form on P through the use of the I-map (see appendix B). We define the 2-form as

$$\Omega \equiv I_5^2(\omega). \tag{3.29}$$

That Ω is closed $\mathrm{d}\Omega = 0$ follows from the properties of the I-map (see (B.7)) and the fact that Σ is closed ($\partial\Sigma = \emptyset$). In fact, Ω is not only closed, but also exact, $\Omega = \mathrm{d}\Theta$, with

$$\Theta \equiv I_4^1(\theta), \tag{3.30}$$

because then it follows (see (B.7)) that (since $\partial\Sigma = \emptyset$)

$$\mathrm{d}\Theta = \mathrm{d}I_4^1(\theta) = I_5^2(\mathrm{d}\theta) = I_5^2(\omega) = \Omega. \tag{3.31}$$

Thus the multisymplectic 5-form ω on P induces the presymplectic 2-form Ω on $\mathcal{P}$ and the multisymplectic potential θ on P induces the presymplectic potential Θ on $\mathcal{P}$.

To be more explicit, consider a map $f \in \mathcal{P}$ with coordinates $(\phi(\sigma), \pi^\mu(\sigma), x^\mu(\sigma))$ and a vector $X \in T_f\mathcal{P}$, written

$$X = \int_\Sigma \mathrm{d}^3\sigma\, \delta\phi \frac{\delta}{\delta\phi} + \delta\pi^\mu \frac{\delta}{\delta\pi^\mu} + \delta x^\mu \frac{\delta}{\delta x^\mu} \tag{3.32}$$

as in (3.24), except here we allow for variations in $x^\mu(\sigma)$ as well. This vector represents an infinitesimal shift of both the image of Σ as well as the fields over the image. We then find the following expression for Θ by inserting X into (B.5):

$$\Theta|_f(X) = \int_\Sigma \left\{\pi^\mu\, \delta\phi\, \mathrm{d}^3x_\mu - \left[\mathcal{H}\, \delta x^\mu\, \mathrm{d}^3x_\mu + \pi^\mu\, \delta x^\nu\, \mathrm{d}\phi \wedge \mathrm{d}^2x_{\mu\nu}\right]\right\}. \tag{3.33}$$

Similarly, for Ω we use two vectors X, Y and distinguish their corresponding variations with subscripts:

$$\begin{aligned}\Omega|_f(X,Y) = \int_\Sigma &\Bigg\{(\delta\pi^\mu)_X(\delta\phi)_Y \mathrm{d}^3x_\mu - (\delta_X\mathcal{H})(\delta x^\mu)_Y\, \mathrm{d}^3x_\mu \\ &+ \left[(\delta\phi)_X(\delta x^\lambda)_Y\, \mathrm{d}\pi^\mu - (\delta\pi^\mu)_X(\delta x^\lambda)_Y\, \mathrm{d}\phi - \frac{1}{2}(\delta x^\mu)_X(\delta x^\lambda)_Y\, \mathrm{d}\mathcal{H}\right] \wedge \mathrm{d}^2x_{\mu\lambda} \\ &+ \left[\frac{1}{2}(\delta x^\lambda)_X(\delta x^\kappa)_Y\, \mathrm{d}\pi^\mu \wedge \mathrm{d}\phi\right] \wedge \mathrm{d}x_{\mu\lambda\kappa}\Bigg\} - \{X \leftrightarrow Y\},\end{aligned} \tag{3.34}$$

where

$$\delta_X\mathcal{H} = \frac{\partial\mathcal{H}}{\partial\phi}(\delta\phi)_X + \frac{\partial\mathcal{H}}{\partial\pi^\mu}(\delta\pi^\mu)_X + \frac{\partial\mathcal{H}}{\partial x^\mu}(\delta x^\mu)_X, \tag{3.35}$$

and similarly for Y. Note that the differentials such as $\mathrm{d}\phi$ and $\mathrm{d}x^\mu$ in these integrals are forms on Σ, so $\mathrm{d}\phi = (\partial\phi/\partial\sigma^i)\mathrm{d}\sigma^i$, $\mathrm{d}x^\mu = (\partial x^\mu/\partial\sigma^i)\mathrm{d}\sigma^i$, and so forth (whereas on P they were coordinate differentials). We have grouped the terms so that the symplectic parts (having the form $p\,\delta q$ for Θ and $\delta p \wedge \delta q$ for Ω) come first. The remaining terms are analogous to the $H\delta t$ and $\delta H \wedge \delta t$ terms in particle mechanics. The remaining term in Θ can be written as $H^\mu{}_\nu\, \delta x^\nu\, \mathrm{d}^3 x_\mu$ where $H^\mu{}_\nu$ is related to the stress-energy tensor of the theory.

This construction on $\mathcal{P}$ can be pulled back to $\mathcal{S}$. In principle, this involves pulling back by the bundle map : $\mathcal{S} \to \mathcal{P}$, but in practice all we need to do is set $\delta x^\mu = N^\mu\, \delta\lambda$, where $N = N^\mu \partial/\partial x^\mu$ is the lapse-shift vector field along γ_λ (see 3.3) connecting nearby surfaces Σ_λ and $\Sigma_{\lambda+\delta\lambda}$. Explicitly, we use the curve $\gamma : \lambda \mapsto \gamma_\lambda$, which defines $\mathcal{S}$, to construct $N \equiv (\gamma_\lambda)_*[\partial/\partial\lambda]$. For a given λ and a point $\sigma \in \Sigma$ we thus have a vector $N^\mu(\sigma,\lambda)\partial/\partial x^\mu \in T_{\gamma_\lambda(\sigma)}M$, which defines the functions $N^\mu(\sigma,\lambda)$. We can thus write Θ on $\mathcal{S}$ as

$$\Theta|_f\,(X) = \int_\Sigma \delta\phi\, \pi^\mu \mathrm{d}^3 x_\mu - \delta\lambda\, \left[\mathrm{d}\phi \wedge \pi^\mu N^\nu \mathrm{d}^2 x_{\mu\nu} + \mathcal{H} N^\mu \mathrm{d}^3 x_\mu\right], \tag{3.36}$$

where the functions $x^\mu(\sigma)$ are now the (λ-dependent) components of $\gamma_\lambda : \Sigma \to M$.

In the special case where each γ_λ is an embedding and furthermore when the family $\{\gamma_\lambda\}$ generates a foliation, we can take the hitherto arbitrary coordinates x^μ and specialize them into coordinates adapted to Σ. These satisfy $x^0 = \lambda$ and $x^i = \sigma^i$ and so $N^\mu = \delta^\mu_0$. Adapted coordinates are common in the literature [16, 30]. In adapted coordinates (3.36) becomes

$$\Theta|_f\,(X) = \int_\Sigma \left[\pi^0\, \delta\phi - \left(-\pi^i \partial_i \phi + \mathcal{H}\right)\, \delta\lambda\right] \mathrm{d}^3\sigma \tag{3.37}$$

The coefficient of $\delta\lambda$ is the canonical Hamiltonian for the flow along N. Thus we see that Θ has the standard structure of the presymplectic potential on an extended phase space:

$$\Theta|_f\,(X) = \int_\Sigma \pi^0\, \delta\phi\, \mathrm{d}^3\sigma - H\, \delta\lambda, \tag{3.38}$$

and similarly for Ω:

$$\Omega|_f\,(X,Y) = \left[\int_\Sigma (\delta\pi^0)_X (\delta\phi)_Y\, \mathrm{d}^3\sigma - (\delta H)_X (\delta\lambda)_Y\right] - [X \leftrightarrow Y], \tag{3.39}$$

with the usual definition of δH (see for example (3.35) or below (3.27)).

This presymplectic structure on $\mathcal{S}$ is similar to the presymplectic (homogeneous) structure presented in section 3.3. However, our small boundary phase space (and the Hamiltonian) has the momenta π^i, which are not present in the usual presymplectic formulations. Typically, we can eliminate these extra momenta to reproduce the standard approach, as we will show in section 3.11. In cases when the π^i cannot be eliminated (as can occur when the image of Σ has both spacelike and null parts), the small boundary phase space provides a cleaner description than traditional presymplectic approaches.

3.6 Multisymplectic Equations of Motion and the Variational Principle

Before discussing the equations of motion on $\mathcal{P}$ and $\mathcal{S}$, we prove our previous claim that (3.7) yields the equations of motion. The proof is simplified by using the I-map (see appendix B), so we first rewrite (3.7) using the I-map and then prove that it is equivalent to the Euler-Lagrange equations for the action defined in (3.1). The I-map then allows us to derive the 3+1-decomposed equations of motion on $\mathcal{P}$ and $\mathcal{S}$ from the covariant equations, (3.7), on P (see section 3.7).

To use the I-map, we need to consider maps from a standard region into P. Previously in sections 3.3-3.5, we have considered a 3-dimensional standard region Σ and maps $f : \Sigma \to P$. In this section, we consider a 4-dimensional standard region or manifold Σ_4 and maps $f : \Sigma_4 \to P$. We associate the submanifold $\bar{R} \subset P$ used in (3.7) with a map $f : \Sigma_4 \to P$, which is the composition of the embedding : $\Sigma_4 \to R$ and the section $\sigma : R \to P$. Thus $\bar{R} = \mathrm{img}(f)$. We refer to such compositions as "sectional" maps.

For a given sectional map f, (3.7) is equivalent to $f^*(i_Y\omega) = 0$, which in turn is equivalent to

$$\int_{\Sigma_4} f^*(i_Y\omega) = 0. \tag{3.40}$$

That is, (3.7) is true for all vector fields Y if and only if (3.40) is true for all vector fields Y. We recognize the left-hand side of (3.40) as $I_4^0(i_Y\omega)\big|_f$ (see (B.5)). Using (B.31a), we commute i_Y past the I-map and write $i_{\bar{Y}} I_5^1(\omega)\big|_f$, where $\bar{Y}$ is the vector field on $C^\infty(\Sigma_4, P)$ corresponding to the vector field Y on P (see discussion above (B.31)). Note that $i_{\bar{Y}} I_5^1(\omega)\big|_f$ depends on the vector field $\bar{Y}$ only through its value at f, which we denote by Y_f for simplicity. The map : $Y \mapsto Y_f$ from the space of vector fields on P to the tangent space $T_f[C^\infty(\Sigma_4, P)]$ is surjective (unlike the map : $Y \to \bar{Y}$ considered in appendix B). This is because a tangent vector Y_f at f corresponds to variations of the fields $(\delta\phi, \delta\pi^\mu, \delta x^\mu)$, which in turn correspond to a vector at each point of $\bar{R}$. Since f is an embedding, this gives a vector field in P defined on $\bar{R}$. The set of all possible vector fields Y on P produces the set of all possible vector fields defined on $\bar{R}$ and thus all of $T_f[C^\infty(\Sigma_4, P)]$. Thus (3.40) is true for all vector fields Y if and only if

$$I_5^1(\omega)\big|_f (Y_f) = 0, \tag{3.41}$$

for all $Y_f \in T_f[C^\infty(\Sigma_4, P)]$. So, for a given sectional map f, (3.41) is true for all vectors Y_f if and only if (3.7) is true for all vector fields Y on P.

We now show that a sectional $f : \Sigma_4 \to P$ satisfies (3.41) for all Y_f, if and only if f represents a solution to the Euler-Lagrange equations. Actually, there are two sets of Euler-Lagrange equations and we need to show equivalence to both. The first set comes from the action defined in (3.1) and the second comes from the lifted action $S = I_4^0(\theta)$, that is

$$S[f] = \int_f \theta, \tag{3.42}$$

where the integration over a map is defined in appendix B. Note that this is a functional on $C^\infty(\Sigma_4, P)$, as opposed to a functional of ϕ as in (3.1).

We start with the lifted action. According to (B.7) we have

$$\mathrm{d}S\big|_f (Y_f) = I_5^1(\omega)\big|_f (Y_f) + (I_\partial)_4^1(\theta)\big|_f (Y_f), \tag{3.43}$$

where Y_f an arbitrary vector in $T_f[C^\infty(\Sigma_4, P)]$ as before. The left-hand side is the variation of S in the direction of Y_f, which we denote $\delta_Y S$. As for the right-hand side, recall Y_f corresponds to field variations $(\delta\phi, \delta\pi^\mu, \delta x^\mu)$. The first term on the right-hand side engages only the values of these variations in the interior of Σ_4, while the second term engages only the values on the boundary $\partial\Sigma_4$. Thus we call these two terms the bulk and boundary contributions to $\delta_Y S$. Equations of the form (3.43) can be found in [5, 32] where they are derived without the aid of the I-map.

We now show that the lifted Euler-Lagrange equations are true if and only if the bulk term, $I_5^1(\omega)\big|_f (Y_f)$, vanishes for all Y_f. Let ρ label points of Σ_4 and write the components of f as $(\phi(\rho), \pi^\mu(\rho), x^\mu(\rho))$, so that

$$S[\phi(\rho), \pi^\mu(\rho), x^\mu(\rho)] = \int\limits_{\Sigma_4} \pi^\mu \mathrm{d}\phi \wedge \mathrm{d}^3 x_\mu - \mathcal{H}(\phi, \pi^\mu, x^\mu)\, \mathrm{d}^4 x. \tag{3.44}$$

The variation of this action is

$$\begin{aligned} \delta S = \int\limits_{\Sigma_4} \delta\pi^\mu & \left(\mathrm{d}\phi \wedge \mathrm{d}^3 x_\mu - \frac{\partial\mathcal{H}}{\partial\pi^\mu}\, \mathrm{d}^4 x \right) + \\ & \delta\phi \left(-\mathrm{d}\pi^\mu \wedge \mathrm{d}^4 x - \frac{\partial\mathcal{H}}{\partial\phi}\, \mathrm{d}^4 x \right) + \\ & \delta x^\mu(\dots) + \int\limits_{\partial\Sigma_4} (\dots). \end{aligned} \tag{3.45}$$

To get this equation, it was necessary to integrate by parts to obtain the coefficients of $\delta\phi$ and δx^μ. The boundary terms coming from these are not explicitly written, but are represented by the second ellipsis of (3.45). The first ellipsis indicates the remaining variation with respect to δx^μ, which we do not write out because the corresponding equations will be a consequence of the others.

We note that Σ_4 is diffeomorphic to R and so functions $\phi(\rho)$ and $\pi^\mu(\rho)$ can be regarded as functions of x^μ instead. From this it follows that $\mathrm{d}\phi \wedge \mathrm{d}^3 x_\mu = \partial_\mu \phi\, \mathrm{d}^4 x$ and $\mathrm{d}\pi^\mu \wedge \mathrm{d}^3 x_\mu = \partial_\mu \pi^\mu\, \mathrm{d}^4 x$, so the bulk part of the variation simplifies to

$$(\delta S)_{\mathrm{bulk}} = \int\limits_{\Sigma_4} \delta\pi^\mu \left(\partial_\mu\phi - \frac{\partial\mathcal{H}}{\partial\pi^\mu} \right) \mathrm{d}^4 x + \delta\phi \left(-\partial_\mu\pi^\mu - \frac{\partial\mathcal{H}}{\partial\phi} \right) \mathrm{d}^4 x + \delta x^\mu(\dots). \tag{3.46}$$

Requiring this to vanish for all variations $(\delta\phi, \delta\pi^\mu, \delta x^\mu)$ yields

$$\partial_\mu\phi = \frac{\partial\mathcal{H}}{\partial\pi^\mu}, \tag{3.47a}$$

$$\partial_\mu\pi^\mu = -\frac{\partial\mathcal{H}}{\partial\phi}, \tag{3.47b}$$

which are exactly the Euler-Lagrange equations of the S in (3.42). Hence we have shown the vanishing of the bulk term in the variation is equivalent to the Euler-Lagrange equations.

Now we show the equivalence of these lifted Euler-Lagrange equations to the original ones. Since $\mathcal{H}$ is obtained from $\mathcal{L}$ via a Legendre transformation (see (3.5)), we can relate their partial derivatives by starting with

$$\mathrm{d}\mathcal{L}(\phi, \partial_\mu \phi, x^\mu) = \frac{\partial \mathcal{L}}{\partial \phi}\,\mathrm{d}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\mathrm{d}(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial x^\mu}\,\mathrm{d}x^\mu. \tag{3.48}$$

Recall from (3.2) that $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \pi^\mu$, and rearrange the second term so that

$$\mathrm{d}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\,\mathrm{d}\phi - (\partial_\mu \phi)\,\mathrm{d}\pi^\mu + \mathrm{d}(\pi^\mu \partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial x^\mu}\,\mathrm{d}x^\mu, \tag{3.49}$$

hence

$$\mathrm{d}(\pi^\mu \partial_\mu \phi - \mathcal{L}) = \mathrm{d}\mathcal{H} = -\frac{\partial \mathcal{L}}{\partial \phi}\,\mathrm{d}\phi + (\partial_\mu \phi)\,\mathrm{d}\pi^\mu - \frac{\partial \mathcal{L}}{\partial x^\mu}\,\mathrm{d}x^\mu. \tag{3.50}$$

Thus $\frac{\partial \mathcal{H}}{\partial \phi} = -\frac{\partial \mathcal{L}}{\partial \phi}$ and $\frac{\partial \mathcal{H}}{\partial \pi^\mu} = \partial_\mu \phi$. Plugging these relations into (3.47) reproduces the equations of motion of the theory (the Euler-Lagrange equations for (3.1)). Thus the vanishing of the bulk term in (3.43) for all variations is equivalent to the lifted Euler-Lagrange equations, which in turn are equivalent to the original equations of motion. Thus we see that (3.41), and hence (3.7), yields the equations of motion.

In this discussion, we have only considered sectional maps. This is an important restriction on f in order to avoid unphysical solutions of (3.41) (and equations derived from it). Non-sectional maps exist in our large and small boundary phase spaces, but for simplicity we exclude such maps when considering equations of motion and their solutions.

3.7 Equations of Motion On the Boundary Phase Spaces

In the previous section, we connected the multisymplectic equations of motion on P to the Euler-Lagrange equations of the theory. We now show how to connect the equations of motion on $\mathcal{P}$ and $\mathcal{S}$ to the multisymplectic equations of motion on P. Recall that $\mathcal{P}$ consists of maps $f : \Sigma \to P$ (see Figure 3.2) from a closed 3-dimensional region Σ ($\partial\Sigma = \emptyset$), while $\mathcal{S} = \gamma^*\mathcal{P}$ is the pull-back bundle with respect to a curve $\gamma : \mathbb{R} \to C^\infty(\Sigma, M)$ (see Figure 3.4). In standard Hamiltonian mechanics, solutions to the equations of motion are geometrically represented by integral curves of the Hamiltonian vector field (see the discussion in 3.3). This is a basic aspect of symplectic geometry. On $\mathcal{P}$ and $\mathcal{S}$, we will find a presymplectic generalization of this geometry of solutions, where the solutions will again be represented by integral curves of certain vector fields, namely those lying in the kernel of the presymplectic 2-form Ω (as defined by (3.34) and (3.39), respectively).

Let $\lambda \mapsto f_\lambda \in \mathcal{P}$ be a curve in $\mathcal{P}$. The curve may be equivalently represented by the map $f : \mathbb{R} \times \Sigma \to P : (\lambda, \sigma) \mapsto f_\lambda(\sigma)$ from a 4-dimensional standard region into P. Note that f_λ is a map from the 3-dimensional region Σ into P, while f is a map from the 4-dimensional region $\mathbb{R} \times \Sigma$ into P. Thus we can apply (3.41) with $\Sigma_4 = \mathbb{R} \times \Sigma$, which tells us that f represents a solution only if $I^1_5(\omega)\big|_f (Y_f)$ vanishes for all $Y_f \in T_f[C^\infty(\Sigma_4, P)]$. Using the definition of the I-map, (B.5), we can write this condition as

$$0 = I^1_5(\omega)\big|_f (Y_f) = \int\limits_{\mathbb{R}\times\Sigma} \omega\left(Y_f, f_*\left(\frac{\partial}{\partial\lambda}\right), f_*\left(\frac{\partial}{\partial\sigma^1}\right), f_*\left(\frac{\partial}{\partial\sigma^2}\right), f_*\left(\frac{\partial}{\partial\sigma^3}\right)\right) \mathrm{d}\lambda \mathrm{d}^3\sigma. \quad (3.51)$$

All five arguments of ω are vector fields along f (maps : $\Sigma_4 \to TP$) evaluated at the point $(\lambda, \sigma) \in \mathbb{R} \times \Sigma = \Sigma_4$. It is this evaluation that creates vectors on P that can be inserted into ω.

Since this has to be true for all Y_f, the integrand must vanish, and therefore any of its integrals must also vanish. In particular we choose to drop the λ-integration, but leave the Σ integral in place,

$$0 = \int\limits_{\Sigma} \omega\left(Y_f, f_*\left(\frac{\partial}{\partial\lambda}\right), f_*\left(\frac{\partial}{\partial\sigma^1}\right), f_*\left(\frac{\partial}{\partial\sigma^2}\right), f_*\left(\frac{\partial}{\partial\sigma^3}\right)\right) \mathrm{d}^3\sigma, \quad (3.52)$$

as it corresponds to the definition of the I-map. Specifically, since λ has a fixed value in (3.52), we can interpret Y_f as a map : $\Sigma \to TP : \sigma \mapsto Y_f(\lambda, \sigma)$ and hence (by appendix A) a vector $Y_\lambda \in T_{f_\lambda}\mathcal{P} = T_{f_\lambda}[C^\infty(\Sigma, P)]$. Similarly, $f_*(\partial/\partial\lambda)$ determines a vector $X_\lambda \in T_{f_\lambda}\mathcal{P}$, which is tangent (at f_λ) to the curve $\lambda' \mapsto f'_\lambda$. Lastly, the push-forwards $f_*(\partial/\partial\sigma^i)$ for a given λ can be written as $(f_\lambda)_*(\partial/\partial\sigma^i)$. Putting this together, we see that (3.52) can be re-written using the I-map as

$$0 = I^2_5(\omega)\big|_{f_\lambda} (Y_\lambda, X_\lambda) = \Omega|_{f_\lambda} (Y_\lambda, X_\lambda). \quad (3.53)$$

for all $Y \in T_{f_\lambda}\mathcal{P}$. Since this has to hold for all $Y_\lambda \in T_{f_\lambda}\mathcal{P}$ and all λ, the curve $\lambda \mapsto f_\lambda$ represents a solution only if its tangent vectors lie in the kernel of Ω. As mentioned at the end of 3.6, the converse is generally not true.

In particular the map $f : \mathbb{R} \times \Sigma \to P$, which is meant to represent a solution over spacetime (or possibly a subset thereof), may not be sectional. This occurs when the map $f_\lambda = f(\lambda, \cdot)$ is itself not sectional, or when f_λ is sectional but the flow $\lambda \mapsto f_\lambda$ results in f not being sectional. Such a flow occurs, for example, when the tangent vector $X_\lambda \in T_{f_\lambda}\mathcal{P}$ is vertical (thus the flow traces out a vertical surface in P, which is not a section) or when X_λ corresponds to vectors $X_\lambda(\sigma)$ that are tangent to the image $f_\lambda(\Sigma) \subset P$ (for some $\sigma \in \Sigma$). We could choose to avoid non-sectional maps f as we did before, but this is too restrictive. Rather, we look for conditions on maps f_λ and flows X_λ such that if (3.53) is satisfied for all Y_λ, then the flow X_λ determines a solution.

When $f_\lambda : \Sigma \to P$ is not sectional, it does not represent field configurations over some initial value surface, so henceforth we restrict our attention to maps in $\mathcal{P}$ that are sectional. We also need to avoid regions of $\mathcal{P}$ where the only possible flows $X_\lambda \in \ker\Omega$ change the fields without changing the initial value surface. This includes X_λ that are vertical or tangent to the image of $f_\lambda \in \mathcal{P}$. It is impossible to generate a 4-dimensional solution f starting from such $f_\lambda \in \mathcal{P}$. Lastly, we need to be careful with flows $X_\lambda \in \ker\Omega$ that only partially flow the image of f_λ vertically or are tangent to only part of the image of f_λ. Such flows may represent actual physical solutions or they may represent flows satisfying (3.53) that are not physical solutions. When dealing with such flows, it is usually clear how to choose one that really is a physical solution, so we will not discuss this further.

Let us call a vector at f_λ that is (completely) non-vertical and nowhere tangent to the image of f_λ "admissible". We need to make sure we are working on the subspace of $\mathcal{P}$ where there exists at least one admissible vector that lies in the kernel of Ω. In general, admissible vectors that lie in the kernel of Ω may only exist on a subset $C \subset \mathcal{P}$. As stated earlier, we take C to consist only of sectional maps. On C, there might only be a subset $C' \subset C$ where we can find admissible vectors in the kernel of Ω that are also tangent to C. For the rest of C, we only find admissible vectors in the kernel that take us off C. If we were to follow such flows, we would only be able to do so for an infinitesimal amount of time (once we leave C, there are no more admissible vectors to follow). We do not want this from our flows, so we can only start our flows from C'. On C', there might again be only a subset $C'' \subset C'$ where we can find admissible vectors in the kernel of Ω that are also tangent to C'. This further restricts our choice of starting points for our flows. This process continues until we can find some final $\tilde{C} \subset \mathcal{P}$ such that at each point of $\tilde{C}$ we can find an admissible vector in the kernel of Ω that is tangent to $\tilde{C}$. Such an algorithm, which produces a series of "constraint" submanifolds, is the homogeneous analog of the presymplectic geometric constraint algorithm of [18].

