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Asymptotic stability of a class of nonlinear stochastic systems undergoing Markovian jumps

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ABSTRACT

Systems which specifications change abruptly and statistically, referred to as Markovian-jump systems, are considered in this paper. An approximate method is presented to assess the asymptotic stability, with probability one, of nonlinear, multi-degree-of-freedom, Markovian-jump quasi-nonintegrable Hamiltonian systems subjected to stochastic excitations. Using stochastic averaging and linearization, an approximate formula for the largest Lyapunov exponent of the Hamiltonian equations is derived, from which necessary and sufficient conditions for asymptotic stability are obtained for different jump rules. In a Markovian-jump system with unstable operating forms, the stability conditions prescribe limitations on time spent in each unstable form so as to render the entire system asymptotically stable. The validity and utility of this approximate technique are demonstrated by a nonlinear two-degree-of-freedom oscillator that is stochastically driven and capable of Markovian jumps.

1. Introduction

The operation of complex dynamic systems is often accompanied by abrupt changes in their configurations caused by component or interconnection failure, or by the onset of environmental disturbance. When these sudden changes in the operating rules occur in accordance with a Markov process, the associated stochastic system is referred to as a Markovian-jump system [1,2]. Indeed, important examples include most high-integrity or safety-critical systems such as nuclear power plants, integrated communication networks, and large-scale flexible space structures. Since Markovian-jump systems were first introduced a few decades ago, issues concerning stability, optimal control, filtering, and robustness have been examined in the literature. However, most published results are only applicable to linear systems. Far less is known about nonlinear Markovian-jump systems, particularly for multi-degree-of-freedom (MDOF) systems. Development of methodology for the analysis of nonlinear MDOF Markovian-jump systems is thus much deserving.

A basic issue in system design is stability. An unstable system may burn out, disintegrate, or saturate and becomes unusable. Several important criteria for stochastic stability of Markovian-jump systems [3–5] have been established. Using Lyapunov exponents, necessary and sufficient conditions for moment stability of linear systems have been derived [6]. The equivalence of different second-moment stability concepts has been established [7]. Feng et al. [8] proposed conditions for exponential mean-square stability of linear Markovian-jump systems. Costa and Fragoso [9] obtained conditions for mean-square stability of linear systems subjected to additive noise. Stability analysis using Lyapunov functions was considered [10,11]. In addition, sufficient conditions for almost sure stability of linear Markovian-jump systems were derived [12]. Some results for stability of neural Markovian-jump networks were reported in [13,14]. However, these earlier studies focused mainly on the moment stability of linear systems. The purpose of this paper is to investigate the stochastic stability of nonlinear MDOF Markovian-jump systems.

An approximate method is presented in this paper to assess the asymptotic stability of nonlinear, MDOF, Markovian-jump quasi-nonintegrable Hamiltonian systems subjected to stochastic excitations. In a nonintegrable system, energy is the only constant of motion [15]. The organization of this paper is as follows. In Section 2, the equations of Markovian-jump quasi-nonintegrable stochastically-driven Hamiltonian systems are examined. The utility of Lyapunov exponents in determining asymptotic stability with probability one is reviewed in Section 3. Stochastic averaging [16,17] is applied to Markovian-jump systems in Section 4, which permits the reduction of the Hamiltonian equations to a one-dimensional Itô equation governing the approximate energy envelope. In Section 5, the Itô equation of energy is linearized and its Lyapunov exponent is decomposed into a weighted sum of the Lyapunov exponents associated with different forms of the system.

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Conditions for asymptotic stability of nonlinear MDOF Markovian-jump systems are obtained in Section 6, where computation of equilibrium probabilities is also discussed. In Section 7, utility and reliability of the method thus developed are demonstrated by a nonlinear two-degree-of-freedom oscillator driven by Gaussian white noise, wherein comparison with direct system simulation is made and detailed calculations are provided. A summary of findings is given in Section 8. Throughout the paper, an effort is made to clarify the theoretical development in practical terms.

2. Problem formulation

The equations of motion of an n-degree-of-freedom dynamical system are composed of n second-order differential equations in the generalized displacements. These second-order equations can always be recast as $2n$ first-order differential equations, in the usual state space or in the Hamiltonian phase space. Consider an n-degree-of-freedom, stochastically driven, nonlinear Hamiltonian system with Markovian jumps governed by

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{1}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} - \varepsilon c_{ij}^{[s(t)]}(\mathbf{q},\mathbf{p})\frac{\partial H}{\partial p_j} + \varepsilon^{1/2} f_{ik}^{[s(t)]}(\mathbf{q}) w_k(t) \tag{2}$$

where $i, j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$; q_i, p_i are respectively the generalized displacements and momenta; $\mathbf{q} = (q_1, \ldots, q_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$. In accordance with the summation convention, the repeated indices j and k in Eq. (2) are summed over their respective ranges. In the above equations, $H = H(\mathbf{q}, \mathbf{p})$ is the Hamiltonian, $s(t)$ is a finite state Markov jump process, ε is a small parameter, $\varepsilon c_{ij}^{[s(t)]}(\mathbf{q}, \mathbf{p})\frac{\partial H}{\partial p_j}$ denote the jump nonlinear damping, and $\varepsilon^{1/2} f_{ik}^{[s(t)]}(\mathbf{q})$ denote the jump amplitudes of excitations. The stochastic excitations $w_k(t)$ are independent zero-mean Gaussian white noise processes with correlation functions $E[w_k(t)w_l(t+\tau)] = 2D_{kl}\delta(\tau)$.

The Markov process $s(t)$ points to the model or form in which the system operates [18], and it takes discrete values from a finite set $S = \{1, 2, \ldots, l\}$ with transition probability

$$P\left\{j, t+\Delta t | i, t\right\} = P\left\{s(t+\Delta t) = j | s(t) = i\right\} = \begin{cases} \lambda_{ij}\Delta t + o(\Delta t), & i \neq j \\ 1 + \lambda_{ij}\Delta t + o(\Delta t), & i = j \end{cases} \tag{3}$$

where $o(\Delta t)$ represents an infinitesimal expression of order higher than Δt. The conditional probability $P\{s(t+\Delta t) = j | s(t) = i\}$ denotes the probability that the system takes the form j at time $t + \Delta t$ given that it has the form i at time t. $\lambda_{ij} \geq 0$ are the transition densities from model i to model j if $i \neq j$ while

$$\lambda_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{l} \lambda_{ij} \tag{4}$$

Note that $c_{ij}^{(l)}(\mathbf{q}, \mathbf{p})$ and $f_{ik}^{(l)}(\mathbf{q})$ are changed to $c_{ij}^{(u)}(\mathbf{q}, \mathbf{p})$ and $f_{ik}^{(u)}(\mathbf{q})$ as $s(t)$ jumps from l to $u(l, u \in S)$. For each s $(s \in S)$, $c_{ij}^{(s)}(\mathbf{q}, \mathbf{p})$ and $f_{ik}^{(s)}(\mathbf{q})$ have the same function form however different values of parameters.

