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Sufficient Conditions for Passivity and Stability of Interconnections of Hybrid Systems using Sums of Storage Functions

Roberto Naldi and Ricardo G. Sanfelice

Abstract— **Building from recent results on passivity for a class of hybrid systems, we investigate the properties of negative feedback interconnections of such systems. We establish links between the passivity properties of the individual subsystems and passivity, stability, and asymptotic stability of their interconnection. As a main difference to the continuous time counterpart, it is found that the sum of the two storage functions of two individual hybrid subsystems may not be a storage function for their interconnection. This issue motivates exploring additional sufficient conditions that guarantee that passivity and stability of the interconnected system hold using the individual storage functions. Throughout the paper, an application and examples illustrate the definitions and the results obtained.**

I. Introduction

Dissipativity and its special case, passivity, have been successfully employed to design feedback control laws and to investigate the stability properties of closed-loop dynamical systems. In fact, they provide a useful physical intuition, in term of energy supplied and dissipated by a system, which relates nicely to Lyapunov and $\mathcal{L}_2$ stability theories. For continuous-time systems, passivity and dissipativity notions as well as sufficient conditions linking to stability and asymptotic stability are reported in several textbooks, including [1], [2], [3], [4]. Passivity-based control techniques, based on energy considerations, have been also proposed and applied to stabilize different physical systems [5]. More recently, the concept of passivity and dissipativity has been investigated also for switching systems [6], [7], [8], complementary mechanical systems [9], impulsive dynamical systems [10], and hybrid systems [11], [12]. In [12], in particular, the case of hybrid systems in which the energy dissipation may only happen along either the continuous or the discrete time dynamics has been considered, deriving two weak notions of passivity, respectively *flow-passivity*, in which dissipation happens along flows, and *jump-passivity*, in which dissipation happens along jumps.

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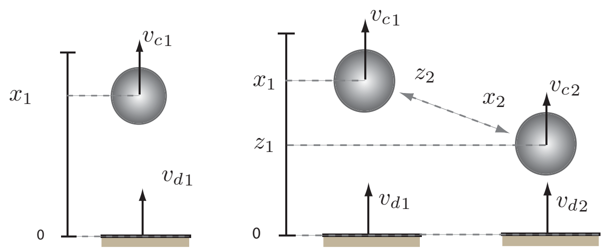
Dissipativity and passivity have also been shown to be useful tools to investigate the stability properties of interconnected systems. More specifically, for continuous-time systems, a fundamental result is the fact that the (negative) feedback interconnection of two passive systems is passive [2], [1]. Interestingly, this property can be established using the sum of the two individual storage functions. More recently, passivity and dissipativity concepts have been employed to analyze interconnections of different classes of systems, including, in particular, hybrid systems. In [10], results pertaining to the (negative) feedback interconnection of dissipative impulsive dynamical systems have been proposed. The definition of interconnected system employed therein, allows for the dissipativity inequalities to hold when considering the sum of the two individual storage functions as a storage function for the interconnection. Asymptotic stability of large-scale interconnections of hybrid systems is investigated in [11] by considering the notions of dissipativity and detectability. In [11], the hybrid system is decomposed into a number of subsystems and then reconstituted by defining interconnection constraints. In [13], the interconnection of hybrid systems is investigated and input-output stability properties, based on the notion of input-to-state stability and the small gain theorem [3], are derived.

In this paper, by considering the hybrid-specific notion of passivity for hybrid systems proposed in [12] and the definition of feedback interconnection of hybrid systems in [13], we derive sufficient conditions linking passivity properties of hybrid systems to passivity and stability of their interconnection. When the passivity properties hold during jumps, it is shown how the sum of the individual storage functions may not be a storage function for the interconnection. This fact, which represents an important difference with respect to continuous-time systems, derives from the fact that the interconnected system may encounter situations in which only one of the two subsystems is allowed to jump. Accordingly, sufficient conditions are derived to establish passivity, 0-input stability and asymptotic stability of the interconnection by considering the sum of the two storage functions. The results are obtained by considering the cases in which, for each individual subsystem, the passivity properties hold either only during flows or jumps (or along both regimes). An application, considering an actuated bouncing ball, and several examples illustrate the main results.



(a) $\mathcal{H}_{BB}$ (b) $\mathcal{H}_{BB,12}$

Fig. 1. Motivational application.

The paper is organized as follows. Section II introduces a motivational application pertaining to an actuated bouncing ball. Section III presents the main definition of passivity and of (negative) feedback interconnection. Passivity properties of the interconnected system are investigated in Section IV. Stability and asymptotic stability are then addressed in Section V.

II. Motivational Application

We consider the mechanical system depicted in Figure 1(a), which consists of a ball bouncing on a fixed horizontal surface. The motion of the ball can be affected by two mechanical actuators, one able to apply a force to the ball while flowing and the other one controlling the speed of the ball at impacts (similar actuation is considered in the systems in [14]). The mechanical system of interest can be described by the following hybrid system:

$$\mathcal{H}_{BB}\left\{\begin{array}{lr} \dot{x} = F(x, v_c) := \begin{bmatrix} x_2 \\ -\gamma + v_c \end{bmatrix} & x \in C \\ x^+ = G(x, v_d) := \begin{bmatrix} x_1 \\ -\varrho x_2 - v_d \end{bmatrix} & (x, v_d) \in D \end{array}\right. \tag{1}$$