We now consider dynamics on $\mathcal{S}$. A curve on $\mathcal{S}$ is a solution to the Euler-Lagrange equations if the corresponding curve on $\mathcal{P}$ (the image under the bundle map) is as well. A vector X tangent to such a curve on $\mathcal{S}$ thus has the property that its push-forward is in the kernel of Ω on $\mathcal{P}$, and so X is in the kernel of Ω on $\mathcal{S}$. Thus, just as on $\mathcal{P}$, a curve on $\mathcal{S}$ is a solution to the Euler-Lagrange equations only if its tangent vector lies in the kernel of Ω (on $\mathcal{S}$). To use the converse of this, we need to again restrict attention to sectional maps $f_\lambda \in \mathcal{S}$ and follow the constraint algorithm outlined above to identify a subspace $\tilde{C} \subset \mathcal{S}$. It is possible to choose $\mathcal{S}$ such that these requirements cannot be satisfied, in which case we need to pick a different $\mathcal{S}$ to look at the dynamics.

3.8 Momentum Maps on the Large and Small Boundary Phase Spaces

We now show how group actions and momentum maps on P (see section 3.2) induce group actions and momentum maps on $\mathcal{P}$ and $\mathcal{S}$. Suppose we have a group action on P as in (3.9). This induces the group action $\tilde{\Phi} : G \to \mathrm{Diff}(\mathcal{P}) : g \mapsto \tilde{\Phi}_g$, where

$$\tilde{\Phi}_g : \mathcal{P} \to \mathcal{P} \tag{3.54a}$$

$$: f \mapsto \Phi_g \circ f. \tag{3.54b}$$

The composition is well-defined since f maps Σ to P and Φ_g maps P to P. This is a left action, which follows from Φ being a left action. Furthermore, if the group action Φ on P leaves ω invariant ($\Phi_g^* \omega = \omega$), then $\tilde{\Phi}_g$ leaves $\Omega = I_5^2 \omega$ invariant ($\tilde{\Phi}_g^* \Omega = \Omega$).

To see this, consider a point $f \in \mathcal{P}$ and two vectors $\tilde{X}, \tilde{Y} \in T_f \mathcal{P}$. From the definition of the pull-back, $\left(\tilde{\Phi}_g^* \Omega\right)\Big|_f (\tilde{X}, \tilde{Y}) = \Omega|_{\tilde{\Phi}_g(f)} (\tilde{\Phi}_{g*} \tilde{X}, \tilde{\Phi}_{g*} \tilde{Y})$. We now want to use the definition of the I-map (B.5), but first we need to see how to represent $\tilde{\Phi}_{g*}\tilde{X}, \tilde{\Phi}_{g*}\tilde{Y}$ as a vector field along f. Let $[f_\lambda]$ be the equivalence class of curves $\tilde{X}$ and $X_\sigma \equiv [f_\lambda(\sigma)]$ denote the corresponding vector field along f. Then $\tilde{\Phi}_{g*}\tilde{X}$ is the equivalence class of curves $[\tilde{\Phi}_g(f_\lambda)] = [\Phi_g \circ f_\lambda]$, but for a given $\sigma \in \Sigma$ this is the equivalence class $[\Phi_g \circ f_\lambda(\sigma)] = \Phi_{g*}X_\sigma$. Thus $\tilde{\Phi}_{g*}\tilde{X}$ corresponds to $\Phi_{g*}X_\sigma$ (similarly for $\tilde{Y}$), and by the definition of the I-map we have

$$\begin{aligned}
&\Omega|_{\tilde{\Phi}_g(f)}(\tilde{\Phi}_{g*}\tilde{X}, \tilde{\Phi}_{g*}\tilde{Y}) \\
&= \int_\Sigma \omega\left(\Phi_{g*}X_\sigma, \Phi_{g*}Y_\sigma, \Phi_{g*}f_*\left(\frac{\partial}{\partial\sigma^1}\right), \Phi_{g*}f_*\left(\frac{\partial}{\partial\sigma^2}\right), \Phi_{g*}f_*\left(\frac{\partial}{\partial\sigma^3}\right)\right) \mathrm{d}^3\sigma, \quad (3.55\text{a}) \\
&= \int_\Sigma (\Phi_g^*\omega)\left(X_\sigma, Y_\sigma, f_*\left(\frac{\partial}{\partial\sigma^1}\right), f_*\left(\frac{\partial}{\partial\sigma^2}\right), f_*\left(\frac{\partial}{\partial\sigma^3}\right)\right) \mathrm{d}^3\sigma, \quad (3.55\text{b}) \\
&= \int_\Sigma \omega\left(X_\sigma, Y_\sigma, f_*\left(\frac{\partial}{\partial\sigma^1}\right), f_*\left(\frac{\partial}{\partial\sigma^2}\right), f_*\left(\frac{\partial}{\partial\sigma^3}\right)\right) \mathrm{d}^3\sigma = \Omega|_f(X, Y), \quad (3.55\text{c})
\end{aligned}$$

where we used $\tilde{\Phi}_g(f) = \Phi_g \circ f$ and $(\Phi_g \circ f)_* = \Phi_{g*}f_*$ in (3.55a) and $\Phi_g^*\omega = \omega$ to go from (3.55b) to (3.55c). This completes the proof.

Since we have a group action with $\tilde{\Phi}_g\Omega = \Omega$, we expect a momentum map on $\mathcal{P}$. This momentum map is closely related to the one on P. To show this, we first discuss the relationship between the induced vector fields of Φ and $\tilde{\Phi}$. Let X_ξ be an induced vector field of Φ. Since X_ξ is a vector field on P, it produces a vector field $\bar{X}_\xi$ on $\mathcal{P}$ as discussed in appendix B. Specifically, viewing all vector fields as maps from the manifold into its tangent bundle, we have $\bar{X}_\xi(f) = X_\xi \circ f$ for all $f \in \mathcal{P}$ and where $X_\xi \circ f : \Sigma \to TP$ is regarded as an element of $T_f\mathcal{P}$. Let $\tilde{X}_\xi$ be an induced vector field of $\tilde{\Phi}$. From the definition of induced vector fields, we have

$$\tilde{X}_\xi F = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{\Phi}^*_{\exp(t\xi)}F, \tag{3.56}$$

for any functional F on $\mathcal{P}$. Evaluating both sides at $f \in \mathcal{P}$ and setting $F[f] = \phi \circ f$, where $\phi : P \to P$ is any smooth map, we have

$$\begin{aligned}
\tilde{X}_\xi F[f] &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F[\tilde{\Phi}_{\exp(t\xi)}(f)] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F[\Phi_{\exp(t\xi)} \circ f] \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi \circ \Phi_{\exp(t\xi)} \circ f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi^*_{\exp(t\xi)}\phi) \circ f = X_\xi\phi \circ f.
\end{aligned} \tag{3.57}$$

Since this is true for all ϕ, we have $\tilde{X}|_f = X_\xi \circ f = \bar{X}|_f$. Thus the induced vector field $\tilde{X}_\xi$ on $\mathcal{P}$ and the vector field $\bar{X}_\xi$ produced by the induced vector field X_ξ are the same ($\bar{X}_\xi = \tilde{X}_\xi$). This lets us use the results of appendix B to relate the momentum maps on $\mathcal{P}$ and P. From now on we drop all tildes and bars for clarity.

The momentum map equation on P is

$$i_{X_\xi}\omega = -\mathrm{d}J_\xi. \tag{3.58}$$

Applying I_4^1 to both sides and commuting the I-map past the interior and exterior derivatives gives

$$i_{X_\xi}\Omega = -\mathrm{d}\mathcal{J}_\xi, \tag{3.59}$$

where $\Omega = I_5^2(\omega)$ and $\mathcal{J}_\xi = I_3^0(J_\xi)$. Equation (3.59) refers to $\mathcal{P}$, and in particular X_ξ is the induced vector field on $\mathcal{P}$ of the lifted group action $\tilde{\Phi}$. Thus, given a momentum map J_ξ on P, the 3+1 momentum map on $\mathcal{P}$ is

$$\mathcal{J}_\xi|_f = \int_f J_\xi \equiv \int_\Sigma f^* J_\xi. \tag{3.60}$$

Equation (3.59) can be pulled back to the small boundary phase space $\mathcal{S}$, as it is just a 1-form equation. This will define a function $\mathcal{J}_\xi$ on $\mathcal{S}$. However, in general the induced vector field X_ξ (on $\mathcal{P}$) may not be pulled back, and hence (3.59) will lose its meaning as a momentum map equation and $\mathcal{J}_\xi$ on $\mathcal{S}$ will not correspond to a momentum map. This occurs when the symmetry generated by X_ξ involves a spacetime symmetry that is incompatible with the curve $\gamma : \mathbb{R} \to \mathcal{M}$ defining $\mathcal{S}$. By incompatible we mean that the projected vector field $\pi_* X_\xi$ (on $\mathcal{M}$) is not everywhere tangent to γ.

We can use (3.20) to give an explicit expression for $\mathcal{J}_\xi$, which will be exactly the explicit form of presymplectic momentum maps, as in (2.20). Before applying I_3^0 to both sides of (3.20), recall that $\mathrm{d}\beta_\xi$ is just the local expression of a closed 3-form we will call γ_ξ. Thus we apply I_3^0 to both sides of $J_\xi = i_{X_\xi}\theta - \alpha_\xi + \gamma_\xi$ to get

$$\mathcal{J}_\xi = i_{X_\xi}\Theta - A_\xi + C_\xi, \tag{3.61}$$

where $A_\xi = I_3^0 \alpha_\xi$ and C_ξ is a closed 0-form (a constant, though it may depend on ξ) since $\mathrm{d}C_\xi = \mathrm{d}I_3^0\gamma_\xi = I_4^1(\mathrm{d}\gamma_\xi) = 0$. Furthermore, α_ξ was determined from $\pounds_{X_\xi}\theta = \mathrm{d}\alpha_\xi$ to which we can apply I_4^1, giving $\pounds_{X_\xi}\Theta = \mathrm{d}A_\xi$. Thus A_ξ is determined as expected for (3.61) to match (2.20). For point transformations $\alpha_\xi = 0 \implies A_\xi = 0$, in which case (3.61) matches (2.21).

3.9 Phase Spaces in the Presence of Boundaries

Now we consider the case when Σ has a boundary ($\partial\Sigma \neq \emptyset$). This includes (through boundary compactification) the case of non-compact Σ with fields that do not go to zero "at infinity". We again consider the space $C^\infty(\Sigma, P)$, but now this space does not carry the correct presymplectic structure, as we shall explain. Let us use the notation $\bar{\mathcal{P}} \equiv C^\infty(\Sigma, P)$. The bar reminds us that this is not a phase space due to problems with the presymplectic structure.

To see the problem, consider again the forms $\Omega = I_5^2(\omega) \in \Omega^2(\bar{\mathcal{P}})$ and $\Theta = I_4^1(\theta) \in \Omega^1(\bar{\mathcal{P}})$. These are well-defined whether or not Σ has a boundary. But notice that using (B.7), we have

$$\mathrm{d}\Omega = \mathrm{d}I_5^2(\omega) = I_6^3(\mathrm{d}\omega) + (I_\partial)_5^3(\omega) = (I_\partial)_5^3(\omega) \neq 0, \tag{3.62}$$

so Ω is not closed and is thus not a presymplectic 2-form. Of course, $\mathrm{d}\Theta$ is still a valid presymplectic 2-form (note $\mathrm{d}\Theta \neq \Omega$ when Σ has a boundary), but there is still a problem. The derivation of the equations of motion in section 3.7 does not depend on Σ being closed and thus remains unchanged. In particular, tangent vectors to solution curves lie in the kernel of Ω. Since $\mathrm{d}\Omega \neq 0$, the kernel of Ω is not an integrable distribution, which poses problems for interpreting the system in terms of presymplectic Hamiltonian dynamics. The presence of an alternative presymplectic structure $\mathrm{d}\Theta$ does not solve the problem because its kernel has no relation to solutions.

To proceed, consider the pull-back bundle $\bar{\mathcal{S}} \equiv \gamma^* \bar{\mathcal{P}}$ constructed by the same method as the small boundary phase space using a curve γ. The bar again reminds us that $\bar{\mathcal{S}}$ is not a true phase space. $\bar{S}$ carries the 2-form Ω whose definition (3.39) is unchanged by the presence of a boundary. As above, we have $\mathrm{d}\Omega \neq 0$. We look for a subspace $\mathcal{S} \subset \bar{\mathcal{S}}$ on which Ω is closed. Explicitly, consider a map $f \in \bar{\mathcal{S}}$ with coordinates $(\phi(\sigma), \pi^\mu(\sigma), \lambda)$. Let $X, Y, Z \in T_f \bar{\mathcal{S}}$ be three vectors at f with coordinate representation similar to (3.32). We have

$$
\begin{aligned}
\mathrm{d}\Omega|_f (X, Y, Z) &= \int_{\partial f} \omega(X, Y, Z, \cdot) \\
&= \int_{\partial\Sigma} (\delta\pi^\mu)_X \, (\delta\phi)_Y \, (\delta\lambda)_Z \, N^\nu \, \mathrm{d}^2 x_{\mu\nu} - (\delta_X \mathcal{H}) \, (\delta\lambda)_Y \, (\delta\lambda)_Z \, N^\mu \, N^\nu \, \mathrm{d}^2 x_{\mu\nu} \\
&- \int_{\partial\Sigma} (\delta\pi^\mu)_Y \, (\delta\phi)_X \, (\delta\lambda)_Z \, N^\nu \, \mathrm{d}^2 x_{\mu\nu} - (\delta_Y \mathcal{H}) \, (\delta\lambda)_X \, (\delta\lambda)_Z \, N^\mu \, N^\nu \, \mathrm{d}^2 x_{\mu\nu} \\
&+ (\text{cyclic}\, X \to Y \to Z),
\end{aligned}
\tag{3.63}
$$

where we have used the definition of the (I_∂)-map in the first line. The antisymmetry in μ, ν of $\mathrm{d}^2 x_{\mu\nu}$ causes the second terms in the integrals to vanish. The sum of cyclic (and anti-cyclic) permutations of X, Y, Z is the same that appears in a wedge product of three 1-forms, so let us define the suggestive notation

$$
(\delta\pi^\mu \curlywedge \delta\phi \curlywedge \delta\lambda)(X, Y, Z) \equiv (\delta\pi^\mu)_X \, (\delta\phi)_Y \, (\delta\lambda)_Z - (\delta\pi^\mu)_Y \, (\delta\phi)_X \, (\delta\lambda)_Z + (\text{cyclic}\, X \to Y \to Z),
\tag{3.64}
$$

which lets us write

$$
\mathrm{d}\Omega|_f = \int_{\partial\Sigma} (\delta\pi^\mu \curlywedge \delta\phi \curlywedge \delta\lambda) \, N^\nu \, \mathrm{d}^2 x_{\mu\nu}.
\tag{3.65}
$$

We now switch to adapted coordinates (see section 3.5) where $N^\nu = \delta_0^\nu$ and $x^i = \sigma^i$. In adapted coordinates,

$$
\mathrm{d}\Omega|_f = \int_{\partial\Sigma} (\delta\pi^i \curlywedge \delta\phi \curlywedge \delta\lambda) \, \mathrm{d}^2 \sigma_i.
\tag{3.66}
$$

We can further adapt the coordinates σ^i to the boundary $\partial\Sigma$ so that the boundary is given by $\sigma^3 = \text{const}$. Then we find

$$
\mathrm{d}\Omega|_f = \int_{\partial\Sigma} (\delta\pi^3 \curlywedge \delta\phi \curlywedge \delta\lambda) \mathrm{d}^2 \sigma,
\tag{3.67}
$$

where we see that only the component of π^i transverse to $\partial\Sigma$ remains. We can now select a presymplectic subspace $\mathcal{S}$ by requiring the restricted 2-form $\Omega|_{\mathcal{S}}$ be closed,

$$
\mathrm{d}\Omega|_{\mathcal{S}} = 0 \implies \int_{\partial\Sigma} (\delta\pi^i \curlywedge \delta\phi \curlywedge \delta\lambda) \, \mathrm{d}^2 \sigma_i = 0 \quad \forall \, (\phi(\sigma), \pi^\mu(\sigma), \lambda) \in \mathcal{S},
\tag{3.68}
$$

and similarly in coordinates adapted to $\partial\Sigma$.

For example, we can select the space $\mathcal{S} = \{(\phi(\sigma), \pi^\mu, \lambda) | \left.\phi\right|_{\partial\Sigma} = \phi_0\}$ where $\phi_0 \in \mathbb{R}$ is a fixed real number. A tangent vector to $\mathcal{S}$ has $\delta\phi = 0$, and thus satisfies (3.68). More generally, we can let $\phi_0 : \partial\Sigma \to \mathbb{R}$ be any fixed function on the boundary (since $\left.\delta\phi\right|_{\partial\Sigma} = 0$ when the function $\phi(\partial\Sigma)$ remain unchanged). This is still not the most general boundary condition we can impose on ϕ to satisfy (3.68). Suppose we specify a function $\phi_0 : \partial\Sigma \times \mathbb{R} \to \mathbb{R} : (\sigma^1, \sigma^2, \lambda) \mapsto \phi_0(\sigma^1, \sigma^2, \lambda)$. Then $\left.\delta\phi\right|_{\partial\Sigma} = (\partial\phi_0/\partial\lambda)\delta\lambda$. Plugging this in, we see (3.68) is still satisfied due to the antisymmetry of the wedge product. We thus see that fairly general Dirichlet-type boundary conditions (conditions on the field ϕ) satisfy the requirement (3.68). We could have constant boundary conditions, fixed non-constant boundary conditions, or even "time-dependent" boundary conditions. Each possible choice of ϕ on the boundary yields a different phase space $\mathcal{S}$. Within each $\mathcal{S}$, ϕ is allowed to take on any value in the interior of Σ (as long as ϕ remains a smooth map), while on the boundary ϕ is set to a fixed value (aside from possible λ-dependence).

According to (3.68), we could equally well choose boundary conditions for π^3 (in coordinates adapted to $\partial\Sigma$). In section 3.11, we show that in simple cases, this is related to a condition on the normal derivative of ϕ. As with the Dirichlet-type conditions, these Neumann-type conditions can be quite general. We could also pick various mixed boundary conditions (part Dirichlet, part Neumann) to satisfy (3.68). The only restriction is that we want the resulting $\mathcal{S}$ to support at least one solution. If we pick boundary conditions that over-determine the system, $\mathcal{S}$ may end up having no solutions.

In the literature, boundary conditions are typically determined either by making the variational principle well-defined (see [33] for the physics view point, [34, 35] for the modern symplectic geometry view point) or by making the functional derivative of the Hamiltonian well-defined (see [10, 24, 36, 37] and references therein). These methods often lead to considering Dirichlet, Neumann, or mixed boundary conditions. But condition (3.68) makes clear that there exists a broad class of boundary conditions that, to our knowledge, is not discussed in the literature.

To understand this class, we use coordinates adapted to the boundary, so that the boundary has coordinates $(\sigma^1, \sigma^2, \sigma^3 = 0)$. Condition (3.68) is satisfied if we introduce a (point-wise) linear dependence among $\delta\phi, \delta\pi^3, \delta\lambda$ on the boundary. We do this by specifying an arbitrary function $f : \mathbb{R}^5 \to \mathbb{R}$ and using it to construct the level set (in $\bar{\mathcal{S}}$) $f(\phi(\sigma^1, \sigma^2, 0), \pi^3(\sigma^1, \sigma^2, 0), \sigma^1, \sigma^2, \lambda) = 0$. On this level set we have $0 = \delta f = (\partial f/\partial\phi)\,\delta\phi + (\partial f/\partial\pi^3)\,\delta\pi^3 + (\partial f/\partial\lambda)\,\delta\lambda$ and the linear dependence among the forms causes the expression in (3.68) to vanish. Different boundary conditions can be obtained with different choices of f. For the inhomogeneous case (no λ parameter) the class of boundary conditions obtained by different choices of f fills out a Lagrangian submanifold with respect to the symplectic form $\int_{\partial\Sigma} \delta\phi \curlywedge \delta\pi^3$ (see [34, 35] and references therein). Our class of boundary conditions can thus be thought to lie on a homogeneous extension of such a Lagrangian manifold.

We give some examples of how to choose f to generate boundary conditions. Choosing $f = \phi(\sigma) - \phi_0(\sigma, \lambda)$ gives general Dirichlet conditions, $f = \pi^3(\sigma) - \pi^3_0(\sigma, \lambda)$ gives Neumann, $f = g(\sigma, \lambda)\,\phi(\sigma) + h(\sigma, \lambda)\,\pi^3(\sigma) - F(\sigma, \lambda)$ with g, h having disjoint support on $\partial\Sigma$ gives mixed boundary conditions (a common support gives Robin boundary conditions), while most other choices of f give genuinely new boundary conditions.

In addition to clarifying the allowed boundary conditions, (3.68) sheds new light on the connection between boundary terms, presymplectic geometry, and multisymplectic geometry (see also [24] for an alternative discussion of boundary terms in a multisymplectic context).

From now on, we assume that a choice of boundary conditions satisfying (3.68) has been made, selecting a presymplectic small boundary phase space $\mathcal{S}$. We will look at what other modifications to presymplectic geometry are necessary in the presence of boundaries, starting with the presymplectic 1-form. Since $\mathrm{d}\Omega = 0$, we can locally find $\tilde{\Theta}$ such that $\Omega = \mathrm{d}\tilde{\Theta}$. We find $\tilde{\Theta}$ simply by applying I_5^2 to $\omega = \mathrm{d}\theta$,

$$\Omega = I_5^2(\omega) = I_5^2(\mathrm{d}\theta) = \mathrm{d}\Theta + (I_\partial)_4^2(\theta). \tag{3.69}$$

Since $\mathrm{d}\Omega = 0$, there must locally exist a 1-form Ψ such that $\mathrm{d}\Psi = (I_\partial)_4^2(\theta)$, so that $\tilde{\Theta} = \Theta + \Psi \implies \Omega = \mathrm{d}(\Theta + \Psi)$. The (relevant) presymplectic 1-form is thus not simply Θ, but has an additional boundary term Ψ. We find Ψ by solving

$$\mathrm{d}\Psi|_f (X,Y) = \int\limits_{\partial\Sigma} \pi^\mu \, (\delta\phi)_X \, (\delta\lambda)_Y \, N^\nu \, \mathrm{d}^2 x_{\mu\nu} - \mathcal{H} \, (\delta\lambda)_X \, (\delta\lambda)_Y \, N^\mu N^\nu \, \mathrm{d}^2 x_{\mu\nu} - (X \leftrightarrow Y), \tag{3.70}$$

which can be simplified to

$$\mathrm{d}\Psi|_f = \int\limits_{\partial\Sigma} \pi^i \, (\delta\phi \curlywedge \delta\lambda) \, \mathrm{d}^2\sigma_i, \tag{3.71}$$

in adapted coordinates and using definition (3.64).

We see that for any Dirichlet boundary conditions, $\Psi = 0$ and no modification to Θ is necessary. Actually, we could choose to use any Ψ such that $\mathrm{d}\Psi = 0$ in this case, but this is just the usual ambiguity in defining the 1-form Θ.

For Neumann boundary conditions, we find

$$\Psi|_f (X) = \left(\int\limits_{\partial\Sigma} \pi^i \phi \, \mathrm{d}^2\sigma_i \right) (\delta\lambda)_X. \tag{3.72}$$

The easiest way to see this is to treat the δ's in (3.71) as exterior derivatives and substitute the product rule $\delta(\pi^i\phi \, \mathrm{d}^2\sigma_i) \curlywedge \delta\lambda = \pi^i \, (\delta\phi \curlywedge \delta\lambda) \, \mathrm{d}^2\sigma_i$, where the term proportional to $\delta\pi^i$ vanishes due to the boundary conditions. Notice that (3.72) is equivalent to a modification of the Hamiltonian,

$$\tilde{\Theta}\Big|_f (X) = \int\limits_\Sigma \pi^0 (\delta\phi)_X \, \mathrm{d}^3\sigma - H(\delta\lambda)_X + \left(\int\limits_{\partial\Sigma} \pi^i \phi \, \mathrm{d}^2\sigma_i \right) (\delta\lambda)_X = \int\limits_\Sigma \pi^0 (\delta\phi)_X \, \mathrm{d}^3\sigma - \tilde{H}(\delta\lambda)_X, \tag{3.73}$$

with

$$\tilde{H} = H - \int\limits_{\partial\Sigma} \pi^i \phi \, \mathrm{d}^2\sigma_i. \tag{3.74}$$

This is the standard addition of a boundary term to the Hamiltonian that occurs under Neumann boundary conditions [10, 16, 24, 33, 36, 37].