It is assumed that the Markov process $s(t)$ is irreducible and ergodic [19]. That means any form represented by $S = \{1, 2, \ldots, l\}$ is accessible and, as time progresses, the system will revisit a previous form with probability one. From a practical viewpoint, this assumption can be made without loss of generality because failure of components in a system is normally followed by repair and restoration at a later time. Under this assumption, the process $s(t)$ has an equilibrium distribution as $t \to \infty$ irrespective of the initial distribution.

A system governed by Eqs. (1) and (2) is referred to as a Markovian-jump quasi-Hamiltonian system. To be sure, an exact solution of these Hamiltonian equations is preferred. Owing to a lack of general methods to solve nonlinear equations, the primary concern herein is an assessment of asymptotic stability, with probability one, of the system. A secondary purpose is to look into restrictions for stabilizing a system that possesses some unstable operating forms.

3. Mathematical preliminaries

In the investigation of stochastic stability, the concept of "stability with probability one" or "almost sure stability" is frequently used. There is a connection between Lyapunov exponents and asymptotic stability with probability one. Let $\mathbf{x} = (\mathbf{q}, \mathbf{p})$ denote the $2n$-dimensional vector of canonical variables. The trivial solution $\mathbf{x} = \mathbf{0}$ of the Markovian-jump system is stable with probability one if, for any $\mathbf{x}_0$ and initial distribution of $s(t)$

$$\lim_{\|\mathbf{x}_0\| \to 0} P\left\{\sup_{t \geq 0} \|\mathbf{x}(t)\| < \varepsilon\right\} = 1 \tag{5}$$

The trivial solution $\mathbf{x} = \mathbf{0}$ is asymptotically stable with probability one if Eq. (5) is satisfied and, in addition,

$$\lim_{\|\mathbf{x}_0\| \to 0} P\left\{\lim_{t \to \infty} \|\mathbf{x}(t)\| = 0\right\} = 1 \tag{6}$$

In the absence of Markovian jumps, a theorem by Oseledec [20] provides a means for determining asymptotic stability with probability one. Specifically, let the Markov process be arbitrarily fixed at $s(t) = u$ where $1 \leq u \leq l$. When the system is linearized about the trivial solution $\mathbf{x}^{(u)} = \mathbf{0}$, the Lyapunov exponents are generated by

$$\Lambda = \lim_{t \to \infty} \frac{1}{t} \ln \|\mathbf{x}^{(u)}(t)\| \tag{7}$$

where $\mathbf{x}^{(u)}(t)$ is solution of the linearized system operating only in the form u. Lyapunov exponents are generalizations of eigenvalues. A necessary and sufficient condition [20] for asymptotic stability, with probability one, of the trivial solution $\mathbf{x} = \mathbf{0}$ is that the largest Lyapunov exponent be negative.

Both Eq. (7) and the stability condition can be applied to Markovian-jump systems. However, the important result by Oseledec [20] cannot be used easily. The barrier lies indetermination of the largest Lyapunov exponent, which is not straightforward even for a system without Markovian jumps. In this paper, the $2n$ quasi-nonintegrable Hamiltonian equations are first reduced by stochastic averaging to a one-dimensional Itô equation governing the energy envelope [21,22]. The Lyapunov exponent associated with the linearized Itô equation of energy is then used as an approximation of the largest Lyapunov exponent of the original system.

4. Energy envelope by stochastic averaging

The Hamiltonian system governed by Eqs. (1) and (2) is assumed nonintegrable. There is only one independent integral of the motion, i.e., the Hamiltonian H which, in general, is equal to the energy envelope or total energy of the system. Theoretically speaking, Eqs. (1) and (2) are equivalent to the following Itô stochastic differential equations:

$$dq_i = \frac{\partial H}{\partial p_i}dt \tag{8}$$

$$dp_i = -\left[\frac{\partial H}{\partial q_i} + \varepsilon c_{ij}^{[s(t)]}(\mathbf{q},\mathbf{p})\frac{\partial H}{\partial p_j}\right]dt + \varepsilon^{1/2}\sigma_{ik}^{[s(t)]}(\mathbf{q})dB_k(t) \tag{9}$$

where $B_k(t)$ are standard Werner processes such that $\sigma_{ik}^{[s(t)]}dB_k(t) = f_{ik}^{[s(t)]}w_k(t)dt$. Let the Markov jump process be arbitrarily fixed at $s(t) = u$ where $1 \le u \le l$. Then, a stochastic differential for H can be derived from Eqs. (8) and (9) using the Itô differential rule [23] so that

$$dH = \varepsilon\left[-c_{ij}^{(u)}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j} + \frac{1}{2}\sigma_{ik}^{(u)}\sigma_{jk}^{(u)}\frac{\partial^2 H}{\partial p_i \partial p_j}\right]dt + \varepsilon^{1/2}\frac{\partial H}{\partial p_i}\sigma_{ik}^{(u)}dB_k(t) \tag{10}$$

Owing to the small parameter ε, the above relation indicates that H is a slowly varying process while the generalized displacements $q_1,\ldots, q_n$ and generalized momenta $p_2,\ldots, p_n$ are usually rapidly varying processes with respect to time. By a theorem of Khasminskii [24], H converges to a one-dimensional diffusion process as $\varepsilon \to 0$. The Itô equation for this diffusion process is obtained by time averaging of Eq. (10). The effect of stochastic averaging is to average out the rapidly varying processes so as to yield an equation for the slowly varying process H, which is essential for describing the long-term behavior of the system.

Time averaging of Eq. (10) can be conducted by traditional methods [16,17] because the system only takes the form u. Upon stochastic averaging, the Itô equation for the slowly varying process H is obtained as

$$dH = m^{(u)}(H)dt + \bar{\sigma}^{(u)}(H)dB(t) \tag{11}$$

In the above expression, $B(t)$ is a standard Werner process and the drift coefficient $m^{(u)}(H)$ and diffusion coefficient $\bar{\sigma}^{(u)}(H)$ are given by

$$m^{(u)}(H) = \frac{1}{T(E)}\int_{\Omega}\left[-c_{ij}^{(u)}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j} + \frac{1}{2}\sigma_{ik}^{(u)}\sigma_{jk}^{(u)}\frac{\partial^2 H}{\partial p_i \partial p_j}\right]\left(\frac{\partial H}{\partial p_1}\right)^{-1}d\mathbf{z} \tag{12}$$

$$\left\{\bar{\sigma}^{(u)}(H)\right\}^2 = \frac{1}{T(E)}\int_{\Omega}\left[\sigma_{ik}^{(u)}\sigma_{jk}^{(u)}\frac{\partial^2 H}{\partial p_i \partial p_j}\right]\left(\frac{\partial H}{\partial p_1}\right)^{-1}d\mathbf{z} \tag{13}$$

where $\mathbf{z} = (q_1, \ldots, q_n, p_2, \ldots, p_n)$ is of order $2n - 1$; the region of integration is $\Omega = \{\mathbf{z}: H(q_1, \ldots, q_n, 0, p_2, \ldots, p_n) \le H\}$; and the parameter

$$T(H) = \int_{\Omega}\left(\frac{\partial H}{\partial p_1}\right)^{-1}d\mathbf{z} \tag{14}$$

Note that Eq. (11) is only valid when $s(t) = u$. As Markovian jumps are allowed, $s(t)$ takes values from $S = \{1, 2, \ldots, l\}$. Then, Eq. (11) can be extended so that

$$dH = m^{[s(t)]}(H)dt + \bar{\sigma}^{[s(t)]}(H)dB(t) \tag{15}$$

where the drift and diffusion coefficients $m^{[s(t)]}(H)$, $\bar{\sigma}^{[s(t)]}(H)$ change as $s(t)$ jumps. In this interpretation, the Markovian-jump system governed by Eqs. (8) and (9) possesses an energy envelope given approximately by the solution H of Eq. (15). In the next section, the Lyapunov exponent of the linearized equation of energy is evaluated.

