Title

3-9 node isoparametric planar or axisymmetric finite element

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3-9 NODE ISOPARAMETRIC
PLANAR OR AXISYMMETRIC
FINITE ELEMENT

by

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1. SUMMARY

The element presented here is a two-dimensional planar triangle or quadrilateral which may be arbitrarily orientated in three-dimensional space. In its own plane the element may represent:

1) Plane Stress: - i.e., the stress normal to the plane of the element is negligible

2) Plane Strains: - i.e., the strain normal to the plane of the element if zero

3) Axisymmetric idealizations

The element is described by 4 to 9 nodes as shown in Figure 1. A triangular element may be specified by defining two of the corner nodes of the element to be coincidental. This is treated as a degenerate quadrilateral and requires no special formulation.

Temperature dependent, isotropic or orthotropic material properties may be specified.

The following inplane element loads are possible

1) Temperature: with specification of nodal point temperatures
2) Pressure: with specification of nodal point pressures
3) Gravity: out of plane gravity also generated
4) Centrifugal: axisymmetric idealization only

Stresses may be requested at up to six points in the plane of the element.

2. FORMULATION

2.1 Local Coordinates

A system of local coordinates (x, y) is uniquely defined for each element by the location of its corner nodes (1, 3, 7, 9). The program during input checks to ensure the element is planar before initiating formulation. The relationship between local (x, y) and global (X, Y, Z) coordinate systems is shown in Figure 1.

2.2 Element Material Properties

Orthotropic temperature dependent material properties are possible. The principal axis for the orthotropic materials (n, s) are related to the local element coordinate system (s, y) by the angle (β) as shown in Figure 2.

2.3 Displacement Functions

A two-dimensional isoparametric formulation is used for this element. For the general quadrilateral element shown in Figure 3, the local coordinate system (x, y) is transformed onto the natural coordinate system (r, s). The coordinates of any point are then given by:

$$x = \sum_{i=1}^{n} h_i(r,s)x_i \tag{1a}$$

$$y = \sum_{i=1}^{n} h_i(r,s)x_i \tag{1b}$$

where n is the number of nodes used to define the element; x_i, y_i the local coordinates of node "i" and h_i the nodal interpolation functions as follows:

Define

$$\begin{aligned} R &= 1 + r \\ S &= 1 + s \\ \bar{R} &= 1 - r \\ \bar{S} &= 1 - s \\ R^* &= 1 - r^2 \\ S^* &= 1 - s^2 \end{aligned} \tag{2}$$

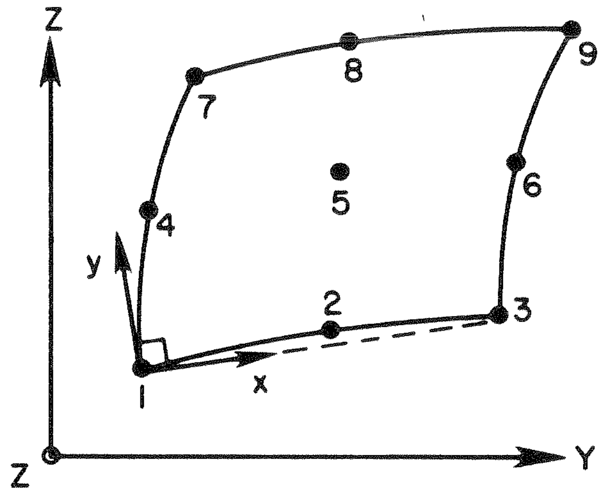


FIGURE 1 LOCAL (x,y) AND GLOBAL (X,Y,Z) COORDINATE SYSTEM FOR QUADRILATERAL

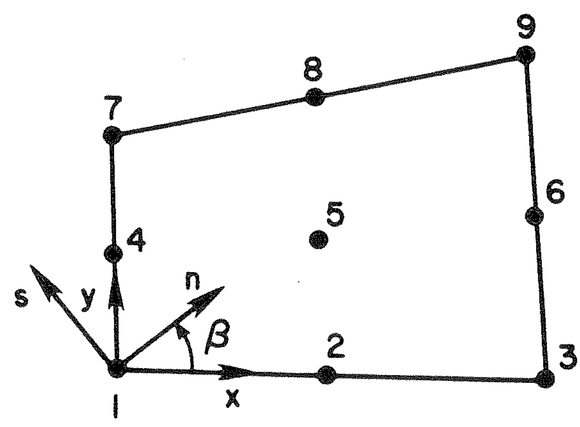
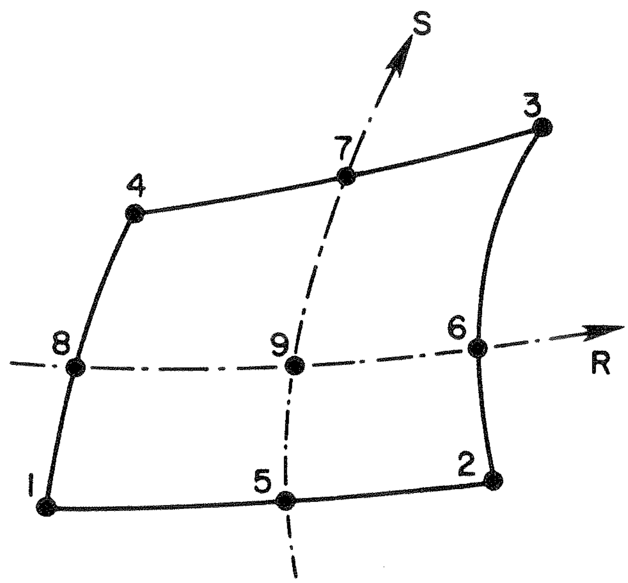