For the general class of boundary conditions specified by $f(\phi(\sigma^3 = 0), \pi^3(\sigma^3 = 0), \sigma^1, \sigma^2, \lambda) = 0$, Ψ is obtained by performing the integral

$$\Psi|_f(X) = \left(\int\limits_{\partial\Sigma} \int\limits_{\phi'=0}^{\phi'=\phi} \pi^3(\phi', \sigma, \lambda)\, \mathrm{d}\phi'\, \mathrm{d}^2\sigma \right) (\delta\lambda)_X, \tag{3.75}$$

where $\pi^3(\phi, \sigma, \lambda)$ is a solution to $f = 0$. The corresponding modified Hamiltonian is

$$\tilde{H} = H - \int\limits_{\partial\Sigma} \int\limits_{\phi'=0}^{\phi'=\phi} \pi^3(\phi', \sigma, \lambda)\, \mathrm{d}\phi'\, \mathrm{d}^2\sigma. \tag{3.76}$$

3.10 Boundary Terms of Momentum Maps, the Action, and Hamilton's Principal Function

We now look at what happens to momentum maps in the presence of boundaries. The arguments of section 3.8 remain unchanged until (3.58). However, in the presence of boundaries we will be working directly with the small boundary phase space $\mathcal{S}$, bypassing the large boundary phase space. In order to have true momentum maps, we only consider symmetries that are compatible with $\mathcal{S}$, as mentioned at the end of section 3.8 (in particular, the symmetry must preserve the boundary conditions defining $\mathcal{S}$).

For any $\xi \in \mathfrak{g}$ compatible with $\mathcal{S}$, there is an induced vector field X_ξ on $\mathcal{S}$. To see this, consider the curve $\gamma : \mathbb{R} \to \mathcal{M}$ responsible for $\mathcal{S}$ and the natural bundle map $F : \mathcal{S} \to F(\mathcal{S}) \subset \bar{\mathcal{P}}$ coming from the construction of the pull-back bundle. For ξ compatible with $\mathcal{S}$, X_ξ at $F(\mathcal{S})$ is tangent to $F(\mathcal{S})$ and so can be pulled back to $\mathcal{S}$ in a natural way to produce an induced vector field X_ξ on $\mathcal{S}$. For such compatible symmetries we also have $\tilde{\Phi}^*_{\exp(t\xi)}\Omega = \Omega \implies \pounds_{X_\xi}\Omega = 0$ on $\mathcal{S}$. Since $\mathrm{d}\Omega = 0$ on $\mathcal{S}$, we end up with the usual momentum map equation

$$i_{X_\xi}\Omega = -\mathrm{d}\tilde{\mathcal{J}}_\xi, \tag{3.77}$$

for some function $\tilde{\mathcal{J}}_\xi$. We would like to express $\tilde{\mathcal{J}}_\xi$ in terms of J_ξ as we did in section 3.8. Starting with (3.58), we again apply I^1_4 to both sides, but in the presence of a boundary the result is

$$i_{X_\xi}\Omega = -dI^0_3(J_\xi) + (I_\partial)^1_3(J_\xi). \tag{3.78}$$

From (3.77) we know that the right hand side of (3.78) is (locally) exact, and so there must exist a function $\mathcal{K}_\xi$ that makes the second term exact,

$$\mathrm{d}\mathcal{K}_\xi = (I_\partial)^1_3(J_\xi). \tag{3.79}$$

In the presence of a boundary, we thus have the modified momentum map $\tilde{\mathcal{J}}_\xi = \mathcal{J}_\xi - \mathcal{K}_\xi$, where $\mathcal{J}_\xi \equiv I^0_3(J_\xi)$ was the 3+1 momentum map of section 3.8.

We now look for an equation of the form (3.61) in the presence of boundaries. The most direct method is to proceed as in the (pre)symplectic case (2.20). Since $\pounds_{X_\xi}\Omega = 0$ and

$\Omega = \mathrm{d}\tilde{\Theta}$, we have $\mathrm{d}\pounds_\xi \tilde{\Theta} = 0 \implies \pounds_{X_\xi} \tilde{\Theta} = \mathrm{d}\tilde{A}_\xi$ for some function $\tilde{A}_\xi : \mathcal{S} \to \mathbb{R}$. Then (3.77) becomes

$$\mathrm{d}\tilde{\mathcal{J}}_\xi = -i_{X_\xi}\Omega = -i_{X_\xi}\mathrm{d}\tilde{\Theta} = -\pounds_{X_\xi}\tilde{\Theta} + \mathrm{d}i_{X_\xi}\tilde{\Theta} = -\mathrm{d}\tilde{A}_\xi + \mathrm{d}i_{X_\xi}\tilde{\Theta}. \tag{3.80}$$

Solving for the momentum map, we get

$$\tilde{\mathcal{J}}_\xi = i_{X_\xi}\tilde{\Theta} - \tilde{A}_\xi + \tilde{C}_\xi, \tag{3.81}$$

where $\tilde{C}_\xi$ is a (possibly ξ-dependent) constant. Note how the modified momentum map $\tilde{\mathcal{J}}_\xi$ uses the modified 1-form $\tilde{\Theta}$. This shows that we do not need to calculate the correction $\mathcal{K}_\xi$ once we know the correction Ψ from solving (3.71), though now we do need to compute $\tilde{A}_\xi$.

We look to relate $\tilde{A}_\xi$ to α_ξ from (3.20) as we did in the absence of boundaries. It follows from the definition of $\tilde{A}_\xi$ that

$$\mathrm{d}\tilde{A}_\xi = \pounds_{X_\xi}(\Theta + \Psi) = \pounds_{X_\xi} I^1_4(\theta) + \pounds_{X_\xi}\Psi = I^1_4(\mathrm{d}\alpha_\xi) + \pounds_{X_\xi}\Psi, \tag{3.82}$$

where we used $\pounds_{X_\xi}\theta = \mathrm{d}\alpha_\xi$ after commuting the I-map past the Lie derivative. Commuting the I-map back past the exterior derivative and rearranging gives

$$\pounds_{X_\xi}\Psi - (I_\partial)^1_3(\alpha_\xi) = \mathrm{d}\left(\tilde{A}_\xi - I^0_3(\alpha_\xi)\right) \equiv \mathrm{d}V_\xi. \tag{3.83}$$

Thus (3.81) becomes

$$\tilde{\mathcal{J}}_\xi = i_{X_\xi}\tilde{\Theta} - I^0_3(\alpha_\xi) + V_\xi + C_\xi. \tag{3.84}$$

Comparing with (3.61), we see that V_ξ is an additional boundary modification to the momentum map. This has important ramifications for point transformations (see below).

We now have two ways to compute the modified momentum map $\tilde{\mathcal{J}}_\xi$. We can either solve (3.79) for $\mathcal{K}_\xi$ and substitute into $\tilde{\mathcal{J}}_\xi = \mathcal{J}_\xi - \mathcal{K}_\xi$ or we can solve (3.83) for V_ξ and substitute into (3.84) (assuming all other quantities in (3.84) have already been computed).

Now suppose we have a point transformation on the multisymplectic phase space P, so that $\alpha_\xi = 0$. In the presence of boundaries we end up with

$$\tilde{\mathcal{J}}_\xi = i_{X_\xi}\tilde{\Theta} + V_\xi + C_\xi, \tag{3.85}$$

which is not the expected form for a point transformation (see (2.21)). We can see the reason for this discrepancy by noting the equation for V_ξ in this case: $\mathrm{d}V_\xi = \pounds_{X_\xi}\Psi$. The modification V_ξ to the momentum map occurs when the point transformation fails to preserve the boundary modification Ψ. The modification V_ξ tends to lead to central extensions of the Poisson algebra (see [38] and references therein for examples).

We now look at the boundary terms in the action and in Hamilton's principal function. Recall from (3.43) that the variation of the nominal action $S = I^0_4(\theta)$ is

$$\mathrm{d}S\big|_f(Y) = I^1_5(\omega)\big|_f(Y) + (I_\partial)^1_4(\theta)\big|_f(Y), \tag{3.86}$$

where $f : \Sigma_4 \to P$ is a map from a 4-dimensional standard region. We can choose Σ_4 such that $\Sigma_4 = \Sigma \times \mathbb{R}$ and define $f_\lambda : \Sigma \to P : \sigma \mapsto f(\sigma, \lambda)$. We can then use the methods of section 3.7 to write (3.86) as

$$\mathrm{d}S\big|_f(Y) = \int \mathrm{d}\lambda\ I^2_5(\omega)\big|_{f_\lambda}(Y, X_\lambda) + \int \mathrm{d}\lambda\ (I_\partial)^2_4(\theta)\big|_{f_\lambda}(Y, X_\lambda), \tag{3.87}$$

where $X_\lambda = f_*(\partial/\partial\lambda)|_\lambda$ is the vector field along f_λ giving the infinitesimal motion $f_\lambda \mapsto f_{\lambda+\delta\lambda}$. Note that the terms on the right hand side of (3.86) can be written as $\int d\lambda\ \Omega|_{f_\lambda}(Y, X_\lambda)$ and $\int d\lambda\ d\Psi|_{f_\lambda}(Y, X_\lambda)$ respectively, where Ψ is the boundary modification to Θ (see (3.69) and the discussion surrounding it). If f represents a solution, then the first term on the right hand side of (3.86),(3.87) vanishes by the multisymplectic equations of motion. We thus see that the variation of the action at a solution is non-zero due to the presence of the boundary term Ψ. We can, however, define a modified action of the form $\tilde{S} = S + K$ such that solutions will be extrema of this new action. The modification K solves the equation

$$dK|_f(Y) = \int d\lambda\ d\Psi|_{f_\lambda}(X_\lambda, Y), \tag{3.88}$$

so K is closely related to an integral of the boundary term Ψ. The modification K is important if we want the classical equations to follow from a stationary phase approximation to a quantum path integral. In the context of gravity, K is known as the Gibbons-Hawking-York term. [33]

To analyze Hamilton's principal function, consider two small boundary phase spaces (with boundary conditions) $\mathcal{S}_1$, $\mathcal{S}_2$ with presymplectic forms Ω_1, Ω_2. On the doubled phase space $\mathcal{S}_{12} = \mathcal{S}_1 \times \mathcal{S}_2$ we introduce the presymplectic 2-form $\Omega_{12} = \Omega_2 - \Omega_1$. Consider the set of points $L = \{(f_1, f_2)\} \subset \mathcal{S}_{12}$ that lie on the same solution. That is, there exists a solution f_λ with $f_1 = f_{\lambda_1}$, $f_2 = f_{\lambda_2}$ for some λ_1, λ_2.

The manifold L is isotropic with respect to Ω_{12}. To see this, consider any point $(f_1, f_2) \in L$ and two nearby points $(f_1 + \delta_X f_1, f_2 + \delta_X f_2)$, $(f_1 + \delta_Y f_1, f_2 + \delta_Y f_2)$. We can think of X and Y as labeling vectors pointing to the nearby points. The nearby points are associated with solutions $f_\lambda + \delta_X f_\lambda$ and $f_\lambda + \delta_Y f_\lambda$. Now consider the small boundary phase space $\mathcal{S}$ of the theory (with appropriate boundary conditions) with presymplectic 2-form Ω. The solutions $f_\lambda, f_\lambda + \delta_X f_\lambda, f_\lambda + \delta_Y f_\lambda$ produce a 3-dimensional tubular region T. We can integrate $0 = d\Omega$ over T to get

$$0 = \int_T d\Omega = \int_{\partial T} \Omega = \left(\int_{\text{end 1}} + \int_{\text{end 2}} + \int_{\text{walls}} \right) \Omega. \tag{3.89}$$

The integral over the walls vanishes because there is always a direction parallel to a solution and such directions are in the kernel of Ω. The integrals over the ends become (in the limit of small displacements δf_λ) $\Omega_2(X, Y) - \Omega_1(X, Y)$, and thus $0 = \Omega_{1,2}(X, Y)$. From this it follows that $\Omega|_L = 0$. On the manifold L, we thus have $d\tilde{\Theta}_{12} = 0 \implies \tilde{\Theta}_{12} = dS$, where $\tilde{\Theta}_{12} = \tilde{\Theta}_2 - \tilde{\Theta}_1$ and S is Hamilton's principal function. We thus see it is the modified $\tilde{\Theta}$ (with boundary term) that enters into Hamilton's principal function. The boundary modification to Hamilton's principal function involves integrating Ψ and is directly related to the boundary modification in the action above. That is, to get the additional boundary term in Hamilton's principal function we need to evaluate the modified action $\tilde{S}$ on solutions.

3.11 Examples

In the following, we apply the multisymplectic formalism to a scalar field 3.11.1, electromagnetism 3.11.2, and non-Abelian Yang-Mills theories. Although many aspects of the scalar field have already been presented above, we find it useful to collect all the results here. For each example, we will first present the multisymplectic structure of the theory, including multisymplectic momentum maps, before proceeding to a discussion of 3+1 decompositions and then boundary terms.

3.11.1 Scalar Field

We assume the same action as in (3.1), so the Lagrangian is $\mathcal{L}(\phi, \partial_\mu\phi, x^\mu) = \big(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\big)\sqrt{|g|}$, with explicit coordinate dependence coming from a choice of background metric. The multimomenta are simply $\pi^\mu = \partial\mathcal{L}/\partial(\partial_\mu\phi) = -\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi$. The DeDonder-Weyl Hamiltonian is

$$\mathcal{H} = \pi^\mu\partial_\mu\phi - \mathcal{L} = -\frac{\pi^\mu\pi^\nu g_{\mu\nu}}{\sqrt{|g|}} + \frac{1}{2}\frac{\pi^\mu\pi^\nu g_{\mu\nu}}{\sqrt{|g|}} + V(\phi)\sqrt{|g|} = -\frac{\pi^\mu\pi_\mu}{2\sqrt{|g|}} + V(\phi)\sqrt{|g|}. \tag{3.90}$$

The multisymplectic phase space has coordinates (ϕ, π^μ, x^μ) with multisymplectic 4-form

$$\theta = \pi^\mu\,\mathrm{d}\phi \wedge \mathrm{d}^3x_\mu - \mathcal{H}\,\mathrm{d}^4x. \tag{3.91}$$

For a single, real scalar field, there is no internal gauge symmetry, so the only interesting group action to look at is spacetime diffeomorphisms or a subset thereof. Notice that for constructing momentum maps, we need a group action on P (the multisymplectic phase space), but we only have an action of diffeomorphisms on M (spacetime). In appendix C we describe how to lift the diffeomorphism action from M onto P. We see that P is a form bundle over spacetime with fibers consisting of a 0-form (ϕ) and a 3-form (π^μ). That π^μ refers to a 3-form $\pi^\mu\mathrm{d}^3x_\mu$ is clear both from the 4-form θ (where the $\pi^\mu\mathrm{d}^3x_\mu$ must play the role of a 3-form on P) and the definition in terms of the Legendre transformation $\pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$ (where $\mathcal{L}$ is a scalar density, not a true scalar, meaning the π^μ is a densitized vector, hence 3-form). This lets us immediately conclude that $\phi \in \Omega^0(P)$ and $\pi = \pi^\mu\mathrm{d}^3x_\mu = \Omega^3(P)$ are left invariant by the action of lifted diffeomorphisms (see appendix C). Since $\omega = \mathrm{d}\phi \wedge \mathrm{d}\pi - \mathrm{d}\mathcal{H} \wedge \mathrm{d}^4x$, we are left to ask whether lifted diffeomorphisms preserve $\mathrm{d}\mathcal{H} \wedge \mathrm{d}^4x$.

Let $\Phi : M \to M$ be a diffeomorphism and $\tilde{\Phi} : P \to P$ be its lift. We want to know if

$$\tilde{\Phi}^*(\mathrm{d}\mathcal{H} \wedge \mathrm{d}^4x) = \mathrm{d}\mathcal{H} \wedge \mathrm{d}^4x. \tag{3.92}$$

If $\mathcal{H}$ is fully general, there is not much we can say, but in practice we will be interested in theories where any explicit dependence on spacetime enters through a background metric $\mathcal{H} = \mathcal{H}(\phi, \pi^\mu, g_{\mu\nu}(x))$ as in (3.90). Furthermore, for such theories $\mathcal{H}$ transforms as a scalar density $\mathcal{H} \mapsto \mathcal{H}\det(\partial x/\partial x')$ when $g_{\mu\nu}$ transforms as a (0,2) tensor $g_{\mu\nu} \mapsto g_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}$. The problem is that our $\tilde{\Phi}$ knows nothing about the metric and simply transforms $g_{\mu\nu}(x) \mapsto g_{\mu\nu}(x')$. Denote the transformed variables by primes, so that $\tilde{\Phi}$ maps $\phi \mapsto \phi', \pi^\mu \mapsto \pi'^\mu, x^\mu \mapsto x'^\mu$. Let $g'_{\mu\nu} = g_{\rho\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}$ be the properly transformed metric (here the prime does *not* denote action by $\tilde{\Phi}$). Then we have

$$\tilde{\Phi}^*(\mathrm{d}\mathcal{H} \wedge \mathrm{d}^4x) = \mathrm{d}\mathcal{H}(\phi', \pi'^\mu, g_{\mu\nu}(x')) \wedge \mathrm{d}^4x', \tag{3.93a}$$

$$\mathrm{d}\mathcal{H}(\phi', \pi'^\mu, g'_{\mu\nu}(x)) \wedge \mathrm{d}^4x' = \mathrm{d}\mathcal{H}(\phi, \pi^\mu, g_{\mu\nu}(x)) \wedge \mathrm{d}^4x, \tag{3.93b}$$

where (3.93a) is simply substituting the definitions of the primed variables and (3.93b) comes from the invariance of $\mathrm{d}\mathcal{H}\wedge \mathrm{d}^4x$ when the metric is properly transformed by diffeomorphism. Combining (3.93a) and (3.93b) with the requirement (3.92) gives $\mathrm{d}\mathcal{H}(\phi', (\pi^\mu)', g_{\mu\nu}(x')) \wedge \mathrm{d}^4x' = \mathrm{d}\mathcal{H}(\phi', (\pi^\mu)', g'_{\mu\nu}(x)) \wedge \mathrm{d}^4x'$, leading to

$$\frac{\partial\mathcal{H}(\phi', (\pi^\mu)', g_{\mu\nu}(x'))}{\partial\phi} = \frac{\partial\mathcal{H}(\phi', (\pi^\mu)', g'_{\mu\nu}(x))}{\partial\phi} \tag{3.94a}$$

$$\frac{\partial\mathcal{H}(\phi', (\pi^\nu)', g_{\mu\nu}(x'))}{\partial\pi^\mu} = \frac{\partial\mathcal{H}(\phi', (\pi^\nu)', g'_{\mu\nu}(x))}{\partial\pi^\mu} \tag{3.94b}$$

or simply

$$\frac{\partial\mathcal{H}(\phi, \pi, g(x'))}{\partial\phi} = \frac{\partial\mathcal{H}(\phi, \pi, g'(x))}{\partial\phi} \tag{3.95a}$$

$$\frac{\partial\mathcal{H}(\phi, \pi, g(x'))}{\partial\pi^\mu} = \frac{\partial\mathcal{H}(\phi, \pi, g'(x))}{\partial\pi^\mu} \tag{3.95b}$$

We now take the transformation to be an infinitesimal diffeomorphism generated by the vector field ξ so that

$$x'^\mu = x^\mu + \xi^\mu \tag{3.96a}$$

$$g_{\mu\nu}(x') = g_{\mu\nu} + \xi^\lambda\partial_\lambda g_{\mu\nu} \tag{3.96b}$$

$$g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu\xi^m)g_{m\nu} - (\partial_\nu\xi^n)g_{\mu n}, \tag{3.96c}$$

where the expression for $g_{\mu\nu}(x')$ is obtained by Taylor expansion, while the expression for $g'_{\mu\nu}(x)$ is an infinitesimal push-forward. Plugging into (3.95), we get

$$\frac{\partial^2\mathcal{H}}{\partial\phi\partial g_{\mu\nu}}\left[\xi^\lambda\partial_\lambda g_{\mu\nu} + (\partial_\mu\xi^m)g_{m\nu} + (\partial_\nu\xi^n)g_{\mu n}\right] = 0 \tag{3.97a}$$

$$\frac{\partial^2\mathcal{H}}{\partial\pi^\kappa\partial g_{\mu\nu}}\left[\xi^\lambda\partial_\lambda g_{\mu\nu} + (\partial_\mu\xi^m)g_{m\nu} + (\partial_\nu\xi^n)g_{\mu n}\right] = 0 \tag{3.97b}$$

We see that the term in the bracket is just the Lie derivative of $g_{\mu\nu}$:

$$\begin{aligned}(\pounds_\xi g)_{\mu\nu}(x) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(\Phi_t^* g)\bigg|_x\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g\bigg|_{\Phi_t(x)}\left(\Phi_{t*}\frac{\partial}{\partial x^\mu}, \Phi_{t*}\frac{\partial}{\partial x^\nu}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g|_{x'}\left(\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial}{\partial x^\alpha}, \frac{\partial x'^\beta}{\partial x^\nu}\frac{\partial}{\partial x^\beta}\right),\end{aligned} \tag{3.98}$$

and plugging in $x'^\mu = x^\mu + t\xi^\mu$ reproduces the term in brackets. We thus conclude that the diffeomorphisms that are symmetries of our theory must satisfy

$$\frac{\partial^2\mathcal{H}}{\partial\phi\partial g_{\mu\nu}}(\pounds_\xi g)_{\mu\nu} = 0 \tag{3.99a}$$

$$\frac{\partial^2\mathcal{H}}{\partial\pi^\lambda\partial g_{\mu\nu}}(\pounds_\xi g)_{\mu\nu} = 0 \tag{3.99b}$$

This includes all symmetries of the background metric (for which the generator is a Killing field of the metric), but may include a larger class. For example, if the field theory is topological, (3.99) will be satisfied for all diffeomorphisms because $\partial\mathcal{H}/\partial g = 0$.

With the Hamiltonian (3.90) we have

$$\frac{\partial^2\mathcal{H}}{\partial g_{\mu\nu}\partial\pi^\lambda} = \frac{\delta_\lambda^{(\mu}\pi^{\nu)}}{4\sqrt{-g}} + \frac{g_{\lambda\kappa}\pi^\kappa}{4(-g)^{3/2}}g^{\mu\nu}g = \frac{\delta_\lambda^\mu\pi^\nu + \delta_\lambda^\nu\pi^\mu - \pi_\lambda g^{\mu\nu}}{4\sqrt{-g}} \tag{3.100a}$$

$$\frac{\partial^2\mathcal{H}}{\partial g_{\mu\nu}\partial\phi} = -\frac{g^{\mu\nu}g}{2\sqrt{-g}}\frac{\partial V}{\partial\phi} = g^{\mu\nu}\sqrt{-g}\frac{\partial V}{\partial\phi}, \tag{3.100b}$$

and thus (3.99) become

$$\pi^\nu\nabla_{(\lambda}\xi_{\nu)} - \pi_\lambda\nabla_\nu\xi^\nu = 0 \tag{3.101a}$$

$$\frac{\partial V}{\partial\phi}\nabla_\nu\xi^\nu = 0, \tag{3.101b}$$

where we have used $(\pounds_\xi g)_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$. Both equations must be satisfied everywhere in the multisymplectic phase space, in particular (3.101a) must hold for all π^μ. Hence

$$\nabla_{(\lambda}\xi_{\nu)} - g_{\lambda\nu}\nabla_\mu\xi^\mu = 0, \tag{3.102}$$

and tracing over λ, ν, we get $-2\nabla_\mu\xi^\mu = 0$. Plugging this back in, we find $\nabla_{(\lambda}\xi_{\nu)} = 0$, which also solves (3.101b). Thus the only diffeomorphisms which are symmetries of the theory are isometries of the background metric (solutions to Killing's equation).