5. Linearization and Lyapunov exponents

The Itô Eq. (15) can be linearized about $H = 0$, which yields

$$dH = m_H^{[s(t)]}(0)Hdt + \bar{\sigma}_H^{[s(t)]}(0)HdB(t) \tag{16}$$

where $m_H^{[s(t)]}(0)$, $\bar{\sigma}_H^{[s(t)]}(0)$ depend only on $s(t)$ and are given by

$$m_H^{[s(t)]}(0) = \left.\frac{\partial m^{[s(t)]}(H)}{\partial H}\right|_{H=0}, \quad \bar{\sigma}_H^{[s(t)]}(0) = \left.\frac{\partial \bar{\sigma}^{[s(t)]}(H)}{\partial H}\right|_{H=0} \tag{17}$$

Observe that the solution of Eq. (16) is

$$H(t) = H(0)\exp\left\{\int_0^t\left[m_H^{[s(\tau)]}(0) - \frac{1}{2}(\bar{\sigma}_H^{[s(\tau)]}(0))^2\right]d\tau + \int_0^t \bar{\sigma}_H^{[s(\tau)]}(0)dB(\tau)\right\} \tag{18}$$

Since Eq. (16) is one-dimensional, it has only one Lyapunov exponent, which will be constructed in two steps.

In the first step, let the Markov jump process be arbitrarily fixed at $s(t) = u$ where $1 \le u \le l$. Then, Eq. (18) simplifies to

$$H(t) = H(0)\exp\left\{\left[m_H^{(u)} - \frac{1}{2}(\bar{\sigma}_H^{(u)})^2\right]t + \bar{\sigma}_H^{(u)}B(t)\right\} \tag{19}$$

The definition of Lyapunov exponent in Eq. (7) involves the norm $\|\mathbf{x}\|$, which intuitively is the distance of the system from the origin $\mathbf{x} = \mathbf{0}$ in the Hamiltonian phase space. In general, the Hamiltonian H is equal to the total energy, and therefore $H^{1/2}$ is also a measure of the distance of the system from the trivial solution $\mathbf{x} = \mathbf{0}$. Thus Eq. (7) may be modified as [21,22]

$$\Lambda = \lim_{t\to\infty}\frac{1}{t}\ln H^{1/2} \tag{20}$$

while $H^{1/2} = H^{1/2}(\mathbf{q}, \mathbf{p})$ need not be a homogeneous function of degree one as required for a norm, the above equation provides a streamlined and fairly accurate method for generating Lyapunov exponents.

Define $\Lambda^{(u)}$ as the Lyapunov exponent of Eq. (16) when $s(t) = u$. Substitute Eq. (19) into Eq. (20) to obtain

$$\Lambda^{(u)} = \lim_{t\to\infty}\frac{1}{2t}\left\{\ln H(0) + \left[m_H^{(u)} - \frac{1}{2}(\bar{\sigma}_H^{(u)})^2\right]t + \bar{\sigma}_H^{(u)}B(t)\right\} \tag{21}$$

As $t \to \infty$, the first and the last terms in the above expression vanish [21,22] and, as a consequence,

$$\Lambda^{(u)} = \frac{1}{2}\left[m_H^{(u)} - \frac{1}{2}(\bar{\sigma}_H^{(u)})^2\right] \tag{22}$$

In the second step of evaluating the Lyapunov exponent of Eq. (16), the Markov process $s(t)$ is permitted to have transitions. In the time window $[0, t]$, let $s(t) = u$ for a total duration of T_u. Since the system must operate in one of the forms represented by $S = \{1, 2, \ldots, l\}$, $t = \sum_{u=1}^{l} T_u$ and Eq. (18) can be expressed as

$$H(t) = H(0)\exp\left\{\sum_{u=1}^{l}\left[m_H^{(u)} - \frac{1}{2}(\bar{\sigma}_H^{(u)})^2\right]T_u + \int_0^t \bar{\sigma}_H^{[s(\tau)]}(0)dB(\tau)\right\} \tag{23}$$

Substitute Eq. (23) into Eq. (20) to yield, upon simplification,

$$\Lambda = \lim_{t\to\infty}\frac{1}{2t}\sum_{u=1}^{l}\left[m_H^{(u)} - \frac{1}{2}(\bar{\sigma}_H^{(u)})^2\right]T_u \tag{24}$$

Combine Eqs. (22) and (24) to obtain

$$\Lambda = \sum_{u=1}^{l}\left(\lim_{t\to\infty}\frac{T_u}{t}\right)\Lambda^{(u)} \tag{25}$$

Thus the Lyapunov exponent Λ of Eq. (16) has been expressed as a weighted sum of the Lyapunov exponents $\Lambda^{(u)}$ associated with different forms of the system. The coefficient of $\Lambda^{(u)}$ is the fraction of time that $s(t) = u$ in the long run. Are these coefficients convergent?

6. Stability and stationary probabilities

As described earlier, the Markov process $s(t)$ can be assumed irreducible and ergodic without practical loss of generality. Under this assumption, the transition probabilities of $s(t)$ reach an equilibrium distribution as $t \to \infty$ independent of the initial distribution [19]. Denote by $P_e(u)$ the stationary or equilibrium probability that $s(t) = u$, where $1 \le u \le l$. Recall from Eq. (3) that $P(u, t + t_0|i, t_0)$ denotes the transition probability that the system takes the form u at time $t + t_0$ given that it has the form i at t_0. Existence of an equilibrium distribution implies that

$$\lim_{t\to\infty} P(u, t + t_0|i, t_0) = P_e(u), \quad \sum_{u=1}^{l} P_e(u) = 1 \tag{26}$$

Since $P_e(u) = \lim_{t\to\infty}(T_u/t)$, Eq. (25) can now be written as

$$\Lambda = \sum_{u=1}^{l} P_e(u)\Lambda^{(u)} \tag{27}$$

As a result, the Lyapunov exponent Λ of Eq. (27) is a convex linear combination [25] of the Lyapunov exponents associated with different forms of the system, weighted by the stationary probabilities of the Markov jump process $s(t)$.

How is Λ related to the Lyapunov exponents of the system governed by Eqs. (1) and (2)? Since Eq. (16) is the linearized equation of energy envelope, Λ is an approximation of the largest Lyapunov exponent of the original nonlinear MDOF system. Likewise, $\Lambda^{(u)}$ is an approximation of the largest Lyapunov exponent of the original system when it operates only in the form $s(t) = u$ [21]. Thus a necessary and sufficient condition for asymptotic stability with probability one [20] of the trivial solution $\mathbf{x} = \mathbf{0}$ of the system governed by Eqs. (1) and (2) is

$$\Lambda = \sum_{u=1}^{l} P_e(u)\Lambda^{(u)} < 0 \tag{28}$$

Numerical simulations suggest that the above stability condition is fairly reliable.