with state $x = [x_1, x_2]^\top \in \mathbb{R}^2$, inputs $v_c, v_d \in \mathbb{R}$, fixed restitution coefficient $\varrho \in [0, 1)$, gravity constant $\gamma > 0$ and sets C and D given by

$$\begin{array}{ll} C & := \left\{x \in \mathbb{R}^2 \ : \ x_1 \geq 0 \right\} \\ D & := \left\{(x, v_d) \in \mathbb{R}^2 \times \mathbb{R} \ : \ x_1 = 0, x_2 \leq 0, v_d \in U(x_2) \right\} \end{array} \tag{2}$$

where $U : \mathbb{R} \to \mathbb{R}$ defines the constraint set for the input v_{d1} which is given by $x_2 \mapsto U(x_2) := \{v_d \in \mathbb{R} : {v_d}^2 \leq (1 - \varrho^2)x_2^2\}$; namely, the applied input at impacts is upper bounded by the ball's velocity.

For the above hybrid system, by considering a natural notion of passivity (see Definition 1 in the upcoming section), it is possible to show the following property.

Proposition 1 *System (1) with inputs v_c and v_d, and outputs*

$$y_c = h_c(x) := x_2, \ y_d = h_d(x) := \varrho x_2 \tag{3}$$

is passive with respect to the compact set $\mathcal{A} = \{(0, 0)\}$ with storage function $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by $V(x) := \frac{1}{2}x_2^2 + \gamma x_1$.

Motivated by the passivity properties shown in Proposition 1, we consider now the negative feedback interconnection of two actuated bouncing balls (1)-(3), denoted as $\mathcal{H}_{BB,1}$ and $\mathcal{H}_{BB,2}$, respectively. For notational simplicity, in the following we denote with $z = [z_1, z_2]^\top \in \mathbb{R}^2$ the state of $\mathcal{H}_{BB,2}$ and we use the subscripts 1 and 2 to denote inputs and outputs of $\mathcal{H}_{BB,1}$ and $\mathcal{H}_{BB,2}$, respectively.

The negative feedback interconnection of $\mathcal{H}_{BB,1}$ and $\mathcal{H}_{BB,2}$ can be then obtained by choosing $v_{c1} = \tilde{v}_{c1} - y_{c2}$, $v_{c2} = \tilde{v}_{c2} + y_{c1}$, $v_{d1} = \tilde{v}_{d1} - y_{d2}$ and $v_{d2} = \tilde{v}_{d2} + y_{d1}$ where $\tilde{v}_{c1}$, $\tilde{v}_{c2}$, $\tilde{v}_{d1}$, $\tilde{v}_{d2}$ are additional inputs.[1] By considering the results in [13] for the interconnection of hybrid systems, the interconnected system can be written as a new hybrid system $\mathcal{H}_{BB,12}$ (see Figure 1(b)) given by

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\gamma - z_2 + \tilde{v}_{c1} \\ \dot{z}_1 = z_2 \\ \dot{z}_2 = -\gamma + x_2 + \tilde{v}_{c2} \end{array}\right\} & x \in C \text{ and } z \in C \\ \left.\begin{array}{l} x_1^+ = x_1 \\ x_2^+ = -\varrho x_2 + \varrho z_2 - \tilde{v}_{d1} \\ z_1^+ = z_1 \\ z_2^+ = -\varrho z_2 - \varrho x_2 - \tilde{v}_{d2} \end{array}\right\} & \begin{array}{c}(x, \tilde{v}_{d1} - y_{d2}) \in D \text{ and} \\ (z, \tilde{v}_{d2} + y_{d1}) \in D\end{array} \\ \left.\begin{array}{l} x_1^+ = x_1 \\ x_2^+ = -\varrho x_2 + \varrho z_2 - \tilde{v}_{d1} \\ z_1^+ = z_1 \\ z_2^+ = z_2 \end{array}\right\} & \begin{array}{c}(x, \tilde{v}_{d1} - y_{d2}) \in D \text{ and} \\ (z, \tilde{v}_{d2} + y_{d1}) \notin D\end{array} \\ \left.\begin{array}{l} x_1^+ = x_1 \\ x_2^+ = x_2 \\ z_1^+ = z_1 \\ z_2^+ = -\varrho z_2 - \varrho x_2 - \tilde{v}_{d2} \end{array}\right\} & \begin{array}{c}(x, \tilde{v}_{d1} - y_{d2}) \notin D \text{ and} \\ (z, \tilde{v}_{d2} + y_{d1}) \in D\end{array} \end{array}\right. \tag{4}$$

Let V_1 and V_2 be given by $V_1(x) = \frac{1}{2}x_2^2 + \gamma x_1$ and $V_2(z) = \frac{1}{2}z_2^2 + \gamma z_1$. According to Proposition 1, $\mathcal{H}_{BB,1}$ is passive with respect to $\mathcal{A}$ with inputs v_{c1} and v_{d1}, outputs y_{c1} and y_{d1}, and storage function V_1, while $\mathcal{H}_{BB,2}$ is passive with respect to $\mathcal{A}$ with inputs v_{c2} and v_{d2}, outputs y_{c2} and y_{d2} with storage function V_2. Following results for continuous-time systems [2], [1], which guarantee that the negative feedback interconnection of two passive systems is passive (property that can be established using the sum of the individual storage functions) the hybrid system $\mathcal{H}_{BB,12}$ is expected to be passive with inputs $v_c = [\tilde{v}_{c1}^\top, \tilde{v}_{c2}^\top]^\top$ and $v_d = [\tilde{v}_{d1}^\top, \tilde{v}_{d2}^\top]^\top$, outputs $y_c = [{y_{c1}}^\top, {y_{c2}}^\top]^\top$ and $y_d = [{y_{d1}}^\top, {y_{d2}}^\top]^\top$ and storage function $V(x, z) = V_1(x) + V_2(z)$. However, this property does not seem to hold for $\mathcal{H}_{BB,12}$. In fact, along jumps, when $(x, \tilde{v}_{d1} - y_{d2}) \in D$ and $(z, \tilde{v}_{d2} + y_{d1}) \notin D$, since $v_{d1} = \tilde{v}_{d1} - y_{d2}$, we obtain[2]