We now construct the momentum map for the isometries. Note that the lifted diffeomorphism action preserves the 4-form θ. This is because the first term in θ can be written $\mathrm{d}\phi \wedge \pi$, which consist of the canonical invariant forms on the scalar and 3-form fibers of P, while $\mathcal{H} = \frac{g_{\mu\nu}\pi^\mu\pi^\nu}{2\sqrt{-g}} + \sqrt{-g}V(\phi)$ becomes

$$\mathcal{H}' = \frac{1}{2\left[\det\left(\frac{\partial x'}{\partial x}\right)\right]^2\sqrt{-g}}g_{\mu\nu}\frac{\partial x'^\mu}{\partial x^m}\frac{\partial x'^\nu}{\partial x^n}\pi^m\pi^n + \sqrt{-g}V(\phi), \tag{3.103}$$

where we have transformed π^μ as a vector density. Since we have an isometry, it follows that

$$g_{\mu\nu}\frac{\partial x'^\mu}{\partial x^m}\frac{\partial x'^\nu}{\partial x^n} = g_{mn} \implies \left[\det\left(\frac{\partial x'}{\partial x}\right)\right]^2 = 1, \tag{3.104}$$

which shows $\mathcal{H}$ is invariant. Additionally, d^4x is invariant since it transforms by the determinant of the isometry. Preservation of the 4-form implies that we can find the momentum map simply as

$$J_\xi = i_{X_\xi}\theta = -(\pi^\mu\mathrm{d}\phi \wedge \mathrm{d}^2x_{\mu\nu} + \mathcal{H}\mathrm{d}^3x_\nu)\xi^\nu. \tag{3.105}$$

We can pull this back by a field configuration $f : \Sigma_4 \to P$ to get a cleaner expression

$$f^*J_\xi = -(\pi^\mu\partial_\nu\phi + \delta_\nu^\mu[\pi^\lambda(\partial_\lambda\phi(x^\mu,\phi,\pi^\mu) - \partial_\lambda\phi) - \mathcal{L}(x^\mu,\phi,\pi^\mu)])\xi^\nu\mathrm{d}^3x_\mu = -H^\mu{}_\nu\xi^\nu\mathrm{d}^3x_\mu, \tag{3.106}$$

where we have introduced the Hamiltonian tensor $H^{\mu}{}_{\nu}$. Note that $\partial\phi$ appears both as a derivative of the field ϕ and as the solution to the Legendre transformation $\partial\phi(\phi,\pi,x)$. If f represents a solution, then $H^{\mu}{}_{\nu}$ becomes $\pi^{\mu}\partial_{\nu}\phi-\mathcal{L}\delta^{\mu}_{\nu}=T^{\mu}{}_{\nu}$, the stress-energy tensor of the theory. Thus the momentum map for isometries, evaluated on solutions, is $J_{\xi}=T^{\mu}{}_{\nu}\xi^{\nu}\,\mathrm{d}^3x_{\mu}$.

The 3+1 decomposition proceeds as described in 3.4-3.8. The large boundary phase space $\mathcal{P}$ is defined as the space of maps $f:\Sigma\to P$. The presymplectic 1-form on $\mathcal{P}$ is

$$\Theta=\int\limits_{\Sigma}\pi^{\mu}\,\delta\phi\,\mathrm{d}^3x_{\mu}-\delta x^{\nu}\left(\mathcal{H}\mathrm{d}^3x_{\nu}+\pi^{\mu}\mathrm{d}\phi\wedge\mathrm{d}^2x_{\mu\nu}\right)=\int\limits_{\Sigma}\mathrm{d}^3\sigma\,\pi^{\mu}C_{\mu}\delta\phi-\left(\mathcal{H}C_{\nu}+\pi^{\mu}C^{i}_{\mu\nu}\frac{\partial\phi}{\partial\sigma^{i}}\right)\delta x^{\nu}, \tag{3.107}$$

where we have introduced the cofactor tensors

$$C_{\mu}=\frac{1}{3!}\epsilon_{\mu\alpha\beta\gamma}\epsilon^{ijk}\frac{\partial x^{\alpha}}{\partial\sigma^{i}}\frac{\partial x^{\beta}}{\partial\sigma^{j}}\frac{\partial x^{\gamma}}{\partial\sigma^{k}} \tag{3.108a}$$

$$C^{i}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\epsilon^{ijk}\frac{\partial x^{\alpha}}{\partial\sigma^{j}}\frac{\partial x^{\beta}}{\partial\sigma^{k}}, \tag{3.108b}$$

to highlight the explicit dependence on σ. On the small boundary phase space, we can use adapted coordinates to simplify Θ to

$$\Theta=\int\limits_{\Sigma}\mathrm{d}^3x\,\pi^{0}\delta\phi-H\delta\lambda, \tag{3.109}$$

with

$$H=-\pi^{i}\partial_{i}\phi+\frac{g_{00}(\pi^{0})^2+2g_{0i}\pi^{0}\pi^{i}+g_{ij}\pi^{i}\pi^{j}}{2\sqrt{-g}}+\sqrt{-g}V(\phi). \tag{3.110}$$

Applying the constraint algorithm (see 2.2.3) on the small boundary phase space gives the constraint

$$\partial_{i}\phi=\frac{g_{i0}\pi^{0}+g_{ij}\pi^{j}}{\sqrt{-g}}. \tag{3.111}$$

When g_{ij} is full-rank (as occurs when the surfaces given by $x^{\mu}(\Sigma,\lambda)$ are spacelike), (3.111) is the only constraint and we can restrict our theory to the constraint manifold to eliminate the π^{i}. That is, the constraint manifold is second-class (see appendix D). When g_{ij} is not full rank (as occurs when dealing with null surfaces or surfaces that have at least some null tangent planes), there may be additional constraints produced by the constraint algorithm. We do not consider that case.

The momentum map for isometries (3.105) becomes

$$\mathcal{J}_{\xi}=\int\limits_{\Sigma}J_{\xi}=$$

$$-\int\limits_{\Sigma}(\pi^{\mu}\partial_{\nu}\phi+\delta^{\mu}_{\nu}[\pi^{\lambda}(\partial_{\lambda}\phi(x^{\mu},\phi,\pi^{\mu})-\partial_{\lambda}\phi)-\mathcal{L}(x^{\mu},\phi,\pi^{\mu})])\xi^{\nu}\mathrm{d}^3x_{\mu}=-\int\limits_{\Sigma}H^{\mu}{}_{\nu}\xi^{\nu}\,\mathrm{d}^3x_{\mu} \tag{3.112}$$

on the large boundary phase space. Spacetime transformations (such as isometries) are not symmetries on the small boundary phase space $\mathcal{S}$ as they change the curve γ used in the definition of $\mathcal{S}$. Nevertheless, the function $\mathcal{J}_\xi$ is well-defined on $\mathcal{S}$, and in adapted coordinates is

$$\mathcal{J}_\xi = -\int_\Sigma \mathrm{d}^3x\, H^0{}_\nu \xi^\nu = -\int_\Sigma \mathrm{d}^3x(\pi^0\partial_\nu\phi + \delta^0_\nu[\pi^\lambda(\partial_\lambda\phi(x^\mu,\phi,\pi^\mu) - \partial_\lambda\phi) - \mathcal{L}(x^\mu,\phi,\pi^\mu)])\xi^\nu \mathrm{d}^3x_\mu \tag{3.113}$$

3.11.2 Electromagnetic Field

We will be interested in Lagrangians of the form $\mathcal{L}(A_\mu, F_{\mu\nu}, g_{\mu\nu}(x))$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This means the derivatives of the electromagnetic potential A_μ enter only in their anti-symmetric combination. The Legendre transformation defines the multi-momenta

$$\pi^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial\mathcal{L}}{\partial F_{\alpha\beta}}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = 2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}. \tag{3.114}$$

Note that $\pi^{\mu\nu}$ is antisymmetric:

$$\pi^{\nu\mu} = 2\frac{\partial\mathcal{L}}{\partial F_{\nu\mu}} = -2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} = -\pi^{\mu\nu}. \tag{3.115}$$

This means that the multisymplectic phase space P is a fiber bundle over spacetime with fiber coordinates A_μ and *antisymmetric* $\pi^{\mu\nu}$. We may naturally regard $\pi^{\mu\nu}$ as the components of 2-form $\pi = \frac{1}{2}\pi^{\mu\nu}\,\mathrm{d}^2x_{\mu\nu}$ and so P consists of 1-form and 2-form fibers over spacetime. The multisymplectic 4-form is

$$\theta = \pi^{\mu\nu}\mathrm{d}A_\nu \wedge \mathrm{d}^3x_\mu - \mathcal{H}\mathrm{d}^4x = \pi \wedge \mathrm{d}A - \mathcal{H}\mathrm{d}^4x, \tag{3.116}$$

where $\mathcal{H} = \pi^{\mu\nu}(\partial_\mu A_\nu)(A,\pi,g) - \mathcal{L}(A,\pi,g)$.

We consider both the internal gauge transformations and the (external) spacetime diffeomorphisms (or some appropriate subset thereof) as symmetries acting on our multisymplectic phase space. Let us begin with gauge transformations. The gauge group of electromagnetism is the $U(1)$ gauge group, $G = C^\infty(M,(U(1)))$ with point-wise multiplication and inversion. It has the Lie algebra $\mathfrak{g} = C^\infty(M,\mathfrak{u}(\mathfrak{1}))$, with point-wise commutation. The action of $\gamma \in G$ on P is given by

$$(A_\mu, \pi^{\mu\nu}, x^\mu) \mapsto (A'_\mu(x^\mu), \pi^{\mu\nu}, x^\mu), \tag{3.117a}$$
$$A'_\mu(x) = \gamma(x)A_\mu\gamma^{-1}(x) - \gamma^{-1}(x)(\partial_\mu\gamma)(x). \tag{3.117b}$$

The multiplication $\gamma A_\mu \gamma^{-1}$ refers to the adjoint action of the group $U(1)$ on its Lie algebra $\mathfrak{u}(1) = \mathbb{R}$ (since everything is done point-wise). The infinitesimal version of the action is $A_\mu \mapsto A_\mu + \partial_\mu\chi$, which implies the induced vector field

$$X_\chi = (\partial_\mu\chi)\frac{\partial}{\partial A_\mu}. \tag{3.118}$$

Here $\chi : M \to \mathfrak{u}(1)$ and so it is just a function on spacetime. We check that this transformation is indeed a symmetry by computing

$$\pounds_{X_\chi}\theta = \pi^{\mu\nu}\mathrm{d}\pounds_{X_\chi}A_\nu \wedge \mathrm{d}^3x_\mu - \pounds_{X_\chi}(\mathcal{H})\mathrm{d}^4x, \tag{3.119}$$

where we have used the fact that $\pi^{\mu\nu}$ and x^μ are invariant under the transformation. In order to continue the calculation, we need an explicit form for $\mathcal{H}$. Let us use the standard Maxwell Lagrangian (with coupling to a background current)

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g}g^{\mu\alpha}g^{\mu\beta}F_{\mu\nu}F_{\alpha\beta} + \sqrt{-g}j^\mu A_\mu. \tag{3.120}$$

The Legendre transformation gives

$$\pi^{\mu\nu} = 2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} = -\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} = -\sqrt{-g}F^{\mu\nu}, \tag{3.121}$$

from which we get

$$\begin{aligned}\mathcal{H} = \pi^{\mu\nu}\partial_\mu A_\nu - \mathcal{L} &= \frac{1}{2}\pi^{\mu\nu}F_{\mu\nu} + \frac{1}{4\sqrt{-g}}g_{\mu\alpha}g_{\nu\beta}\pi^{\mu\nu}\pi^{\alpha\beta} - \sqrt{-g}j^\mu A_\mu \\ &= -\frac{1}{4\sqrt{-g}}g_{\mu\alpha}g_{\nu\beta}\pi^{\mu\nu}\pi^{\alpha\beta} - \sqrt{-g}j^\mu A_\mu.\end{aligned} \tag{3.122}$$

For this choice of $\mathcal{H}$ we have

$$\pounds_{X_\chi}\mathcal{H} = -\sqrt{-g}j^\mu\pounds_{X_\chi}A_\mu = -\sqrt{-g}j^\mu\partial_\mu\chi \tag{3.123}$$

Now j^μ is a background current 4-vector. We will assume that it is conserved. This means that there is a 3-form $J = \sqrt{-g}j^\mu\mathrm{d}^3x_\mu$ with $\mathrm{d}J = 0 \implies \partial_\mu(\sqrt{-g}j^\mu) = 0$. Thus $\pounds_{X_\chi}\mathcal{H} = -\partial_\mu(\sqrt{-g}j^\mu\chi)$ and

$$\pounds_{X_\chi}\theta = \pi^{\mu\nu}\mathrm{d}(\partial_\nu\chi)\wedge\mathrm{d}^3x_\mu + \partial_\mu(\sqrt{-g}j^\mu\chi)\mathrm{d}^4x = \pi^{\mu\nu}\partial_\mu\partial_\nu\chi\mathrm{d}^4x + \mathrm{d}(\sqrt{-g}j^\mu\chi\mathrm{d}^3x_\mu). \tag{3.124}$$

The first term vanishes due to the antisymmetry of $\pi^{\mu\nu}$ (we can see that it is important to use the antisymmetric object in our fiber, otherwise the gauge transformation would not be a symmetry). We thus have a symmetry since

$$\pounds_{X_\chi} = \mathrm{d}(\sqrt{-g}j^\mu\chi\mathrm{d}^3x_\mu) \implies \mathrm{d}(\pounds_{X_\chi}\theta) = \pounds_{X_\chi}\omega = 0. \tag{3.125}$$

Furthermore, we can easily extract the momentum map using

$$i_{X_\chi}\omega = -\mathrm{d}J_\chi = \pounds_{X_\chi}\theta - \mathrm{d}i_{X_\chi}\theta \implies \mathrm{d}(J_\chi + \sqrt{-g}j^\mu\chi\mathrm{d}^3x_\mu - i_{X_\chi}\theta) = 0. \tag{3.126}$$

We solve for the momentum map to get

$$J_\chi = i_{X_\chi}\theta - \sqrt{-g}j^\mu\chi\mathrm{d}^3x_\mu = \pi^{\mu\nu}\partial_\nu\chi\mathrm{d}^3x_\mu - \sqrt{-g}j^\mu\chi\mathrm{d}^3x_\mu = \pi\wedge\mathrm{d}\chi - \chi J, \tag{3.127}$$

where $\pi = \frac{1}{2}\pi^{\mu\nu}\mathrm{d}^2x_{\mu\nu}, \mathrm{d}\chi = \partial_\mu\chi\mathrm{d}x^\mu, J = \sqrt{-g}j^\mu\mathrm{d}^3x_\mu$.

In general, we could check what subgroup of spacetime diffeomorphisms result in symmetries using the equations analogous to (3.99), but for simplicity we will just take the isometries of the background metric. This will ensure invariance as far as the background metric is concerned, but $\mathcal{H}$ also contains the background current $j^\mu(x)$. In general, this will further restrict the allowed spacetime transformations. For simplicity, set $j^\mu(x) = 0$ (pure Maxwell), for which all isometries of the background metric are symmetries. The momentum map corresponding to isometries is

$$J_\xi = i_{X_\xi}\theta = -\pi^{\mu\nu} A_\lambda \partial_\nu \xi^\lambda \mathrm{d}^3 x_\mu - \pi^{\mu\nu} \mathrm{d}A_\nu \wedge \mathrm{d}^2 x_{\mu\lambda} \xi^\lambda - \mathcal{H} \mathrm{d}^3 x_\mu \xi^\mu, \tag{3.128}$$

which we can pull back by a field configuration $f : \Sigma_4 \to P$,

$$\begin{aligned} f^* J_\xi &= -\pi^{\mu\nu} A_\lambda \partial_\nu \xi^\lambda \mathrm{d}^3 x_\mu - \pi^{\mu\nu} \partial_\kappa A_\nu \delta^\alpha_{[\mu} \delta^\kappa_{\lambda]} \xi^\lambda \mathrm{d}^3 x_\alpha - \mathcal{H} \mathrm{d}^3 x_\mu \xi^\mu \\ &= -\partial_\nu(\pi^{\mu\nu} A_\lambda \xi^\lambda) \mathrm{d}^3 x_\mu + \partial_\nu(\pi^{\mu\nu} A_\lambda) \xi^\lambda \mathrm{d}^3 x_\mu - \pi^{\mu\nu} \partial_\lambda A_\nu \xi^\lambda \mathrm{d}^3 x_\mu + \pi^{\mu\nu} \partial_\mu A_\nu \xi^\lambda \mathrm{d}^3 x_\lambda - \mathcal{H} \mathrm{d}^3 x_\mu \xi^\mu. \end{aligned} \tag{3.129}$$

The first term is a total divergence which we can drop since the momentum map is always defined only up to a total divergence. The rest of the terms become

$$\begin{aligned} &f^* J_\xi = \\ &- \left[\pi^{\mu\lambda} F_{\nu\lambda} - \mathcal{L}\delta^\mu_\nu + \pi^{\alpha\beta}(\partial_\alpha A_\beta - (\partial_\alpha A_\beta)(A, \pi, g))\delta^\mu_\nu - (\partial_\lambda \pi^{\mu\lambda}) A_\nu\right] \xi^\nu \mathrm{d}^3 x_\mu = -H^\mu{}_\nu \xi^\nu \mathrm{d}^3 x_\mu, \end{aligned} \tag{3.130}$$

where in the last equality, we define the Hamiltonian tensor for electromagnetism. On solutions, this Hamiltonian tensor becomes

$$H^\mu{}_\nu = (T_{\mathrm{sym}})^\mu{}_\nu, \tag{3.131}$$

where we have used both the Legendre transformation $(\partial_\alpha A_\beta = (\partial_\alpha A_\beta)(A, \pi, g))$ and Maxwell's equations $(\partial_\lambda \pi^{\mu\lambda} = 0)$. The tensor (density) $(T_{\mathrm{sym}})^\mu{}_\nu = \sqrt{-g} F^{\mu\lambda} F_{\nu\lambda} - \mathcal{L}\, \delta^\mu_\nu$ is the symmetric stress-energy tensor for electromagnetism. We have arrived at the symmetric rather than canonical stress-energy tensor by carefully tracking all isometries (including rotations) in the derivation of the momentum map.

The presymplectic structure of the large boundary phase space is similar to the scalar field case (3.107),

$$\Theta = \int_\Sigma \mathrm{d}^3\sigma\, \pi^{\mu\nu} C_\mu \delta A_\nu - \left(\mathcal{H} C_\nu + \pi^{\mu\lambda} C^i_{\mu\nu} \frac{\partial A_\lambda}{\partial \sigma^i}\right) \delta x^\nu, \tag{3.132}$$

and on the small boundary phase space we have

$$\Theta = \int_\Sigma \mathrm{d}^3 x\, \pi^{0i} \delta A_i - H \delta\lambda, \tag{3.133}$$

with

$$H = \pi^{0i} \partial_i A_0 - \pi^{ij} \partial_i A_j + \frac{1}{4\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \pi^{\mu\nu} \pi^{\alpha\beta} + \sqrt{-g} j^\mu A_\mu. \tag{3.134}$$

Notice the absence of π^{00} and δA_0 in the presymplectic structure. This is natural consequence of using a form bundle for our multisymplectic phase space. Writing H explicitly we get

$$H = \pi^{0i}\partial_i A_0 - \pi^{ij}\partial_i A_j + \frac{1}{4\sqrt{-g}}\left[2\pi^{0i}\pi^{0j}(g_{00}g_{ij} - g_{0i}g_{0j}) + 4\pi^{0i}\pi^{jk}g_{0j}g_{ik} + \pi^{ij}\pi^{k\ell}g_{ik}g_{j\ell}\right] + \sqrt{-g}j^0 A_0 + \sqrt{-g}j^i A_i \tag{3.135}$$

Using the constraint algorithm, we find two constraints. One is analogous to the scalar field (3.111),

$$\partial_{[i}A_{j]} = \frac{\pi^{0k}g_{0[i}g_{j]k} + \pi^{k\ell}g_{ik}g_{j\ell}}{\sqrt{-g}}, \tag{3.136}$$

and as before when the induced metric g_{ij} is non-degenerate, we can restrict to this constraint submanifold and eliminate π^{ij} from our theory. On the constraint manifold,

$$H = \pi^{0i}\partial_i A_0 + \frac{1}{2\sqrt{-g}}\left[\pi^{0i}\pi^{0j}(g_{00}g_{ij} - g_{0i}g_{0j}) - h^{ik}h^{j\ell}(\sqrt{-g}\partial_{[i}A_{j]} - \pi^{0m}g_{0[i}g_{j]n})(\sqrt{-g}\partial_{[k}A_{\ell]} - \pi^{0n}g_{0[k}g_{\ell]n})\right] + \sqrt{-g}j^0 A_0 + \sqrt{-g}j^i A_i. \tag{3.137}$$

The second constraint is Gauss's law,

$$(\mathrm{d}\pi - J)|_\Sigma = 0. \tag{3.138}$$

We will show later that this is the 0 level set of the momentum map for gauge transformations. As such, this constraint produces a first-class constraint manifold that we cannot restrict to without producing spurious solutions (see appendix D for details). We can eliminate the Gauss constraint by reducing by the gauge group, leaving us with the unconstrained, gauge-invariant variables $A_\perp, E_\perp$.

We now look at the momentum maps for gauge and spacetime symmetries in the 3+1 split. We first work with the momentum map for gauge transformations. Working in the large boundary phase space, we have the momentum map

$$\mathcal{J}_\chi\Big|_f = \int_f J_\chi = \int_f \pi \wedge \mathrm{d}\chi - J\chi = \int_f (-\mathrm{d}\pi - J)\chi. \tag{3.139}$$

We see that

$$\pi = \frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\pi^{\mu\nu}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta = -\frac{\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}}{4}\epsilon_{\mu\nu\alpha\beta}F_{\rho\sigma}\mathrm{d}x^\alpha\mathrm{d}x^\beta = - * F, \tag{3.140}$$

So the vanishing of the 3+1 momentum map on the large boundary phase space is equivalent to $\mathrm{d}{*F} = J$, which are the equations of motion. For sufficiently simple systems, the vanishing of the momentum map on the large boundary phase space is equivalent to the equations of motion. Moving to the small boundary phase space, we get

$$\mathcal{J}_\chi = \int_\Sigma (-\partial_i\pi^{0i} - \rho)\chi\, \mathrm{d}^3\sigma, \tag{3.141}$$

where $\pi^{0i} = -\sqrt{-g}F^{0i} = -E^i$. We thus see that the 3+1 momentum map on the small boundary phase space implies the Gauss constraint.

For the momentum map for isometries, we have on the large boundary phase space

$$\mathcal{J}_\xi\bigg|_f = \int\limits_f -\pi^{\mu\nu}A_\lambda\partial_\nu\xi^\lambda\,\mathrm{d}^3x_\mu - \pi^{\mu\nu}\mathrm{d}A_\nu\wedge\mathrm{d}^2x_{\mu\lambda}\xi^\lambda - \mathcal{H}\mathrm{d}^3x_\mu\xi^\mu, \tag{3.142}$$

and on the small boundary phase space in adapted coordinates,

$$\mathcal{J}_\xi\bigg|_\mathcal{S} = \int\limits_\Sigma \left(-\pi^{0i}A_\lambda\partial_i\xi^\lambda - \pi^{0i}\xi^j\partial_jA_i + \pi^{i\nu}\partial_iA_\nu\xi^0 - \mathcal{H}\xi^0\right)\mathrm{d}^3x. \tag{3.143}$$

Integrating the first term by parts and dropping the boundary term, we get

$$\mathcal{J}_\xi\bigg|_\mathcal{S} = \int\limits_U (\partial_i\pi^{0i})\xi^\lambda A_\lambda - \xi^0(\mathcal{H} - \pi^{ij}\partial_iA_j) - \xi^j\pi^{0i}\partial_{[j}A_{i]}. \tag{3.144}$$

We now clearly see that on the final constraint manifold, the first term will vanish, while the second two terms will become the symmetric versions of the Hamiltonian and linear momentum of the field, respectively. We thus write the whole momentum map as

$$\mathcal{J}_\xi\bigg|_\mathcal{S} = -\int\limits_U P_\mu\xi^\mu, \tag{3.145}$$

with

$$P_0 = -A_0\partial_i\pi^{0i} + \mathcal{H} - \pi^{ij}\partial_iA_j \tag{3.146}$$
$$P_i = -A_i\partial_j\pi^{0j} + \pi^{0j}\partial_{[i}A_{j]}. \tag{3.147}$$

On solutions, we recover $P_\mu = (T_{\text{sym}})^0{}_\mu$.

Chapter 4

Multisymplectic Approach to General Relativity

We apply our multisymplectic approach from chapter 3 to general relativity (GR). Several approaches to multisymplectic GR have been proposed [14,24,39], along with other covariant symplectic approaches [8]. The novel aspects of our approach involve careful definitions of the relevant phase spaces and their multisymplectic and presymplectic structure, identifying symmetries with their corresponding momentum maps, and explicitly constructing the symmetry reduction that links our approach to the traditional (symplectic) ADM approach [16].

In order to define the multisymplectic structure of GR, we first change from the usual metric formulation of GR to a tetrad formulation. The metric is a function of the tetrad, but the tetrad contains extra degrees of freedom. This introduces an additional gauge symmetry, local Lorentz transformations, into the theory. We further increase the degrees of freedom by treating the tetrad and connection as independent fields. This increased freedom makes the theory much easier to work with and its multisymplectic structure particularly clear.

We perform a 3+1 decomposition as described in sections 3.4-3.8. The associated large boundary phase space is interesting in that it still supports full diffeomorphism invariance, despite providing a 3+1 split. This allows us to clearly see the connection between the momentum map for diffeomorphisms and constraints, even when we move to the small boundary phase space. The additional local Lorentz symmetry gives us an additional momentum map that helps us more clearly understand the full constraint structure of the 3+1-decomposed theory. Lastly, we show how to obtain the traditional ADM approach through reduction by the local Lorentz symmetry. This provides us with a clear path from fully covariant GR to the traditional ADM decomposition.