FIGURE 2 ELEMENT (x,y) AND MATERIAL (n,s) COORDINATES



NODE	r	s
1	−1	−1
2	+1	−1
3	+1	+1
4	−1	+1
5	0	−1
6	+1	0
7	0	+1
8	−1	0
9	0	0

FIGURE 3 NATURAL COORDINATE SYSTEM

then

$$
\begin{array}{llcllllll}
\text{Order**} & & & & \text{(delete if node I not included)} & & & & \\
 & & & & I=5 & I=6 & I=7 & I=8 & I=9 \\
6: & h_1 & = & (1/4)\ RS & -(1/2)\ h_5 & & & -(1/2)\ h_8 & -(1/4)\ h_9 \\
7: & h_2 & = & (1/4)\ \overline{R}S & -(1/2)\ h_5 & -(1/2)\ h_6 & & & -(1/4)\ h_9 \\
8: & h_3 & = & (1/4)\ \overline{R}\overline{S} & & -(1/2)\ h_6 & -(1/2)\ h_7 & & -(1/4)\ h_9 \\
9: & h_4 & = & (1/4)\ R\overline{S} & & & -(1/2)\ h_7 & -(1/2)\ h_8 & -(1/4)\ h_9 \\
2: & h_5 & = & (1/2)\ R^*S & & & & & -(1/2)\ h_9 \\
3: & h_6 & = & (1/2)\ \overline{R}S^* & & & & & -(1/2)\ h_9 \\
4: & h_7 & = & (1/2)\ R^*\overline{S} & & & & & -(1/2)\ h_9 \\
5: & h_8 & = & (1/2)\ RS^* & & & & & -(1/2)\ h_9 \\
1: & h_9 & = & (1/2)\ R^*S^* & & & & & -(1/2)\ h_9
\end{array}
\qquad (3)
$$

** the interpolations functions must be established in the order given.

The above functions have been written for all 9 nodes. If any one node from 5 to 9 is omitted, the corresponding interpolation function is deleted and the interpolation functions h_1 - h_8 are modified accordingly. These interpolation functions are also used to express the assumed displacement field for the element.

$$
\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} H & \cdot \\ \cdot & H \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{v} \end{Bmatrix} \qquad (4)
$$

2.4 Constitutive Relationship

Hooke's law in two dimensions is given by:

$$
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix} =
\begin{bmatrix}
\frac{1}{Ex} & -\frac{\nu xy}{Ex} & -\frac{\nu xz}{Ex} & 0 & 0 & 0 \\
-\frac{\nu yx}{Ey} & \frac{1}{Ey} & -\frac{\nu yz}{Ey} & 0 & 0 & 0 \\
-\frac{\nu xz}{Ez} & -\frac{\nu zy}{Ez} & \frac{1}{Ez} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{Gxy} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{Gxz} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{Gyz}
\end{bmatrix}
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}
$$

$$+ \Delta T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5}$$

or

$$\underline{\varepsilon} = \underline{D}\,\underline{\tau} + \underline{\varepsilon}_0 \tag{6}$$

For orthotropic material properties this relationship may be transformed from the material coordinate system (n, t) to the local system (x, y) using:

$$\begin{Bmatrix} n \\ t \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}}_{\underline{a}} \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{7}$$

then

$$\underline{\varepsilon}(x,y) = \underline{a}^T\, \underline{\varepsilon}(n,t)\, \underline{a} \tag{8}$$

The constitutive relationship given by Eq. 8 may be further modified to represent the conditions of plane strain and plain stress by specifying:

$$\begin{aligned} \sigma_{zz} &= 0 \quad - \text{ plane stress} \\ \varepsilon_{zz} &= 0 \quad - \text{ plain strain} \end{aligned} \tag{9}$$

The stress-strain relationship is obtained by inverting Eq. 6.

$$C = \underline{D}^{-1} \tag{10}$$

2.5 Strain-Displacement Relationship

The local strains are computed from the assumed displacements using:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \tag{11}$$

and using the relationship given in Eq. 4

$$\underline{\varepsilon} = \underbrace{\begin{bmatrix} H,x & 0 \\ & H,y \\ H,y & H,x \end{bmatrix}}_{\underline{B}} \begin{Bmatrix} \underline{u}_x \\ \underline{u}_y \end{Bmatrix} \tag{12}$$

Since the functions $\underline{H}$ are given in terms of the natural coordinates (r, s) a series of intermediate calculations are performed to establish $\underline{B}$ using the chain rule (inverted):

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{Bmatrix} \tag{13}$$

where

$$\begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y} \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial t} & -\frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \end{bmatrix} \tag{14}$$

and $|J|$ is the determinant of the Jacobian matrix

$$|J| = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \tag{15}$$

This term $|J|$ represents the conversion for the volume differential between the local and natural coordinate space, i.e.

$$dv = tdA = tdxdy = t|J|drds \tag{16}$$

and is useful in detecting modelling errors as it must always be positive. Non-positive $|J|$ can be caused by:

1) node sequence errors during input
2) the angle between adjacent sides of element being greater than 180°
3) element having zero area.

2.6 Stiffness Matrix

The stiffness matrix is given by

$$\underline{K} = \int_V \underline{B}^T \underline{C} \, \underline{B} \, dv \tag{17}$$

where $\underline{B}$ is the strain-displacement relationship and C the stress-strain matrix.

