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Krstic, MIROSLAV

Publication Date

2004-10-01

Peer reviewed

Feedback Linearizability and Explicit Integrator Forwarding Controllers for Classes of Feedforward Systems

Miroslav Krstic, *Fellow, IEEE*

Abstract—We identify a class of feedforward nonlinear systems that are linearizable by a coordinate change. Then we develop explicit expressions for the Lyapunov-based integrator forwarding recursive procedure of Sepulchre, Jankovic, and Kokotovic, which has its roots in a coordinate transformation proposed by Mazenc and Praly. The explicit expressions that we develop allow us to also find closed-form control laws for several classes of systems that are not feedback linearizable, including some that are in the feedforward form and others that are in what we refer to as the "block-feedforward" form. Performance advantages of Lyapunov-based forwarding controllers over nested saturation controllers have been well illustrated in the literature on examples. The analytical expressions for the Lyapunov functions and the control laws allow us to give quantitative performance bounds.

Index Terms—Backstepping, feedback linearization, forwarding, Lyapunov function, stabilization.

I. INTRODUCTION

A. History and Summary of the Literature

IN THE WORLD of recursive control designs for nonlinear systems, two basic classes of systems are the most easily recognizable—the systems with (strict-)feed*back* structure and the systems with (strict-)feed*forward* structure. The strict-feedback systems, which occupied the attention of the nonlinear control community in the first half of the 1990s, are controlled using *backstepping*, a method that employs aggressive controls[1] necessary to suppress finite escape instabilities inherent (in open loop) to strict-feedback systems. In contrast, the strict-feedforward systems, which were studied intensively in the mid- and late-1990s, can be only marginally unstable in open loop,[2] and permit (and in many cases call for) cautious controllers.

The theoretical foundation of how to exercise "caution" in the control design for feedforward systems was laid out by Teel in his 1992 dissertation [41], where he introduced the technique of nested saturations whose parameters are carefully selected to essentially achieve robustness of linear controllers to nonlinearities (of superlinear and other types). Soon after this first design, Teel [43] developed a series of results that, among other things,

interpreted and generalized [41] in the light of nonlinear small gain techniques that he developed in [43]. The earliest use of the nonlinear small gain techniques as a design tool appears in [13].

The next major spurt of progress on feedforward systems came with [26], which introduced a Lyapunov approach for stabilization of feedforward systems. This approach, initially conceived in March 1993, has roots that go further back to Praly's 1991–1992 designs for adaptive nonlinear control [31] and output feedback stabilization [32] where he was designing forwarding like coordinate changes involving a stable manifold that can be written as a graph of a function. A related idea was used by Sontag and Sussmann [38] for stabilization of linear systems with saturated controls. Recently, Praly *et al.* [35] relaxed the conditions under which such manifolds can be found.

Jankovic *et al.* [11] developed a different Lyapunov solution to the problem of forwarding (and stabilization of a broad class of cascade systems), which, rather than a coordinate change or domination of (certain) "cross terms" (as Mazenc and Praly), employs an exact cross term in the Lyapunov function. In [37], they presented an algorithmic, inverse optimal design for a class of feedforward systems and provided a detailed insight into the structure of the target system in the forwarding recursion.

Further developments on feedforward systems have gone in several directions. The nested saturation ideas have been expanded upon by Lin and Li [18], Arcak *et al.* [2], Marconi and Isidori [21], and Xudong [46]. Implicit (or explicit) in the first three papers are robustness results with respect to certain classes of unmodeled dynamics. The Lyapunov approach has been developed further by Sepulchre *et al.* [37], [36], Mazenc *et al.* [29], and Mazenc and Praly [28]. Lin and Qian [19] proposed designs for systems satisfying certain growth conditions.

In [44], Teel designed $\mathcal{L}_2$ stabilizing controllers for feedforward systems ($\mathcal{L}_\infty$ disturbance attenuation, while impossible in general, remains a problem of interest for subclasses of feedforward systems). Trajectory tracking, while hard to achieve for arbitrary trajectories, has been solved under reasonable conditions by Mazenc and Praly [27] and Mazenc and Bowong [24]. Extensions to nonlinear integrator chains have been proposed by Mazenc [22] and Tsinias and Tzamtzi [45]. Even a generalization to feedforward systems with exponentially unstable linearizations has been reported by Grognard *et al.* [7]. Discrete-time feedforward systems have also been studied, in [25]. Linear low-gain semiglobal stabilization of feedforward systems was proposed by Grognard *et al.* [8]. An output feedback problem for feedforward systems was recently solved by Mazenc and Vivalda [30]. Feedforward systems do not lend themselves easily

[1]As measured by the growth of their nonlinearities.

[2]With solutions growing only polynomially in time.

to adaptive control—one related result is by Jankovic *et al.*[12]. Nonparametric robust control, i.e., disturbance attenuation in the style of [16] (for example) with disturbances entering through a nonlinear vector field, has so far remained intractable (except in the case when the vector field is constant).

Starting with Teel's original interest in the ball-and-beam problem [41] and Mazenc and Praly's design for the pendulum-cart problem [26], the research on forwarding has continuously been driven by applications. The following papers on forwarding are fully (or almost fully) dedicated to applications: [39] (pole-cart), [3] (ball-and-beam), [1] (spherical inverted pendulum), [35] (inverted pendulum with disk inertia), [23] (pendulum-cart), and [34] (satellite orbit transfer with weak but continuous thrust).

Differential geometric characterization of feedforward systems has eluded researchers until recent major progress was reported by Tall and Respondek [40].

For tutorial coverage of forwarding, the reader is referred to [36] and [34]. Some coverage of forwarding is also available in [5] and [15].

B. Contribution and Organization of the Present Paper

The idea of exact forwarding coordinate transformations as a Lyapunov avenue toward performance improvement relative to the "cautious" saturation-based approaches first appeared in [26, Sec. IV]. However, it is not until the result of [37], which considers a special subclass of the systems studied in [26] and [11], that this idea crystalized into a conceptually transparent, elegant recursive procedure, which is easy to compare with backstepping. Still, the crucial element that remained lacking in the procedure was computability. In principle, one has to solve (analytically) a series of nonlinear systems and compute (again analytically) a series of integrals. This paper is dedicated to providing closed-form solutions to these nonlinear systems and integrals.

We start in Section II by reviewing the Sepulchre–Jankovic–Kokotovic (SJK) [37] design procedure. While it has been long believed that feedforward systems are "generically not feedback linearizable," in Section III we show that many of them are and provide a parametrization of linearizable feedforward systems. For those systems, the SJK procedure provides the needed change of coordinates, which is given explicitly in Section IV. The coordinate change does not require the solution of a series of nonlinear systems (as in the general SJK procedure) but does require analytical computation of a series of integrals. For two important subclasses of linearizable feedforward systems, those integrals are calculated explicitly in Sections V and VI. Second- and third-order examples of those classes of systems are presented in some detail in Section VII, shedding light on how typical, or atypical, linearizability is for feedforward systems. In Section VIII, we exploit the closed-form nature of the designs in Sections V and VI to develop closed-form SJK formulas for two classes of feedforward systems that are *not* linearizable, followed, in Section IX, by an example similar to (but more challenging than) the celebrated Kokotovic–Teel third-order "benchmark" example. The SJK procedure is extended to a class of "block-feedforward" systems in Section X, where closed-form feedback laws are also developed for two subclasses. Following an idea in [36], interlacing of forwarding and backstepping is formalized for two classes of systems for which feedback linearization formulas are given in Section XI. Block-forwarding and interlacing are then all illustrated on an example (which is not feedback linearizable) in Section XII. Bounds on control effort are given in Section XIII. Finally, in Section XIV, we pose a question of how generic linearizability is within the feedforward class.

To keep this paper at reasonable length, we give the proofs very concisely. Throughout the paper, all of the plant nonlinearities are assumed to be Lipschitz continuous (or smoother, if specified).

II. SJK Algorithm

Consider the class of *strict-feedforward systems*

$$\dot{x}_i = x_{i+1} + \psi_i(\underline{x}_{i+1}) + \phi_i(\underline{x}_{i+1})u, \qquad i = 1, 2, \ldots, n \quad (1)$$

where $\underline{x}_j = [x_j, \ x_{j+1}, \ldots, x_n]^T$, $x_{n+1} = u$, $\phi_n = 1$, $\partial \psi_i(0)/\partial x_j = \phi_i(0) = 0$, and

$$\psi_i(x_{i+1}, 0, \ldots, 0) \equiv 0 \quad (2)$$

for $i = 1, 2, \ldots, n-1, j = i+1, \ldots, n$. (This notation implies that $\psi_n = 0$.)

Relative to the class of systems in [37] we make a trade of generality for conceptual clarity by requiring that the drift term be of the form $x_{i+1} + \psi_i(\underline{x}_{i+1})$, where the ψ_i's, in addition to being higher order, vanish whenever $x_{i+2}, \ldots, x_n$ vanish. In Section IX we show that this restriction can be relaxed in some cases, however we keep it throughout most of the paper for notational and conceptual convenience. We note that (2) means, in particular, that $\psi_{n-1}(x_n) \equiv 0$.

The control law for this class of systems is designed as follows. Let $\beta_{n+1} = \alpha_{n+1} = 0$. For $i = n, n-1, \ldots, 2, 1$, see (3)–(6) as shown at the bottom of the page, where the

$$z_i = x_i - \beta_{i+1} \tag{3}$$

$$w_i(\underline{x}_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial \beta_{i+1}}{\partial x_j} \phi_j - \frac{\partial \beta_{i+1}}{\partial x_n} \tag{4}$$

$$\alpha_i(\underline{x}_i) = \alpha_{i+1} - w_i z_i \tag{5}$$

$$\beta_i(\underline{x}_i) = -\int_0^\infty \left[\xi_i^{[i]}(\tau, \underline{x}_i) + \psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right) + \phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right) \alpha_i\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right) \right] d\tau \tag{6}$$

notation in the integrand of (6) refers to the solutions of the (sub)system(s)

$$\frac{d}{d\tau}\xi_j^{[i]} = \xi_{j+1}^{[i]} + \psi_j\left(\underline{\xi}_{j+1}^{[i]}\right) + \phi_j\left(\underline{\xi}_{j+1}^{[i]}\right)\alpha_i\left(\underline{\xi}_i^{[i]}\right) \quad (7)$$

for $j = i, i+1, \ldots, n$, at time τ, starting from the initial condition $\underline{x}_i$. The control law is

$$u = \alpha_1. \quad (8)$$

It is important to first understand the meaning of the integral in (6). Clearly, the solution $\underline{\xi}_i(\tau, \underline{x}_i)$ is impossible to obtain analytically in general. Dealing with this issue is the main subject of this paper. Note that the last of the β_i's that need to be computed is β_2 (β_1 is not defined).