$$\begin{array}{rl} V(G(x, v_{d1}), z) - V(x, z) & = V_1(G(x, v_{d1})) - V_1(x) \\ & = \frac{1}{2}\left(-\varrho x_2 - (\tilde{v}_{d1} - \varrho z_2)\right)^2 - \frac{1}{2}x_2^2 \\ & \leq \varrho x_2(\tilde{v}_{d1} - \varrho z_2) = \tilde{v}_{d1}y_{d1} - y_{d1}y_{d2} \end{array}$$

[1] Such a negative interconnection could emerge in a feedback control setting where the speed of the two balls are measured quantities, or in the hypothetical setting where ball $\mathcal{H}_{BB,1}$ introduces viscous friction on the motion of $\mathcal{H}_{BB,2}$, while $\mathcal{H}_{BB,2}$ introduces negative viscous friction on the motion of $\mathcal{H}_{BB,1}$ both along flows and jumps; see Figure 1(b).

[2] When both systems jump simultaneously, namely $(x, \tilde{v}_{d1} - y_{d2}) \in D$ and $(z, \tilde{v}_{d2} + y_{d1}) \in D$, passivity can be established by considering the sum of the two storage functions. In fact, in such a case, $V(G(x, v_{d1}), G(z, v_{d2})) - V(x, z) = {v_d}^\top y_d$.

while, when $(x,\, \tilde{v}_{d1} - y_{d2}) \notin D$ and $(z,\, \tilde{v}_{d2} + y_{d1}) \in D$,

$$\begin{aligned} V(x, G(z, v_{d2})) - V(x, z) &= V_2(G(z, v_{d2})) - V_2(z) \\ &= \tfrac{1}{2}\left(-\varrho z_2 - (\tilde{v}_{d2} + \varrho x_2)\right)^2 - \tfrac{1}{2} z_2^2 \\ &\leq \varrho z_2(\tilde{v}_{d2} + \varrho x_2) = \tilde{v}_{d2} y_{d2} + y_{d1} y_{d2} \,. \end{aligned}$$

Now assume $\tilde{v}_{d1} = \tilde{v}_{d2} = 0$ and note that, from the definition of D, when $(x,\, -y_{d2}) \notin D$ and $(z,\, y_{d1}) \in D$, the product $y_{d1} y_{d2}$ is allowed to be larger or equal than zero and hence the function V may increase at jumps. Note that the fact that $V_1(x) + V_2(z)$ is not a storage function for $\mathcal{H}_{BB,\,12}$ does not imply that passivity may not follow by considering a different storage function. It is worth to note, however, that this fact represents an important difference with respect to continuous-time systems for which such a property holds true [2]. In this paper, we present conditions under which the interconnection has passivity properties certified by the sum of the systems storage functions.[3]

III. Preliminaries

A. Notation

Throughout this paper, $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the field of real and positive real numbers, respectively. For $x \in \mathbb{R}^n$, $|x|$ and $|x|_\infty$ denote respectively the Euclidean and the infinity norm and, given a closed set $\mathcal{A}$, subset of $\mathbb{R}^n$, $|x|_{\mathcal{A}} = \min_{y \in \mathcal{A}} |x - y|$ denotes the distance to $\mathcal{A}$ from x. Given a set $\mathcal{S}$, $\bar{\mathcal{S}}$ denotes its closure. At times, for convenience, we may refer to a vector $[x^\top,\, y^\top]^\top$ as $(x,\, y)$. Given a set $S \subset \mathbb{R}^n \times \mathbb{R}^m$, we denote $\Pi_0(S) := \{x \in \mathbb{R}^n \ : \ (x, 0) \in S\}$ and $\Pi(S) := \{x \in \mathbb{R}^n \ : \ \exists u \in \mathbb{R}^m \text{ s.t. } (x, u) \in S\}$.

B. Passivity Definitions

In this work, we consider hybrid systems $\mathcal{H}$ as in [15] given by

$$\mathcal{H} \qquad \begin{cases} \dot{x} & \in \ F(x, v_c) \qquad (x, v_c) \in C \\ x^+ & \in \ G(x, v_d) \qquad (x, v_d) \in D \\ y_c & = \ h_c(x) \\ y_d & = \ h_d(x) \end{cases} \tag{5}$$

with state $x \in \mathbb{R}^n$, input $v = \left[{v_c}^\top,\, {v_d}^\top\right]^\top \in \mathbb{R}^{m_c + m_d}$ in which $v_c \in \mathbb{R}^{m_c}$ and $v_d \in \mathbb{R}^{m_d}$ are respectively the inputs for flows and jumps. The sets $C \subset \mathbb{R}^n \times \mathbb{R}^{m_c}$ and $D \subset \mathbb{R}^n \times \mathbb{R}^{m_d}$ define the flow and jump sets, respectively; the set-valued mappings $F : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightrightarrows \mathbb{R}^n$ define the flow map and jump map, respectively. Finally, we let $y = \left[{y_c}^\top,\, {y_d}^\top\right]^\top \in \mathbb{R}^{m_c + m_d}$ be the output where y_c, y_d are assigned via functions of the state $h_c : \mathbb{R}^n \to \mathbb{R}^{m_c}$ and $h_d : \mathbb{R}^n \to \mathbb{R}^{m_c}$.