This is evaluated using numerical integration in the natural coordinate system (r, s) at the Gauss points as shown in Figure 4.

$$\underline{K} = \int_{-1}^{1} \int_{-1}^{1} \underline{B}^T \underline{C} \underline{B} |J| \text{ tdrds} \tag{18}$$

where t is the element thickness and

$$\underline{K} = \sum_{j} \sum_{k} w_j w_k |J| t \underline{B}^T (r_j, s_k) \underline{C} \underline{B} (r_j, s_k) \tag{19}$$

in which r_j, s_k are the coordinates of the integration points and w_j, w_k are the appropriate weighting functions.

2.7 Mass Matrix

Both lumped and consistant mass matrices are possible. The consistent mass matrix is given by:

$$M_{ij} = \sum_{\ell}^{n} \sum_{k}^{n} w_\ell \cdot w_k |J| t H_i^T (r_\ell, s_k) H_j(r_\ell, s_k) \rho \tag{20}$$

where ρ is the density of the material.

The lumped mass matrix is formed by scaling the diagonal of the consistent mass matrix such that the sum is equivalent to the total element mass.

$$M_{ii} = \sum_{\ell}^{n} \sum^{n} w_\ell w_k |J| t H_i^T (r_\ell, s_k) H_i (r_\ell, s_k)\rho * FAC \tag{21}$$

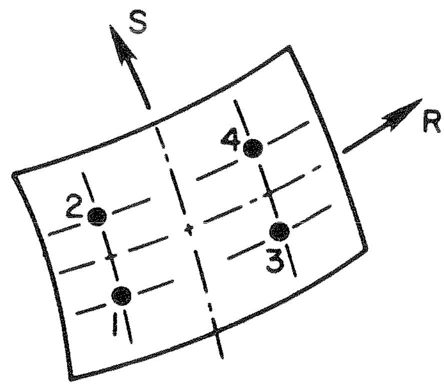
where

$$FAC = \sum_{i}^{n} \frac{MASS}{M_{ii}}$$

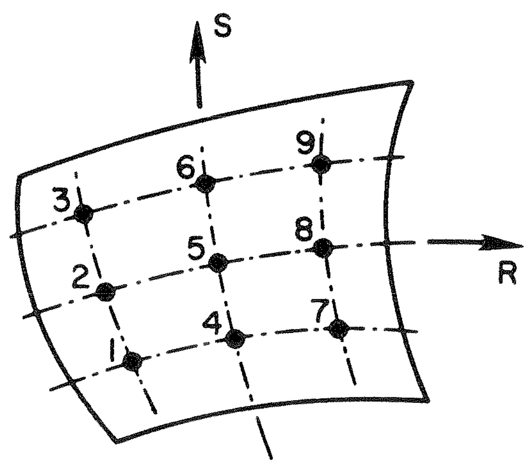
where MASS is the total element mass.

2.8 Element Loads

Thermal, pressure and gravity loadings are possible at the element level. Thermal and pressure loadings are inplane actions derived from specified nodal point values of temperature and pressure respectively.



2 * 2 INTEGRATION POINTS



3 * 3 INTEGRATION POINTS

FIGURE 4 INTEGRATION POINTS FOR QUADRILATERAL

Gravity loadings are assigned to the global degress of freedom at each node based on tributary areas. It should be noted that gravity loadings normal to the plane at the element must be resisted by additional elements having stiffness in this direction.

1) Temperature loading:

The distribution of temperature within the element is given by

$$T = \sum_{i}^{n} H_i T_i \tag{20}$$

where T_i is the temperature of node "i".

The initial stresses due to this temperature are given by

$$\underline{\tau}_o = \underline{C}\,\underline{\alpha}\,(\underline{T} - T_R)^T \tag{21}$$

and the nodal loads due to these initial stresses.

$${}^{T}\underline{R}_i = \sum_{\ell}^{n} \sum_{k}^{n} w_\ell\, w_k\, t\, |J|\, \underline{B}_i^T (r_\ell, s_k)\, \underline{\tau}_o \tag{22}$$

2) Pressure loadings:

Pressure loadings are treated as body forces for the calculation of element loads. The pressure distribution in the element is given by

$$P_x = P_{,x} \tag{23}$$

where P_z is the pressure in the x direction. Using the same approximation as made for the displacements:

$$P_x = \sum_{i}^{n} H_{i,x} P_i \tag{24}$$

where P_i is the pressure at node "i". The element loads are therefore given by

$${}^{P}R_{ix} = \sum_{\ell}^{n} \sum_{k}^{n} W_\ell\, W_k\, |J|\, t\, H_i (r_\ell, s_k)\, H_{i,x} (r_\ell, s_k)\, P_i \tag{25}$$

3) Gravity loading:

Identically to above the gravity forces may be calculated by substituting for P_x the weight of the element per unit volume (WG)

$$^{G}R_{iz} = \sum_{\ell}^{n} \sum_{k}^{n} W_{\ell} W_{k} \; |J| \; t \; H_i \; (r_{\ell}, s_k) \; WG \qquad (26)$$

2.9 Stress Calculations

Stresses may be calculated at up to six arbitary points in the element. These points are located by specifying natural coordinate pairs. All stress output is in terms of the local coordinate system as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \qquad (27)$$

Stresses are computed by

$$\underline{\tau}_i = \sum_{j}^{n} B_j^T (r_i, s_i) \, \underline{C} \, u_j + \tau_o^i \qquad (28)$$

where u_j is the displacements at node "j" and τ_o^i are the initial stresses at output point "i".

3. FORMULATION (AXISYMMETRIC)

3.1 Stiffness Modifications

The axisymmetric problem is formulated in a similar manner. The above description holds if the following modifications are considered:

1) the strain displacement relationship becomes

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{bmatrix} H,x & 0 \\ 0 & H,y \\ H,y & H,x \\ H,x/R & 0 \end{bmatrix} \begin{Bmatrix} \underline{u}_z \\ \underline{u}_y \end{Bmatrix} \tag{29}$$

where R is the radius along the x axis.