The stability analysis of the closed-loop system is straightforward. Starting with the observation that $x_{i+1}+\psi_i+\phi_i\alpha_{i+1} = \sum_{j=i+1}^n \partial\beta_{i+1}/\partial x_j(x_{j+1}+\psi_j+\phi_j\alpha_{j+1})$, it is easy to verify that $\dot{z}_i = w_i\left(u + \sum_{j=i+1}^n w_j z_j\right)$. Noting from (8) and (5) that

$$u = -\sum_{i=1}^n w_i z_i \quad (9)$$

we get $\dot{z}_i = -w_i^2 z_i - \sum_{j=1}^{i-1} w_i w_j z_j$ (note that this notation implies that $\dot{z}_1 = -w_1^2 z_1$). Taking the Lyapunov function $V = (1/2)\sum_{i=1}^n z_i^2$, one obtains

$$\dot{V} = -\frac{1}{2}\sum_{i=1}^n w_i^2 z_i^2 - \frac{1}{2}\left(\sum_{i=1}^n z_i w_i\right)^2. \quad (10)$$

Theorem 1: [37] The feedback system (1), (8) is globally asymptotically stable at the origin.

Although the proof of this theorem is available in [37], we provide some of its elements here for two reasons—one is to ease a nonexpert reader into the topic of forwarding, and the other is that some of our further arguments mimic those used in the proof of this theorem (and we will not repeat them). First, a careful inspection of the design algorithm reveals that $\beta_i(0) = 0$, which means that the triangular coordinate transformation $z(x)$ is a global diffeomorphism with $z(0) = 0$. From (10) it then follows that the equilibrium $x = 0$ is globally stable. LaSalle's theorem guarantees that $z_i w_i \to 0$ as $t \to \infty$. Since $w_n \equiv 1$ and $z_n = x_n$, it follows that $x_n(t) \to \infty$. One can verify recursively that $w_i(0) = 1$ for all i [this is a consequence of the fact that $x_{n+1} = u$ and of the presence of the linear term $\xi_i^{[i]}$ in (6)]. Thus, it follows that $w_{n-1}(x_n(t)) \to 1$, which, along with $\beta_n(0) = 0$, implies that $x_{n-1}(t) \to \infty$. Continuing in this fashion, one recursively shows that $w_i(t) \to 1, \beta_{i+1}(t) \to 0$ for each i and, thus, that $x(t) \to 0$ as $t \to \infty$.

III. LINEARIZABILITY OF FEEDFORWARD SYSTEMS

The main interest in this paper is in making the computation of the integral in (6) tractable. Toward that end, let us start by

noting that (7), which needs to be solved analytically, can be written in the z-coordinates[3] as

$$\frac{d}{d\tau}\zeta_j^{[i]} = -w_j^2\zeta_j^{[i]} - \sum_{l=1}^{j-1} w_j w_l \zeta_l^{[i]}, \qquad j = i, i+1, \ldots, n \quad (11)$$

which is obtained with $\dot{\zeta}_j^{[i]} = w_j\alpha_i$. Suppose now that (somehow, miraculously,...) all of the w_l's were equal to 1 (for all values of their arguments, rather than just $w_l(0) = 1$). We would have a lower triangular linear system $(d/d\tau)\zeta_j^{[i]} = -\zeta_j^{[i]} - \sum_{l=1}^{j-1}\zeta_l^{[i]}, j = i, i+1, \ldots, n$, which is easily solvable in closed form. Then, the only difficulty remaining would be the integration with respect to τ of the integral (6) (using an appropriate coordinate change from $\underline{\zeta}_i^{[i]}$ to $\underline{\xi}_i^{[i]}$). Calculating the integral is by no means trivial, but it is a much easier task than solving the nonlinear ordinary differential equation (ODE) (7) *and* calculating the integral.

Before we start exploring the conditions under which one would get

$$w_i(\underline{x}_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1}\frac{\partial\beta_{i+1}}{\partial x_j}\phi_j - \frac{\partial\beta_{i+1}}{\partial x_n} = 1 \quad (12)$$

let us note another consequence of this. In this case, the coordinate change, before applying the feedback, would yield

$$\dot{z} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & & \vdots \\ \vdots & 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u. \quad (13)$$

We refer to this as the Teel [42] canonical form. This is a completely controllable linear system. Hence, the systems that satisfy condition (12) are linearizable (into this linear form, and, ultimately, into the Brunovsky canonical form).

Thus, the exploration of analytical computability of control laws for strict-feedforward systems that we undertake in this paper amounts, to a large extent, to a study of linearizability. Clearly, merely checking the coordinate-free conditions for linearizability [10] will not get us any closer to actually finding the control laws. Such a test would lead to conditions on the ϕ_i's in the form of partial differential equations that they have to satisfy (these conditions would arise from the involutivity test).

Up until now we have used the word "linearizable" somewhat loosely. In the next definition, we make this notion precise.

Definition 1: If there exists a diffeomorphism

$$y_i = x_i - \theta_{i+1}(\underline{x}_{i+1}), \qquad i = 1, \ldots, n-1 \quad (14)$$

$$y_n = x_n \quad (15)$$

[3]We point out that, analogous to (7), we use ζ, a Greek version of z, to denote the solution of the $\underline{z}_i$ subsystem, under the control α_i, starting from initial condition $\underline{z}_i$. It should be also self understood that w_j stands for $w_j\left(\underline{\xi}_{j+1}^{[i]}\right)$, where $\xi_k^{[i]} = \zeta_k^{[i]} + \beta_{k+1}\left(\underline{\xi}_{k+1}^{[i]}\right)$, and so on (i.e., expressing w_j as a function of $\underline{\zeta}_{j+1}^{[i]}$).

where

$$\theta_i(0) = \frac{\partial \theta_i(0)}{\partial x_j} = 0, \qquad i = 2, \ldots, n, \quad j = i, \ldots, n \quad (16)$$

transforming the strict-feedforward system (1)–(2) into a system of the form

$$\dot{y}_i = y_{i+1}, \qquad i = 1, 2, \ldots, n-1 \quad (17)$$
$$\dot{y}_n = u \quad (18)$$

system (1)–(2) is said to be *diffeomorphically equivalent to a chain of integrators (DECI)*.

We point out that the term DECI does not reflect that (14) and (15) restrict the class of admissible diffeomorphisms to a "triangular" form. In the next theorem, we give sufficient conditions for characterizing DECI strict-feedforward systems.

Theorem 2: All strict-feedforward systems (1)–(2) with $\psi_i(\underline{x}_{i+1}), \phi_i(\underline{x}_{i+1})$ that can be written as $\phi_{n-1}(x_n) = \theta'_n(x_n), \psi_{n-1}(x_n) = 0$, and

$$\phi_i(\underline{x}_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(\underline{x}_{i+1})}{\partial x_j} \phi_j(\underline{x}_{j+1}) + \frac{\partial \theta_{i+1}(\underline{x}_{i+1})}{\partial x_n} \quad (19)$$

$$\psi_i(\underline{x}_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(\underline{x}_{i+1})}{\partial x_j} \left(x_{j+1} + \psi_j(\underline{x}_{j+1})\right) - \theta_{i+2}(\underline{x}_{i+2}) \quad (20)$$

for $i = n-2, \ldots, 1$, using some C^1 scalar-valued functions $\theta_i(\underline{x}_i)$ satisfying (16), are DECI.

Proof: Straightforward to verify using (14) and (15). ∎

Theorem 2 is not a substitute for a geometric test of linearizability, nor is it a control design tool. It is just a *parametrization* of a subclass of strict-feedforward systems that are DECI.

For instance, all third-order strict-feedforward systems of the form

$$\dot{x}_1 = x_2 + \frac{\partial \theta_2(x_2, x_3)}{\partial x_2} x_3 - \theta_3(x_3)$$
$$+ \left(\frac{\partial \theta_2(x_2, x_3)}{\partial x_2} \theta'_3(x_3) + \frac{\partial \theta_2(x_2, x_3)}{\partial x_3} \right) u \quad (21)$$
$$\dot{x}_2 = x_3 + \theta'_3(x_3)u \quad (22)$$
$$\dot{x}_3 = u \quad (23)$$

are linearizable, where any two locally quadratic C^1 functions $\theta_2(x_2, x_3)$ and $\theta_3(x_3)$ are the "parameters." Take, for instance, $\theta_2(x_2, x_3) \equiv 0$ and $\theta_3(x_3) = \cosh(x_3) - 1$, which is locally quadratic. We get that the strict-feedforward system

$$\dot{x}_1 = x_2 + \cosh(x_3) - 1 \quad \dot{x}_2 = x_3 + \sinh(x_3)u \quad \dot{x}_3 = u \quad (24)$$

is linearizable using the coordinate change

$$y_1 = x_1 \quad y_2 = x_2 + \cosh(x_3) - 1 \quad y_3 = x_3. \quad (25)$$

Unfortunately, there is no easy systematic way to obtain this coordinate change (we know what it is because we started with

$\theta_3(x_3) = \cosh(x_3) - 1$ and constructed the system). The only systematic way to arrive at it is the SJK procedure. In the next section we show that the SJK procedure greatly simplifies for DECI strict-feedroward systems, and, in particular, directly leads to (25) for (24) without having to solve nonlinear ODEs of the form (7).

Before we move on, it is interesting to note that the equations in Theorem 2, if viewed as partial differential equations in the θ_i's, fit the single-step feedback linearization framework of [14].

IV. ALGORITHM FOR ALL LINEARIZABLE FEEDFORWARD SYSTEMS

For linearizable strict-feedforward systems we present the following design algorithm, which eliminates the requirement to solve the ODEs (7) and reduces the problem to calculating a set of integrals with respect to time. Let $\beta_{n+1} = \alpha_{n+1} = 0$. For $i = n, n-1, \ldots, 2, 1$

$$\alpha_i(\underline{x}_i) = -\sum_{j=i}^{n} \left(x_j - \beta_{j+1}(\underline{x}_{j+1})\right) \quad (26)$$

$$\xi_n^{[i]}(\tau, \underline{x}_i) = e^{-\tau} \sum_{k=0}^{n-i} \frac{(-\tau)^k}{k!} \left(x_{n-k} - \beta_{n-k+1}(\underline{x}_{n-k+1})\right) \quad (27)$$

$$\xi_j^{[i]}(\tau, \underline{x}_i) = e^{-\tau} \sum_{k=0}^{j-i} \frac{(-\tau)^k}{k!} \left(x_{j-k} - \beta_{j-k+1}(\underline{x}_{j-k+1})\right)$$
$$+ \beta_{j+1}\left(\xi_{j+1}^{[i]}(\tau, \underline{x}_i)\right),$$
$$j = n-1, \ldots, i+1, i \quad (28)$$

$$\beta_i(\underline{x}_i) = -\int_0^\infty \left[\xi_i^{[i]}(\tau, \underline{x}_i) + \psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right) \quad (29)\right.$$
$$\left. + \phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right) \alpha_i\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right) \right] d\tau. \quad (30)$$

The control law is

$$u = \alpha_1. \quad (31)$$

We stress that, due to linearizability, the ODEs (7) are solved in closed form, and the only calculation remaining is the integrals (30), which can be obtained with symbolic software (coded in Mathematica or Maple/Matlab). This calculation is particularly straightforward (and can be done, in principle, by hand) when the nonlinearities $\psi_i(\cdot), \phi_i(\cdot)$ are polynomial. In that case, the following identity is useful in calculating (30):

$$\int_0^\infty \tau^p e^{-q\tau} d\tau = \frac{p!}{q^{p+1}} \qquad \forall p, q \in N. \quad (32)$$

Theorem 3: If the strict-feedforward plant (1)–(2) is DECI, then the feedback system (1), (31) is globally asymptotically stable at the origin.