If all operating forms of the system are asymptotically stable with probability one, then $\Lambda^{(u)} < 0$ for all u and, from Eq. (27), $\Lambda < 0$ and the entire system is also asymptotically stable. This is an obvious deduction using Eq. (28). A Markovian-jump system can be asymptotically stable even when some operating forms are unstable. The linear inequality (28) specifies a convex subspace [25] for the allowable stationary probabilities $P_e(u)$ (but not for λ_{ij}) for $1 \le u \le l$. Because $P_e(u) = \lim_{t\to\infty}(T_u/t)$, the allowable stationary probabilities impose limitations on occupancy time in each unstable form so as to render the entire system asymptotically stable. This is just one example demonstrating the utility of Eq. (28). In optimization of Markovian-jump nonlinear systems requiring asymptotic stability, the functional form of Eq. (28) as a convex linear inequality allows it to be conveniently used as a constraint.

The use of Eq. (27) requires evaluation of $\Lambda^{(u)}$ and $P_e(u)$ for $1 \le u \le l$. The Lyapunov exponents $\Lambda^{(u)}$ can be systematically calculated by Eqs. (18) and (22). It remains to describe how the stationary probabilities $P_e(u)$ can be computed. In an irreducible and ergodic Markov process $s(t)$, it can be shown [19, 26] that the stationary probabilities governing equilibrium behavior are given by the algebraic equations

$$\sum_{u=1}^{l} P_e(u)\lambda_{ui} = 0, \quad 1 \le i \le l \tag{29}$$

where λ_{ui} are the transition densities of the process $s(t)$. This completes the discussion on evaluation of Λ in Eq. (28). From a different perspective, Eq. (28) can be used to construct stability boundaries by imposing $\Lambda = 0$. In fact, stability boundaries can be generated numerically as a function of various system parameters rather than just the stationary probabilities. This will be illustrated by a numerical example in the next section.

It should be pointed out that rigorous analysis of the errors committed by stochastic averaging has not been reported in the open literature. As such, analytical quantification of the degree of approximation associated with Eq. (28) cannot be made. While it is believed that Eq. (28) provides a reliable formula for determining asymptotic stability with probability one, its accuracy can only be examined by comparison with direct simulations.

7. Illustrative example

To demonstrate the utility and perhaps reliability of the method presented in this paper, consider a nonlinear two-degree-of-freedom oscillator that is capable of independent Markovian jumps and governed by

$$\begin{aligned}&\ddot{x}_1 + \beta_1^{[s(t)]}\dot{x}_1 + \omega_1^2 x_1 + \alpha_1^{[s(t)]}x_1^2\dot{x}_1 + ax_2 + b(x_1 - x_2)^3 \\ &\quad = f_1^{[s(t)]}x_1 w_1(t)\end{aligned} \tag{30}$$

$$\begin{aligned}&\ddot{x}_2 + \beta_2^{[s(t)]}\dot{x}_2 + \omega_2^2 x_2 + \alpha_2^{[s(t)]}x_2^2\dot{x}_2 + ax_1 + b(x_2 - x_1)^3 \\ &\quad = f_2^{[s(t)]}x_2 w_2(t)\end{aligned} \tag{31}$$

where $\beta_k^{[s(t)]}$, $\alpha_k^{[s(t)]}$, $f_k^{[s(t)]}$ $(k = 1, 2)$ are coefficients with Markov jump; $w_k(t)$ are independent zero-mean Gaussian white noise processes with power spectral density $2D_k$. The Markov process $s(t)$ takes discrete values from a finite set $S = \{1, 2, \dots, l\}$ with transition probability defined in Eq. (3). By inspection, the system kinetic energy is $T_2 = (\dot{x}_1^2 + \dot{x}_2^2)/2$ and the dynamic potential is $U = (\omega_1^2 x_1^2 + \omega_2^2 x_2^2)/2 + ax_1x_2 + b(x_1 - x_2)^4/4$.

Let $q_i = x_i$ and $p_i = \dot{x}_i$. The Hamiltonian is

$$\begin{aligned}H = T_2 + U = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2) + aq_1q_2 \\ + \frac{b}{4}(q_1 - q_2)^4\end{aligned} \tag{32}$$

As a Hamiltonian system, Eqs. (30) and (31) can be expressed as

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{p}_1\\ \dot{p}_2\end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ -\dfrac{\partial H}{\partial q_1} + \beta_1^{[s(t)]}p_1 + \alpha_1^{[s(t)]}q_1^2 p_1 + f_1^{[s(t)]}q_1 w_1(t) \\ -\dfrac{\partial H}{\partial q_2} + \beta_2^{[s(t)]}p_2 + \alpha_2^{[s(t)]}q_2^2 p_2 + f_2^{[s(t)]}q_2 w_2(t)\end{bmatrix} \tag{33}$$

Since $U(q_1, q_2)$ is not separable, the Hamiltonian system is nonintegrable. Upon stochastic averaging, Eq. (15) for the energy envelope of system (33) is obtained, for which $\Omega = \{(q_1, q_2, p_2): p_2^2/2 + U(q_1, q_2) \le H\}$, and

$$\begin{aligned}m^{(u)}(H) &= m^{[s(t)=u]}(H) \\ &= \frac{1}{T(H)}\int_{\Omega}\Big[-\left\{\beta_1^{(u)} + \alpha_1^{(u)}q_1^2\right\}p_1^2 - \left\{\beta_2^{(u)} + \alpha_2^{(u)}q_2^2\right\}p_2^2 \\ &\quad + \left\{f_1^{(u)}\right\}^2 D_1 q_1^2 + \left\{f_2^{(u)}\right\}^2 D_2 q_2^2\Big]\frac{1}{p_1}dq_1 dq_2 dp_2\end{aligned} \tag{34}$$

$$\left[\bar{\sigma}^{(u)}(H)\right]^2 = \left[\bar{\sigma}^{[s(t)=u]}(H)\right]^2 = \frac{1}{T(H)}\int_{\Omega}\left[2\left\{f_1^{(u)}\right\}^2 D_1 q_1^2 p_1^2 + 2\left\{f_2^{(u)}\right\}^2 D_2 q_2^2 q_2^2\right]\frac{1}{p_1}dq_1 dq_2 dp_2 \tag{35}$$

$$T(H) = \int_{\Omega}\frac{1}{p_1}dq_1 dq_2 dp_2 \tag{36}$$

Evaluate the above integrals by using the change of variables:

$$q_1 = \frac{r}{\omega_1}\cos\theta, \quad q_2 = \frac{r}{\omega_2}\sin\theta \tag{37}$$