2) the volume intergral is amended whre appropriate from

$$dv = t\,|J|\,drds$$

to (30)

$$dv = 2\pi\,|J|\,R\,drds$$

These modifications along with the appropriate change in the constitutive relationship allow the axisymmetric idealization to be included in this element.

An additional element load case is possible for the axisymmetric problem.

3.2 Additional Load Case:

1) Centrifugal loading:
This is formulated as a gravity loading where the weight of the element is assigned the value

$$WG = \rho\, W^2\, R \tag{31}$$

where ρ is the density, w the angular velocity and R the radius.

4. ELEMENT CHARACTERISTICS

4.1 Accuracy

Several papers exist on the relative accuracy of different order isoparametric quadrilateral elements under a variety of strain states (1, 2) The majority of these papers have limited their comparisons too:

1) Eight node quadrilateral; 2 x 2 and 3 x 3 integration
2) Two and four element linear strain triangles

In addition to straight line boundary models, the four node quadrilateral with non-conforming modes (3) is also compared. For convenience and generality the eight node isoparametric element is commonly used, though somewhat less efficient than the elements assembled from linear stress triangles (for 3 x 3 integration).

Papers by Stricklin, et al and Thomas (5) have shown that geometric distortion may substantially effect the accuracy of these elements. For the isoparametric element formulation the effects are a direct results of mapping the displacement functions by the same transformation functions as are used to map the geometry of the element. The assumed displacement polynomials are no longer simple after mapping into the new coordinate system and hence may not accurately represent the imposed strain fields.

Stricklin showed that the eight node element for the distorted model shown in Figure 9 gives completely erroneous displacements.

The addition of the ninth node to the center of this (eight node) element corrected this failure. The results for the Stricklin test plus the new 9 node are given in Figure 8.

4.2 Efficiency

The coding for the element formulation attempted to minimize storage and repeated operations. Where possible repeated expressions were coded explicitly and use made of symmetry. The comparison of formulation times for the eight and nine node element with 3 x 3 integration showed negligible cost increase.

A significant portion of the overall cost for the element is devoted to the stress-displacement matrix formulation. As such the number of output points should be carefully selected for economy.

Timing comparisons for the four node derived from this general three to nine node formulation and the simple four node quadrilateral showed little cost variation.

The use of the higher 3 x 3 compared to the 2 x 2 integration order increased the overall formulation times by 40-50%.

4.3 Stability

The zero energy mode shapes for the eight node element are shown in Figure 5. In general for the planar element the number of zero energy modes is a function of the number of nodes and the order of intergration. Addition of the center node to the eight node element (2 x 2 intergration) adds two degress of freedom and two additional zero energy nodes. Fortunately, however, increasing the order of intergration to 3 x 3 removes these additional modes. This form of the element is therefore very dependent on the nature of the boundary conditions, and therefore in general the use of the 2 x 2 intergration order on the nine node element is not advised due to the increased possibility of generating a non-singular system.