For this class of hybrid systems, we consider the following concept of passivity (see [12]). Below, the functions h_c, h_d, and a compact set $\mathcal{A} \subset \mathbb{R}^n$ satisfy $h_c(\mathcal{A}) = h_d(\mathcal{A}) = 0$.

[3] In [10], the passivity of interconnections of impulsive dynamical systems can be certified by considering the sum of the individual storage functions. In fact the notion of interconnection used therein, allows only simultaneous jumps of the individual systems, which ensures that such a property holds.

Definition 1 (see Def. 2 in [12]) *A hybrid system $\mathcal{H}$ for which there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, called a "storage function,"*

- *continuous on $\mathbb{R}^n$;*
- *continuously differentiable on a neighborhood of $\Pi(\overline{C})$;*
- *satisfying for some functions $\omega_c : \mathbb{R}^{m_c} \times \mathbb{R}^n \to \mathbb{R}$ and $\omega_d : \mathbb{R}^{m_c} \times \mathbb{R}^n \to \mathbb{R}$*

$$\langle \nabla V(x), \xi \rangle \leq \omega_c(v_c, x) \qquad \forall (x, v_c) \in C,\ \xi \in F(x, v_c) \tag{6}$$

$$V(\xi) - V(x) \leq \omega_d(v_d, x) \qquad \forall (x, v_d) \in D,\ \xi \in G(x, v_d) \tag{7}$$

is said to be

- passive with respect to a compact set $\mathcal{A}$ *if*

$$(v_c, x) \mapsto \omega_c(v_c, x) \ = \ v_c^\top y_c \tag{8}$$

$$(v_d, x) \mapsto \omega_d(v_d, x) \ = \ v_d^\top y_d. \tag{9}$$

  It is then called flow-passive (respectively, jump-passive) if it is passive with $\omega_d \equiv 0$ (respectively, $\omega_c \equiv 0$).
- strictly passive with respect to a compact set $\mathcal{A}$ *if*

$$\begin{aligned} (v_c, x) \mapsto \omega_c(v_c, x) &= v_c^\top y_c - \rho_c(x) \\ (v_d, x) \mapsto \omega_d(v_d, x) &= v_d^\top y_d - \rho_d(x), \end{aligned}$$

  where $\rho_c, \rho_d : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are positive definite with respect to $\mathcal{A}$. It is then called flow-strictly passive (respectively, jump-strictly passive) if it is strictly passive with $\omega_d \equiv 0$ (respectively, $\omega_c \equiv 0$).
- output strictly passive with respect to $\mathcal{A}$ *if*

$$\begin{aligned} (v_c, x) \mapsto \omega_c(v_c, x) &= v_c^\top y_c - y_c^\top \rho_c(y_c) \\ (v_d, x) \mapsto \omega_d(v_d, x) &= v_d^\top y_d - y_d^\top \rho_d(y_d), \end{aligned}$$

  where $\rho_c : \mathbb{R}^{m_c} \to \mathbb{R}^{m_c}$, $\rho_d : \mathbb{R}^{m_d} \to \mathbb{R}^{m_d}$ are functions such that $y_c^\top \rho_c(y_c) > 0$ for all $y_c \neq 0$ and such that $y_d^\top \rho_d(y_d) > 0$ for all $y_d \neq 0$, respectively. It is then called flow-output strictly passive (respectively, jump-output strictly passive) if it is output strictly passive with $\omega_d \equiv 0$ (respectively, $\omega_c \equiv 0$).

C. Stability Notions

Given an input (v_c, v_d), a solution to $\mathcal{H}$ is defined by a state trajectory ϕ that satisfies the differential and difference inclusions with constraints in (5). Both the input and the state trajectory are functions of $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} := [0, \infty) \times \{0, 1, 2, \ldots\}$, where t keeps track of the amount of flow while j counts the number of jumps of the solution. These functions are given by *hybrid arcs* and *hybrid inputs*, which are defined on *hybrid time domains*. A hybrid time domain is a subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ that, for each $(T, J) \in E$, $E \ \cap \ ([0, T] \times \{0, 1, \ldots J\})$ can be written in the form $\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_J$. A hybrid arc ϕ is a function on a hybrid time domain. The hybrid time domain of ϕ is denoted by $\operatorname{dom} \phi$. A hybrid arc is such that, for each $j \in \mathbb{N}$, $t \mapsto \phi(t, j)$ is absolutely continuous on intervals of flow $I^j := \{t \ : \ (t, j) \in \operatorname{dom} \phi \ \}$ with nonzero Lebesgue

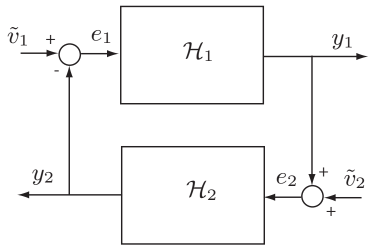


Fig. 2. The hybrid system $\mathcal{H}_{12}$ resulting from the negative feedback interconnection between $\mathcal{H}_1$ and $\mathcal{H}_2$.

measure. A hybrid input u is a function on a hybrid time domain that, for each $j \in \mathbb{N}$, $t \mapsto u(t,j)$ is Lebesgue measurable and locally essentially bounded on the interval I^j. A solution ϕ is maximal if it cannot be further extended and complete if its hybrid time domain is unbounded.