It can be checked that

$$\begin{aligned} m^{(u)}(H) = \frac{2\pi}{\omega_1\omega_2 T(H)}\int_0^{\pi}\Bigg[& -\left\{\beta_1^{(u)} + \beta_2^{(u)}\right\}F(H,\theta) \\ & -\left(\frac{\alpha_1^{(u)}}{\omega_1^2}\cos^2\theta + \frac{\alpha_2^{(u)}}{\omega_2^2}\sin^2\theta\right)G(H,\theta) \\ & + \frac{r^4}{2}\left(\frac{\left\{f_1^{(u)}\right\}^2 D_1}{\omega_1^2}\cos^2\theta + \frac{\left\{f_2^{(u)}\right\}^2 D_2}{\omega_2^2}\sin^2\theta\right)\Bigg]d\theta \end{aligned} \tag{38}$$

$$\left[\bar{\sigma}^{(u)}(H)\right]^2 = \frac{4\pi}{\omega_1\omega_2 T(H)}\int_0^{\pi}\left[\frac{\left\{f_1^{(u)}\right\}^2 D_1}{\omega_1^2}\cos^2\theta + \frac{\left\{f_2^{(u)}\right\}^2 D_2}{\omega_2^2}\sin^2\theta\right] \times G(H,\theta)\,d\theta \tag{39}$$

$$T(H) = \frac{2\pi}{\omega_1\omega_2}\int_0^{\pi} r^2 d\theta \tag{40}$$

for which

$$F(H,\theta) = Hr^2 - \frac{r^4}{2}\left(1 + \frac{a}{\omega_1\omega_2}\sin 2\theta\right) - \frac{br^6}{4}\left(\frac{\cos\theta}{\omega_1} - \frac{\cos\theta}{\omega_2}\right)^4 \tag{41}$$

$$G(H,\theta) = Hr^4 - \frac{r^6}{2}\left(1 + \frac{a}{\omega_1\omega_2}\sin 2\theta\right) - \frac{br^8}{4}\left(\frac{\cos\theta}{\omega_1} - \frac{\cos\theta}{\omega_2}\right)^4 \tag{42}$$

and r is the solution of the equation

$$H - \frac{r^2}{2}\left(1 + \frac{a}{\omega_1\omega_2}\sin 2\theta\right) - \frac{br^4}{4}\left(\frac{\cos\theta}{\omega_1} - \frac{\cos\theta}{\omega_2}\right)^4 = 0 \tag{43}$$

It follows from Eq. (22) that

$$\Lambda^{(u)} = -\frac{1}{4}\left\{\beta_1^{(u)} + \beta_2^{(u)}\right\} + \frac{1}{6}\left(\frac{\left\{f_1^{(u)}\right\}^2 D_1}{\omega_1^2} + \frac{\left\{f_2^{(u)}\right\}^2 D_2}{\omega_2^2}\right)\eta \tag{44}$$

where η is independent of the form u and is given by

$$\eta = \int_0^{\pi}\frac{1}{\left[1 + (a/\omega_1\omega_2)\sin 2\theta\right]^2}d\theta\left\{\int_0^{\pi}\frac{1}{1 + (a/\omega_1\omega_2)\sin 2\theta}d\theta\right\}^{-1} \tag{45}$$

Given the transition densities λ_{ij} of the inherent Markov process $s(t)$, stationary probabilities $P_e(u)$ can be computed from Eq. (29). An approximation of the largest Lyapunov exponent Λ of system (33) is then obtained by Eq. (27). Numerical results are given explicitly for a 2-form jump process and a 3-form jump process.

8. Two-form system

In this case, $l = 2$ and $S = \{1, 2\}$. In Eqs. (30) and (31), assume that $\omega_1 = 1$, $\omega_2 = 1.5$, $D_1 = 0.01$, $D_2 = 0.01$, and

$$\beta_1^{(1)} = -0.01, \quad \beta_2^{(1)} = 0.001, \quad \beta_1^{(2)} = 0.02, \quad \beta_2^{(2)} = 0.002 \tag{46}$$

$$f_1^{(1)} = 1, \quad f_2^{(1)} = 1, \quad f_1^{(2)} = 0.5, \quad f_2^{(2)} = 0.5 \tag{47}$$

The above specifications indicate an increase in damping and a decrease in the amplitudes of random excitations when the system takes the form $s(t) = 2$. From Eq. (44),

$$\Lambda^{(1)} = 0.0047 > 0, \quad \Lambda^{(2)} = -0.0049 < 0 \tag{48}$$

Thus the form $s(t) = 2$ is asymptotically stable with probability one while the form $s(t) = 1$ is unstable.

Prescribe the transition densities of the Markov jump process $s(t)$ by a transition matrix $\mathbf{A} = [\lambda_{ij}]$ which, because of Eq. (4), must have the form

$$\mathbf{A} = \begin{bmatrix}\lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22}\end{bmatrix} = \begin{bmatrix}-\nu_1 & \nu_1 \\ \nu_2 & -\nu_2\end{bmatrix} \tag{49}$$

There are only two parameters $\nu_1 > 0$ and $\nu_2 > 0$ to be specified, and the stationary probabilities $P_e(1)$, $P_e(2)$ can be readily computed from Eq. (29). Thereafter the largest Lyapunov exponent Λ of the system is obtained numerically by Eq. (27). It is found that Λ is a monotonic decreasing function of ν_1 and a monotonic increasing function of ν_2. In Fig. 1, the variation of Λ with ν_1 is shown for four arbitrarily fixed values of ν_2. In every case, $\Lambda \to \Lambda_2$ as $\nu_1 \to \infty$ while $\partial\Lambda/\partial\nu_1 < 0$. This is not surprising: the transition density $\lambda_{12} = \nu_1$ out of the unstable form $s(t) = 1$ and into the stable form $s(t) = 2$ increases as ν_1 is increased. Systems represented by points A, B, C and D will be used in Fig. 3 for direct simulations. In Fig. 2, the variation of Λ with ν_2 is displayed for four arbitrarily fixed values of ν_1. It is observed that $\Lambda \to \Lambda_1$ as $\nu_2 \to \infty$ while $\partial\Lambda/\partial\nu_2 > 0$, and the observation can be explained by analogous reasoning.

In Fig. 1, it is noted that points A, B, C and D are associated with (ν_1, ν_2) given respectively by (2.5, 1), (3, 2), (1.5, 3), and (2, 5). Using Eq. (49), specification of (ν_1, ν_2) defines a transition matrix $\mathbf{A}$ for the 2-form system. Based upon the analysis of this paper, $\Lambda < 0$ for systems A, B and they are predicted to be asymptotically stable. Similarly, $\Lambda > 0$ for systems C, D and they are predicted unstable. Direct simulations are conducted on the four systems represented by A, B, C and D. Indeed, the displacement-time graphs for q_1 shown in Fig. 3 confirm the analytical predictions.

In the previous section, it has been mentioned that stability boundaries can be constructed by imposing $\Lambda = 0$. Instead of graphing the stability boundaries as a function of the transition probabilities, it is decided to plot the stability boundaries in the $(\beta_1^{(1)}, \beta_2^{(1)})$ plane for different jump rules. Recall that $s(t) = 1$ is unstable and $s(t) = 2$ is stable in the 2-form system. As the damping coefficients $\beta_1^{(1)}, \beta_2^{(1)}$ are increased, it is more likely for the entire Markovian-jump system to become asymptotically stable. In Fig. 4, stability boundaries are shown for three different transition matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, specified in accordance with Eq. (50) by $(\nu_1, \nu_2) = (1, 1)$, $(\nu_1, \nu_2) = (5, 1)$, $(\nu_1, \nu_2) = (1, 5)$ respectively, and for the form $s(t) = 1$. It is seen that the stability boundaries and stability regions are intimately connected with the jump rules.