4.1 Einstein-Cartan Theory

An important step in setting up the multisymplectic formalism for GR is to use the Einstein-Cartan formulation (also known as Einstein-Cartan-Sciama-Kibble theory or tetrad gravity). Details can be found in [14, 15, 40]. Given a spacetime M with Lorentzian metric g, we can introduce a field of orthonormal frames e_I ($I = 0, 1, 2, 3$) called the tetrad. By orthonormality, $g(e_I, e_J) = \eta_{IJ}$ where $\eta = \text{diag}(-1, 1, 1, 1)$ is the usual Minkowski metric. The dual frame field or co-tetrad β^I satisfies $\beta^I(e_J) = \delta^I_J$. It is conventional in the literature to use e for both the tetrad and co-tetrad, but here we prefer to distinguish between vectors e and 1-forms β. We of course have $g^{-1}(\beta^I, \beta^J) = \eta^{IJ}$. We will often refer to the co-tetrad as the tetrad, since it will be clear from context which one is meant.

If we introduce coordinates x^μ on M, we can write the (co)tetrad and the above relations as

$$e_I = e^\mu_I \frac{\partial}{\partial x^\mu}, \tag{4.1a}$$

$$\beta^I = \beta^I_\mu \, \mathrm{d}x^\mu, \tag{4.1b}$$

$$\beta^I(e_J) = \beta^I_\mu e^\mu_J = \delta^I_J, \tag{4.1c}$$

$$g(e_I, e_J) = g_{\mu\nu} e^\mu_I e^\nu_J = \eta_{IJ}. \tag{4.1d}$$

Note that (4.1c) shows that the matrices e^μ_I and β^I_μ are inverses. With this, we can invert (4.1) as

$$\frac{\partial}{\partial x^\mu} = \beta^I_\mu \, e_I, \tag{4.2a}$$

$$\mathrm{d}x^\mu = e^\mu_I \, \beta^I, \tag{4.2b}$$

$$\mathrm{d}x^\mu(\partial_\nu) = \beta^I_\nu \, e^\mu_I = \delta^\mu_\nu, \tag{4.2c}$$

$$g(\partial_\mu, \partial_\nu) = \eta_{IJ} \beta^I_\mu \beta^J_\nu = g_{\mu\nu}. \tag{4.2d}$$

The last equation is particularly useful as it shows how to express the metric in terms of the tetrad. Throughout, we follow the convention that uppercase Latin indices refer to the tetrad frame while lowercase Greek indices refer to the coordinate frame. Greek indices are raised and lowered with the metric $g_{\mu\nu}$ while Latin indices are raised and lowered using the Minkowski metric η_{IJ}. As we see from (4.1),(4.2) the tetrad can be used to convert tensors from Greek to Latin indices and vice versa.

A spacetime in GR also carries a covariant derivative ∇ with associated connection $\Gamma^\lambda_{\mu\nu} \equiv \mathrm{d}x^\lambda(\nabla_\mu \partial_\nu)$. We express the connection coefficients with respect to the tetrad as $\omega^I_{\mu\,J} = \beta^I(\nabla_\mu e_J)$. When expressing the connection in the tetrad frame it is customary to denote it ω and refer to it as the spin connection (due to its relevance in coupling fermions to gravity). We can use the spin connection to introduce the covariant exterior derivative D, which takes a tensor-valued p-form $T^{I_1 \ldots I_m}{}_{J_1 \ldots J_n}$ to the tensor-valued $(p+1)$-form

$$\begin{aligned} DT^{I_1 \ldots I_m}{}_{J_1 \ldots J_n} &= \mathrm{d}T^{I_1 \ldots I_m}{}_{J_1 \ldots J_n} + \\ \sum_k \omega^{I_k}{}_I \wedge T^{I_1 \ldots I_{k-1} I I_{k+1} \ldots I_m}{}_{J_1 \ldots J_n} &- \sum_k \omega^J{}_{J_k} \wedge T^{I_1 \ldots I_m}{}_{J_1 \ldots J_{k-1} J J_{k+1} \ldots J_n}. \end{aligned} \tag{4.3}$$

We take the spin connection to be metric-compatible, $0 = D\eta_{IJ} = -\omega^K{}_I \eta_{KJ} - \omega^K{}_J \eta_{IK} = -\omega_{JI} - \omega_{IJ}$. The spin connection is thus anti-symmetric in its orthonormal indices and may be regarded as an $\mathfrak{so}(3,1)$-valued 1-form. The Riemann curvature tensor is then naturally regarded as an $\mathfrak{so}(3,1)$-valued 2-form,

$$R^I{}_J = \mathrm{d}\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J. \tag{4.4}$$

The torsion of the connection is a vector-valued 2-form,

$$T^I = D\beta^I = \mathrm{d}\beta^I + \omega^I{}_J \wedge \beta^J. \tag{4.5}$$

If the torsion vanishes, we can solve for the spin connection explicitly in terms of the tetrad and the result is the Levi-Civita connection. We will not assume that the torsion vanishes *a priori*, but its vanishing will emerge from the Euler-Lagrange equations. In the Einstein-Cartan framework, the Bianchi identities are obtained by applying the covariant exterior derivative to (4.4) and (4.5). Applying D to (4.4) gives the differential Bianchi identity $DR^I{}_J = 0$, while applying D to (4.5) gives the algebraic Bianchi identity $DT^I = R^I{}_J \wedge \beta^J$ (or $R^I{}_J \wedge \beta^J = 0$ when torsion vanishes).

With these basic definitions in hand, we are ready to describe the dynamics of Einstein-Cartan theory. We work in units where $16\pi G = 1$ so that the Einstein-Hilbert action is

$$S_{\mathrm{EH}} = \int \sqrt{|g|}R\, \mathrm{d}^4x, \tag{4.6}$$

where R is the Ricci scalar. We recognize $\sqrt{|g|}\mathrm{d}^4x$ as the volume form Ω, which can be written $\Omega = \beta^0 \wedge \beta^1 \wedge \beta^2 \wedge \beta^3$ using the tetrad. The integrand can then be written $R^{IJ}{}_{IJ}\beta^0 \wedge \beta^1 \wedge \beta^2 \wedge \beta^3 = R^{IJ}{}_{KL}\delta^K_I\delta^L_J\beta^0 \wedge \beta^1 \wedge \beta^2 \wedge \beta^3 = \frac{1}{2}R^{IJ}{}_{KL}\delta^{[K}_I\delta^{L]}_J\beta^0 \wedge \beta^1 \wedge \beta^2 \wedge \beta^3$. We recognize the anti-symmetrized $\delta^{[K}_I\delta^{L]}_J = \frac{1}{2}\epsilon_{IJAB}\epsilon^{KLAB}$ and $\epsilon^{KLAB}\beta^0 \wedge \beta^1 \wedge \beta^2 \wedge \beta^3 = \beta^K \wedge \beta^L \wedge \beta^A \wedge \beta^B$. Putting this together gives $\frac{1}{4}\epsilon_{IJAB}R^{IJ}{}_{KL}\beta^K \wedge \beta^L \wedge \beta^A \wedge \beta^B$. The curvature 2-form is $R^{IJ} = \frac{1}{2}R^{IJ}{}_{KL}\beta^K \wedge \beta^L$, so we can write the action in the simplified form,

$$S[e, \omega] = \int \frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge R^{KL}. \tag{4.7}$$

Rather than regard this action as a functional of the metric (or tetrad) only, we treat the tetrad and spin connection as independent fields. We will refer to the action (4.7) and associated variational principle as the Cartan-Palatini approach, as Cartan was the first to introduce an orthonormal frame approach to GR and Palatini [41] pioneered the idea of using the connection as an independent field.

We now show how the Cartan-Palatini variational principle leads to the usual Einstein equations. The variation of (4.7) is

$$\delta S = \int (\delta\beta^I) \wedge \epsilon_{IJKL}\, \beta^J \wedge R^{KL} + \frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge D(\delta\omega)^{KL}, \tag{4.8}$$

where the variation of β produced two identical terms and we used

$$\begin{aligned} \delta R^{KL} &= \mathrm{d}\delta\omega^{KL} + \delta\omega^K{}_M \wedge \omega^{ML} + \omega^K{}_M \wedge \delta\omega^{ML} = \\ &\mathrm{d}\delta\omega^{KL} + \omega^K{}_M \wedge \delta\omega^{ML} + \omega^L{}_M \wedge \delta\omega^{KM} = D(\delta\omega)^{KL}. \end{aligned} \tag{4.9}$$

Since $\delta\omega$ is a tensor, its covariant exterior derivative is well-defined. We can integrate the second term of (4.8) by parts using

$$\begin{aligned} &\mathrm{d}\left(\frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge \delta\omega^{KL}\right) = D\left(\frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge \delta\omega^{KL}\right) \\ &= \frac{1}{2}\epsilon_{IJKL}\, D(\beta^I \wedge \beta^J) \wedge \delta\omega^{KL} + \frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge D(\delta\omega)^{KL}, \end{aligned} \tag{4.10}$$

to get

$$\delta S = \int (\delta\beta^I) \wedge \epsilon_{IJKL}\, \beta^J \wedge R^{KL} - \epsilon_{IJKL}\, T^I \wedge \beta^J \wedge \delta\omega^{KL}. \tag{4.11}$$

This leads to the Euler-Lagrange equations,

$$\epsilon_{IJKL}\, \beta^J \wedge R^{KL} = 0 \tag{4.12a}$$

$$\epsilon_{IJKL}\, T^I \wedge \beta^J = 0. \tag{4.12b}$$

Equation (4.12a) is formally the Einstein equation, but using the curvature of a connection that is not necessarily torsion-free. Equation (4.12b) implies the vanishing of torsion. Taken together, (4.12) are equivalent to the usual Einstein equations. We now show this in detail.

Taking the wedge product of (4.12a) with β^M gives

$$\epsilon_{IJKL}\, R^{KL}{}_{AB}\beta^M \wedge \beta^J \wedge \beta^A \wedge \beta^B = 0 \implies \epsilon_{IJKL}\epsilon^{MJAB}\, R^{KL}{}_{AB} = \delta_I^{[M}\delta_K^A\delta_L^{B]} R^{KL}{}_{AB} = 0. \tag{4.13}$$

The antisymmetric product of δ's (and R) simplifies to three terms using the antisymmetry of $R^{KL}{}_{AB}$,

$$\delta_I^M R + R^{BM}{}_{IB} + R^{MA}{}_{AI} = 0 \implies R^M{}_I - \frac{1}{2}\delta_I^M R = 0, \tag{4.14}$$

where the R without indices is the Ricci scalar and the R with two indices is the Ricci tensor. The result is the familiar Einstein equations, but without assuming torsion vanishes. Similarly taking the wedge product of (4.12b) with β^M gives

$$\epsilon_{IJKL}\, T^I_{AB}\beta^M \wedge \beta^A \wedge \beta^B \wedge \beta^J = 0 \implies \epsilon_{IJKL}\epsilon^{MABJ}\, T^I_{AB} = \delta_I^{[M}\delta_K^A\delta_L^{B]} T^I_{AB} = 0. \tag{4.15}$$

As before, the antisymmetry of T^I_{AB} allows us to reduce to three terms,

$$T^M_{KL} + T^A_{AK}\delta^M_L - T^A_{AL}\delta^M_K = 0. \tag{4.16}$$

Contracting M and K gives

$$T^A_{AL} + T^A_{AL} - 4T^A_{AL} = 0 \implies T^A_{AL} = 0. \tag{4.17}$$

Plugging this back gives the vanishing of the full torsion tensor.

4.2 Multisymplectic Formulation of General Relativity

We can develop the multisymplectic formalism for GR by using the Cartan-Palatini formulation as a starting point. We do not need to perform a covariant Legendre transform, since the Cartan-Palatini formulation is essentially multisymplectic already. We define the multisymplectic phase space of general relativity, P, as a bundle over spacetime M with fiber coordinates $(\beta^I_\mu, \omega^I_{\mu J})$. Thus a section of $P \to M$ is a tetrad field and connection field. The multisymplectic 4-form is just the Cartan-Palatini integrand,

$$\theta = \frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge R^{KL} \in \Omega^4(P). \tag{4.18}$$

More precisely, $\beta^I = \beta^I_\mu \, \mathrm{d}x^\mu \in \Omega^1(P)$, where $\mathrm{d}x^\mu$ is interpreted as a 1-form on P via pull-back, $R^{KL} = \mathrm{d}\omega^{KL} + \omega^K{}_M \wedge \omega^{ML}$, with $\omega^{KL} = \omega^{KL}_\mu \, \mathrm{d}x^\mu \in \Omega^1(P)$, and $\mathrm{d}\omega^{KL} = \mathrm{d}\omega^{KL}_\mu \wedge \mathrm{d}x^\mu \in \Omega^2(P)$. With this definition of θ, $S = I^0_4(\theta)$ is exactly the Cartan-Palatini action, which guarantees the multisymplectic equations of motion will agree with (4.12) by the logic in section 3.6.

We can also write $\theta = \pi^{\mu\nu}{}_{IJ} \mathrm{d}\omega^{IJ}_\nu \wedge \mathrm{d}^3 x_\mu - \mathcal{H} \, \mathrm{d}^4 x$ with $\pi^{\mu\nu}{}_{IJ} = \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\alpha\beta} \, \beta^K_\alpha \beta^J_\beta$ and $\mathcal{H} = \frac{1}{2}\epsilon_{IJKL}\epsilon^{\alpha\beta\rho\sigma} \, \beta^I_\alpha \beta^J_\beta \omega^K_{\rho M} \omega^{ML}_\sigma$. We see that the multisymplectic structure of GR naturally picks out the connection as the fundamental field with the tetrad serving as a (multi)momentum. We also note that the De Donder-Weyl Hamiltonian for GR is non-vanishing, in contrast to the ordinary (3+1) Hamiltonian, which is a combination of constraints and vanishes on solutions.

4.3 Symmetries and Momentum Maps

There are two types of symmetries acting on the multisymplectic GR phase space: local Lorentz transformations and diffeomorphisms. Local Lorentz transformations belong to the group $C^\infty(M, SO(3,1))$ (with pointwise multiplication and inversion). The action of this group on P is given by

$$(\beta^I_\mu, \omega^I_{\mu J}, x^\mu) \mapsto (\Lambda^I{}_J \beta^J_\mu, \Lambda^I{}_K \omega^K_{\mu L} (\Lambda^{-1})^L{}_J + \Lambda^I{}_K \mathrm{d}(\Lambda^{-1})^K{}_J, x^\mu), \tag{4.19}$$

with all Λ, Λ^{-1} evaluated at x^μ. The diffeomorphism group $\mathrm{Diff}(M)$ acts on P by the usual lift (see appendix C),

$$(\beta^I_\mu, \omega^I_{\mu J}, x^\mu) \mapsto \left(\frac{\partial x^\nu}{\partial x'^\mu} \beta^I_\nu, \frac{\partial x^\nu}{\partial x'^\mu} \omega^I_{\nu J}, x'^\mu \right). \tag{4.20}$$

The full symmetry group is the semi-direct product $C^\infty(M, SO(3,1)) \rtimes \mathrm{Diff}(M)$, since diffeomorphisms have a natural action on local Lorentz transformations (moving the base point). The semi-direct product structure is important for composition of symmetry transformations, commutativity of flows, and hence questions of Poisson brackets and co-adjoint equivariance of momentum maps. The Lie algebra consists of pairs $(\chi^I{}_J(x), \xi^\mu(x))$, with $\chi^I{}_J(x)$ a local $\mathfrak{so}(3,1)$ element and $\xi^\mu(x)$ a vector field on M. The induced vector field for a pure local Lorentz transformation is

$$X_\chi|_{(\beta,\omega,x)} = \chi^I{}_J(x) \beta^J_\mu \frac{\partial}{\partial \beta^I_\mu} - (D\chi)^I_{\mu J}(x) \frac{\partial}{\partial \omega^I_{\mu J}}, \tag{4.21}$$

while for a pure diffeomorphism it is

$$X_\xi|_{(\beta,\omega,x)} = -\beta^I_\nu \partial_\mu \xi^\nu(x) \frac{\partial}{\partial \beta^I_\mu} - \omega^I_{\nu J} \partial_\mu \xi^\nu(x) \frac{\partial}{\partial \omega^I_{\mu J}} - \xi^\mu(x) \frac{\partial}{\partial x^\mu}. \tag{4.22}$$

We can see that these symmetry transformations preserve θ, as it is clearly gauge-invariant and involves only diffeomorphism-invariant 1-forms β, ω (see appendix C for details).

Since the symmetries preserve θ and not just $\omega = \mathrm{d}\theta$, we can obtain the momentum map explicitly as $J_{(\chi,\xi)} = i_{X_{(\chi,\xi)}} \theta$. We proceed with this computation in parts. The only non-vanishing term in $i_{X_\chi}\theta$ comes from $i_{X_\chi}\mathrm{d}\omega^{KL} = i_{X_\chi}\mathrm{d}\omega^{KL}_\mu \wedge \mathrm{d}x^\mu = -D\chi^{KL}$. The result is

$$i_{X_\chi}\theta = -\frac{1}{2}\epsilon_{IJKL}\,\beta^I \wedge \beta^J \wedge D\chi^{KL}, \tag{4.23}$$

which is of the typical form $\pi D\chi$ for local gauge transformations (see section 3.11). The insertion $i_{X_\xi}\theta$ involves terms of the form $i_{X_\xi}\beta^I, i_{X_\xi}\mathrm{d}\omega^{KL}, i_{X_\xi}\omega^K{}_M$. The first of these, $i_{X_\xi}\beta^I = \beta^I_\mu \xi^\mu = \xi^I$ gives the components of the vector ξ with respect to the tetrad frame. Next,

$$i_{X_\xi}\mathrm{d}\omega^{KL} = i_{X_\xi}(\mathrm{d}\omega^{KL}_\mu \wedge \mathrm{d}x^\mu) = -\omega^{KL}_\nu \mathrm{d}\xi^\nu - \xi^\mu \mathrm{d}\omega^{KL}_\mu = -\mathrm{d}(\xi^\mu \omega^{KL}_\mu) \equiv -\mathrm{d}\omega^{KL}_\xi, \tag{4.24}$$

where we introduce $\omega^{KL}_\xi = \xi^\mu \omega^{KL}_\mu$ as the connection along the vector ξ. Together with $i_{X_\xi}\omega^K{}_M = \omega^K_{\mu M}\xi^\mu = \omega^K_{\xi M}$, we have

$$i_{X_\xi}R^{KL} = -\mathrm{d}\omega^{KL}_\xi + \omega^K_{\xi M}\omega^{ML} - \omega^K{}_M \omega^{ML}_\xi = -\tilde{D}\omega^{KL}_\xi, \tag{4.25}$$

where $\tilde{D}$ acts just like D but on non-tensors. In particular, ω^{KL}_ξ is not a tensor with respect to local Lorentz transformations. There is no well-defined action by the covariant exterior derivative D, but we may pretend ω^{KL}_ξ is a rank two tensor based on its index structure and apply $\tilde{D}$ accordingly. A lot of the properties of D carry over to $\tilde{D}$, which makes $\tilde{D}$ useful for calculations. The final result for the full momentum map is thus

$$J_{(\chi,\xi)} = i_{X_{(\chi,\xi)}}\theta = \xi^I \left(\epsilon_{IJKL}\,\beta^J \wedge R^{KL}\right) - \frac{1}{2}\epsilon_{IJKL}\,\beta^I \wedge \beta^J \wedge \left(D\chi^{KL} + \tilde{D}\omega^{KL}_\xi\right). \tag{4.26}$$

We can interpret the two terms in the momentum map as follows. Consider the (dual) vector-valued 3-form

$$G_I = -\frac{1}{2}\epsilon_{IJKL}\,\beta^J \wedge R^{KL}. \tag{4.27}$$

The manipulations involved in simplifying (4.12a) show that this 3-form is related to the Einstein tensor. Explicitly, $G_I = -\frac{1}{4}\epsilon_{IJKL}\,R^{KL}{}_{AB}\beta^J \wedge \beta^A \wedge \beta^B = -\frac{1}{4}\epsilon_{IJKL}\epsilon^{MJAB}\,R^{KL}{}_{AB}\Omega_M$, where $\Omega_M = \frac{1}{3!}\epsilon_{MABC}\,\beta^A \wedge \beta^B \wedge \beta^C$ is the tetrad version of $\sqrt{|g|}\mathrm{d}^3x_\mu$. Following the manipulations used in (4.13),(4.14) we find $G_I = G^J{}_I\,\Omega_J$ where $G_{JI} = R_{JI} - \frac{1}{2}\eta_{JI}R$ is the Einstein tensor expressed with respect to the tetrad. This shows that the Einstein tensor should naturally be regarded as the dual-vector-valued 3-form G_I, which we call the Einstein 3-form. This interpretation of G_I in terms of the Einstein tensor does not apply directly to the momentum map, as we used $R^{KL} = \frac{1}{2}R^{KL}{}_{AB}\beta^A \wedge \beta^B$, which is not possible on P due to the presence of $\mathrm{d}\omega^{KL}$ in R^{KL}. In order to interpret the G_I in the momentum map as the Einstein 3-form, we must pull the momentum map back by test fields $f : \Sigma_4 \to P$.

For the second term of (4.26), we use

$$\begin{aligned}\mathrm{d}\left(\frac{1}{2}\epsilon_{IJKL}\,\beta^I \wedge \beta^J \wedge (\chi^{KL} + \omega^{KL}_\xi)\right) &= \tilde{D}\left(\frac{1}{2}\epsilon_{IJKL}\,\beta^I \wedge \beta^J \wedge (\chi^{KL} + \omega^{KL}_\xi)\right)\\ &= \epsilon_{IJKL}\,T^I \wedge \beta^J \wedge (\chi^{KL} + \omega^{KL}_\xi) + \frac{1}{2}\epsilon_{IJKL}\,\beta^I \wedge \beta^J \wedge \left(D\chi^{KL} + \tilde{D}\omega^{KL}_\xi\right).\end{aligned} \tag{4.28}$$

The replacement $d \to \tilde{D}$ and the Leibniz rule can be proved by explicit computation, or by simply noting that if ω^{KL}_ξ was a tensor we could do all the above manipulations using D

and using $\tilde{D}$ consists of doing exactly the same manipulations. We can plug this into $J_{(\chi,\xi)}$ (assuming we have pulled back by test fields) and drop an exact 3-form to write

$$J_{(\chi,\xi)} = -2\xi^I G_I + \epsilon_{IJKL}\, T^I \wedge \beta^J \wedge (\chi^{KL} + \omega_\xi^{KL}). \tag{4.29}$$

This momentum map vanishes on solutions (using (4.12)) because the symmetry group is local. That is, if we have a local Lie algebra element $\xi(x)$ and momentum map $J_\xi = \langle J, \xi \rangle$, we can use that $\mathrm{d}J = 0$ on solutions (from section 3.2) to conclude $\langle \mathrm{d}J, \xi \rangle + \langle J, \mathrm{d}\xi \rangle = 0$. Since $\xi(x)$ and $\mathrm{d}\xi(x)$ are independent, we can conclude that $J = 0$ (and $\mathrm{d}J = 0$).

4.4 3+1 Decomposition of Multisymplectic General Relativity

We can use the machinery developed in sections 3.4-3.8 to introduce a space+time split into multisymplectic GR. The large boundary phase space $\mathcal{P}$ consists of maps $f : \Sigma \to P$, so has local coordinates $(\beta_\mu^I(\sigma), \omega_\mu^{IJ}(\sigma), x^\mu(\sigma))$. It carries the presymplectic 1-form $\Theta = I_4^1(\theta) = I_4^1\left(\frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge R^{KL}\right)$. For simplicity and to facilitate comparisons with standard approaches such as ADM [10, 16, 30], we will work with the small boundary phase space. We assume all maps $f : \Sigma \to P$ in the small boundary phase space $\mathcal{S}$ are embeddings. Then a vector $X \in T_f\mathcal{S}$ can be represented as a vector field at the image of f,

$$X = (\delta\beta_\mu^I)_X \frac{\partial}{\partial \beta_\mu^I} + (\delta\omega_\mu^{IJ})_X \frac{\partial}{\partial \omega_\mu^{IJ}} + (\delta\lambda)_X\, N^\mu \frac{\partial}{\partial x^\mu}. \tag{4.30}$$

For embeddings, $\Theta = I_4^1(\theta)$ simplifies to

$$\begin{aligned} \Theta|_f(X) = \int_f i_X\theta = \int_\Sigma \frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge (\delta\omega^{KL})_X \\ - (\delta\lambda)_X \int_\Sigma \left(-N^I\, \epsilon_{IJKL}\, \beta^J \wedge R^{KL} + \frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge \tilde{D}(i_N\omega^{KL})\right), \end{aligned} \tag{4.31}$$

where $\delta\omega^{KL} = (\delta\omega_\mu^{KL})\mathrm{d}x^\mu + (\delta\lambda)\, \omega_\mu^{KL}\, \mathrm{d}N^\mu$, $N^I = \beta_\mu^I N^\mu$, and $i_N\omega^{KL} = N^\mu\omega_\mu^{KL}$. All forms in the integrand of (4.31) are taken to be functions on Σ. From this we see that the 3+1 Hamiltonian of GR is

$$\begin{aligned} H &= -\int_\Sigma N^I\, \epsilon_{IJKL}\, \beta^J \wedge R^{KL} - \frac{1}{2}\epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge \tilde{D}\omega_0^{KL} \\ &= -\int_\Sigma N^I\, \epsilon_{IJKL}\, \beta^J \wedge R^{KL} - \frac{1}{2}\epsilon_{IJKL}\, D(\beta^I \wedge \beta^J)\, \omega_0^{KL}, \end{aligned} \tag{4.32}$$

where we have integrated $\tilde{D}$ by parts and dropped a boundary term to get the final expression.