We consider the following stability definitions for hybrid systems when their input is set to zero.

Definition 2 (see Def. 3 in [12]) *A compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be*

- 0-input stable *if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each maximal solution pair $(\phi, 0)$ to $\mathcal{H}$ and $\phi(0,0) = \xi$, $|\xi|_{\mathcal{A}} \le \delta$, satisfies $|\phi(t,j)|_{\mathcal{A}} \le \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$;*
- 0-input pre-attractive *if there exists $\mu > 0$ such that every maximal solution pair $(\phi, 0)$ to $\mathcal{H}$ and $\phi(0,0) = \xi$, $|\xi|_{\mathcal{A}} \le \mu$, is bounded and if it is complete satisfies*
$$\lim_{(t,j)\in \operatorname{dom}\phi,\, t+j\to\infty} |\phi(t,j)|_{\mathcal{A}} = 0;$$
- 0-input pre-asymptotically stable *if it is 0-input stable and 0-input pre-attractive.*

When every maximal solution is complete, the prefix "pre" can be removed. Asymptotic stability is said to be global when the attractivity property holds in $\overline{C} \cup D$.

IV. Passivity of Feedback Interconnections of Hybrid Systems

Given two hybrid systems, $\mathcal{H}_1$ and $\mathcal{H}_2$, we are interested in investigating the properties of the hybrid system $\mathcal{H}_{12}$ obtained as the negative feedback interconnection of $\mathcal{H}_1$ and $\mathcal{H}_2$, having[4] $m_{c1} = m_{c2} = m_c$, $m_{d1} = m_{d2} = m_d$. Following the definition of a typical negative feedback interconnection (see [2] for interconnections of two continuous-time systems), $\mathcal{H}_{12}$ is defined as in Figure 2 by considering the following assignments:

$$\begin{array}{ll} v_{c1} = e_{c1} := \tilde v_{c1} - y_{c2}, & v_{c2} = e_{c2} := \tilde v_{c2} + y_{c1}, \\ v_{d1} = e_{d1} := \tilde v_{d1} - y_{d2}, & v_{d2} = e_{d2} := \tilde v_{d2} + y_{d1} \end{array} \tag{10}$$

in which $\tilde v_{c1}, \tilde v_{c2} \in \mathbb{R}^{m_c}$, $\tilde v_{d1}, \tilde v_{d2} \in \mathbb{R}^{m_d}$ are the inputs for the interconnection.

[4]In the following, we will denote with subscript i vectors (state, inputs and outputs) and other objects (functions, parameters, etc.) belonging to the hybrid system $\mathcal{H}_i$, $i \in \{1, 2\}$.

Accordingly, for the interconnection $\mathcal{H}_{12}$ we denote with $x = [x_1^\top, x_2^\top]^\top \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ the state, with $v_c = [\tilde v_{c1}^\top, \tilde v_{c2}^\top]^\top \in \mathbb{R}^{2m_c}$ and $v_d = [\tilde v_{d1}^\top, \tilde v_{d2}^\top]^\top \in \mathbb{R}^{2m_d}$ the inputs and with $y_c = [y_{c1}^\top, y_{c2}^\top]^\top \in \mathbb{R}^{2m_c}$ and $y_d = [y_{d1}^\top, y_{d2}^\top]^\top \in \mathbb{R}^{2m_d}$ the outputs.

Following [13], the interconnection $\mathcal{H}_{12}$ can be written as a hybrid system $\mathcal{H}$ with data (C, F, D, G, h_c, h_d) given by

$$\begin{aligned} C &:= \{(x,\eta) \ : \ (x_1,\eta_1) \in C_1,\, (x_2,\eta_2) \in C_2 \ \}, \\ F(x,\eta) &:= [F_1(x_1,\eta_1)^\top, F_2(x_2,\eta_2)^\top]^\top, \\ D &:= \{(x,\eta) \ : \ (x_1,\eta_1) \in D_1 \ \} \cup \\ &\qquad\qquad \{(x,\eta) \ : \ (x_2,\eta_2) \in D_2 \ \}, \\ G(x,\eta) &:= [\tilde G_1(x_1,\eta_1)^\top, \tilde G_2(x_2,\eta_2)^\top]^\top, \end{aligned}$$

where

$$\begin{aligned} \tilde G_1(x_1,\eta_1) &:= \begin{cases} G_1(x_1,\eta_1) & (x_1,\eta_1) \in D_1 \\ x_1 & \text{otherwise} \end{cases} \\ \tilde G_2(x_2,\eta_2) &:= \begin{cases} G_2(x_2,\eta_2) & (x_2,\eta_2) \in D_2 \\ x_2 & \text{otherwise} \end{cases} \end{aligned}$$

and $h_c(x) := [h_{c1}(x_1)^\top, h_{c2}(x_2)^\top]^\top$, $h_d(x) := [h_{d1}(x_1)^\top, h_{d2}(x_2)^\top]^\top$.

A. Passivity

For two hybrid systems that are flow-passive as in Definition 1, we have the following passivity result of the interconnection $\mathcal{H}_{12}$.