Proof: One can verify that in the coordinates

$$z_i = x_i - \beta_{i+1}(\underline{x}_{i+1}) \quad (33)$$

the control system becomes (13), and under the feedback control (31), the resulting system is

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & & \vdots \\ \vdots & -1 & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & -1 \end{bmatrix} z. \tag{34}$$

The rest of the proof is as in Theorem 1. ∎

As we indicated in Section III, checking the geometric conditions for linearizability is easy, whereas actually constructing the linearizing coordinates is not. The algorithm (26)–(30) constructs the coordinate change into the (non-Brunovsky) Teel canonical form (13). The next theorem gives the coordinate change into the Brunovsky/chain-of-integrators form.

Theorem 4: If the strict-feedforward plant (1)–(2) is DECI, it has a relative degree[4] n with respect to the output

$$y_1 = \sum_{j=1}^{n} \binom{n-1}{j-1} (-1)^{j-1} \left(x_j - \beta_{j+1}(\underline{x}_{j+1}) \right). \tag{35}$$

Furthermore, the coordinate change (26)–(30), (33), and

$$y_i = \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} z_j, \qquad i = 1, 2, \ldots, n \tag{36}$$

converts system (1) into the chain of integrators (17)–(18).

Proof: By verification. ∎

Inverse optimality, proved for the general case in [37], becomes particularly meaningful in the linearizable case.

Theorem 5: The control law

$$u^* = 2\alpha_1(x) = -2 \sum_{j=1}^{n} \left(x_j - \beta_{j+1}(\underline{x}_{j+1}) \right) \tag{37}$$

where $\alpha_1(x)$ is defined via (26)–(30), minimizes the cost functional $J = \int_0^\infty \left(l(x(t)) + u(t)^2 \right) dt$ along the solutions of (1), where

$$l(x) = \sum_{j=1}^{n} \left(x_j - \beta_{j+1}(\underline{x}_{j+1}) \right)^2$$

$$+ \left(\sum_{j=1}^{n} \left(x_j - \beta_{j+1}(\underline{x}_{j+1}) \right) \right)^2 \tag{38}$$

is a positive–definite, radially unbounded function. Furthermore, the control law (37) remains globally asymptotically stabilizing at the origin in the presence of input unmodeled dynamics of the form $a(I + \mathcal{P})$, where $a \geq 1/2$ is a constant, $\mathcal{P}u$ is the output of any strictly passive nonlinear system[5] with u as its input, and I denotes the identity operator.

Proof: It follows from [16, Th. 2.8, Th. 2.17, Cor. 2.18]. ∎

[4]As defined in [10].

[5]With possibly nonzero initial conditions.

The main result of this section was a control algorithm that eliminates the requirement to solve the ODEs (7) and reduces the problem to calculating only the integrals (30). In the next two sections, we present algorithms that eliminate even the need to calculating the integrals (30) for two subclasses of DECI strict-feedforward systems.

V. LINEARIZABLE FEEDFORWARD SYSTEMS OF TYPE I

Consider the class of strict-feedforward systems given by

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j) x_{j+1} + \pi_n(x_n) u \tag{39}$$

$$\dot{x}_i = x_{i+1}, \qquad i = 2, \ldots, n-1 \tag{40}$$

$$\dot{x}_n = u \tag{41}$$

where $\pi_j(0) = 0$. Any system in this class is DECI.

Theorem 6: The diffeomorphic transformation

$$y_1 = x_1 - \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s) ds \tag{42}$$

$$y_i = x_i, \qquad i = 2, \ldots, n \tag{43}$$

converts the strict-feedforward system (39)–(41) into the chain of integrators (17)–(18). The feedback law

$$u = \alpha_1(x) = -\sum_{i=1}^{n} \binom{n}{i-1} y_i \tag{44}$$

globally asymptotically stabilizes the origin of (39)–(41).

Proof: The first part by verification. In the second part, we note that the y system has n closed-loop poles at -1 and use that fact that the coordinate change is diffeomorphic. ∎

We note that in the design (42)–(44) we have completely circumvented the SJK procedure. It is therefore worth noting that, following the SJK procedure, one would have obtained

$$\alpha_i(\underline{x}_i) = -\sum_{j=i}^{n} \binom{n-i+1}{j-i} x_j + \delta_{i,1} \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s) ds \tag{45}$$

$$w_i = 1 \tag{46}$$

where $\delta_{i,1}$ denotes the Kronecker delta.[6] However, the most important product of the SJK procedure is the coordinate shift β_i (from x to z), which is given in the context of the following result.

Corollary 1: The control law (37), with $\alpha_1(x)$ defined in (44), applied to the plant (39)–(41) achieves the result of Theorem 5 with

$$\beta_{i+1}(\underline{x}_{i+1}) = -\sum_{j=i+1}^{n} \binom{n-i}{j-i} x_j + \delta_{i,1} \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s) ds \tag{47}$$

for $i = 1, \ldots, n-1$.

While in Section IV we showed that one can avoid having to solve the nonlinear ODEs (7), in Theorem 6 we showed that, for

[6]Note that (45) for $i = 1$ is the same as (44).

the feedforward subclass (39)–(41), one can also avoid having to calculate the integrals (30). In the next result, we go even further and show that, not only does one have a closed-form formula for the control law (44) but one can even get a closed-form formula for the *solutions* of the system under that control law. This is not just an aesthetically pleasing result—it will allow us, in Section VIII, to extend the constructive methodology to a class of strict-feedforward systems that are *not linearizable*.

To prevent confusion about the notation in the theorem, before its statement we emphasize that x, which denotes the initial condition, is constant. This notation is important for a seamless use of the theorem in subsequent results. We also point out that, relative to the notation in Sections II and III, $\xi(\tau, x)$ and $\zeta(\tau, z)$ should be understood, respectively, as $\xi^{[1]}(\tau, x)$ and $\zeta^{[1]}(\tau, z)$.

Lemma 1: Starting from the initial condition denoted by x, the solution $\xi_i(\tau, x)$ of the feedback system (39)–(41), (42)–(44) at time τ is

$$
\xi_i(\tau, x) = e^{-\tau} \left[\sum_{j=i}^{n} \binom{n-i}{j-i}(-1)^{j-i} \right.
$$
$$
\times \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} x_l
$$
$$
+ (-1)^i \sum_{j=i}^{n} \binom{n-i}{j-i} \frac{\tau^{j-1}}{(j-1)!}
$$
$$
\left. \times \left(\sum_{m=2}^{n} \int_0^{x_m} \pi_m(s)ds \right) \right],
$$
$$
i = 2, \dots, n \tag{48}
$$
$$
\tag{49}
$$

for $i = 2, \dots, n$ and

$$
\xi_1(\tau, x) = e^{-\tau} \left[\sum_{j=1}^{n} \binom{n-1}{j-1}(-1)^{j-1} \right.
$$
$$
\times \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} x_l
$$
$$
- \sum_{j=1}^{n} \binom{n-1}{j-1} \frac{\tau^{j-1}}{(j-1)!}
$$
$$
\left. \times \left(\sum_{m=2}^{n} \int_0^{x_m} \pi_m(s)ds \right) \right]
$$
$$
+ \sum_{j=2}^{n} \int_0^{\xi_j(\tau, x)} \pi_j(s)ds \tag{50}
$$

whereas the control signal is

$$
u = \tilde{\alpha}_1(\tau, x) = -e^{-\tau} \sum_{i=1}^{n} \binom{n}{i-1} \left[\sum_{j=i}^{n} \binom{n-i}{j-i}(-1)^{j-i} \right.
$$
$$
\times \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} x_l
$$
$$
+ (-1)^i \sum_{j=i}^{n} \binom{n-i}{j-i} \frac{\tau^{j-1}}{(j-1)!}
$$
$$
\left. \times \left(\sum_{m=2}^{n} \int_0^{x_m} \pi_m(s)ds \right) \right]. \tag{51}
$$

Proof: By using (42), (43), their inverse, $x_1 = y_1 - \sum_{j=2}^{n} \int_0^{y_j} \pi_j(s)ds, x_i = y_i, i = 2, \dots, n$, the transformation (36), and its inverse $z_i = \sum_{j=i}^{n} \binom{n-i}{j-i} y_j, i = 1, 2, \dots, n$, the explicit form of the solution of (34), $\zeta_j(\tau, z) = \sum_{k=0}^{j-1} ((-\tau)^k/k!) e^{-\tau} z_{j-k}$, and (44). ∎

VI. Linearizable Feedforward Systems of Type II

Consider the subclass of the strict-feedforward systems (1) given by

$$
\dot{x}_i = x_{i+1} + \phi_i(\underline{x}_{i+1})u, \qquad i = 1, \dots, n-1 \tag{52}
$$
$$
\dot{x}_n = u \tag{53}
$$

where $\phi_i(0) = 0$. In this section we construct control laws for a linearizable *subclass* of (52) and (53).

To characterize the linearizable subclass, let us consider the functions $\phi_{n-1}(x_n)$ and $\phi_i(0, \dots, 0, x_n), i = 1, \dots, n-2$, as given and introduce the sequence of functions shown in (54)–(55) at the bottom of the page, for $i = n-1, n-2, \dots, 2$, and

$$
\gamma_1(x_n) = \mu'_n(x_n) \tag{56}
$$
$$
\gamma_k(x_n) = \sum_{l=1}^{k-1} \gamma_l(x_n)\mu_{l+n+1-k}(x_n) + \frac{d\mu_{n+1-k}(x_n)}{dx_n} \tag{57}
$$

for $k = 2, \dots, n-2$.