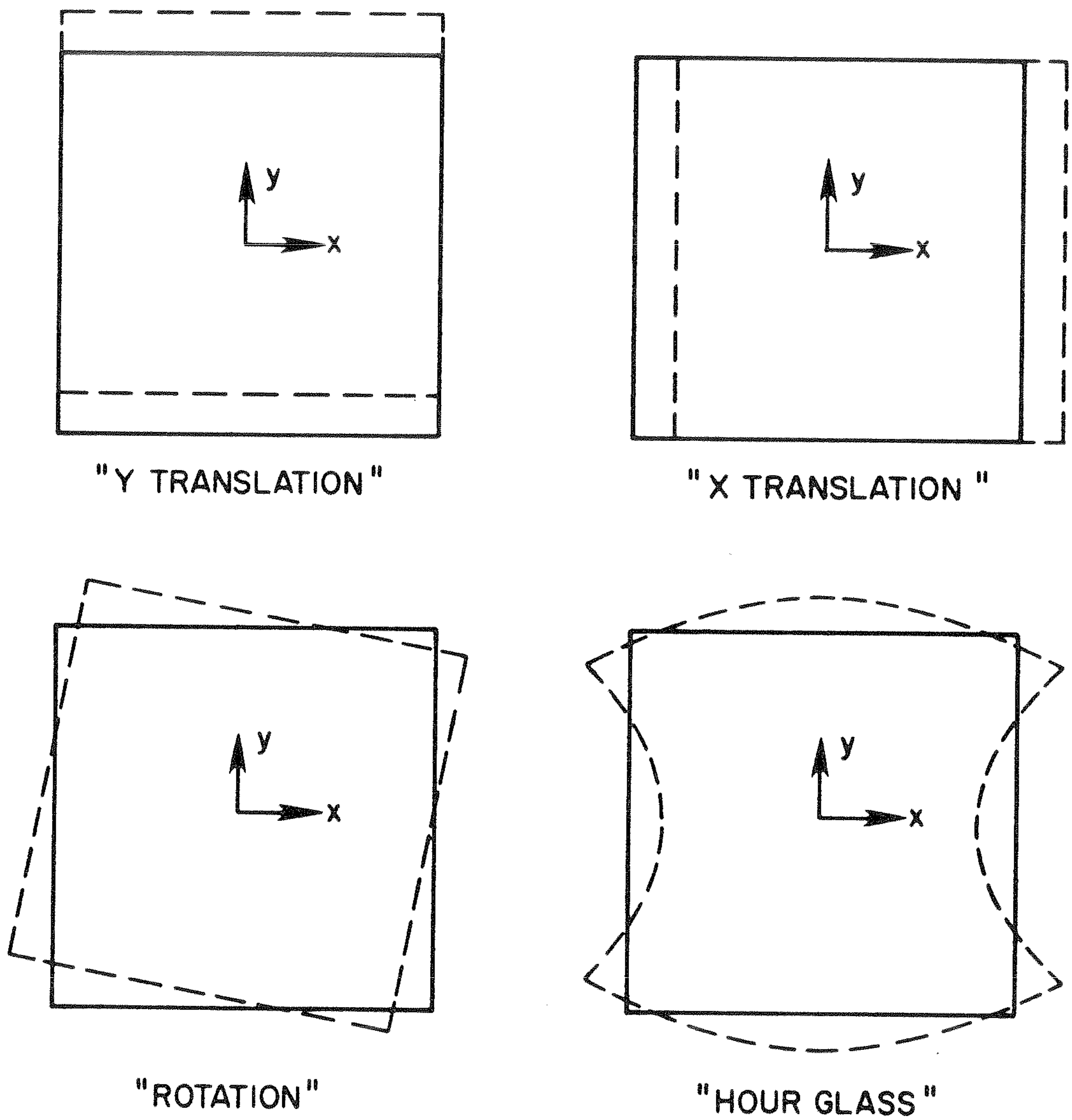


FIGURE 5 ZERO ENERGY MODE SHAPES FOR 8 NODE ELEMENT

5. MODELLING

5.1 Orientation

In its own plane the element equations are expressed in terms of two components of displacement per node. In final form however these are transformed to three global translations per node and may therefore introduce singularities during equation solving due to linear dependency of the global equations. To avoid this problem the following notes should be kept in mind during modelling:

a) The planar element (non-axisymmetric idealization) has no stiffness normal to its local plane. Therefore unless other elements contribute to these normal degrees of freedom the equations should be elemimated from the analysis during specification at the boundary conditions.

b) The planar elements do not contribute stiffness to the rotational degrees of freedom at each node to which they are attached.

5.2 Order of Integration

Care should be exercised in the selection of the order of intergration especially for the eight and nine node element. Both 2 x 2 and 3 x 3 integration may be used for the eight node. However it is recommended that only 3 x 3 be used for the nine node for reasons discussed in the previous section.

5.3 Consistent Nodal Loads

Work equivalent nodel loads for a given displacement expansion are equivalent to the actual distributed load. For load intensities of quadratic variation or less on any face (see Figure 6) the work equaivalent loads $\underline{R}$ are given by:

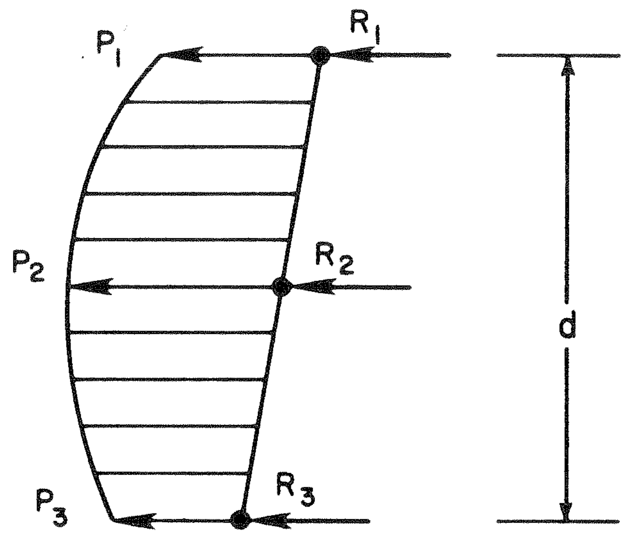


FIGURE 6 CONSISTANT NODAL LOADS

$$R_1 = \frac{d}{15}(2P_1 - \frac{1}{2}P_3 + P_2)$$

$$R_2 = \frac{d}{15}(P_1 - P_3 + 8P_2)$$

$$R_3 = \frac{d}{15}(-\frac{1}{2}P_1 + 2P_3 + P_2)$$

where R_1, R_2, R_3 are the nodal loads at node 1, node 2 (midside) and node 3 respectively and P_1, P_2, P_3 are the corresponding loading intensities at these points.

5.4 Zero Radius

For axisymmetric examples where the internal radius is zero, the stress-displacement matrix becomes undefined for radial stresses. To avoid this problem stresses should not be requested at zero radius. The radial stress at this may be approximated by the hoop stress.

5.5 Pressure Loads

The formulation used in determing pressure loads produces loads parallel to both planar axis. Hence for linearly varying pressure loads applied across a face a net force may result in the orthogonal axis (see Figure 7).