This approach exhibits some parallels to the traditional ADM approach for performing 3+1 decompositions of GR. Recall that $\mathcal{S}$ is the pull-back bundle $\gamma^*\mathcal{P}$ with respect to a curve $\gamma : \mathbb{R} \to C^\infty(\Sigma, M)$. The curve γ corresponds to a 1-parameter family of surfaces $\Sigma_\lambda \equiv \mathrm{img}(\gamma(\lambda)) \subset M$. In traditional ADM, each surface Σ_λ is assumed to be spacelike and the surfaces taken together form a foliation of spacetime. We will not assume our Σ_λ form a foliation as it is unnecessary and the requirement that Σ_λ be spacelike will emerge in the course of our analysis. The vector field N tangent to $\mathrm{img}(\gamma) \subset C^\infty(\Sigma, M)$ can be regarded as the vector field N^μ in M at Σ_λ. This vector field describes the infinitesimal motion $\Sigma_\lambda \mapsto \Sigma_{\lambda+\mathrm{d}\lambda}$ and is analogous to the vector field $\partial/\partial t$ in ADM. From now on, we use adapted coordinates for which $N = \partial/\partial t$ exactly. If we select our tetrad such that e_0 is always the unit normal of our surfaces Σ_λ then the orthonormal components N^0, N^a are respectively the lapse function and shift vector used in ADM. We will not make this choice of gauge, but will often refer to N by analogy to ADM as the lapse-shift vector (field).

In adapted coordinates, $N^\mu = \delta^\mu_0$, $\delta\omega^{KL} = (\delta\omega^{KL}_\mu)\mathrm{d}x^\mu$, $N^I = \beta^I_0$, and $i_N\omega^{KL} = \omega^{KL}_0$. We will use the notations interchangeably, as needed. To obtain the 3+1 decomposed equations of motion, we use the 2-form $\Omega = \mathrm{d}\Theta$. The equations are given by the kernel of Ω, but we must first identify constraints as they select the region of phase space where Ω has an appropriate kernel. To find the constraints we look at $\Omega(X_H, Y) = 0$ with

$$X_H = \dot{N}^I \frac{\delta}{\delta N^I} + \dot{\beta}^I_i \frac{\delta}{\delta \beta^I_i} + \dot{\omega}^{IJ}_0 \frac{\delta}{\delta \omega^{IJ}_0} + \dot{\omega}^{IJ}_i \frac{\delta}{\delta \omega^{IJ}_i} + \frac{\partial}{\partial t}, \tag{4.33a}$$

$$Y = (\delta N^I) \frac{\delta}{\delta N^I} + (\delta\beta^I_i) \frac{\delta}{\delta \beta^I_i} + (\delta\omega^{IJ}_0) \frac{\delta}{\delta \omega^{IJ}_0} + (\delta\omega^{IJ}_i) \frac{\delta}{\delta \omega^{IJ}_i} + (\delta\lambda) \frac{\partial}{\partial t}, \tag{4.33b}$$

where the dots in (4.33a) are the usual time derivatives since $\mathrm{d}t/\mathrm{d}\lambda = 1$. We get

$$0 = \Omega(X_H, Y) = \int_\Sigma \epsilon_{IJKL}\, \dot{\beta}^I \wedge \beta^J \wedge \delta\omega^{KL} - \epsilon_{IJKL}\, (\delta\beta^I) \wedge \beta^J \wedge \dot{\omega}^{KL} - X_H(H)\,(\delta\lambda) + \delta H, \tag{4.34}$$

where

$$\begin{aligned}
\delta H = &- \int_\Sigma \delta N^I\, \beta^J \wedge R^{KL} - \int_\Sigma \epsilon_{IJKL} N^I\, \delta\beta^J \wedge R^{KL} \\
&- \int_\Sigma \epsilon_{IJKL}\, N^I\, \beta^J \wedge D\delta\omega^{KL} - \frac{1}{2} \int_\Sigma D(\epsilon_{IJKL}\, \beta^I \wedge \beta^J)\, \delta\omega^{KL}_0 \\
&- \int_\Sigma D(\epsilon_{IJKL}\, \delta\beta^I \wedge \beta^J)\, \omega^{KL}_0 + \int_\Sigma \delta\omega^M{}_K \wedge \epsilon_{IJML}\, \beta^I \wedge \beta^J\, \omega^{KL}_0.
\end{aligned} \tag{4.35}$$

By letting the variations be arbitrary we get

$$\delta N^I : \epsilon_{IJKL}\, \beta^J \wedge R^{KL} = 0, \tag{4.36a}$$

$$\delta\omega^{KL}_0 : D(\epsilon_{IJKL}\, \beta^I \wedge \beta^J) = 0, \tag{4.36b}$$

$$\delta\beta^I : -\epsilon_{IJKL}\, \beta^J \wedge \dot{\omega}^{KL} + \epsilon_{IJKL}\, N^J \wedge R^K L + \epsilon_{IJKL}\, \beta^J \wedge \tilde{D}\omega^{KL}_0 = 0, \tag{4.36c}$$

$$\delta\omega^{KL} : \epsilon_{IJKL}\, \dot{\beta}^I \wedge \beta^J - D(\epsilon_{IJKL}\, N^I \wedge \beta^J) - \frac{1}{2}\epsilon_{IJKM}\, \beta^I \wedge \beta^J\, \omega^M_{0L} + \frac{1}{2}\epsilon_{IJLM}\, \beta^I \wedge \beta^J\, \omega^M_{0K} = 0, \tag{4.36d}$$

with all forms implicitly restricted to Σ. Recall that since we are working through the constraint algorithm, we do not regard (4.36) as equations of motion, but rather use them to identify constraints. We immediately see that (4.36a) and (4.36b) are constraints. There are further constraints in (4.36d) as it is 18 equations for only 12 unknowns ($\dot{\beta}^I$).

To manipulate (4.36) and obtain these additional constraints, it is convenient to work with true tensors. Noting that time derivatives can be expressed as Lie derivatives $\mathcal{L}_N$ along N, we introduce

$$U^I = i_N T^I = i_N(\mathrm{d}\beta^I + \omega^I{}_J \wedge \beta^J) = \dot{\beta}^I - \mathrm{d}N^I + \omega^I_{0J}\beta^J - \omega^I{}_J N^J = \dot{\beta}^I - DN^I + \omega^I_{0J}\beta^J, \tag{4.37a}$$

$$S^{KL} = i_N R^{KL} = i_N(\mathrm{d}\omega^{KL} + \omega^K{}_M \wedge \omega^M L) = \dot{\omega}^K L - \tilde{D}\omega_0^{KL}, \tag{4.37b}$$

where we used the Cartan identity $\mathcal{L}_N = i_N\mathrm{d} + \mathrm{d}i_N$ to obtain the time derivatives. Notice that we can solve for $\dot{\beta}$ and $\dot{\omega}$ in terms of the tensors U and S. With these definitions, we can rewrite (4.36c)-(4.36d) as

$$\delta\beta^I : -\epsilon_{IJKL}\,\beta^J \wedge S^{KL} + \epsilon_{IJKL}\,N^J \wedge R^{KL} = 0, \tag{4.38a}$$

$$\delta\omega^{KL} : \epsilon_{IJKL}(U^I \wedge \beta^J + T^I N^J) = 0. \tag{4.38b}$$

We now work to extract constraints from (4.38b). We will frequently convert from tetrad to coordinate indices in the course of the following computations and will represent coordinate indices with lowercase Greek letters, spatial coordinate indices with lowercase Latin letters, and the time index with a 0. Converting (4.38b) to coordinate indices gives

$$\Omega_{\mu\nu\alpha\beta}(U^\mu \wedge\, \delta^\nu_i \mathrm{d}x^i + T^\mu \delta^\nu_0) = 0, \tag{4.39}$$

where $\Omega_{\mu\nu\alpha\beta} = \sqrt{|g|}\epsilon_{\mu\nu\alpha\beta}$ is the ϵ tensor in a coordinate frame. Substituting $T^\mu = \frac{1}{2}T^\mu_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$ and setting the two free indices α, β to either $0, k$ or k, l gives

$$\Omega_{lj0k}U^l_i - \Omega_{li0k}U^l_j = 0, \tag{4.40a}$$

$$\Omega_{0jkl}U^0_i - \Omega_{0ikl}U^0_j + \Omega_{m0kl}T^m_{ij} = 0. \tag{4.40b}$$

The first equation is just $\epsilon_{ljk}U^l_i - (i \leftrightarrow j) = 0$. Multiplying this by ϵ^{ijm} gives $\epsilon_{ljk}\epsilon^{ijm}U^l_i = \delta^i_{[l}\delta^m_{k]}U^l_i = 0$. This further simplifies to $\mathrm{tr}(U)\delta^m_k - U^m_k = 0$, in which we can contract m and k to get $\mathrm{tr}(U) = 0$ and conclude $U^m_k = 0$. We can similarly multiply the second equation by ϵ^{ijn}, which gives

$$-2U^0_k\delta^n_l + 2U^0_l\delta^n_k - \epsilon_{mkl}\epsilon^{ijn}T^m_{ij} = 0. \tag{4.41}$$

Contracting n with k lets us solve for U^0_l,

$$U^0_l = -\frac{1}{2}T^m_{ml}, \tag{4.42}$$

which we can plug back in to find the constraints

$$2\epsilon^{kna}T^m_{mk} - \epsilon^{ijn}T^a_{ij} = 0. \tag{4.43}$$

This together with (4.36a)-(4.36b) define the constraint manifold at this step of the constraint algorithm. We can rearrange (4.36b) and (4.43) into the simpler equation $T^I\big|_\Sigma = 0$. To see this, first introduce

$$V^{an} \equiv \epsilon^{ijn} T^a_{ij} \implies T^a_{ij} = \frac{1}{2}\epsilon_{ijn} V^{an}. \tag{4.44}$$

Plugging this into (4.43) we find that the symmetric part of V vanishes, $V^{(ab)} = 0$. Converting (4.36b) to coordinate indices gives

$$\Omega_{\mu k \alpha\beta} T^\mu_{ij} \epsilon^{ijk} = 0. \tag{4.45}$$

By setting $\alpha, \beta = 0, n$ and plugging in V, we get $\epsilon_{mkn} V^{mk} = 0$, which reduces to $V^{mk} = 0$ since V is already anti-symmetric. Thus we have $T^i_{jk} = 0$ from (4.44). Setting α, β in (4.45) equal to a, b, we get $\epsilon_{kab}\epsilon^{ijk} T^0_{ij} = 0 \implies T^0_{ij} = 0$. All together we have $T^I_{ij} = 0$, as stated.

The full system of constraints obtained in the first step of the constraint algorithm is thus

$$\epsilon_{IJKL}\, \beta^J \wedge R^{KL}\big|_\Sigma = 0, \tag{4.46a}$$

$$T^I\big|_\Sigma = 0. \tag{4.46b}$$

Let C be the constraint manifold given by (4.46). To see if there are additional constraints, we need to check whether a suitable kernel of Ω exists tangent to C. We can find the kernel of Ω at C directly from (4.38). We just need to check whether intersecting this kernel with TC produces any further constraints. A simpler approach is to note that both the kernel of Ω and the constraints (4.46) can be obtained from the covariant equations

$$\epsilon_{IJKL}\, \beta^J \wedge R^{KL} = 0, \tag{4.47a}$$

$$T^I = 0, \tag{4.47b}$$

by either directly restricting to Σ (constraints) or first inserting N and then restricting to Σ (kernel). The kernel at C is thus

$$\epsilon_{IJKL}\, N^J \wedge R^{KL} - \epsilon_{IJKL}\, \beta^J \wedge S^{KL} = 0, \tag{4.48a}$$

$$U^I = 0, \tag{4.48b}$$

with all forms restricted to Σ. Intersecting with TC to find additional constraints is equivalent to checking whether (4.48) preserve the constraints (4.46). To check whether $\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon_{IJKL}\, \beta^J \wedge R^{KL}\right)\big|_\Sigma = 0$ follows from the equations, we act $i_N D$ in two different ways:

$$\begin{aligned} i_N D(\epsilon_{IJKL}\beta^J \wedge R^{KL}) = \frac{\mathrm{d}}{\mathrm{d}t}(\epsilon_{IJKL}\beta^J \wedge R^{KL}) - \mathrm{d}\left[\epsilon_{IJKL}(N^J R^{KL} - \beta^J \wedge S^{KL})\right] + \\ \omega^M_{0I}\epsilon_{MJKL}\, \beta^J \wedge R^{KL} - \omega^M{}_J \wedge \epsilon_{IJKL}(N^J R^{KL} - \beta^J \wedge S^{KL}) \\ = \epsilon_{IJKL}\,(U^J \wedge R^{KL} + T^J \wedge S^{KL}). \end{aligned} \tag{4.49}$$

We can eliminate almost every term using either the equations (4.48) or constraints (4.46), leaving just $\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon_{IJKL}\, \beta^J \wedge R^{KL}\right)\big|_\Sigma = 0$. Similarly, to check whether $\dot{T}^I\Big|_\Sigma = 0$ follows from the equations, act $i_N D$ in two different ways:

$$i_N D T^I = \dot{T}^I - \mathrm{d}U^I + \omega^I_{0J} T^J - \omega^I{}_J \wedge U^J = i_N(R^I{}_J \wedge \beta^J). \tag{4.50}$$

Clearly $U^I = 0, T^I\big|_\Sigma = 0$ from equations of motion and constraints, respectively. Taken together, these imply that the full torsion vanishes and hence $0 = DT^I = R^I{}_J \wedge \beta^J$. Thus $i_N(R^I{}_J \wedge \beta^J) = 0$, and plugging into (4.50) gives $\dot{T}^I\Big|_\Sigma = 0$. Thus $\dot{T}^I\Big|_\Sigma = 0$ follows from the equations of motion and constraints and there are no additional constraints. The final constraint manifold is C, given by (4.46). Hamilton's equations are (4.48) evaluated along C.

4.5 Interpretations of the Constraints and their Relation to Momentum Maps

The constraints (4.46) have interesting interpretations. The constraint (4.46b) is the vanishing of restricted torsion. This implies that the 3-torsion vanishes and hence the 3-connection on a spatial slice is Levi-Civita. The vanishing of the remaining component of the restricted torsion is related to the symmetry of the second fundamental form of our spatial slices (it would exactly imply the symmetry of the second fundamental form if e_0 were aligned with the normal to the surface). The other constraint (4.46a) is the vanishing of the restricted Einstein 3-form $G_I|_\Sigma = 0$. In terms of the usual Einstein tensor, we have $G^{0\mu} = 0$. This constraint appears in ADM as well, where $G^{00} = 0$ is called the energy, Hamiltonian, or scalar constraint and $G^{0i} = 0$ is called the momentum, spatial diffeomorphism, or vector constraint. Our constraint is in a different phase space from ADM, so we should not make a direct comparison; however, the constraints do become the same after the symmetry reduction discussed in section 4.6.

A different interpretation can be made by breaking up the torsion constraint into its original form,

$$\epsilon_{IJKL}\,\beta^J \wedge R^{KL} = 0, \tag{4.51a}$$

$$\epsilon_{IJKL}\,T^I \wedge \beta^J = 0, \tag{4.51b}$$

$$2\epsilon^{kna}T^m_{mk} - \epsilon^{ijn}T^a_{ij} = 0. \tag{4.51c}$$

The first two equations are actually the vanishing of a momentum map. Recall the multisymplectic momentum map $J_{(\chi,\xi)}$ given by (4.26). The 3+1 decomposition induces the 3+1 momentum map

$$\mathcal{J}_{(\chi,\xi)} = I^0_3 J_{(\chi,\xi)} = \int\limits_\Sigma \xi^I \left(\epsilon_{IJKL}\,\beta^J \wedge R^{KL}\right) - \frac{1}{2}\epsilon_{IJKL}\,\beta^I \wedge \beta^J \wedge \left(D\chi^{KL} + \tilde{D}\omega^{KL}_\xi\right). \tag{4.52}$$

This 3+1 momentum maps will vanish on solutions just as the multisymplectic $J_{(\chi,\xi)}$ did. The vanishing of $\mathcal{J}_{(\chi,\xi)}$ for $SO(3,1)$ gauge transformations implies (4.51b) while its vanishing for diffeomorphisms (modulo $SO(3,1)$ transformations) implies (4.51a). This holds even though diffeomorphisms are not symmetry transformations on the small boundary phase space. Diffeomorphisms are a true symmetry on the large boundary phase space, leading to the vanishing of the momentum map on solutions. The components of the momentum map are well-defined functions on the small boundary phase space and remain zero on solutions despite no longer being the components of a true momentum map.

The final constraint (4.51c) does not have an interpretation in terms of a momentum map, but rather can be restricted to without introducing spurious solutions and so is second-class (see appendix D for further discussion). The easiest way to see this is to restrict the theory to $T^I\big|_\Sigma = 0$ and work out the consequences. To perform this restriction, we introduce Lagrange multipliers into a 3+1 decomposition of the action

$$S = \int \mathrm{d}t \int_\Sigma \frac{1}{2}\epsilon_{IJKL}\,\beta^I \wedge \beta^J \wedge \dot{\omega}^{KL} + N^I \epsilon_{IJKL}\,\beta^J \wedge R^{KL} + \epsilon_{IJKL}\,T^I \wedge \beta^J\,\omega_0^{KL} + \mu_I \wedge T^I, \quad (4.53)$$

where μ_I is a 1-form of Lagrange multipliers and $\mu_I = 0$ corresponds to our original 3+1 equations (4.36). Varying with respect to all the fields, we get

$$\delta N^I : \epsilon_{IJKL}\,\beta^J \wedge R^{KL} = 0, \quad (4.54a)$$

$$\delta\omega_0^{KL} : D(\epsilon_{IJKL}\,\beta^I \wedge \beta^J) = 0, \quad (4.54b)$$

$$\delta\beta^I : -\epsilon_{IJKL}\,\beta^J \wedge \dot{\omega}^{KL} + \epsilon_{IJKL}\,N^J \wedge R^K L + \epsilon_{IJKL}\,\beta^J \wedge \tilde{D}\omega_0^{KL} + D\mu_I = 0, \quad (4.54c)$$

$$\begin{aligned}\delta\omega^{KL} : &\ \epsilon_{IJKL}\,\dot{\beta}^I \wedge \beta^J - D(\epsilon_{IJKL}\,N^I \wedge \beta^J) \\ &- \frac{1}{2}\epsilon_{IJKM}\,\beta^I \wedge \beta^J\,\omega_{0L}^M + \frac{1}{2}\epsilon_{IJLM}\,\beta^I \wedge \beta^J\,\omega_{0K}^M - \frac{1}{2}\mu_K \wedge \beta_L + \frac{1}{2}\mu_L \wedge \beta_K = 0,\end{aligned} \quad (4.54d)$$

$$\delta\mu_I : T^I = 0. \quad (4.54e)$$

As before we can rewrite (4.54d) using U^I,

$$\epsilon_{IJKL}(U^I \wedge \beta^J + T^I N^J) - \frac{1}{2}\mu_K \wedge \beta_L + \frac{1}{2}\mu_L \wedge \beta_K = 0. \quad (4.55)$$

Converting to coordinate indices gives

$$\Omega_{\mu\nu\alpha\beta}\epsilon^{ijk}\left(U_i^\mu \delta_j^\nu + \frac{1}{2}\delta_0^\nu T_{ij}^\mu\right) - \frac{1}{2}\epsilon^{ijk}[(\mu_\alpha)_i g_{\beta j} + (\mu_\beta)_i g_{\alpha j}] = 0, \quad (4.56)$$

where $\mu_\alpha = (\mu_\alpha)_i \mathrm{d}x^i$ and $g_{\mu\nu}$ is the metric with respect to a coordinate frame. We recognize that $T_{ij}^\mu = 0$ due to the constraint (4.54e). Setting $\alpha, \beta = a, b$ and multiplying through by ϵ^{abl}, we get

$$\sqrt{|g|}U_i^0 \epsilon^{ilk} = -(\mu_a)_i g_{bj}\epsilon^{ijk}\epsilon^{abl} = -h(h^{ia}h^{kl} - h^{il}h^{ak})(\mu_a)_i, \quad (4.57)$$

where we have introduced the 3-metric $h_{ij} \equiv g_{ij}$, its 3-determinant h, and its inverse h^{ij} ($h^{ij}h_{jk} = \delta_k^i$). Dropping the parentheses on μ for convenience, we have

$$\sqrt{|g|}U_i^0 \epsilon^{ilk} = h(\mu^{kl} - h^{kl}\mathrm{tr}(\mu)), \quad (4.58)$$

which we can symmetrize on k, l to show that the symmetric part $\mu^{(kl)}$ vanishes. We can check what constraint $\mu^{(kl)}$ is enforcing by looking at $\mu_I \wedge T^I = (\mu_\alpha)_i T_{jk}^\alpha \epsilon^{ijk}\mathrm{d}^3x$. Using V from (4.44), we see that the Lagrange multiplier terms are $(\mu_0)_i T_{jk}^0 \epsilon^{ijk}\mathrm{d}^3x + (\mu_a)_i V^{ai}\mathrm{d}^3x$. Thus, the symmetric part of $(\mu_a)_i$ is enforcing the vanishing of the symmetric part of V^{ai}, which is exactly the constraint (4.51c). If we wish to restrict only to the constraint (4.51c), we can set all of $\mu_I = 0$ except $\mu_{(ai)}$. The resulting restricted system is equivalent to the original unrestricted one by virtue of $\mu_{(ai)}$ dropping out. Hence, we can restrict to (4.51c) without introducing any spurious solutions.

4.6 Reducing by Local Lorentz Transformations

We now show how to obtain the traditional ADM approach to GR from our 3+1 decomposition of multisymplectic GR. The goal is to reduce by the local $SO(3,1)$ gauge symmetry. In a symplectic setting, reducing by a symmetry involves either symplectic reduction or Poisson reduction (see for example [2,3]). Since we are working in a presymplectic space, we need to modify symplectic reduction to suit our needs.

The action of the local Lorentz group $C^\infty(M, SO(3,1))$ leaves the constraint manifold C given by $G_I|_\Sigma = 0$, $T^I\big|_\Sigma = 0$ invariant since $G_I \mapsto (\Lambda^{-1})^J{}_I G_J = 0$ and $T^I \mapsto \Lambda^I{}_J T^J = 0$. We will focus on the larger invariant constraint manifold $\tilde{C} : T^I\big|_\Sigma = 0$. This manifold (since it is invariant under the group action) is foliated by group orbits of $C^\infty(M, SO(3,1))$ and we can perform the reduction $\tilde{C}/C^\infty(M, SO(3,1)) \equiv \tilde{\mathcal{S}}$. The reason for using $\tilde{C}$ rather than C in the reduction is that the remaining constraints $G_I|_\Sigma = 0$ can be shown to be constant along the group orbits and thus project down to $\tilde{\mathcal{S}}$. Furthermore, these constraints (in the form $G^0{}_\mu = 0$) are present in ADM, so it is useful to keep them through the reduction. Note that in symplectic reduction, we start with a level set of a momentum map and reduce by the group orbit along the level set (which is also the kernel of the symplectic form restricted to the level set). In Poisson reduction, we reduce the phase space directly using the group orbits. In our case we have a hybrid approach where we use the group orbits along a submanifold that is not a level set of a momentum map, though $\tilde{C}$ does lie inside the level set of a momentum map, and this is important to the geometry of reduction.

Just as in symplectic reduction, we would like to project $\Omega|_{\tilde{C}}$ down to $\tilde{\mathcal{S}}$, so we first check if this is possible. In order for Ω to be projectable, $\Omega|_f(X,Y)$ must be invariant as $f \in \tilde{C}$ moves along a group orbit or if X or Y are changed by a vector tangent to the group orbit. So if Z is a vector field tangent to the group orbits on $\tilde{C}$, we must have $\pounds_X \Omega = 0$ and $\Omega(X+Z, Y) = \Omega(X, Y+Z) = \Omega(X,Y) \implies i_Z\Omega = 0$. These two conditions apply to projecting a form of any degree, so for example Θ is projectable if $\pounds_Z\Theta = 0, i_Z\Theta = 0$ for any Z tangent to the group orbits. A vector field tangent to the group orbits can be written as a linear combination of induced vector fields X_χ associated with the group action of $C^\infty(M, SO(3,1))$. The conditions then reduce to $\pounds_{X_\chi}\Omega = 0, i_{X_\chi}\Omega = 0$ for all X_χ. We have already shown in section 4.3 that the multisymplectic forms ω, θ are invariant under the flow of X_χ. Since the I-map and Lie derivative commute, it follows that $\pounds_{X_\chi}\Omega = 0$ and $\pounds_{X_\chi}\Theta = 0$. Furthermore we have $i_{X_\chi}\Omega = -\mathrm{d}\mathcal{J}_\chi$, $i_{X_\chi}\Theta = \mathcal{J}_\chi$ from applying the I-map to the multisymplectic equations $i_{X_\chi}\omega = -\mathrm{d}J_\chi, i_{X_\chi}\theta = J_\chi$. Since

$$\mathcal{J}_\chi = \int_\Sigma \epsilon_{IJKL}\, T^I \wedge \beta^J \chi^{KL}, \tag{4.59}$$

we have that $J_\chi = 0$ on $\tilde{C}$. Hence $i_{X_\chi}\Omega = 0, i_{X_\chi}\Theta = 0$ and both Ω, Θ can be projected down to $\tilde{\mathcal{S}}$.