Theorem 7: If

$$
\phi_i(\underline{x}_{i+1}) = \sum_{j=i+1}^{n-1} \gamma_{j-i}(x_n)x_j + \phi_i(0, \dots, 0, x_n) \tag{58}
$$

$$
\mu_n(x_n) = \frac{\int_0^{x_n} \phi_{n-1}(s)ds}{x_n} \tag{54}
$$

$$
\mu_i(x_n) = \frac{1}{x_n} \int_0^{x_n} \left[\phi_{i-1}(0, \dots, 0, s) - \sum_{j=i+1}^{n} \mu_j(s)\phi_{i+n-j}(0, \dots, 0, s) \right] ds \tag{55}
$$

$\forall x, i = 1, \ldots, n-2$, then the diffeomorphic transformation

$$y_i = x_i - \sum_{j=i+1}^{n} \mu_{i+1+n-j}(x_n)x_j, \qquad i = 1, \ldots, n-1 \tag{59}$$

$$y_n = x_n \tag{60}$$

converts the strict-feedforward system (52)–(53) into the chain of integrators (17)–(18). The feedback law

$$u = \alpha_1(x) = -\sum_{i=1}^{n} \binom{n}{i-1} y_i \tag{61}$$

globally asymptotically stabilizes the origin of (52)–(53).

Proof: First part by (lengthy) verification. The rest as in the proof of Theorem 6. ∎

As in Section V, we point out that, following the SJK procedure, one would have obtained

$$\alpha_i(\underline{x}_i) = -x_i - \sum_{m=i+1}^{n} x_m \left[\binom{n-i+1}{m-i} \right.$$
$$\left. - \sum_{j=i}^{m} \binom{n-i+1}{j-i} \mu_{j+1+n-m}(x_n) \right] \tag{62}$$

$$w_i = 1 \tag{63}$$

and the coordinate shift β_i is given in the context of the following result.

Corollary 2: The control law (37), with $\alpha_1(x)$ defined in (61), applied to the plant (52)–(53), (54)–(55), (56)–(57), (58) achieves the result of Theorem 5 with (64), as shown at the bottom of the page.

Example 1: To illustrate the aforementioned concepts (and notation), let us consider a fourth-order example of a Type II feedforward system:

$$\dot{x}_1 = x_2 + \left(\frac{x_2}{2} - \frac{x_3 x_4}{12} \right) u \tag{65}$$

$$\dot{x}_2 = x_3 + \frac{x_3}{2} u \tag{66}$$

$$\dot{x}_3 = x_4 + x_4 u \tag{67}$$

$$\dot{x}_4 = u. \tag{68}$$

The control law $u = -y_1 - 4y_2 - 6y_3 - 4y_4 = -z_1 - z_2 - z_3 - z_4$, where

$$y_1 = x_1 - \frac{x_4 x_2}{2} + \frac{x_4^2 x_3}{6} - \frac{x_4^4}{24} \quad y_2 = x_2 - \frac{x_4 x_3}{2} + \frac{x_4^3}{6}$$

$$y_3 = x_3 - \frac{x_4^2}{2} \quad y_4 = x_4 \tag{69}$$

which is obtained with $\mu_2 = x_4^3/24$, $\mu_3 = -x_4^2/6$, $\mu_4 = x_4/2$, and $z_i = x_i - \beta_{i+1}$ with

$$\beta_4 = \left(\frac{x_4}{2} - 1 \right) x_4 \tag{70}$$

$$\beta_3 = \left(\frac{x_4}{2} - 2 \right) x_3 - x_4 + x_4^2 - \frac{x_4^3}{6} \tag{71}$$

$$\beta_2 = \left(\frac{x_4}{2} - 3 \right) x_2 + \left(-3 + \frac{3}{2}x_4 - \frac{x_4^2}{6} \right) x_3$$
$$- x_4 - \frac{3}{2}x_4^2 + \frac{1}{2}x_4^3 - \frac{1}{24}x_4^4 \tag{72}$$

achieves (13) for $n = 4$ and $(s+1)^4 y_1(s) = 0$. ◇

For using the results of this section for control designs beyond the Type II class of systems, we need the inverse of the coordinate transformation (59). The explicit form of the inverse transformation is given in the following theorem.

Lemma 2: Consider the series of functions

$$\lambda_n(x_n) = \mu_n(x_n) \tag{73}$$

$$\lambda_i(x_n) = \frac{1}{x_n} \int_0^{x_n} \left(s \sum_{l=i+1}^{n} \gamma_{l-i}(s)\lambda_l(s) + \phi_{i-1}(0, \ldots, 0, s) \right) ds \tag{74}$$

for $i = n-1, \ldots, 2$. The inverse of the diffeomorphic transformation (59) is

$$x_i = y_i + \sum_{j=i+1}^{n} \lambda_{i+1+n-j}(y_n)y_j, \qquad i = 1, \ldots, n-1 \tag{75}$$

$$x_n = y_n. \tag{76}$$

Proof: By induction, using the intermediate step that $\gamma_{n-i+1}(x_n) = -\sum_{m=1}^{n-i} \gamma_m(x_n)\lambda_{m+i}(x_n) + \lambda_i'(x_n)$ for $i = n-1, \ldots, 3$. ∎

As in Lemma 1, in the next result we give a closed-form formula for the *solutions* of the feedback system from Theorem 7, which will allow us, in Section VIII, to extend the constructive methodology to a class of strict-feedforward systems that are *not linearizable*.

Lemma 3: Starting from the initial condition x, the solution of the feedback system (52)–(58), (61) at time τ is shown in (77) at the bottom of the next page, where $i = 1, \ldots, n$, and the control signal is

$$u = \tilde{\alpha}_1(\tau, x) = -e^{-\tau} \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i}$$
$$\times \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k}$$
$$\times \left(x_l - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n)x_m \right). \tag{78}$$

Proof: Analogous to the proof of Lemma 1, employing also Lemma 2. ∎

$$\beta_{i+1}(\underline{x}_{i+1}) = -\sum_{m=i+1}^{n} x_m \left[\binom{n-i}{m-i} - \sum_{j=i}^{m} \binom{n-i}{j-i} \mu_{j+1+n-m}(x_n) \right], \qquad i = 1, \ldots, n-1. \tag{64}$$

VII. Type-I and -II Systems in Dimensions Two and Three

We start by pointing out that in dimension two all strict-feedforward systems are simultaneously of Types I and II. This implies that all second order strict-feedforward systems are linearizable.

Theorem 8: Consider the system

$$\dot{x}_1 = x_2 + \phi_1(x_2)u \tag{79}$$
$$\dot{x}_2 = u \tag{80}$$

where $\phi_1(x_1)$ is continuous and $\phi_1(0) = 0$. The control law

$$u = -x_1 - 2x_2 + \int_0^{x_2} \phi_1(s)ds \tag{81}$$

ensures global asymptotic stability of the origin.

Proof: By verification that $\dot{z}_1 = x_2 + u, \dot{z}_2 = u$, where $z_1 = x_1 - \beta_2(x_2)$, $\beta_2(x_2) = -x_2 + \int_0^{x_2} \phi_1(s)ds$, and $u = -z_1 - x_2$. ∎

Example 2: Let us now consider an example with $\phi_1(x_2) = -x_2^2$. This example was worked out in [36]. In this case the formula (81) gives[7]

$$u = -x_1 - 2x_2 - \frac{x_2^3}{3}. \tag{82}$$

One should recognize that the "$-x_1 - 2x_2$" portion of the control law (82) is responsible for exponential stabilization of the linearized system. To see that this linear controller is not sufficient for global stabilization, we plug it back into the plant and obtain a closed-loop system, written in the form of a second-order equation, as

$$\ddot{x}_2 + (2 - x_2^2)\dot{x}_2 + x_2 = 0. \tag{83}$$

This is a Van der Pol equation with an unstable limit cycle, which exhibits a finite escape instability. Hence, the nonlinear term "$-x_2^3/3$," designed to accommodate the input nonlinearity $\phi_1(x_2) = -x_2^2$, is crucial for global stabilization. ◇

The possibilities, as well as the limits, of Type I/II linearizability for strict-feedforward systems are best understood in dimension three. For the following class of systems, which represents a union of all Type I and Type II feedforward systems in dimension three, a linearizing coordinate change and a stabilizing control law are designed in the next theorem.

[7]A reader checking back the details in [36] will notice that this control law differs from [36, eq. (6.2.12)]. This is due to an extra "x_3^2" term that has crept into the calculations in [36, eq. (6.2.7)].

Theorem 9: Consider the class of systems

$$\dot{x}_1 = x_2 + \pi_2(x_2)x_3$$
$$+ \left(\frac{x_3\phi_2(x_3) - \int_0^{x_3} \phi_2(s)ds}{x_3^2} x_2 + \pi_3(x_3) \right) u \tag{84}$$
$$\dot{x}_2 = x_3 + \phi_2(x_3)u \tag{85}$$
$$\dot{x}_3 = u \tag{86}$$

where $\pi_2(\cdot), \pi_3(\cdot) \in C^0$ and $\phi_2(\cdot) \in C^1$ are vanishing at the origin and

$$\pi_2(x_2)\phi_2(x_3) \equiv 0. \tag{87}$$

Then, the control law $u = -y_1 - 3y_2 - 3y_3$, where

$$y_1 = x_1 - \int_0^{x_2} \pi_2(s)ds - \mu_3(x_3)x_2 - \int_0^{x_3} \pi_3(s)ds$$
$$+ \frac{1}{2}x_3 (\mu_3(x_3))^2 + \frac{1}{2}\int_0^{x_3} (\mu_3(s))^2 ds \tag{88}$$
$$y_2 = x_2 - \int_0^{x_3} \phi_2(s)ds \tag{89}$$
$$y_3 = x_3 \tag{90}$$

and $\mu_3(x_3) = \int_0^{x_3} \phi_2(s)ds/x_3$, achieves global asymptotic stability of the origin.

Proof: One can verify that $\dddot{y}_1 + 3\ddot{y}_1 + 3\dot{y}_1 + y_1 = 0$. ∎

A Type II example of a system from this class is

$$\dot{x}_1 = x_2 + \left(\frac{1}{2}x_2 + x_3 \sin x_3\right)u \quad \dot{x}_2 = x_3 + x_3 u \quad \dot{x}_3 = u \tag{91}$$

which is stabilized (and feedback linearized) using

$$u = -x_1 - 3x_2 - 3x_3 + \frac{x_2 x_3}{2} + \frac{3}{2}x_3^2 - \frac{1}{6}x_3^3$$
$$+ x_3 \sin x_3 + \cos x_3 - 1. \tag{92}$$

We point out that the key restriction in this example is the bold-faced 1/2. If this value were anything else (say, 1 or 0), this system would not be linearizable. It would, however, be stabilizable using the procedure we present in Section VIII.