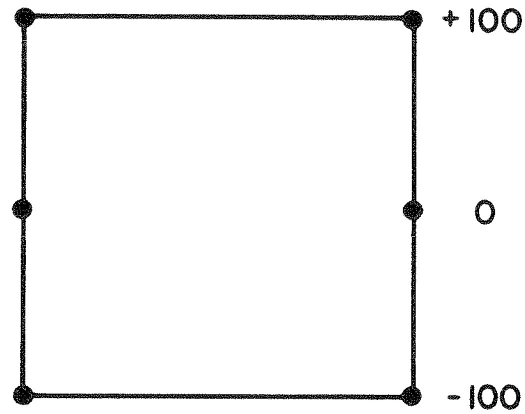


FIGURE 7.a PRESSURE AS SPECIFIED ON ELEMENT FACE

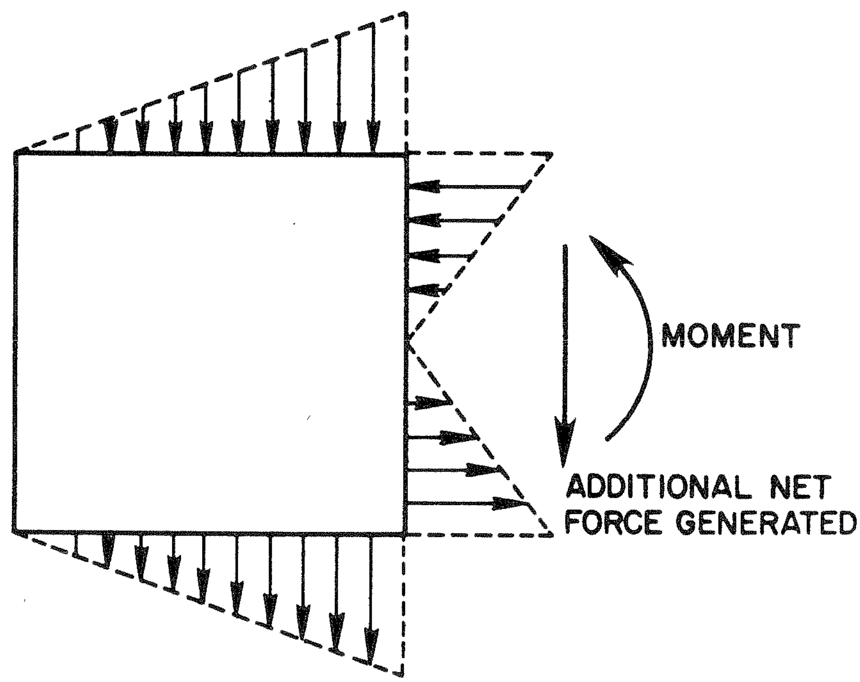


FIGURE 7.b RESULTANT ELEMENT PRESSURE
DISTRIBUTION THROUGH ELEMENT

6. CONCLUSIONS

This element offers a flexible, efficient and reliable means of analyzing planar and axisymmetric structures. The isoparametric formulation allows convenient modelling of curved boundaries. The extension of the standard eight node isoparametric element to allow a midpoint node has increased the elements reliability under geometric distortion.

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4. J. A. Stricklin, W. S. Ho., E. Q. Richardson and W. E. Haisler, "On Isoparametric versus Linear Strain Elements," International Journal for Numerical Methods in Engineering, Vol. 11, pp. 1041-1055.

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APPENDIX A

EXAMPLES

ELEMENT	STRICKLIN ET. AL	HOLLINGS / WILSON
2 ELEMENT LST	0.03001	———
4 ELEMENT LST	0.03087	———
8 NODE (2x2)	———	0.04341
8 NODE (3x3)	0.03054	0.03054
9 NODE (2x2)	———	0.04342
9 NODE (3x3)	———	0.03102

BEAM THEORY 0.040

$E = 1.0 \times 10^7$

$\nu = 0.3$

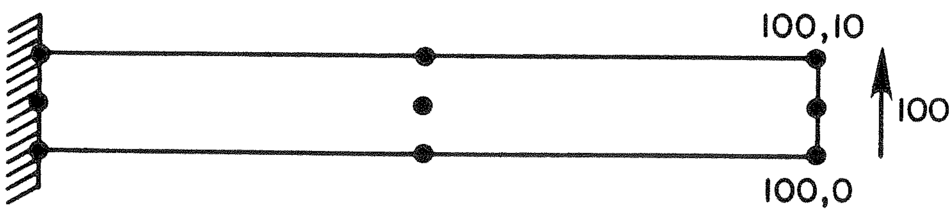

FIGURE 8 COMPARISON OF TIP DEFLECTIONS FOR CANTILEVER BEAM

	STRICKLIN ET. AL	HOLLINGS/WILSON
8 NODE 3*3	0.00644	0.00644
8 NODE 2*2	———	0.01449
9 NODE 3*3	———	0.03165
9 NODE 2*2	———	0.04434