To complete the projection, we find coordinates on $\tilde{\mathcal{S}}$ and use these to write out the reduced Θ. The easiest way to do this is to find functions on $\tilde{C}$ that are invariant under the group flow. It is easy to check that the functions

$$g_{\mu\nu} = \beta^I_\mu \beta^J_\nu \eta_{IJ}, \tag{4.60a}$$

$$\Gamma^m u_{i\nu} = e^\mu_I \partial_i \beta^I_\nu + e^\mu_I \beta^J_\nu \omega^I_{iJ}, \tag{4.60b}$$

are invariant under local Lorentz transformations. The functions $g_{\mu\nu}$ are 10 independent functions on $\tilde{C}$, while $\Gamma^\mu_{i\nu}$ consist of only 6 independent functions. The reason is that the constraint $T^I\big|_\Sigma = 0$ implies $\Gamma^\mu_{[ij]} = 0$ and metric compatibility $\mathrm{d}g_{\mu\nu} - \Gamma_{(\mu\nu)} = 0$ implies the relations $(\Gamma_{(\mu\nu)})_i = \partial_i g_{\mu\nu}$. A convenient way of expressing the 6 independent Γ functions is to introduce the normal vector (to the surfaces Σ_λ) n^μ and second fundamental form $K_{ij} = (\nabla_i n)_j = \partial_i n_j - \Gamma^\mu_{ij} n_\mu$. The normal vector satisfies $n_i = 0, n^\mu n_\mu = -1$ and so

$$n^\mu = \frac{g^{\mu 0}}{\sqrt{-g^{00}}} \tag{4.61a}$$

$$K_{ij} = -\frac{\Gamma^0_{ij}}{\sqrt{-g^{00}}}. \tag{4.61b}$$

We may regard (4.61) as convenient functions on $\tilde{C}$, independent of their relationship to the geometry of the embeddings Σ_λ. All of the functions (4.60), (4.61) are invariant under the group orbits and so project down to $\tilde{C}$. We express Θ in terms of these functions, from which the projection of Θ will be clear.

On $\tilde{C}$ we have

$$\Theta\big|_f (X) = \int_\Sigma \frac{1}{2} \epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge (\delta\omega^{KL})_X + (\delta\lambda)_X \int_\Sigma N^I \epsilon_{IJKL}\, \beta^J \wedge R^{KL}, \tag{4.62}$$

where we have dropped a boundary term and a term proportional to $T^I\big|_\Sigma$. The Hamiltonian $H = -\int_\Sigma N^I \epsilon_{IJKL}\, \beta^J \wedge R^{KL}$ is proportional to the integral of the component ${G^0}_0$ of the Einstein tensor. In turn, ${G^0}_0$ can be expressed in terms of (4.60). Since Θ is invariant under the group flow and its second term is also invariant, it follows that $\epsilon_{IJKL}\beta^I \wedge \beta^J \wedge \delta\omega^{KL}$ (restricted to $\tilde{C}$) must be invariant under local Lorentz transformations. If we pick a gauge and in that gauge re-express $\epsilon_{IJKL}\beta^I \wedge \beta^J \wedge \delta\omega^{KL}$ using the manifestly gauge-invariant (4.60), (4.61), then our relationship must hold independent of the gauge choice. We choose our gauge such that $e_0 = n$. In this gauge, $\beta^I_i = 0$ and $K_{ij} = (\nabla_i e_0)_j = \beta_{jI}\omega^I_{i0} = \omega^0_{ij}$ (here the 0 index refers to a tetrad frame). We then have

$$\frac{1}{2}\, \epsilon_{IJKL}\, \beta^I \wedge \beta^J \wedge \delta\omega^{KL}\big|_\Sigma = \epsilon_{abc}\epsilon^{ijk}\beta^a_i \beta^b_j \delta\omega^{0c}_k \,\mathrm{d}^3x, \tag{4.63}$$

where indices a, b, c refer to non-zero tetrad indices. Introducing the 3-metric h_{ij} and noting $\omega^{0c}_k = \omega^0_{kl} e^l_d \eta^{dc}$, we get

$$2\sqrt{h} e^k_c \delta(K_{kl} e^l_d \eta^{dc}) \mathrm{d}^3x = 2\left[\sqrt{h} h^{kl} \delta(K_{kl}) + \sqrt{h} K_{kl} \eta^{dc} e^k_c \delta e^l_d\right] \mathrm{d}^3x, \tag{4.64}$$

where we have used $\det(\beta^a_i) = \sqrt{h}$. Since $h^{kl} = \eta^{dc} e^k_d e^l_c$, $\delta h^{kl} = \eta^{dc} e^k_c \delta e^l_d + (k \leftrightarrow l)$ and thus $K_{kl}\delta h^{kl} = 2K_{kl}\eta^{dc} e^k_c \delta e^l_d$. We can thus express Θ in a manifestly gauge-invariant way,

$$\Theta\big|_f (X) = \int_\Sigma \mathrm{d}^3x\, 2\sqrt{h} h^{kl} \delta(K_{kl})_X + \sqrt{h} K_{kl} (\delta h^{kl})_X - (\delta\lambda)_X \int_\Sigma \mathrm{d}^3x\, 2{G^0}_0. \tag{4.65}$$

We can further reduce this to the standard ADM presymplectic structure using a Legendre transform to move the δ from K to h

$$\Theta|_f(X) = \int_\Sigma \sqrt{h} \mathrm{d}^3x\, \pi^{ij} \delta h_{ij} - (\delta\lambda)_X \int_\Sigma \mathrm{d}^3x\, 2G^0{}_0, \tag{4.66}$$

with $\pi^{ij} = K^{ij} - h^{ij}\mathrm{tr}(K)$.

Chapter 5

Conclusion

We have described the geometrical foundations necessary for understanding covariant field theories and, in particular, gravity. We used particle mechanics to build our understanding of symplectic and presymplectic geometry along with inhomogeneous and homogeneous formulations. In these relatively simple cases, we discussed momentum maps and constraints, which are critically important for understanding field theories.

We constructed the multisymplectic approach to field theories, building upon the presymplectic approach to particle mechanics. The key difference being that field theories use a multisymplectic 5-form in place of the presymplectic 2-form. Furthermore, field theories lend themselves to a 3+1 decomposition, leading to an infinite-dimensional presymplectic system. The path from multisymplectic geometry to traditional approaches based on infinite-dimensional symplectic geometry involves two intermediate phase spaces: the large and small boundary phase spaces. We developed a simple method (leveraging the I-map) for moving geometrical objects from the multisymplectic phase space to the large boundary phase space and then onto the small boundary phase space. In this way, the 3+1 decomposition becomes straightforward and clear. We took further advantage of the properties of the I-map to analyze 3+1 decompositions in the case when the spatial slices have boundaries. This produced interesting results relating boundary conditions to the presymplectic structure of the theory, as well as producing boundary modifications to Hamiltonians and momentum maps.

We detailed how these abstract ideas play out in typical examples, such as the scalar field and electromagnetism. Gravity was our final example, though its uniqueness warranted a separate chapter. After reformulating general relativity in tetrad language, we could apply many of our multisymplectic ideas directly. In particular, the 3+1 decomposition went over smoothly. We noted the interesting relationships between diffeomorphisms, momentum maps, and constraints, before proceeding to a reduction that would recover the standard ADM 3+1 structure. Lastly, we pointed out the role our boundary formulation played in understanding the ADM momentum and black hole entropy.

The geometric ideas presented here provide a powerful tool for analyzing field theories both in traditional regimes that have been extensively studied and in brand new regimes that are now much more accessible.

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Appendix A

Tangent Vectors to Spaces of Maps as Maps into Tangent Bundles

In this appendix, we describe elements of $T\left[C^{\infty}(\Sigma, M)\right]$, the tangent bundle to $C^{\infty}(\Sigma, M) = \{\text{smooth maps} : \Sigma \to M\}$. The notation of this appendix is independent of the rest of the paper, but Σ in this appendix roughly corresponds to Σ in the main body, while M corresponds to either M or P. The primary goal of this appendix is to demonstrate the dual role of tangent vectors to $C^{\infty}(\Sigma, M)$. On the one hand, they are simply tangent vectors to $C^{\infty}(\Sigma, M)$, while on the other they are "vector fields along maps," [42, 43] a term we will define below.

First, we give the intuition behind this duality. A tangent vector $\tilde{X} \in T_f[C^{\infty}(\Sigma, M)]$ is intuitively an infinitesimal motion from the map $f \in C^{\infty}(\Sigma, M)$ to the nearby map $f + \delta f$. But we can visualize this infinitesimal motion in M. For $\sigma \in \Sigma$, the infinitesimal motion from $f(\sigma)$ to $(f + \delta f)(\sigma)$ is a tangent vector to M. When f is an embedding (f is full rank and its image has no self-intersections), the collection of these tangent vectors for all $\sigma \in \Sigma$ produces a vector field defined on the image of f. Intuitively, it points from the image of f to the image of $f + \delta f$ (see Figure A.1). When f is not an embedding, the collection of vectors does not form a vector field defined over the image of f (see Figure A.2 with caption). The vectors in the collection are, however, parametrized by σ and so can be viewed as a map X from Σ into TM, where $X(\sigma)$ is the vector pointing from $f(\sigma)$ to $(f + \delta f)(\sigma)$. This map $X : \Sigma \to TM$ which corresponds to the vector $\tilde{X} \in T_f[C^{\infty}(\Sigma, M)]$ is called the "vector field along f" (see [42, 43] and references therein for more details).

This correspondence is one-to-one, as we now show by a more formal analysis. A tangent vector to $C^{\infty}(\Sigma, M)$ at $f_0 \in C^{\infty}(\Sigma, M)$ is an equivalence class of curves, $\tilde{X} = [f_\lambda]$ where f^1_λ and f^2_λ are equivalent if the directional (functional) derivatives along the curves are the same at f_0. More explicitly, $f^1_\lambda \sim f^2_\lambda$ if and only if for all smooth functionals $F : C^{\infty}(\Sigma, M) \to \mathbb{R}$ we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} F[f^1_\lambda] = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} F[f^2_\lambda]. \tag{A.1}$$

We now show that for each vector $\tilde{X}$ tangent to $C^{\infty}(\Sigma, M)$, there is a map $X \in C^{\infty}(\Sigma, TM)$.

To each parametrized curve $\lambda \mapsto f_\lambda \in C^{\infty}(\Sigma, M)$ is associated a family of parametrized curves $\lambda \mapsto f_\lambda(\sigma) \in M$, with $\sigma \in \Sigma$. Each member of the family belongs to an equivalence class $[f_\lambda(\sigma)]$ defining a vector $X_\sigma \in T_{f_0(\sigma)}M$. Thus, given a representative of $\tilde{X} = [f_\lambda]$, we

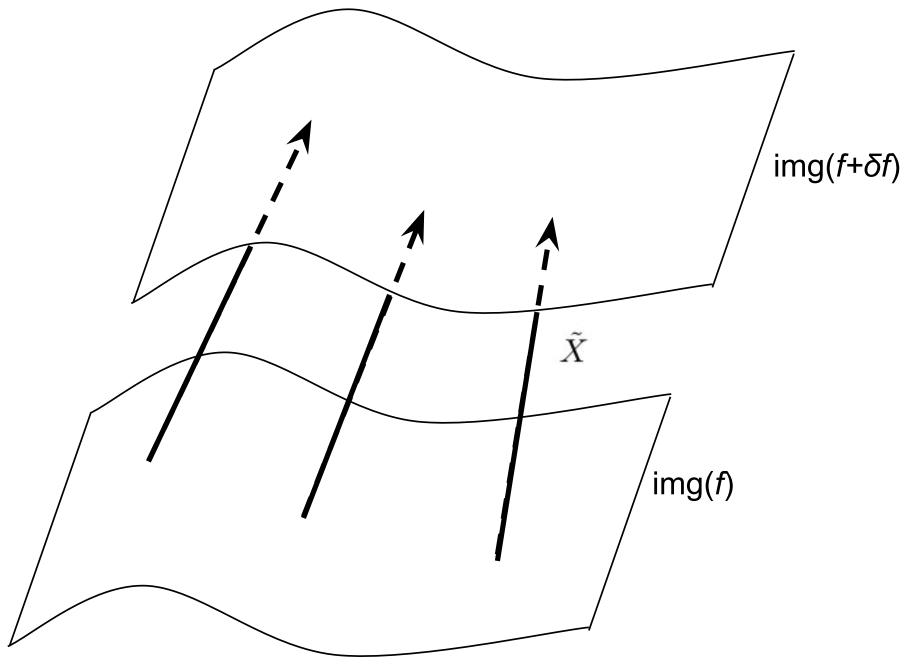

Figure A.1: The vector field along f ($\tilde{X}$) represented in M, the target space of f. In this case f is an embedding, so $\tilde{X}$ is a vector field on M, but defined only on the image of f.

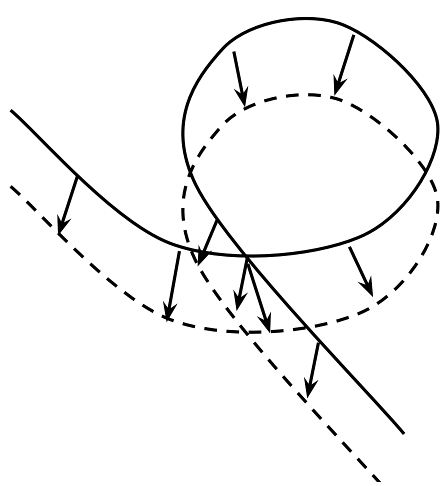

Figure A.2: Vector field along a map whose image has a self-intersection. Note the two vectors assigned to the intersection point.

can identify a map $X : \Sigma \to TM : \sigma \mapsto X_\sigma$. We now need to show that picking a different representative of $\tilde{X} = [f_\lambda]$ yields the same map X. That is, given any two curves $f^1_\lambda, f^2_\lambda \in \tilde{X}$ satisfying (A.1), we want $f^1_\lambda(\sigma), f^2_\lambda(\sigma)$ to be in the equivalence class X_σ. This means the curves $f^1_\lambda(\sigma), f^2_\lambda(\sigma)$ need to satisfy

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \phi(f^1_\lambda(\sigma)) = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \phi(f^2_\lambda(\sigma)), \tag{A.2}$$

for all smooth $\phi : M \to \mathbb{R}$ and all $\sigma \in \Sigma$. In fact (A.2) follows from (A.1) by inserting the functionals $F_{\phi,\sigma}$ defined by $F_{\phi,\sigma}[f] = (\phi \circ f)(\sigma)$ into (A.1). Thus the entire equivalence class $\tilde{X} = [f_\lambda]$ has a set of corresponding equivalence classes $\{X_\sigma = [f_\lambda(\sigma)] | \sigma \in \Sigma\}$. Hence, given any vector $\tilde{X} \in T_{f_0}[C^\infty(\Sigma, M)]$, we obtain a unique map $X : \Sigma \to TM : \sigma \mapsto X_\sigma$.

To show the correspondence is one-to-one, we now want to show that every $X \in C^\infty(\Sigma, TM)$ corresponds to a unique $\tilde{X} \in T[C^\infty(\Sigma, M)]$. First, we define $f_0 = \pi \circ X$ where $\pi : TM \to M$ is the bundle projection map. This then specifies the point of $C^\infty(\Sigma, M)$ at which the vector $\tilde{X}$ will be attached. Next, note that $X(\sigma) \equiv X_\sigma$ is an equivalence class of curves $[\gamma_\lambda]_\sigma$ where two curves $(\gamma^1_\lambda)_\sigma, (\gamma^2_\lambda)_\sigma \in [\gamma_\lambda]_\sigma$ have equal directional derivatives along them at $\lambda = 0$. Given local coordinates x^μ on M (we will not need coordinates on Σ) there is a curve $(\gamma_\lambda)_\sigma \in [\gamma_\lambda]_\sigma$ with coordinate representation $f^\mu_0(\sigma) + \lambda X^\mu_\sigma$, where f^μ_0 is the coordinate representation of $f_0 = \pi \circ X$ and $X_\sigma = X^\mu_\sigma \partial/\partial x^\mu$. We can then define a curve $\lambda \mapsto f_\lambda \in C^\infty(\Sigma, M)$ via $f^\mu_\lambda(\sigma) = f^\mu_0(\sigma) + \lambda X^\mu_\sigma$. This curve belongs to an equivalence class $\tilde{X} = [f_\lambda]$ for some $\tilde{X} \in T_{f_0}[C^\infty(\Sigma, M)]$, and so we have identified a vector $\tilde{X}$. We now show this identification is coordinate-invariant. This is needed so that when covering the manifold in coordinate patches, we get agreement in the overlap regions. In a different set of coordinates x'^μ we would have picked the curve $f'^\mu_\lambda(\sigma) = f'^\mu_0(\sigma) + \lambda X'^\mu_\sigma = f'^\mu_0(\sigma) + \lambda \frac{\partial x'^\mu}{\partial x^\nu} X^\nu_\sigma$ where f'^μ_0 is the coordinate representation of f_0 in the new coordinates. We need to show that f_λ and f'_λ belong to the same equivalence class $\tilde{X}$ by showing they satisfy (A.1). We first note that they both pass through the same point f_0. We now compute

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} F[f'_\lambda] = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} F\left[f'^\mu_0(\sigma) + \lambda \frac{\partial x'^\mu}{\partial x^\nu} X^\nu_\sigma\right] = \frac{\delta F}{\delta f'^\mu}\frac{\partial x'^\mu}{\partial x^\nu} X^\nu_\sigma = \frac{\delta F}{\delta f^\nu} X^\nu_\sigma = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} F[f_\lambda], \tag{A.3}$$

and so (A.1) is satisfied. Furthermore, we see this is the inverse of the identification in the previous paragraph. This follows since taking the specific element $f_\lambda \in \tilde{X}$ given by $f^\mu_\lambda(\sigma) = f^\mu_0(\sigma) + \lambda X^\mu_\sigma$, gives a family of curves on M produced by $f_\lambda(\sigma)$, each of which belongs to an equivalence class given by the corresponding X_σ. Thus each map $X : \Sigma \to TM$ can be used to uniquely identify a tangent vector to $C^\infty(\Sigma, M)$. So in mapping $\tilde{X}$ to X_σ we showed that the equivalence classes of curves $\tilde{X}$ maps to the set of equivalence classes X_σ. In mapping X_σ to $\tilde{X}$, we showed a set of curves in the equivalence classes X_σ can be used to identify a $\tilde{X}$, but this confirms that the entire set of equivalence classes X_σ must map to $\tilde{X}$ because the equivalence class $\tilde{X}$ maps to the set of equivalence classes X_σ.

Appendix B

Insertion-Integration Map

As with the previous appendix, we will be interested in generic manifolds Σ and M, except that now we require $\dim(\Sigma) \leq \dim(M)$. In the previous appendix, we related vectors tangent to $C^\infty(\Sigma, M)$ to vector fields along maps into M (which are "vector field-like" objects on M). In this appendix, we will define a mapping from r-forms on M to s-forms on $C^\infty(\Sigma, M)$, with $s = r - \dim(\Sigma)$, and explore its interplay with interior, Lie, and exterior derivatives.

We will sometimes write $\int_f$ for an integral over the map $f \in C^\infty(\Sigma, M)$. We now explain what this notation means. In physics it is common to integrate over submanifolds, for example, of M. But in the theory of integration, the objects one integrates over are chains, which are linear combinations of maps from a simplex $T \subset \mathbb{R}^r$ (for some $r < \dim M$) into M. To integrate a differential form $\alpha \in \Omega^r(M)$ over a chain $C = \sum_i a_i f_i$ (a_i are the coefficients, f_i are the maps) one defines

$$\int_C \alpha \equiv \sum_i a_i \int_T f_i^* \alpha, \tag{B.1}$$

where the integrals on the right are standard Riemann integrals over T (well-defined since T is in $\mathbb{R}^r$). Similarly, we can integrate $\alpha \in \Omega^r(M)$ over $f \in C^\infty(\Sigma, M)$ with $r = \dim \Sigma$ by defining

$$\int_f \alpha \equiv \int_\Sigma f^* \alpha, \tag{B.2}$$

by analogy with (B.1) and where the integration of the differential form $f^*\alpha$ over the manifold Σ is defined in the usual way (we assume an orientation for Σ is given). Furthermore, for $\alpha \in \Omega^{r-1}(M)$, we define

$$\int_{\partial f} \alpha \equiv \int_{\partial\Sigma} f^* \alpha, \tag{B.3}$$

by analogy with integration over boundaries of chains. Notice the important distinction between these integrals over f and integrals over the image of f in M. For example, suppose $\dim \Sigma = 3$ and the image of f is a 2-dimensional submanifold of M. One does not integrate 2-forms over this image, instead the integral over the *map* f requires a 3-form and indeed is zero (due to linear dependence of the push-forwards of the coordinate basis vectors on Σ). By defining integration over maps, we achieve a more general framework for integration, allowing us to handle cases when the image of f is not a submanifold of M, while reducing to standard integration over submanifolds when f is an embedding.

Returning to the relationship between r-forms on M and s-forms on $C^\infty(\Sigma, M)$, let us define what we call the insertion-integration map, or simply I-map, by

$$\begin{aligned} I_r^s &: \Omega^r(M) \to \Omega^s(C^\infty(\Sigma, M)) \\ &: \qquad \alpha \mapsto A \equiv I_r^s(\alpha), \end{aligned} \tag{B.4}$$

where A is given by

$$A|_f(\tilde{X}_1, \ldots, \tilde{X}_s) = \int\limits_\Sigma \alpha\left((X_1)_\sigma, \ldots, (X_s)_\sigma, f_*\left(\frac{\partial}{\partial\sigma^1}\right), \ldots, f_*\left(\frac{\partial}{\partial\sigma^{r-s}}\right)\right) \mathrm{d}^{r-s}\sigma \quad \forall f \in C^\infty(\Sigma, M). \tag{B.5}$$

The integral on the right-hand side is really a sum of Riemann integrals (one for each coordinate patch on Σ). The sum is independent of the choice of a coordinate atlas on Σ because the integrand is coordinate-invariant just like an integral of a differential form (in special cases it is an integral of a differential form, see (B.6) below). The $\partial/\partial\sigma^i$ must satisfy $\Omega(\partial/\partial\sigma^1, \ldots, \partial/\partial\sigma^{r-s}) > 0$ where $\Omega \in \Omega^{r-s}(\Sigma)$ is a volume form specifying the orientation of Σ.

Notice how (B.5) makes explicit the dual role of the X_i. On the left-hand side they are vectors $\tilde{X}_i$ tangent to $C^\infty(\Sigma, M)$, while on the right-hand side the $(X_i)_\sigma$ come from the images of X_i, which are vector fields along the map f (see appendix A for details on the dual role and the notation).

The definition of the I-map simplifies in the case when f is an embedding. In this case, the image of f is a submanifold and the vector fields along f (the X_i) are true vector fields on M defined over this submanifold. We may then write (B.5) as

$$A|_f(\tilde{X}_1, \ldots, \tilde{X}_s) = \int\limits_f \alpha(X_1, \ldots, X_s, \cdot). \tag{B.6}$$

The notation $\alpha(X_1, \ldots, X_s, \cdot)$ means to insert the vector fields X_i in the first s slots of α, producing an $(r-s)$-form, which is then integrated over f.

Given $A = I_r^s(\alpha)$, $\mathrm{d}A$ and $\mathrm{d}\alpha$ are related by the following theorem.

Theorem 1 *Let $\alpha \in \Omega^r(M)$. The I-map satisfies the following relation:*

$$\mathrm{d}I_r^s = I_{r+1}^{s+1}\mathrm{d} + (-1)^s (I_\partial)_r^{s+1}, \tag{B.7}$$

where

$$(I_\partial)_r^{s+1} : \Omega^r(M) \to \Omega^{s+1}(C^\infty(\Sigma, M)) \tag{B.8}$$

$$\left[(I_\partial)_r^{s+1}(\alpha)\right]\Big|_f(\tilde{X}_1, \ldots, \tilde{X}_{s+1}) = \int\limits_{\partial\Sigma} \alpha\left((X_1)_\tau, \ldots, (X_{s+1})_\tau, f_*\left(\frac{\partial}{\partial\tau^1}\right), \ldots, f_*\left(\frac{\partial}{\partial\tau^{r-s-1}}\right)\right) \mathrm{d}^{r-s-1}\tau, \tag{B.9}$$

and τ^i are coordinates on $\partial\Sigma$ with coordinate basis vectors $\partial/\partial\tau^i$.