The focus on third-order systems is partly motivated by the fact that the celebrated "benchmark problem"

$$\dot{x}_1 = x_2 + x_3^2 \tag{93}$$
$$\dot{x}_2 = x_3 \tag{94}$$
$$\dot{x}_3 = u \tag{95}$$

$$\xi_i(\tau, x) = e^{-\tau} \left[\sum_{j=i}^{n} \binom{n-i}{j-i}(-1)^{j-i} \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k}\left(x_l - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n)x_m \right) \right.$$
$$+ \sum_{p=i+1}^{n} \lambda_{i+1+n-p} \left(e^{-\tau} \sum_{k=0}^{n-1} \frac{(-\tau)^k}{k!} \sum_{l=n-k}^{n} \binom{k}{l-n+k}\left(x_l - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n)x_m \right) \right)$$
$$\left. \times \sum_{j=p}^{n} \binom{n-p}{j-p}(-1)^{j-p} \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} \times \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k}\left(x_l - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n)x_m \right) \right] \tag{77}$$

first solved by Teel [41] using his method of nested saturations, is of third order. The system (93)–(95) is not feedback linearizable. However, the following similar (at least visually) systems, are linearizable. The system

$$\dot{x}_1 = x_2 + x_3^2 u \quad \dot{x}_2 = x_3 \quad \dot{x}_3 = u \qquad (96)$$

is linearizable, as it is of both Type I and II. The system

$$\dot{x}_1 = x_2 + x_2^2 x_3 \quad \dot{x}_2 = x_3 \quad \dot{x}_3 = u \qquad (97)$$

is of Type I, and therefore linearizable. Other such systems exist, outside of Types I or II, that are linearizable. For example

$$\dot{x}_1 = x_2 + x_3^2 + x_2 u \quad \dot{x}_2 = x_3 \quad \dot{x}_3 = u \qquad (98)$$

(which is temptingly close in appearance to Type I but is not in that class), is linearizable using the coordinate change

$$y_1 = x_1 - x_2 x_3 \quad y_2 = x_2 \quad y_3 = x_3. \qquad (99)$$

The previous examples all had the last two equations actually linear. The neither-Type-I-nor-II feedforward system

$$\dot{x}_1 = x_2 + x_2^2 x_3 + \frac{x_3^3}{3} - x_2^2 x_3^2 u \quad \dot{x}_2 = x_3 - x_3^2 u \quad \dot{x}_3 = u \qquad (100)$$

which includes nonlinearities in both of the first two equations, is linearizable using

$$y_1 = x_1 - \frac{x_2^3}{3} \quad y_2 = x_2 + \frac{x_3^3}{3} \quad y_3 = x_3. \qquad (101)$$

Clearly, since (98) and (100) are neither of Type I nor II, the coordinate changes (99) and (101) cannot be obtained from the explicit formulae in Sections V and VI. However, they can be obtained following the simplified SJK procedure in Section IV, which, we remind the reader, avoids the requirement to solve the nonlinear ODEs (7).

VIII. ALGORITHMS FOR NONLINEARIZABLE FEEDFORWARD SYSTEMS

In this section, we expand upon the Type I and II feedforward systems, to develop algorithms for feedforward systems that are not linearizable. Two classes of systems that we consider consist of a linearizable subsystem $[x_1, \ldots, x_n]^T$ and a scalar equation x_0 that is (possibly) not linearizable. This structure belongs to the class of nonflat *Liouvillian systems* of defect equal to one, see [4] (especially Example 2).

Consider the following extension of the Type I strict-feedforward systems:

$$\dot{x}_0 = x_1 + \psi_0(x) + \phi_0(x)u \qquad (102)$$

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u \qquad (103)$$

$$\dot{x}_i = x_{i+1}, \qquad i = 2, \ldots, n-1 \qquad (104)$$

$$\dot{x}_n = u \qquad (105)$$

where x denotes $[x_1, \ldots, x_n]^T$ (i.e., x_0 is not included in x), $\psi_0(0) = \phi_0(0) = \pi_j(0) = 0, j = 2, \ldots, n$, and $\partial\psi_0(0)/\partial x_i = 0, i = 1, \ldots, n$. Subsystem (103)–(105) is linearizable. This makes it possible to develop a closed-form formula for a globally stabilizing SJK-type control law.

We propose the following design algorithm. Start by computing the expressions in Lemma 1. Then, calculate

$$\beta_1(x) = -\int_0^\infty [\xi_1(\tau, x) + \psi_0(\xi(\tau, x)) + \phi_0(\xi(\tau, x))\tilde{\alpha}_1(\tau, x)]d\tau \qquad (106)$$

$$w_0(x) = \phi_0(x) - \frac{\partial\beta_1(x)}{\partial x_1}\pi_n(x_n) - \frac{\partial\beta_1(x)}{\partial x_n} \qquad (107)$$

and

$$u = \alpha_0(x_0, x) = -w_0(x)(x_0 - \beta_1(x)) - \sum_{i=1}^n \binom{n}{i-1}x_i + \sum_{i=2}^n \int_0^{x_i}\pi_i(s)ds. \qquad (108)$$

Theorem 10: The feedback system (102)–(105), (108) is globally asymptotically stable at the origin.

Proof: Lengthy calculations verify that $(d/dt)\sum_{i=0}^n z_i^2 = -w_0^2 z_0^2 - \sum_{i=1}^n z_i^2 - (w_0 z_0 + \sum_{i=1}^n z_i)^2$, where $w_0(0) = 1$ and $z_0 = x_0 - \beta_1, z_i = \sum_{j=i}^n \binom{n-i}{j-i}x_j - \delta_{i,1}\sum_{j=2}^n \int_0^{x_j}\pi_j(s)ds$ for $i = 1, \ldots, n$. ∎

Next, consider the following extension of the Type II strict-feedforward systems:

$$\dot{x}_0 = x_1 + \psi_0(x) + \phi_0(x)u \qquad (109)$$

$$\dot{x}_1 = x_2 + \phi_i(\underline{x}_{i+1})u, \qquad i = 1, \ldots, n-2 \qquad (110)$$

$$\dot{x}_{n-1} = x_n + \phi_{n-1}(x_n)u \qquad (111)$$

$$\dot{x}_n = u \qquad (112)$$

where the ϕ_i's satisfy the conditions of Theorem 7.

We propose the following design algorithm. Start by computing the expressions in Theorem 3. Then, calculate

$$\beta_1(x) = -\int_0^\infty [\xi_1(\tau, x) + \psi_0(\xi(\tau, x)) + \phi_0(\xi(\tau, x))\tilde{\alpha}_1(\tau, x)]d\tau \qquad (113)$$

$$w_0(x) = \phi_0(x) - \sum_{i=1}^{n-1}\frac{\partial\beta_1(x)}{\partial x_i}\phi_i(\underline{x}_{i+1}) - \frac{\partial\beta_1(x)}{\partial x_n} \qquad (114)$$

and

$$u = \alpha_0(x_0, x) = -w_0(x)(x_0 - \beta_1(x)) - x_1 - \sum_{m=2}^n x_m\left[\binom{n}{m-1} - \sum_{j=1}^m \binom{n}{j-1}\mu_{j+1+n-m}(x_n)\right]. \qquad (115)$$

Theorem 11: The feedback system (109)–(112), (115) is globally asymptotically stable at the origin.

Proof: The same as the proof of Theorem 10, except that $z_i = x_i + \sum_{m=i+1}^n x_m\left[\binom{n-i}{m-i} - \sum_{j=i}^m \binom{n-i}{j-i}\mu_{j+1+n-m}(x_n)\right]$ for $i = 1, \ldots, n-1$. ∎

IX. THIRD-ORDER EXAMPLE (NOT FEEDBACK LINEARIZABLE)

To illustrate the construction in Section VIII, consider the following example:

$$\dot{x}_1 = x_2 + x_3^2 \tag{116}$$
$$\dot{x}_2 = x_3 + x_3 u \tag{117}$$
$$\dot{x}_3 = u. \tag{118}$$

The second-order (x_2, x_3) subsystem is linearizable and is of both Type I and Type II. Like the "benchmark problem" (93)–(95), the overall system (116)–(118) is not feedback linearizable.

While the benchmark system (93)–(95) requires only two steps of forwarding because the (x_2, x_3) subsystem is linear, the system (116)–(118) requires three steps. The first two steps are already precomputed in Lemma 1

$$\xi_3 = \left(x_3 - \tau \left(x_2 + x_3 - \frac{x_3^2}{2} \right) \right) e^{-\tau} \tag{119}$$

$$\xi_2 = \left((1 + \tau) \left(x_2 + x_3 - \frac{x_3^2}{2} \right) - x_3 \right) e^{-\tau}$$
$$+ \frac{1}{2} \left(x_3 - \tau \left(x_2 + x_3 - \frac{x_3^2}{2} \right) \right)^2 e^{-2\tau} \tag{120}$$

and $\tilde{\alpha}_2 = -\xi_2 - \xi_3 + \xi_3^2/2$. The third step of forwarding is about calculating (106), $\beta_2 = -2x_2 - x_3 + (5/8)x_3^2 - (3/8)\left(x_2 - x_3^2/2\right)^2$, (107), $w_1 = 1 + (3/4)x_3$, and the final control law $u = -w_1(x_1 - \beta_2) - \left(x_2 + x_3 - x_3^2/2\right) - x_3$, i.e.,

$$u = -x_1 - 3x_2 - 3x_3 - \frac{3}{8}x_2^2 + \frac{3}{4}x_3 \left(-x_1 - 2x_2 + \frac{1}{2}x_3 \right.$$
$$\left. + \frac{x_2 x_3}{2} + \frac{5}{8}x_3^2 - \frac{1}{4}x_3^3 - \frac{3}{8} \left(x_2 - \frac{x_3^2}{2} \right)^2 \right). \tag{121}$$

In the remainder of this section, we show that the restriction (2) can be lifted in some cases. Consider the example

$$\dot{x}_1 = x_2 + x_3^2 \quad \dot{x}_2 = \sinh x_3 + x_3 u \quad \dot{x}_3 = u \tag{122}$$

which, although only a slight variation from (116)–(118), is not represented in the class (102)–(105). The difference in the second equation of (122) is easily accommodated by the coordinate/prefeedback change $X_3 = \sinh x_3$, $v = \sqrt{1 + (\sinh x_3)^2}u$, which converts (122) into

$$\dot{x}_1 = x_2 + \left(\sinh^{-1}(X_3) \right)^2 \quad \dot{x}_2 = X_3 + \frac{\sinh^{-1}(X_3)}{\sqrt{1 + X_3^2}} v$$
$$\dot{X}_3 = v. \tag{123}$$

This system fits the forms in Section VIII.