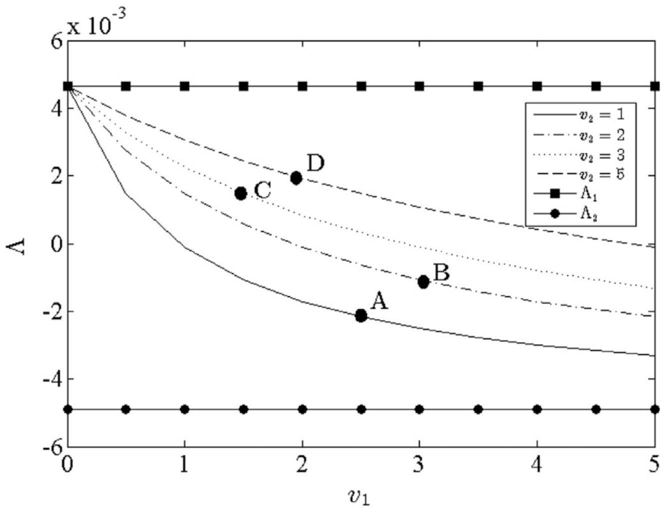

Fig. 1. The largest Lyapunov exponent Λ of 2-form system (33) versus v_1 for $v_2 = 1$, $v_2 = 2$, $v_2 = 3$, and $v_2 = 5$. Systems represented by points A, B, C and D are used in Fig. 3 for direct simulations.

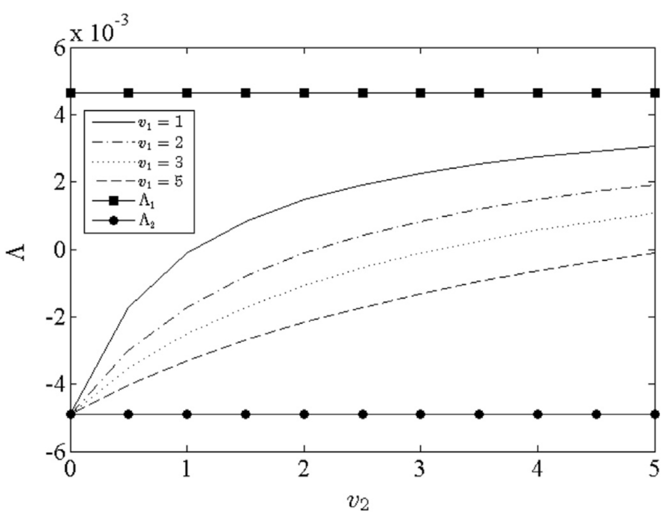

Fig. 2. The largest Lyapunov exponent Λ of 2-form system (33) versus v_2 for $v_1 = 1$, $v_1 = 2$, $v_1 = 3$, and $v_1 = 5$.

9. Three-form system

In this case, $l = 3$ and $S = \{1,\ 2,\ 3\}$, the Markov process may be regarded as an extension of the 2-form jump process considered earlier, and all previous parameters except the damping coefficients remain unchanged. The new damping coefficients and additional parameters are:

$$\beta_1^{(1)} = -0.02,\quad \beta_2^{(1)} = 0.001,\quad \beta_1^{(2)} = -0.005, \beta_2^{(2)} = 0.002 \tag{50}$$

$$\beta_1^{(3)} = 0.02,\quad \beta_2^{(3)} = 0.003,\quad f_1^{(3)} = 0.5,\quad f_1^{(3)} = 0.5 \tag{51}$$

From Eq. (44),

$$\Lambda^{(1)} = 0.0072 > 0,\quad \Lambda^{(2)} = 0.0014 > 0,\quad \Lambda^{(3)} = -0.0051 < 0 \tag{52}$$

Thus the form $s(t) = 3$ is asymptotically stable with probability one while the forms $s(t) = 1$ and $s(t) = 2$ are unstable.

Prescribe the transition densities of the Markov jump process $s(t)$ by the transition matrix

$$\mathbf{A} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} = \begin{bmatrix} -2v_1 & v_1 & v_1 \\ v_2 & -2v_2 & v_2 \\ v_3 & v_3 & -2v_3 \end{bmatrix} \tag{53}$$

where, for simplicity, it is assumed that $\lambda_{ij} = \lambda_{ik} = v_i$ so that the transition densities into different forms are equal. There are only three parameters v_1, v_2, v_3 to be specified, and computation of the largest Lyapunov exponent Λ of the system is streamlined. In Fig. 5, the variation of Λ with v_1 is shown for four arbitrarily fixed values of $v_2 = v_3$. It is found that Λ is a monotonic decreasing function of v_1. However, Λ converges to a limit between Λ_2 and Λ_3 as $v_1 \to \infty$ while $\partial\Lambda/\partial v_1 < 0$. This is not surprising: the Markov jump process reaches an equilibrium state going between the forms $s(t) = 2$ and $s(t) = 3$ as v_1 is increased because $\lambda_{12} = \lambda_{13} = v_1$. Systems represented by points A, B, C and D will be used in Fig. 7 for direct simulations. In Fig. 6 the variation of Λ with v_3 is displayed for four arbitrarily fixed values of $v_1 = v_2$. It is observed that Λ converges to a limit between Λ_1 and Λ_2 as $v_3 \to \infty$ while $\partial\Lambda/\partial v_3 > 0$, and the observation can be explained in a similar way.

In Fig. 5, it is noted that points A, B, C and D are associated with (v_1, v_2, v_3) given respectively by $(4,\ 1,\ 1)$, $(1,\ 2,\ 2)$, $(1,\ 3,\ 3)$, and $(1.5,\ 4,\ 4)$. Using Eq. (53), specification of (v_1, v_2, v_3) defines a transition matrix $\mathbf{A}$ for the 3-form system. Based upon the analysis of this paper, $\Lambda < 0$ for system A and it is predicted to be asymptotically stable. Similarly, $\Lambda > 0$ for systems B, C, D and they are predicted unstable. Direct simulations are conducted on the four systems represented by A, B, C and D. The displacement-time graphs for q_1 shown in Fig. 7 clearly confirm the analytical predictions. In Fig. 8, stability boundaries are shown for three different transition matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, specified in accordance with Eq. (53) by $(v_1, v_2, v_3) = (1,\ 1,\ 1)$, $(v_1, v_2, v_3) = (3,\ 1,\ 1)$, $(v_1, v_2, v_3) = (1,\ 1,\ 3)$ respectively, and for the form $s(t) = 1$. It is again observed that the stability boundaries and stability regions are rather sensitive to changes in the jump rule.