As with the I-map, the definition of the (I_∂)-map is coordinate-invariant. When f is an embedding, the (I_∂)-map has the simpler definition:

$$\left[(I_\partial)^{s+1}_r(\alpha)\right]\Big|_f(\tilde{X}_1,\ldots,\tilde{X}_{s+1}) = \int\limits_{\partial f} \alpha(X_1,\ldots,X_{s+1},\cdot). \tag{B.10}$$

Proof: Evaluate (B.7) on a collection of vectors:

$$\mathrm{d}A\big|_F(\tilde{X}_1,\ldots,\tilde{X}_{s+1}) = I^{s+1}_{r+1}(\mathrm{d}\alpha)\Big|_f(\tilde{X}_1,\ldots,\tilde{X}_{s+1}) + (-1)^s\,(I_\partial)^{s+1}_r(\alpha)\Big|_f(\tilde{X}_1,\ldots,\tilde{X}_{s+1}). \tag{B.11}$$

First, consider the $I^{s+1}_{r+1}\mathrm{d}\alpha$ term. By the definition of the I-map, we can write this term as

$$\int\limits_\Sigma \mathrm{d}\alpha\left((X_1)_\sigma,\ldots,(X_{s+1})_\sigma, f_*\left(\frac{\partial}{\partial\sigma^1}\right),\ldots,f_*\left(\frac{\partial}{\partial\sigma^{r-s}}\right)\right)\mathrm{d}^{r-s}\sigma. \tag{B.12}$$

We now work to simplify the integral. Introduce the standard region $\Sigma\times[0,1]^{s+1}$. Let λ^i be the coordinate along the i^{th} copy of $[0,1]$, $\sigma\in\Sigma$, and thus (σ,λ^i) labels a point in $\Sigma\times[0,1]^{s+1}$. Let x^μ be the coordinates on M.

Using these coordinates, define the map

$$\begin{aligned} C &: \Sigma\times[0,1]^{s+1}\to M \\ &: (\sigma,\lambda^i)\mapsto x^\mu = f^\mu(\sigma) + \sum_k \lambda^k (X_k)^\mu_\sigma, \end{aligned} \tag{B.13}$$

where $f^\mu(\sigma)$ is the coordinate representation of $f:\Sigma\to M$ and $(X_k)_\sigma \equiv X_k(\sigma) = (X_k)^\mu_\sigma\,\partial/\partial x^\mu$. This map satisfies two properties,

$$C\circ j = f \tag{B.14a}$$

$$C_*\left(\frac{\partial}{\partial\lambda^i}\right)\bigg|_{(\sigma,\lambda^i=0)} = X_i(\sigma), \tag{B.14b}$$

where $j:\Sigma\to\Sigma\times[0,1]^{s+1}:\sigma\mapsto(\sigma,\lambda^i=0)$ is the inclusion map.

In a different set of coordinates λ'^i, x'^μ, (B.13) would lead to a map C', but we can see that it would still satisfy (B.14), and so these properties of C are coordinate-invariant. In what follows, we will only need to rely on the properties (B.14) and so our proof is coordinate-invariant.

Using (B.14), we write (B.12) as

$$\int\limits_\Sigma \mathrm{d}^{r-s}\sigma$$

$$\mathrm{d}\alpha\left(C_*\left(\frac{\partial}{\partial\lambda^1}\right)\bigg|_{(\sigma,\lambda^i=0)},\ldots,C_*\left(\frac{\partial}{\partial\lambda^{s+1}}\right)\bigg|_{(\sigma,\lambda^i=0)}, C_*\circ j_*\left(\frac{\partial}{\partial\sigma^1}\right),\ldots,C_*\circ j_*\left(\frac{\partial}{\partial\sigma^{r-s}}\right)\right). \tag{B.15}$$

Using the definition of the pull-back, we rewrite this as

$$\int_{\Sigma} (C^* \mathrm{d}\alpha) \left(\left(\frac{\partial}{\partial \lambda^1} \right) \bigg|_{\lambda^i=0}, \ldots, \left(\frac{\partial}{\partial \lambda^{s+1}} \right) \bigg|_{\lambda^i=0}, j_* \left(\frac{\partial}{\partial \sigma^1} \right), \ldots, j_* \left(\frac{\partial}{\partial \sigma^{r-s}} \right) \right) \mathrm{d}^{r-s}\sigma = \int_{j(\Sigma)} (C^* \mathrm{d}\alpha) \left(\left(\frac{\partial}{\partial \lambda^1} \right) \bigg|_{\lambda^i=0}, \ldots, \left(\frac{\partial}{\partial \lambda^{s+1}} \right) \bigg|_{\lambda^i=0}, \cdot \right), \tag{B.16}$$

where we have taken a sum of Riemannian integrals on the left (with $\int_\Sigma$ simply reminding us to patch together over Σ and that the integrand is a function of σ) and turned it into an integral of a differential form ($\in \Omega^{r-s}(\Sigma \times [0,1]^{s+1})$) over the submanifold $j(\Sigma) \subset \Sigma \times [0,1]^{s+1}$. From now on it will be understood that the vectors $\partial/\partial\lambda^j$ are evaluated at $\lambda^i = 0$.

Pull-backs commute with exterior derivatives, so $C^*(\mathrm{d}\alpha) = \mathrm{d}(C^*\alpha)$. Defining $\tilde{\alpha} = C^*\alpha$, the result in (B.16) becomes

$$\int_{j(\Sigma)} (\mathrm{d}\tilde{\alpha}) \left(\frac{\partial}{\partial \lambda^1}, \ldots, \frac{\partial}{\partial \lambda^{s+1}}, \cdot \right). \tag{B.17}$$

The $\partial/\partial\lambda^i$ define an infinitesimal region $R_\epsilon \equiv [0,\epsilon]^{s+1}$, so we can replace the insertions of the vectors by integration over the small region:

$$\int_{j(\Sigma)} (\mathrm{d}\tilde{\alpha}) \left(\frac{\partial}{\partial \lambda^1}, \ldots, \frac{\partial}{\partial \lambda^{s+1}}, \cdot \right) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{s+1}} \int_{R_\epsilon \times j(\Sigma)} \mathrm{d}\tilde{\alpha}, \tag{B.18}$$

where the product region $R_\epsilon \times j(\Sigma)$ is assumed to have orientation consistent with the original integration. We can now use Stokes' theorem so that the result in (B.18) becomes

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{s+1}} \int_{\partial(R_\epsilon \times j(\Sigma))} \tilde{\alpha}. \tag{B.19}$$

The boundary decomposes as

$$\partial(R_\epsilon \times j(\Sigma)) = \partial(R_\epsilon) \times j(\Sigma) + (-1)^{s+1} R_\epsilon \times \partial j(\Sigma), \tag{B.20}$$

where the sign of the second term comes from ensuring the proper orientation of the boundary. Thus the original integral becomes

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{s+1}} \left[\int_{\partial(R_\epsilon) \times j(\Sigma)} \tilde{\alpha} + (-1)^{s+1} \int_{R_\epsilon \times \partial j(\Sigma)} \tilde{\alpha} \right]. \tag{B.21}$$

We write the first integral explicitly as a (sum of) Riemann integrals by introducing coordinates σ^i on Σ (and hence on $j(\Sigma)$) and coordinates κ^i on ∂R_ϵ. Remembering $\tilde{\alpha} = C^*(\alpha)$ and using the definition of the pull-back, we have

$$\int_{\partial(R_\epsilon) \times j(\Sigma)} \tilde{\alpha} = \int_{\partial R_\epsilon} \int_{\Sigma} \alpha \left(C_* \frac{\partial}{\partial \kappa^1}, \ldots, C_* \frac{\partial}{\partial \kappa^s}, C_* \frac{\partial}{\partial \sigma^1}, \ldots, C_* \frac{\partial}{\partial \sigma^{r-s}} \right) \mathrm{d}^{r-s}\sigma \, \mathrm{d}^s\kappa. \tag{B.22}$$

At fixed $\kappa \in \partial R_\epsilon$, each $C_*(\partial/\partial\kappa^i)\big|_\kappa$ can be viewed as a map

$$\begin{aligned} C_* \frac{\partial}{\partial \kappa^i}\bigg|_\kappa &: \Sigma \to TM \\ &: \sigma \mapsto C_* \frac{\partial}{\partial \kappa^i}\bigg|_{(\sigma,\kappa)}, \end{aligned} \tag{B.23}$$

and hence there is a corresponding vector $\tilde{C}_*(\partial/\partial\kappa^i) \in T[C^\infty(\Sigma, M)]$. This means we can use the definition of the I-map (equation (B.5)) to write the integral as

$$\int_{\partial R_\epsilon} A|_{\tilde{C}(\kappa)} \left(\tilde{C}_* \frac{\partial}{\partial \kappa^1}, \ldots, \tilde{C}_* \frac{\partial}{\partial \kappa^s} \right) \mathrm{d}^s\kappa, \tag{B.24}$$

where we have introduced the map $\tilde{C} : R_\epsilon \to C^\infty(\Sigma, M) : \lambda \mapsto C(\cdot, \lambda)$. One can verify that the vector $\tilde{C}_*(\partial/\partial\kappa^i)$ is indeed the push-forward of $\partial/\partial\kappa^i$ under the map $\tilde{C}$. We thus rewrite the integral as

$$\int_{\partial R_\epsilon} \tilde{C}^* A = \int_{R_\epsilon} \mathrm{d}\tilde{C}^* A = \int_{R_\epsilon} \tilde{C}^* \mathrm{d}A = \int_{\tilde{C}(R_\epsilon)} \mathrm{d}A. \tag{B.25}$$

The small region $\tilde{C}(R_\epsilon)$ is formed by the vectors $\tilde{C}_*(\partial/\partial\lambda^i)\big|_{\lambda^j=0}$, which by (B.14b) are the vectors $\tilde{X}_i$ (based at f). Therefore,

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^{s+1}} \left[\int_{\partial(R_\epsilon)\times j(\Sigma)} \tilde{\alpha} \right] = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{s+1}} \left[\int_{\tilde{C}(R_\epsilon)} \mathrm{d}A \right] = \mathrm{d}A|_f (\tilde{X}_1, \ldots, \tilde{X}_{s+1}). \tag{B.26}$$

This takes care of the first integral of (B.21). For the second integral, we reverse the transition from vectors to small regions:

$$(-1)^{s+1} \int_{\partial j(\Sigma)} \tilde{\alpha} \left(\frac{\partial}{\partial \lambda^1}, \ldots, \frac{\partial}{\partial \lambda^{s+1}}, \cdot \right). \tag{B.27}$$

Introducing coordinates τ^i on $\partial\Sigma$ (and hence on $j(\partial\Sigma)$), the integral becomes

$$\int_{\partial j(\Sigma)} \tilde{\alpha} \left(\frac{\partial}{\partial \lambda^1}, \ldots, \frac{\partial}{\partial \lambda^{s+1}}, \cdot \right) = \int_{\partial\Sigma} \alpha \left(C_* \frac{\partial}{\partial \lambda^1}, \ldots, C_* \frac{\partial}{\partial \lambda^{s+1}}, C_* \frac{\partial}{\partial \tau^1}, \ldots, C_* \frac{\partial}{\partial \tau^{r-s-1}} \right) \mathrm{d}^{r-s-1}\tau, \tag{B.28}$$

from which we can use the (I_∂)-map (and (B.14b)) to write

$$\int_{\partial\Sigma} \alpha \left(C_* \frac{\partial}{\partial \lambda^1}, \ldots, C_* \frac{\partial}{\partial \lambda^{s+1}}, C_* \frac{\partial}{\partial \tau^1}, \ldots, C_* \frac{\partial}{\partial \tau^{r-s-1}} \right) \mathrm{d}^{r-s-1}\tau = (I_\partial)^{s+1}_r \alpha\big|_f (\tilde{X}_1, \ldots, \tilde{X}_{s+1}). \tag{B.29}$$

Putting everything together, we conclude

$$I^{s+1}_{r+1}(\mathrm{d}\alpha)\big|_f(\tilde{X}_1,\ldots,\tilde{X}_{s+1}) = \mathrm{d}A\big|_f(\tilde{X}_1,\ldots,\tilde{X}_{s+1}) + (-1)^{(s+1)}\,(I_\partial)^{s+1}_r\alpha\big|_f(\tilde{X}_1,\ldots,\tilde{X}_{s+1}), \tag{B.30}$$

which proves the theorem.

Knowing $A = I^s_r(\alpha)$, we would like to know what the I-map does to $i_Y\alpha$ and $\pounds_Y\alpha$. Of course to evaluate $\pounds_Y\alpha$, Y must be a vector field on M. To map $i_Y\alpha$ under the I-map, Y also need to be a vector field because the I-map involves integration and we need a vector defined at all points of the region of integration.

So suppose we are given a vector field Y on M. This is a section $Y : M \to TM$, and thus for every $f \in C^\infty(\Sigma, M)$, we have the map $Y \circ f : \Sigma \to TM$. Due to the correspondence in appendix A, we thus have a vector in $T_f[C^\infty(\Sigma, M)]$ for each f, and so a vector field on $C^\infty(\Sigma, M)$ which we call $\bar{Y} : C^\infty(\Sigma, M) \to T[C^\infty(\Sigma, M)] : f \mapsto Y \circ f$. We then have

Theorem 2 *Let $\alpha \in \Omega^r(M)$ and $A = I^s_r\alpha$. The I-map satisfies the following relations:*

$$i_{\bar{Y}}A = I^{s-1}_{r-1}(i_Y\alpha) \tag{B.31a}$$

$$\pounds_{\bar{Y}}A = I^s_r(\pounds_Y\alpha) \tag{B.31b}$$

Proof:

(B.31a): Working from the right-hand side,

$$[I^{s-1}_{r-1}(i_Y\alpha)]\big|_f(\tilde{X}_1,\ldots,\tilde{X}_{s-1}) = \int_\Sigma (i_Y\alpha)\left((X_1)_\sigma,\ldots,(X_{s-1})_\sigma, f_*\frac{\partial}{\partial\sigma^1},\ldots,f_*\frac{\partial}{\partial\sigma^{r-s}}\right)\mathrm{d}^{r-s}\sigma$$

$$= \int_\Sigma \alpha\left(Y_\sigma,(X_1)_\sigma,\ldots,(X_{s-1})_\sigma, f_*\frac{\partial}{\partial\sigma^1},\ldots,f_*\frac{\partial}{\partial\sigma^{r-s}}\right)\mathrm{d}^{r-s}\sigma = (I^s_r\alpha)\big|_f(\bar{Y},\tilde{X}_1,\ldots,\tilde{X}_{s-1})$$

$$= (i_{\bar{Y}}A)\big|_f(\tilde{X},\ldots,\tilde{X}_{s-1}). \tag{B.32}$$

(B.31b): Working from the right hand side and applying (B.7) and (B.31a),

$$I^s_r(\pounds_Y\alpha) = I^s_r(i_Y\mathrm{d}\alpha + \mathrm{d}i_Y\alpha) = i_{\bar{Y}}I^{s+1}_{r+1}(\mathrm{d}\alpha) + \mathrm{d}I^{s-1}_{r-1}(i_Y\alpha) - (-1)^{s-1}(I_\partial)^s_{r-1}(i_Y\alpha). \tag{B.33}$$

The (I_∂)-map satisfies (B.31a), since the proof is the same as for the I-map. Thus we find,

$$I^s_r(\pounds_Y\alpha) = i_{\bar{Y}}\mathrm{d}A - i_{\bar{Y}}(-1)^s(I_\partial)^{s+1}_r\alpha + \mathrm{d}i_{\bar{Y}}A - (-1)^{s-1}i_{\bar{Y}}(I_\partial)^{s+1}_r\alpha = \pounds_{\bar{Y}}A, \tag{B.34}$$

which completes the proof.

Appendix C

Lifting Diffeomorphisms into the Multisymplectic Bundle

Suppose we have a tensor bundle over spacetime. A diffeomorphism maps one point x in the base space to another x'. We would like to know where we should map a point in the fiber over x. First, it is clear that we should map it to a point in the fiber over x', otherwise the bundle transformation will not agree with the transformation of the base space. Second, a point in the fiber over x gives us a tensor at that point, and tensors transform under diffeomorphisms via push-forward (specifically, their contravariant components transform by push-forward while their covariant components transform by inverse pull-back). Thus, for a point in the fiber over x specified by the tensor $T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots,\nu_s}$, the lifted diffeomorphism maps it into a point of the fiber over x' specified by the tensor

$$(T')^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots,\nu_s} = \left(\frac{\partial x'^{\mu_1}}{\partial x^{m_1}}\right)\cdots\left(\frac{\partial x'^{\mu_r}}{\partial x^{m_r}}\right)\left(\frac{\partial x^{n_1}}{\partial x'^{\nu_1}}\right)\cdots\left(\frac{\partial x^{n_s}}{\partial x'^{\nu_s}}\right)T^{m_1,\ldots,m_r}{}_{n_1,\ldots,n_s} \tag{C.1}$$

For infinitesimal diffeomorphisms, if the generator of the spacetime diffeomorphism is $\xi = \xi^\mu \frac{\partial}{\partial x^\mu}$, then the generator of the corresponding lifted diffeomorphism is

$$X_\xi = \xi^\mu \frac{\partial}{\partial x^\mu} + (\delta T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots,\nu_s})_\xi \frac{\partial}{\partial T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots,\nu_s}} \tag{C.2}$$

where $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots,\nu_s}}$ are coordinate basis vectors in the bundle, while $(\delta T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots,\nu_s})_\xi$ is obtained by inserting $x'^\mu = x^\mu + \xi^\mu$ into (C.1), and computing $T' - T$ to first order in ξ,

$$(\delta T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots,\nu_s})_\xi = \sum_{\mu=\mu_1}^{\mu_r}(\partial_m \xi^\mu) T^{\mu_1,\ldots,(\mu\to m),\ldots,\mu_r}{}_{\nu_1,\ldots,\nu_s} - \sum_{\nu=\nu_1}^{\nu_s}(\partial_\nu \xi^n) T^{\mu_1,\ldots,\mu_r}{}_{\nu_1,\ldots,(\nu\to n),\ldots,\nu_s} \tag{C.3}$$

Consider the special case of a p-form bundle $\pi : P \to M$. On this bundle there is a canonical p-form $\theta \in \Omega^p(P)$ given by

$$\theta|_{(x,\alpha)}(X_1,\ldots,X_p) = \alpha|_x(\pi_* X_1,\ldots,\pi_* X_p). \tag{C.4}$$

The canonical 1-form on the cotangent (1-form) bundle is a special case of this construction. Taking coordinates x^μ on M, we get coordinates $(x^\mu, \alpha_{\mu_1 \ldots \mu_p})$ on P. In these coordinates, $theta = \frac{1}{p!}\alpha_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$. We can see that this canonical p-form is invariant under the action of lifted diffeomorphisms. Under a lifted diffeomorphism, $\theta \mapsto \frac{1}{p!}\alpha'_{\mu_1 \ldots \mu_p} \mathrm{d}x'^{\mu_1} \wedge \cdots \wedge \mathrm{d}x'^{\mu_p}$, with α' given by (C.1). The inverse Jacobians from (C.1) cancel the Jacobians from $\mathrm{d}x'^\mu = (\partial x'^\mu / \partial x^\nu)\mathrm{d}x^\nu$ and thus θ is invariant. Since most of our multisymplectic phase spaces are form bundles, we can use this invariance to easily show that multisymplectic 4- and 5-forms are preserved by lifted diffeomorphisms.

Appendix D

Constraints, Momentum Maps, and Kernels

When analyzing field theories in a 3+1 decomposition, we often end up with constraint submanifolds of our phase space P. A natural question to ask is whether we can restrict our theory to the constraint submanifold, thereby simplifying the system. In general, this is not possible because the restricted theory will have additional solutions not present in the original theory. To see this, note that the Hamiltonian vector field that determines solutions lies inside the kernel of the presymplectic 2-form ω (we are assuming a homogeneous formulation). If we have a constraint manifold C, then the Hamiltonian vector field will lie inside $\ker \omega \cap TC$. If we restrict the theory, we would conclude that the Hamiltonian vector field lies inside $\ker \omega|_C$, but

$$\ker \omega|_C \subseteq \ker \omega \cap TC. \tag{D.1}$$

This is because if $X \in \ker \omega \cap TC$ then $\omega(X, Y) = 0$ for all vectors Y on P (at C). This is a stronger condition than $\omega(X, Y) = 0$ for all vectors Y on C, which is the condition for $X \in \ker \omega|_C$. It is thus important to understand the relationship between $\ker \omega \cap TC$ and $\ker \omega|_C$.

In practice we frequently have $C = L \cap C'$, where L is the level set of a momentum map and C' has the property that $\ker \omega|_C = \ker \omega \cap TC$. In these cases we can say exactly how and why the kernel of the restricted form enlarges. Let us start with the case $C = L$.

Theorem 3 *Let G be a symmetry group with associated momentum map J. Let $I \subset G$ be the isotropy subgroup, whose flow leaves $C = L$ invariant. At a point $x \in C$, let $B_I \subset T_x C$ be the tangent plane to the isotropy subgroup orbit. Then at x*

$$\ker \omega|_C = \ker \omega \cap T_x C + B_I. \tag{D.2}$$

To prove this, let ξ_i be a basis in the Lie algebra of I, X_i be the induced vector field of ξ_i, and $J_i \equiv J_{\xi_i}$. Then X_i is tangent to C and

$$i_{X_i}\omega = -\mathrm{d}J_i. \tag{D.3}$$

Restricting both sides to C and recognizing $\mathrm{d}J_i|_C = 0$ (C is a level set of the momentum map), we get $\omega(X_i, Y) = 0$ for all vectors Y on C. Hence $X_i \in \ker \omega|_C$. This proves

$\ker\omega \cap TC + B_I \subset \ker\,\omega|_C$. Conversely, if $X \in \ker\,\omega|_C$, then $i_X\omega|_C = 0 \implies i_X\omega = \sum_\mu c_\mu \mathrm{d}J_\mu$ where J_μ are components of the momentum map. This follows because the J_μ are constraint functions defining C. Pick J_μ such that a subset is J_i and call the rest J_I. The solution to $i_X\omega = \sum_i c_i \mathrm{d}J_i + \sum_I c_I \mathrm{d}J_I$ is a linear combination of a particular solution and an element of $\ker\omega$. We may write the full solution as $X = \sum_i c_i X_i + \sum_I c_I X_I + \ker\omega$, where the X_I are induced vector fields that are necessarily transverse to C (as only flows along X_i preserve C). Since $X \in T_xC$ we must intersect our full solution with T_xC, which leaves $X = \sum_i X_i + \ker\omega \cup T_xC$. Thus $\ker\,\omega|_C \subset \ker\omega \cap TC + B_I$, which completes the proof.

Let us now tackle the case $C = L \cup C'$. Write (D.2) as

$$\ker\,\omega|_L = \ker\omega \cap T_xL + B_I, \tag{D.4}$$

and intersect both sides with TC'. We use the fact that restricting to C' doesn't enlarge the kernel ($\ker\,\omega|_L \cup TC' = \ker\,\omega|_{C=L\cup C'}$) to write

$$\ker\,\omega|_C = \ker\omega \cap T_xC + B_{\tilde{I}}, \tag{D.5}$$

where $B_{\tilde{I}} = B_I \cup T_xC'$ is the tangent plane to the orbit of the subgroup $\tilde{I} \subset G$ whose flow leaves $C = L \cup C'$ invariant.

When dealing with gauge theories, we often have that the final constraint manifold is gauge-invariant (the full group leaves it invariant). In that case, (D.5) tells us that upon restriction, the kernel will be enlarged by (infinitesimal) gauge transformations.

Lastly, we discuss how our theorem relates to the classification of constraint manifolds. Dirac [6] originally classified constraints based on whether they were first-class (Poisson-commuted on the constraint surface) or second-class (did not Poisson-commute on the constraint surface). Several geometric variants of these classifications have been explored [18,22]. We follow the literature in defining first-class constraint manifolds as coisotropic (even in a homogeneous, presymplectic formulation). This is a natural generalization of Dirac (a manifold in symplectic geometry given by Poisson-commuting functions is coisotropic) and fits nicely with the first case $C = L$ considered above (level sets of momentum maps are coisotropic submanifolds). There is some disagreement as to the best way to generalize second-class constraint manifolds to a presymplectic setting. We choose our own variant: a constraint manifold is second-class if it can be restricted to ($\ker\,\omega|_C = \ker\omega \cup TC$). This reduces to Dirac's classification in a symplectic setting (submanifolds given by non-Poisson-commuting functions are symplectic) and gives the nice result that most constraint manifolds encountered in field theory are intersections of first-class and second-class manifolds.