However, the system

$$\dot{x}_i = \sin(x_{i+1}), \qquad i = 1, \ldots, n - 1 \tag{124}$$
$$\dot{x}_n = u \tag{125}$$

suggested to us by Teel, (very) remotely motivated by the ball-and-beam problem [41], cannot be brought into those forms, except in the case $n = 2$ where the resulting control law is

$$u = -x_2 - \frac{\sin x_2}{x_2} \left(x_1 - \int_0^{x_2} \frac{\sin \xi}{\xi} d\xi \right). \tag{126}$$

X. BLOCK-FORWARDING

In this section, we extend the class of systems to which the SJK forwarding procedure is applicable. Then we present our explicit controller formulas for this class of systems.

Consider the class of *block-strict-feedforward* systems in (130)–(131). The blocks considered here are less general than those in [43], [26], and [11]. We can generalize the idea we are presenting (even somewhat beyond the classes considered [43], [26], [11]), to include blocks q_i that are merely input-to-state stable with respect to $\left(\underline{x}_i, \underline{q}_{i+1} \right)$, rather than being linear in q_i. A simple example is the system

$$\dot{q} = -q^3 + x_2 \tag{127}$$
$$\dot{x}_1 = x_2 + qu \tag{128}$$
$$\dot{x}_2 = u. \tag{129}$$

This generalization would, however, preclude closed-form solvability of the problem; the result would be only an extension of [37].

$$\dot{x}_i = x_{i+1} + \psi_i \left(\underline{x}_{i+1}, \underline{q}_{i+1} \right) + \phi_i \left(\underline{x}_{i+1}, \underline{q}_{i+1} \right) u \tag{130}$$
$$\dot{q}_i = A_i q_i + \omega_i \left(\underline{x}_i, \underline{q}_{i+1} \right) \tag{131}$$

where $i = 1, 2, \ldots, n$, each x_i is scalar valued, each q_i is r_i-vector valued, $\underline{x}_i = [x_i, x_{i+1}, \ldots, x_n]^T$, $\underline{q}_i = [q_i^T, q_{i+1}^T, \ldots, q_n^T]^T$, A_i is a Hurwitz matrix for all $i = 1, 2, \ldots, n$, $x_{n+1} = u, q_{n+1} = 0, \phi_n = 0$, and $\partial\psi_i(0)/\partial x_j = \phi_i(0) = \omega_i(0) = 0$ for $i = 1, 2, \ldots, n - 1, j = i + 1, \ldots, n$. This class of systems should be understood as a dual of the *block-strict-feedback* systems in [17, Sec. 4.5.2].

The control law for this class of systems is designed as follows. Let $\beta_{n+1} = \alpha_{n+1} = 0$. For $i = n, n - 1, \ldots, 2, 1$, see (132)–(135) as shown at the bottom of the page, where the no-

$$z_i = x_i - \beta_{i+1} \tag{132}$$

$$w_i(\underline{x}_{i+1}, \underline{q}_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial\beta_{i+1}}{\partial x_j} \phi_j - \frac{\partial\beta_{i+1}}{\partial x_n} \tag{133}$$

$$\alpha_i(\underline{x}_i, \underline{q}_{i+1}) = \alpha_{i+1} - w_i z_i \tag{134}$$

$$\beta_i(\underline{x}_i, \underline{q}_i) = -\int_0^\infty \left[\xi_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i) + \psi_{i-1} \left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i), \underline{\eta}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i) \right) \right.$$
$$\left. + \phi_{i-1} \left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i), \underline{\eta}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i) \right) \alpha_i \left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i) \underline{\eta}_{i+1}^{[i]}(\tau, \underline{x}_i, \underline{q}_i) \right) \right] d\tau \tag{135}$$

tation in the integrand of (135) refers to the solutions of the (sub)system(s)

$$\frac{d}{d\tau}\xi_j^{[i]} = \xi_{j+1}^{[i]} + \psi_j\left(\underline{\xi}_{j+1}^{[i]}, \underline{\eta}_{j+1}^{[i]}\right)$$
$$+ \phi_j\left(\underline{\xi}_{j+1}^{[i]}, \underline{\eta}_{j+1}^{[i]}\right)\alpha_i\left(\underline{\xi}_i^{[i]}, \underline{\eta}_{i+1}^{[i]}\right) \quad (136)$$

$$\frac{d}{d\tau}\eta_j^{[i]} = A_j\eta_j^{[i]} + \omega_j\left(\underline{\xi}_j^{[i]}, \underline{\eta}_{j+1}^{[i]}\right) \quad (137)$$

for $j = i - 1, i, \ldots, n$, at time τ, starting from the initial condition $(\underline{x}_i, \underline{q}_i)$. The control law is

$$u = \alpha_1. \quad (138)$$

Theorem 12: The feedback system (130), (131), (138) is globally asymptotically stable at the origin.

Proof: As in the proof of Theorem 1, the Lyapunov function $V = (1/2)\sum_{i=1}^{n} z_i^2$, has a negative–definite derivative

$$\dot{V} = -\frac{1}{2}\sum_{i=1}^{n} w_i^2 z_i^2 - \frac{1}{2}\left(\sum_{i=2}^{n} z_i w_i\right)^2. \quad (139)$$

This implies that $x_n(t)$ converges to zero. Since $\omega_n(0) = 0$, we have that $\omega_n(x_n(t))$ converges to zero. Because A_n is Hurwitz, $q_n(t)$ converges to zero. One can show recursively that $w_i(0) = 1$ and $\beta_i(0) = 0$. It then follows that $w_{n-1}(x_n(t), q_n(t))$ converges to one. Since (139) guarantees that $w_{n-1}z_{n-1}$ goes to zero, $z_{n-1}(t)$ also goes to zero. Hence, $x_{n-1}(t) = z_{n-1}(t) + \beta_n(x_n(t), q_n(t))$ converges to zero. Continuing in the same fashion, one shows that $x(t), q(t) \to 0$ as $t \to \infty$. This establishes that the equilibrium $x = 0, q = 0$ is (uniformly) attractive. Global stability is argued in a similar, recursive fashion, using (139) and the fact that the subsystems (131) are input-to-state stable. In conclusion, the origin is globally asymptotically stable. ∎

As in Section II, the solution $\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i), \underline{\eta}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i)\right)$, needed in the integral (135), is impossible to obtain analytically in general. For this reason, we consider two classes of block-feedforward systems, inspired by feedforward systems of Types I and II, for which a closed-form controller can be obtained.

Consider the class of systems we refer to as *Type I block-feedforward* systems:

$$\dot{x}_0 = x_1 + \psi_0(x, q) + \phi_0(x, q)u \quad (140)$$
$$\dot{q}_0 = A_0 q_0 + \omega_0(x_0, x, q) \quad (141)$$
$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1}\pi_j(x_j)x_{j+1} + \pi_n(x_n)u \quad (142)$$
$$\dot{q}_1 = A_1 q_1 + \omega_1\left(x, \underline{q}_2\right) \quad (143)$$
$$\dot{x}_i = x_{i+1}, \qquad i = 2, \ldots, n-1 \quad (144)$$
$$\dot{q}_i = A_i q_i + \omega_i\left(\underline{x}_i, \underline{q}_{i+1}\right) \quad (145)$$
$$\dot{x}_n = u \quad (146)$$
$$\dot{q}_n = A_n q_n + \omega_n(x_n) \quad (147)$$

where x denotes $[x_1, \ldots, x_n]^T$, q denotes $[q_1^T, \ldots, q_n^T]^T$ (i.e., it does not include q_0), $\psi_0(0) = \phi_0(0) = \omega_0(0) = \omega_1(0) = \omega_j(0) =$

$\pi_j(0) = 0$, $j = 2, \ldots, n$, and $\partial\psi_0(0)/\partial x_i = 0, i = 1, \ldots, n$. The subsystem $(x_1, \ldots, x_n)$ is linearizable, which makes it possible to develop a closed-form formula. The first step in the design algorithm is to compute the expressions in Lemma 1. It is worth noting that $\xi(\tau, x)$ and $\tilde{\alpha}_1(\tau, x)$ are both independent of q. Then, for $i = n, n-1, \ldots, 2$, we calculate

$$\eta_i\left(\tau, \underline{q}_i, x\right) = e^{A_i\tau}q_i$$
$$+ \int_0^\tau e^{A_i(\tau-\sigma)}\omega_i\left(\underline{\xi}_i(\sigma, x), \underline{\eta}_{i+1}\left(\sigma, \underline{q}_{i+1}, x\right)\right)d\sigma \quad (148)$$

followed by (149)–(150), shown at the bottom of the page, and

$$u = \alpha_0(x_0, x, q) = -w_0(x, q)(x_0 - \beta_1(x, q))$$
$$- \sum_{i=1}^{n}\binom{n}{i-1}x_i + \sum_{i=2}^{n}\int_0^{x_i}\pi_i(s)ds. \quad (151)$$

Theorem 13: The feedback system (140)–(147), (151) is globally asymptotically stable at the origin.

Proof: Lengthy calculations verify that the same expressions hold as in the proof of Theorem 10. In the present proof, however, z_0 depends not only on x_0, x but also on $q_n, q_{n-1}, \ldots, q_1$. Thus, convergence to the origin is proved in the following order: $x_n, x_{n-1}, \ldots, x_1, q_n, q_{n-1}, \ldots, q_1, x_0, q_0$. Global stability is argued similarly. Hence, the equilibrium $x_0 = q_0 = 0, x = 0, q = 0$ is globally asymptotically stable. ∎

Finally, consider the class of systems we refer to as *Type II block-feedforward* systems

$$\dot{x}_0 = x_1 + \psi_0(x, q) + \phi_0(x, q)u \quad (152)$$
$$\dot{q}_0 = A_0 q_0 + \omega_0(x_0, x, q) \quad (153)$$
$$\dot{x}_i = x_{i+1} + \phi_i(x_{i+1})u, \qquad i = 1, \ldots, n-1 \quad (154)$$
$$\dot{q}_i = A_i q_i + \omega_i\left(\underline{x}_i, \underline{q}_{i+1}\right) \quad (155)$$
$$\dot{x}_n = u \quad (156)$$
$$\dot{q}_n = A_n q_n + \omega_n(x_n) \quad (157)$$

where the ϕ_i's satisfy the conditions of Theorem 7. With $\xi(\tau, x)$ and $\tilde{\alpha}_1(\tau, x)$ calculated as in Theorem 3, and η_i's and β_1 calculated as in (148), (149), respectively, the algorithm's final step is to calculate

$$w_0(x, q) = \phi_0(x) - \sum_{i=1}^{n-1}\frac{\partial\beta_1(x, q)}{\partial x_i}\phi_i(\underline{x}_{i+1}) - \frac{\partial\beta_1(x, q)}{\partial x_n} \quad (158)$$

and

$$u = \alpha_0(x_0, x, q)$$
$$= -w_0(x, q)(x_0 - \beta_1(x, q)) - x_1 - \sum_{m=2}^{n}x_m\left[\binom{n}{m-1}\right.$$
$$\left. - \sum_{j=1}^{m}\binom{n}{j-1}\mu_{j+1+n-m}(x_n)\right]. \quad (159)$$

Theorem 14: The feedback system (152)–(157), (159) is globally asymptotically stable at the origin.