Theorem 1 *If, for each $i \in \{1, 2\}$, $\mathcal{H}_i$ is flow-passive with respect to the compact set $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ with storage function $V_i(x_i)$, then the hybrid system $\mathcal{H}_{12}$ is flow-passive with respect to the compact set $\mathcal{A}_{12} = \mathcal{A}_1 \times \mathcal{A}_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with storage function $V(x) = V_1(x_1) + V_2(x_2)$.*

The above result is showing how, for the special case of flow-passive hybrid systems, passivity of the negative feedback interconnection can be established by considering the sum of the two individual storage functions. This fact reveals an interesting analogy between flow-passive hybrid systems and their continuous-time counterpart. Next, we show how Theorem 1 can be applied to establish passivity for a modified version of the motivational application.

Example 1 *(Motivational Application (revisited)) Consider the actuated bouncing ball $\mathcal{H}_{BB}$ defined in (1). Denote with $\mathcal{H}_{BB^\star}$ the hybrid system $\mathcal{H}_{BB}$ when the input $v_d = 0$. Using Proposition 1, it can be easily shown that $\mathcal{H}_{BB^\star}$ is flow-passive according to Definition 1 with respect to the compact set $\mathcal{A} = \{(0, 0)\}$ by considering the input v_c and the output y_c defined in (3). Applying Theorem 1, the negative feedback interconnection $\mathcal{H}_{BB^\star,12}$, which corresponds to (4) (in which the interconnection along jump has been removed due to $ujump = 0$), is flow-passive with input $v_c = [\tilde v_{c1}, \tilde v_{c2}]^\top$, output $y_c = [y_{c1}, y_{c2}]^\top = [x_2, z_2]^\top$ and storage function $V(x,z) = \frac{1}{2}x_2^2 + \gamma x_1 + \frac{1}{2}z_2^2 + \gamma z_1$.* □

Unfortunately, as suggested in Section II, the jumps of the individual subsystems may not allow for their interconnection to be passive. The following conditions will be employed hereafter to derive additional passivity results for the hybrid system $\mathcal{H}_{12}$[5].

(C1) For all $x_1,\ x_2,\ \tilde{v}_{d1},\ \tilde{v}_{d2}$ such that $(x_1, \tilde{v}_{d1} - h_{d2}(x_2)) \in D_1$ and $(x_2, \tilde{v}_{d2} + h_{d1}(x_1)) \notin D_2$

$$-h_{d1}(x_1)^\top h_{d2}(x_2) \leq \tilde{v}_{d2}^\top h_{d2}(x_2)\,; \tag{11}$$

(C2) For all $x_1,\ x_2,\ \tilde{v}_{d1},\ \tilde{v}_{d2}$ such that $(x_1, \tilde{v}_{d1} - h_{d2}(x_2)) \notin D_1$ and $(x_2, \tilde{v}_{d2} + h_{d1}(x_1)) \in D_2$

$$h_{d1}(x_1)^\top h_{d2}(x_2) \leq \tilde{v}_{d1}^\top h_{d1}(x_1)\,. \tag{12}$$

These conditions permit establishing the following result for the interconnection of hybrid systems that are passive or jump-passive.

Theorem 2 *If, for each $i \in \{1,2\}$, $\mathcal{H}_i$ is passive (jump-passive) with respect to $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ with storage function $V_i(x_i)$, and (C1)-(C2) hold true for $\mathcal{H}_{12}$, then the hybrid system $\mathcal{H}_{12}$ resulting from the negative feedback interconnection of $\mathcal{H}_1$ and $\mathcal{H}_2$ is passive (respectively jump-passive) with respect to the compact set $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with storage function $V(x) = V_1(x_1) + V_2(x_2)$.*

Theorem 2 exploits (C1)-(C2) to establish passivity of the negative feedback interconnection of hybrid systems for which, in particular, the passivity property holds during jumps. In fact, conditions (C1)-(C2) ensure that the passivity inequalities continue to hold in all those situations in which only one of the two subsystems jumps.

V. 0-Input Stability Properties of Feedback Interconnections of Hybrid Systems

We relate different forms of passivity to stability and asymptotic stability with zero input, that is, for the hybrid system $\mathcal{H}_{12}$ with $v_c = 0$, $v_d = 0$

$$\mathcal{H}_{12}^0 \quad \begin{cases} \dot{x} & \in & F(x,0) & (x,0) \in C \\ x^+ & \in & G(x,0) & (x,0) \in D \\ y_c & = & h_c(x) & \\ y_d & = & h_d(x) & \end{cases} \tag{13}$$

Below, let $\mathcal{X}$ be defined as $\mathcal{X} := \Pi_0(C) \cup \Pi_0(D) \cup G(\Pi_0(D), 0)$.

The following conditions will be employed hereafter to derive stability results for the hybrid system $\mathcal{H}_{12}^0$.

(Cs1) $h_{c1}(x_1)^\top h_{c2}(x_2) \geq 0$ for all $x_1,\ x_2$ such that $(x_1,\ -h_{c2}(x_2)) \in C_1$ and $(x_2,\ h_{c1}(x_1)) \in C_2$;
(Cs2) $h_{d1}(x_1)^\top h_{d2}(x_2) \leq 0$ for all $x_1,\ x_2$ such that $(x_2,\ h_{d1}(x_1)) \in D_2$;
(Cs3) $h_{c1}(x_1)^\top h_{c2}(x_2) \leq 0$ for all $x_1,\ x_2$ such that $(x_1,\ -h_{c2}(x_2)) \in C_1$ and $(x_2,\ h_{c1}(x_1)) \in C_2$;
(Cs4) $h_{d1}(x_1)^\top h_{d2}(x_2) \geq 0$ for all $x_1,\ x_2$ such that $(x_1,\ -h_{d2}(x_2)) \in D_1$.