10. Conclusions

Markovian-jump systems are practically significant since they include many industrial plants and communication networks. In this paper, an approximate method has been presented to assess the asymptotic stability with probability one of nonlinear MDOF stochastic systems which equations of motion, when cast in first-order form, are Markovian-jump Hamiltonian equations. Using stochastic averaging, the Hamiltonian equations of quasi-non-integrable system are first reduced to a one-dimensional Itô equation governing the energy envelope, from which the largest Lyapunov exponent of the original system is estimated. Important results reported in the paper are summarized in the following statements.

1. An approximate formula for the largest Lyapunov exponent of the Markovian-jump Hamiltonian equations has been derived as a convex linear combination of the Lyapunov exponents associated with different forms of the system, weighted by the stationary probabilities of the irreducible and ergodic Markov jump process.
2. Approximate necessary and sufficient conditions for asymptotic stability with probability one have been obtained as a linear inequality. In a Markovian-jump system with unstable operating forms, the stability conditions prescribe limitations on occupancy

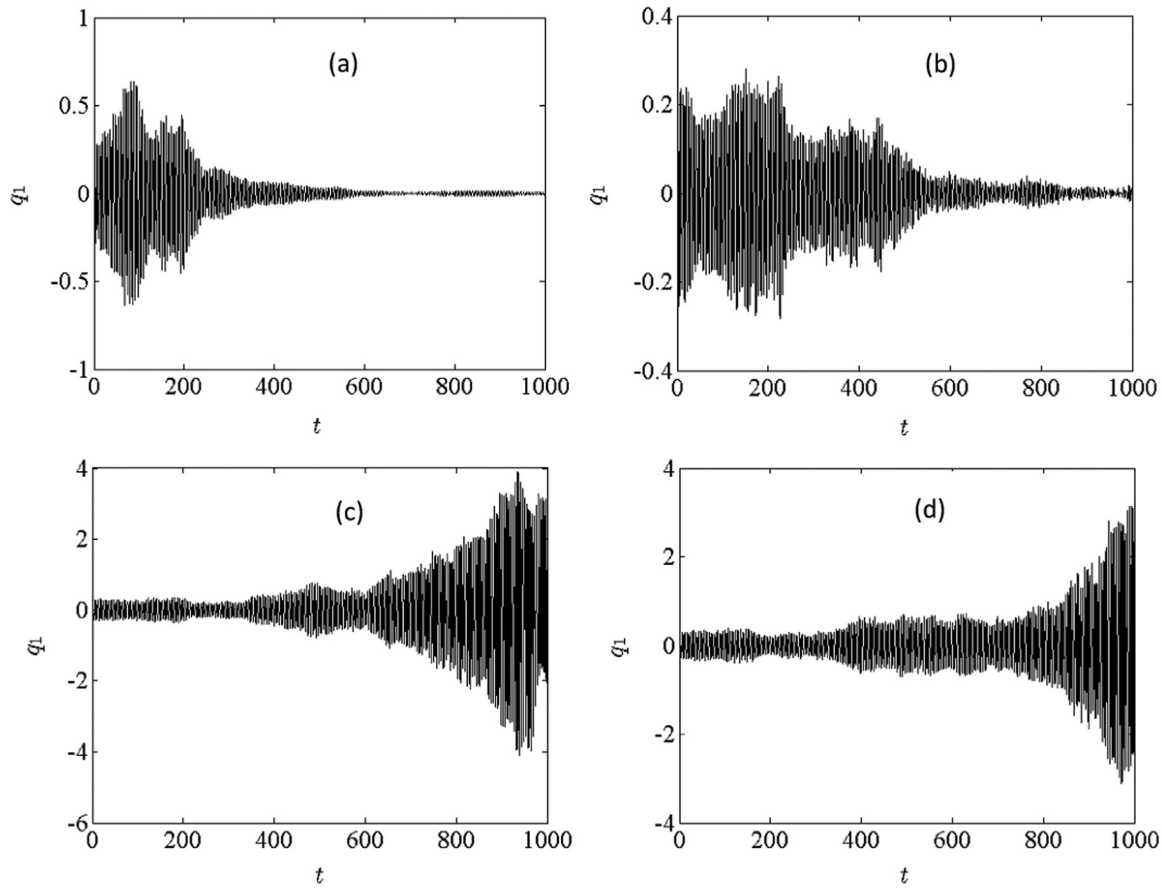

Fig. 3. Displacement-time graphs for q_1 obtained by direct simulations of systems represented by points A, B, C and D in Fig. 1: (a) for system A; (b) for system B; (c) for system C; and (d) for system D.

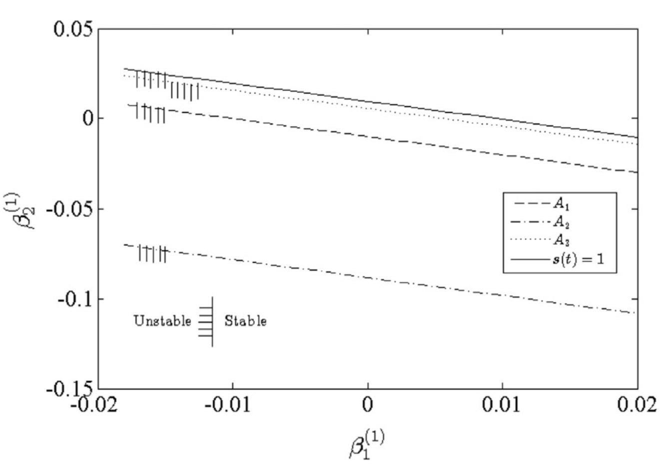

Fig. 4. Stability boundaries of 2-form system (33) in the $(\beta_1^{(1)}, \beta_2^{(1)})$ plane for three different transition matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ specified respectively by $(\nu_1, \nu_2) = (1,\ 1)$, $(\nu_1, \nu_2) = (5,\ 1)$, $(\nu_1, \nu_2) = (1,\ 5)$ and for the form $s(t) = 1$.

time in each unstable form so as to render the entire system asymptotically stable.

3. The validity and utility of the approximate method presented herein have been demonstrated by a nonlinear two-degree-of-freedom oscillator that is capable of independent Markovian jumps. By comparison with direct system simulations, it has been observed that the method is fairly reliable in assessing

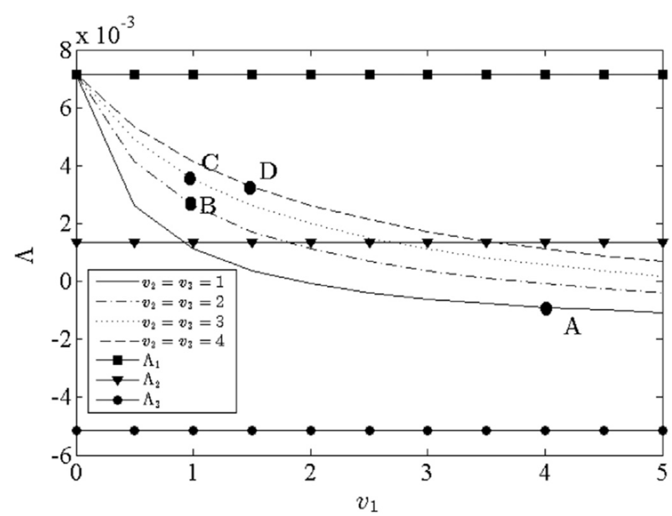

Fig. 5. The largest Lyapunov exponent Λ of 3-form system (33) versus ν_1 for $\nu_2 = \nu_3 = 1$, $\nu_2 = \nu_3 = 2$, $\nu_2 = \nu_3 = 3$, and $\nu_2 = \nu_3 = 4$. Systems represented by points A, B, C and D are used in Fig. 7 for direct simulations.

asymptotic stability.