Proof: Analogous to the proof of Theorem 13. ∎

$$\beta_1(x, q) = -\int_0^\infty\left[\xi_1(\tau, x) + \psi_0\left(\xi(\tau, x), \eta(\tau, q, x)\right) + \phi_0\left(\xi(\tau, x), \eta(\tau, q, x)\right)\tilde{\alpha}_1(\tau, x)\right]d\tau \quad (149)$$

$$w_0(x, q) = \phi_0(x) - \frac{\partial\beta_1(x, q)}{\partial x_1}\pi_n(x_n) - \frac{\partial\beta_1(x, q)}{\partial x_n} \quad (150)$$

XI. INTERLACED FEEDFORWARD-FEEDBACK SYSTEMS

The ability to stabilize systems that are neither in the strict-feedback nor in the strict-feedforward form was nicely illustrated in [36]. In this section, we present designs for two classes of systems obtained by interlacing strict-feedback systems [17] with feedforward systems of Type I and II.

First, consider the class of *interlaced systems of Type I*

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u \tag{160}$$

$$\dot{x}_i = x_{i+1}, \qquad i = 2, \ldots, n \tag{161}$$

$$\dot{x}_{n+1} = x_{n+2} + f_1(\underline{x}_1, x_{n+1}) \tag{162}$$

$$\dot{x}_{n+j} = x_{n+j+1} + f_j(\underline{x}_1, \bar{x}_{n+j}), \qquad j = 2, \ldots, N \tag{163}$$

where $x_{n+N+1} = u$. In this system, $\bar{x}_{n+j}$ denotes $[x_{n+1}, \ldots, x_{n+j}]^T$, and, as before, $\underline{x}_j$ denotes $[x_j, x_{j+1}, \ldots, x_n]^T$ (which means, in particular, that $\underline{x}_1 = [x_1, \ldots, x_n]^T$). It is clear from the aforementioned notation that the overall system order is $n + N$, where the feedforward part (top) is of order n and the feedback part (bottom) is of order N. We assume that $\pi_i(0) = 0, i = 2, \ldots, n$ and $f_i(0) = 0, i = 1, \ldots, N$. The control synthesis for this system is given in the following theorem.

Theorem 15: The control law given by

$$z_i = x_i + \sum_{j=i+1}^{n} \binom{n-i}{j-i} x_j$$

$$- \delta_{i,1} \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s)ds \tag{164}$$

$$\alpha_1(\underline{z}_1) = -\sum_{i=1}^{n} z_i \tag{165}$$

for $i = 1, \ldots, n$

$$z_{n+1} = x_{n+1} - \alpha_1 \tag{166}$$

$$\alpha_{n+1}(\underline{z}_1, z_{n+1}) = -(n+1)z_{n+1}$$
$$+ \sum_{l=1}^{n}(n-l)z_l - f_1(\underline{x}_1, x_{n+1}) \tag{167}$$

$$z_{n+j} = x_{n+j} - \alpha_{n+j-1}(\underline{z}_1, z_{n+j-1}) \tag{168}$$

$$\alpha_{n+j} = -z_{n+j-1} - z_{n+j} - f_j(\underline{x}_1, \bar{x}_{n+j})$$
$$+ \sum_{l=1}^{n} \frac{\partial \alpha_{n+j-1}}{\partial z_l}\left(-\sum_{k=1}^{i} z_k + z_{n+1}\right)$$
$$+ \frac{\partial \alpha_{n+j-1}}{\partial z_{n+1}}\left(-\sum_{k=1}^{n+1} z_k + z_{n+2}\right)$$
$$+ \sum_{l=2}^{j-1} \frac{\partial \alpha_{n+j-1}}{\partial z_{n+l}}(-z_{n+l} + z_{n+l+1}) \tag{169}$$

for $j = 2, \ldots, n$, and

$$u = \alpha_{n+N} \tag{170}$$

globally asymptotically stabilizes the system (160)–(163) at the origin.

Proof: It can be verified that the closed-loop system in the z coordinates is

$$\dot{z}_i = -\sum_{k=1}^{i} z_k + z_{n+1}, \qquad i = 1, \ldots, n \tag{171}$$

$$\dot{z}_{n+1} = -\sum_{k=1}^{n} z_k - z_{n+1} + z_{n+2} \tag{172}$$

$$\dot{z}_{n+j} = -z_{n+j} + z_{n+j+1}, \qquad j = 2, \ldots, N \tag{173}$$

where $z_{n+N+1} = 0$. The Lyapunov function $V = \sum_{i=1}^{n+N} z_i^2$ satisfies $\dot{V} = -\sum_{i=1}^{n+N} z_i^2 - \sum_{i=n+2}^{n+N}(z_i - z_{i-1})^2 - \left(\sum_{i=1}^{n} z_i\right)^2$ which proves the result. ∎

Next, consider the class of *interlaced systems of Type II*

$$\dot{x}_1 = x_2 + \phi_i(\underline{x}_{i+1})u, \qquad i = 1, \ldots, n-2 \tag{174}$$

$$\dot{x}_{n-1} = x_n + \phi_{n-1}(x_n)u \tag{175}$$

$$\dot{x}_n = x_{n+1} \tag{176}$$

$$\dot{x}_{n+1} = x_{n+2} + f_1(\underline{x}_1, x_{n+1}) \tag{177}$$

$$\dot{x}_{n+j} = x_{n+j+1} + f_j(\underline{x}_1, \bar{x}_{n+j}), \qquad j = 2, \ldots, N \tag{178}$$

where $x_{n+N+1} = u$. We assume that $\phi_i(0) = f_j(0) =$ and the ϕ_i's satisfy the conditions of Theorem 7.

Theorem 16: The control law given by (179)–(180), as shown at the bottom of the page, and (165)–(170) globally asymptotically stabilizes the system (174)–(178) at the origin.

Proof: The same as Theorem 15. ∎

Since the interlaced systems of both Types I and II are feedback linearizable, one does not have to necessarily commit to the integrator forwarding plus integrator backstepping design procedure. It suffices to define an output with respect to which one has a relative degree equal to the order of the system, with which one can pursue full-state feedback linearization by conversion to the Brunovsky canonical form. This is spelled out in the next theorem.

Theorem 17: Systems (160)–(163) and (174)–(178) are of relative degree $n + N$ from u to the respective outputs $y_1 = x_1 - \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s)ds$ and $y_1 = x_1 - \sum_{j=2}^{n} \mu_{2+n-j}(x_n)x_j$.

XII. EXAMPLE: COMBINING BLOCK-BACKSTEPPING AND BLOCK-FORWARDING

In this section, we show that *block*-backstepping and *block*-forwarding can be combined in a similar manner on an example that is outside of the forms considered in Section XI (and also outside of those in [36])

$$\dot{q} = -2q + x_2^2 \tag{181}$$

$$\dot{x}_1 = x_2 + qx_3 \tag{182}$$

$$z_i = x_i + \sum_{m=i+1}^{n} x_m\left[\binom{n-i}{m-i} - \sum_{j=i}^{m}\binom{n-i}{j-i} \times \mu_{j+1+n-m}(x_n)\right], \qquad i = 1, \ldots, n-1 \tag{179}$$

$$z_n = x_n \tag{180}$$

$$\dot{x}_2 = x_3 + q \tag{183}$$

$$\dot{x}_3 = u + qx_1. \tag{184}$$

This system is neither in the block-strict-feedforward form (because of qx_1 in the x_3-equation) nor in the block-strict-feedback form (because of qx_3 in the x_1-equation). However, the x_1, x_2, q-subsystem is block-strict-feedforward if one views x_3 as control, and the x_2, x_3, q-subsystem is block-strict-feedback with u as control. Hence, we will derive a controller for this system using one step of forwarding, followed by one step of backstepping.

Following the design from Section X, we first calculate $\xi_2^{[2]}(\tau, x_2) = x_2 e^{-\tau}$ and $\eta^{[2]}(\tau, x_2, q) = (q + \tau x_2^2) e^{-\tau}$ Then, we derive

$$\beta_2(x_2, q) = -x_2 + \frac{qx_2}{3} + \frac{qx_2^2}{8} + \frac{q^2}{4} + \frac{x_2^3}{9} + \frac{x_2^4}{32} \tag{185}$$

$$w_1(x_2, q) = 1 + \frac{2}{3}q - \frac{qx_2}{4} - \frac{x_2^2}{3} - \frac{x_2^3}{8}. \tag{186}$$

The system is converted from the x_1, x_2, x_3 coordinates into z_1, x_2, z_3 (note that x_2 is unaltered), where

$$z_1 = x_1 - \beta_1 \tag{187}$$

$$z_3 = x_3 + q + w_1 z_1 + x_2. \tag{188}$$

Note that (187) corresponds to one step of forwarding, resulting in a 'virtual control' $-q - w_1 z_1 - x_2$ for x_3 as a control input, whereas (188) corresponds to one step of backstepping. The control law

$$\begin{aligned} u = & -z_3 - x_2 - w_1 z_1 - x_1 q + 2q - x_2^2 - w_1^2 (x_3 + q + x_2) \\ & - (x_3 + q) + z_1 \left[\left(\frac{x_2}{4} - \frac{2}{3} \right) \left(-2q + x_2^2 \right) \right. \\ & \left. + \left(\frac{q}{4_{2/3}} x_2 + \frac{3}{8} x_2^2 \right) (x_3 + q) \right] \end{aligned} \tag{189}$$

results in the system being transformed into

$$\dot{z}_1 = -w_1^2 z_1 + w_1 z_3 \tag{190}$$

$$\dot{x}_2 = -w_1 z_1 - x_2 + z_3 \tag{191}$$

$$\dot{z}_3 = -w_1 z_1 - x_2 - z_3. \tag{192}$$

The stability of this system follows from the Lyapunov function $V(x, q) = z_1(x_1, x_2, q)^2 + x_2^2 + z_3(x_1, x_2, x_3, q)^2$ because

$$\dot{V} = -w_1^2 z_1^2 - x_2^2 - (w_1 z_1 + x_2)^2 - 2z_3^2. \tag{193}$$

The convergence to zero can be seen in the following order: x_2 [from (193)], q [from (181)], x_1 [from (187) and (185)], x_3 [from (188)].