[5]Note that for the case in which inputs of $\mathcal{H}_{12}$ are zero, inequalities (11)-(12) in (C1)-(C2) become $h_{d1}(x_1)^\top h_{d2}(x_2) \geq 0$ and $h_{d1}(x_1)^\top h_{d2}(x_2) \leq 0$. Those inequalities have to be evaluated for different values of $x_1,\ x_2$ which depend on the definition of the sets D_1 and D_2.

Theorem 3 *Given a compact set $\mathcal{A}_{12} = \mathcal{A}_1 \times \mathcal{A}_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, if at least one of the following conditions is satisfied for each $i \in \{1,2\}$, then $\mathcal{A}_{12}$ is 0-input stable for $\mathcal{H}_{12}^0$:*

1) *The hybrid system $\mathcal{H}_i$ is flow-passive with respect to $\mathcal{A}_i$ with storage function V_i that is positive definite on $\mathcal{X}_i$ with respect to $\mathcal{A}_i$;*
2) *The hybrid system $\mathcal{H}_i$ is passive (jump-passive) with respect to $\mathcal{A}_i$ with storage function V_i that is positive definite on $\mathcal{X}_i$ with respect to $\mathcal{A}_i$ and (C1)-(C2) hold with $\tilde{v}_{d1} = 0$ and $\tilde{v}_{d2} = 0$ for $\mathcal{H}_{12}^0$;*
3) *The hybrid system $\mathcal{H}_1$ is flow-passive (jump-passive) with respect to $\mathcal{A}_1$ with storage function V_1 that is positive definite on $\mathcal{X}_1$ with respect to $\mathcal{A}_1$ and $\mathcal{H}_2$ is jump-passive (flow-passive) with respect to $\mathcal{A}_2$ with storage function V_2 that is positive definite on $\mathcal{X}_2$ with respect to $\mathcal{A}_2$ and (Cs1)-(Cs2) (respectively, (Cs3)-(Cs4)) hold true;*
4) *The hybrid system $\mathcal{H}_1$ is passive (flow-passive) with respect to $\mathcal{A}_1$ with storage function V_1 that is positive definite on $\mathcal{X}_1$ with respect to $\mathcal{A}_1$ and $\mathcal{H}_2$ is flow-passive (passive) with respect to $\mathcal{A}_2$ with storage function V_2 that is positive definite on $\mathcal{X}_2$ with respect to $\mathcal{A}_2$ and (Cs2) (respectively, (Cs4)) holds true;*
5) *The hybrid system $\mathcal{H}_1$ is passive (jump-passive) with respect to $\mathcal{A}_1$ with storage function V_1 that is positive definite on $\mathcal{X}_1$ with respect to $\mathcal{A}_1$ and $\mathcal{H}_2$ is jump-passive (passive) with respect to $\mathcal{A}_2$ with storage function V_2 that is positive definite on $\mathcal{X}_2$ with respect to $\mathcal{A}_2$ and (Cs1) (respectively, (Cs3)) holds true.*

The results in Theorem 3 are linking the passivity properties of each individual subsystems to 0-input stability of the interconnection. In particular, regarding item 1), note once again how the case of flow-passive hybrid systems does not require any additional conditions. On the other hand, when passivity properties involve jumps, appropriate conditions are required to take into account situations in which only one of the two subsystems jumps.

The following example shows how Theorem 3 can be employed to establish 0-input stability of the interconnection.

Example 2 *Consider two hybrid systems $\mathcal{H}_i$, $i \in \{1,2\}$, given by*

$$\mathcal{H}_i \begin{cases} \dot{x}_i & = & -a_i x_i + b_i u_i & (x_i, u_i) \in C_i \\ x_i^+ & = & u_i & (x_i, u_i) \in D_i \\ y_{c_i} & = & y_{d_i} \ = \ x_i, & \end{cases} \tag{14}$$

where $C_i := \{(x_i, u_i) \ : \ u_i\,(x_i - \varepsilon_i u_i) \leq 0\ \}$, $D_i := \{(x_i, u_i) \ : \ u_i\,(x_i - \varepsilon_i u_i) \geq 0\ \}$, $a_i, b_i, \varepsilon_i > 0$ and $x_i, u_i \in \mathbb{R}$. Let $V_i(x_i) = \frac{1}{2}x_i^2$ and note that, on C_i,

$$\begin{array}{r}\langle \nabla V_i(x_i), f_i(x_i, u_i)\rangle = \\ -a_i x_i^2 + b_i x_i u_i \ = \ b_i y_{c_i} u_i - a_i x_i^2\end{array} \tag{15}$$

and that, on D_i,

$$\begin{array}{r}V_i(g_i(x_i)) - V_i(x_i)\rangle \leq \\ \frac{1}{2\varepsilon_i} u_i x_i - \frac{1}{2}x_i^2 \ = \ \frac{1}{2\varepsilon_i} u_i y_{d_i} - \frac{1}{2}x_i^2\end{array} \tag{16}$$

since, for points in D_i, we have $u_i x_i \geq \varepsilon_i u_i^2$. Then, according to Definition 1, each $\mathcal{H}_i$ is strictly passive with respect to $\mathcal{A}_i = \{0\}$ by defining $v_{ci} = b_i u_i$, $v_{di} = \frac{1}{2\varepsilon_i} u_i$, $\rho_{ci}(s) = a_i s^2$, $\rho_{di}(s) = \frac{1}{2} s^2$ for all $s \geq 0$. In particular, each system is flow-passive when $v_{di} = 0$, which, in turn, using item 1) of Theorem 3 implies that $\mathcal{A} = \{0\} \times \{0\}$ is 0-input stable for $\mathcal{H}_{12}^0$. □

A. 0-*Input Asymptotic Stability*

The following theorem states that asymptotic stability of the interconnection can be established by considering either the strict passivity property or the flow-strict passivity property of the two individual subsystems.