BEAM THEORY 0.040

$E = 1.0 \times 10^7$

$\nu = 0.3$

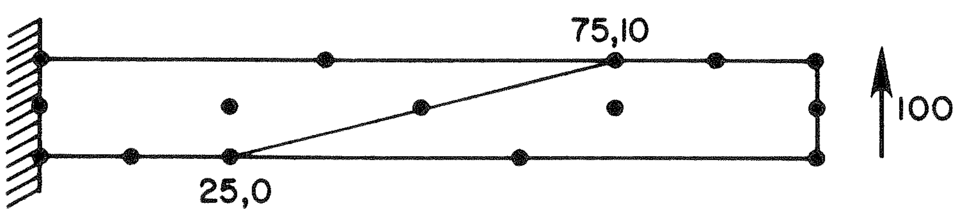

FIGURE 9 COMPARISON OF TIP DEFLECTIONS FOR A CANTILEVER BEAM USING 8 AND 9 NODE ELEMENTS

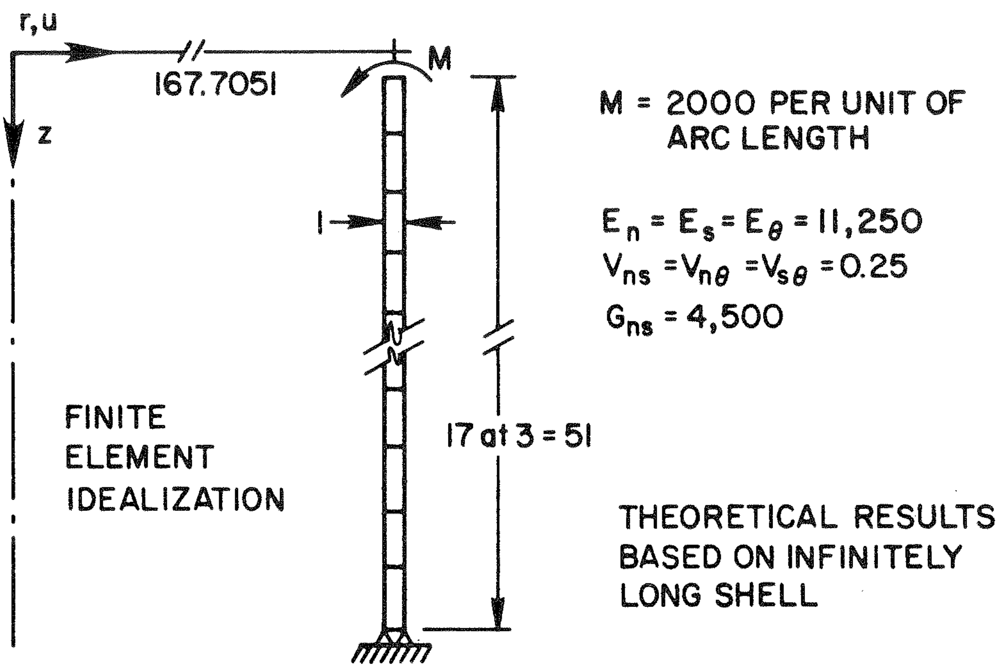

FIGURE 10 CYLINDRICAL SHELL ANALYSIS

LATERAL DISPLACEMENTS u						
Z	THEORY	04CST	QM5	Q4	Q6	3-9Q
0	100.00	39.97	98.56	46.47	100.01	99.75
3	48.88	26.04	47.87	29.17	48.98	49.20
6	14.31	14.98	13.49	15.69	14.19	14.55
9	-6.57	6.56	-7.29	5.69	-6.54	6.49
12	-17.16	0.47	-17.77	-1.31	-17.15	-17.20
15	-20.68	-3.65	-21.17	-5.82	-20.70	-20.90
18	-19.85	-6.16	-20.21	-8.35	-19.88	-20.07
21	-16.75	-7.40	-16.97	-9.39	-16.83	-16.98
24	-12.82	-7.68	-12.92	-9.33	-12.85	-12.98
27	-8.95	-7.27	-8.98	-8.55	-9.00	- 9.10
30	-5.63	-6.40	-5.65	-7.32	-5.72	- 5.76
33	-3.06	-5.27	-3.12	-5.87	-3.23	- 3.24

HOOP STRESSES						
Z	THEORY	Q4CST	QM5	Q4	Q6	3-9Q
1.5	4846	2210	4903	2536	4837	4845
4.5	1991	1369	2050	1507	1986	2024
7.5	159	716	201	718	154	156
10.5	-868	230	-846	147	-873	-855
13.5	-1316	-120	-1311	-238	-1320	-1337
16.5	-1386	-334	-1392	-475	-1390	-1383
19.5	-1240	-459	-1250	-595	-1243	-1261
22.5	-994	-510	-1006	-627	-996	- 1004
25.5	-727	-505	-738	-599	-729	- 739
28.5	-483	-462	-493	-532	-487	- 492
31.5	-285	-395	-297	-442	-293	- 295
34.5	-138	-315	-155	-344	153	-150