It should be mentioned that, due to the lack of general techniques to treat nonlinear systems, rigorous analysis of the errors of approximation of the method thus presented cannot be made.

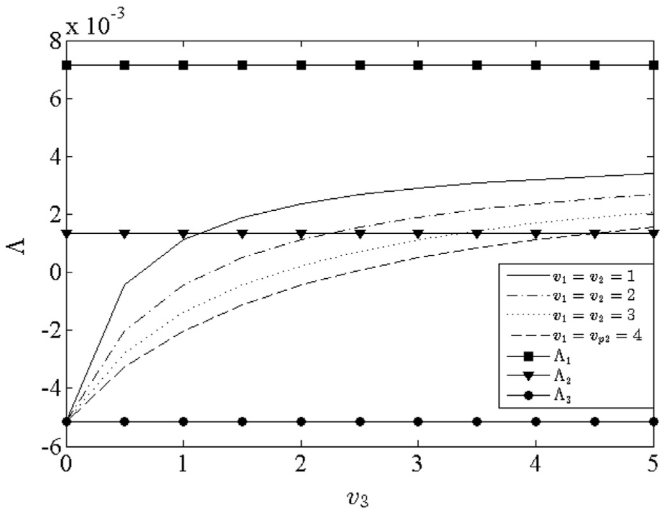

Fig. 6. The largest Lyapunov exponent Λ of 3-form system (33) versus v_3 for $v_1 = v_2 = 1$, $v_1 = v_2 = 2$, $v_1 = v_2 = 3$, and $v_1 = v_2 = 4$.

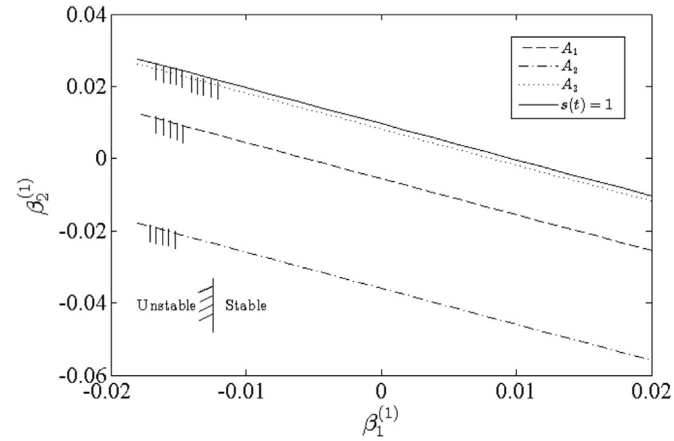

Fig. 8. Stability boundaries of 3-form system (33) in the $(\beta_1^{(1)}, \beta_2^{(1)})$ plane for three different transition matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ specified respectively by $(v_1, v_2, v_3) = (1, 1, 1)$, $(v_1, v_2, v_3) = (3, 1, 1)$, $(v_1, v_2, v_3) = (1, 1, 3)$ and for the form $s(t) = 1$.

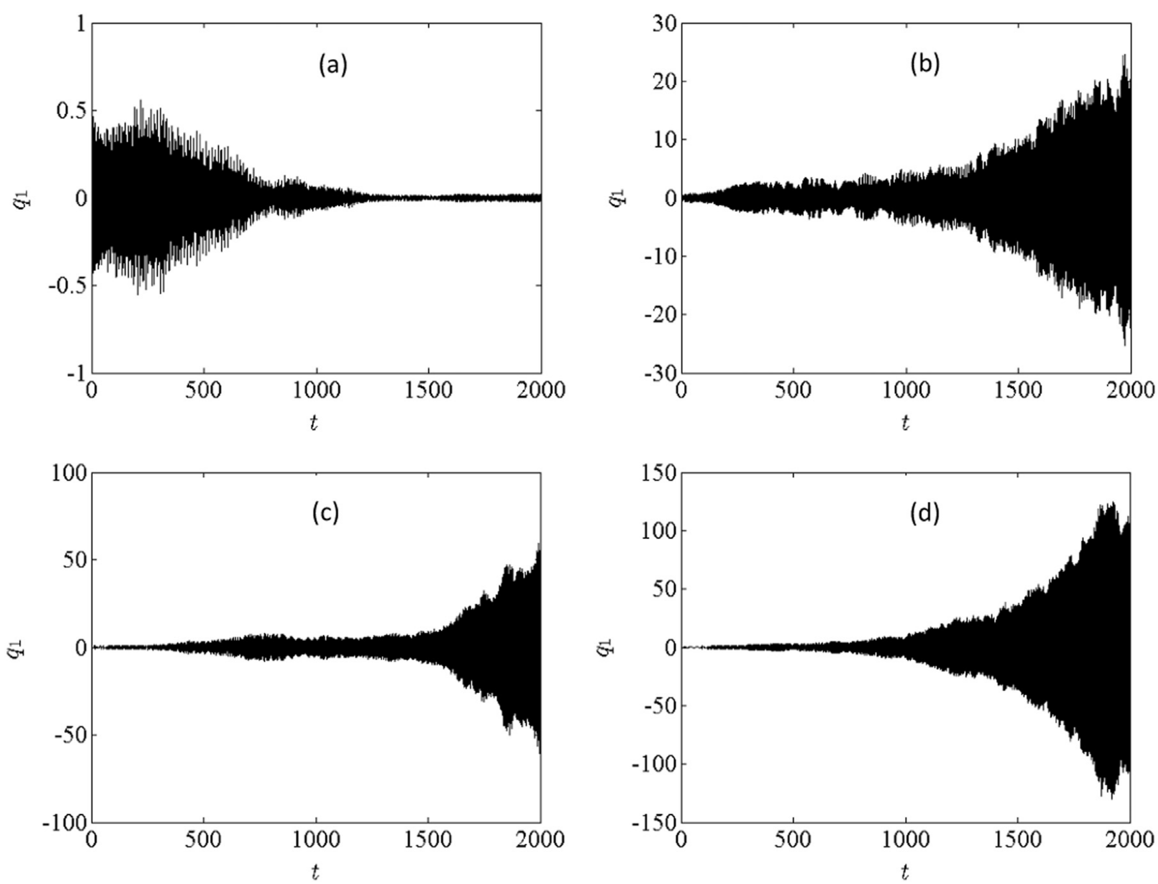

Fig. 7. Displacement-time graphs for q_1 obtained by direct simulations of systems represented by points A, B, C and D in Fig. 5: (a) for system A; (b) for system B; (c) for system C; and (d) for system D.

Among other things, it is hoped that this paper would point to directions along which further research efforts can be made. One application, however, appears attractive. In the optimization of Markovian-jump nonlinear systems requiring asymptotic stability, the linear-inequality stability conditions presented herein can be conveniently used as a constraint.

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