Theorem 4 *Given a compact set $\mathcal{A}_{12} = \mathcal{A}_1 \times \mathcal{A}_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ if either of the following conditions hold:*

1) *There exist $\rho'_{d1} : \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$ positive definite with respect to $\mathcal{A}_1$, $\rho'_{d2} : \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ positive definite with respect to $\mathcal{A}_2$ such that, for each $i \in \{1, 2\}$, $\mathcal{H}_i$ is strictly passive with respect to $\mathcal{A}_i$ with storage function V_i that is positive definite on $\mathcal{X}_i$ with respect to $\mathcal{A}_i$, and*

$$-h_{d1}(x_1)^\top h_{d2}(x_2) \leq -\rho'_{d2}(x_2) \quad \forall x_1, x_2 : (x_1, -h_{d2}(x_2)) \in D_1 \text{ and } (x_2, h_{d1}(x_1)) \notin D_2 , \tag{17}$$

$$h_{d1}(x_1)^\top h_{d2}(x_2)^\top \leq -\rho'_{d1}(x_1) \quad \forall x_1, x_2 : (x_1, -h_{d2}(x_2)) \notin D_1 \text{ and } (x_2, h_{d1}(x_1)) \in D_2 ; \tag{18}$$

2) *For each $i \in \{1, 2\}$, $\mathcal{H}_i$ is flow-strictly passive with respect to $\mathcal{A}_i$ with storage function V_i that is positive definite on $\mathcal{X}_i$ with respect to $\mathcal{A}_i$, and*
   - 2.a) *for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$, $N_r > 0$ such that for every solution ϕ to $\mathcal{H}_{12}^0$, $|\phi(0,0)|_\mathcal{A} \in (0, r]$, $(t, j) \in \operatorname{dom}\phi$, $t + j \geq T$ imply $t \geq \gamma_r(T) - N_r$;*

then $\mathcal{A}_{12}$ is 0-input pre-asymptotically stable for $\mathcal{H}_{12}^0$. Furthermore, if, for each $i \in \{1, 2\}$, V_i is radially unbounded, then the 0-input pre-asymptotic stability property of $\mathcal{A}_{12}$ holds globally.

Example 3 *For each $i \in \{1, 2\}$, consider the hybrid system*

$$\mathcal{H}_i \quad \left\{ \begin{array}{ll} \dot{x}_i = F_i(x_i, v_{ci}) = A x_i + B v_{ci} & x_i \in C_i \\ x_i^+ = G_i(x_i) = R x_i & x_i \in D_i \end{array} \right. \tag{19}$$

with state $x_i = [p_i, v_i]^\top \in \mathbb{R}^2$, input $v_{ci} \in \mathbb{R}$,

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & -e_R \end{bmatrix},$$

sets

$$C_i = \begin{array}{l} \{x_i \in \mathbb{R}^2 \ : \ p_i \leq 0 \ \} \cup \\ \{x_i \in \mathbb{R}^2 \ : \ p_i \geq 0, \ v_i \leq \bar{v} \ \} , \end{array}$$

$$D_i = \begin{array}{l} \{x_i \in \mathbb{R}^2 \ : \ p_i \geq 0, v_i \geq \bar{v} \ \} \cap \\ \{x_i \in \mathbb{R}^2 \ : \ v_i(\beta v_i - \alpha p_i) \leq 0 \ \} \end{array}$$

with $\bar{v}, a_1, a_2 \in \mathbb{R}_{>0}$, $e_R \in [0, 1]$, $\alpha = e_R/a_2$, $\beta = (e_R^2 - 1)(1 + a_1)/(2 a_1 a_2)$. For the above system the following passivity result holds true.

Proposition 2 *Let P be the solution to the Lyapunov equation $A^\top P + PA = -I_2$. Then, for each $i \in \{1, 2, \}$, system (19) is flow-strictly passive with respect to the compact set $\mathcal{A}_i = \{(0, 0)\}$ with storage function $V_i(x_i) := \frac{1}{2} x_i^\top P x_i$, input v_{ci}, output $y_{ci} = h_c(x_i) := 2B^\top P x_i$, and function $\rho_c(x_i) := x_i^\top x_i$.*

Now, consider the negative feedback interconnection $\mathcal{H}_{12}$ obtained by considering the assignments in (10). Using Theorem 4, we have the following result.

Proposition 3 *The set $\mathcal{A}_{12} = \mathcal{A}_1 \times \mathcal{A}_2$ is 0-input pre-asymptotically stable for $\mathcal{H}_{12}$.*

□

VI. Conclusion

By considering hybrid specific notions of passivity, where dissipation is allowed to happen only along flows or jumps, we presented sufficient conditions to establish passivity, stability, and asymptotic stability of the (negative) feedback interconnection of hybrid systems using, as a storage function, the sum of storage functions of the individual subsystems. The differences and analogies with respect to continuous-time systems have been pointed out. An application, considering a mechanical system, and different examples have been employed to illustrate the results.

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