FIGURE 11 RESULTS OF CYLINDRICAL SHELL ANALYSIS

APPENDIX B

SELECTED LISTING OF ELEMENTS

```
      SUBROUTINE ELSTF(S,RSI,WG,INR,INS,NTYPE)
C.......................................................................
C                                                                      .
C          FORMS ELEMENT STIFFNESS IN LOCAL COORDINATE SYSTEM          .
C.......................................................................
C
      DIMENSION S(27,27),RSI(3,3),WG(3,3)
      COMMON/TEMP/NTMP,REFT,ETMP,MA,BETA,THK,DJ,NN,DEN,WGHT,
     .             NODE(9),H(3,9),XY(2,9),TP(9),C(4,4),ALP(4),EMUL(4,6),
     .             TL(27),PL(27),GL(9),RL(9),NNODE(9),PR(9),TAU(4),
     .             AA(4),BB(4),CC(4),SS(27,27)
C
C-----STIFFNESS MATRIX--------------------------------------------------
      ND=2*NN
      DO 25 J=1,18
      DO 25 I=1,18
   25 S(I,J)=0.
      DO 500 IR=1,INR
      DO 500 IS=1,INS
      CALL FORMH(RSI(IR,INR),RSI(IS,INS))
      FAC=THK
      IF(NTYPE.NE.0) GO TO 100
      FAC=0.
      DO 50 I=1,NN
   50 FAC=FAC+H(1,I)*XY(1,I)
  100 WGT=WG(IR,INR)*WG(IS,INS)*DJ*FAC
      DO 400 I=1,NN
      I2=2*I
      I1=I2-1
      DO 390 J=1,I
      J2=2*J
      J1=J2-1
      HXHX=H(2,I)*H(2,J)*WGT
      HYHY=H(3,I)*H(3,J)*WGT
      HXHY=H(2,I)*H(3,J)*WGT
      HYHX=H(3,I)*H(2,J)*WGT
      S(I1,J1)=S(I1,J1)+HXHX*C(1,1)+HXHY*C(1,4)+HYHX*C(4,1)+HYHY*C(4,4)
      S(I2,J2)=S(I2,J2)+HYHY*C(2,2)+HYHX*C(2,4)+HXHY*C(4,2)+HXHX*C(4,4)
      S(I1,J2)=S(I1,J2)+HXHY*C(1,2)+HXHX*C(1,4)+HYHY*C(4,2)+HYHX*C(4,4)
      S(I2,J1)=S(I2,J1)+HYHX*C(2,1)+HYHY*C(2,4)+HXHX*C(4,1)+HXHY*C(4,4)
C-------MODIFY FOR AXISYMMETRIC ANALYSIS--------------------------------
      IF(NTYPE.NE.0) GO TO 390
      HH =H(1,I)*H(1,J)*WGT/(FAC*FAC)
      HHX=H(1,I)*H(2,J)*WGT/FAC
      HXH=H(2,I)*H(1,J)*WGT/FAC
      HHY=H(1,I)*H(3,J)*WGT/FAC
      HYH=H(3,I)*H(1,J)*WGT/FAC
      S(I1,J1)=S(I1,J1)+HXH*C(1,3)+HYH*C(4,3)+HHX*C(3,1)+HHY*C(3,4)+
     .                     HH*C(3,3)
      S(I1,J2)=S(I1,J2)+HHY*C(3,2)+HHX*C(3,4)
      S(I2,J1)=S(I2,J1)+HYH*C(2,3)+HXH*C(4,3)
  390 CONTINUE
  400 CONTINUE
  500 CONTINUE
      DO 450 J=1,ND
      DO 450 I=1,J
  450 S(I,J)=S(J,I)
C
      RETURN
      END
```

```
      SUBROUTINE FORMH(R,S)
C.......................................................................
C                                                                      .
C        FORMS 3 - 9 NODE SHAPE FUNCTIONS AND DERIVATIVES IN R - S SPACE
C.......................................................................
C
      COMMON/CNTRL/NIN,NOUT,IPRT
      COMMON/TEMP/NTMP,REFT,ETMP,MA,BETA,THK,DJ,NN,IFILT(2),
     .               NODE(9),H(3,9),XY(2,9),TP(9),C(4,4),ALP(4),EMUL(4,6),
     .              TL(18),PL(18),GL(9),RL(9),NNODE(9),PR(9),TAU(4)
     .             ,AA(4),BB(4),CC(4),SS(27,27)
      DIMENSION N(2,9),D(2,2)
      DATA N/1,1,-1,1,-1,-1,1,-1,0,1,-1,0,0,-1,1,0,0,0/
C
C-------DEFINE INITIAL FUNCTIONS AND DERIVATIVES-----------------------
      DO 200 I=1,9
      CALL GD(R,N(1,I),GR,GRD)
      CALL GD(S,N(2,I),GS,GSD)
      H(1,I) = GR *GS
      H(2,I) = GRD*GS
  200 H(3,I) = GR *GSD
C-----MODIFY FUNCTIONS FOR CENTER NODE (9)---------
      IF(NODE(9)) 325,325,310
  310 DO 315 K=1,3
      X=H(K,9)/2.
      DO 320 I=1,4
  320 H(K,I)=H(K,I)-X/2.
      DO 321 I=5,8
  321 H(K,I)=H(K,I)-X
  315 CONTINUE
  325 CONTINUE
C-----MODIFY FOR NODES 5 TO 8 ---------------------------
      M=4
      DO 300 I=5,8
      IF(NODE(I)) 300,300,250
  250 M=M+1
      DO 290 K=1,3
      X=H(K,I)/2.
      H(K,M)=H(K,I)
      II=I-4
      H(K,II)=H(K,II)-X
      II=II+1
      IF(II.GT.4) II=1
  290 H(K,II)=H(K,II)-X
  300 CONTINUE
      IF(NODE(9).LE.0) GO TO 295
      DO 294 K=1,3
  294 H(K,M+1)=H(K,9)
  295 CONTINUE
C------CALCULATE JACOBIAN MATRIX-----------------
      DO 400 I=1,2
      DO 400 J=1,2
      SUM=0.0
      DO 350 K=1,NN
  350 SUM=SUM+H(I+1,K)*XY(J,K)
  400 D(I,J)=SUM
C-----INVERT JACOBIAN MATRIX --------------------
      DJ=D(1,1)*D(2,2)-D(1,2)*D(2,1)
      IF(DJ.LE.0.0) WRITE(NOUT,2000) DJ
      SUM=D(1,1)/DJ
      D(1,1)=D(2,2)/DJ
      D(1,2)=-D(1,2)/DJ
      D(2,1)=-D(2,1)/DJ
      D(2,2)=SUM
C-----CALCULATE H,Y AND H,Z ---------------------
      DO 500 K=1,NN
      HR=H(2,K)
      HS=H(3,K)
      H(2,K)=D(1,1)*HR+D(1,2)*HS
  500 H(3,K)=D(2,1)*HR+D(2,2)*HS
```

```
C
  700 RETURN
 2000 FORMAT (10H JACOBIAN=  E15.6)
      END
```

```
      SUBROUTINE GD(B,IB,G,D)
      IF(IB) 100,200,300
  100 G=(1.-B)/2.
      D=-.5
      RETURN
  200 G=1.-B*B
      D=-2.*B
      RETURN
  300 G=(1.+B)/2.
      D=.5
      RETURN
      END
```

$\frac{1}{2}(1-\xi)$

$-\frac{1}{2}$

$(1-\xi^2)$

-2ξ

$\frac{1}{2}(1+\xi)$

$\frac{1}{2}$