XIII. PERFORMANCE

The general performance advantages of the SJK-type integrator forwarding were thoroughly illuminated in [36, Sec. 6.2.6]. It was shown there that "overly cautious" nested saturation designs, whose form is in many cases the same irrespective of the sign of the plant nonlinearities, don't perform as well as Lyapunov-based designs. In these, and various other simulations presented in the literature, the nested saturation (and other

bounded) controllers display the trademark linear (nonexponential) decay resulting from saturating the control. Saturation itself leads to large overall control effort (at least in $\mathcal{L}_2$) by letting the states linger at large values for extended periods of time. On the other hand, there is an inherent engineering merit in having control laws that are robust to actuator saturation (by means of "caution"), which the nested saturation controllers are.

The Lyapunov-function equipped SJK algorithm not only shows good performance in simulations, this performance can be quantified. This is already implicit in the inverse optimality result in [37] for $u^* = 2\alpha_1$, but is actually true even for $u = \alpha_1$.

Theorem 18: The control effort for the feedback system (1), (8) satisfies the following bound:

$$\|u\|_{\mathcal{L}_2} \le \sqrt{\sum_{j=1}^{n} \left(x_j(0) - \beta_{j+1}(\underline{x}_{j+1}(0)) \right)^2}. \tag{194}$$

Proof: By rewriting (10) as $\dot{V} = -(1/2)\sum_{i=1}^{n} w_i^2 z_i^2 - (1/2)u^2$ with the help of (9), and by integrating it in time from 0 to ∞. ∎

An additional desirable property arises from linearizability—that the performance and control effort can be quantified in terms of the original problem data (in terms of the plant vector fields). In the case of Type I and II systems, we have closed-form solutions for the state and control which allow such quantification. In the next two theorems, which are proved using (32), the identity

$$\tau^p e^{-q\tau} \le \left(\frac{p}{qe} \right)^p \qquad \forall p, q, \tau \ge 0 \tag{195}$$

and, respectively, Lemmas 1 and 3, we calculate explicit $\mathcal{L}^1$ and $\mathcal{L}^\infty$ bounds on the control effort in stabilizing feedforward systems of Types I and II.

Theorem 19: The control law (44) applied to the plant (39)–(41), (42), (43) expends the control effort in the amount bounded by

$$\begin{aligned} \|u\|_{\mathcal{L}^1} \le & \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i} \\ & \times \left[\sum_{k=0}^{j-1} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} |x_l(0)| \right. \\ & \left. + \left| \sum_{m=2}^{n} \int_0^{x_m(0)} \pi_m(s) ds \right| \right] \end{aligned} \tag{196}$$

and

$$\begin{aligned} \|u\|_{\mathcal{L}^\infty} \le & \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i} \\ & \times \left[\sum_{k=0}^{j-1} \frac{k^k}{e^k k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} |x_l(0)| \right. \\ & \left. + \frac{(j-1)^{j-1}}{e^{j-1}(j-1)!} \left| \sum_{m=2}^{n} \int_0^{x_m(0)} \pi_m(s) ds \right| \right] \end{aligned} \tag{197}$$

where $x_i(0)$ are the initial conditions of the state.

Theorem 20: The control law (61) applied to the plant (52)–(58) expends the control effort in the amount bounded by

$$
\|u\|_{\mathcal{L}^1} \le \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i}
$$
$$
\times \sum_{k=0}^{j-1} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k}
$$
$$
\times \left| x_l(0) - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n(0)) x_m(0) \right|
$$
$$(198)$$

and

$$
\|u\|_{\mathcal{L}^\infty} \le \sum_{i=1}^{n} \binom{n}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i}
$$
$$
\times \sum_{k=0}^{j-1} \frac{k^k}{e^k k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k}
$$
$$
\times \left| x_l(0) - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n(0)) x_m(0) \right|
$$
$$(199)$$

where $x_i(0)$ are the initial conditions of the state.

Stabilization by bounded controls is unquestionably a major accomplishment (of [41] and the papers it directly inspired, [2], [7], [18], [19], [21], [22], [43], [45], [46]) especially from the engineering point of view. However, given the character of open loop instability in feedforward systems, it should be less surprising that one can stabilize them with bounded controls than that one can actually indulge in controls with large nonlinear growth (in quest of performance), like those represented by the SJK Lyapunov procedure. For strict-feedback systems, due to their finite escape instabilities, the challenge was to design bounded stabilizing controls [6]. By analogy, for feedforward systems, a (theoretically) worthy future challenge would be to design high-performance controllers with unrestricted nonlinear growth.

XIV. (IN LIEU OF) CONCLUSIONS: MORE ON TYPE I AND II SYSTEMS

How generic, or nongeneric, is linearizability within the class of strict-feedforward systems? It is hard to quantitatively state what "percentage" of feedforward system are linearizable, or, how close (in some metric) a feedforward system is to a linearizable feedforward system. However, it is clear from the results of this paper that one should not expect the majority of feedforward systems to be linearizable.

The Type II class is particularly interesting because of its structural peculiarity (recall (54)–(58)). Based on the third-order case where any linearizable system with $\psi_1(x_2, x_3) \equiv 0$ is of Type II, one might be tempted to conjecture that all linearizable systems (of any order) with $\psi_i(\underline{x}_{i+1}) \equiv 0, i = 1, \dots, n-2$ are of Type II.[8] A fourth-order counter-example to this conjecture is

$$
\dot{x}_1 = x_2 + \left(2x_2^2 x_4 - x_2 x_3^2\right) u \tag{200}
$$
$$
\dot{x}_2 = x_3 \quad \dot{x}_3 = x_4 \quad \dot{x}_4 = u \tag{201}
$$

[8]We remind the reader that $\psi_{n-1}(x_n) \equiv 0$.

which is linearizable via coordinate change

$$
y_1 = x_1 - \left(x_2 x_4 - \frac{x_3^2}{2}\right)^2
$$
$$
y_2 = x_2 \quad y_3 = x_3 \quad y_4 = x_4 \tag{202}
$$

but is not of Type II.

It is not clear at this point what the avenues for possible generalization of the results of this paper might be. The most immediate idea would be to start by exploring the possibilities for combining the systems of Type I and Type II. Theorem 9 does this, at least notationally, for systems of order three. Condition (87) shows actually that these two classes do not mix well, i.e., that Theorem 9 is a concise statement of two results, not a statement for a mixed Type I/II class. However, while mixing is impossible in order three, it is not impossible in higher orders. For example, the fourth-order system

$$
\dot{x}_1 = x_2 + \frac{x_3 a(x_3) - \int_0^{x_3} a(s)ds}{x_3^2} x_2 x_4 + b(x_4) u \tag{203}
$$
$$
\dot{x}_2 = x_3 + a(x_3) x_4 \tag{204}
$$
$$
\dot{x}_3 = x_4 \tag{205}
$$
$$
\dot{x}_4 = u \tag{206}
$$

where $a(\cdot)$ and $b(\cdot)$ are any nonlinearities vanishing at zero (a also must be C^1), is a system that mixes the features of Types I and II and is linearizable via

$$
y_1 = x_1 - \mu(x_3) x_2 + \frac{1}{2} x_3 \left(\mu(x_3)\right)^2
$$
$$
+ \frac{1}{2} \int_0^{x_3} \left(\mu(s)\right)^2 ds - \int_0^{x_4} b(s) ds \tag{207}
$$
$$
y_2 = x_2 - \int_0^{x_3} a(s) ds \tag{208}
$$
$$
y_3 = x_3 \tag{209}
$$
$$
y_4 = x_4 \tag{210}
$$

where $\mu(x_3) = \int_0^{x_3} a(s) ds / x_3$.

As a final comment, we do concede that linearization (by coordinate change) may be viewed as a step backward, if seen as a procedure that eliminates all the nonlinearities—the "harmful," as well as the "useful" ones—and applies controls with high nonlinear growth, in contrast to the nested saturation designs. To clarify what we mean by "useful" nonlinearities in the case where 'feedback linearization' amounts to just a coordinate transformation (without feedback, i.e., without direct cancellation), consider the system from Theorem 8. In Example 2, we presented a case of a harmful nonlinearity that had to be eliminated. However, if $\phi = x_2^2$ (just a sign change), the linear control law $u = -x_1 - 2x_2$, resulting in $\ddot{x}_2 + (2 + x_2^2)\dot{x}_2 + x_2 = 0$, would be more sensible than the linear*izing* control law $u = -x_1 - 2x_2 + x_2^3/3$ resulting in $\ddot{x}_2 + 2\dot{x}_2 + x_2 = 0$. Since the linear control law can be viewed as a close cousin of the nested saturation controllers, we point out that linearization (by forwarding coordinate change) and nested saturation need not be regarded as alternatives. Linearization can be performed first, followed by a nested saturation implementation of a linear controller, like in [42], or various saturation related techniques covered in [9] and [20]. In this way, the performance advantage of the linearizing design would be lost but the robustness to magnitude saturation would be achieved.

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Miroslav Krstic (F'01) received the Ph.D. degree in electrical engineering from the University of California, Santa Barbara (UCSB), under the advisement of Petar Kokotovic.

He is a Professor and Vice Chair in the Department of Mechanical and Aerospace Egineering at University of California (UCSD), San Diego. Prior to moving to UCSD, he was an Assistant Professor in the Department of Mechanical Engineering and the Institute of Systems Research, the University of Maryland, College Park, from 1995 to 1997. He is a coauthor of the books *Nonlinear and Adaptive Control Design* (New York: Wiley, 1995), *Stabilization of Nonlinear Uncertain Systems* (New York: Springer-Verlag, 1998), *Flow Control by Feedback* (New York: Springer-Verlag, 2002), and *Real Time Optimization by Extremum Seeking Control* (New York: Wiley, 2003). He is a coauthor of two patents on control of aeroengine compressors and combustors. His research interests include nonlinear, adaptive, robust, and stochastic control theory for finite-dimensional and distributed parameter systems, and applications to fluid flows and fusion.

Dr. Krstic received the UCSB Best Dissertation Award, the National Science Foundation Career Award, the Office of Naval Research Young Investigator Award, the Presidential Early Career Award for Scientists and Engineers (PECASE), the George S. Axelby Outstanding Paper Award of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and the O. Hugo Schuck Award for the Best Paper at the American Control Conference. He has served as Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, the *International Journal of Adaptive Control and Signal Processing*, *Systems and Control Letters*, and the *Journal for Dynamics of Continuous, Discrete, and Impulsive Systems*. He is a Vice President for Technical Activities and a Member of the Board of Governors of the IEEE Control Systems Society.